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Ordering signed graphs with large index

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Abstract

The index of a signed graph is the largest eigenvalue of its adjacency matrix. We establish the first few signed graphs ordered decreasingly by the index in classes of connected signed graphs, connected unbalanced signed graphs and complete signed graphs with a fixed number of vertices.

Keywords: Adjacency matrix, largest eigenvalue, edge relocation, unbalanced signed graph, complete signed graph.

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1 Introduction

A *signed graph* $\dot{G}$ is a pair (G, σ), where $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{1, -1\}$ is the *sign function* or the *signature*. The number of vertices of $\dot{G}$ is called the *order* and denoted by n. The edge set of $\dot{G}$ is composed of the subset E^+ of positive edges and the subset E^- of negative edges. We interpret an unsigned graph as a signed graph with the all positive signature, that is the signature which assigns 1 to every edge.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is obtained from the standard adjacency matrix of its underlying graph by switching the sign of all 1's which correspond to negative edges. The *eigenvalues* of $\dot{G}$ are identified to be the eigenvalues of its adjacency matrix; they form

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the *spectrum* of $\dot{G}$. The largest eigenvalue of $\dot{G}$ is called the *index* and denoted by λ_1 (or $\lambda_1(\dot{G})$).

If S is a set of vertices of $\dot{G}$, the switched signed graph $\dot{G}^S$ is obtained from $\dot{G}$ by reversing the signs of the edges in the cut $[S, V(\dot{G}) \setminus S]$. The signed graphs $\dot{G}$ and $\dot{G}^S$ are said to be *switching equivalent*. The switching equivalence is an equivalence relation that preserves the eigenvalues, and the *switching class* of $\dot{G}$ is denoted by $[\dot{G}]$.

A signed graph is said to be *balanced* if it switches to the signed graph with all positive signature. Otherwise, it is said to be *unbalanced*. Equivalently, $\dot{G}$ is balanced if every cycle contained in $\dot{G}$ is balanced [15].

Ordering of unsigned graphs by the largest eigenvalue of some associated matrix has received a great deal of attention in literature. Many results can be found in [12]. More recently, there has been a growing interest for extremal problems in the framework of signed graphs. For instance, in [9] Koledin and the second author studied connected signed graphs of fixed order, size and number of negative edges that maximize the index. In the wake of that paper, signed graph maximizing the index in suitable subsets of complete signed graphs have been studied in [2]. Let $\mathfrak{U}_n$ (resp. $\mathfrak{B}_n$) denote the class of unbalanced unicyclic (resp. bicyclic) signed graphs of order n. Akbari et al. [1] determined the signed graphs attaining the extremal indices in $\mathfrak{U}_n$. Some of the same authors studied in [10] signed graphs achieving the maximum index among signed graphs in $\mathfrak{U}_n$ of fixed girth. The first five largest indices among signed graphs in $\mathfrak{B}_n$ with $n \geqslant 36$ are detected by He et al. [8]. Signed graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with extremal spectral radius were identified in [4]. Finally, extremal graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with respect to the least Laplacian eigenvalue were studied in [5] and [3], respectively.

In [6] we determined the unbalanced signed graph with largest index, for every order n. In this paper we continue this research by presenting a general method for ordering the signed graphs with a fixed number of vertices by their index. We demonstrate the method by determining the first few signed graphs ordered by the index in the class of connected signed graphs, or connected unbalanced signed graphs, or complete signed graphs with n vertices.

The paper is organized as follows. Section 2 contains a preliminary setting related to the graphical representations of signed graphs in this paper along with terminology, notation, a few known results and the proofs of two preliminary lemmas. The main result that provides the subsequent orderings is formulated in Theorem 3.2 of Section 3. Orderings in the mentioned classes are considered in Sections 3–5. Further computations, including orderings of signed graphs with a comparatively small number of vertices, are given in Section 6.

2 Preparatory

We introduce a way of depicting signed graphs that will be used in the subsequent sections. For a signed graph of order n, we draw only the negative edges and the non-edges, along with the assumption that all non-depicted edges are positive. By convention, a negative edge is represented by a full line and a non-edge is represented by a dotted line. Accordingly, the complete signed graph with the all positive signature (i.e. the complete unsigned graph) is represented by an empty figure, the complete signed graph with a single negative edge is represented by a negative edge, and so on.

The following lemmas are taken from [13, 14].

Lemma 2.1 ([14]). *For a connected signed graph $\dot{G} = (G, \sigma)$, we have $\lambda_1(\dot{G}) \leq \lambda_1(G)$ with equality if and only if $\dot{G}$ switches to G.*

Lemma 2.2 ([13]). *For an eigenvalue λ of a signed graph $\dot{G}$, there is a switching equivalent signed graph for which the λ-eigenspace contains an eigenvector whose non-zero coordinates are of the same sign.*

For the sake of completeness we say that a signed graph of the previous lemma can be constructed by taking $\dot{G}$ with $A_{\dot{G}}\mathbf{x} = \lambda\mathbf{x}$ and considering $D^{-1}A_{\dot{G}}D$ where D is the diagonal matrix of ± 1s whose negative entries correspond to negative coordinates of $\mathbf{x}$.

We proceed with some notation. For a signed graph $\dot{G}$ we denote by $\mathcal{R}(\dot{G})$ the set of signed graphs obtained by taking a positive edge e of some signed graph of the switching class $[\dot{G}]$, and then either removing e or reversing its sign.

Let $\mathcal{S} = (\dot{G}_1, \dot{G}_2, \dots, \dot{G}_g)$ be a sequence which consists of the representatives of all switching equivalence classes of connected signed graphs with n vertices such that the representatives are ordered non-increasingly by the index and chosen in such a way that, for $1 \leq i \leq g$, the λ_1-eigenspace of $\dot{G}_i$ contains an eigenvector whose non-zero coordinates are positive. (The existence of $\dot{G}_i$ is provided by Lemma 2.2.)

We now prove the following lemmas. They generalize known results for unsigned graphs that can be found in [12, Lemma 1.28].

Lemma 2.3 (**Changing an edge**). *Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ be an eigenvector associated with the index of a signed graph $\dot{G}$ and let r, s be fixed vertices of $\dot{G}$.*

(i) *If $x_r x_s \geq 0$ and rs is a non-edge (resp. rs is a negative edge), then for a signed graph $\dot{G}'$ obtained by inserting a positive edge between r and s (resp. deleting rs or reversing its sign) we have $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. If at least one of x_r, x_s is non-zero, the previous inequality is strict.*

(ii) *If $x_r x_s < 0$ and rs is a non-edge (resp. rs is a positive edge), then for a signed graph $\dot{G}'$ obtained by inserting a negative edge between r and s (resp. deleting rs or reversing its sign) we have $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

Proof. We only demonstrate the proof of (i), as (ii) is proved analogously. If $\mathbf{y}$ is an eigenvector associated with $\lambda_1(\dot{G}')$, using the Rayleigh principle we get

$$\begin{aligned}
&\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \\
&\quad = \mathbf{y}^\intercal A_{\dot{G}'}\mathbf{y} - \mathbf{x}^\intercal A_{\dot{G}}\mathbf{x} \geq \mathbf{x}^\intercal A_{\dot{G}'}\mathbf{x} - \mathbf{x}^\intercal A_{\dot{G}}\mathbf{x} = \mathbf{x}^\intercal (A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} \\
&\quad = \begin{cases} 2x_r x_s & \text{if } \big(rs \notin E(\dot{G}) \wedge rs \in E^+(\dot{G}')\big) \vee \big(rs \in E^-(\dot{G}) \wedge rs \notin E(\dot{G}')\big), \\ 4x_r x_s & \text{if } rs \in E^-(\dot{G}) \wedge rs \in E^+(\dot{G}'). \end{cases}
\end{aligned}$$

Hence, $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$.

Assume that $x_r \neq 0$ and, by way of contradiction, that $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$. In this case, the inequality in the previous chain reduces to equality, which means that $\mathbf{x}$ is an eigenvector afforded by $\lambda_1(\dot{G}')$. Using the eigenvalue equations at vertex s in $\dot{G}$ and $\dot{G}'$, we get

$$0 = \big(\lambda_1(\dot{G}') - \lambda_1(\dot{G})\big)x_s = \sum_{i\,:\,is \in E(\dot{G}')} \sigma_{\dot{G}'}(is)x_i - \sum_{i\,:\,is \in E(\dot{G})} \sigma_{\dot{G}}(is)x_i = \alpha x_r, \tag{2.1}$$

where α depends on (r,s)-entries in $A_{\dot{G}}$ and $A_{\dot{G}'}$, but it is always non-zero; for example, if $rs \notin E(\dot{G}) \wedge rs \in E^+(\dot{G}')$ then $\alpha = \sigma_{\dot{G}'}(rs) = 1$, and similarly for the remaining possibilities listed in the statement formulation. Together with (2.1), this leads to $x_r = 0$, which in turn contradicts the initial assumption and we are done. □

Let r, s, t, u be fixed vertices of a signed graph. A relocation $\mathrm{Rot}(r,s,t)$ (called a *rotation*) is realised in the adjacency matrix by replacing the entries a_{rs}, a_{sr} with the entries a_{rt}, a_{tr}, and vice versa. In simple words, this relocation is realised by taking the object (which can be a positive edge, or a negative edge, or a non-edge) located between r and s and the object located between r and t and then inserting the first object between r and t and the second object between r and s.

A relocation $\mathrm{Shift}(r,s,t,u)$ (called a *shifting*) is realised in the adjacency matrix by replacing the entries a_{rs}, a_{sr} with a_{tu}, a_{ut}, and vice versa.

Lemma 2.4 (Rotation and shifting). *Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ be an eigenvector associated with the index of a signed graph $\dot{G}$ and let r, s, t, u be fixed vertices of $\dot{G}$.*

(i) *Let $\dot{G}'$ be obtained from $\dot{G}$ by the relocation $\mathrm{Rot}(r,s,t)$. If $\big(x_r(x_s - x_t) > 0 \vee (x_s = x_t \wedge x_r \neq 0)\big) \wedge \big((rs$ is a non-edge $\wedge$ rt is a positive edge$) \vee (rs$ is a negative edge and rt is a positive edge$) \vee (rs$ is a negative edge and rt is a non-edge$)\big)$ then $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

(ii) *Let $\dot{G}'$ be obtained from $\dot{G}$ by the relocation $\mathrm{Shift}(r,s,t,u)$. If $\big(x_t x_u > x_r x_s \vee (x_t x_u = x_r x_s \wedge$ at least one of x_r, x_s, x_t, x_u is non-zero$)\big) \wedge \big((rs$ is a positive edge $\wedge$ tu is a non-edge$) \vee (rs$ is a positive edge and tu is a negative edge$) \vee (rs$ is a non-edge and tu is a negative edge$)\big)$ then $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$.*

Proof. This proof is similar to the proof of the previous lemma. If rs and rt are as in (i), then we compute

$$\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \geq \mathbf{x}^\intercal (A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} = 2\alpha x_r(x_s - x_t),$$

where $\alpha = 1$ for the first and the third assumption on rs and rt, and $\alpha = 2$ for the second assumption. Now, for $x_r(x_s - x_t) > 0$ we get $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$. For $x_s = x_t$ we have $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. In case of equality, we have that $\mathbf{x}$ is afforded by $\lambda_1(\dot{G}')$. Considering the eigenvalue equation at the vertex s in $\dot{G}$ and $\dot{G}'$, we get $x_r = 0$, which completes (i).

If rs and rt are as in (ii), then we compute

$$\lambda_1(\dot{G}') - \lambda_1(\dot{G}) \geq \mathbf{x}^\intercal (A_{\dot{G}'} - A_{\dot{G}})\mathbf{x} = 2\alpha(x_t x_u - x_r x_s),$$

with $\alpha \in \{1, 2\}$, as before. We are done for $x_t x_u > x_r x_s$, while for $x_t x_u = x_r x_s$, using the previous reasoning we get that the equality between the indices necessarily leads to the conclusion that $\mathbf{x}$ takes zero at the corresponding four vertices. □

3 Ordering signed graphs by the index

We start our considerations with an example.

Example 3.1. Clearly, there are just 3 connected signed graphs of order 3, up to switching: the positive triangle (with index 2), the 3-vertex path (with index $\sqrt{2}$) and the negative triangle (with index 1). We know from [11] that there are exactly 12 connected signed graphs of order 4 (again, up to switching). Their indices are computed directly, and the corresponding ordering is given in Figure 1.

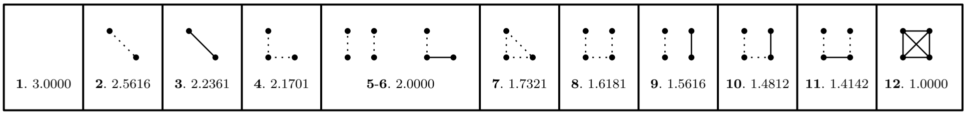


Figure 1: Connected signed graphs with 4 vertices ordered by the index. Here and in the subsequent graphical representations, signed graphs are depicted according to the convention explained in Section 2.

In what follows, we determine the first 5 connected signed graphs with n vertices ordered by the index, for every $n \geq 5$. In other words, we determine the signed graphs $\dot{G}_1 - \dot{G}_5$ of the sequence $\mathcal{S}$ defined in the previous section. First, $\dot{G}_1$ is the complete signed graph with the all positive signature, which follows from the well-known Perron-Frobenius Theorem and Lemma 2.1. We now prove the following theorem, crucial for our considerations.

Theorem 3.2. *Let $\mathcal{S}' = (\dot{G}_{k+1}, \dot{G}_{k+2}, \dots, \dot{G}_{k+\ell})$ be a subsequence of $\mathcal{S}$ such that*

$$\lambda_1(\dot{G}_k) > \lambda_1(\dot{G}_{k+1}) = \lambda_1(\dot{G}_{k+2}) = \cdots = \lambda_1(\dot{G}_{k+\ell}) > \lambda_1(\dot{G}_{k+\ell+1}). \tag{3.1}$$

Then, for every $\dot{G} \in \mathcal{S}'$, we have $\dot{G} \in \mathcal{R}(\dot{H})$ where $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_{k+\ell}\} \setminus \dot{G}$.

In addition:

(a) *For at least one $\dot{G} \in \mathcal{S}'$ we have $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$;*

(b) *If $\dot{H} \notin \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$ then a non-negative λ_1-eigenvector for $\dot{G}$ has at least two zero coordinates and the same eigenvector is afforded by $\lambda_1(\dot{H})$.*

Proof. Assume by way of contradiction that for some $\dot{G} \in \mathcal{S}'$, $\dot{G} \notin \mathcal{R}(\dot{H})$, for every $\dot{H}$ that belongs to the set given in the statement formulation. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\intercal$ be an eigenvector with non-negative coordinates afforded by the index of $\dot{G}$.

Assume that at least one of x_r, x_s is non-zero for some vertices r, s of $\dot{G}$. If rs is not a positive edge, then Lemma 2.3(i) produces a signed graph $\dot{G}'$ that differs from $\dot{G}$ only in the positive edge rs, along with $\lambda_1(\dot{G}') > \lambda_1(\dot{G})$. Since $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we have $\dot{G}' \notin \bigcup_{i=1}^{k} [\dot{G}_i]$, i.e. there are at least $k+1$ signed graphs whose index is larger than $\lambda_1(\dot{G})$, which contradicts (3.1). Hence, rs is a positive edge.

Further, if $x_r = x_s = 0$, then reversing the sign of rs, or removing rs, or adding rs do not affect the existence of the corresponding eigenvalue; indeed, it is afforded by the same eigenvector. Thus if for any such r, s there is no positive edge between them, as before we get $\dot{G}'$ with $\lambda_1(\dot{G}') \geq \lambda_1(\dot{G})$. Since $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we have $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$, but then $\dot{G} \in \bigcup_{i=1}^{\ell} \mathcal{R}(\dot{G}_{k+i})$ which together with the initial assumption leads to $\dot{G} \in \mathcal{R}(\dot{G})$, i.e. $\dot{G}$ is isomorphic to the signed graph obtained by inserting a positive edge rs. Replacing $\dot{G}$ with this signed graph we get that rs is positive.

Amalgamating the previous conclusions we get that $\dot{G}$ is the complete signed graph with the all positive signature, which together with Lemma 2.1 contradicts (3.1) (since we assumed in (3.1) that $\dot{G}$ is not $\dot{G}_1$).

Consider now (a). Take an arbitrary $\dot{G} \in \mathcal{S}'$. If $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$, we are done. Assume that $\dot{G} \notin \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. If $\mathbf{x}$ is the previously defined eigenvector, then there is a positive edge rs for every pair r, s such that at least one of x_r, x_s is non-zero, as otherwise by inserting a positive edge between such vertices we get $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. If at most one coordinate of $\mathbf{x}$ is zero, then $\dot{G}$ switches to a complete unsigned graph, which contradicts (3.1). Therefore there exist at least 2 vertices at which $\mathbf{x}$ takes zero. In addition, there is a negative edge between at least one such a pair, since otherwise $\dot{G}$ has the all positive signature and then $\mathbf{x}$ has no zero coordinates by the Perron-Frobenius Theorem. If $\dot{G}'$ is obtained by switching the sign of such a negative edge, then $\lambda_1(\dot{G}') = \lambda_1(\dot{G})$, as otherwise we get $\dot{G} \in \bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. Moreover, $\lambda_1(\dot{G}')$ is afforded by the same eigenvector, so we may repeat the previous consideration with $\dot{G}'$ in the role of $\dot{G}$. In this way we necessarily arrive at some $\dot{G} \in \mathcal{S}' \cap \left(\bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)\right)$ since the number of negative edges strictly decreases in passing from $\dot{G}$ to $\dot{G}'$.

It remains to consider (b). Let $\dot{G} \in \mathcal{R}(\dot{H})$. If at most one coordinate of $\mathbf{x}$ is zero, then $\lambda_1(\dot{H}) > \lambda_1(\dot{G})$ (by Lemma 2.3(i)), which implies $\dot{H} \in \{\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k\}$. Further, the assumption that $\dot{G} \in \mathcal{R}(\dot{H})$ together with $\lambda_1(\dot{H}) = \lambda_1(\dot{G})$ leads to the conclusion that $x_r = x_s = 0$ for a non-positive (resp. positive) edge rs in $\dot{G}$ (resp. $\dot{H}$), and thus $A_{\dot{H}}\mathbf{x} = A_{\dot{G}}\mathbf{x} = \lambda_1(\dot{H})\mathbf{x}$. □

Remark 3.3. Theorem 3.2 gives a method for the ordering of signed graphs by the index. Let $\dot{G}_1, \dot{G}_2, \dots, \dot{G}_k$ be the first k signed graphs ordered by the index such that all signed graphs (if any) sharing the index with $\dot{G}_k$ are listed before it (so, as in the theorem). Then the sequence is extended by the signed graph(s) belonging to $\bigcup_{i=1}^{k} \mathcal{R}(\dot{G}_i)$. The candidates must be connected and the λ_1-eigenspace for each of them must contain an eigenvector with non-negative coordinates. They are compared on the basis of an algebraic computation that relies on Lemmas 2.3 and 2.4. As long as we deal with signed graphs whose λ_1-eigenspaces do not contain an eigenvector with at least two zero coordinates, there are no other candidates. In case of such eigenvectors, signed graphs with equal indices sharing the same eigenvector may appear.

Remark 3.4. To determine the set $\mathcal{R}(\dot{G})$ we need to consider the entire switching class of $\dot{G}$. For example, if $\dot{G}$ is the complete signed graph with exactly one negative edge, say e, then the signed graphs obtained from $\dot{G}$ by removing a positive edge or reversing its sign are the following 4:

By making a switch at a vertex incident with e and either removing e or reversing its sign, we get 2 additional members of $\mathcal{R}(\dot{G})$ that are not switching equivalent to the previous ones. But in both cases the λ_1-eigenspace does not contain a non-negative eigenvector (the condition required in Remark 3.3). A method for computing the λ_1-eigenvectors is demonstrated in the proof of the forthcoming Lemma 3.5.

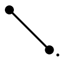
Now we proceed with the ordering.

Lemma 3.5. *$\dot{G}_2$ is* .

Proof. There are exactly two candidates for $\dot{G}_2$: $\dot{F}$ obtained by removing an edge of $\dot{G}_1$ and $\dot{H}$ obtained by reversing the sign of an edge of $\dot{G}_1$. If the vertices joined by the unique negative edge of $\dot{H}$ are labelled by 1 and 2, using the eigenvalue equation for $\lambda_1(\dot{H})$ we get

$$\begin{aligned} \lambda_1 a &= -a + (n-2)b \\ \lambda_1 b &= \ \ 2a + (n-3)b \end{aligned}$$

which leads to the $\lambda_1(\dot{H})$-eigenvector $b(\frac{\lambda_1+1}{\lambda_1+3}, \frac{\lambda_1+1}{\lambda_1+3}, 1, 1, \ldots, 1)^\intercal$, $b \neq 0$. By virtue of Lemma 2.3(i) (applied to $\dot{H}$), we have $\lambda_1(\dot{F}) > \lambda_1(\dot{H})$. Hence $\dot{G}_2 \cong \dot{F}$. □

Lemma 3.6. *$\dot{G}_3$ is* .

Proof. The candidates for $\dot{G}_3$ are illustrated in Figure 2. They are obtained by considering $\mathcal{R}(\dot{G}_1) \cup \mathcal{R}(\dot{G}_2)$; we also include the transposes of the corresponding positive eigenvectors afforded by the index.

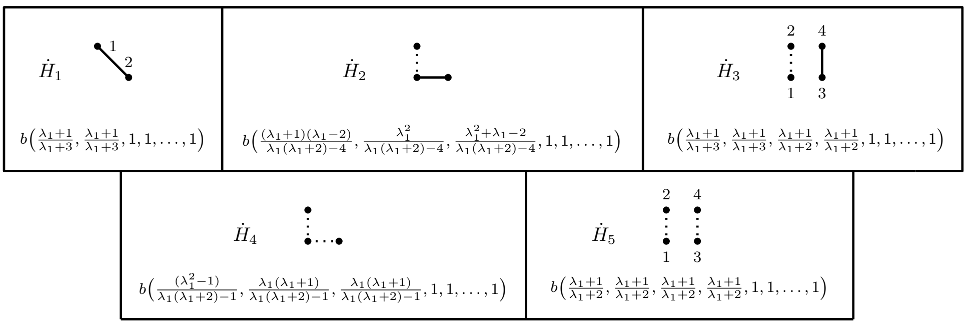


Figure 2: The candidates for $\dot{G}_3$.

From Lemma 2.3(i), we have $\lambda_1(\dot{H}_1) > \max\{\lambda_1(\dot{H}_2), \lambda_1(\dot{H}_3)\}$. We further apply Lemma 2.4(i) to $\dot{H}_5$ with $(r, s, t) = (1, 2, 3)$ to conclude that $\lambda_1(\dot{H}_4) > \lambda_1(\dot{H}_5)$. To show that $\dot{G}_3 \cong \dot{H}_1$ it remains prove that $\lambda_1(\dot{H}_1) > \lambda_1(\dot{H}_4)$. The adjacency matrix of $\dot{H}_1$ is

$$A_{\dot{H}_1} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} & J^\intercal \\ J & A_{K_{n-2}} \end{pmatrix},$$

(where J is the all-1 matrix) which leads to the quotient matrix (i.e. the matrix of row sums in the corresponding blocks of $A_{\dot{H}_1}$):

$$Q_{\dot{H}_1} = \begin{pmatrix} -1 & n-2 \\ 2 & n-3 \end{pmatrix}.$$

We know from [7] that every eigenvalue whose eigenspace does not contain an eigenvector orthogonal to the all-1 vector $\mathbf{j}$ belongs to the spectrum of the quotient matrix. In our case, this means that $\lambda_1(\dot{H}_1)$ is an eigenvalue of $Q_{\dot{H}_1}$, i.e. $\lambda_1(\dot{H}_1)$ is the largest root of

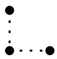
$$x^2 + (4-n)x - 3n + 7. \tag{3.2}$$

In the same way, we get that $\lambda_1(\dot{H}_4)$ is the largest root of

$$f(x) = x^3 - (n-3)x^2 - (2n-5)x + n - 3. \tag{3.3}$$

Now, computing the largest root of (3.2) we get $\lambda_1(\dot{H}_1) = \frac{n+\sqrt{(n-2)(n+6)}}{2} - 2$. Inserting it in (3.3), we get $f\big(\frac{n+\sqrt{(n-2)(n+6)}}{2} - 2\big) = \sqrt{(n-2)(n+6)} - n > 0$, which leads to the conclusion that either $\lambda_1(\dot{H}_1) > \lambda_1(\dot{H}_4)$ or the two roots of f are larger than $\lambda_1(\dot{H}_1)$. The latter is not true because $f(0) > 0$, $f(1) < 0$ which means that f has a negative root and a root in $(0, 1)$. □

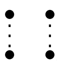
To avoid repetitive proofs, in the remainder of this section and the next two sections we omit the parts in which we compute the λ_1-eigenvectors of potential candidates since these are technical algebraic computations performed in exactly the same way as in the previous proof.

Lemma 3.7. *$\dot{G}_4$ is* .

Proof. The 6 candidates for $\dot{G}_4$ are (those of Figure 2 that have not passed for $\dot{G}_3$ are included, of course):

We note that there are two additional members of $\mathcal{R}(\dot{G}_3)$ mentioned in Remark 3.3, but the non-existence of a required λ_1-eigenvector eliminates them.

The 3rd and the 5th candidate are eliminated since their indices are dominated by the index of the 1st one, while the 4th and the 6th are eliminated by the 2nd one in the same way – all this by virtue of Lemma 2.3(i). Finally, the fact that the index of the 1st signed graph is larger than that of the 2nd one is established in the proof of Lemma 3.6, and we are done. □

Lemma 3.8. *$\dot{G}_5$ is* .

Proof. The candidates for $\dot{G}_5$ are the 5 signed graphs that are eliminated in the proof of the previous lemma (when we considered $\dot{G}_4$) and the following 8 signed graphs:

Lemma 2.3(i) eliminates all except the 2nd and the 3rd of the previous proof; in Figure 2 they are denoted by $\dot{H}_5$ and $\dot{H}_2$) and the 1st and the 3rd of the additional candidates (we denote them by $\dot{F}_1$ and $\dot{F}_2$). By Lemma 2.4, $\lambda_1(\dot{F}_2)$ dominates $\lambda_1(\dot{F}_1)$; we already had this in the proof of Lemma 3.6.

Thus, it remains to prove that $\lambda_1(\dot{H}_5) > \max\{\lambda_1(\dot{H}_2), \lambda_1(\dot{F}_2)\}$. As in the proof of Lemma 3.6 we deduce that these indices are the largest roots of characteristic polynomials of the corresponding quotient matrices. These polynomials are:

$$h_5(x) = x^2 - (n-3)x - 2(n+3)$$
$$h_2(x) = x^4 - (n-4)x^3 - (3n-7)x^2 + 2(n-4)x + 4(n-3)$$
$$f_2(x) = x^3 - (n-3)x^2 - 2(n-3)x + 2(n-4)$$

The largest root of h_5 is $\frac{1}{2}(n-3+\sqrt{(n-3)(n+5)})$. Concerning h_2 we get $h_2(-4) = 12(n+11) > 0$, $h_2(-2) = -4(n-4) < 0$, $h_2(0) = 4(n-3) > 0$ and $h_2(n-2) = -(n-1)(n-4)^2 < 0$, which together with $n-2 < \lambda_1(\dot{H}_5)$ leads to the conclusion that at least 3 roots of h_2 are less than $\lambda_1(\dot{H}_5)$. Since $h_2(\lambda_1(\dot{H}_5)) = (n-4)(n-3+\sqrt{(n-3)(n+5)}) > 0$, we conclude that the fourth root of h_2 is also less than $\lambda_1(\dot{H}_5)$.

Similarly, we have $f_2(-3) = -n-26 < 0$, $f_2(0) = 2(n-4) > 0$ and $f_2(1) = 2-n < 0$, which means that that two roots of f_2 are less than $\lambda_1(\dot{H}_5)$, while $f_2(\lambda_1(\dot{H}_5)) = 2(n-4) > 0$ confirms the same for the third root, and we are done. □

Amalgamating the previous results we arrive at the following theorem.

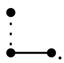
Theorem 3.9. *The first* 5 *connected signed graphs with* $n \geq 5$ *vertices ordered by their indices are:*

$\dot{G}_1$ $\dot{G}_2$ $\dot{G}_3$ $\dot{G}_4$ $\dot{G}_5$

4 Unbalanced signed graphs

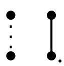
Let now $(\dot{U}_1, \dot{U}_2, \dots, \dot{U}_u)$ be the subsequence of $\mathcal{S}$ (defined in Section 2) containing only unbalanced signed graphs. In other words, the previous sequence ignores the balanced ones. In what follows, we determine $\dot{U}_1 - \dot{U}_4$ for $n \geq 6$.

We know from [6] that $\dot{U}_1$ is obtained by reversing the sign of a single edge in the complete graph of order n.

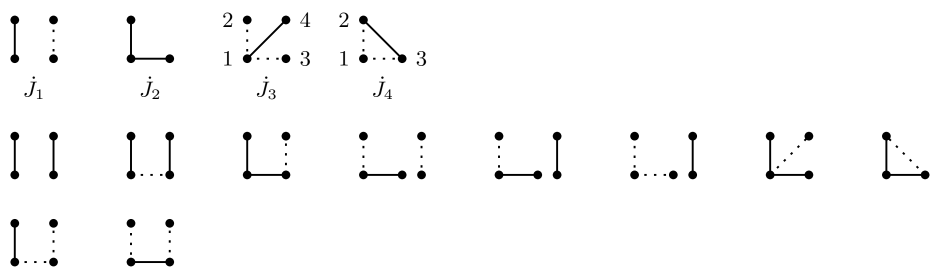
Lemma 4.1. $\dot{U}_2$ *is* .

Proof. The candidates for $\dot{U}_2$ are the last 4 signed graphs considered as the candidates in the proof of Lemma 3.7. (As before, it is not complicated to show that these are the only candidates with positive λ_1-eigenvectors).

The latter two candidates are eliminated by Lemma 2.3(i) – they are dominated by the 1st candidate. Observe that the λ_1-eigenvector for the 2nd candidate is given in Figure 2. Using the same vertex labelling and applying the relocation $\mathrm{Rot}(3,4,1)$ we arrive at the result formulated in this statement. □

Lemma 4.2. $\dot{U}_3$ *is* .

Proof. The candidates for $\dot{U}_3$ are the following 14 signed graphs:



All in the second row are easily eliminated on the basis of Lemma 2.3(i). The two in the third row are eliminated by Lemma 2.4(i). Namely, if we denote the vertices in the representing path by $1, 2, 3, 4$ (in the natural order) and if $x_2 \geq x_3$, then $\mathrm{Rot}(1, 2, 3)$ implies that the index of the signed graph under consideration is less than that of $\dot{J}_3$. Otherwise, we can apply $\mathrm{Rot}(4, 3, 2)$ with the same result.

It remains to consider the indices of $\dot{J}_1 - \dot{J}_4$. We first show that $\lambda_1(\dot{J}_2) > \max\{\lambda_1(\dot{J}_3), \lambda_1(\dot{J}_4)\}$. As in the proof of Lemma 3.5 we can show that $\dot{J}_3$ and $\dot{J}_4$ have a positive λ_1-eigenvector. (Namely, we compute $b\big(\frac{(\lambda_1+1)(\lambda_1-3)}{\lambda_1(\lambda_1+2)-5}, \frac{(\lambda_1^2+\lambda_1-2)}{\lambda_1(\lambda_1+2)-5}, \frac{(\lambda_1^2+\lambda_1-2)}{\lambda_1(\lambda_1+2)-5}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-5}, 1, 1, \dots, 1\big)^\intercal$ for $\dot{J}_3$ which is positive for every $b > 0$, as $\lambda_1 > 3$ when $n \geq 6$. Similarly, we get $b\big(\frac{(\lambda_1+1)^2}{\lambda_1(\lambda_1+4)+1}, \frac{\lambda_1(\lambda_1+1)}{\lambda_1(\lambda_1+4)+1}, \frac{\lambda_1(\lambda_1+1)}{\lambda_1(\lambda_1+4)+1}, 1, 1, \dots, 1\big)^\intercal$ for $\dot{J}_4$, which is positive for $b > 0$, as well.) Set $\dot{J}_* \in \{\dot{J}_3, \dot{J}_4\}$, and let $\mathbf{x}$ be a positive eigenvector afforded by $\lambda_1(\dot{J}_*)$. Observe that $\dot{J}_2$ is obtained by inserting a positive edge 12 and a negative edge 13 in $\dot{J}_*$. Therefore, we have

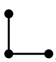
$$\lambda_1(\dot{J}_2) - \lambda_1(\dot{J}_*) \geq \mathbf{x}^\intercal(A_{\dot{J}_2} - A_{\dot{J}_*})\mathbf{x} = 2(x_1x_2 - x_1x_3) = 0,$$

where the last equality follows since $x_2 = x_3$ (by the symmetry in $\dot{J}_*$). Hence, $\lambda_1(\dot{J}_2) \geq \lambda_1(\dot{J}_*)$. If $\lambda_1(\dot{J}_2) = \lambda_1(\dot{J}_*)$, then $\mathbf{x}$ is afforded by $\lambda_1(\dot{J}_2)$, but this is impossible since the eigenvalue equation at the vertex 2 cannot hold in $\dot{J}_2$ and $\dot{J}_*$.

Characteristic polynomials of quotient matrices of $\dot{J}_1$ and $\dot{J}_2$ are:

$$\begin{aligned} j_1(x) &= x^3 - (n-6)x^2 - (5n-17)x - 6n + 20 \\ j_2(x) &= x^3 - (n-3)x^2 - (2n-3)x + 7n - 23 \end{aligned}$$

We compute $j(x) = j_1(x) - j_2(x) = 3x^2 - (3n-14)x - 13n + 43$, with roots: $x_1, x_2 = \frac{1}{6}(3n - 14 \pm \sqrt{9n(8+n) - 320})$. It follows that $j_1(x) < j_2(x)$ for $x \in (x_1, x_2)$. For the larger root x_2 we have $j_1(x_2) = j_2(x_2) = \frac{1}{27}\big((3n-16)\sqrt{9n(8+n) - 320} + 9n^2 - 96n + 248\big) > 0$ where the inequality follows since $9n^2 - 96n + 248 > 0$ for $n \geq 7$, while for $n = 6$ it is confirmed directly. Taking into account that x_1 is negative (the easiest way to see this is to compute $j(0)$), we conclude that $\lambda_1(\dot{J}_1), \lambda_1(\dot{J}_2) \in (x_1, x_2)$. Together with $j_1(x) < j_2(x)$ on the same interval, this leads to $\lambda_1(\dot{J}_1) > \lambda_1(\dot{J}_2)$. □

Lemma 4.3. *$\dot{U}_4$ is* .

Proof. Besides the 13 signed graphs listed in the previous lemma, we have other 4 candidates for $\dot{U}_4$ (that arise from $\dot{U}_3$ but not from $\dot{U}_1$ or $\dot{U}_2$):

The former two are eliminated by Lemma 2.3(i), the latter two by Lemma 2.4(i). Therefore, it remains to consider $\dot{J}_2 - \dot{J}_4$, but they have been already considered in the proof of the previous lemma, when we proved that $\lambda_1(\dot{J}_2) > \max\{\lambda_1(\dot{J}_3), \lambda_1(\dot{J}_4)\}$, as desired. □

The previous results lead to the following theorem.

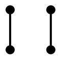
Theorem 4.4. *The first 4 connected unbalanced signed graphs with $n \geq 6$ vertices ordered by their indices are:*

$\dot{U}_1$ $\dot{U}_2$ $\dot{U}_3$ $\dot{U}_4$

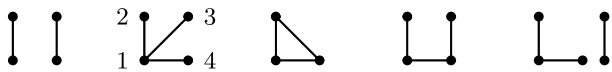
5 Complete signed graphs

As before, let $(\dot{C}_1, \dot{C}_1, \ldots, \dot{C}_c)$ be the subsequence of $\mathcal{S}$ containing complete signed graphs. Clearly, the complete signed graph with the largest index switches to the one with the all positive signature. The next one contains exactly one negative edge. There are 2 candidates for $\dot{C}_3$, both with 2 negative edges. By Lemma 2.4(i), $\dot{C}_3$ is the one in which negative edges are adjacent.

In what follows we set $n \geq 10$.

Lemma 5.1. *$\dot{C}_4$ is* .

Proof. The candidates are:



$\dot{L}_1$ $\dot{L}_2$ $\dot{L}_3$

The latter two are eliminated by Lemma 2.4(i). By inserting the largest eigenvalue of the quotient matrix $Q_{\dot{L}_3}$ into the characteristic polynomial ℓ_2, we get

$$\ell_2\big(\tfrac{1}{2}(n-6+\sqrt{n(n+8)-32})\big) = 2(-n-4+\sqrt{n(n+8)-32}) < 0$$

as $n(n+8)-32 < (n-4)^2$. The latter inequality implies that the largest root of ℓ_2 is larger than the largest eigenvalue of $Q_{\dot{L}_3}$, i.e. $\lambda_1(\dot{L}_2) > \lambda_1(\dot{L}_3)$.

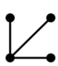
If the vertices of $\dot{L}_2$ are labelled as above then the λ_1-eigenvector has the form

$$\mathbf{b} = b\Big(\frac{(\lambda_1+1)(\lambda_1-5)}{\lambda_1(\lambda_1+2)-11}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}, \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}, 1, 1, \ldots, 1\Big)^{\intercal},$$

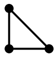
for $b > 0$. Now, $\dot{L}_1$ is obtained by reversing the sign of edges 12, 13 and 23, and thus we have

$$\begin{aligned}\lambda_1(\dot{L}_1) - \lambda_1(\dot{L}_2) &\geq \mathbf{b}^{\intercal}(A_{\dot{L}_1} - A_{\dot{L}_2})\mathbf{b}\\ &= \frac{4(\lambda_1^2+1)b^2}{\lambda_1(\lambda_1+2)-11}\Big(\frac{2(\lambda_1+1)(\lambda_1-5)}{\lambda_1(\lambda_1+2)-11} - \frac{\lambda_1^2+1}{\lambda_1(\lambda_1+2)-11}\Big)\\ &= \frac{4(\lambda_1^2+1)b^2}{\lambda_1(\lambda_1+2)-11}\cdot\frac{\lambda_1^2-8\lambda_1-9}{\lambda_1(\lambda_1+2)-11} > 0 \text{ for } \lambda_1 > 9.\end{aligned}$$

We compute $\lambda_1(\dot{L}_2) > 9$ for $n = 11$, and then by eigenvalue interlacing we have the same inequality for $n \geq 12$. For $n = 10$, the inequality $\lambda_1(\dot{L}_1) > \lambda_1(\dot{L}_2)$ is confirmed directly, and we are done. □

Lemma 5.2. *$\dot{C}_5$ is* .

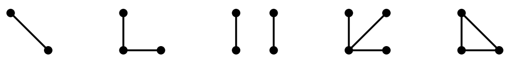
Proof. Apart from the signed graphs faced in the proof of the previous lemma, there is exactly one additional candidate: it contains exactly 3 non-adjacent negative edges. This candidate is eliminated on the basis of Lemma 2.4(i), while the remaining ones are already considered in the previous proof. In particular, we know that $\lambda_1(\dot{L}_2) > \lambda_1(\dot{L}_3)$, and the proof is completed. □

Lemma 5.3. *$\dot{C}_6$ is* .

Proof. The only critical case is the comparison of the indices of $\dot{L}_3$ and the signed graph, say $\dot{L}$, containing 4 negative edges that share the same vertex. Computing the λ_1-eigenvector for $\dot{L}$ and following the proof of Lemma 5.1, we get $\lambda_1(\dot{L}_3) > \lambda_1(\dot{L})$ for $\lambda_1^2 - 12\lambda_1 - 13 > 0$, i.e. for $\lambda_1 = \lambda_1(\dot{L}) > 13$. This proves this lemma for $n \geq 15$ (as there $\lambda_1(\dot{L}) > 13$). The case $10 \leq n \leq 14$ is considered directly, and we are done. □

We arrive at the following result.

Theorem 5.4. *The first* 6 *complete signed graphs with* $n \geq 10$ *vertices ordered by their indices are:*



$\dot{C}_1$ $\dot{C}_2$ $\dot{C}_3$ $\dot{C}_4$ $\dot{C}_5$ $\dot{C}_6$

Remark 5.5. With a slight modification in which a full line represents a positive edge and an unpictured line represents a non-edge, the result of Theorem 5.4 remains valid for the ordering of unsigned graphs by the index of the Seidel matrix. Indeed, the Seidel matrix of an unsigned graph G coincides with the adjacency matrix of the complete signed graph in which negative edges are induced by the edges of G.

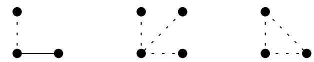
6 Further computations

We complete the results of Sections 4 and 5 by determining the 6 signed graphs with largest indices for every order that is not covered by Theorem 4.4 and the 7 signed graphs with largest indices for every order that is not covered by Theorem 5.4.

There is exactly one connected unbalanced signed graph with 3 vertices (the unbalanced triangle), while the ordering for $n \in \{4, 5\}$ is given in the first part of Figure 3. We note that there are exactly 6 connected unbalanced signed graphs for $n = 4$, so in this case the given list is complete.

There are exactly 3 complete signed graphs with 4 vertices and their ordering does not deviate from the general case considered in Theorem 5.4. For $5 \leq n \leq 9$ the one with the largest index switches to the signed graph with all positive signature, while the remaining 6 are given in the second part of Figure 3. Again, for $n = 5$ the list is complete.

In this paper our idea was to give a general method for the ordering by the index and to demonstrate its use by determining the lists of the first few signed graphs as reported in the previous sections. Of course, these results can be extended, but the theoretical approach is becoming more complicated as the number of candidates increases and comparison of their indices requires more sophisticated methods. However, it occurs that the list of Theorem 3.9 continues with:



$\dot{G}_6$ $\dot{G}_7$ $\dot{G}_8$

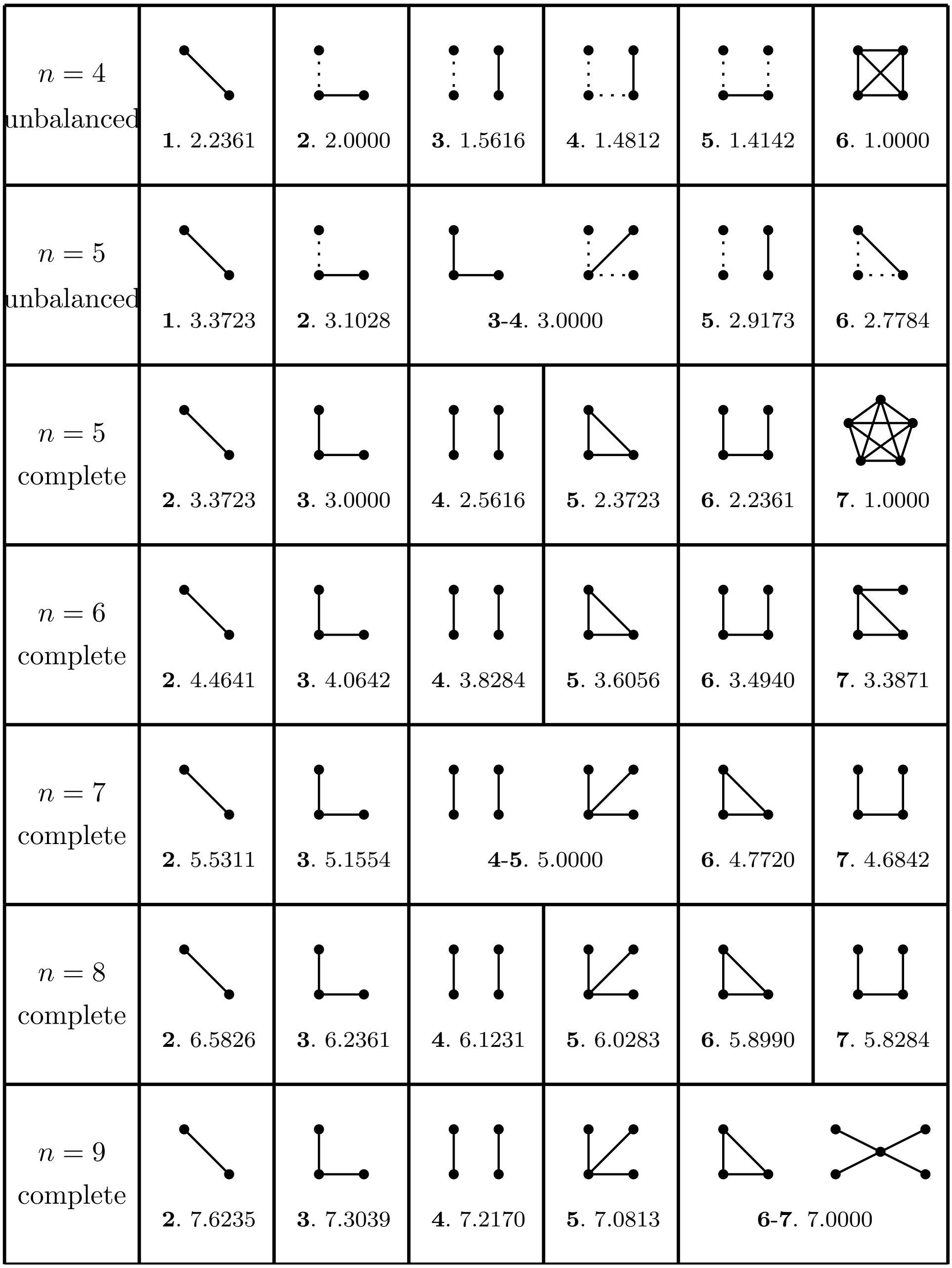


Figure 3: Orderings of small signed graphs that are uncovered by Theorem 4.4 or Theorem 5.4.

We skip the details and note that the proof relies on an intensive algebraic computation that basically does not deviate from those of the previous sections.

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