



Convergence of natural p -means for the p -Laplacian in the Heisenberg group

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ABSTRACT

In this paper we prove uniform convergence of approximations to p -harmonic functions by using natural p -mean operators on bounded domains of the Heisenberg group \mathbb{H} which satisfy an intrinsic exterior corkscrew condition. These domains include Euclidean $C^{1,1}$ domains.

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1. Homage to Emmanuele

Bianca and Juan had the pleasure of knowing Emmanuele personally. Juan was fortunate to have Emmanuele as a mentor. He learnt from Emmanuele to read original sources and to write down explicitly

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all details in the proofs. They worked on regularity for p -Laplacian systems and wrote one of the first papers on the ∞ -Laplacian. Emmanuele set very high expectations and, at the same time, offered his continuous support. He was very generous with his time. Bianca met him at several conferences and was always impressed by the clarity and accessibility of his talks. Diego and András never had the honor of meeting Emmanuele, but learned and continue to learn a lot from his papers and books.

Riposa in pace, Emmanuele.

2. Introduction

Solutions to a large class of elliptic and parabolic equations can be characterized by asymptotic mean value properties (see for example [4,11]). Consider the case of p -harmonic functions for $1 < p < \infty$. A smooth function with non-vanishing gradient satisfies at a point $x \in \mathbb{R}^n$ the p -harmonic equation

$$\sum_{i,j=1}^n \left\{ \delta_{ij} + (p-2) \frac{u_{x_i}(x)u_{x_j}(x)}{|\nabla u(x)|^2} \right\} u_{x_i x_j}(x) = 0,$$

if and only if

$$u(x) = \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u + \inf_{B_\varepsilon(x)} u \right) + \beta \int_{B_\varepsilon(x)} u(y) dy + o(\varepsilon^2), \tag{2.1}$$

where $\alpha = \frac{p-2}{n+p}$ and $\beta = \frac{n+2}{n+p}$. For general viscosity solutions the same characterization prevails provided that we interpret (2.1) in the viscosity sense.

The expansion (2.1) suggests the consideration of solutions u_ε to the Dynamic Programming Principle (DPP)

$$u_\varepsilon(x) = \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right) + \beta \int_{B_\varepsilon(x)} u_\varepsilon(y) dy, \tag{2.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, the function $u_\varepsilon : \Omega \rightarrow \mathbb{R}$ and $B_\varepsilon(x) \subset \Omega$. Suppose that $g : \partial\Omega \rightarrow \mathbb{R}$ is continuous. In order to consider (2.2) for all points $x \in \overline{\Omega}$ and for all $0 < \varepsilon \leq 1$, consider the ε -boundary strip

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \Omega) \leq \varepsilon\}.$$

We extend the function g continuously to a function G to this boundary strip and consider the problem

$$\begin{cases} u_\varepsilon(x) &= \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right) + \beta \int_{B_\varepsilon(x)} u_\varepsilon(y) dy, & x \in \Omega, \\ u_\varepsilon(x) &= G(x), & x \in \Gamma_\varepsilon. \end{cases} \tag{2.3}$$

Since we are requiring that $u_\varepsilon = G$ in Γ_ε , the expression (2.2) is now well defined for $x \in \overline{\Omega}$. When the domain Ω is Lipschitz, one can solve the Dirichlet problem (2.3) and obtain a family $\{u_\varepsilon\}_{0 < \varepsilon \leq 1}$ of functions such that $u_\varepsilon \rightarrow u$ uniformly in $\overline{\Omega}$, where u is the unique viscosity solution to the Dirichlet problem

$$\begin{cases} \Delta_p^N u = \sum_{i,j=1}^n \left\{ \delta_{ij} + (p-2) \frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \right\} u_{x_i x_j} &= 0, & \text{in } \Omega \\ u &= g, & \text{in } \partial\Omega. \end{cases} \tag{2.4}$$

We note that viscosity solutions of the problem (2.4) are also weak solutions, when the normalized p -Laplacian is replaced by the variational p -Laplacian, $\text{div} \left(|\nabla u|^{p-2} \nabla u \right) = 0$, [8]. For a mean value property that applies directly to the variational p -Laplacian see [18].

The nonlinear mean value expression in the right-hand side of (2.2) is a *tug-of-war with noise* mean, suggested by connections with probability developed in [16,17].

Let $\Omega_E = \Omega \cup \Gamma_1$ and $\mathcal{B}(\overline{\Omega})$, $\mathcal{B}(\Omega_E)$ be the class of bounded real measurable functions defined on $\overline{\Omega}$ and Ω_E , respectively.

Definition 2.1. We say that an operator $A : \mathcal{B}(\Omega_E) \rightarrow \mathcal{B}(\overline{\Omega})$ is an *average operator* if it satisfies the following properties:

- (Stability) $\inf_{\{y \in \Omega_E\}} \phi(y) \leq A[\phi](x) \leq \sup_{\{y \in \Omega_E\}} \phi(y), \forall x \in \overline{\Omega}$;
- (Monotonicity) If $\phi \leq \psi$ in Ω_E then $A[\phi] \leq A[\psi]$ in $\overline{\Omega}$;
- (Affine invariance) $A[\lambda\phi + \xi] = \lambda A[\phi] + \xi, \forall \lambda > 0, \forall \xi \in \mathbb{R}$.

Definition 2.2. We say that a family of averages $\{A_\varepsilon\}_{\varepsilon > 0}$ satisfies the **asymptotic mean value property (AMVP)** for the p -Laplacian if for every $\phi \in C^\infty(\Omega_E)$ such that $\nabla\phi \neq 0$, we have

$$A_\varepsilon[\phi](x) = \phi(x) + c\varepsilon^2 (\Delta_p^N \phi(x)) + o(\varepsilon^2)$$

for some constant $c > 0$ independent of ε and ϕ , and $o(\varepsilon^2)$ can be taken uniformly for all $x \in \overline{\Omega}$.

Associated to an average operator A_ε , we have a dynamic programming principle (DPP) at scale ε given by

$$\begin{cases} u_\varepsilon(x) &= A_\varepsilon[u_\varepsilon](x) & \text{in } \Omega, \\ u_\varepsilon(x) &= G(x) & \text{on } \Gamma_1. \end{cases} \tag{2.5}$$

Later in Section 3 we will discuss existence and uniqueness for the DPP (2.5) associated to the natural p -means in the Heisenberg group.

We will say that a function $u \in \mathcal{B}(\Omega_E)$ is a subsolution (resp. supersolution) of (2.5) in Ω with boundary datum G , if $u \leq G$ (resp. $u \geq G$) on Γ_1 and $u(x) \leq A_\varepsilon[u](x)$ (resp. $u(x) \geq A_\varepsilon[u](x)$) for $x \in \Omega$.

Consider the following conditions on the family of averages $\{A_\varepsilon\}_{\varepsilon > 0}$:

Uniform Stability:

For all $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{B}(\Omega_E)$, a solution of (2.5) with a bound on $\|u_\varepsilon\|_{L^\infty(\Omega_E)}$ uniform in ε . (US)

Uniform Boundedness:

For all $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{B}(\Omega_E)$, a solution of (2.5), and $\inf_{\Gamma_1} G \leq u_\varepsilon(x) \leq \sup_{\Gamma_1} G$ for all $x \in \Omega$. (UB)

Comparison Principle:

Let u_ε^1 and u_ε^2 be a subsolution and a supersolution of (2.5) with boundary data G_1 and G_2 respectively. (CP)
 If $G_1 \leq G_2$ on Γ_1 , then $u_\varepsilon^1 \leq u_\varepsilon^2$ in Ω_E .

Theorem 2.3 (Convergence Of General Mean Approximations in the Euclidean Case, $1 < p \leq \infty$, [19]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $g \in C(\partial\Omega)$. Let the family of averages $\{A_\varepsilon\}_{\varepsilon > 0}$ satisfy the **AMVP** with respect to the p -Laplacian. Let also $\{u_\varepsilon\}_{\varepsilon > 0}$ be a sequence of solutions of the corresponding DPP (2.5), where G is a continuous extension of g to Γ_1 . Then, we have that*

$$u_\varepsilon \rightarrow u \text{ uniformly in } \overline{\Omega} \text{ as } \varepsilon \rightarrow 0,$$

- when the domain Ω is of class C^2 and the family of averages $\{A_\varepsilon\}_{\varepsilon > 0}$ satisfies (US), or
- when the domain Ω is Lipschitz and the family of averages $\{A_\varepsilon\}_{\varepsilon > 0}$ satisfies the uniform boundedness (UB) and the comparison principle (CP) properties,

where u is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u &= 0 & \text{in } \Omega \\ u &= g & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

The proof of this theorem for C^2 domains is based on an extension of the method in [2], once we have the *strong uniqueness theorem* for the p -Laplacian (Proposition 3.2 in [19]). The key observation is that the convergence of approximations that satisfy the asymptotic mean value property, the uniform boundedness property (UB), and the comparison principle (CP) depends only on the strong uniqueness principle for the limit operator, which is the p -Laplacian in this case.

To apply the Barles–Souganidis method in smooth domains of the Heisenberg group \mathbb{H} , we need to establish the *strong uniqueness principle* for the normalized p -Laplacian in the Heisenberg group $\Delta_{\mathbb{H},p}^N$. In [13] we were able to establish it only when the domain $\Omega = B$, the Euclidean unit ball in \mathbb{R}^3 . Thus, we concluded the convergence of general mean approximations only in this case.

For more general domains in \mathbb{H} an obstruction to the application of this method is the presence of characteristic points. In the current paper, we pursue a different approach that does not rely on the strong uniqueness principle. First, we use good properties of the fundamental solution of the p -Laplacian to establish convergence in smooth ring domains with p -harmonic boundary data. Second, we set up a boundary iteration suggested by the proof of sufficiency for the Wiener condition of boundary regularity. These steps are independent of each other. We will establish them for the case of the natural p -means in the Heisenberg group.

The notion of natural p -means in general topological measure spaces was introduced in [7]. Let X be a compact topological measure space endowed with a positive Radon measure ν . Given a function $u \in L^p(X)$ and $1 < p \leq \infty$, there exists a unique real value $\mu_p^X(u)$ such that

$$\|u - \mu_p^X(u)\|_p = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_p. \tag{2.7}$$

We will call $\mu_p^X(u)$ the natural p -mean of u in X . Note that the above definition extends to the case $p = 1$, provided that u is assumed to be continuous on X . In this case $\mu_1^X(u)$ is the median $\text{med}(u)$ on X , which is the unique solution λ of the equation

$$\nu(\{y \in X : u(y) \geq \lambda\}) = \nu(\{y \in X : u(y) \leq \lambda\}).$$

Existence, uniqueness, and several useful properties of the natural p -means were studied in [7], where the AMVP for the natural p -means is established in the Euclidean case.

While for general p there is no explicit formula for $\mu_p^X(u)$, for the cases $p = 1, 2$, and $p = \infty$ we have:

$$\begin{aligned} \mu_1^X(u) &= \text{med}(u), \\ \mu_2^X(u) &= \int_X u(y) d\nu, \text{ and} \\ \mu_\infty^X(u) &= \frac{1}{2}(\text{ess inf}_{y \in X} u(y) + \text{ess sup}_{y \in X} u(y)). \end{aligned}$$

Consider the family of natural p -mean operators $\{\mu_p(u, \varepsilon)\}_{0 < \varepsilon < 1}$ defined on functions $u \in \mathcal{B}(\Omega_E)$ as follows. For $x \in \overline{\Omega} \subset \mathbb{R}^n$ and $B_\varepsilon(x)$ the Euclidean ball of radius ε centered at x , we set

$$\mu_p(u, \varepsilon)(x) = \mu_p^{\overline{B_\varepsilon(x)}}(u). \tag{2.8}$$

Observe that for any $u \in L^p(\Omega_E)$ the function $x \mapsto \mu_p(u, \varepsilon)(x)$ is continuous in $\overline{\Omega}$. This property is not shared by the tug-of-war means of type (2.2).

We can now rephrase Theorem 3.2 in [7]:

Theorem 2.4. *The family of natural p -mean operators $\{\mu_p(\cdot, \varepsilon)\}_{0 < \varepsilon < 1}$ satisfies the asymptotic mean value property relative to the p -Laplacian; that is, for every $\phi \in C^\infty(\Omega_E)$ such that $\nabla\phi \neq 0$, we have*

$$\mu_p(\phi, \varepsilon)(x) = \phi(x) + \frac{\varepsilon^2}{2(n+p)} \Delta_p^N \phi(x) + o(\varepsilon^2), \text{ as } \varepsilon \rightarrow 0, \tag{2.9}$$

where $o(\varepsilon^2)$ can be taken uniformly for all $x \in \overline{\Omega}$.

Here Δ_p^N is the operator defined in (2.4).

The uniform stability (US) property, the uniform boundedness (UB) property, and the comparison principle (CP) of the natural p -means in the Euclidean space were established in [13]. Thus, Theorems 2.3 and 2.4 can be combined to get the following theorem.

Theorem 2.5 (Convergence Of Natural p -Mean Approximations in the Euclidean Case, [13]). *Assume $p \in (1, \infty]$, $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $g \in C(\partial\Omega)$. For $0 < \varepsilon \leq 1$ and $A_\varepsilon = \mu_p(\cdot, \varepsilon)$, let u_ε be the solution to the DPP (2.5), where G is a continuous extension of g to Γ_1 . Then, we have*

$$u_\varepsilon \rightarrow u \text{ uniformly in } \overline{\Omega} \text{ as } \varepsilon \rightarrow 0,$$

where u is the unique solution of the Dirichlet problem (2.6).

In a recent paper, Chandra, Ishiwata, Magnanini, and Wadade [6] have also proved the convergence of the natural p -means in the Euclidean case. Their approach and our approach differ in the treatment at the boundary, but the main results are essentially the same in the case of \mathbb{R}^n .

Recall that for the first Heisenberg group, $\mathbb{H} = (\mathbb{R}^3, *)$, the group operation is given as

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1) \right).$$

The vector fields

$$X_1 = \partial_{x_1} - \frac{x_2}{2}\partial_{x_3}, \quad X_2 = \partial_{x_2} + \frac{x_1}{2}\partial_{x_3} \quad \text{and} \quad T = \partial_{x_3}$$

form a basis of the associated Lie algebra. We denote the horizontal gradient of a smooth function u by $\nabla_{\mathbb{H}}u = (X_1u)X_1 + (X_2u)X_2$, the horizontal Laplacian by

$$\Delta_{\mathbb{H}}u = X_1^2u + X_2^2u,$$

the horizontal p -Laplacian by

$$\Delta_{\mathbb{H},p}u = X_1(|\nabla_{\mathbb{H}}u|^{p-2}X_1u) + X_2(|\nabla_{\mathbb{H}}u|^{p-2}X_2u),$$

and the normalized horizontal ∞ -Laplacian by

$$\Delta_{\mathbb{H},\infty}u = \left\langle D_{\mathbb{H}}^{2,*}u \frac{\nabla_{\mathbb{H}}u}{|\nabla_{\mathbb{H}}u|}, \frac{\nabla_{\mathbb{H}}u}{|\nabla_{\mathbb{H}}u|} \right\rangle,$$

where $D_{\mathbb{H}}^{2,*}u$ denotes the symmetrized Hessian $(D_{\mathbb{H}}^{2,*}u)_{ij} = (X_iX_j + X_jX_i)/2$. For a smooth function u , with non-vanishing horizontal gradient, we define the normalized p -Laplacian as

$$\Delta_{\mathbb{H},p}^N u = |\nabla_{\mathbb{H}}u|^{2-p} \Delta_{\mathbb{H},p}u = (p-2)\Delta_{\mathbb{H},\infty}u + \Delta_{\mathbb{H}}u.$$

The Korányi smooth gauge, given by

$$|x|_{\mathbb{H}} = |(x_1, x_2, x_3)|_{\mathbb{H}} = ((x_1^2 + x_2^2)^2 + 16x_3^2)^{\frac{1}{4}},$$

induces the left-invariant metric

$$d_{\mathbb{H}}(x, y) = |x^{-1} * y|_{\mathbb{H}}.$$

The Korányi ball with center x_0 and radius r will be denoted by

$$B_r^{\mathbb{H}}(x_0) = \{x \in \mathbb{H} \mid d_{\mathbb{H}}(x_0, x) < r\}.$$

The Heisenberg group \mathbb{H} is unimodular and its Haar measure is the Lebesgue measure in \mathbb{R}^3 . We also have a family of anisotropic dilations $(\rho_\lambda)_{\lambda>0}$, that are group isomorphisms

$$\rho_\lambda(x) = \rho_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3). \tag{2.10}$$

The homogeneous dimension of \mathbb{H} is $Q = 4$.

The natural p -means in the Heisenberg group \mathbb{H} were studied in [13]. Suppose that $\Omega \subset \mathbb{H}$ is a bounded domain. Fix $0 \leq \varepsilon < 1$ and consider the ε -boundary strip given by

$$\Gamma_\varepsilon^{\mathbb{H}} = \{x \in \mathbb{H} \setminus \Omega : \text{dist}_{\mathbb{H}}(x, \Omega) \leq \varepsilon\}.$$

Define $\Omega_E = \Omega \cup \Gamma_1^{\mathbb{H}}$ and denote by $\mathcal{B}(\Omega_E)$ and $\mathcal{B}(\overline{\Omega})$ the set of real-valued bounded Lebesgue measurable functions on Ω_E and $\overline{\Omega}$, respectively. We define the natural p -mean average operator in the Heisenberg group,

$$\mu_p^{\mathbb{H}}(\cdot, \varepsilon) : \mathcal{B}(\Omega_E) \longrightarrow \mathcal{B}(\overline{\Omega}),$$

given by

$$\mu_p^{\mathbb{H}}(\phi, \varepsilon)(y) = \overline{\mu_p^{\mathbb{H}}(\phi, \varepsilon)(y)} \quad \text{for all } \phi \in \mathcal{B}(\Omega_E) \text{ and } y \in \overline{\Omega},$$

where $\overline{\mu_p^{\mathbb{H}}(\phi, \varepsilon)(y)}$ is defined as in (2.7), using as Radon measure ν the Lebesgue measure on $X = \overline{B_\varepsilon^{\mathbb{H}}(y)}$.

Let us now describe the DPP associated to the natural p -means in \mathbb{H} . Let $G : \Gamma_\varepsilon^{\mathbb{H}} \rightarrow \mathbb{R}$ be a continuous function and consider the solutions (in the point-wise sense) of the following boundary value problem

$$\begin{cases} u_\varepsilon(x) &= \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x) & \text{if } x \in \Omega, \\ u_\varepsilon(x) &= G(x) & \text{if } x \in \Gamma_\varepsilon^{\mathbb{H}}. \end{cases} \tag{2.11}$$

The stability and monotonicity of the natural p -means in \mathbb{H} was established in [13], as well as the following AMVP with respect to the p -Laplacian

Lemma 2.6 ([13]). *For $1 < p < \infty$ define the constant*

$$c_p = \frac{2}{(p+2)(p+4)} \left(\frac{\Gamma(\frac{p}{4} + \frac{3}{2})}{\Gamma(\frac{p}{4} + 1)} \right)^2,$$

where Γ is the Euler Gamma function. Let u be a smooth function in Ω_E with $\nabla_{\mathbb{H}} u \neq 0$ in Ω_E . Then, we have the expansion

$$\mu_p^{\mathbb{H}}(u, \varepsilon)(x_0) = u(x_0) + c_p \varepsilon^2 \Delta_{\mathbb{H},p}^N u(x_0) + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0$$

for all $x_0 \in \overline{\Omega}$.

When $p = \infty$, the lemma also holds with $c_\infty = \lim_{p \rightarrow \infty} c_p = 1/2$. We remark that this lemma has been obtained independently and in the case of general Carnot groups in [1].

To show convergence, we first study the case when the boundary data is itself the restriction of a p -harmonic function with non-vanishing gradient. For tug-of-war means

$$T(u, \varepsilon)(x) = \frac{\alpha}{2} \left(\sup_{B_\varepsilon(x)} u_\varepsilon + \inf_{B_\varepsilon(x)} u_\varepsilon \right) + \beta \int_{B_\varepsilon(x)} u_\varepsilon(y) dy,$$

this was first done using probability in [12] in \mathbb{R}^n . The proofs of our results in this paper are analytic and do not rely on probabilistic techniques.

To go from continuous to semi-discrete, we build sub- and super-solutions of the DPP (2.11) from solutions to the continuous problem by using the following perturbation lemma.

Lemma 2.7 (Perturbations For Natural p -Means, $p \geq 2$). *Let Ω' be an open set containing Ω , and let U be a function such that $\Delta_{\mathbb{H},p}^N U = 0$ and $\nabla_{\mathbb{H}} U \neq 0$ in Ω' . Then, there exist $\hat{\varepsilon} > 0$, $s \geq 4$ and $q_0 \in \mathbb{H}$ such that, denoting $v(x) = |q_0^{-1} * x|_{\mathbb{H}}^s$, we have*

$$U + \varepsilon v \text{ is a subsolution of (2.11)}$$

and

$$U - \varepsilon v \text{ is a supersolution of (2.11)}$$

in Ω_ε for all $0 < \varepsilon < \hat{\varepsilon}$.

For tug-of-war means the Perturbation Lemma above is valid for $1 < p < \infty$, and it is due to Lewicka [9,14] in \mathbb{R}^n and to [10] in \mathbb{H} . The key to proving this lemma in \mathbb{H} is a strengthening of the expansion in ε of $\mu_p^{\mathbb{H}}(u, \varepsilon)(x_0)$ that we are able to prove for $p \geq 2$

Proposition 2.8. *Let $p \geq 2$ and u be a smooth function in Ω_E with $\nabla_{\mathbb{H}} u \neq 0$ in Ω_E . Then, there exist $C > 0$ and $\hat{\varepsilon} > 0$ such that*

$$|\mu_p^{\mathbb{H}}(u, \varepsilon)(x_0) - u(x_0) - c_p \varepsilon^2 \Delta_{\mathbb{H},p}^N u(x_0)| \leq C \varepsilon^3$$

for all $0 < \varepsilon < \hat{\varepsilon}$ and $x_0 \in \overline{\Omega}$. In particular, the constants C and $\hat{\varepsilon}$ are uniform in $x_0 \in \overline{\Omega}$ and depend only on p and the derivatives of u .

Note that we have replaced $o(\varepsilon^2)$ by $O(\varepsilon^3)$ when $p \geq 2$. We remark that the new argument in the proof of this Proposition can also be used to give an alternative proof of the second order expansion in Lemma 2.6 in the case $1 < p < 2$, different than those in [1,7].

From Proposition 2.8 the convergence when the boundary data is itself the restriction of a p -harmonic function with non-vanishing gradient follows.

Proposition 2.9. *Let Ω' be an open set containing Ω . Let U be a function such that $\Delta_{\mathbb{H},p}^N U = 0$ and $\nabla_{\mathbb{H}} U \neq 0$ in Ω' and let u_ε be the solution of (2.11) with boundary datum U , for $\varepsilon > 0$ sufficiently small. Then*

$$u_\varepsilon \rightarrow U \quad \text{uniformly in } \overline{\Omega}$$

This proposition gives convergence in arbitrary domains Ω in the special case when the boundary values are the restriction of a p -harmonic function with non vanishing gradient in a slightly larger domain. To study the convergence of solutions to the DPP (2.5) when $\varepsilon \rightarrow 0$ with arbitrary continuous boundary values, we define the following boundary regularity condition.

Definition 2.10. We say that a domain $\Omega \subset \mathbb{H}$ satisfies the exterior \mathbb{H} -corkscrew condition if there exists $\bar{\delta} > 0$ and $\mu \in (0, 1)$ such that for every $\delta \in (0, \bar{\delta})$ and $y \in \partial\Omega$ there exists a ball $B_{\mu\delta}^{\mathbb{H}}(z)$ strictly contained in $B_\delta^{\mathbb{H}}(y) \setminus \Omega$.

It is known that domains with $C^{1,1}$ boundary in the Euclidean sense satisfy the exterior \mathbb{H} -corkscrew condition (see [5], Theorem 14 for domains in the Heisenberg group and [15], Theorem 1.3 for the more general case of domains in step 2 Carnot groups). This regularity is optimal in the sense that for every $\alpha \in [0, 1)$ there exist domains with $C^{1,\alpha}$ boundary in the Euclidean sense that do not satisfy the condition in Definition 2.10 (see Example 8.2 in [10]).

For domains satisfying the exterior \mathbb{H} -corkscrew condition we first prove the following boundary estimate.

Theorem 2.11. *Let Ω be an open, bounded subset of \mathbb{H} satisfying the exterior \mathbb{H} -corkscrew condition in Definition 2.10 and $G \in C(\Gamma_1^{\mathbb{H}})$. For $2 \leq p < \infty$ let u_ε be the solution of (2.11) in Ω with boundary value G on $\Gamma_\varepsilon^{\mathbb{H}}$, for $0 < \varepsilon < 1$. Given $\eta > 0$ there exist $\delta_0 = \delta_0(\eta, \mu, p)$ and $\varepsilon_0 = \varepsilon_0(\eta, \delta, \mu)$ such that*

$$|u_\varepsilon(x) - G(y)| \leq \eta,$$

for all $y \in \partial\Omega$, $x \in B_{\delta_0}^{\mathbb{H}}(y) \cap \Omega$ and $\varepsilon \leq \varepsilon_0$.

Once we have Theorem 2.11, we are ready to state our main result.

Theorem 2.12 (For The Range $2 \leq p < \infty$). *Let Ω be an open bounded subset of \mathbb{H} satisfying the exterior \mathbb{H} -corkscrew condition and let $\Gamma_\varepsilon^{\mathbb{H}}$ be its outer ε -boundary, for $0 < \varepsilon < 1$. Let $g \in C(\partial\Omega)$, $G \in C(\Gamma_1^{\mathbb{H}})$ be a continuous extension of g and u_ε be the solution of the DPP (2.11) with boundary datum G . Then u_ε converges to u uniformly on $\overline{\Omega}$ as $\varepsilon \rightarrow 0$, where u is the solution to the Dirichlet problem*

$$\begin{cases} -\Delta_{\mathbb{H},p}^N u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The plan of the paper is as follows. In Section 3 we present various properties of the natural p -means in \mathbb{H} . The proofs of Lemma 2.6 and of Proposition 2.8 are in Section 4, while the proofs of the perturbation Lemma 2.7 and Proposition 2.9 are in Section 5. The proof of the key boundary continuity estimate Theorem 2.11 is in Section 6, which is the most technical section of the paper, and the proof of the main result is in Section 7.

The major technical difference between the results in [13], this article, and [6] is that in the latter the DPP is modified at points close to $\partial\Omega$. As a consequence the solution of the DPP exists and is continuous provided that Ω satisfies a regularity condition implied by the exterior sphere property. Thus, we both prove convergence of slightly different approximations to p -harmonic functions. In our case, we get existence of possibly discontinuous solutions of the DPP for a general bounded domain Ω , but need boundary regularity to prove convergence.

Finally, we note that the validity of Proposition 2.8 for $1 < p < 2$ would imply that Theorem 2.12 also holds for $1 < p < 2$. However, our current proof of Proposition 2.8 requires $p \geq 2$.

3. Natural p -means in \mathbb{H}

We now collect several results on the natural p -means in \mathbb{H} that we will need later. For all $u, v \in \mathcal{B}(\Omega_E)$, $0 < \varepsilon \leq 1$ and $y \in \Omega$, the following properties hold:

- Continuity in the L^p -norm (Theorem 2.4 in [7]):

$$\left| \|u - \mu_p^{\mathbb{H}}(u, \varepsilon)(y)\|_{L^p(B_\varepsilon^{\mathbb{H}}(y))} - \|v - \mu_p^{\mathbb{H}}(v, \varepsilon)(y)\|_{L^p(B_\varepsilon^{\mathbb{H}}(y))} \right| \leq \|u - v\|_{L^p(B_\varepsilon^{\mathbb{H}}(y))}.$$

In particular, the function $y \mapsto \mu_p^{\mathbb{H}}(u, \varepsilon)(y)$ is continuous in $\overline{\Omega}$.

- Monotonicity (Theorem 2.5 in [7]):

$$\text{if } u \leq v \text{ a.e. on } B_\varepsilon^{\mathbb{H}}(y), \text{ then } \mu_p^{\mathbb{H}}(u, \varepsilon)(y) \leq \mu_p^{\mathbb{H}}(v, \varepsilon)(y).$$

- Affine invariance (Proposition 2.7 in [7]): for $c, \alpha \in \mathbb{R}$ it holds

$$\mu_p^{\mathbb{H}}(\alpha u + c, \varepsilon)(y) = \alpha \mu_p^{\mathbb{H}}(u, \varepsilon)(y) + c.$$

- Rescaling (Corollary 2.3 in [7]): Let $x \in \Omega$. Defining $u_{\varepsilon, x}(z) = u(x + \varepsilon z)$ for $z \in B_1^{\mathbb{H}}(0)$, we have

$$\mu_p(u, \varepsilon)(x) = \mu_p(u_{\varepsilon, x}, 1)(0).$$

- Integral characterization (Theorem 2.1 in [7]): For $1 < p < \infty$, $\mu_p^{\mathbb{H}}(u, \varepsilon)(y)$ is the unique solution of

$$\int_{B_\varepsilon^{\mathbb{H}}(y)} |u(z) - \mu_p^{\mathbb{H}}(u, \varepsilon)(y)|^{p-2} (u(z) - \mu_p^{\mathbb{H}}(u, \varepsilon)(y)) \, dz = 0,$$

with the convention that $0^{p-2}0 = 0$ if $1 < p < 2$.

All the properties above extend to the case $p = 1$ if we assume that u and v are continuous on Ω_E .

Existence and uniqueness of the solution of the DPP (2.11) can be proved as in the Euclidean case. Solutions are automatically continuous in the interior (see Lemma 5.6 and Theorem 3.4 in [13]):

Lemma 3.1. *Let $1 < p \leq \infty$. There exists a unique $u_\varepsilon \in \mathcal{B}(\Omega_\varepsilon)$ that satisfies (2.11). Moreover, we have that $u_\varepsilon \in C(\Omega)$.*

Note, however, that u_ε might not be continuous on $\overline{\Omega}$.

Next, we provide a version of the comparison principle which will be needed later. This is a slight modification of Theorem 4.3 in [13].

Lemma 3.2. *Let $u_\varepsilon \in \mathcal{B}(\Omega_\varepsilon)$ be a subsolution of (2.11) in Ω with boundary datum F , and $v_\varepsilon \in \mathcal{B}(\Omega_\varepsilon)$ be a supersolution of (2.11) in Ω with boundary datum G . Then*

$$u_\varepsilon \leq v_\varepsilon + \sup_{\Gamma_\varepsilon^{\mathbb{H}}} (F - G) \quad \text{on } \Omega_\varepsilon.$$

Proof. Let $\phi_\varepsilon = u_\varepsilon - v_\varepsilon$ be defined on Ω_ε . Note that $\phi_\varepsilon \leq F - G$ on $\Gamma_\varepsilon^{\mathbb{H}}$ and therefore $\sup_{\Gamma_\varepsilon^{\mathbb{H}}} \phi_\varepsilon \leq \sup_{\Gamma_\varepsilon^{\mathbb{H}}} (F - G)$. Assume by contradiction that

$$M_\varepsilon := \sup_{\Omega} \phi_\varepsilon > \sup_{\Gamma_\varepsilon^{\mathbb{H}}} (F - G).$$

Note that this implies $\sup_{\Omega_\varepsilon} \phi_\varepsilon = \sup_{\Omega} \phi_\varepsilon = M_\varepsilon$. By definition of M_ε , there exists a sequence $x_n \in \Omega$ such that $\phi_\varepsilon(x_n)$ converges to M_ε . We can assume that, up to a subsequence, x_n converges to some $x_0 \in \overline{\Omega}$. Passing to the limit in the inequality

$$\phi_\varepsilon(x_n) \leq \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_n) - \mu_p^{\mathbb{H}}(v_\varepsilon, \varepsilon)(x_n),$$

we obtain

$$M_\varepsilon \leq \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0) - \mu_p^{\mathbb{H}}(v_\varepsilon, \varepsilon)(x_0),$$

because the function $x \rightarrow \mu_p^{\mathbb{H}}(f, \varepsilon)(x)$ is continuous in $\overline{\Omega}$ for $f \in L^p(\Omega_\varepsilon)$. Therefore,

$$\begin{aligned} \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0) &\geq \mu_p^{\mathbb{H}}(v_\varepsilon, \varepsilon)(x_0) + M_\varepsilon \\ &= \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0). \end{aligned} \tag{3.1}$$

Using the notation $h(u, \lambda) = |u - \lambda|^{p-2}(u - \lambda)$, we have that h is increasing in u for fixed λ , and decreasing in λ for fixed u . Therefore, in Ω_ε , we have

$$\begin{aligned} h(u_\varepsilon, \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0)) &\leq h(u_\varepsilon, \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0)) \\ &\leq h(v_\varepsilon + M_\varepsilon, \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0)), \end{aligned}$$

because $u_\varepsilon - v_\varepsilon \leq M_\varepsilon$ on Ω_ε . We can rewrite the previous inequality as

$$h(u_\varepsilon, \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0)) - h(v_\varepsilon + M_\varepsilon, \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0)) \leq 0$$

in Ω_ε . The integral characterization of the natural p -means implies that

$$\int_{B_\varepsilon^{\mathbb{H}}(x_0)} h(u_\varepsilon, \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0)) - h(v_\varepsilon + M_\varepsilon, \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0)) = 0,$$

and therefore

$$h(u_\varepsilon, \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0)) - h(v_\varepsilon + M_\varepsilon, \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0)) = 0 \quad \text{a.e. in } B_\varepsilon^{\mathbb{H}}(x_0).$$

Since $s \rightarrow |s|^{p-2}s$ is injective, we obtain

$$u_\varepsilon - \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0) = v_\varepsilon + M_\varepsilon - \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0) = 0 \quad \text{a.e. in } B_\varepsilon^{\mathbb{H}}(x_0),$$

which rewrites as

$$u_\varepsilon - v_\varepsilon = M_\varepsilon + \mu_p^{\mathbb{H}}(u_\varepsilon, \varepsilon)(x_0) - \mu_p^{\mathbb{H}}(v_\varepsilon + M_\varepsilon, \varepsilon)(x_0).$$

Recalling (3.1), we get $u_\varepsilon - v_\varepsilon \geq M_\varepsilon = \sup_{\Omega_\varepsilon}(u_\varepsilon - v_\varepsilon)$ a.e. in $B_\varepsilon^{\mathbb{H}}(x_0)$, hence

$$u_\varepsilon - v_\varepsilon = M_\varepsilon \quad \text{a.e. in } B_\varepsilon^{\mathbb{H}}(x_0).$$

The previous argument shows that if $x \in \bar{\Omega}$ is such that $\phi_\varepsilon(x) = M_\varepsilon$, then $\phi_\varepsilon = M_\varepsilon$ a.e. in $B_\varepsilon^{\mathbb{H}}(x)$. Since Ω is bounded and connected, it is possible to find a finite chain of balls $\{B_\varepsilon^{\mathbb{H}}(x_i)\}_{i=0}^N$ starting from x_0 , such that x_{i+1} is a point in $B_\varepsilon^{\mathbb{H}}(x_i)$ with $\phi_\varepsilon(x_{i+1}) = M_\varepsilon$ for all $i = 0, \dots, N - 1$ and $B_\varepsilon^{\mathbb{H}}(x_N) \cap \Gamma_\varepsilon^{\mathbb{H}} \neq \emptyset$. This means that $\phi_\varepsilon = M_\varepsilon = \sup_\Omega \phi_\varepsilon$ a.e. on $B_\varepsilon^{\mathbb{H}}(x_N) \cap \Gamma_\varepsilon^{\mathbb{H}}$, contradicting $\phi_\varepsilon \leq F - G$ on $\Gamma_\varepsilon^{\mathbb{H}}$. \square

4. Taylor expansions

The goal of this section is to prove Proposition 2.8. The case $p = 2$ follows directly from the Taylor expansion of u around x_0 , so that we focus on the case $p > 2$.

First, we include details of the proof of Lemma 2.6 in the relevant case for our analysis $p > 2$, because similar arguments will be used later to prove the stronger version Proposition 2.8. We pay particular attention to the dependence of the estimate on $x_0 \in \bar{\Omega}$.

Proof. Let $p > 2$ and denote $h(s) = |s|^{p-2}s$. Let $x_0 \in \Omega$ and $0 < \varepsilon < 1$. Define $u_{\varepsilon, x_0}(z) = u(x_0 * \rho_\varepsilon(z))$ for all $z = (z_1, z_2, z_3) = (z_h, z_3) \in B$, where $B = B_1^{\mathbb{H}}(0)$ denotes the Korányi ball centered at the origin with radius 1. The Heisenberg Taylor expansion gives

$$u_{\varepsilon, x_0}(z) = u(x_0) + \varepsilon \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + \varepsilon^2 T u(x_0) z_3 + \frac{1}{2} \varepsilon^2 \langle D_{\mathbb{H}}^2 u(x_0) z_h, z_h \rangle + O(\varepsilon^3),$$

where the constant in $O(\varepsilon^3)$ is uniform in $x_0 \in \Omega$ and depends on the $L^\infty(\Omega)$ norm of the second and third derivatives of u . Therefore, by the affine invariance and monotonicity of the natural means, it is enough to show the expansion for the quadratic function

$$q_{\varepsilon, x_0}(z) = u(x_0) + \varepsilon \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + \varepsilon^2 T u(x_0) z_3 + \frac{1}{2} \varepsilon^2 \langle D_{\mathbb{H}}^2 u(x_0) z_h, z_h \rangle.$$

1. First order term. Let

$$v_{\varepsilon, x_0}(z) = \frac{q_{\varepsilon, x_0}(z) - q_{\varepsilon, x_0}(0)}{\varepsilon} = \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + \varepsilon Tu(x_0)z_3 + \frac{1}{2}\varepsilon \langle D_{\mathbb{H}}^2 u(x_0)z_h, z_h \rangle,$$

and note that v_{ε, x_0} converges uniformly on B to the function $v_{x_0}(z) = \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle$ as $\varepsilon \rightarrow 0$. As a consequence,

$$\frac{\mu_p^{\mathbb{H}}(q_{\varepsilon, x_0}, 1)(0) - q_{\varepsilon, x_0}(0)}{\varepsilon} = \mu_p^{\mathbb{H}}(v_{\varepsilon, x_0}, 1)(0) \rightarrow \mu_p^{\mathbb{H}}(v_{x_0}, 1)(0) = 0,$$

where the last equality is due to

$$\int_B h(v_{x_0}(z)) \, dz = \int_B |\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle|^{p-2} \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle \, dz = 0,$$

which holds by symmetry of the integrand on B .

2. Second order term. Let

$$\delta_{\varepsilon}(x_0) = \frac{\mu_p^{\mathbb{H}}(q_{\varepsilon, x_0}, 1)(0) - q_{\varepsilon, x_0}(0)}{\varepsilon^2} = \frac{\mu_p^{\mathbb{H}}(v_{\varepsilon, x_0}, 1)(0)}{\varepsilon}.$$

By the integral characterization of the natural means we have

$$\begin{aligned} 0 &= \int_B h(v_{\varepsilon, x_0}(z) - \mu_p^{\mathbb{H}}(v_{\varepsilon, x_0}, 1)) \, dz \\ &= \int_B h \left(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + \varepsilon Tu(x_0)z_3 + \frac{1}{2}\varepsilon \langle D_{\mathbb{H}}^2 u(x_0)z_h, z_h \rangle - \varepsilon \delta_{\varepsilon}(x_0) \right) \, dz \\ &= \int_B h(v_{x_0}(z)) \, dz + \varepsilon \int_B \left(\int_0^1 h'(F(z, x_0, \varepsilon, t)) \, dt \right) \psi(z, x_0, \varepsilon) \, dz, \end{aligned} \tag{4.1}$$

where we used a first order Taylor expansion and denoted

$$\psi(z, x_0, \varepsilon) = Tu(x_0)z_3 + \frac{1}{2}\langle D_{\mathbb{H}}^2 u(x_0)z_h, z_h \rangle - \delta_{\varepsilon}(x_0) \tag{4.2}$$

and

$$F(z, x_0, \varepsilon, t) = \langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + t\varepsilon\psi(z, x_0, \varepsilon).$$

Manipulating (4.1), we can explicitly compute

$$\delta_{\varepsilon}(x_0) = \frac{\int_B \int_0^1 h'(F(z, x_0, \varepsilon, t)) \, dt \left(Tu(x_0)z_3 + \frac{1}{2}\langle D_{\mathbb{H}}^2 u(x_0)z_h, z_h \rangle \right) \, dz}{\int_B \int_0^1 h'(F(z, x_0, \varepsilon, t)) \, dt \, dz},$$

which implies the uniform bound

$$|\delta_{\varepsilon}(x_0)| \leq \|Tu\|_{L^{\infty}(\Omega)} + \|D_{\mathbb{H}}^2 u\|_{L^{\infty}(\Omega)}$$

for all $0 < \varepsilon < 1$ and $x_0 \in \overline{\Omega}$. Since $p > 2$, by the dominated convergence theorem, up to a subsequence, we obtain that

$$\delta_{\varepsilon}(x_0) \rightarrow \frac{\int_B \left(Tu(x_0)z_3 + \frac{1}{2}\langle D_{\mathbb{H}}^2 u(x_0)z_h, z_h \rangle \right) h'(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle) \, dz}{\int_B h'(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle) \, dz} =: \delta_0(x_0).$$

This integral can be explicitly computed [1,7] to obtain

$$\delta_0(x_0) = c_p \Delta_{\mathbb{H}, p}^N u(x_0). \quad \square$$

We show now the proof of Proposition 2.8.

Proof. Using the notation in the proof of Lemma 2.6, it is enough to show that the quantity

$$\frac{\delta_\varepsilon(x_0) - \delta_0(x_0)}{\varepsilon}$$

is uniformly bounded in $0 < \varepsilon < 1$ and $x_0 \in \bar{\Omega}$.

To this end, in (4.1) use a second order Taylor expansion to get

$$\begin{aligned} 0 &= \int_B h'(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle) \psi(z, x_0, \varepsilon) \, dz \\ &\quad + \varepsilon \int_B \left(\int_0^1 h''(F(z, x_0, \varepsilon, t)) \, dt \right) \psi^2(z, x_0, \varepsilon) \, dz. \end{aligned}$$

Therefore,

$$\frac{\delta_\varepsilon(x_0) - \delta_0(x_0)}{\varepsilon} = \frac{\int_B \left(\int_0^1 h''(F(z, x_0, \varepsilon, t)) \, dt \right) \psi^2(z, x_0, \varepsilon) \, dz}{\int_B h'(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle) \, dz}.$$

First, we establish a lower bound on the denominator. Let R_{x_0} be a rotation in the horizontal plane such that $R_{x_0}^\tau \nabla_{\mathbb{H}} u(x_0) = |\nabla_{\mathbb{H}} u(x_0)|(1, 0) = (|\nabla_{\mathbb{H}} u(x_0)|, 0)$. Observe that B is invariant under the change of variables $z = R_{x_0} y$, so

$$\begin{aligned} \int_B h'(\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle) \, dz &= (p-1) \int_B |\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle|^{p-2} \, dz \\ &= (p-1) |\nabla_{\mathbb{H}} u(x_0)|^{p-2} \int_B |y_1|^{p-2} \, dy \\ &\geq c_p \min_{\bar{\Omega}} |\nabla_{\mathbb{H}} u|^{p-2}, \end{aligned}$$

because $0 < \int_B |y_1|^{p-2} \, dy < \infty$ for $p > 2$.

Now we establish an upper bound on the numerator. Note that

$$\psi^2(z, x_0, \varepsilon) \leq 4 \left(\|Tu\|_{L^\infty(\Omega)} + \|D_{\mathbb{H}}^2 u\|_{L^\infty(\Omega)} \right)^2$$

for all $0 < \varepsilon < 1$, $z \in B$, and $x_0 \in \bar{\Omega}$. Since $h''(s) = (p-1)(p-2)|s|^{p-4}s$, it is enough to estimate the integral

$$\int_B \int_0^1 |\langle \nabla_{\mathbb{H}} u(x_0), z_h \rangle + t\varepsilon \psi(z, x_0, \varepsilon)|^{p-3} \, dt \, dz.$$

After performing the change of variables $z = R_{x_0} y$, where R_{x_0} is the same rotation as above, the integral becomes

$$\begin{aligned} &\int_B \int_0^1 \left| |y_1| |\nabla_{\mathbb{H}} u(x_0)| + t\varepsilon \psi(R_{x_0} y, x_0, \varepsilon) \right|^{p-3} \, dt \, dy \\ &= |\nabla_{\mathbb{H}} u(x_0)|^{p-3} \int_B \int_0^1 |y_1 + t\varepsilon \Psi(y, x_0, \varepsilon)|^{p-3} \, dt \, dy, \end{aligned}$$

where we denoted

$$\begin{aligned} \Psi(y, x_0, \varepsilon) &= |\nabla_{\mathbb{H}} u(x_0)|^{-1} \psi(R_{x_0} y, x_0, \varepsilon) \\ &= |\nabla_{\mathbb{H}} u(x_0)|^{-1} \left(Tu(x_0) y_3 + \frac{1}{2} \langle R_{x_0}^\tau D_{\mathbb{H}}^2 u(x_0) R_{x_0} y_h, y_h \rangle - \delta_\varepsilon(x_0) \right). \end{aligned}$$

Observe that

$$|\Psi(y, x_0, \varepsilon)| \leq \frac{2 \|Tu\|_{L^\infty(\Omega)} + \|D_{\mathbb{H}}^2 u\|_{L^\infty(\Omega)}}{\min_{\overline{\Omega}} |\nabla_{\mathbb{H}} u|} = c_\Psi$$

for all $y \in B$, $0 < \varepsilon < 1$ and $x_0 \in \overline{\Omega}$.

For fixed ε , $t \in [0, 1]$ and $x_0 \in \overline{\Omega}$, consider the change of variables $\zeta = \Phi_{\varepsilon,t,x_0}(y) = \Phi_\varepsilon(y)$, given by

$$\begin{cases} \zeta_1 = y_1 + \varepsilon t \Psi(y, x_0, \varepsilon) \\ \zeta_i = y_i \quad i = 2, 3 \end{cases}$$

for $y \in B$.

We claim that there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ the map $\Phi_\varepsilon : B \rightarrow \Phi_\varepsilon(B)$ is a diffeomorphism with $\min_B \mathcal{J} \Phi_\varepsilon \geq \frac{1}{2}$. More exactly, $\hat{\varepsilon}$ can be taken to be

$$\hat{\varepsilon} = \frac{1}{2} \left(\frac{\|Tu\|_{L^\infty(\Omega)} + \|D_{\mathbb{H}}^2 u\|_{L^\infty(\Omega)}}{\min_{\overline{\Omega}} |\nabla_{\mathbb{H}} u|} + 1 \right)^{-1},$$

so it is independent of $x_0 \in \overline{\Omega}$. Indeed, the Euclidean Jacobian of the transformation is

$$\mathcal{J} \Phi_\varepsilon(y) = I_{3 \times 3} + \varepsilon t \begin{pmatrix} \nabla_y \Psi(y, x_0, \varepsilon) \\ 0 \\ 0 \end{pmatrix},$$

which implies

$$|\mathcal{J} \Phi_\varepsilon(y)| = |1 + \varepsilon t \partial_{y_1} \Psi(y, x_0, \varepsilon)| \geq \frac{1}{2}$$

for all $y \in B$, $0 < \varepsilon < \hat{\varepsilon}$ and $x_0 \in \overline{\Omega}$, because

$$\|\partial_{y_1} \Psi(\cdot, x_0, \varepsilon)\|_{L^\infty(\Omega)} \leq \frac{\|Tu\|_{L^\infty(\Omega)} + \|D_{\mathbb{H}}^2 u\|_{L^\infty(\Omega)}}{\min_{\overline{\Omega}} |\nabla_{\mathbb{H}} u|}.$$

Moreover, for all $0 < \varepsilon < \hat{\varepsilon}$ and $\eta, \xi \in B$ we have

$$|\Phi_\varepsilon(\xi) - \Phi_\varepsilon(\eta)| \geq |\xi - \eta| - \varepsilon t |\Psi(\xi, x_0, \varepsilon) - \Psi(\eta, x_0, \varepsilon)| \geq \frac{1}{2} |\xi - \eta|,$$

because $|\Psi(\xi, x_0, \varepsilon) - \Psi(\eta, x_0, \varepsilon)| \leq \|\nabla_y \Psi(\cdot, x_0, \varepsilon)\|_{L^\infty(\Omega)} |\xi - \eta|$. This concludes the proof of the claim.

Now, for $y \in B$, $\zeta \in \Phi_\varepsilon(B)$ we have $|\zeta_2| = |y_2| \leq 1$, $|\zeta_3| = |y_3| \leq 1$ and $|\zeta_1| = |y_1 + \varepsilon t \Psi(y, x_0, \varepsilon)| \leq |y_1| + c_\Psi \varepsilon \leq 1 + c_\Psi \varepsilon$, therefore

$$\begin{aligned} \int_B |y_1 + \varepsilon t \Psi(y, x_0, \varepsilon)|^{p-3} dy &= \int_{\Phi_\varepsilon(B)} |\zeta_1|^{p-3} |\mathcal{J} \Phi_\varepsilon(\Phi_\varepsilon^{-1}(\zeta))|^{-1} d\zeta \\ &\leq 2 \int_{\Phi_\varepsilon(B)} |\zeta_1|^{p-3}, d\zeta \\ &\leq 2 \int_{\{|\zeta_1| \leq 1 + c_\Psi \varepsilon\}} |\zeta_1|^{p-3} d\zeta_1 \\ &= 4(1 + c_B \varepsilon)^{p-2}, \end{aligned}$$

since $p > 2$. \square

5. Convergence in the case of p -harmonic data

In this section we prove [Lemma 2.7](#) and [Proposition 2.9](#).

In [Lemma 2.7](#) we show that we can obtain supersolutions and subsolutions of the DPP (2.11) by small perturbations of p -harmonic functions with non-vanishing horizontal gradient. This will be used as a first step towards proving convergence of the approximation scheme to the appropriate solution of the Dirichlet problem for the p -Laplace equation, in the case its boundary datum is p -harmonic.

First we show the proof of [Lemma 2.7](#).

Proof. Up to a left translation, we can assume that Ω does not intersect the cylinder

$$\{(x_1, x_2, x_3) \in \mathbb{H} \mid x_1^2 + x_2^2 < 1\},$$

so we can choose $q_0 = 0$ and $v(x) = |x|_{\mathbb{H}}^s$. From the calculations in the proof of Theorem 12.1 in [10] there exist $s \geq 4$ and $\hat{\varepsilon}$ such that

$$\Delta_{\mathbb{H},p}^N(U + \varepsilon v) \geq s\varepsilon \quad \text{in } \Omega$$

for all $0 < \varepsilon < \hat{\varepsilon}$. From the expansion in [Proposition 2.8](#), there exists $C > 0$ such that

$$\begin{aligned} (U + \varepsilon v)(x) &\leq \mu_p^{\mathbb{H}}(U + \varepsilon v, \varepsilon)(x) - c_p \varepsilon^2 \Delta_{\mathbb{H},p}^N(U + \varepsilon v)(x) + C\varepsilon^3 \\ &\leq \mu_p^{\mathbb{H}}(U + \varepsilon v, \varepsilon)(x) - \varepsilon^3(c_p s - C) \\ &\leq \mu_p^{\mathbb{H}}(U + \varepsilon v, \varepsilon)(x) \end{aligned}$$

for all $x \in \Omega$ and $0 < \varepsilon < \hat{\varepsilon}$, provided we choose $s > C/c_p$ and further restrict $\hat{\varepsilon}$.

Analogous computations give $\Delta_{\mathbb{H},p}^N(U - \varepsilon v) \leq -s\varepsilon$ in Ω , which, again by [Proposition 2.8](#), implies $(U + \varepsilon v)(x) \leq \mu_p^{\mathbb{H}}(U + \varepsilon v, \varepsilon)(x)$ for all $x \in \Omega$ and an appropriate choice of s . \square

Next comes the proof of [Proposition 2.9](#).

Proof. By [Lemma 2.7](#), there exist $\hat{\varepsilon} > 0$, $s \geq 4$ and $q_0 \in \mathbb{H}$ such that, denoting $v(x) = |q_0^{-1} * x|_{\mathbb{H}}^s$ we have that $U + \varepsilon v$ is a subsolution and $U - \varepsilon v$ is a supersolution of (2.11) in Ω_ε for all $0 < \varepsilon < \hat{\varepsilon}$. By the comparison principle in [Lemma 3.2](#) we get

$$U + \varepsilon v \leq u_\varepsilon + \varepsilon \sup_{\Gamma_\varepsilon^{\mathbb{H}}}(v)$$

and

$$u_\varepsilon \leq U - \varepsilon v + \varepsilon \sup_{\Gamma_\varepsilon^{\mathbb{H}}}(v)$$

on Ω_ε , for all $0 < \varepsilon < \hat{\varepsilon}$. Therefore,

$$|u_\varepsilon - U| \leq -\varepsilon v + \varepsilon \sup_{\Gamma_\varepsilon^{\mathbb{H}}}(v) \leq 2\varepsilon \sup_{\Omega_1} v$$

on Ω , so

$$\|u_\varepsilon - U\|_{L^\infty(\Omega)} \leq C_\Omega \varepsilon,$$

where $C_\Omega = 2 \sup_{\Omega_1} v$. This uniform bound concludes the proof. \square

6. Boundary estimate

In this section we prove [Theorem 2.11](#).

Proof. Fix $\eta > 0$. We prove the upper bound $u_\varepsilon(x) \leq G(y) + \eta$, the proof of the lower one being analogous. By uniform continuity, there exists $\delta \in (0, \bar{\delta})$ such that

$$G(x) \leq G(y) + \frac{\eta}{2} \quad \text{for all } y \in \partial\Omega, x \in B_{5\delta}^{\mathbb{H}}(y) \cap \Gamma_\varepsilon^{\mathbb{H}} \text{ and } 0 < \varepsilon < 1. \tag{6.1}$$

Introducing the notations

$$N^\varepsilon(y) = N^{\varepsilon, \delta}(y) := \sup_{B_{5\delta}^{\mathbb{H}}(y) \cap \Gamma_\varepsilon^{\mathbb{H}}} G \quad \text{and} \quad M^\varepsilon := \sup_{\Gamma_\varepsilon^{\mathbb{H}}} G, \tag{6.2}$$

inequality [\(6.1\)](#) rewrites as

$$N^\varepsilon(y) \leq G(y) + \frac{\eta}{2} \quad \text{for all } y \in \partial\Omega \text{ and } 0 < \varepsilon < 1. \tag{6.3}$$

Set $\xi = \frac{4-p}{p-1}$ and

$$\theta = \frac{1 - \frac{1}{2} \left(\frac{\mu}{2-\mu}\right)^\xi - \frac{1}{2} \left(\frac{\mu}{2}\right)^\xi}{1 - \left(\frac{\mu}{2}\right)^\xi} \in (0, 1). \tag{6.4}$$

Note that θ depends only on μ and p . For $k \geq 0$ define

$$\delta_k = \frac{\delta}{4^{k-1}} \tag{6.5}$$

and

$$M_k^\varepsilon(y) = N^\varepsilon(y) + \theta^k (M^\varepsilon - N^\varepsilon(y)). \tag{6.6}$$

We have the following

Claim Let $\varepsilon_k > 0$ and suppose that for all $\varepsilon \in (0, \varepsilon_k)$ we have

$$u_\varepsilon \leq M_k^\varepsilon(y) \quad \text{in } B_{\delta_k}^{\mathbb{H}}(y) \cap \Omega.$$

Then, there exists $\varepsilon_{k+1} = \varepsilon_{k+1}(\eta, \mu, \delta, p) \in (0, \varepsilon_k)$ such that

$$u_\varepsilon \leq M_{k+1}^\varepsilon(y) \quad \text{in } B_{\delta_{k+1}}^{\mathbb{H}}(y) \cap \Omega$$

for all $\varepsilon \in (0, \varepsilon_{k+1})$.

Using the claim above, since $u_\varepsilon \leq M^\varepsilon = M_0^\varepsilon$ in Ω_ε for all $\varepsilon \in (0, 1)$, we can find $\varepsilon_1 > 0$ such that $u_\varepsilon \leq M_1^\varepsilon(y)$ in $B_{\delta_1}^{\mathbb{H}}(y) \cap \Omega$ for $\varepsilon \leq \varepsilon_1$. We now repeat this process, and after k_0 iterations we find $\varepsilon_{k_0} > 0$ such that $u_\varepsilon \leq M_{k_0}^\varepsilon(y)$ in $B_{\delta_{k_0}}^{\mathbb{H}}(y) \cap \Omega$ for $\varepsilon \in (0, \varepsilon_{k_0})$. Choosing $k_0 \in \mathbb{N}$ such that

$$k_0 > \log_\theta \left(\frac{\eta}{2} \left(\sup_{\Gamma_1^{\mathbb{H}}} G - \inf_{\Gamma_1^{\mathbb{H}}} G + 1 \right)^{-1} \right)$$

we have

$$M_{k_0}^\varepsilon(y) - N^\varepsilon(y) = \theta^{k_0} (M^\varepsilon - N^\varepsilon(y)) \leq \frac{\eta}{2}, \tag{6.7}$$

for all $y \in \partial\Omega$, because $M^\varepsilon \leq \sup_{\Gamma_\varepsilon^{\mathbb{H}}} G$ and $N^\varepsilon(y) \geq \inf_{\Gamma_\varepsilon^{\mathbb{H}}} G$. Combining with [\(6.3\)](#) we get

$$u_\varepsilon \leq G(y) + \eta \quad \text{in } B_{\delta_{k_0}}^{\mathbb{H}}(y) \cap \Omega,$$

for all $\varepsilon \in (0, \varepsilon_{k_0})$.

To conclude, we now provide a sketch of the proof of the claim. The key properties of having radial fundamental solutions in annular domains for $\Delta_{\mathbb{H},p}^N$ with non-vanishing horizontal gradient, together with the comparison principle for solutions of (2.11), the uniform convergence in Proposition 2.9 and the fact that a genuine triangle inequality holds for the Korányi norm, allow us to adapt the proof of Lemma 4.4 of [19] to the current setting with minor modifications.

Proof of claim:

1. Since Ω satisfies the exterior \mathbb{H} -corkscrew condition, there exists a sequence of balls $B_{\mu\delta_{k+1}}^{\mathbb{H}}(z_k)$ strictly contained in $B_{\delta_{k+1}}^{\mathbb{H}}(y) \setminus \Omega$ for all $k \in \mathbb{N}$. To simplify the notation, let $N = N^\varepsilon(y)$ and $M_k = M_k^\varepsilon(y)$. The following boundary value problem

$$\begin{cases} -\Delta_{\mathbb{H},p}^N U_k = 0 & \text{in } B_{\delta_k}^{\mathbb{H}}(z_k) \setminus \overline{B_{\mu\delta_{k+1}}^{\mathbb{H}}(z_k)} \\ U_k = N & \text{on } \partial B_{\mu\delta_{k+1}}^{\mathbb{H}}(z_k) \\ U_k = M_k & \text{on } \partial B_{\delta_k}^{\mathbb{H}}(z_k) \end{cases}$$

has the radial solution

$$U_k(x) = \begin{cases} \frac{a_k}{|z_k^{-1} * x|_{\mathbb{H}}^\xi} + b_k & \text{if } p \neq 4 \\ a_k \log(|z_k^{-1} * x|_{\mathbb{H}}) + b_k & \text{if } p = 4 \end{cases},$$

for suitable coefficients a_k and b_k . Denote by $\tilde{\Omega}$ the annulus $B_{\delta_k}^{\mathbb{H}}(z_k) \setminus \overline{B_{\mu\delta_{k+1}}^{\mathbb{H}}(z_k)}$. Choosing ε_{k+1} sufficiently small, for $0 < \varepsilon < \varepsilon_{k+1}$ we let U_k^ε be the solution of the problem (2.11) in $\tilde{\Omega}$ with boundary value U_k . Since $\nabla_{\mathbb{H}} U_k \neq 0$ in $\tilde{\Omega}_\varepsilon$, Proposition 2.9 implies that U_k^ε converges uniformly to U_k in $\tilde{\Omega}$.

2. Consider now $B_{\delta_k/2}^{\mathbb{H}}(z_k)$. Due to its radial nature, we have that $U_k \geq \alpha M_k + \beta N$ on $B_{\delta_k/2}^{\mathbb{H}}(z_k)$, for appropriate coefficients satisfying $\alpha + \beta = 1$. By hypothesis, $\alpha u_\varepsilon + \beta N \leq \alpha M_k + \beta N$ on the ε -boundary of $B_{\delta_k/2}^{\mathbb{H}}(z_k) \cap \Omega$. Using the fact that U_k^ε is uniformly close to U_k and the comparison principle for solutions of (2.11), given $\gamma > 0$ we get that

$$\alpha u_\varepsilon + \beta N \leq U_k + 2\gamma \quad \text{on } B_{\delta_k/2}^{\mathbb{H}}(z_k) \cap \Omega, \tag{6.8}$$

for $\varepsilon \in (0, \varepsilon_{k+1})$ sufficiently small.

3. Now consider $B_{\delta_{k+1}}^{\mathbb{H}}(y)$. Because a genuine triangle inequality holds for the Korányi gauge, we have $B_{\delta_{k+1}}^{\mathbb{H}}(y) \subset B_{\delta_k/2}^{\mathbb{H}}(z_k)$ (so (6.8) holds in this ball) and $B_{\delta_{k+1}}^{\mathbb{H}}(y) \subset B_{(2-\mu)\delta_{k+1}}^{\mathbb{H}}(z_k)$. We have that $U_k \leq \beta' N + \alpha' M_k$ on $B_{(2-\mu)\delta_{k+1}}^{\mathbb{H}}(z_k)$ for suitable coefficients satisfying $\alpha' + \beta' = 1$. Combining with (6.8), in $B_{\delta_{k+1}}^{\mathbb{H}}(y) \cap \Omega$ and for $\varepsilon \in (0, \varepsilon_{k+1})$ we have

$$\begin{aligned} u_\varepsilon &\leq \frac{\beta' - \beta}{\alpha} N + \frac{\alpha'}{\alpha} M_k + \frac{2\gamma}{\alpha} \\ &= N + \theta(M_k - N) = M_{k+1}, \end{aligned}$$

with an appropriate choice of γ . Keeping track of the coefficients, whose expressions can be explicitly computed, we find that θ turns out to be as in (6.4). \square

7. The proof of Theorem 2.12

Recall the notion of viscosity solutions of the p -Laplace equation

$$-\Delta_{\mathbb{H},p}^N v = 0 \quad \text{in } \Omega. \tag{7.1}$$

Definition 7.1. Let $\Omega \subset \mathbb{H}$ be open and bounded. A bounded upper semicontinuous function $v : \Omega \rightarrow \mathbb{R}$ is a viscosity subsolution of (7.1) if for every $x_0 \in \Omega$ and for every $\phi \in C^2(\Omega)$ such that $v - \phi$ has a strict maximum at x_0 , $v(x_0) = \phi(x_0)$ and $\nabla_{\mathbb{H}}\phi(x_0) \neq 0$, it holds

$$-\Delta_{\mathbb{H},p}^N \phi(x_0) \leq 0.$$

A bounded lower semicontinuous function $v : \Omega \rightarrow \mathbb{R}$ is a viscosity supersolution of (7.1) if for every $x_0 \in \Omega$ and for every $\phi \in C^2(\Omega)$ such that $v - \phi$ has a strict minimum at x_0 , $v(x_0) = \phi(x_0)$ and $\nabla_{\mathbb{H}}\phi(x_0) \neq 0$, it holds

$$-\Delta_{\mathbb{H},p}^N \phi(x_0) \geq 0.$$

Finally, a viscosity solution of (7.1) is a function which is both a viscosity subsolution and a viscosity supersolution of (7.1).

We now study the convergence of u_ε as ε goes to zero, where u_ε is the solution of (2.11) with boundary value $G \in C(I_1^{\mathbb{H}})$. For $x \in \Omega$ define

$$\begin{aligned} \bar{u}(x) &:= \limsup_{\substack{\varepsilon \rightarrow 0^+ \\ q \rightarrow x}} u_\varepsilon(q), \\ \underline{u}(x) &:= \liminf_{\substack{\varepsilon \rightarrow 0^+ \\ q \rightarrow x}} u_\varepsilon(q) \end{aligned} \tag{7.2}$$

Note that \bar{u} is upper semicontinuous and \underline{u} is lower semicontinuous.

Theorem 7.2. Let \bar{u} and \underline{u} be as in (7.2). Then \bar{u} is a viscosity subsolution and \underline{u} is a viscosity supersolution of $-\Delta_{\mathbb{H},p}^N u = 0$ in Ω .

Proof. We give a proof for \bar{u} , the one for \underline{u} being analogous. Take $x_0 \in \Omega$ and a test function $\phi \in C^2(\Omega)$ with $\nabla_{\mathbb{H}}\phi(x_0) \neq 0$ that touches \bar{u} at x_0 from above, i.e. $\bar{u}(x_0) = \phi(x_0)$ and $\phi - \bar{u}$ has a strict minimum at x_0 . By definition of \bar{u} there exist $\varepsilon_n \rightarrow 0^+$ and $x_n \rightarrow x_0$ such that

$$u_{\varepsilon_n}(x_n) \rightarrow \bar{u}(x_0). \tag{7.3}$$

Fix $B_r^{\mathbb{H}}(x_0) \subset \Omega$ so that $\nabla_{\mathbb{H}}\phi \neq 0$ on this set. By definition of infimum, for every $n \in \mathbb{N}$ there exists $\bar{x}_n \in B_r^{\mathbb{H}}(x_0)$ such that

$$\phi(\bar{x}_n) - u_{\varepsilon_n}(\bar{x}_n) \leq \inf_{B_r^{\mathbb{H}}(x_0)} (\phi - u_{\varepsilon_n}) + \varepsilon_n^3. \tag{7.4}$$

By possibly passing to a subsequence, we can assume $\bar{x}_n \rightarrow \bar{x}_0$ for some $\bar{x}_0 \in \overline{B_r^{\mathbb{H}}(x_0)}$. We get

$$\begin{aligned} \phi(\bar{x}_0) - \bar{u}(\bar{x}_0) &= \liminf_{\substack{\varepsilon \rightarrow 0^+ \\ q \rightarrow \bar{x}_0}} (\phi(q) - u_\varepsilon(q)) \\ &\leq \liminf_{n \rightarrow \infty} (\phi(\bar{x}_n) - u_{\varepsilon_n}(\bar{x}_n)) \\ &\leq \liminf_{n \rightarrow \infty} \left(\inf_{B_r^{\mathbb{H}}(x_0)} (\phi - u_{\varepsilon_n}) + \varepsilon_n^3 \right) \\ &\leq \liminf_{n \rightarrow \infty} (\phi(x_n) - u_{\varepsilon_n}(x_n) + \varepsilon_n^3) \\ &= \phi(x_0) - \bar{u}(x_0), \end{aligned}$$

where we used (7.3) and (7.4) respectively in the third and second lines. As a consequence $\bar{x}_0 = x_0$, because $\phi - \bar{u}$ has a strict minimum at x_0 . Note that inequality (7.4) rewrites as

$$u_{\varepsilon_n}(x) - u_{\varepsilon_n}(\bar{x}_n) \leq \phi(x) - \phi(\bar{x}_n) + \varepsilon_n^3 \quad \text{for all } x \in B_r^{\mathbb{H}}(x_0).$$

For n large enough, we have $\varepsilon_n < r/2$ and $\bar{x}_n \in B_{r/2}^{\mathbb{H}}(x_0)$ so that $B_{\varepsilon_n}^{\mathbb{H}}(\bar{x}_n) \subseteq B_r^{\mathbb{H}}(x_0)$. Therefore

$$\begin{aligned} \mu_p(u_{\varepsilon_n}, \varepsilon_n)(\bar{x}_n) - u_{\varepsilon_n}(\bar{x}_n) &\leq \mu_p(\phi, \varepsilon_n)(\bar{x}_n) - \phi(\bar{x}_n) + \varepsilon_n^3 \\ &= c\varepsilon_n^2 \Delta_{\mathbb{H},p}^N \phi(\bar{x}_n) + o(\varepsilon_n^2), \end{aligned}$$

where we used Lemma 2.6. We remark that the proof of this Lemma shows that the error in the above expansion is uniform on the set where ϕ has nonvanishing horizontal gradient. Since $\nabla_{\mathbb{H}}\phi \neq 0$ on $B_r^{\mathbb{H}}(x_0)$, such error is independent of x_n . By definition of u_{ε_n} we get

$$0 \leq \Delta_{\mathbb{H},p}^N \phi(\bar{x}_n) + o(1),$$

which, after taking the limit as $n \rightarrow \infty$, concludes the proof for \bar{u} . \square

We finalize now the proof of Theorem 2.12.

Proof. By Theorem 2.11, for $y \in \partial\Omega$ we have

$$\limsup_{\Omega \ni x \rightarrow y} \bar{u}(x) \leq G(y) \leq \liminf_{\Omega \ni x \rightarrow y} \underline{u}(x).$$

Using the comparison principle for viscosity solutions of the p -Laplace equation in \mathbb{H} (see [3]), we get $\bar{u} \leq \underline{u}$ in Ω . Since trivially $\bar{u} \geq \underline{u}$, we have that $\bar{u} = \underline{u} := u$ is a viscosity solution of $-\Delta_{\mathbb{H},p}^N u = 0$ in Ω . Moreover, u attains the appropriate boundary value and therefore, by a compactness argument, Theorem 2.12 follows. \square

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References

- [1] Tomasz Adamowicz, Antoni Kijowski, Andrea Pinamonti, Ben Warhurst, Variational approach to the asymptotic mean-value property for the p -Laplacian on Carnot groups, *Nonlinear Anal.* 198 (2020) 111893, 22.
- [2] Guy Barles, Panagiotis E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptot. Anal.* 4 (3) (1991) 271–283.
- [3] Thomas Bieske, Equivalence of weak and viscosity solutions to the p -Laplace equation in the Heisenberg group, *Ann. Acad. Sci. Fenn. Math.* 31 (2) (2006) 363–379.
- [4] Pablo Blanc, Fernando Charro, Juan J. Manfredi, Julio D. Rossi, A nonlinear mean value property for the Monge-Ampère operator, *J. Convex Anal.* 28 (2) (2021) 353–386.
- [5] Luca Capogna, Nicola Garofalo, Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics, *J. Fourier Anal. Appl.* 4 (4–5) (1998) 403–432.
- [6] Evan W. Chandra, Michinori Ishiwata, Rolando Magnanini, Hidemitsu Wadade, Variational p -harmonious functions: existence and convergence to p -harmonic functions, 2021, <https://arxiv.org/pdf/2101.02662.pdf>.
- [7] Michinori Ishiwata, Rolando Magnanini, Hidemitsu Wadade, A natural approach to the asymptotic mean value property for the p -Laplacian, *Calc. Var. Partial Differential Equations* 56 (4) (2017) 97, 22.
- [8] Petri Juutinen, Peter Lindqvist, Juan J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation, *SIAM J. Math. Anal.* 33 (3) (2001) 699–717.
- [9] Marta Lewicka, A Course on Tug-of-War Games with Random Noise, in: Universitext, Springer, Cham, 2020, p. 254.
- [10] Marta Lewicka, Juan Manfredi, Diego Ricciotti, Random walks and random tug of war in the Heisenberg group, *Math. Ann.* 377 (1–2) (2020) 797–846.
- [11] Juan J. Manfredi, Mikko Parviainen, Julio D. Rossi, An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games, *SIAM J. Math. Anal.* 42 (5) (2010) 2058–2081.
- [12] Juan J. Manfredi, Mikko Parviainen, Julio D. Rossi, On the definition and properties of p -harmonious functions, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 11 (2) (2012) 215–241.
- [13] Juan J. Manfredi, Bianca Stroffolini, Convergence of the natural p -Means for the p -Laplacian, *ESAIM Control Optim. Calc. Var.* 27 (33) (2021) 18.

- [14] Lewicka Marta, A course on tug-of-war games with random noise, in: Introduction and Basic Constructions, in: Universitext, Springer, Cham, 2020.
- [15] Roberto Monti, Daniele Morbidelli, Regular domains in homogeneous groups, *Trans. Amer. Math. Soc.* 357 (8) (2005) 2975–3011.
- [16] Yuval Peres, Oded Schramm, Scott Sheffield, David B. Wilson, Tug-of-war and the infinity Laplacian, *J. Amer. Math. Soc.* 22 (1) (2009) 167–210.
- [17] Yuval Peres, Scott Sheffield, Tug-of-war with noise: a game-theoretic view of the p -Laplacian, *Duke Math. J.* 145 (1) (2008) 91–120.
- [18] Félix del Teso, Erik Lindgren, A mean value formula for the variational p -Laplacian, *NoDEA Nonlinear Differential Equations Appl.* 28 (3) (2021) 33, Paper No. 27.
- [19] Félix del Teso, Juan J. Manfredi, Mikko Parviainen, Convergence of dynamic programming principles for the p -Laplacian, *Adv. Calc. Var.* (2020) <https://doi.org/10.1515/acv-2019-0043>.