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ORIGINAL RESEARCH

New conditions for finite-time stability of impulsive dynamical systems via piecewise quadratic functions

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Abstract
In this paper, the use of time-varying piecewise quadratic functions is investigated to characterize the finite-time stability of state-dependent impulsive dynamical linear systems. Finite-time stability defines the behavior of a dynamic system over a bounded time interval. More precisely, a system is said to be finite-time stable if, given a set of initial conditions, its state vector does not exit a predefined domain for a certain finite interval of time. This paper presents new sufficient conditions for finite-time stability based on time-varying piecewise quadratic functions. These conditions can be reformulated as a set of Linear Matrix Inequalities that can be efficiently solved through convex optimization solvers. Different numerical analysis are included in order to prove that the presented conditions are able to improve the results presented so far in the literature.

1 | INTRODUCTION

The definition of finite-time stability (FTS) was introduced to characterize the evolution of a dynamical system, starting from a specified set of initial condition, over an assigned and finite interval of time. More specifically, in 1953 Kamenkov [1] defined as finite-time stable an autonomous nonlinear system $\dot{x} = f(x, t)$ if, given three scalars α, β and T, with $0 < \alpha < \beta$,

$$\|x(t_0)\| < \alpha \Rightarrow \|x(t)\| < \beta \quad \text{for all } t \in [t_0, t_0 + T]. \tag{1}$$

Since the definition by Kamenkov does not consider the system operation over an infinite time interval and, moreover, it introduces bounds on the state variables, it is different and not directly correlated to Lyapunov stability and other classical stability concepts.

A key point in the definition of FTS in [1] is the fact that it depends on the specific norm used for the bounds. More generically, as discussed in Section 2, the FTS definition can be stated assuming the initial state to belong to a certain set, defined *initial domain*, whereas the state trajectory is requested to remain within another set, defined *trajectories domain*, for a finite time interval. For this reason FTS is a more practical concept than Lyapunov stability, that is, it can be used to verify than the state trajectories of the system remain inside a specified domain over the considered time-interval. Indeed, FTS has been recently adopted for the solution of control problem related to different applications, such as the design of a collision avoidance system for a vehicle [2] and the design of a control system for a missile [3].

In the literature, the term finite-time stability has been used also with a different meaning than the one considered here. In particular, in [4, 5] the authors refers to FTS as the property of the system state of converging to zero in finite time.

After Kamenkov's introduction, the concept of FTS was developed in the sixties in [6–11]. In the following years, several other results were proposed on this topic using the alternative terms of *stability over finite interval* or *practical stability* [12–15]. However, most of the techniques presented in that period both for the analysis [16–18] and for the design of finite-time stable control systems [19, 20] were computationally cumbersome.

More recently, the concept of FTS has been applied to different classes of dynamic systems and novel conditions for FTS analysis and control design have been formulated exploiting the theories of linear matrix inequalities (LMIs) [21] and differential LMIs (DLMIs) [22]. The proposed conditions are more computationally tractable than the previous ones.

The most relevant results on the FTS of the class of linear time-varying systems, possibly uncertain, have been collected in the volume [23].

Extensions to the class of hybrid systems [24–28] and to the class of nonlinear quadratic systems [29] have been proposed, as well. Moreover, the *stochastic*-FTS property has been recently introduced and it has been applied to the class of Itô stochastic linear time-varying systems [30, 31] and to the class of Markovian jump linear time-varying systems [32–34].

In all cited papers the initial and trajectories domains are assumed to be ellipsoidal, which is consistent with the natural choice of the Euclidean weighted norm. As a consequence, the machinery used for finding the sufficient conditions for FTS is based on quadratic Lyapunov functions.

In [35] the FTS problem assuming polytopic initial and trajectories domains has been considered. This case is interesting since polytopes enable to deal with common bounds on the state variables in the form $x_{i_{\min}} \leq x_i \leq x_{i_{\max}}$. However, the FTS sufficient conditions proposed in [35] can be solved only by means of a nonconvex algorithm and therefore they are much less attractive than the conditions proposed for ellipsoidal domains.

In this paper we focus on the FTS problem for the more general class of state-dependent impulsive dynamical linear systems (SD-IDLS) [36]. Early results regarding the FTS of SD-IDLS, based on the use of quadratic Lyapunov functions, are presented in [25]. The new relevant contribution of this paper is given by the fact that the class of *Piecewise Quadratic Functions* (PQFs) [37–39] is considered for the FTS analysis. The use of PQFs for the FTS problem of linear time-varying system has been preliminary proposed in [40] showing the effectiveness of the approach.

The main advantages given by the use of PQFs for FTS purposes as it is proposed in this paper are:

- it is possible to recover as particular cases the ellipsoidal and polytopic domains, and, moreover, it is also possible to extend the FTS theory to any *Piecewise Quadratic Domain* (PQD) without introducing any sort of approximation of the domains;
- the conditions can be implemented by means of convex optimization problems, even in the particular case of polytopic domains, and therefore they are very efficient from the computational point of view. This is possible adopting an original reparameterization of the optimization matrices, so as to remove the equality constraints, without adding further conservativeness;
- the use of PQFs fits particularly well with the class of SD-IDLS giving the opportunity to dedicate a particular partition of the piecewise quadratic Lyapunov function to the jump state region.

The paper is structured as follows. In Section 2 some preliminaries notions on FTS and on the class of PQFs and PQDs are presented. The novel sufficient conditions for the FTS of SD-IDLS, obtained exploiting the class of piecewise quadratic Lyapunov function are provided in Sections 3; then some numerical examples are presented in Section 4.

2 | PRELIMINARIES

2.1 | Problem statement

In this section the class of linear time-varying systems characterized by finite state jump is first introduced. Indeed, the class of SD-IDLS is considered, for which the state jumps take place when the trajectory achieves a specified subset of the state space defined *resetting set*.

A time-varying SD-IDLS is characterized by both a continuous-time and a discrete-time dynamics

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \quad x(t) \in \mathbf{R}^n \setminus \bigcup_{k=1}^{N} S_k, \tag{2a}$$

$$x(t^+) = J_k x(t), \quad x(t) \in S_k, \quad k = 1, \dots, N, \tag{2b}$$

where $A(\cdot) : \mathbf{R}_0^+ \to \mathbf{R}^{n\times n}$, $J_k \in \mathbf{R}^{n\times n}$, $k = 1, \dots, N$. The sets S_k, $k = 1, \dots, N$, are connected and closed pairwise disjoint sets, such that $\mathbf{0} \notin S_k$.

The following assumption allows only a finite number of state jumps in any finite interval of time, that is, no Zeno behaviour is allowed.

Assumption 1. For all $t \in \mathbb{R}_0^+$ such that $x(t) \in S_k$, there exists $\epsilon > 0$ such that $x(t+\delta) \notin S_k$, $\forall \delta \in (0, \epsilon)$, $k = 1, \dots, N$.

In this paper we derive novel sufficient conditions for the FTS of SD-IDLS (2) in case that both the initial and the trajectories domains can be modelled as PQDs, that is, there exist positive definite PQFs such that their unitary level curves model the boundaries of the considered domains [37, 38].

For sake of clarity, before introducing a definition about FTS property via PQDs, we recall some preliminary definitions about PQFs and PQDs [39].

2.2 | Cones and conical partitions

A set G is a polyhedral cone in $\mathbf{R}^n$ if it is defined by means of the *conical* combination of p vectors, $\bar{x}_i \in \mathbf{R}^n$, $i = 1, \dots, p$, that is

$$\begin{aligned} G &= \text{cone}\left(\{\bar{x}_1, \dots, \bar{x}_p\}\right) \\ &:= \left\{ \bar{x} \,:\, \bar{x} = \sum_{i=1}^{p} \mu_i \bar{x}_i,\ \mu_i \geq 0,\ i = 1, \dots, p \right\}. \end{aligned} \tag{3}$$

The *dimension* of a cone G in $\mathbf{R}^n$ is the column rank of the matrix $\mathcal{M}_G := [\bar{x}_1 \quad \dots \quad \bar{x}_p \quad]$. A set of *normalized extremal rays* of G is defined as any minimal set of $q \leq p$ vectors $\{\hat{x}_1, \dots, \hat{x}_q\}$, with $||\hat{x}_i||_2 = 1, i = 1, \dots, q$, such that

$$G = \text{cone}(\{\hat{x}_1, \dots, \hat{x}_q\}). \tag{4}$$

If a cone G in $\mathbf{R}^n$ has dimension $\dim(G) = n$, the set of normalized extremal rays is univocally determined.

The definition of cones allows us to divide the overall state space in conical regions, as proposed in the following definition.

Definition 1 (Conical partition of $\mathbf{R}^n$). A collection of cones G_i, $i = 1, \dots, r$, defines a conical partition $\mathcal{Y} = \{G_1, \dots, G_r\}$ of $\mathbf{R}^n$ if and only if:

- the dimension of each conical subset G_i is n
- the whole state space $\mathbf{R}^n$ is covered by the union of the conical sets G_i, $i = 1, \dots, r$
- the intersection between the interior of two adjacent cones G_i and G_j is empty.

$\diamondsuit$

Given a conical partition $\mathcal{Y}$, the *set of generating rays* $R_\mathcal{Y}$ is defined as the union of the normalized extremal rays of each cone in $\mathcal{Y}$

$$\mathcal{R}_\mathcal{Y} = \bigcup_{G_i \in \mathcal{Y}} \text{extr}(G_i) = \{\hat{x}_1, \dots, \hat{x}_v\}. \tag{5}$$

Finally, from union of two conical partitions $\mathcal{Y}^1$ and $\mathcal{Y}^2$ of $\mathbf{R}^n$ we obtain a conical partition $\mathcal{Y}$ whose set of generating rays is given by

$$\mathcal{R}_\mathcal{Y} = \mathcal{R}_{\mathcal{Y}^1} \cup \mathcal{R}_{\mathcal{Y}^2}. \tag{6}$$

2.3 | Piecewise quadratic functions and domains over a conical partition

The definitions of PQFs and PQDs, originally introduced in [39], are recalled in the following.

Definition 2 (PQFs over a conical partition [39]). A time–varying PQF, defined over a conical partition $\mathcal{Y} = \{G_1, \dots, G_r\}$ of $\mathbf{R}^n$, is a space-continuous, piecewise continuously time-differentiable, positive definite function in the form

$$H_\mathcal{Y}(x, t) = x^T H_i(t) x, \quad x \in G_i, \quad i = 1, \dots, r, \tag{7}$$

where $H_i(t) \in \mathbf{R}^{n \times n}$, $i = 1, \dots, r$, are symmetric matrix-valued functions positive definite in the cone G_i, such that

$$x^T H_i(t) x > 0, \quad x \in G_i \backslash \{0\} \tag{8}$$

for $t \geq 0$ and $i = 1, \dots, r$. $\diamondsuit$

The condition

$$x^T H_i(t) x = x^T H_j(t) x, \quad x \in G_i \cap G_j, \tag{9}$$

has to be satisfied for $i, j = 1, \dots, r$, $i \neq j$, $t \geq 0$ in order to ensure the space-continuity of the PQF $H_\mathcal{Y}(x, t)$.

Definition 3 (PQD over a conical partition [39]). Given the conical partition $\mathcal{Y} = \{G_1, \dots, G_r\}$ of $\mathbf{R}^n$, a time-varying PQD defined over $\mathcal{Y}$ is a compact domain whose boundary is the unitary level curve of a PQF $H_\mathcal{Y}(x, t)$, that is

$$\begin{aligned} \mathcal{X}_{H_\mathcal{Y}}(t) &:= \{x : H_\mathcal{Y}(x, t) \leq 1\} \\ &= \{x : x^T H_i(t) x \leq 1, \quad x \in G_i, \quad i = 1, 2, \dots r\}. \end{aligned} \tag{10}$$

$\diamondsuit$

In what follows a time-invariant PQF will be denoted by $H_\mathcal{Y}(x)$, while a time-invariant PQD will be denoted by $\mathcal{X}_{H_\mathcal{Y}}$.

2.4 | Finite-time stability via PQFs

In [25] the authors analyzed the FTS property of system (2) assuming ellipsoidal domains. Now, the FTS property is generalized to the case of piecewise quadratic initial and the trajectories domains. Indeed, let us consider an initial domain $\mathcal{X}_{R_\mathcal{Y}}$ and a trajectory domain $\mathcal{X}_{\Gamma_\mathcal{Y}}$, being $R_\mathcal{Y}$ and $\Gamma_\mathcal{Y}(t)$ two PQFs defined over the same partition $\mathcal{Y} = (G_1, \dots, G_r)$ of $\mathbf{R}^n$ [1]:

$$R_\mathcal{Y} = x^T R_i x \quad \text{for } x \in G_i \text{ and } i = 1, \dots, r, \tag{11}$$

$$\Gamma_\mathcal{Y}(t) = x^T \Gamma_i(t) x \quad \text{for } x \in G_i \text{ and } i = 1, \dots, r. \tag{12}$$

Definition 4 (FTS of SD-IDLS via PQDs). Given an initial time t_0, a positive scalar T, a time invariant PQD $\mathcal{X}_{R_\mathcal{Y}}$, a time varying PQD $\mathcal{X}_{\Gamma_\mathcal{Y}}$, system (2) is said to be finite-time stable wrt $(t_0, T, \mathcal{X}_{R_\mathcal{Y}}, \mathcal{X}_{\Gamma_\mathcal{Y}})$ if

$$x_0 \in \mathcal{X}_{R_\mathcal{Y}} \Rightarrow x(t, x_0) \in \mathcal{X}_{\Gamma_\mathcal{Y}}, \forall t \in [t_0,\ t_0 + T]. \tag{13}$$

where $x(t, x_0)$ is the trajectory of system starting from the initial state x_0. $\diamondsuit$

Since the level curves of a positive definite PQF are described by the union of portion of ellipsoids, we are able to tackle FTS problems characterized to any PQDs without introducing any

[1] The definition can be also generalized to the case of initial and trajectories domains defined over different partitions considering that a reparameterized on a common partition can be easily obtained from the union of the different partitions (see [39])

sort of approximation of the domains; in particular, ellipsoidal and polytopic domains can be recovered as particular cases.

3 | MAIN RESULTS

In the following, a novel sufficient conditions for the FTS of the SD-IDLS system (2) will be obtained by assuming PQDs domains. After that, the presented condition will be recast as a DLMIs feasibility problem, in order to formulate more computationally tractable sufficient conditions.

Theorem 1. *Let us consider the SD-IDLS system (2), the time interval* $[t_0, t_0 + T]$ *and two PQDs* $\mathcal{X}_{R_\mathcal{Y}}$ *and* $\mathcal{X}_{\Gamma_\mathcal{Y}}(t)$, $t \in [t_0, t_0 + T]$, *defined over the partition* $\mathcal{Y} = (G_1, \dots, G_r)$ *of* $\mathbf{R}^n$. *System (2) is FTS with respect to* $(t_0, T, \mathcal{X}_{R_\mathcal{Y}}, \mathcal{X}_{\Gamma_\mathcal{Y}})$ *if there exists a PQF* $P_\mathcal{Y}(x, t)$, *defined over the partition* $\mathcal{Y}$, *verifying the following equality conditions for space-continuity*

$$x^T P_i(t)x = x^T P_j(t)x, \quad x \in G_i \cap G_j, \tag{14}$$

for $i, j = 1, \dots, r, i \neq j, \forall t \in [t_0, t_0 + T]$, *and such that*

$$x^T \left(\dot{P}_i(t) + A^T(t)P_i(t) + P_i(t)A(t)\right)x < 0, x \in G_i \setminus \bigcup_{k=1}^{N} \mathcal{S}_k, \tag{15a}$$

$$x^T \left(A_{d,k}^T P_j(t)J_k - P_i(t)\right)x < 0, \quad x \in G_i \cap \mathcal{S}_k, \tag{15b}$$

$$x^T \left(P_i(t) - \Gamma_i(t)\right)x \geq 0, \quad x \in G_i, \tag{15c}$$

$$x^T \left(P_i(t_0) - R_i\right)x \leq 0, \quad x \in G_i, \tag{15d}$$

for $\forall t \in [t_0, t_0 + T]$, $i, j = 1, \dots, r$ *and* $k = 1, \dots, N$.

Proof. Let consider a PQF $P_\mathcal{Y}(x, t)$ which verifies the equality conditions for space-continuity (14).

Given a system trajectory which does not reach any resetting set $\mathcal{S}_k$ for $k = 1, \dots, N$, the time derivative of $P_\mathcal{Y}(x, t)$ is defined and it yields

$$\dot{P}_\mathcal{Y}(t, x) = x^T \left(\dot{P}_i(t) + A^T(t)P_i(t) + P_i(t)A(t)\right)x,$$
$$x \in G_i \setminus \bigcup_{k=1}^{N} \mathcal{S}_k, \quad i = 1, \dots, , r, \tag{16}$$

which is negative by virtue of (15a).

Moreover, when the system trajectory touches a reset set, moving from the cone G_i to G_j, the variation of $P_\mathcal{Y}(x, t)$ is written as

$$P_\mathcal{Y}(t^+, x) - P_\mathcal{Y}(t, x) = x^T \left(A_{d,k}^T P_j(t)J_k - P_i(t)\right)x(t)$$
$$\in G_i \cap \mathcal{S}_k, \quad x(t^+) \in G_j, \tag{17}$$

which is negative by virtue of (15a) for any pair of cones (G_j, G_j), for $i, j = 1, \dots, r$, such that $G_i \cap \mathcal{S}_k \neq \emptyset$ for $k = 1, \dots, N$.

From the above considerations we obtain that $P_\mathcal{Y}(x, t)$ is strictly decreasing along the trajectories of system (2); hence, we have

$$P_\mathcal{Y}(x(t, x_0), t) \leq P_\mathcal{Y}(x(t_0, x_0), t_0), \quad t \in [t_0, t_0 + T]. \tag{18}$$

Now, consider an initial state $x_0 \in \mathcal{X}_{R_\mathcal{Y}}$, the following chain of inequalities holds, for $t \in [t_0, t_0 + T]$,

$$\Gamma_\mathcal{Y}(x(t, x_0), t) \leq P_\mathcal{Y}(x(t, x_0), t) \text{ in view of (15c)} \tag{19}$$

$$\leq P_\mathcal{Y}(x(t_0, x_0), t_0) \tag{20}$$

$$\leq R_\mathcal{Y}(x_0) < 1 \text{ in view of (15d).} \tag{21}$$

We can conclude that $x_0 \in \mathcal{X}_{R_\mathcal{Y}}$ implies $\Gamma_\mathcal{Y}(x(t, x_0), t) < 1$, for $t \in [t_0, t_0 + T]$, that is, $x(t, x_0) \in \mathcal{X}_{\Gamma_\mathcal{Y}}$, for $t \in [t_0, t_0 + T]$. □

Now, by considering the results in [39] and taking advantage from the S-Procedure arguments [21], we investigate how to recast the infinite-dimensional inequalities (14) and (15) to a DLMIs feasibility problem.

Theorem 2. *Let us consider the SD-IDLS system (2), the time interval* $[t_0, t_0 + T]$ *and two PQDs* $\mathcal{X}_{R_\mathcal{Y}}$ *and* $\mathcal{X}_{\Gamma_\mathcal{Y}}$, $t \in [t_0, t_0 + T]$, *defined over the partition* $\mathcal{Y} = (G_1, \dots, G_r)$ *of* $\mathbf{R}^n$. *Given any matrices* $V_i, L_{i,k} \in \mathbf{R}^{n \times n}$ *verifying*

$$x^T V_i x \leq 0, \quad x \in G_i, \quad i = 1, \dots, r, \tag{22}$$

$$x^T L_{i,k} x \leq 0, \quad x \in G_i \cap \mathcal{S}_k, \quad i = 1, \dots, r, \quad k = 1, \dots, N, \tag{23}$$

system (2) is FTS with respect to $(t_0, T, \mathcal{X}_{R_\mathcal{Y}}, \mathcal{X}_{\Gamma_\mathcal{Y}})$ *if there exists a PQF* $P_\mathcal{Y}(x, t)$, *defined over the partition* $\mathcal{Y}$, *verifying the following equality conditions for space-continuity*

$$\hat{x}_h^T P_i(t)\hat{x}_h = \hat{x}_h^T P_j(t)\hat{x}_h, \tag{24a}$$

$$\hat{x}_k^T P_i(t)\hat{x}_k = \hat{x}_k^T P_j(t)\hat{x}_k, \tag{24b}$$

$$\hat{x}_h^T P_i(t)\hat{x}_k = \hat{x}_h^T P_j(t)\hat{x}_k, \tag{24c}$$

for all pairs of vectors $\hat{x}_h, \hat{x}_k$ *taken from* $extr(G_i \cap G_j)$ *for* $i, j = 1, \dots, r$, $i \neq j$, $\forall t \in [t_0, t_0 + T]$, *and there exist positive scalar functions* $a_i(t)$, $b_{i,k}(t)$, $c_i(t)$ *and positive scalars* d_i *satisfying the DLMIs conditions*

$$\dot{P}_i(t) + A^T(t)P_i(t) + P_i(t)A(t) - a_i(t)V_i < 0, \tag{25a}$$

$$A_{d,k}^T P_j J_k - P_i(t) - b_{i,k}(t) L_{i,k} < 0, \quad if\, G_i \cap \mathcal{S}_k \neq \emptyset, \tag{25b}$$

$$P_i(t) - \Gamma_i(t) + c_i(t) V_i \geq 0, \tag{25c}$$

$$P_i(t_0) - R_i - d_i V_i \leq 0, \tag{25d}$$

for $\forall t \in [t_0, t_0 + T]$*,* $i, j = 1, \dots, r$ *and* $k = 1, \dots, N$*.*

Proof. The proof is obtained by making use of S–Procedure arguments following the approach in [39]. □

Finally, we consider a reparameterization of the quadratic form $P_{\mathcal{Y}}(x, t)$, which allows us to obtain a convex optimization problem from the sufficient conditions (24)-(25). In particular, we consider for matrix functions $P_i(\cdot)$, $i = 1, \dots, r$ a structure that allows us to verify the equality constraints (24) without explicitly account them into the theorem statement. For this reason, we consider the next technical lemma proved in [39].

Lemma 1 ([39]). *Consider the quadratic form* $P_{\mathcal{Y}}(x, t)$ *defined over the partition* $\mathcal{Y} = (G_1, \dots, G_r)$ *of* $\mathbf{R}^n$ *with the set of generating rays* $[\hat{x}_1 \dots \hat{x}_v]$*. The quadratic form* $P_{\mathcal{Y}}(x, t)$ *verifies the equality conditions for space-continuity if and only if there exists a symmetric matrix function* $\Theta(t) = \{\theta_{ij}(t)\} \in \mathbf{R}^{v \times v}$ *such that*

$$\hat{x}_i^T P_k(t) \hat{x}_i = \theta_{ii}(t), \tag{26a}$$

$$\hat{x}_i^T P_k(t) \hat{x}_j = \theta_{ij}(t), \tag{26b}$$

for $i, j = 1, \dots, v$ *and* $\hat{x}_i, \hat{x}_j \in \mathrm{extr}(G_k)$*,* $k = 1, \dots, r$*.*

In the following we assume to consider only PQFs defined over a partition of $\mathbf{R}^n$ characterized by cone with dimension n. Since any cone with more than n extremal rays, can be partitioned in a collection of cones of dimension n, the above assumption does not introduce any loss of generality.

In order to take advantage from the result of Lemma 1, we consider that given the matrix containing all the generating rays of a conical partition $\mathcal{M}_{\mathcal{R}_{\mathcal{Y}}} = [\hat{x}_1 \dots \hat{x}_v]$, we can built a *selection matrices* Λ_i that allows us to compute the matrix with the extremal rays of G_i, namely $\mathcal{M}_{extr(G_i)} = [\hat{x}_{i_1} \dots \hat{x}_{i_n}]$, as

$$\mathcal{M}_{extr(G_i)} = \mathcal{M}_{\mathcal{R}_{\mathcal{Y}}} \Lambda_i. \tag{27}$$

From its definition, we obtain that Λ_i is a $v \times n$ matrix and its columns have all zero terms except a unitary value in the row associated to one of the extremal rays of the cone G_i.

Starting from the matrix function $\Theta(\cdot)$ defined in Lemma 1, we define the matrix functions $\Theta_i(\cdot)$ for $i = 1, \dots, r$, which are characterized by the parameters associated with the i-th cone as

$$\Theta_i(t) = \begin{bmatrix} \theta_{i_1 i_1}(t) & \theta_{i_1 i_2}(t) & \cdots & \theta_{i_1 i_n}(t) \\ * & \theta_{i_2 i_2}(t) & \cdots & \cdots \\ * & * & \cdots & \theta_{i_{n-1} i_n}(t) \\ * & * & * & \theta_{i_n i_n}(t) \end{bmatrix}, \quad i = 1, \dots, r. \tag{28}$$

By taking into account their definitions, we obtain that the matrix functions $\Theta(\cdot)$ and $\Theta_i(\cdot)$ are related by the selection matrix Λ_i as follows

$$\Theta_i(\cdot) = \Lambda_i^T \Theta(\cdot) \Lambda_i. \tag{29}$$

Moreover, form results of Lemma 1 and the definition of the matrix function $\Theta_i(\cdot)$ in (28), we obtain that the equality constraints (24) for the i-th cone can be rewritten as

$$\begin{bmatrix} \hat{x}_{i_1}^T \\ \vdots \\ \hat{x}_{i_n}^T \end{bmatrix} P_i(t) \left[\hat{x}_{i_1} \dots \hat{x}_{i_n}\right] = \Theta_i(t), \tag{30}$$

or equivalently

$$\mathcal{M}_{extr(G_i)}^T P_i(t) \mathcal{M}_{extr(G_i)} = \Theta_i(t). \tag{31}$$

Finally, the assumption on the dimensions of the cones that define the partition ensures that the matrices $\mathcal{M}_{R_{\mathcal{Y}}} \Lambda_i$ for $i = 1, \dots, r$ are invertible. Hence,

$$P_i(t) = \left(\mathcal{M}_{R_{\mathcal{Y}}} \Lambda_i\right)^{-T} \Lambda_i^T \Theta(t) \Lambda_i \left(\mathcal{M}_{R_{\mathcal{Y}}} \Lambda_i\right)^{-1}, \quad i = 1, \dots, r, \tag{32}$$

Reparameterization (32) of the functions $P_i(\cdot)$ for $i = 1, \dots, r$ as a function of $\Theta(\cdot)$ allows us to verify the equality constraints (24). In this way, condition for FTS of system (2) can be rewritten as feasibility problem based on DLMIs/LMIs.

Theorem 3. *Let us consider the SD-IDLS system (2), the time interval* $[t_0, t_0 + T]$ *and two PQDs* $\mathcal{X}_{R_{\mathcal{Y}}}$ *and* $\mathcal{X}_{\Gamma_{\mathcal{Y}}}$*,* $t \in [t_0, t_0 + T]$*, defined over the partition* $\mathcal{Y} = (G_1, \dots, G_r)$ *of* $\mathbf{R}^n$*.*

Given any matrices $V_i, L_{i,k} \in \mathbf{R}^{n \times n}$ *verifying*

$$x^T V_i x \leq 0, \quad x \in G_i, \quad i = 1, \dots, r, \tag{33a}$$

$$x^T L_{i,k} x \leq 0, x \in G_i \cap \mathcal{S}_k, \quad i = 1, \dots, r, \quad k = 1, \dots, N. \tag{33b}$$

System (2) is FTS with respect to $(t_0, T, \mathcal{X}_{R_{\mathcal{Y}}}, \mathcal{X}_{\Gamma_{\mathcal{Y}}})$ *if there exists a PQF* $P_{\mathcal{Y}}(x, t)$*, defined over the partition* $\mathcal{Y}$*, the positive scalar functions* $a_i(t)$*,* $b_{i,k}(t)$*,* $c_i(t)$ *and positive scalars* d_i *satisfying the DLMIs* *conditions*

$$\dot{P}_i(t) + A^T(t)P_i(t) + P_i(t)A(t) - a_i(t)V_i < 0, \tag{34a}$$

$$A_{d,k}^T P_j J_k - P_i(t) - b_{i,k}(t)L_{i,k} < 0, \quad \text{if } G_i \cap S_k \neq \emptyset, \tag{34b}$$

$$P_i(t) - \Gamma_i(t) + c_i(t)V_i \geq 0, \tag{34c}$$

$$P_i(t_0) - R_i - d_i V_i \leq 0, \tag{34d}$$

for $\forall t \in [t_0, t_0 + T]$, $i, j = 1, \dots, r$ *and* $k = 1, \dots, N$, *where the matrices* $P_i(\cdot)$ *for* $i = 1, \dots, r$ *are defined as*

$$P_i(t) = \left(\mathcal{M}_{R_y}\Lambda_i\right)^{-T}\Lambda_i^T\Theta(t)\Lambda_i\left(\mathcal{M}_{R_y}\Lambda_i\right)^{-1}, \quad i = 1, \dots, r. \tag{35}$$

The analysis conditions (34) require to be rewritten in terms of LMIs, defining a convex optimization problem. To this aim, the following remarks are considered.

Remark 1. The use of piecewise quadratic functions reduces the conservativeness of the LMI conditions for the FTS analysis with respect to the case of an unique quadratic function. Moreover, in the example section, it is shown that the number of conical regions of the Lyapunov piecewise quadratic function also affects the performance of the method. On the other hand, the dimension of the conical partition influences the computational complexity of Theorem 3. Indeed, assuming that each resetting set belongs to a single cone, the number of time-dependent optimization variables is given by $\frac{v(v+1)}{2} + 3r + N$, where

- $\frac{v(v+1)}{2}$ is the number of elements of the symmetric matrix $\Theta(t) \in R^{v \times v}$, where v is the number of generating rays;
- $3r$ is the total number of variables $a_i(t)$, $c_i(t)$, d_i, for $i = 1, \dots, r$ being r the number of cones in the partition;
- N is the number of resetting sets, and hence of variables $b_{i,k}(t)$ since $k = 1, \dots, N$;

while the number of inequality conditions is $(3 + N)r$.

Remark 2. According to the previous literature (e.g. see [23], [25]), a possible way of recasting the DLMIs conditions (34) in terms of LMIs, is to assume a piecewise linear structure for the scalar functions $a_i(t)$, $b_{i,k}(t)$, $c_i(t)$ and for the matrix function $\Theta(\cdot) \in \mathbf{R}^{v \times v}$. For instance, the matrix function $\Theta(\cdot)$ can be assumed in the form

$$\Theta(t) = \begin{cases} \Theta_0 + \Psi_1(t - t_0), & t \in [t_0, t_0 + T_s], \\ \Theta_0 + \sum_{h=1}^{j}\Psi_h T_s + \Psi_{j+1}(t - jT_s - t_0), & t \in [t_0 + jT_s, t_0 + (j+1)T_s] \\ & j = 1, \dots, J \end{cases} \tag{36}$$

where $J = \max\{j \in \mathbb{N} : j < T/T_s\}$, $T_s \ll T$, and Θ_0 and Ψ_l, $l = 1, \dots, J + 1$, are the new optimization variables. ◊

4 | EXAMPLES

Four numerical examples have been developed with the purpose of investigating on the improvement introduced by the analysis conditions (34). In particular, these examples analyze the advantages related to the time-varying piecewise quadratic Lyapunov functions and their level curves, which represent the bounds of PQDs and can conform the shape of the bounds of different class of domains.

In the first three examples, we consider the SD-IDLS system considered in [25], which has been characterized by the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 + 0.3\sin 10t & 0.5 \end{bmatrix}, \quad A_{d,1} = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.75 \end{bmatrix}, \tag{37a}$$

$$A_{d,2} = \begin{bmatrix} -0.72 & 0.16 \\ 0.13 & -0.78 \end{bmatrix}, \tag{37b}$$

and by the following two resetting sets S_1 and S_2

$$S_1 = conv\left(\begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix}, \begin{pmatrix} 0.8 \\ 0.7 \end{pmatrix}\right), \tag{38a}$$

$$S_2 = conv\left(\begin{pmatrix} -0.8 \\ 0.3 \end{pmatrix}, \begin{pmatrix} 0.4 \\ -0.6 \end{pmatrix}\right). \tag{38b}$$

In the first example, we propose a comparison between proposed approach and previous literature one for the case of ellipsoidal domains [25]. In particular, results point out that conditions (34) are less conservative that the one in [25] in terms of FTS analysis.

In the second example, we tackle the FTS analysis where the domains are assumed to be polytopic. In this case, the approaches proposed in the previous literature cannot be applied since they are restricted to ellipsoidal domains. The main goal of this example is to show that the proposed technique allows us to take into account a more general class of domains (piecewise quadratic).

In the third example, we tackle the FTS analysis where the initial domains is assumed to be ellipsoidal, while the trajectory domains is polytopic. In this way, we stress the capability of the proposed approach to be used for FTS analysis problems characterized by different class of domains.

The last example deals with the FTS analysis problem of a bouncing pendulum over a vertical wall. The main goal of this example is to show the effectiveness of the proposed approach for an application.

Example 1. In this example we perform a comparison with the results shown in [25], where the authors have proven the FTS property for the SD-IDLS system characterized by the

TABLE 1 Maximization procedure of the time interval

Number of cones of partition	Time interval
$r = 1$	T = 2.5 s
$r = 2$	T = 2.6 s
$r = 4$	T = 2.8 s
$r = 8$	T = 3.5 s

matrices (37) and the reset sets (38), with respect to

$$t_0 = 5s, \quad T = 2.5s, \tag{39}$$

$$\Gamma = \begin{bmatrix} 0.20 & -0.10 \\ -0.10 & 0.18 \end{bmatrix}, \quad R = \begin{bmatrix} 7.5 & 4.5 \\ 4.5 & 9.0 \end{bmatrix}. \tag{40}$$

A general FTS problem requires the definition of an initial domain, a trajectories domain and a time interval. To the aim of evaluating the improvement of the proposed approach, we achieve a maximization procedure of the time interval, without change both initial and trajectories domains.

In particular, after setting the conical partition of both the initial and trajectories domains, we look for the maximum time interval, that is, the constant T, for which the solution of the analysis conditions (34) can be computed. This optimization problem has been solved by considering the following iterative procedure: starting from the initial value $\bar{T} = 2.5$ s, we increase its value by $\Delta T = 0.1$ s until we are able to compute a solution of the analysis conditions (34).

Table 1 shows the solution of the optimization problems for different symmetric partitions $\mathcal{Y}$ of $\mathbf{R}^2$ characterized by different numbers of cones. It can be noted that we are able to verify the FTS of the considered SD-IDLS system for a longer time interval by increasing the number of cones of the partition. In particular, we are able to improve the fitting capability of the level curves of the Lyapunov functions by increasing the number of cones of the partition. Finally, Figure 1 shows some system trajectories that are FTS with respect to $T = 3.5\,s$ and the considered ellipsoidal domains.

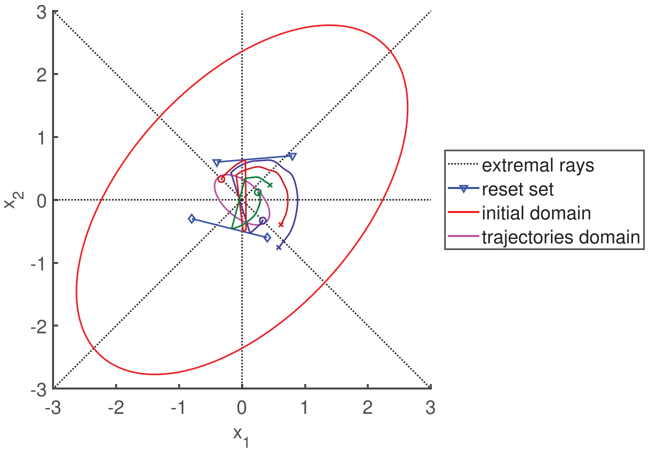


FIGURE 1 Example 1: FTS analysis problem characterized by ellipsoidal domains. Figure 3 shows some state trajectories, which remain confined into the ellipsoidal trajectories domain for the whole time interval, starting from some different points inside the ellipsoidal initial domain

Example 2. In this example we check the FTS property for the SD-IDLS system characterized by the matrices (37) and the reset sets (38). We consider polytopic bounds on both the initial and trajectories domains. In particular, the initial domain is defined by the PQD $\mathcal{X}_{R_{\mathcal{Y}_0}}$ and the trajectories domain is defined by the PQD $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$, where the conical partition is $\mathcal{Y}_0 = \{G_1, G_2, G_3, G_4\}$ with

$$G_1 = \text{cone}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T\right\}\right), \tag{41}$$

$$G_2 = \text{cone}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & -1\end{pmatrix}^T\right\}\right), \tag{42}$$

$$G_3 = \text{cone}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}-1 & -1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\end{pmatrix}^T\right\}\right), \tag{43}$$

$$G_4 = \text{cone}\left(\left\{\frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T\right\}\right); \tag{44}$$

and the PQF $R_{\mathcal{Y}}(x)$ and $\Gamma_{\mathcal{Y}}(x)$ are defined by the following diadic matrices

$$R_1 = 2.8571\begin{pmatrix}1 & 0\end{pmatrix}^T\begin{pmatrix}1 & 0\end{pmatrix}, \quad \Gamma_1 = 0.1273\,R_1, \tag{45}$$

$$R_2 = 2.8571\begin{pmatrix}0 & -1\end{pmatrix}^T\begin{pmatrix}0 & -1\end{pmatrix}, \quad \Gamma_2 = 0.1273\,R_2, \tag{46}$$

$$R_3 = 2.8571\begin{pmatrix}-1 & 0\end{pmatrix}^T\begin{pmatrix}-1 & 0\end{pmatrix}, \quad \Gamma_3 = 0.1273\,R_3, \tag{47}$$

$$R_4 = 2.8571\begin{pmatrix}0 & 1\end{pmatrix}^T\begin{pmatrix}0 & 1\end{pmatrix}, \quad \Gamma_4 = 0.1273\,R_4. \tag{48}$$

By considering the above domains $\mathcal{X}_{R_{\mathcal{Y}_0}}$ and $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$ and the initial time $t_0 = 0$, we have maximized the time interval $[t_0\ T]$, that is, we have maximized the time instant T, in which we are able to solve the FTS conditions (34). In particular, by using the same maximization procedure defined in previous example, and by considering a symmetric partition $\mathcal{Y}$ of $\mathbf{R}^2$ composed by 8 cones, we are able to compute a solution of the FTS conditions (34) for $T = 4\,s$. Figure 2 shows some state trajectories of the considered system starting from some different points inside the initial polytopic domain.

Example 3. In this example we check the FTS property for the SD-IDLS system characterized by the matrices (37) and the reset sets (38). In this case, the FTS analysis problem is characterized by an initial domain with an ellipsoidal bound and a trajectory domain with a polytopic bound. The main goal of

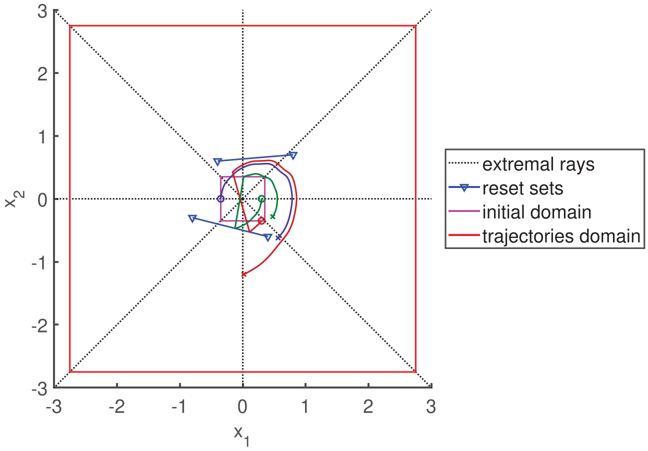


FIGURE 2 Example 2: FTS analysis problem characterized by polytopic domains. Figure shows some state trajectories, which remain confined into the polytopic domain $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$ for the whole time interval, starting from some different points inside the polytopic initial domain

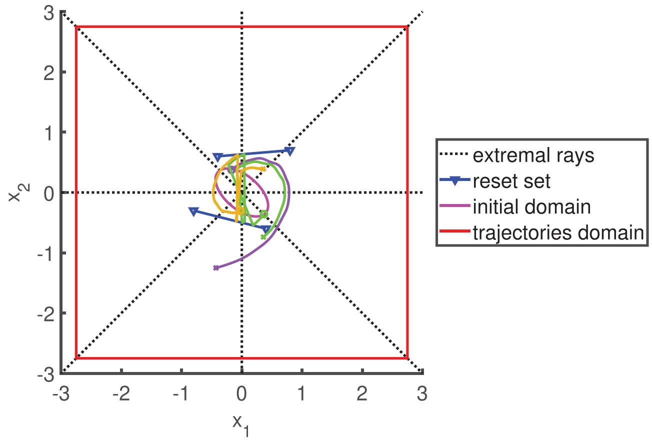


FIGURE 3 Example 3: FTS analysis problem characterized by an ellipsoidal initial domain and a polytopic trajectories domain. Figure shows some state trajectories, which remain confined into the polytopic domain $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$ for the whole time interval, starting from some different points inside the ellipsoidal initial domain.

this example is to show the capability of the proposed approach to work with different classes of domain. In particular, previous literature approaches do not allow us to solve this kind of FTS analysis problem.

To this purpose, we investigate on the FTS property of the the SD-IDLS system characterized by the matrices (37) and the reset sets (38), with respect to the initial domain defined by the positive defined matrix

$$R = \begin{bmatrix} 7.5 & 4.5 \\ 4.5 & 9.0 \end{bmatrix}, \tag{49}$$

and the trajectories domain defined by the PQD $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$, where the conical partition is $\mathcal{Y}_0 = \{G_1, G_2, G_3, G_4\}$ with

$$G_1 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T \right\}\right), \tag{50}$$

$$G_2 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & -1\end{pmatrix}^T \right\}\right), \tag{51}$$

$$G_3 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & -1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\end{pmatrix}^T \right\}\right), \tag{52}$$

$$G_4 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\end{pmatrix}^T; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T \right\}\right); \tag{53}$$

and the PQF $\Gamma_{\mathcal{Y}}(x)$ is characterized by the following diadic matrices

$$\Gamma_1 = \begin{pmatrix}\sqrt{2.75} & 0\end{pmatrix}^T \begin{pmatrix}\sqrt{2.75} & 0\end{pmatrix}, \tag{54}$$

$$\Gamma_2 = \begin{pmatrix}0 & -\sqrt{2.75}\end{pmatrix}^T \begin{pmatrix}0 & -\sqrt{2.75}\end{pmatrix}, \tag{55}$$

$$\Gamma_3 = \begin{pmatrix}-\sqrt{2.75} & 0\end{pmatrix}^T \begin{pmatrix}-\sqrt{2.75} & 0\end{pmatrix}, \tag{56}$$

$$\Gamma_4 = \begin{pmatrix}0 & \sqrt{2.75}\end{pmatrix}^T \begin{pmatrix}0 & \sqrt{2.75}\end{pmatrix}. \tag{57}$$

By considering the above ellipsoidal the PQD $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$ and the initial time $t_0 = 0$, we have maximized the time interval $\begin{bmatrix}t_0 & T\end{bmatrix}$, that is, we have maximized the time instant T, in which we are able to solve the FTS conditions (34). In particular, by using the same maximization procedure defined in Example 1, and by considering a symmetric partition $\mathcal{Y}$ of $\mathbf{R}^2$ composed by 8 cones, we are able to compute a solution of the FTS conditions (34) for $T = 4\,s$. Figure 3 shows some state trajectories, which remain confined into the polytopic domain $\mathcal{X}_{\Gamma_{\mathcal{Y}_0}}$ for the whole time interval, starting from some different points inside the ellipsoidal initial domain.

Example 4. In this example we consider a pendulum bouncing on a vertical wall (see Figure 4). Due to the vertical wall, the dynamic of the pendulum has been modelled as a SD-IDLS.

The continuous-time dynamic of the pendulum has been modelled by considering its linearized model for small oscillations. By following the hybrid model of a bouncing ball proposed in [41], the wall has been modelled as a resetting set, which instantly reverses the speed of the pendulum, moreover a positive coefficient $\mu < 1$ has been introduced to take into account the loss of energy during the impact with the wall. In this way, the SD-IDLS system (2) has been characterized by the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{C_d}{mL^2} \end{bmatrix}, A_d = \begin{bmatrix} 1 & 0 \\ 0 & -\mu \end{bmatrix}, \tag{58}$$

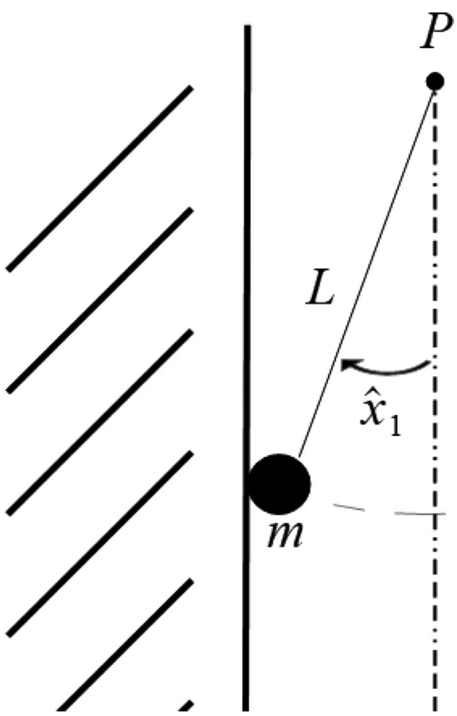


FIGURE 4 Example 4: Schematic representation of a pendulum, characterized by the mass m and the length L, bouncing on a vertical wall. The angular position of the pendulum $\hat{x}_1$ at the impact with the wall can be obtained from the distance between the pivot point P and the wall

TABLE 2 Simulation parameters

Parameters	Value
g	9.81 m/s^2
m	0.5 kg
C_d	0.01 kg m^2/ s
L	0.5 m
μ	0.9
$\hat{x}_1$	0.1745 rad

and by the following resetting set

$$\mathcal{S} = \left\{ x \in \mathbf{R}^2 : x_1 = \hat{x}_1 \right\}, \tag{59}$$

where the state variables x_1 and x_2 are the angular position and the angular speed of the pendulum, respectively, m and L are the mass and the length of the pendulum, respectively, g is the gravity acceleration, C_d is the air friction coefficient and $\hat{x}_1$ is the angular position of the pendulum at the impact with the wall. The considered values of the above the coefficients are reported in Table 2.

In the following we provide the results of the FTS analysis with respect to the initial domain defined by the PQD $\mathcal{X}_{R_{\mathcal{Y}_0}}$, where the conical partition is $\mathcal{Y}_0 = \{G_1, G_2, G_3, G_4\}$ with

$$G_1 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T ; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T \right\}\right), \tag{60}$$

$$G_2 = \text{cone}\left(\left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}1 & -1\end{pmatrix}^T ; \begin{pmatrix}-0.40 & -0.92\end{pmatrix}^T \right\}\right), \tag{61}$$

$$G_3 = \text{cone}\left(\left\{ \begin{pmatrix}-0.40 & -0.92\end{pmatrix}^T ; \begin{pmatrix}-0.40 & 0.92\end{pmatrix}^T \right\}\right), \tag{62}$$

$$G_4 = \text{cone}\left(\left\{ \begin{pmatrix}-0.40 & 0.92\end{pmatrix}^T ; \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\end{pmatrix}^T \right\}\right); \tag{63}$$

and the PQF $R_{\mathcal{Y}}(x)$ is defined by the following diadic matrices

$$R_1 = 0.1745\begin{pmatrix}1 & 0\end{pmatrix}^T\begin{pmatrix}1 & 0\end{pmatrix}, \tag{64}$$

$$R_2 = 0.2000\begin{pmatrix}0 & -1\end{pmatrix}^T\begin{pmatrix}0 & -1\end{pmatrix}, \tag{65}$$

$$R_3 = 0.0873\begin{pmatrix}-1 & 0\end{pmatrix}^T\begin{pmatrix}-1 & 0\end{pmatrix}, \tag{66}$$

$$R_4 = 0.2000\begin{pmatrix}0 & 1\end{pmatrix}^T\begin{pmatrix}0 & 1\end{pmatrix}. \tag{67}$$

Note that the initial domain constrains the initial angle to satisfy the condition

$$\hat{x}_1 \leq x_1(t_0) \leq 10 \text{ deg}, \tag{68}$$

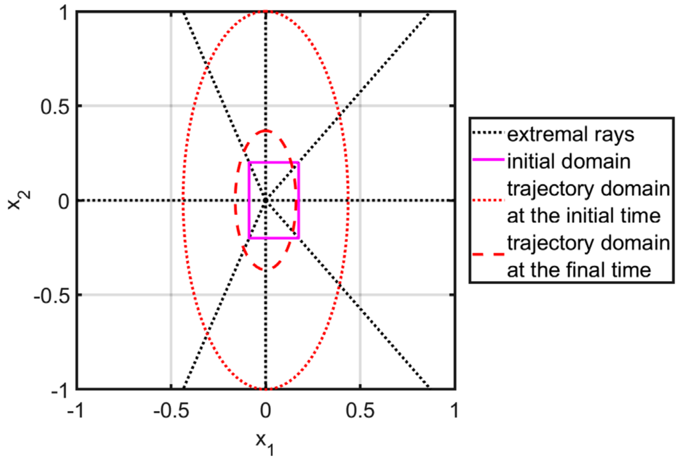
in order to take into account the minimum position related to the wall and the small oscillations assumption; the trajectory domain is defined by a time-varying ellipsoidal domain by considering the following matrix function

$$\Gamma(t) = \Gamma_0 e^{\epsilon t} \tag{69}$$

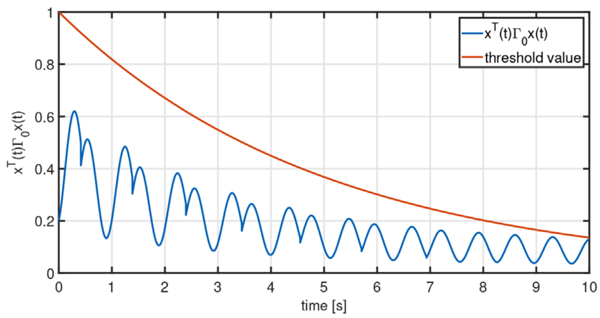
where

$$\Gamma_0 = \begin{bmatrix} 5.2525 & 0 \\ 0 & 1.0000 \end{bmatrix}, \qquad \epsilon = 0.2. \tag{70}$$

By considering the initial and final time $t_0 = 0\,s$ and $T = 10\,s$, respectively, we are able to compute a solution of the FTS conditions (34) over a partition $\mathcal{Y}$ of $\mathbf{R}^2$ composed by 8 cones. Figure 5a shows the extremal rays of the partition $\mathcal{Y}$, the polytopic initial domain $\mathcal{X}_{R_{\mathcal{Y}_0}}$ and the trajectory domains at the initial and final time, respectively. Finally, Figure 5a shows the weighted norm of the system state vector starting from



(a)



(b)

FIGURE 5 Example 4: (a) shows a polytopic initial domain and a time-vaying ellipsoidal trajectories domain that characterize the FTS analysis problem; (b) shows that, starting from the point $x_0 = [0.1745 \;\; -0.2]^T$, the weighted norm of the state vector is less than the time-varying threshold value for the whole time interval $[t_0,\; t_0 + T]$, that is, $x^T(t)\Gamma_0 x(t) < e^{-\varepsilon t}, \forall t \in [t_0,\; t_0 + T]$; this proves that the state trajectory remains confined into the time-varying trajectory domain for the whole time interval

the point $x_0 = \left[0.1745 \; -0.2\right]^T$ and the time-varying threshold value; in particular, since

$$x^T(t)\Gamma_0 x(t) < e^{-\varepsilon t}, \quad \forall t \in [t_0,\; t_0 + T], \tag{71}$$

it can be verified that the state trajectory remains confined into the time-varying trajectory domain for the whole time interval.

5 | CONCLUSIONS

Novel analysis conditions for the FTS property have been derived for the class of time-varying SD-IDLSs. The proposed approach makes use of the class of time-varying piecewise quadratic Lyapunov functions. The main advantage of this class of Lyapunov functions is related to their level curves, which represent the bounds of PQDs and they can conform the shape of the bounds of different class of domains. From the practical point of view, this approach has led us to a condition based on an optimization problem involving infinite dimensional inequalities, that has been recast as feasibility problem based on LMIs. Different numerical examples have been developed and their results prove the improvements of the proposed approach. The novel analysis condition results to be less conservative with respect to the existing literature and it allows us to deal with FTS analysis problem in which the initial and/or the trajectories domain can be modelled as PQDs, such as polytopic and ellipsoidal ones.

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CONFLICT OF INTEREST

The authors have declared no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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