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A thorough description of one-dimensional steady open channel flows using the notion of viscosity solution

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ABSTRACT

Determining water surface profiles of steady open channel flows in a one-dimensional bounded domain is one of the well-trodden topics in conventional hydraulic engineering. However, it involves Dirichlet problems of scalar first-order quasilinear ordinary differential equations, which are of mathematical interest. We show that the notion of viscosity solution is useful in thoroughly describing the characteristics of possibly non-smooth and discontinuous solutions to such problems, achieving the conservation of momentum and the entropy condition. Those viscosity solutions are the generalized solutions in the space of bounded measurable functions. Generalized solutions to some Dirichlet problems are not always unique, and a necessary condition for the non-uniqueness is derived. A concrete example illustrates the non-uniqueness of discontinuous viscosity solutions in a channel of a particular cross-sectional shape.

1. Introduction

Open channel hydraulics is a branch of fluid mechanics to discuss the liquid water that flows over a course with a free surface subject to atmospheric pressure. Such a flow is referred to as an open channel flow, and the course is termed as the channel. Open channel flows in dominantly one-dimensional (1D) channels are of practical importance in civil engineering applications. The governing equations of 1D open channel flows are derived from the shallow water equations (SWEs), constituting a hyperbolic system of first-order partial differential equations (PDEs) to represent the conservation laws of mass and momentum in fluid mechanics. The pioneering work of de Saint-Venant established the SWEs, including a non-conservative form of momentum equation [15]. While, the SWEs in a thoroughly conservative form imply that a discontinuity of water depths, referred to as a hydraulic jump or shock, may occur under the Rankine–Hugoniot condition.

Tremendous efforts have been devoted to initial-boundary value problems of the SWEs in bounded domains, primarily for the numerical solution of unsteady states. Toro and Garcia-Navarro [30] thoroughly reviewed Godunov-type methods applied to SWEs, mentioning the critical points such as the jump conditions and the presence of source terms. As far as numerical methods for 1D problems are concerned, careful considerations on moving fronts [27] and on balancing source terms and flux gradients [23] have resulted in significant advances [31]. However, there remain mathematical fundamentals involving the 1D SWEs, and several recent papers highlighted the approximation of the 1D SWEs. Chentouf and Smaoui [9] discussed

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the well-posedness and stability of solutions in a diffusion wave approximation of the 1D SWEs. A significant advantage of such a stable diffusive wave equation is the applicability to closed-loop control systems for rivers [10,11], and Ammari and Chentouf [2] achieved state-of-the-art stabilization and regulation results. Mean et al. [25] explored the applicability of the level-set methods to kinematic wave approximation of the 1D SWEs. Sukhtayev et al. [29] investigated water surface profiles, which may be non-smooth or discontinuous, in the framework of a generalized Sturm-Liouville problem reduced from the stability problem of the 1D SWEs.

More emphasis should be placed on analyzing 1D open channel flows in steady states, as the hydraulic design of rivers, irrigation canals, and sewer systems is often based on them with the possible inclusion of control devices such as gates and weirs to regulate the water flows. Simply dropping the unsteady terms of the 1D SWEs results in a scalar first-order (SFO) ordinary differential equation (ODE), on which we should focus in this study. A significant difficulty is that determining 1D steady open channel flows in a domain bounded by two water level control devices involves a Dirichlet problem of the SFO ODE constraining water depths at the two boundary points. Such a two-point boundary value problem is mathematically ill-posed unless admitting a discontinuity in the water depths.

Adding a virtual unsteady term to the SFO ODE might be a viable approach, as it constitutes a first-order quasilinear (FOQL) PDE representing a scalar conservation law. There are several mathematical notions to deal with non-smooth or discontinuous solutions of FOQL PDEs. Oleĭnik [26] established generalized solutions of FOQL PDEs as the limit functions of the solutions to parabolic PDEs, namely, using the method of vanishing viscosity. Kružkov [22] refined that notion of generalized solution in the space of functions of bounded variation (BV). Jerez and Arciga [19] called such a generalized solution as the BV entropy weak solution, which is required to achieve the uniqueness of a physically reasonable solution to a FOQL PDE representing a scalar conservation law. Glaubitz [17] developed a shock-capturing procedure for the stable numerical approximation of BV entropy weak solutions to FOQL PDEs. Functions of BV are also appropriate for different applications, such as value functions of optimal control problems [33] and total variation flows [16]. However, it is challenging to clarify whether such a BV entropy solution exists globally in time and converges to a steady state as time goes by or not.

An option comprehending SFO ODEs and FOQL PDEs is to regard them as equations of Hamilton-Jacobi type (HJ equations). In the 1980s, Crandall and Lions [14] introduced the notion of viscosity solution for HJ equations, considering both steady Dirichlet problems and unsteady Cauchy problems. A viscosity solution is obtained not only via the method of vanishing viscosity but also via the finite difference method [13], via the Perron's method [18], and as the value function of the associated optimal control problem [24]. Barles [5] proved existence results for those two problems of HJ equations, whose viscosity solutions are possibly discontinuous. Barles and Perthame [6] focused on possibly discontinuous viscosity solutions of the Dirichlet problems associated with deterministic optimal stopping time problems, which can be approached via the method of vanishing viscosity [7]. However, as revisited in the 1990s [4,28], well-posed Dirichlet problems of HJ equations require certain structural assumptions on the Hamiltonian functions, such as proper dependence on the unknown variable. In such well-structured cases, the link between viscosity solutions of HJ equations and BV entropy solutions of FOQL PDEs representing scalar conservation laws has been known [20]. On the other hand, the HJ equation accommodated to the SFO ODE for the 1D steady open channel flows has an essentially improper Hamiltonian function, where relaxation as in Unami and Mohawesh [32] is not applicable. Therefore, we cannot expect a comparison principle to guarantee the uniqueness and stability of a viscosity solution, although we have the advantage of already knowing several types of 1D steady open channel flows, which can be found in the standard textbooks of open channel hydraulics [8,12].

In this study, we use the notion of viscosity solution to describe the characteristics of the 1D steady open channel flows as the solutions to Dirichlet problems of SFO ODEs, even though the method of vanishing viscosity does not work due to the structure of the Hamiltonian function hindering the solution of a relevant two-point boundary value problem of a second-order ODE [21]. It is shown that the discontinuous viscosity solutions to the Dirichlet problems satisfy the entropy condition, with which the BV entropy weak solutions accompany, and are generalized solutions in the Oleĭnik [26]'s sense. Viscosity solutions to some Dirichlet problems are indeed not unique, and a concrete illustrative example is presented.

The remainder of this paper is organized as follows. In Section 2, preliminaries are given, including conventional governing equations of 1-D open channel flows, functional spaces, and the notion of viscosity solutions. In Section 3, three problems are formulated in the first subsection to address gradually varied flow solutions, viscosity solutions, and generalized solutions involving 1-D steady open channel flows, mathematically analyzed in the subsequent subsections. In Section 4, examples are given to illustrate the results of Section 3. Section 5 provides conclusions of this study, suggesting some relevant open problems.

2. Preliminaries

We consider open channel flows subject to the following physical assumptions **A1**-**A5**.

A1. The channel is prismatic, having a straight alignment and a constant cross-sectional shape.

A2. The channel bed slope $S_0 = \tan\theta$ is so small that the approximations $\theta = \sin\theta = \tan\theta$ and $\cos\theta = 1$ are acceptable.

A3. The pressure distribution is hydrostatic.

A4. The velocity distribution in a channel cross-section is uniform.

A5. The friction force is the same as in uniform flows and represented as the friction slope

$$S_f = S_f(h) = \frac{n^2 Q|Q|}{A^2 R^{2p}}, \quad (1)$$

where h is the water depth, $A = A(h)$ is the wetted cross-sectional area, Q is the discharge, $R = A/P$ is the hydraulic radius with the wetted perimeter $P = P(h)$, n is the roughness coefficient, and p is the exponent as in the uniform flow formula.

2.1. Conventional governing equations of 1-D open channel flows

The conservation laws of mass and momentum under the assumption of **A1**-**A5** are summarized as the 1D SWEs

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ Q \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} Q \\ M \end{pmatrix} = \begin{pmatrix} 0 \\ gA\left(S_0 - S_f\right) \end{pmatrix}, \tag{2}$$

where t is the time, x is the horizontal axis along the channel, g is the acceleration due to gravity, and M is the specific force given by

$$M = \frac{Q^2}{A} + \int_0^h g\frac{\partial A}{\partial z}(h - z)\mathrm{d}z, \tag{3}$$

where z is the upward vertical axis, and h is the water depth. The 1D SWEs (2) in steady states indicate that the discharge Q is constant and that

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Q^2}{A} + \int_0^h g\frac{\partial A}{\partial z}(h - z)\mathrm{d}z\right) = gA\left(S_0 - S_f\right), \tag{4}$$

which is formally rewritten as a non-conservative form

$$\left(gA - \frac{\partial A}{\partial h}\frac{Q^2}{A^2}\right)\frac{\mathrm{d}h}{\mathrm{d}x} = gA\left(S_0 - S_f\right). \tag{5}$$

A water depth h_{uni} achieving $S_f(h_{\mathrm{uni}}) = S_0$ is referred to as the uniform flow depth. The commonly used governing equation of 1D steady open channel flows is obtained from (5) as

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_f}{1 - Fr^2}, \tag{6}$$

where Fr is the Froude number defined by

$$Fr = \sqrt{\frac{\partial A}{\partial h}\frac{Q^2}{gA^3}}. \tag{7}$$

A water depth h_{cri} achieving $Fr(h_{\mathrm{cri}}) = 1$ is referred to as the critical flow depth. It is known that the water depths h gradually varying along the channel satisfy the SFO ODE (6), which does not explain the occurrence of a hydraulic jump.

2.2. Functional spaces

Henceforth, we utilize several spaces of functions of a generic independent variable a. Functions are collectively denoted by $v = v(a)$. Let Ω be an open a-domain in $\mathbb{R}$. The space $C_B(\Omega)$ consisting of all bounded and continuous functions on Ω is complete with respect to the uniform norm $\|v\|_\infty = \sup_{a\in\Omega}|v(a)|$. The space $C(\overline{\Omega})$ consisting all bounded and uniformly continuous functions on Ω is a closed sub-space of $C_B(\Omega)$. The space $C^1(\Omega)$ consists of all continuous functions whose first derivatives are also continuous on Ω. The space $C_B^1(\Omega)$ consists of all bounded and continuous functions whose first derivatives are also bounded and continuous on Ω and is complete with respect to the norm $\|v\|_{C_B^1} = \max(\|v\|_\infty, \|\mathrm{d}v/\mathrm{d}a\|_\infty)$. The spaces of upper and lower semi-continuous functions defined on $\overline{\Omega}$ are denoted by $USC(\overline{\Omega})$ and $LSC(\overline{\Omega})$, respectively. The space $L^1(\Omega)$ is the Lebesgue space of all integrable functions v on Ω, equipped with the norm

$$\|v\|_{L^1} = \int_\Omega |v|\mathrm{d}\Omega. \tag{8}$$

The space $L^\infty(\Omega)$ is the Lebesgue space of all essentially bounded functions v on Ω, equipped with the norm

$$\|v\|_{L^\infty} = \underset{a\in\Omega}{\mathrm{esssup}}|v(a)|. \tag{9}$$

The space $W^{1,1}(\Omega)$ is the Sobolev space, which is the completion of $C^1(\Omega)$ with respect to the norm

$$\|v\|_{1,1} = \|v\|_{L^1} + \|\frac{\mathrm{d}v}{\mathrm{d}a}\|_{L^1}. \tag{10}$$

The space $W^{1,\infty}(\mathbb{R})$ is the Sobolev space, which is the completion of $C^1(\mathbb{R})$ with respect to the norm

$$\|v\|_{1,\infty} = \max\left(\underset{a\in\mathbb{R}}{\mathrm{esssup}}|v(a)|, \underset{a\in\mathbb{R}}{\mathrm{esssup}}\left|\frac{\mathrm{d}v}{\mathrm{d}a}(a)\right|\right). \tag{11}$$

2.3. The notion of viscosity solution

Firstly, we transform the unknown variable, the water depth h, as

$$u = \log h \tag{12}$$

so that the range of the unknown u becomes $\mathbb{R}$. Then, we regard (5) as an HJ equation

$$H\left(u, \frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0, \tag{13}$$

where H is the Hamiltonian function, in a bounded open x-domain Ω. The definition of viscosity solution here is adapted from Bardi and Capuzzo-Dolcetta [3]. The upper semi-continuous envelop (U-env) $u^*(x)$ of a function $u(x)$ is defined as

$$u^*(x) = \lim_{y \to x} sup u(y). \tag{14}$$

The lower semi-continuous envelope (L-env) $u_*(x)$ of a function $u(x)$ is defined as

$$u_*(x) = \lim_{y \to x} inf u(y). \tag{15}$$

There is a trivial comparison principle

$$u_*(x) \le u(x) \le u^*(x) \tag{16}$$

at any $x \in \Omega$. The U-env $u^*(x) \in USC(\overline{\Omega})$ is called a viscosity sub-solution (sub-S) of (13) if, for any weight $w \in C^1(\Omega)$,

$$H\left(u^*(x), \frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) \le 0 \tag{17}$$

holds at any point x where $u^* - w$ achieves a local maximum. The L-env $u_*(x) \in LSC(\overline{\Omega})$ is called a viscosity super-solution (super-S) of (13) if, for any weight $w \in C^1(\Omega)$,

$$H\left(u_*(x), \frac{\mathrm{d}w(x)}{\mathrm{d}x}\right) \ge 0 \tag{18}$$

holds at any point x where $u_* - w$ achieves a local minimum. When the U-env u^* and the -L-env u_* of a function $u(x)$ are a sub-S and a super-S, respectively, $u(x)$ is called a viscosity solution (VS) of (13) in Ω.

3. Problem formulation and mathematical analysis

3.1. Statement of problems

According to the transformation (12), the uniform flow depth and the critical flow depth are transformed as $u_{\mathrm{uni}} = \log h_{\mathrm{uni}}$ and $u_{\mathrm{cri}} = \log h_{\mathrm{cri}}$, respectively. The flux $\varphi(u)$ is the negative specific force

$$\phi(u) = -M = -\frac{Q^2}{A} - \int_0^h g\frac{\partial A}{\partial z}(h - z)\mathrm{d}z, \tag{19}$$

and it is assumed that $\phi(u) \in W^{1,\infty}(\mathbb{R})$, φ_u is strictly monotone decreasing, and thus

$$\phi_{uu}(u) < 0 \tag{20}$$

almost everywhere. The source term $\psi(u)$ is the external force

$$\psi(u) = gA\left(S_0 - S_f\right), \tag{21}$$

which is assumed to be in $W^{1,\infty}(\mathbb{R})$.

We pose the following three problems considering different senses of solutions.

Problem 1. Find a gradually varied flow solution (GVFS) $u(x) \in C_B^1(\Omega)$ of

$$H\left(u, \frac{\mathrm{d}u}{\mathrm{d}x}\right) = \phi_u(u)\frac{\mathrm{d}u}{\mathrm{d}x} + \psi(u) = \left(Fr^2(u) - 1\right)e^u\frac{\mathrm{d}u}{\mathrm{d}x} + S_0 - S_f(u) = 0, \tag{22}$$

in $\Omega = (0, X) = (0, |X|)$ or $\Omega = (X, 0) = (- |X|, 0)$ with a free endpoint X, satisfying the Dirichlet boundary condition

$$u(0) = u_{\mathrm{up}}, \quad u(X) = u_{\mathrm{down}}, \tag{23}$$

where u_{up} and u_{down} are specified boundary values.

Problem 2. Find a VS $u(x) \in L^1(\Omega) \cap L^\infty(\Omega)$ of (22) in $\Omega = (0, X)$ with a specified downstream endpoint X, satisfying the momentum conservation

$$\phi(x) \in C_B(\Omega) \tag{24}$$

and the Dirichlet boundary condition

$$u^*(0) = u(0) = u_*(0) = u_{\mathrm{up}}, \quad u^*(X) = u(X) = u_*(X) = u_{\mathrm{down}}, \tag{25}$$

where u_{up} and u_{down} are specified boundary values.
Problem 3. Find a generalized solution (GS) $u(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, where $\Omega = (0, X)$, such that

$$\int_0^X \left(\phi(u)\frac{\mathrm{d}f}{\mathrm{d}x} - \psi(u)f\right)\mathrm{d}x + f(0)\phi(u_{\mathrm{up}}) - f(X)\phi(u_{\mathrm{down}}) = 0, \tag{26}$$

for any $f \in W^{1,1}(\Omega)$.

3.2. Unique existence of GVFSs

Firstly, we clarify the unique existence of the conventional GVFSs as solutions to Problem 1.
Theorem 1. Let $\Omega = (u_{\mathrm{up}}, u_{\mathrm{down}})$. If the closed interval $\overline{\Omega}$ does not contain any u_{uni} and if $\varphi_u(u)$ is continuous in Ω, then there exists a unique $x(u) \in C(\overline{\Omega})$ such that

$$x(u) = \int_{u_{\mathrm{up}}}^{u} -\frac{\phi_u(u)}{\psi(u)}\mathrm{d}u = \int_{u_{\mathrm{up}}}^{u} \frac{1 - Fr^2}{S_0 - S_f}\frac{\mathrm{d}h}{\mathrm{d}u}\mathrm{d}u. \tag{27}$$

Proof. As $\psi(u) \neq 0$, (22) can be rewritten as $\mathrm{d}x/\mathrm{d}u = e^u(1 - Fr^2)/(S_0 - S_f)$ and then

$$\frac{\mathrm{d}x}{\mathrm{d}h} = \frac{1 - Fr^2}{S_0 - S_f}, \tag{28}$$

whose right hand side is Lipschitz continuous with respect to x with any Lipschitz constant. Then, the unique existence of $x(u) \in C(\overline{\Omega})$ such that (27) is a direct consequence of the well-known result for initial value problems of ODEs (Theorem 1.1.1 of [21]). □
Remark 1. In $\Omega = (0, X) = (0, |X|)$ or $\Omega = (X, 0) = (-|X|, 0)$, there exists unique monotone GVFS $u(x) \in C_B^1(\Omega)$ solving Problem 1 with $X = x(u_{\mathrm{down}})$ determined by (27), if the closed interval $[u_{\mathrm{up}}, u_{\mathrm{down}}]$ does not contain any u_{uni} or u_{cri} and if $\varphi_u(u)$ is continuous in $(u_{\mathrm{up}}, u_{\mathrm{down}})$. There is a comparison principle that the solution $u(x)$ is bounded as

$$\min(u_{\mathrm{up}}, u_{\mathrm{down}}) \le u(x) \le \max(u_{\mathrm{up}}, u_{\mathrm{down}}). \tag{29}$$

Remark 2. The uniform flow $u(x) = u_{\mathrm{uni}} \in C_B^1(\Omega)$ solves Problem 1 with the boundary value specified as $u_{\mathrm{up}} = u_{\mathrm{down}} = u_{\mathrm{uni}}$.

3.3. Properties of VSs

Next, properties of VSs, which are possibly non-smooth or discontinuous, are explored.
It is observed that there are continuous VSs.
Remark 3. A GVFS $u(x) \in C_B^1(\Omega)$ solving Problem 1 is a VS solving Problem 2.
Remark 4. Suppose $u_{\mathrm{cri}} = u_{\mathrm{uni}}$, that $\lim_{u \to u_{\mathrm{cri}}^{\pm}} \phi_{uu}(u)$ exists and is finite for either sign, and $\lim_{u \to u_{\mathrm{uni}}^{\pm}} \psi_u(u) \neq 0$ for the sign. Then,

$$\lim_{u_{\mathrm{down}} \to u_{\mathrm{uni}}^{\pm}} \frac{1 - Fr^2}{S_0 - S_f}\frac{\mathrm{d}h}{\mathrm{d}u} = -\frac{\lim_{u_{\mathrm{down}} \to u_{\mathrm{cri}}^{\pm}} \phi_{uu}(u_{\mathrm{down}})}{\lim_{u_{\mathrm{down}} \to u_{\mathrm{uni}}^{\pm}} \psi_u(u_{\mathrm{down}})}, \tag{30}$$

for the sign becomes finite, allowing $x(u_{\mathrm{cri}})$ in (27) to converge to a finite $\xi \in \mathbb{R}$. Let $u_{\mathrm{GV}}(x)$ denote such a GVFS approaching to $u_{\mathrm{cri}} = u_{\mathrm{uni}}$ at ξ. With another boundary $X \in \mathbb{R}$ such that $0 < \xi X$ and $|\xi| < |X|$,

$$u(x) = \begin{cases} \begin{cases} u_{\mathrm{GV}}(x) & \text{in } [0, \xi) \\ u_{\mathrm{uni}} & \text{in } [\xi, X] \end{cases} & \text{if } 0 < \xi \\ \begin{cases} u_{\mathrm{uni}} & \text{in } [X, \xi] \\ u_{\mathrm{GV}}(x) & \text{in } (\xi, 0] \end{cases} & \text{if } \xi < 0 \end{cases} \tag{31}$$

becomes a VS in $C_B(\Omega)$.
Discontinuities in VSs solving Problem 2, which are indeed GSs solving Problem 3, are characterized in the following theorems.
Theorem 2. Let $u(x)$ be a VS solving Problem 2. Suppose $u^*(\xi) \neq u_*(\xi)$ at a point $\xi \in \Omega$ and that $\varphi_u(u)$ is continuous at $u^*(\xi)$ and $u_*(\xi)$. For the sub-S $u^*(x) \in USC(\overline{\Omega})$ and the super-S $u_*(x) \in LSC(\overline{\Omega})$,

$$\phi_u(u^*(\xi)) < 0, \quad \phi_u(u_*(\xi)) > 0, \tag{32}$$

and $u(x) \in C_B(0,\xi) \cap C_B(\xi,X)$, implying $\varphi_u(u_{\mathrm{up}}) \ge 0$ and $\varphi_u(u_{\mathrm{down}}) \le 0$.

Proof. Let w be any weight in $C^1(\Omega)$. If $\varphi_u(u^*) \geq 0$ and $u^* - w$ achieves a local maximum at ξ, then $u^*(\xi) < u_*(\xi)$. This contradicts the comparison principle (16). If $\varphi_u(u_*) \leq 0$ and $u_* - w$ achieves a local minimum at ξ, then $u^*(\xi) < u_*(\xi)$. This contradicts the comparison principle (16). Therefore, (32) holds. The assumption (20) implies that φ_u is monotone decreasing, and thus there can be at most one point ξ achieving (32). With Remark 1 and Remark 4, it is concluded that $u(x) \in C_B(0,\xi) \cap C_B(\xi,X)$, $\varphi_u(u_{\mathrm{up}}) \geq 0$ and $\varphi_u(u_{\mathrm{down}}) \leq 0$. □

Theorem 3. Let $u(x)$ be a VS solving Problem 2. Suppose that $\varphi_u(u)$ is continuous except at the points of non-smoothness mentioned in Remark 4 and Theorem 2, if any. Then, $u(x)$ is a GS satisfying (26) for any weight $f \in W^{1,1}(\Omega)$, and

$$\phi(u^*(\xi)) = \phi(u_*(\xi)) \tag{33}$$

if there is any discontinuity at $\xi \in \Omega$ such that $u^*(\xi) \neq u_*(\xi)$.

Proof. Because of $u(x) \in C_B(0,\xi) \cap C_B(\xi,X)$,

$$\begin{aligned} 0 &= \int_0^\xi f\left(\phi_u \frac{\mathrm{d}u}{\mathrm{d}x} + \psi\right)\mathrm{d}x + \int_\xi^X f\left(\phi_u \frac{\mathrm{d}u}{\mathrm{d}x} + \psi\right)\mathrm{d}x \\ &= \int_0^\xi \left(f\phi_u \frac{\mathrm{d}u_*}{\mathrm{d}x} + f\psi\right)\mathrm{d}x + \int_\xi^X \left(f\phi_u \frac{\mathrm{d}u^*}{\mathrm{d}x} + f\psi\right)\mathrm{d}x \\ &= \int_0^\xi \left(f\frac{\mathrm{d}\phi(u_*)}{\mathrm{d}x} + f\psi(u_*)\right)\mathrm{d}x + \int_\xi^X \left(f\frac{\mathrm{d}\phi(u^*)}{\mathrm{d}x} + f\psi(u^*)\right)\mathrm{d}x \\ &= f(\xi)\phi(u_*(\xi)) - f(0)\phi(u_*(0)) - \int_0^\xi \left(\phi(u_*)\frac{\mathrm{d}f}{\mathrm{d}x} - \psi(u_*)f\right)\mathrm{d}x \\ &\quad - f(\xi)\phi(u^*(\xi)) + f(X)\phi(u^*(X)) - \int_\xi^X \left(\phi(u^*)\frac{\mathrm{d}f}{\mathrm{d}x} - \psi(u^*)f\right)\mathrm{d}x \\ &= f(X)\phi(u_{\mathrm{down}}) - f(0)\phi(u_{\mathrm{up}}) - \int_0^X \left(\phi(u)\frac{\mathrm{d}f}{\mathrm{d}x} - \psi(u)f\right)\mathrm{d}x \\ &\quad + f(\xi)(\phi(u_*(\xi)) - \phi(u^*(\xi))), \end{aligned} \tag{34}$$

for any weight $f \in W^{1,1}(\Omega)$. The Rankine-Hugoniot condition (33) stems from (24). Then, (34) with (33) turns out (26). □

3.4. Non-uniqueness of GSs

Lastly, we show that there is a case where GSs are not unique, using auxiliary functions stemming from two different functions $u_1(x)$ and $u_2(x)$ with their mollification. A mollifier $J_\rho(x)$ is defined on $\mathbb{R}$ as

$$J_\rho(x) = 0 \text{ if } |x| \geq \rho, \quad \int_{\mathbb{R}} J_\rho(x)\mathrm{d}x = 1, \tag{35}$$

and a function $u(x)$ defined on $\mathbb{R}$ is mollified as

$$u^\rho = J_\rho * u = \int_{\mathbb{R}} J_\rho(x-y)u(y)\mathrm{d}y. \tag{36}$$

A well-known concrete example of a mollifier is

$$J_\rho(x) = \begin{cases} \frac{k}{\rho}\exp\left(-\frac{1}{1-|x/\rho|^2}\right) & \text{if } |x| < \rho \\ 0 & \text{if } |x| \geq \rho \end{cases}, \tag{37}$$

where k is the constant such that $\int_{\mathbb{R}} J_1(x)\mathrm{d}x = 1$ [1]. If $u(x)$ is defined only on a closed interval $[x_a,x_b]$, then it is extended as

$$u(x) = \begin{cases} u(x_a) & \text{if } x < x_a \\ u(x_b) & \text{if } x_b < x \end{cases}. \tag{38}$$

Assume that $\varphi_u(u)$ and $\psi_u(u)$ are continuous in the set

$$U = \left\{u \middle| u = u_1(x) \text{ or } u = u_2(x), \exists x \in \overline{\Omega}\right\} \tag{39}$$

as well as in the set $[\min_{x\in\Omega}(u_1(x), u_2(x)), \max_{x\in\Omega}(u_1(x), u_2(x))] \backslash U$. Then, auxiliary functions of x are defined as

$$\Phi = \int_0^1 \phi_u(u_1 + \tau(u_2 - u_1))\mathrm{d}\tau = \begin{cases} \frac{\phi(u_1)-\phi(u_2)}{u_1-u_2} & \text{if } u_1 \neq u_2 \\ \phi_u(u_1) & \text{if } u_1 = u_2 \end{cases}, \tag{40}$$

$$\Phi_\rho = \int_0^1 \phi_u\left(u_1^\rho + \tau\left(u_2^\rho - u_1^\rho\right)\right)\mathrm{d}\tau = \begin{cases} \frac{\phi(u_1^\rho)-\phi(u_2^\rho)}{u_1^\rho-u_2^\rho} & \text{if } u_1^\rho \neq u_2^\rho \\ \phi_u\left(u_1^\rho\right) & \text{if } u_1^\rho = u_2^\rho \end{cases}, \tag{41}$$

$$\Psi = \int_0^1 \psi_u(u_1 + \tau(u_2 - u_1))\mathrm{d}\tau = \begin{cases} \frac{\psi(u_1)-\psi(u_2)}{u_1-u_2} & \text{if } u_1 \neq u_2 \\ \psi_u(u_1) & \text{if } u_1 = u_2 \end{cases}, \tag{42}$$

and

$$\Psi_\rho = \int_0^1 \psi_u\left(u_1^\rho + \tau\left(u_2^\rho - u_1^\rho\right)\right)\mathrm{d}\tau = \begin{cases} \frac{\psi(u_1^\rho)-\psi(u_2^\rho)}{u_1^\rho-u_2^\rho} & \text{if } u_1^\rho \neq u_2^\rho \\ \psi_u\left(u_1^\rho\right) & \text{if } u_1^\rho = u_2^\rho \end{cases}. \tag{43}$$

The following lemma is the core involving the uniqueness of a GS.

Lemma 1. Assume that both of

$$\lim_{y\to x^-}\Psi(y)\lim_{y\to x^+}\Psi(y)>0 \tag{44}$$

and

$$\lim_{y\to x^-}\left(\frac{\mathrm{d}\Phi(y)}{\mathrm{d}x}-\Psi(y)\right)\lim_{y\to x^+}\left(\frac{\mathrm{d}\Phi(y)}{\mathrm{d}x}-\Psi(y)\right)>0 \tag{45}$$

hold for any $x\in\overline{\Omega}$ such that $\lim_{y\to x^-}\Phi(y)\lim_{y\to x^+}\Phi(y)\le 0$. Then the linear SFO ODE

$$\Phi_\rho\frac{\mathrm{d}f}{\mathrm{d}x}=\Psi_\rho f+F \tag{46}$$

has a solution $f\in W^{1,1}(\Omega)$ for any $F\in C_B^1(\Omega)$, when ρ is taken small enough.

Proof. If $\lim_{y\to x^-}\Phi(y)\lim_{y\to x^+}\Phi(y)>0$ for any $x\in\overline{\Omega}$, then there exists a constant c such that $|\Phi_\rho|\ge c>0$ for any $x\in\overline{\Omega}$ when ρ is taken small enough. Then, an initial value problem of (46) with the initial condition $f(0)=0$ has a solution $f\in C_B^1(\Omega)\subset W^{1,1}(\Omega)$. With the assumption of the lemma, there exists a point ξ in the ρ-neighborhood of x such that $\Phi_\rho(\xi)=0$ and $|\Psi_\rho(\mathrm{d}\Phi_\rho/\mathrm{d}x-\Psi_\rho)(\xi)|>0$ when ρ is taken small enough. Then, solving $\Psi_\rho f+F=0$ and applying the L'Hôpital's rule to $(\Psi_\rho f+F)/\Phi_\rho$ yields

$$f=-\frac{F}{\Psi_\rho} \tag{47}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\Psi_\rho\frac{\mathrm{d}F}{\mathrm{d}x}-\frac{\mathrm{d}\Psi_\rho}{\mathrm{d}x}F}{\Psi_\rho\left(\frac{\mathrm{d}\Phi_\rho}{\mathrm{d}x}-\Psi_\rho\right)} \tag{48}$$

at such a ξ. If $\Phi_\rho\ne 0$ in $[0,x_r)$ and $\Phi_\rho(x_r)=0$, then an initial value problem of (46) with the initial condition $f(x_r)=-F(x_r)/\Psi_\rho(x_r)$ has a solution $f\in C_B^1(0,x_r)$. If $\Phi_\rho\ne 0$ in $(x_l,X]$ and $\Phi_\rho(x_l)=0$, then an initial value problem of (46) with the initial condition $f(x_l)=-F(x_l)/\Psi_\rho(x_l)$ has a solution $f\in C_B^1(x_l,X)$. If $\Phi_\rho\ne 0$ in $(x_l,x_r)\subset\Omega$ and $\Phi_\rho(x_l)=\Phi_\rho(x_r)=0$, then two initial value problems of (46) with the initial conditions $f(x_l)=-F(x_l)/\Psi_\rho(x_l)$ and $f(x_r)=-F(x_r)/\Psi_\rho(x_r)$ have respective solutions $f=f_l$ and $f=f_r$ in $C_B^1(x_l,x_r)$. Then, $\Delta f=f_r-f_l\in C_B^1(x_l,x_r)$ solves

$$\frac{\mathrm{d}\Delta f}{\mathrm{d}x}=\frac{\Psi_\rho}{\Phi_\rho}\Delta f \tag{49}$$

in (x_l,x_r) with the Neumann boundary condition $\mathrm{d}\Delta f/\mathrm{d}x=0$ at $x=x_l$ and $x=x_r$, implying that $\mathrm{d}\Delta f/\mathrm{d}x=0$ and then $\Delta f=0$ in (x_l,x_r). Therefore, f_l and f_r are identical. Then, $f=f_l=f_r\in C_B^1(x_l,x_r)$ solves (46) with the Dirichlet boundary condition consisting of $f(x_l)=-F(x_l)/\Psi_\rho(x_l)$ and $f(x_r)=-F(x_r)/\Psi_\rho(x_r)$. The procedure above completes the construction of a solution $f\in C_B^1(\Omega)\subset W^{1,1}(\Omega)$ of (46). □

A necessary condition so that GSs are not unique is stated as follows.

Theorem 4. Suppose that there are two GSs $u_1(x)$ and $u_2(x)$ satisfying (26) for any $f\in W^{1,1}(\Omega)$. If $u_1(x)$ and $u_2(x)$ are different in the sense that there exists $F\in C_B^1(\Omega)$ such that $\int_0^X F(u_1-u_2)\mathrm{d}x\ne 0$, then there exists $x\in\overline{\Omega}$ such that $\lim_{y\to x^-}\Phi(y)\lim_{y\to x^+}\Phi(y)\le 0$ and either (44) or (45), or both, does not hold.

Proof. If the assertion is false, then Lemma 1 holds. Then,

$$\begin{aligned}\int_0^X F(u_1-u_2)\mathrm{d}x&=\int_0^X\left(\Phi_\rho\tfrac{\mathrm{d}f}{\mathrm{d}x}-\Psi_\rho f\right)(u_1-u_2)\mathrm{d}x\\&=\int_0^X\left(\left(\Phi_\rho-\Phi\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}-\left(\Psi_\rho-\Psi\right)f\right)(u_1-u_2)\mathrm{d}x+\int_0^X\left(\Phi\tfrac{\mathrm{d}f}{\mathrm{d}x}-\Psi f\right)(u_1-u_2)\mathrm{d}x\\&=\int_0^X\left(\left(\Phi_\rho-\Phi\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}-\left(\Psi_\rho-\Psi\right)f\right)(u_1-u_2)\mathrm{d}x\end{aligned} \tag{50}$$

because of (38), (42), and (26). Let $\Delta u=u_1-u_2$, $\Delta u^\rho=u_1^\rho-u_2^\rho$, $\Delta\varphi=\varphi(u_1)-\varphi(u_2)$, and $\Delta\phi^\rho=\phi(u_1^\rho)-\phi(u_2^\rho)$. Applying the triangle inequality to (50) and then using (40) result in

$$\begin{aligned}\left|\int_0^X F(u_1-u_2)\mathrm{d}x\right|&\le\left|\int_0^X\left(\Phi_\rho-\Phi\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}(u_1-u_2)\mathrm{d}x\right|+\left|\int_0^X\left(\Psi_\rho-\Psi\right)f(u_1-u_2)\mathrm{d}x\right|\\&\le\left|\int_0^X\left(\Phi_\rho(u_1-u_2)-(\phi(u_1)-\phi(u_2))\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x\right|+\|\Psi_\rho-\Psi\|_{L^1}\|f\|_{L^1}\|u_1-u_2\|_{L^1}\end{aligned}. \tag{51}$$

With (41) and (46), the first term of the right hand side of (51) is evaluated as

$$\begin{aligned}&\left|\int_0^X\left(\Phi_\rho(u_1-u_2)-(\phi(u_1)-\phi(u_2))\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x\right|\\&=\left|\int_0^X\left(\left(\Delta u\Phi_\rho-\Delta u^\rho\Phi_\rho\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}+\left(\Delta u^\rho\Phi_\rho-\Delta\phi\right)\tfrac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x\right|\\&\le\left|\int_0^X(\Delta u-\Delta u^\rho)\left(\Psi_\rho f+F\right)\mathrm{d}x\right|+\left|\int_0^X\left((\Delta\phi^\rho-\Delta\phi)\tfrac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x\right|.\end{aligned} \tag{52}$$

With (26), the second term of the right hand side of (52) is evaluated as

$$\begin{aligned}
&\left| \int_0^X \left(\left(\Delta\phi^\rho - \Delta\phi \right) \frac{\mathrm{d}f}{\mathrm{d}x} \right) \mathrm{d}x \right| \\
&\leq \left| \int_0^X \left(\phi\left(u_1^\rho\right) \frac{\mathrm{d}f}{\mathrm{d}x} - \phi\left(u_2^\rho\right) \frac{\mathrm{d}f}{\mathrm{d}x} - \psi(u_1) f + \psi(u_2) f \right) \mathrm{d}x \right| \\
&\leq \left| \int_0^X \left(\phi\left(u_1^\rho\right) \frac{\mathrm{d}f}{\mathrm{d}x} - \psi\left(u_1^\rho\right) f \right) \mathrm{d}x - \int_0^X \left(\phi\left(u_2^\rho\right) \frac{\mathrm{d}f}{\mathrm{d}x} - \psi\left(u_2^\rho\right) f \right) \mathrm{d}x \right| \\
&\quad + \left| \int_0^X f\left(\psi\left(u_1^\rho\right) - \psi(u_1) \right) \mathrm{d}x - \int_0^X f\left(\psi\left(u_2^\rho\right) - \psi(u_2) \right) \mathrm{d}x \right|
\end{aligned} \tag{53}$$

From these inequalities (51), (52), and (53), it is concluded that $\left|\int_0^X F(u_1 - u_2)\mathrm{d}x\right|$ approaches to zero for any $F \in C_B^1(\Omega)$ as passing the limit $\rho \to 0^+$, contradicting the hypothesis of $u_1 \neq u_2$. □

4. Illustrative examples

Numerical approximations to the solutions of (28) for several different cases illustrate characteristics of the 1D steady open channel flows. The acceleration due to gravity, the roughness coefficient, and the exponent are fixed as $g = 9.80665$, $n = 0.01$, and $p = 2/3$, respectively. Firstly, we consider rectangular cross-sectional channels with different slopes to revisit the conventional classification of flows in open channel hydraulics. Then, we address the non-uniqueness of solutions in a modified circular cross-sectional channel.

4.1. Unique solutions in rectangular cross-sectional channels

The constant width of the rectangular cross-sectional channels is set as $\partial A/\partial h = 0.6$. The discharge is given as $Q = 0.01$. Then, the value of the critical flow depth is in the range $0.0304830 < h_{\mathrm{cri}} < 0.0304831$. There is a single uniform flow depth h_{uni} if the bed slope S_0 is positive, whereas a real h_{uni} does not exist if $S_0 \leq 0$. The value of the bed slope $S_0 = S_{\mathrm{cri}}$ achieving $h_{\mathrm{uni}} = h_{\mathrm{cri}}$ is in the range $1/139.991 < S_{\mathrm{cri}} < 1/139.990$. Channels with $S_0 > S_{\mathrm{cri}}$, $S_0 = S_{\mathrm{cri}}$, $0 < S_0 < S_{\mathrm{cri}}$, $S_0 = 0$, and $S_0 < 0$ are referred to as the steep slope channel, the critical slope channel, the mild slope channel, the horizontal slope channel, and the adverse slope channel, respectively. Fig. 1 shows the computed water surface profiles of the GVFSs, with appropriate Galilean transformations, for the channels of $S_0 = 2S_{\mathrm{cri}}$, $S_0 = S_{\mathrm{cri}}$, $S_0 = S_{\mathrm{cri}}/2$, $S_0 = 0$, and $S_0 = -S_{\mathrm{cri}}$. According to the conventional classification of flows in open channel hydraulics, the GVFSs as per Remark 1 are labeled as S1, S2, etc. The GVFSs as per Remark 2 are the uniform flow depths (lines in blue color) appearing in the steep, the critical, and the mild slope channels. In the critical slope channel, Remark 4 is indeed the case. As per Theorem 3, a hydraulic jump between two different GVFSs across the critical flow depth occurs if and only if an appropriate Galilean transformation is applied to one of the GVFSs so that the specific forces $M = -\varphi$ (represented as different colors) coincide at a point ξ.

4.2. Non-unique solutions in a modified circular cross-sectional channel

There are non-unique VSs of a Dirichlet problem, and thus GSs, in the channel having a modified circular cross-section whose shape is shown in the top-left of Fig. 2; the top part of the circle with a diameter 0.10 above $z = z_{\mathrm{cri}}$ is replaced with the symmetric two chords. The discharge Q, the bed slope S_0, and the height z_{cri} of the chords' feet are set so that there is a single uniform depth h_{uni} achieving $h_{\mathrm{uni}} = h_{\mathrm{cri}} = z_{\mathrm{cri}}$ under the fixed conditions of the other parameters. The resulting values are estimated as $0.00955282 < Q < 0.00955283$, $1/57.1774 < S_0 < 1/57.1773$, and $0.0938181 < z_{\mathrm{cri}} < 0.0938182$. Note that there can be two uniform depths in general in a circular cross-sectional channel.

The Dirichlet problem consists of the governing Eq (22) in the viscosity sense, the specified downstream endpoint $X = 10$, and the Dirichlet boundary condition (25) with the specified boundary values $u_{\mathrm{up}} = \log 0.08$ and $u_{\mathrm{down}} = u_{\mathrm{uni}} = u_{\mathrm{cri}}$. As $\varphi_u(u_{\mathrm{up}}) > 0$ and $\varphi_u(u_{\mathrm{down}}) = 0$, a VS can exist according to Theorem 2. Indeed, there is an S3-like GVFS below $u_{\mathrm{uni}} = u_{\mathrm{cri}}$ satisfying the upstream boundary condition, and there is an infinite number of VSs in $C_B(\Omega)$ above or at $u_{\mathrm{uni}} = u_{\mathrm{cri}}$ satisfying the downstream boundary condition as mentioned in Remark 4 with non-smoothness at arbitrary $\xi \in \overline{\Omega}$. Therefore, there is an infinite number of VSs of the Dirichlet problem, having a discontinuity across $u_{\mathrm{uni}} = u_{\mathrm{cri}}$ at an arbitrary point in a certain range in Ω. Here, as plotted in the top-right of Fig. 2, we choose two VSs $u_1(x)$ (thick lines) and $u_2(x)$ (thin lines) having discontinuities at $x = \xi_1 = 4$ and at $x = \xi_2 = 7$, respectively. The non-smooth points of $u_1(x)$ and $u_2(x)$ in the $C_B(\Omega)$ parts are close to $x = 5.90$ and $x = 8.02$, respectively. The auxiliary functions Φ (in red color), Ψ (in green color), and $\Phi_x - \Psi = \mathrm{d}\Phi/\mathrm{d}x - \Psi$ (in blue color), are plotted in the bottom-right of Fig. 2. Spurious oscillations in $\Phi_x - \Psi$ around the non-smooth points of the VSs are due to numerical differentiation of Φ. The points $x \in \overline{\Omega}$ such that $\lim_{y \to x^-} \Phi(y) \lim_{y \to x^+} \Phi(y) \leq 0$ are $x = \xi_1 = 4$ and $x = \xi_2 = 7$. At each of those points x, both of (44) and (45) are violated, and thus the assertion of Theorem 4 is confirmed.

Lastly, we address the stability of the VSs. The S3-like GVFS below $u_{\mathrm{uni}} = u_{\mathrm{cri}}$ with the upstream boundary condition $u_{\mathrm{up}} = \log 0.08$ satisfies a downstream boundary condition $u_{\mathrm{down}} = \log h_X$, where $0.0924501 < h_X < 0.0924640 < z_{\mathrm{cri}}$. Therefore, the Dirichlet boundary problem changing u_{down} from u_{cri} to $u_{\mathrm{cri}} - \varepsilon$ with sufficiently small positive ε does not have any solution, implying the instability of the VSs of the original Dirichlet problem.

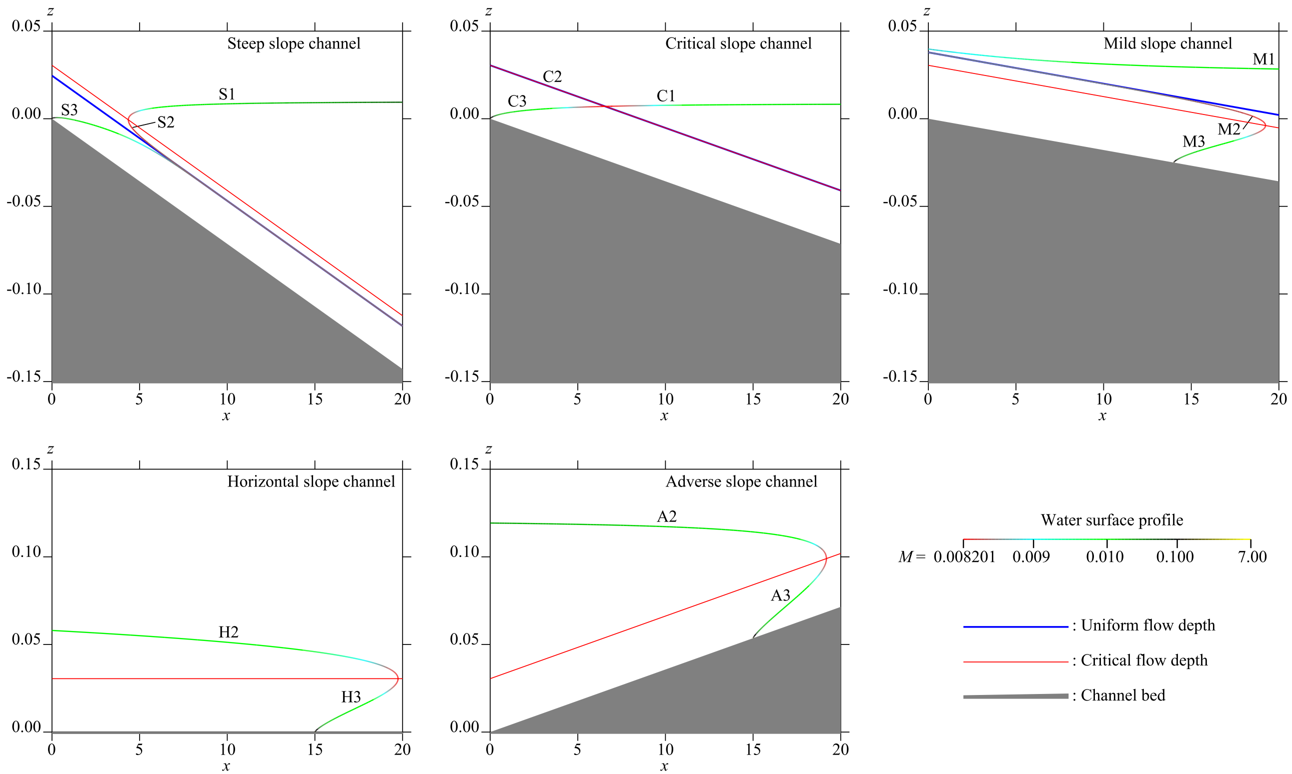

Fig. 1. Numerically reproduced all possible types of GVFSs in rectangular cross-sectional channels with different slopes.

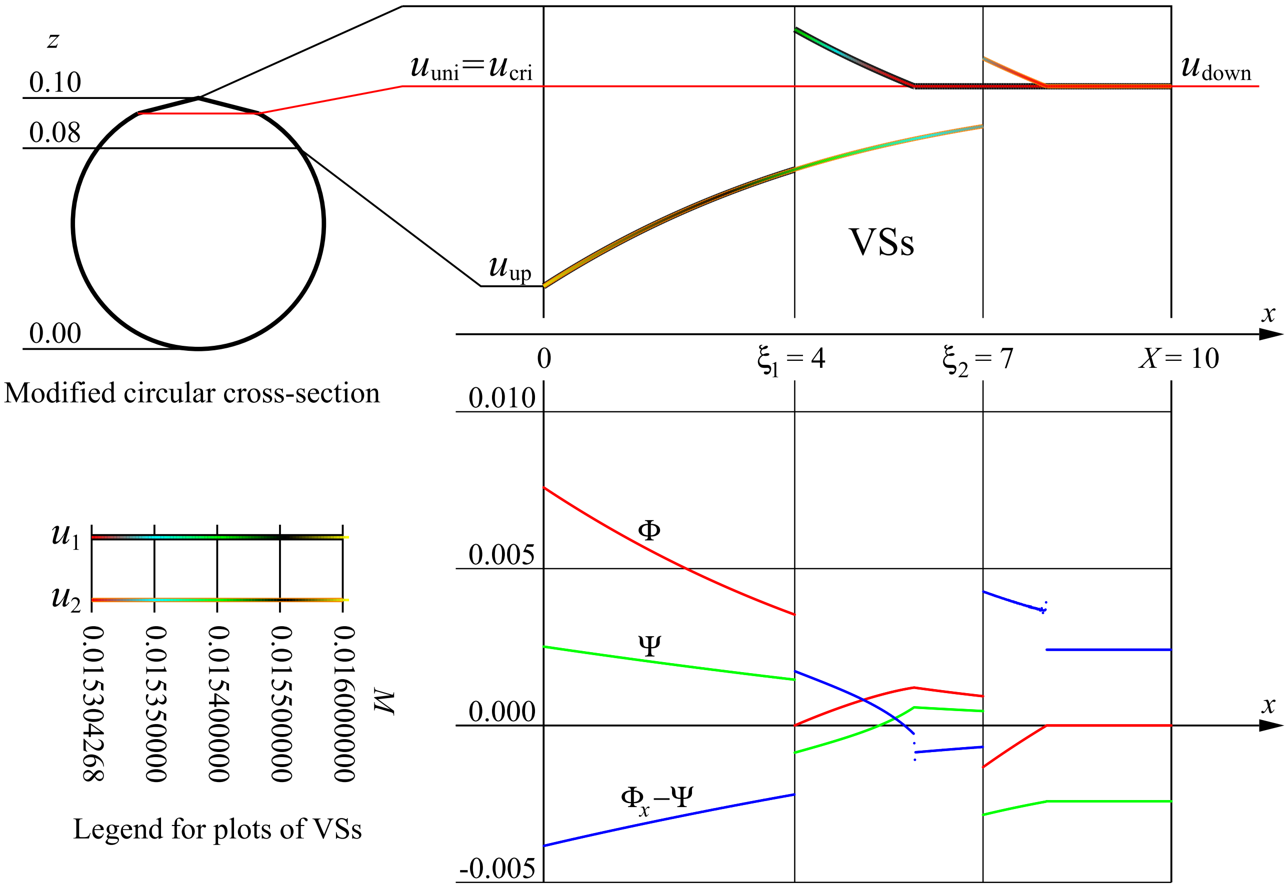

Fig. 2. Non-unique VSs of the Dirichlet problem in the modified circular cross-sectional channel with the auxiliary functions.

Conclusions

This study has presented a thorough description of 1D steady open channel flows, whose water surface profiles are represented by the VSs of the Dirichlet problems in a bounded open x-domain Ω. The governing HJ equation is FOQL with an improper Hamiltonian function. The VSs include the GVFSs and the hydraulic jumps satisfying the entropy condition, which have been known in conventional open channel hydraulics. The U-env and the L-env of a VS belong to $USC(\overline{\Omega})$ and $LSC(\overline{\Omega})$, respectively, but the VS is identical with a GS in the Sobolev space $W^{1,1}(\Omega)$. The non-uniqueness of GSs and thus of VSs involving discontinuities depends on the regularity of the Hamiltonian function, determined by the channel's cross-sectional shape. The necessary condition of non-uniqueness is described in terms of the auxiliary functions. The illustrative examples include the unique solutions in the rectangular cross-sectional channel and the non-unique solutions in the modified circular cross-sectional channel. The implication of non-uniqueness shall be researched for further understanding of 1D open channel flows; an open problem is regarding the well-posedness of initial-boundary value problems for the 1D unsteady SWEs in that modified circular cross-sectional channel. We shall also drop the assumption A1 to deal with more challenging problems of non-prismatic channels in the follow-up studies.

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CRediT authorship contribution statement

Sovanna Mean: Methodology, Validation, Investigation, Writing - Original Draft, Writing - Review & Editing, Visualization. **Koichi Unami:** Methodology, Software, Investigation, Data Curation, Writing - Original Draft, Writing - Review & Editing, Visualization, Funding acquisition, Project administration. **Hisashi Okamoto:** Conceptualization, Resources, Writing - Review & Editing. **Masayuki Fujihara:** Writing - Review & Editing, Supervision, Funding acquisition.

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