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COLLECTIVE DISCUSSIONS FOR THE DEVELOPMENT OF INTERPRETATIVE KNOWLEDGE IN MATHEMATICS TEACHER EDUCATION

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We start from the assumption that teachers need a deep and broad mathematical knowledge —called Interpretative Knowledge (IK)—that allows them to support students in building their mathematical knowledge from their own reasoning and productions. In the present study, we aimed to ascertain how collective discussions focusing on the interpretation of students' productions engage Prospective Teachers (PTs) and impact their IK development. In particular, we observe how this form of collaborative work among PTs allows for the emergence of novel insights into the mathematical aspects of students' productions that were not considered during previous individual work, and produce changes in PTs' attitudes towards students' productions.

Introduction

During class work, mathematics teachers often have to interpret and give meaning to their students' productions. Empirical evidence indicates that the effectiveness of their practice is strongly linked to the quality of these interpretation processes. Indeed, teachers' ability to correctly interpret students' productions can support them in developing class mathematics activities that are based on what students actually know and perceive about mathematics. The "interpretation activity" of students' productions is, therefore, a crucial task for teachers (e.g., Hallman-Thrasher, 2017; Ribeiro, Mellone, & Jakobsen, 2016). It is important to design professional development (PD) program activities around students' productions in order to allow prospective (and in-service) teachers to explore the diversity of students' reasoning in a more reflective context, allowing them to be better prepared to respond to a large (and sometimes unexpected) variety of students' ideas in real and highly interactive classroom settings (e.g., Franke, Carpenter, Fennema, Ansell, & Behrend, 1998; Putnam & Borko, 2000).

In previous research, we have been using tasks containing students' productions to analyze and develop in prospective teachers the so-called *Interpretative Knowledge* (IK, see, for example, Di Martino, Mellone, & Ribeiro, 2019). Following the design research approach (Cobb, Zhao, & Dean, 2009), and adopting the process of continuous revision, we have been exploring the ways of designing and implementing these tasks as a part of training activities included in both initial and continuous PD programs. Indeed, gradually, after several cycles of design and redesign, we chose to focus on developing particular implementations of what we call *interpretative tasks*. As a part of the task, the participants are required to solve a mathematical problem and then, individually and collectively, give meaning and feedback to some students' productions pertaining to the same problem. This initiative has revealed the central role that the collective mathematical discussion among PTs, led by the educators (EDs), can play during and after the implementation of these tasks.

In this study, we focus on the role played by the collective discussions among teachers on their different interpretations of students' productions, aiming to elucidate how these discussions can affect their ability to recognize mathematical aspects and potentialities embodied in students' productions they were unable to see previously. The aim is to support the development of PTs' specialized knowledge, IK in particular. So, our research question is: *How do collective discussions on the interpretation of students' production engage PTs and impact their IK development?*

Theoretical Background

In recent years, taking cue from Shulman's (1986) perspective, the role played by teachers' knowledge in the process of teaching mathematics has been the focus of numerous studies in the teacher education field. For the purpose of meeting the present study aims, we refer to Mathematics Teacher Specialized Knowledge (MTSK, Carrillo et al., 2018) that places teachers' beliefs at the core of the model and considers all the mathematical knowledge for teaching as specialized.

We believe that teachers should possess mathematical knowledge that allows them to support students in building mathematical knowledge from their own productions—including those containing mathematical ambiguities, errors, and non-standard reasoning—assuming that they can be used in the class practice as learning opportunities (Borasi, 1996). Therefore, we state that mathematics teachers should have positive beliefs regarding students' productions and should be able to explore the mathematical potentialities embedded in them. However, in order to achieve this goal, teachers not only need to have a positive attitude toward students' reasoning, but also a broad knowledge of the possible students' strategies, representations, and errors related to a mathematical problem, as this would enable them to make sense of unanticipated solving processes adopted by their students. In this frame, we have been developing the notion of IK (Jakobsen, Ribeiro, & Mellone, 2014), referring to a deep and wide mathematical knowledge that enables teachers to support students in building their mathematical knowledge by starting from their own reasoning and productions (Di Martino et al., 2019). IK includes the ability to expand one's own space of solutions by looking at situations from a wide range of different points of view. Moreover, it includes the capacity for developing specific feedback based on the meaning ascribed to individual students' reasoning (Jakobsen et al., 2014).

During the course of our research, we have studied and measured the level of IK in PTs. We have designed and experimented on different interpretative tasks as a part of various studies, aiming to create effective activities for the development of IK in teacher education (e.g., Jakobsen et al., 2014; Ribeiro et al., 2016). The analysis of the PTs' written answers to the interpretative task allowed us to identify the crucial role that the content knowledge plays in the IK level. However, our findings also revealed that, in isolation, this cannot guarantee a high level of IK. Moreover, our research has indicated that the PTs' individual work on the interpretative is not enough for teachers to develop the IK. As a consequence, following the design study methodology defined as a "family of methodological approaches in which instructional design and research are interdependent" (Cobb et al., 2009, p.169), we designed an Individual−Collective−Individual (ICI) cycle. Thus, after measuring the initial level of PTs' IK through their individual work, all participating PTs were involved in a collective mathematical discussion about the mathematical problem and the students' productions. We used the collective mathematical discussion as a collaborative and knowledge-generating activity, in which students' productions are placed at the center of interpretations and feedback construction (see Cobb et al., 2009; Franke, Carpenter, Levi, & Fennema, 2001).

In mathematics education research, collective discussion is seen as an important tool for constructing knowledge in a social way (e.g., Bartolini Bussi, 1996) and its important role for developing new awareness about errors and nonstandard strategies is also well known in teacher education (Levin, 1995). Spillane (2005) observed that, in mathematics education PD, knowledge is usually gained by attending lectures given by an expert ED, whereby teachers are merely passive listeners, and "when they do talk they ask clarifying questions or acknowledge that they agree or understand" (p. 394). Thus, a further aim of the design and the implementation of the interpretative task (using an ICI cycle) is to disrupt this vicious cycle of teachers being passive listeners, prompting them to assume an active role in their learning—a strategy they should implement when working with their students. Moreover, it is widely established that, when teachers work and learn through collaboration, this can have a crucial positive effects on their practices (Jaworski et al., 2017; Robutti et al., 2016).

In our approach, focusing on the PTs' collective discussion on students' productions related to a specific mathematical problem, the aim is to develop PTs' IK based on social interactions among peers under the guidance of the ED. The basic idea is that the interpretative task involving students' productions, used to measure the PTs' IK level, can stimulate subsequent peer discussions in the teacher group. Owing to its nature and structure, the task should prompt the PTs to develop novel insights into the mathematical aspects of students' productions. Consequently, IK development is transformed from an individual to a collective activity within the teacher group—a transformation characterized by the evolution of community's norms. This evolution is facilitated by the social setting, which is seen as crucial for the development of PTs' IK. The diversity of reasoning, reflecting, and participating in collective discussions of each individual PT represents a resource for the ED who orchestrates collective discussions, in order to identify mathematical and pedagogical issues and develop the IK. The ultimate goal of this process is an evolution of a teacher group into a professional teaching community (Cobb et al., 2009). Cobb et al. (2009) identified four types of norms pertaining to the evolution of a teacher group into a professional teaching community: (a) *norms of general participation*, (b) *norms of pedagogical reasoning,* (c) *norms of mathematical reasoning,* and (d) *norms of institutional reasoning*. The authors further argued that the evolution of norms of one type creates conditions within the teacher group for the evolution of norms of another type:

> For example, shifts that occurred in norms of mathematical reasoning appeared to make possible subsequent developments in general norms of participation. In particular, the norm of challenging others' thinking in mathematics discussions did not emerge until it had become normative for the teachers to develop more sophisticated arguments for justifying their mathematical reasoning. (Cobb at al., 2009, p. 185)

Methodology

Sample

The sample for this study consisted of 34 Master of Mathematics students at an Italian University, who were supposed to have a strong mathematical knowledge, having already completed a Bachelor Degree in Mathematics. All participants were enrolled at a Mathematics Education course, held by one of the authors, which covered several topics, such as number sense, symbol sense, algebraic symbols, etc. As the course is not compulsory, it is typically chosen by students who want to become secondary school teachers. Hence, all study participants are considered PTs.

Task and Activities

The *Interpretative Task*, proposed to this group of PTs by the ED at the beginning of her course, consisted of two parts. Part 1 (Figure 1) involved providing a solution to a problem adapted from the annual Italian National Assessment (2010–2011) for Grade 10 students, released by INVALSI (Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione).

Part I

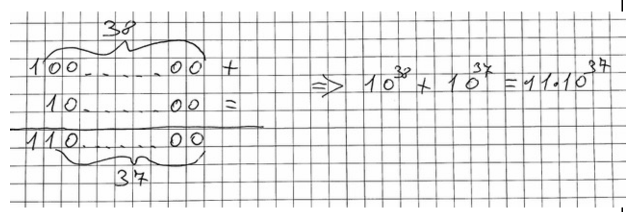
1. Consider the following item taken from the Annual Italian National Assessment (INVALSI) for Grade 10.
The expression. $10^{37}+10^{38}$ is equal to
   A. 20^{75}
   B. 10^{75}
   C. $11\cdot10^{37}$
   D. $10^{37\cdot38}$
2. What is the right answer? Explain why, moreover, discuss why the other ones are not acceptable.

Figure 1: Part I of the Interpretative Task

In Part II (Figure 2), the PTs were asked to interpret seven student productions, chosen by the researchers, as they are deemed suitable for stimulating interesting collective mathematical discussion among the PTs and potentially supporting the development of their IK. Due to the lack of space, only Ciro's production is shown here (for more information, see Mellone, Romano, & Tortora, 2013).

Part II

2. Piera wants to explore with her 9th grade students the above item and she gave the item during a classwork.
   i) For each of the students' productions, indicate if you think it is mathematically correct or not.
   ii) Provide constructive feedback to students, in particular, build possible questions to help them grow in their mathematical knowledge.



Ciro's production

Figure 2: Part II of the Interpretative Task and one of the seven students' productions

For this study, the Interpretative Task involved three activities and was implemented in the *Individual−Collective−Individual* (ICI) cycle format. The first activity lasted 60 minutes and comprised of *Individual Work* by the PTs, who were asked to solve the mathematical problem as well as interpret student productions. The second activity lasted about 90 minutes and involved a *Collective Mathematical Discussion* on the Interpretative Task. It took place in the same class immediately after the individual work and required the PTs to discuss the Interpretative Task, as well as their individual answers to the problem. It was facilitated by the ED, whose aim was to let PTs' different interpretations and perspectives on the students' productions emerge, as well as to prompt the participants to compare their individual productions with those of others in order to identify some mathematical aspects not previously considered. The main phases are: *Opening:* the ED focuses on the topic of discussion with questions such as *What about his production?*; *Dialectic:* the PTs discuss about their interpretations of the students productions, linking them to their own productions, with few interventions of the ED; *Processes Explicitation:* the ED underlines similarities/divergences of solutions/interpretations, and by mirroring techniques (repeating, summarizing or integrating) focus on the error as a resource for building new mathematical knowledge; *Additional Aspects Explicitation*: the ED highlights the emerging mathematical additional aspects. The third activity consisted of requesting that PTs provided an *Individual Extensive Report*, as a part of which they were instructed to describe their experience with the preceding two activities and to give a "final" interpretation of the students' productions. This task took place four weeks after the collective discussion, in order to allow the PTs to reflect on their experience, as well as on the mathematical/pedagogical value of this type of exploration.

Data Collection

The data for the present study was collected in different forms. Specifically, after the individual work, PTs' written answers were collected and were subjected to analysis supporting the design of the collective discussion. The collective discussion was audio-recorded and transcribed, while the PTs' written individual extensive reports were collected, and both sets of data were thematically analyzed.

Analysis

As is often the case in a design research (Cobb et al., 2009), the analysis of data pertaining to each activity allowed us to plan the subsequent activities. In this study, our goal was to understand whether an ICI cycle can be adopted as a methodological approach to support the evolution of the PTs' group in a professional teaching community and the PTs' IK development. In particular, our analyses focused on the effects of norms of general participation, pedagogical reasoning, and mathematical reasoning, and whether these can evolve to support the development of the PTs' IK via the transition from an individual activity to a collective one within the teacher group.

First Activity: Individual Work

Due to limited space, we give only a very brief overview of PTs' individual work. Our analysis revealed that all 34 PTs correctly solved the mathematical problem. However, although the teachers included in our study sample are supposed to have strong mathematical knowledge, the Interpretative Task revealed some gaps in their IK. Indeed, three PTs failed to provide any interpretation of Ciro's production, two PTs considered it incorrect, while 29 PTs' considered it correct. The PTs who considered Ciro's production as incorrect justified this evaluation by stating that a correct production has to incorporate the powers rule, as exemplified by Ornella's comment: "Even if the result is correct the reasoning is wrong because he didn't use the powers rule." Several PTs raised doubts about the range of applicability of Ciro's reasoning. For example, Anna observed: "It is ok, but [what would happen] if the difference between the exponents is larger?".

Moreover, only four PTs provided written feedback. However, even in these cases, their comments were largely limited to a request for a solution by means of the powers rule or algebraic symbols.

Second Activity: Collective Discussion

The analysis pertaining to the second activity focused on the diversity of the teacher group members, of their reasoning and reflections, and of the ways of participating in the collective discussion, orchestrated by the ED, about Ciro's production. In particular, analysis of the discussion transcript revealed presence of steps through which new mathematical awareness is developed as a part of such group interactions. Below, we provide an excerpt exemplifying how PTs' individual interpretations emerge and how these are compared with those of others.

Francesco: He is very visual. He writes each number with the relevant zeros and then he adds them; he doesn't use the properties of the powers.

Antonella: But with other bases different from ten, he can't do that. If for example we have $2^{37}+2^{38}$, he couldn't consider 2 and then a string of zeros. So, it works only in this case.

Samuele: But the request is to argue about this case, not to find a general rule, and he did it. Then, obviously one can prompt him to reflect on the generality or not of his proceeding.

Roberto: Moreover, whatever is the base I can write the number in, this base of representation and the number would appear as "one" followed of a series of many "zeros" [zeri in Italian], so the proceeding doesn't depend on the base.

Educator: What is going on here seems really interesting. Antonella is trying to reflect about the range of applicability of Ciro's proceeding, whereas we are reflecting on the suitability of the chosen representation according to this particular task. In this case, [the question should be] is it better to write the number as decimal alignment or as power? Roberto is prompting us to think that the base of the power is not a limit for Ciro's reasoning because if, for example, we have two powers of 2, if we represent them in base 2 we have the same writings as decimal alignment, and we can use Ciro's strategy. But if we reflect on the range of applicability and the limits of Ciro's proceeding, what can we comment on?

Anna: If the difference between the exponents is very large, it is difficult to use his symbolism and it would be difficult to use it with bases that are not whole.

Educator: Interesting. We are trying to establish the limits of this writing in column with the ellipses in place of the zeros. It has to have some limits in comparison with the compactness and the reification power of the algebraic writing [the ED writes on the board $x^n + x^{n+1} = x^n + x^n x = x^n(x + 1)$]

Francesco: For me it is brilliant because Ciro doesn't apply any rule but he proceeds with his own logic. His uncomfortable, so to speak, representation seems to offer a window into the rules to which we are trying to give sense.

In this excerpt, it is important to highlight the step in which Roberto, who has played a crucial role in the discussion, responds to Antonella's comment regarding the crucial mathematical issue about the generality of Ciro's procedure, by noticing that Ciro's process is also valid when applied to other power bases. This comment reveals that he possesses flexible mathematical knowledge derived from the topic introduced by Antonella. In this sense, we can appreciate the effectiveness of a collective discussion that gives participants the space and time to explore the emerging of issues such as the one that arose in this case, whenever such discussion is designed with such an aim. The norm of challenging others' thinking, put in practice by Roberto, can emerge, as it has become normative for PTs' group to develop more sophisticated arguments to justify their mathematical reasoning. At this point, Anna raises the issue of applicability of Ciro's proceeding when there is a very large difference between the exponents or in case of non-whole bases. However, the ED does not give space to these reflections, because her intention is to lead the discussion toward more meta-mathematical issues about the use of arithmetic representations, in particular the decimal alignment vs. the algebraic ones. The comment following the ED's intervention seems really interesting: Francesco describes Ciro's production as "a window into the rules," pointing toward the possible use of Ciro's production in the class work.

Third Activity: Individual Extensive Report

The analysis at this stage of the study focused on identifying a higher level of PTs' IK, with respect to the first activity, evolution supported by the previous collective discussion. When examining the PTs' extensive reports, we first focused on their "new" interpretations, striving to identify any shifts in their focus of attention. Afterwards, a joint analysis was conducted in order to perceive the changes in PTs' interpretations. Here, we present only a case study regarding the interpretations given by Gennaro. By triangulating information obtained in the course of the three activities, we recognized a radical change in Gennaro's interpretation of Ciro's production. In the following excerpts, denoted as Before (BD) and After (AD) the Discussion, we provide evidence of Gennaro's IK development.

Gennaro (BF): Ciro reached the correct answer by a more practical method than those employed by his peers. In addition, the formalism seems original. He appears to have a strong expertise in the calculations with powers of 10, which highlights their significance and the importance of handling them correctly. Still, his method seems limited to powers of 10. It would be interesting to see how Ciro would proceed if presented with a different base.

Gennaro (AF): Ciro's argument is of an arithmetic character. Nonetheless, it allows us to appreciate some deep algebraic insights. Moreover, although it seems confined to powers of 10, it can actually be generalized to any base, if one represents the number in the base of the power. Hence, from Ciro's production going further, it would be possible to study the tables of operations in different bases, or even the divisibility rules in bases other than 10.

In the BD interpretation, Gennaro acknowledges the originality of Ciro's method, while noting that it is limited to powers with base 10. In the AD comment, Gennaro shifts his focus, whereby his arguments reveal his awareness that Ciro's method can be applied to other bases (indeed, 100…0 always represents the *n*-th power of the base). As indicated in the analysis of the second activity, this fact was noted by Roberto during the collective discussion. For Gennaro, this discovery was so important that it became an essential part of his individual extensive report. Consequently, Gennaro's IK benefitted from the mathematical discussion on Ciro's production. He reconceived the systems of representation of numbers on different bases, motivating him to explore the true meaning of digits, as well as strings of numbers.

Discussion and conclusions

In this study, we aimed to elucidate how an ICI cycle can support the evolution of the PTs' group in a professional teaching community and aid in the development of their IK. Our joint analysis revealed that the ICI cycle adopted in this case promoted an evolution of norms of general participation, as well as pedagogical/mathematical reasoning, supporting the development of the PTs' IK via a transition from an individual activity to collective one within the teacher group.

With respect to the *first activity*, analysis of the individual interpretative task revealed a low level of IK. Indeed, the PTs, in carrying out their individual activities, adopted the norms of pedagogical/ mathematical reasoning (Cobb et al., 2009) based on the "assessment" and not on the "interpretation" of the students' production. They focused on building new mathematical knowledge, asking primarily for a production that is more adherent to their vision of how the task has to be solved, rather than proposing feedback starting from the students' productions. In the context of the *second activity*, we can see a collective development of the PTs' IK. It prompts an evolution of norms of general participation as well as norms of pedagogical/mathematical reasoning in an interdependent way (Cobb et al., 2009). We can also observe in each PT an evolution, with respect to the first activity, of the norms of mathematical/pedagogical reasoning based on what other group members were saying. These changes seem to support an evolution of norms of general participation. Moreover, this collective discussion not only stimulates a deeper reflection on the different students' productions, but also prompts the evolution of norms of general participation, which in turn leads to the evolution of norms of mathematical/pedagogical reasoning, transforming it from evaluative to interpretative. The *third activity* involved institutionalization, as PTs were required to summarize their experience of both individual work and the collective discussion in an Individual Extensive Report, as exemplified by Gennaro's case. In his BD interpretation, we observe the presence of norms of mathematical/pedagogical reasoning based on an evaluative vision about the limits of applicability of Ciro's procedure. The AD interpretation is different, signifying that the evolution of the norms of mathematical/pedagogical reasoning, correlated with the evolution of the norms of general participation (Cobb et al., 2009), has occurred during the collective discussion. The evolution of the norms allowed and supported the development of IK, constructed in a social way through the interactions facilitated by the second activity, based in turn on the first activity.

As future perspective we would like to conduct a longitudinal study evaluating the impact of these PD activities designed around interpretative tasks, on the participating PTs future teaching practices.

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