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Bivariant Class of Degree One

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Abstract. Let $f : X \to Y$ be a projective birational morphism, between complex quasi-projective varieties. Fix a bivariant class $\theta \in H^0(X \xrightarrow{f} Y) \cong Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ (here $\mathbb{A}$ is a Noetherian commutative ring with identity, and $\mathbb{A}_X$ and $\mathbb{A}_Y$ denote the constant sheaves). Let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced Gysin morphism. We say that θ*has degree one* if $\theta_0(1_X) = 1_Y \in H^0(Y)$. This is equivalent to say that θ is a section of the pull-back $f^* : \mathbb{A}_Y \to Rf_*\mathbb{A}_X$, i.e. $\theta \circ f^* = \mathrm{id}_{\mathbb{A}_Y}$, and it is also equivalent to say that $\mathbb{A}_Y$ is a direct summand of $Rf_*\mathbb{A}_X$. We investigate the consequences of the existence of a bivariant class of degree one. We prove explicit formulas relating the (co)homology of X and Y, which extend the classic formulas of the blowing-up. These formulas are compatible with the duality morphism. Using which, we prove that the existence of a bivariant class θ of degree one for a resolution of singularities, is equivalent to require that Y is an $\mathbb{A}$-homology manifold. In this case θ is unique, and the Betti numbers of the singular locus $\mathrm{Sing}(Y)$ of Y are related with the ones of $f^{-1}(\mathrm{Sing}(Y))$.

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1. Introduction

Let $f : X \to Y$ be a projective birational morphism, between complex quasi-projective varieties. Fix a bivariant class

$$\theta \in H^0(X \xrightarrow{f} Y) \cong Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y),$$

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where $\mathbb{A}$ is a Noetherian commutative ring with identity, and $\mathbb{A}_X$ and $\mathbb{A}_Y$ denote the constant sheaves [10]. Let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced Gysin morphism. We say that θ *has degree one* if $\theta_0(1_X) = 1_Y \in H^0(Y)$. This is equivalent to say that θ is a section of the pull-back $f^* : \mathbb{A}_Y \to Rf_*\mathbb{A}_X$, i.e. $\theta \circ f^* = \mathrm{id}_{\mathbb{A}_Y}$ (Remark 2.1, (i)), and it is also equivalent to say that $\mathbb{A}_Y$ is a direct summand of $Rf_*\mathbb{A}_X$ (Theorem 3.1). We investigate the consequences of the existence of a bivariant class of degree one. Specifically, in Sect. 4 we prove explicit formulas relating the (co)homology of X and Y, which extend classic formulas of the blowing-up (Propositions 4.1, 4.3, 4.4, 4.5). These formulas are compatible with the duality morphism (Sect. 5). Using which, we prove that the existence of a bivariant class θ of degree one for a resolution of singularities, is equivalent to require that Y is an $\mathbb{A}$-homology manifold (Theorem 6.1). In this case θ is unique, and the Betti numbers of the singular locus $\mathrm{Sing}(Y)$ of Y are related with the ones of $f^{-1}(\mathrm{Sing}(Y))$ (Remark 6.2, (ii)).

Although elementary, Theorem 3.1 seems to have escaped explicit notice. As far as we know, Theorem 6.1 gives a new characterization of homology manifolds, in terms of their resolution of singularities.

2. Notations

(i) Let $\mathbb{A}$ be a Noetherian commutative ring with identity (e.g. $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$). Every topological space V occurring in this paper will be assumed to be imbeddable as a closed subspace of some $\mathbb{R}^N$ [10, p. 32] (e.g. a complex quasi-projective variety, with the natural topology, and its open subsets). Maps between topological spaces are assumed continuous of finite cohomological dimension [10, p. 83] (e.g. algebraic maps between complex quasi-projective varieties, and their restrictions on open subsets). We denote by $H^i(V)$ and $H_i(V)$ the cohomology and the Borel–Moore homology groups, with $\mathbb{A}$-coefficients, of V [9]. We denote by $\mathrm{Sh}(V)$ the category of sheaves of $\mathbb{A}$-modules on V. Let $D^b_c(V)$ denote the derived category of bounded constructible complexes of $\mathbb{A}$-sheaves $\mathcal{F}^\bullet$ on V [8], [5]. The symbol $IC^\bullet_V$ represents the intersection cohomology complex of V. If V is a smooth, irreducible, quasi-projective complex variety of dimension n, then $IC^\bullet_V \cong \mathbb{A}_V[n]$, where $\mathbb{A}_V$ is the constant sheaf.

(ii) Let $f : X \to Y$ be a continuous and proper map. Fix a bivariant class [10]

$$\theta \in H^0(X \xrightarrow{f} Y) \cong Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y).$$

Let $\theta_0 : H^0(X) \to H^0(Y)$ be the induced Gysin homomorphism. We say that θ *has degree one* (for the map f) if $\theta_0(1_X) = 1_Y \in H^0(Y)$ [16, p. 238].

(iii) Let V be an irreducible, quasi-projective variety of complex dimension n. We say that V is an $\mathbb{A}$-*homology manifold* if for all $y \in Y$ and for all $i \neq 2n$ one has $H_i(Y, Y\backslash\{y\}) = 0$, and $H_{2n}(Y, Y\backslash\{y\}) \cong \mathbb{A}$ [3], [4] (by $H_i(Y, Y\backslash\{y\})$ we denote the singular homology of a pair). This is equivalent to say that $\mathbb{A}_Y[n]$ is self-dual, or that $\mathbb{A}_Y[n] \cong IC_Y^{\bullet}$ [4, p. 804–805].

(iv) An element $\theta \in H^i(X \xrightarrow{f} Y)$ is called a *strong orientation of codimension* i for the morphism $f : X \to Y$ if, for all morphisms $g : Z \to X$, the morphism

$$H^{\bullet}(Z \xrightarrow{g} X) \xrightarrow{\bullet\theta} H^{\bullet}(Z \xrightarrow{f\circ g} Y)$$

is an isomorphism [10, p. 26], [4, p. 803].

Remark 2.1. (i) Observe that θ *has degree one if and only if* θ *is a section of the pull-back* $f^* : \mathbb{A}_Y \to Rf_*\mathbb{A}_X$, i.e.

$$\theta_0(1_X) = 1_Y \iff \theta \circ f^* = \mathrm{id}_{\mathbb{A}_Y}.$$

In fact, assume that θ is of degree one. For every $y \in H^{\bullet}(Y)$, one has ([10, p. 26, (G4), (i)], [16, p. 251, 9]):

$$\theta_*(f^*(y)) = \theta_*(1_X \cup f^*(y)) = \theta_*(1_X) \cup y = 1_Y \cup y = y.$$

By functoriality, this means that the morphism $\theta \circ f^*$ induces the identity on the cohomology groups $\mathrm{id}_{H^{\bullet}(Y)} = \theta_* \circ f^* : H^{\bullet}(Y) \to H^{\bullet}(Y)$. On the other hand, we have $\theta \circ f^* \in \mathrm{Hom}_{D^b_c(Y)}(\mathbb{A}_Y, \mathbb{A}_Y) \cong H^0(Y)$. It follows that $\theta \circ f^* = \mathrm{id}_{\mathbb{A}_Y}$.

Conversely, if $\theta \circ f^* = \mathrm{id}_{\mathbb{A}_Y}$, then the composite $H^0(Y) \xrightarrow{f^*} H^0(X) \xrightarrow{\theta_0} H^0(Y)$ is the identity of $H^0(Y)$. Since $f^*(1_Y) = 1_X$, it follows that $\theta_0(1_X) = 1_Y$, i.e. θ has degree one.

(ii) Let $f : X \to Y$ be a proper map. Let $\theta \in H^0(X \xrightarrow{f} Y)$ be a bivariant class. If $\theta_0(1_X) = d\cdot 1_Y \in H^0(Y)$, and if d is a unit in $\mathbb{A}$, then $d^{-1}\cdot\theta$ is a bivariant class of degree one. Moreover, let $i : W \subseteq Y$ be a non-empty subspace of Y, and let $g : f^{-1}(W) \to W$ be the restriction of f on $f^{-1}(W)$. Denote by $\theta' = i^*(\theta) \in H^0(f^{-1}(W) \xrightarrow{g} W)$ the pull-back of θ. By [10, (G2), *(ii)*, p. 26], we see that $i^*\theta_0(1_X) = \theta'_0 j^*(1_X)$, where $j : f^{-1}(W) \subseteq X$ denotes the inclusion. Therefore, $1_W = \theta'_0(1_{f^{-1}(W)}) \in H^0(W)$. This proves that the pull-back of a bivariant class of degree one, is again of degree one. And, conversely, if Y is path-connected, and θ' is of degree one, then also θ is of degree one.

(iii) Assume that $f : X \to Y$ is a projective, locally complete intersection morphism between complex irreducible quasi-projective varieties, and that f is birational (e.g. f is the blowing-up of Y at a locally complete intersection subvariety $W \subset Y$ [11, p. 114]). Let $\theta \in H^0(X \xrightarrow{f} Y)$ be the *orientation class* of f [11, p. 114], [10, p. 131]. Then θ has degree one. In fact, let U be a non-empty Zariski open set of Y, such that f induces an isomorphism $f^{-1}(U) \cong U$. Let θ' be the restriction of θ on $f^{-1}(U) \to U$.

Since θ' is the orientation class of $f^{-1}(U) \to U$ [11, Lemma 19.2, (a), p. 379], and $f^{-1}(U) \cong U$, it follows that θ' has degree one. By remark (ii) above, also θ has degree one . Compare with [1, p. 137] and [17, p. 12].

(iv) If Y is a quasi-projective $\mathbb{A}$-homology manifold, and $f : X \to Y$ is a resolution of singularities of Y, then there exists a unique bivariant class $\theta \in Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one. See Theorem 6.1 below.

(v) Let $f : X \to Y$ be a projective map between irreducible, complex quasi-projective varieties of the same dimension n. Assume that Y is smooth (or, more generally, that Y is an $\mathbb{A}$-homology manifold). In this case one has (compare with [10, 3.1.4, p. 34], [9, Lemma 2, p. 217], and the proof of Theorem 6.1 below):

$$H^0(X \xrightarrow{f} Y) \cong H_{2n}(X) \cong H^0(X).$$

By remark (i) above, if there exists a bivariant class of degree one for f, then, for every k, $H^k(Y)$ is contained, via pull-back, in $H^k(X)$. Therefore, if $\mathbb{A} = \mathbb{Z}$ and $h^k(Y) > h^k(X)$ for some k, then it happens that $H^0(X \xrightarrow{f} Y) \neq 0$, but $\theta_0 = 0$, for every bivariant class θ. However, if, in addition, f is birational, then the bivariant class θ corresponding to $1_X \in H^0(X)$ is a bivariant class of degree one. In fact, if U is a Zariski open subset of Y such that $f^{-1}(U) \cong U$, the restriction of θ on $f^{-1}(U) \to U$ has degree one. Observe that, if Y is singular, it is no longer true. For instance, let $C \subset \mathbb{P}^3$ be a projective non-singular curve of genus ≥ 1. Let $Y \subset \mathbb{P}^4$ be the cone over C, and let $f : X \to Y$ be the blowing-up of Y at the vertex. Then one has $H^0(X \xrightarrow{f} Y) \neq 0$, but there is no a bivariant class of degree one of f. This is a consequence of Theorem 6.1. For more details, see Remark 6.2, (iii).

(vi) Let $f : X \to Y$ be a projective map between irreducible quasi-projective varieties. Assume there exists a bivariant class θ of degree one. Put $n = \dim X$, and $m = \dim Y$. Since $f_* \circ \theta^* = \mathrm{id}_{\mathrm{H}_\bullet(\mathrm{Y})}$, the Gysin map θ^* induces an inclusion $H_\bullet(Y) \subseteq H_\bullet(X)$. It follows that $m \leq n$. Moreover, f is surjective, otherwise the push-forward $f_* : H_{2m}(X) \to H_{2m}(Y)$ vanishes. Since restricting θ to some special fibre, we obtain again a bivariant class of degree one, in general it may happen that $n > m$. It is clear that, if $n = m$, then f is birational.

3. Bivariant Class of Degree One and Decompositions

The existence of a bivariant class of degree one can be reformulated in terms of certain decompositions in derived category. This is the content of the following Theorem 3.1.

Theorem 3.1. *Let $f : X \to Y$ be a continuous and proper map, with Y path-connected. Let $U \subseteq Y$ be a non-empty open subset such that f induces an* *homeomorphism* $f^{-1}(U) \cong U$*. Set* $W = Y \backslash U$*, and* $\widetilde{W} = f^{-1}(W)$*. The following properties are equivalent.*

(i) *There exists a bivariant class* $\theta \in Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ *of degree one.*
(ii) *In* $D^b_c(Y)$ *there exists a cross isomorphism* $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$*.*
(iii) *In* $D^b_c(Y)$ *there exists a decomposition* $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}$*.*

In this section we are going to prove Theorem 3.1. To this purpose, we need some preliminaries. The first one is the following lemma.

Lemma 3.2. *Let* $\mathcal{T}$ *be a triangulated category, and* $f^* \in \mathrm{Hom}_{\mathcal{T}}(A, B)$ *be a morphism in* $\mathcal{T}$*. Assume that* f^* *if left-invertible, i.e. that there exists* $\theta \in \mathrm{Hom}_{\mathcal{T}}(B, A)$ *such that* $\theta \circ f^* = 1_A$*. Then we have* $B \cong A \oplus C$ *for some* $C \in Ob(\mathcal{T})$*.*

Proof of Lemma 3.2. The axiom TR1 (iii) of triangulated categories implies that f^* can be completed to a distinguished triangle

$$A \xrightarrow{f^*} B \longrightarrow C$$

[12, p. 12]. Thus, combining the hypothesis $\theta \circ f^* = 1_A$ with axioms TR1 and TR3, we have a commutative diagram of distinguished triangles

$$\begin{tikzcd} A \arrow[r, "f^*"] \arrow[d, "1_A"] & B \arrow[r] \arrow[d, "\theta"] & C \arrow[d] \\ A \arrow[r, "1_A"] & A \arrow[r] & 0. \end{tikzcd}$$

The axiom TR2 provides also the following commutative diagram of distinguished triangles

$$\begin{tikzcd} C \arrow[r, "\delta"] \arrow[d] & A[1] \arrow[r] \arrow[d, "1_{A[1]}"] & B[1] \arrow[d, "\theta[1]"] \\ 0 \arrow[r] & A[1] \arrow[r] & A[1], \end{tikzcd}$$

from which we argue that δ vanishes. Then the triangle $A \xrightarrow{f^*} B \longrightarrow C$ splits; see e.g. [12, Exercise 1.38]. $\square$

We are in position to prove that (i) is equivalent to (iii) in Theorem 3.1.

To this purpose, first assume there exists a bivariant class $\theta : Rf_*\mathbb{A}_X \to \mathbb{A}_Y$ of degree one, and let $f^* : \mathbb{A}_Y \to Rf_*\mathbb{A}_X$ be the pull-back morphism. By Remark 2.1, (i), we know that $\theta \circ f^* = 1_{\mathbb{A}_Y}$. Therefore, we may apply previous Lemma 3.2, with $\mathcal{T} = D^b_c(Y)$, $A = \mathbb{A}_Y$, $B = Rf_*\mathbb{A}_X$, with the morphism f^* as the pull-back, and θ as the given bivariant class. It follows a decomposition like $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}$.

Conversely, suppose there exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}$. By projection, it induces a bivariant class $\eta : Rf_*\mathbb{A}_X \to \mathbb{A}_Y$. Let U' be a path-connected component of U. Since the restriction η' of η on U' is an automorphism of $\mathbb{A}_{U'}$, it follows that $\eta'_0(1_{U'}) = d \cdot 1_{U'} \in H^0(U')$, with some unit $d \in \mathbb{A}$. Therefore, $d^{-1} \cdot \eta$ is a bivariant class of degree one (compare with Remark 2.1, (ii)).

This concludes the proof that (i) is equivalent to (iii) in Theorem 3.1.

Remark 3.3. In order to prove that (i) implies (iii), we do not need the existence of U.

Now we are going to prove that (i) is equivalent to (ii).

Observe that the same argument we just used to prove that (iii) implies (i), proves that (ii) implies (i). In fact, suppose there exists a decomposition $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$. By projection, it induces a bivariant class $\eta : Rf_*\mathbb{A}_X \to \mathbb{A}_Y$. Since both $\mathbb{A}_W$ and $Rf_*\mathbb{A}_{\widetilde{W}}$ are supported on W, the restriction of η on U is an automorphism of $\mathbb{A}_U$. And now we may conclude as before.

In order to conclude the proof of Theorem 3.1, we only have to prove that (i) implies (ii). Also in this case, we need some preliminaries.

Consider the following natural commutative diagram

$$\begin{tikzcd} \widetilde{W} \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] & U \arrow[l, hook', "\partial_X"'] \arrow[d, "1"] \\ W \arrow[r, "i"] & Y & U \arrow[l, hook', "\partial_Y"'] \end{tikzcd} \tag{1}$$

where $g : \widetilde{W} \to W$ denotes the restriction of f, and the other maps are the inclusions. Denote by $\mathcal{A}$ (resp. $\mathcal{B}$) the full subcategory of $\mathrm{Sh}(X)$ (resp. $\mathrm{Sh}(Y)$) supported on U.

Lemma 3.4. *On the category* $\mathrm{Sh}(U)$ *we have* $f_* \circ \partial_{X\,!} = \partial_{Y\,!}$. *Furthermore,* f_* *is an exact equivalence between* $\mathcal{A}$ *and* $\mathcal{B}$, *whose inverse is the pull-back* f^*.

Proof. First we prove that $f_* \circ \partial_{X\,!} = \partial_{Y\,!}$ on $\mathrm{Sh}(U)$.

Let $\mathcal{F}$ be a sheaf on U and let $V \subseteq Y$ be an open subset. By [13, Definition 6.1, p. 106], we have

$$f_*(\partial_{X!}(\mathcal{F}))(V) = \{s \in \Gamma\left(f^{-1}(V) \cap U, \mathcal{F}\right) \mid \text{ supp(s) is closed in } f^{-1}(V)\} \tag{2}$$

and

$$\partial_{Y!}(\mathcal{F})(V) = \{s \in \Gamma\left(V \cap U, \mathcal{F}\right) \mid \text{ supp}(s) \text{ is closed in } V\}.$$

Since f is continuous, we have $\partial_{Y!}(\mathcal{F})(V) \subseteq f_*(\partial_{X!}(\mathcal{F}))(V)$. Hence, $\partial_{Y!}(\mathcal{F})$ is a subsheaf of $f_*(\partial_{X!}(\mathcal{F}))$. As for the opposite inclusion, we argue as follows. By the local compactness of Y, we can assume that the closure of V is compact in Y. Fix $s \in f_*(\partial_{X!}(\mathcal{F}))(V)$. Then s is a section of the sheaf $\mathcal{F}$ over the open set $f^{-1}(V) \cap U$ of Y, whose corresponding support

$$C := \text{supp}(s) := \left\{y \in f^{-1}(V) \cap U \mid s_y \neq 0\right\}$$

is closed in $f^{-1}(V)$ (compare with the definition (2)). It suffices to prove that $f(C)$, which is homeomorphic to C, is closed in V. Since f is a proper morphism, $f^{-1}(\overline{V})$ is compact and the map $f^{-1}(\overline{V}) \to \overline{V}$ is closed. Then we have

$$C = f(C) = f(\overline{C} \cap f^{-1}(V)) = f(\overline{C}) \cap V$$

and we are done.

We are left with the proof that f_* induces an exact equivalence between $\mathcal{A}$ and $\mathcal{B}$. By [13, Proposition 6.4, p. 107], we know that $\partial_{X!}$ ($\partial_{Y!}$ resp.) induces an equivalence between $\mathrm{Sh}(U)$ and $\mathcal{A}$ ($\mathcal{B}$ resp.), whose inverse functor is the pull-back ∂_X^* (∂_Y^* resp.). Taking into account what we have just proved, that is $f_* \circ \partial_{X\,!} = \partial_{Y\,!}$, it follows that f_* induces an equivalence between $\mathcal{A}$ and $\mathcal{B}$, whose inverse is the functor $\partial_{X\,!} \circ \partial_Y^* : \mathcal{B} \to \mathcal{A}$. On the other hand, f^* sends $\mathcal{B}$ to $\mathcal{A}$ [13, 4.3, p. 97], and, by functoriality, we have $\partial_X^* \circ f^* = \partial_Y^*$. It follows that $f^* : \mathcal{B} \to \mathcal{A}$ is equal to $\partial_{X\,!} \circ \partial_Y^* : \mathcal{B} \to \mathcal{A}$, which is the inverse of $f_* : \mathcal{A} \to \mathcal{B}$.

As for the exactness, f_* first of all is left-exact by [13, p. 97]. Now, consider an exact sequence of sheaves in $\mathcal{A}$: $\mathcal{D} \to \mathcal{H} \to 0$. By [13, Proposition 6.4, p. 107], we have $\mathcal{D} = \partial_{X!}\mathcal{D}_U$ and $\mathcal{H} = \partial_{X!}\mathcal{H}_U$, with $\mathcal{D}_U = \partial_X^*\mathcal{D}$ and $\mathcal{H}_U = \partial_X^*\mathcal{H}$. Therefore, since $f_* \circ \partial_{X\,!} = \partial_{Y\,!}$, by [13, (6.3) p. 106] we deduce

$$\begin{tikzcd}
f_*\partial_{X!}\mathcal{D}_U \arrow[r] \arrow[d, "="'] & f_*\partial_{X!}\mathcal{H}_U \arrow[r] \arrow[d, "="] & 0 \arrow[d] \\
\partial_{Y!}\mathcal{D}_U \arrow[r] & \partial_{Y!}\mathcal{H}_U \arrow[r] & 0
\end{tikzcd}$$

and we are done. $\square$

Lemma 3.5. *Consider a triangulated category $\mathcal{T}$, and two commutative diagram of distinguished triangles in $\mathcal{T}$*

$$\begin{tikzcd}
A \arrow[r, "\partial"] & B_1 \arrow[r] & C_1 \\
A \arrow[r, "\partial"] \arrow[u, "1_A"] & B \arrow[r] \arrow[u, "f^*"'] & C \arrow[u, "g^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\partial"] \arrow[d, "1_A"'] & B_1 \arrow[r] \arrow[d, "\theta"'] & C_1 \arrow[d, "\eta"'] \\
A \arrow[r, "\partial"] & B \arrow[r] & C.
\end{tikzcd}$$

Assume moreover that $\theta \circ f^* = 1_B$, $\eta \circ g^* = 1_C$, and that $\mathrm{Hom}_{\mathcal{T}}(A, C_1[-1]) = 0$. Then we have a " cross" isomorphism

$$B_1 \oplus C \cong B \oplus C_1.$$

Remark 3.6. If the category $\mathcal{T}$ is the derived category of an abelian category $\mathcal{A}$ with enough injectives (e.g. $D_c^b(Y)$), and $A \in \mathrm{Ob}(\mathcal{A})$, and C_1 is a complex in degree ≥ 0, then the assumption $\mathrm{Hom}_{\mathcal{T}}(A, C_1[-1]) = 0$ is verified.

Proof of Lemma 3.5. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "\partial"] & A \arrow[r] \arrow[d, "\partial"] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d] \\
B \arrow[r, "f^*"] \arrow[d] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B[1] \arrow[d] \\
C \arrow[r, "g^*"] \arrow[d] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C[1] \arrow[d] \\
A[1] \arrow[r, "1_{A[1]}"] & A[1] \arrow[r] & 0 \arrow[r] & A[2]
\end{tikzcd}$$

where the first and second columns are the ones given in the hypothesis, and the fourth column is obtained by the first one by means of TR2. The first row, which gives the fourth one by means of TR2, is given by TR1. The second and third rows are given by completion of f^* and g^*, respectively, by means of TR1. Lastly, the arrows in the third column are given by TR3. Observe that the third column, a priori, is not a distinguished triangle.

Since $\theta \circ f^* = 1_B$ and $\eta \circ g^* = 1_C$, by Lemma 3.2 and its proof, we know that $B_1 \cong B \oplus B_2$, and that $C_1 \cong C \oplus C_2$. Therefore, it suffices to prove that $B_2 \cong C_2$. To this purpose, we are going to use TR4 [8, p. 11] as follows.

Consider the composition $A \xrightarrow{\partial} B \to B_1$ in the top left-square of the diagram, and the distinguished triangles given by the first column, the second row and the second column. Then, TR4 says there exist a distinguished triangle

$$C \xrightarrow{\gamma} C_1 \to B_2 \to C[1] \tag{3}$$

and a triangle morphism:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d, "f^*"] & C \arrow[r] \arrow[d, "\gamma"] & A[1] \arrow[d, "="] \\
A \arrow[r] & B_1 \arrow[r] & C_1 \arrow[r] & A[1].
\end{tikzcd}$$

The same diagram appears in our assumptions, with g^* instead of γ. It follows that $g^* = \gamma$, because $\mathrm{Hom}_{\mathcal{T}}(A, C_1[-1]) = 0$ [2, Proposition 1.1.9., p. 23]. Now, comparing (3) with the third row of the diagram at the beginning of the proof, we see that $B_2 \cong C_2$, because the third object in a distinguished triangle is unique, up to isomorphism. □

We are in position to prove that (i) implies (ii) in Theorem 3.1. We keep the notations introduced in the diagram (1).

First notice that the pull-back induces a natural commutative diagram of distinguished triangles in $D^b_c(Y)$ [8, p. 46]:

$$\begin{tikzcd} Rf_*(\partial_{X\,!}\mathbb{A}_U) \arrow[r, "\partial_X"] & Rf_*\mathbb{A}_X \arrow[r, "j^*"] & Rf_*\mathbb{A}_{\widetilde{W}} \\ \partial_{Y\,!}\mathbb{A}_U \arrow[u, "1"] \arrow[r, "\partial_Y"] & \mathbb{A}_Y \arrow[u, "f^*"] \arrow[r, "i^*"] & \mathbb{A}_W. \arrow[u, "g^*"] \end{tikzcd} \tag{4}$$

In view of Lemma 3.4, the vertical map $\partial_{Y\,!}\mathbb{A}_U \xrightarrow{1} Rf_*(\partial_{X\,!}\mathbb{A}_U)$ on the left is an isomorphism in $D^b_c(Y)$. Now consider the following diagram:

$$\begin{tikzcd} Rf_*(\partial_{X\,!}\mathbb{A}_U) \arrow[r, "\partial_X"] \arrow[d, "1"] & Rf_*\mathbb{A}_X \arrow[r, "j^*"] \arrow[d, "\theta"] & Rf_*\mathbb{A}_{\widetilde{W}} \\ \partial_{Y\,!}\mathbb{A}_U \arrow[r, "\partial_Y"] & \mathbb{A}_Y \arrow[r, "i^*"] & \mathbb{A}_W. \end{tikzcd}$$

Since the pull-back diagram is commutative, and θ has degree one (so $\theta \circ f^* = 1_{\mathbb{A}_Y}$), it follows that previous square commutes. In fact:

$$\theta \circ \partial_X = \theta \circ (f^* \circ \partial_Y \circ 1) = (\theta \circ f^*) \circ \partial_Y \circ 1 = 1_{\mathbb{A}_Y} \circ \partial_Y \circ 1 = \partial_Y \circ 1.$$

Then, by axiom TR3, previous diagram extends to a " Gysin" morphism of triangles, induced by the bivariant class θ:

$$\begin{tikzcd} Rf_*(\partial_{X\,!}\mathbb{A}_U) \arrow[r, "\partial_X"] \arrow[d, "1"] & Rf_*\mathbb{A}_X \arrow[r, "j^*"] \arrow[d, "\theta"] & Rf_*\mathbb{A}_{\widetilde{W}} \arrow[d, "\eta"] \\ \partial_{Y\,!}\mathbb{A}_U \arrow[r, "\partial_Y"] & \mathbb{A}_Y \arrow[r, "i^*"] & \mathbb{A}_W. \end{tikzcd} \tag{5}$$

In this diagram, by [2, loc. cit.] (keep in mind that $Rf_*(\partial_{X\,!}\mathbb{A}_U) \cong \partial_{Y\,!}\mathbb{A}_U$, and compare with Remark 3.6), the morphism η is unique. For the same reason, since composing this diagram with the diagram induced by the pull-back, we get the identity on both $\partial_{Y\,!}\mathbb{A}_U$ and $\mathbb{A}_Y$, we also have $\eta \circ g^* = 1_{\mathbb{A}_W}$. At this point, it is clear that the decomposition appearing in (ii) follows from Lemma 3.5 and Remark 3.6. This concludes the proof of Theorem 3.1.

Remark 3.7. Bivariant Theory provides a pull-back morphism $\eta_1 := i^*(\theta)$ [10, (3), p. 19], with:

$$\eta_1 : Rf_*\mathbb{A}_{\widetilde{W}} \to \mathbb{A}_W.$$

We are not able to prove that $\eta = \eta_1$, i.e. that the Gysin diagram, with η_1 instead of η, commutes. However, we will prove, later, that η and η_1 induce the same morphism in (co)homology. Notice that also η_1 has degree one, and so we also have $\eta_1 \circ g^* = 1_{\mathbb{A}_W}$. Therefore, if a morphism of degree one was unique, then $\eta = \eta_1$.

4. Consequences for the (Co)homology

Keep the same assumption of Theorem 3.1, and suppose there is a bivariant class of degree one for f. Then we have a cross isomorphism $Rf_*\mathbb{A}_X \oplus \mathbb{A}_W \cong Rf_*\mathbb{A}_{\widetilde{W}} \oplus \mathbb{A}_Y$. Taking hypercohomology (hypercohomology with compact support resp.), we deduce isomorphisms in cohomology (Borel–Moore homology resp.):

$$H^\bullet(X) \oplus H^\bullet(W) \cong H^\bullet(\widetilde{W}) \oplus H^\bullet(Y), \quad H_\bullet(X) \oplus H_\bullet(W) \cong H_\bullet(\widetilde{W}) \oplus H_\bullet(Y).$$

Using the triangle morphisms (4) and (5), we can make these isomorphisms explicit as follows.

First, taking hypercohomology [8, p. 46], the triangle morphisms (4) and (5) induce commutative diagrams with exact rows:

$$\begin{tikzcd}
H^k(X,\widetilde{W}) \arrow[r] & H^k(X) \arrow[r, "j^*"] & H^k(\widetilde{W}) \arrow[r, "\partial_X"] & H^{k+1}(X,\widetilde{W}) \\
H^k(Y,W) \arrow[r] \arrow[u, leftrightarrow, "="] & H^k(Y) \arrow[r, "i^*"] \arrow[u, "f^*"] & H^k(W) \arrow[r, "\partial_Y"] \arrow[u, "g^*"] & H^{k+1}(Y,W) \arrow[u, leftrightarrow, "="]
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^k(X,\widetilde{W}) \arrow[r] \arrow[d, leftrightarrow, "="] & H^k(X) \arrow[r, "j^*"] \arrow[d, "\theta_*"] & H^k(\widetilde{W}) \arrow[r, "\partial_X"] \arrow[d, "\eta_*"] & H^{k+1}(X,\widetilde{W}) \arrow[d, leftrightarrow, "="] \\
H^k(Y,W) \arrow[r] & H^k(Y) \arrow[r, "i^*"] & H^k(W) \arrow[r, "\partial_Y"] & H^{k+1}(Y,W)
\end{tikzcd}$$

for every $k \in \mathbb{Z}$. Since these diagrams commute, and $\theta_* \circ f^* = \mathrm{id}_{H^\bullet(Y)}$ and $\eta_* \circ g^* = \mathrm{id}_{H^\bullet(W)}$, a diagram chase shows that the sequence:

$$0 \to H^k(X) \xrightarrow{\alpha^*} H^k(\widetilde{W}) \oplus H^k(Y) \xrightarrow{\beta^*} H^k(W) \to 0,$$

with

$$\alpha^*(x) := (j^*(x),\, -\theta_*(x)), \quad \beta^*(\widetilde{w},\, y) := \eta_*(\widetilde{w}) + i^*(y),$$

is exact (compare with [11, Proposition 6.7, (e), p. 114-115]). Moreover, the map

$$w \in H^k(W) \to (g^*(w), 0) \in H^k(\widetilde{W}) \oplus H^k(Y)$$

is a right section for the sequence, and so we get an explicit isomorphism:

Proposition 4.1. *The map*

$$\varphi^* : H^k(X) \oplus H^k(W) \to H^k(\widetilde{W}) \oplus H^k(Y),$$

with

$$\varphi^*(x,\, w) := (j^*(x) + g^*(w),\, -\theta_*(x)),$$

is an isomorphism.

We may interpret the map φ^* as a matrix product (compare with [14, p. 328]):

$$\begin{bmatrix} \widetilde{w} \\ y \end{bmatrix} = \begin{bmatrix} j^* & g^* \\ -\theta_* & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ w \end{bmatrix}.$$

Since

$$\varphi^*(-f^*y,\, i^*y) = (0, y),$$

the matrix defining the inverse map $(\varphi^*)^{-1}$ has the following form:

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} \lambda_* & -f^* \\ \mu_* & i^* \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w} \\ y \end{bmatrix},$$

where the functions:

$$\lambda_* : H^\bullet(\widetilde{W}) \to H^\bullet(X), \quad \mu_* : H^\bullet(\widetilde{W}) \to H^\bullet(W)$$

are uniquely determined by the condition that the two matrices above are the inverse each other, i.e. by the equations:

$$\begin{cases} \lambda_* \circ j^* + f^* \circ \theta_* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{X})} \\ \lambda_* \circ g^* = 0 \\ \mu_* \circ j^* - i^* \circ \theta_* = 0 \\ \mu_* \circ g^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{W})}, \end{cases} \tag{6}$$

which in turn are equivalent to the equations:

$$\begin{cases} j^* \circ \lambda_* + g^* \circ \mu_* = \mathrm{id}_{\mathrm{H}^\bullet(\widetilde{\mathrm{W}})} \\ \theta_* \circ \lambda_* = 0 \\ j^* \circ f^* = g^* \circ i^* \\ \theta_* \circ f^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{Y})}. \end{cases} \tag{7}$$

Since we also have $\eta_* \circ j^* - i^* \circ \theta_* = 0$ and $\eta_* \circ g^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{W})}$, by the uniqueness, it follows that $\eta_* = \mu_*$.

Remark 4.2. Let $\eta_1 := i^*(\theta)$ be the pull-back of θ on W. By properties of bivariant classes [10, (G2), p. 26], we see that $(\eta_1)_* \circ j^* - i^* \circ \theta_* = 0$ and $(\eta_1)_* \circ g^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{W})}$. As before, this proves that $\eta_* = (\eta_1)_*$. Similarly, for the maps induced in homology, one sees that $\eta^* = (\eta_1)^*$ (see below). Recall that we do not know whether $\eta = \eta_1$ (compare with Remark 3.7).

Using these equations, we are able to explicit also the isomorphism induced in cohomology by the decomposition appearing in (*iii*) of Theorem 3.1. First observe that, since $\eta_* \circ g^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{W})}$, we may see $H^k(W)$, via g^*, as a direct summand of $H^k(\widetilde{W})$ for every integer k. Denote by

$$\frac{H^k(\widetilde{W})}{H^k(W)}$$

the corresponding quotient.

Proposition 4.3. *For every k, the map*

$$x \in H^k(X) \to (\theta_* x,\, j^* x) \in H^k(Y) \oplus \left[\frac{H^k(\widetilde{W})}{H^k(W)}\right]$$

is an isomorphism, whose inverse is the map

$$(y,\, \widetilde{w}) \in H^k(Y) \oplus \left[\frac{H^k(\widetilde{W})}{H^k(W)}\right] \to f^*(y) + \lambda_* \widetilde{w} \in H^k(X).$$

Proof. First observe that the map

$$x \in H^k(X) \to (\theta_* x,\, x - f^* \theta_* x) \in H^k(Y) \oplus \ker \theta_*$$

is an isomorphism. Next, observe that previous Eqs. (6) and (7) imply that j^* induces an isomorphism

$$j^* : \ker \theta_* \to \ker \eta_*,$$

whose inverse acts as λ_*. On the other hand, we also have an isomorphism:

$$\widetilde{w} \in \ker \eta_* \to \widetilde{w} \in \frac{H^k(\widetilde{W})}{H^k(W)}.$$

□

Similarly, taking hypercohomology with compact support, the triangle morphisms (4) and (5) induce commutative diagrams with exact rows involving Borel–Moore homology:

$$\begin{tikzcd}
H_{k+1}(U) \arrow[r, "\partial_X"] \arrow[d, leftrightarrow, "="] & H_k(\widetilde{W}) \arrow[r, "j_*"] \arrow[d, "g_*"] & H_k(X) \arrow[r] \arrow[d, "f_*"] & H_k(U) \arrow[d, leftrightarrow, "="] \\
H_{k+1}(U) \arrow[r, "\partial_Y"] & H_k(W) \arrow[r, "i_*"] & H_k(Y) \arrow[r] & H_k(U)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_{k+1}(U) \arrow[r, "\partial_X"] \arrow[d, leftrightarrow, "="'] & H_k(\widetilde{W}) \arrow[r, "j_*"] & H_k(X) \arrow[r] & H_k(U) \arrow[d, leftrightarrow, "="'] \\
H_{k+1}(U) \arrow[r, "\partial_Y"] & H_k(W) \arrow[r, "i_*"] \arrow[u, "\eta^*"] & H_k(Y) \arrow[r] \arrow[u, "\theta^*"] & H_k(U)
\end{tikzcd}$$

for every $k \in \mathbb{Z}$. Since these diagrams commute, and $f_* \circ \theta^* = \mathrm{id}_{H_\bullet(Y)}$ and $g_* \circ \eta^* = \mathrm{id}_{H_\bullet(W)}$, a diagram chase shows that the sequence:

$$0 \to H_k(W) \overset{\alpha_*}{\to} H_k(\widetilde{W}) \oplus H_k(Y) \overset{\beta_*}{\to} H_k(X) \to 0,$$

with

$$\alpha_*(w) := (\eta^*(w),\, -i_*(w)), \quad \beta_*(\widetilde{w},\, y) := j_*(\widetilde{w}) + \theta^*(y),$$

is exact (compare with [6, pp. 264-266, Proposition 2.5]). Moreover, the map

$$(\widetilde{w},\, y) \in H_k(\widetilde{W}) \oplus H_k(Y) \to g_*\widetilde{w} \in H_k(W)$$

is a left section for the sequence, and so we get an explicit isomorphism:

Proposition 4.4. *The map*

$$\varphi_* : H_k(\widetilde{W}) \oplus H_k(Y) \to H_k(X) \oplus H_k(W),$$

with

$$\varphi_*(\widetilde{w},\, y) := (j_*(\widetilde{w}) + \theta^*(y),\, g_*(\widetilde{w})),$$

is an isomorphism.

We may interpret the map φ_* as a matrix product:

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} j_* & \theta^* \\ g_* & 0 \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w} \\ y \end{bmatrix}.$$

Since

$$\varphi_*(\eta^* w,\, -i_* w) = (0, w),$$

the matrix defining the inverse map $(\varphi_*)^{-1}$ has the following form:

$$\begin{bmatrix} \widetilde{w} \\ y \end{bmatrix} = \begin{bmatrix} \lambda^* & \eta^* \\ \mu^* & -i_* \end{bmatrix} \cdot \begin{bmatrix} x \\ w \end{bmatrix},$$

where the functions:

$$\lambda^* : H_\bullet(X) \to H_\bullet(\widetilde{W}), \quad \mu^* : H_\bullet(X) \to H_\bullet(Y)$$

are uniquely determined by the condition that the two matrices above are the inverse each other, i.e. by the equations:

$$\begin{cases} j_* \circ \lambda^* + \theta^* \circ \mu^* = \mathrm{id}_{\mathrm{H}_\bullet(\mathrm{X})} \\ j_* \circ \eta^* - \theta^* \circ i_* = 0 \\ g_* \circ \lambda^* = 0 \\ g_* \circ \eta^* = \mathrm{id}_{\mathrm{H}_\bullet(\mathrm{W})}, \end{cases} \tag{8}$$

which in turn are equivalent to the equations:

$$\begin{cases} \lambda^* \circ j_* + \eta^* \circ g_* = \mathrm{id}_{\mathrm{H}_\bullet(\widetilde{\mathrm{W}})} \\ \lambda^* \circ \theta^* = 0 \\ \mu^* \circ j_* = i_* \circ g_* \\ \mu^* \circ \theta^* = \mathrm{id}_{\mathrm{H}^\bullet(\mathrm{Y})}. \end{cases} \tag{9}$$

In particular, it follows that $\mu^* = f_*$. Using these equations, we are able to explicit the isomorphism induced in Borel–Moore homology by (*iii*) of Theorem 3.1. First, observe that, since $g_* \circ \eta^* = \mathrm{id}_{\mathrm{H}_\bullet(\mathrm{W})}$, we may see $H_k(W)$, via

η^*, as a direct summand of $H_k(\widetilde{W})$ for every integer k. Denote by

$$\frac{H_k(\widetilde{W})}{H_k(W)}$$

the corresponding quotient.

Proposition 4.5. *For every k, the map*

$$x \in H_k(X) \to (f_*x,\, \lambda^*x) \in H_k(Y) \oplus \left[\frac{H_k(\widetilde{W})}{H_k(W)}\right]$$

is an isomorphism, whose inverse is the map

$$(y,\, \widetilde{w}) \in H_k(Y) \oplus \left[\frac{H_k(\widetilde{W})}{H_k(W)}\right] \to \theta^*(y) + j_*\lambda^* j_*\widetilde{w} \in H_k(X).$$

Proof. First observe that the map

$$x \in H_k(X) \to (f_*x,\, x - \theta^* f_*x) \in H_k(Y) \oplus \ker f_*$$

is an isomorphism. Next, observe that previous Eqs. (8) and (9) imply that λ^* induces an isomorphism

$$\lambda^* : \ker f_* \to \ker g_*,$$

whose inverse acts as j_*. On the other hand, we also have an isomorphism:

$$\widetilde{w} \in \ker g_* \to \widetilde{w} \in \frac{H_k(\widetilde{W})}{H_k(W)}.$$

□

5. Behaviour Under the Duality Morphism

One may ask how previous decompositions given in Propositions 4.3 and 4.5, behave under the cap product with a homology class. In this section we consider only the case of the fundamental class, and algebraic maps.

Consider a map $f : X \to Y$ as in Theorem 3.1, and assume there exists a bivariant class of f of degree one. Moreover, assume that f is onto, and that X and Y are open subsets of complex quasi-projective varieties of the same complex dimension n. Let $[X] \in H_{2n}(X)$ be the fundamental class of X, and consider the map

$$\mathcal{D}_X : x \in H^k(X) \to x \cap [X] \in H_{2n-k}(X) \tag{10}$$

given by the cap product with $[X]$. When X is a compact complex variety (e.g. a projective variety), this map is called *the duality morphism* [15, p. 150] (in this case X is automatically a *circuit* [15, p. 149]). If X is a smooth compact complex variety, then $\mathcal{D}_X$ is the Poincaré Duality isomorphism (the map $\mathcal{D}_X$ remains an isomorphism if X is smooth but not compact, because

$H_{2n-k}(X)$ denotes Borel–Moore homology). In view of the decompositions given in Propositions 4.3 and 4.5, the map $\mathcal{D}_X$ identifies with a map

$$\mathcal{D}_X : H^k(Y) \oplus \left[\frac{H^k(\widetilde{W})}{H^k(W)}\right] \to H_{2n-k}(Y) \oplus \left[\frac{H_{2n-k}(\widetilde{W})}{H_{2n-k}(W)}\right]$$

which acts as follows:

$$\mathcal{D}_X(y, \widetilde{w}) = (f_*([X] \cap (f^*y + \lambda_*\widetilde{w})),\ \lambda^*([X] \cap (f^*y + \lambda_*\widetilde{w}))).$$

The map $\mathcal{D}_X$ induces two projections

$$P_1 : y \in H^k(Y) \to f_*([X] \cap f^*y) \in H_{2n-k}(Y),$$
$$P_2 : \widetilde{w} \in \left[\frac{H^k(\widetilde{W})}{H^k(W)}\right] \to \lambda^*([X] \cap \lambda_*\widetilde{w}) \in \left[\frac{H_{2n-k}(\widetilde{W})}{H_{2n-k}(W)}\right].$$

Observe that, by the projection formula [10, p. 24], we have

$$f_*([X] \cap f^*y) = [Y] \cap y.$$

Therefore, $P_1 = \mathcal{D}_Y$, i.e. P_1 is nothing but the duality morphism on Y.

Corollary 5.1. *The duality morphism $\mathcal{D}_X : H^k(X) \to H_{2n-k}(X)$ is the direct sum of $\mathcal{D}_Y$ and P_2, i.e.*

$$\mathcal{D}_X = \mathcal{D}_Y \oplus P_2.$$

Proof. We have to prove that:

- •) for every $\widetilde{w} \in \frac{H^i(\widetilde{W})}{H^i(W)}$ one has $f_*([X] \cap \lambda_*\widetilde{w}) = 0$, and
- •) for every $y \in H^i(Y)$ one has $\lambda^*([X] \cap f^*y) = 0$.

To this purpose, first observe that $\theta^*([Y]) = [X]$, i.e. the Gysin map sends the fundamental class of Y in the fundamental class of X. In fact, from the Eq. (8) we obtained in homology (recall that $\mu^* = f_*$), we know that $\theta^*([Y]) = \theta^* f_*[X] = [X] - (j_* \circ \lambda^*)([X]) = [X]$ because $\lambda^*[X] = 0 \in H_{2n}(\widetilde{W}) = \{0\}$ for dimensional reasons.

- •) Now, by [10, p. 26, G4, (ii)], we have:

  $$f_*([X] \cap \lambda_*\widetilde{w}) = f_*(\theta^*[Y] \cap \lambda_*\widetilde{w}) = (\theta_*\lambda_*\widetilde{w}) \cap [Y]$$

  which is zero because, from the Eq. (7) we obtained in cohomology, we know that $\theta_* \circ \lambda_* = 0$.
- •) Next, by [10, p. 26, G4, (iii)], we have:

  $$\lambda^*([X] \cap f^*y) = \lambda^*(\theta^*[Y] \cap f^*y) = \lambda^*(\theta^*(Y \cap y))$$

  which is zero because, from the Eq. (9) we obtained in homology, we know that $\lambda^* \circ \theta^* = 0$.

$\square$

6. Resolution of Singularities of a Homology Manifold

As a consequence of previous results, we are going to prove the following Theorem 6.1. Observe that it applies to a resolution of singularities of Y, and gives a characterization of homology manifolds, in terms of their resolution of singularities. In the case Y has only isolated singularities, and the coefficients are in a field, it should be compared with [7, Theorem 3.2 and Remark 6.1].

Theorem 6.1. *Let $f : X \to Y$ be a projective birational morphism between complex, irreducible, and quasi-projective varieties of the same dimension n. Let U be a non-empty Zariski open subset of Y such that f induces an isomorphism $f^{-1}(U) \cong U$. Set $W = Y \backslash U$.*

- *If Y is an $\mathbb{A}$-homology manifold, then there exists a bivariant class θ in $Hom(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one. In this case, θ is unique, and there exists a decomposition $Rf_*\mathbb{A}_X \cong \mathbb{A}_Y \oplus \mathcal{K}$, with $\mathcal{K}$ supported on W. Moreover, if also X is an $\mathbb{A}$-homology manifold, then $\mathcal{K}[n]$ is self-dual.*
- *Conversely, if X is an $\mathbb{A}$-homology manifold and there exists a bivariant class $\theta \in Hom(Rf_*\mathbb{A}_X, \mathbb{A}_Y)$ of degree one, then also Y is an $\mathbb{A}$-homology manifold.*

Proof. First assume that Y is an $\mathbb{A}$-homology manifold.

By [4, Definition 3.1, Theorem 3.7], we know that the fundamental class of Y

$$[Y] \in H_{2n}(Y) \cong H^{-2n}(Y \to pt)$$

is a strong orientation. Therefore, we have

$$Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y) \cong H^0(X \xrightarrow{f} Y) \overset{\bullet[Y]}{\cong} H^{-2n}(X \to pt) \cong H_{2n}(X) \cong H^0(X).$$

Since f is birational, the bivariant class corresponding to $1_X \in H^0(X)$ is a bivariant class of degree one for f, and it is unique (compare with Remark 2.1, (ii) and (v)). By Theorem 3.1, we know there exists a decomposition

$$Rf_*\mathbb{A}_X[n] \cong \mathbb{A}_Y[n] \oplus \mathcal{K}[n]. \tag{11}$$

It is clear that $\mathcal{K}$ is supported on W. Passing to Verdier dual, we get:

$$D\left(Rf_*\mathbb{A}_X[n]\right) \cong D\left(\mathbb{A}_Y[n]\right) \oplus D\left(\mathcal{K}[n]\right). \tag{12}$$

Now let

$$[X] \in H_{2n}(X)$$

be the fundamental class of X. We have [4, p. 804-805]:

$$[X] \in H_{2n}(X) \cong H^{-2n}(X \to pt.) \cong Hom_{D^b_c(X)}(\mathbb{A}_X[n], D\left(\mathbb{A}_X[n]\right)).$$

Therefore, $[X]$ corresponds to a morphism

$$\mathbb{A}_X[n] \to D\left(\mathbb{A}_X[n]\right), \tag{13}$$

whose induced map in hypercohomology is nothing but the duality morphism (10). If we assume that X is an $\mathbb{A}$-homology manifold, the morphism (13) is an isomorphism [4, Proof of Theorem 3.7]. Since $D\left(Rf_*\mathbb{A}_X[n]\right) \cong Rf_* D\left(A_X[n]\right)$ [8, p. 69], it induces an isomorphism

$$Rf_*\mathbb{A}_X[n] \to D\left(Rf_*\mathbb{A}_X[n]\right),$$

which in turn, via the previous decompositions (11) and (12), induces two projections

$$\mathbb{A}_Y[n] \to D\left(\mathbb{A}_Y[n]\right), \quad \mathcal{K}[n] \to D\left(\mathcal{K}[n]\right).$$

By Corollary 5.1, we know that the maps induced in hypercohomology by $\mathcal{K}[n] \to D\left(\mathcal{K}[n]\right)$ are isomorphisms, and this holds true when restricting to every open subset of Y. Therefore, we have $\mathcal{K}[n] \cong D\left(\mathcal{K}[n]\right)$, i.e. $\mathcal{K}[n]$ is self-dual.

Conversely, assume there exists a bivariant class θ of degree one. Arguing as before, by Corollary 5.1, we know that the isomorphism (13) induces an isomorphism $\mathbb{A}_Y[n] \cong D\left(\mathbb{A}_Y[n]\right)$. This is equivalent to say that Y is an $\mathbb{A}$-homology manifold [4, loc. cit.].

This concludes the proof of Theorem 6.1. □

Remark 6.2. (i) With the notations as in Theorem 6.1, assume there exists a bivariant class θ of degree one. When the coefficients are in $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ (i.e. when $\mathbb{A} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$), we may prove that Y is an $\mathbb{A}$-homology manifold in a different manner, using the Decomposition Theorem [8, p. 161]. In fact, by the Decomposition Theorem, there exists a certain decomposition

$$Rf_*\mathbb{A}_X[n] \cong IC_Y^\bullet \oplus \mathcal{H}.$$

Comparing with the decomposition given by Theorem 3.1

$$Rf_*\mathbb{A}_X[n] \cong \mathbb{A}_Y[n] \oplus \mathcal{K}[n],$$

it follows a non-zero endomorphism $IC_Y^\bullet \to \mathbb{A}_Y[n] \to IC_Y^\bullet$. On the other hand, $IC_Y^\bullet$ belongs to the core of $D_c^b(Y)$, which is an abelian subcategory of $D_c^b(Y)$. In this category, $IC_Y^\bullet$ is a simple object. Therefore, by Schur's Lemma, the composition $IC_Y^\bullet \to \mathbb{A}_Y[n] \to IC_Y^\bullet$ is an automorphism. Observe that also the composition $\mathbb{A}_Y[n] \to IC_Y^\bullet \to \mathbb{A}_Y[n]$ is an automorphism, because $Hom_{D_c^b(Y)}(\mathbb{A}_Y, \mathbb{A}_Y) \cong H^0(Y)$. So, $IC_Y^\bullet \cong \mathbb{A}_Y[n]$.

(ii) Let $f : X \to Y$ be a map between $\mathbb{A}$-homology manifolds, of the same complex dimension n, as in Theorem 6.1. Assume $\mathbb{A}$ is a field. Consider the decomposition $Rf_*\mathbb{A}_X[n] \cong \mathbb{A}_Y[n] \oplus \mathcal{K}[n]$ appearing in (11). Taking hypercohomology, we deduce that $\dim_{\mathbb{A}} H^{k+n}(X) - \dim_{\mathbb{A}} H^{k+n}(Y) = \dim_{\mathbb{A}} \mathbb{H}^k(\mathcal{K}[n])$ for every k. On the other hand, from Theorem 6.1, we know that $\mathcal{K}[n]$ is self-dual. Therefore, passing to Verdier dual in (11), and taking hypercohomology, we get $\dim_{\mathbb{A}} H_{n-k}(X) - \dim_{\mathbb{A}} H_{n-k}(Y) = \dim_{\mathbb{A}} \mathbb{H}^k(\mathcal{K}[n])$. And so we have

$$\dim_{\mathbb{A}} H^{k+n}(X) - \dim_{\mathbb{A}} H^{k+n}(Y) = \dim_{\mathbb{A}} H_{n-k}(X) - \dim_{\mathbb{A}} H_{n-k}(Y).$$

Comparing with Propositions 4.3 and 4.5, we deduce, for every $k \in \mathbb{Z}$, the following formula

$$\dim_{\mathbb{A}} H^{k+n}(\widetilde{W}) - \dim_{\mathbb{A}} H_{n-k}(\widetilde{W}) = \dim_{\mathbb{A}} H^{k+n}(W) - \dim_{\mathbb{A}} H_{n-k}(W),$$

which relates the Betti numbers of W with the ones of $\widetilde{W}$.

(iii) The following example shows there exist projective birational maps $f : X \to Y$ such that $H^0(X \xrightarrow{f} Y) \neq 0$, without bivariant classes of degree one. The coefficients are in $\mathbb{Q}$.

Let $C \subset \mathbb{P}^3$ be a projective non-singular curve of genus $g \geq 1$. Let $Y \subset \mathbb{P}^4$ be the cone over C, and let $f : X \to Y$ be the blowing-up of Y at the vertex $y \in Y$. By the Decomposition Theorem (see e.g. [7]) we have

$$Rf_*\mathbb{Q}_X = \mathbb{Q}_y[-2] \oplus IC^{\bullet}_Y[-2].$$

On the other hand, combining [13, 9.13, p. 128] with [8, Remark 2.4.5, (i), p. 46], we have

$$Hom_{D^b_c(Y)}(\mathbb{Q}_y, \mathbb{Q}_Y[2]) \cong H^2(Y, Y\backslash\{y\}) \cong H^1(L),$$

where L is the link of Y at the vertex y. The Hopf fibration $L \to C$ induces a Gysin sequence

$$0 \to H^1(C) \to H^1(L) \to H^0(C) \to H^2(C) \to \dots$$

from which we get $h^1(L) = h^1(C) = 2g \geq 2$. It follows that $H^0(X \xrightarrow{f} Y) \cong Hom_{D^b_c(Y)}(Rf_*\mathbb{A}_X, \mathbb{A}_Y) \neq 0$, and that Y is not a homology manifold. In particular, since X is smooth, in view of Theorem 6.1, there is no a bivariant class of degree one.

Corollary 6.3. *Let $f : X \to Y$ be a projective birational morphism between irreducible and quasi-projective complex varieties of the same complex dimension n. Let $\theta \in H^0(X \xrightarrow{f} Y)$ be a bivariant class. If θ is a strong orientation for f, then θ is a bivariant class of degree one for f, up to multiplication by a unit. Moreover, if X is an $\mathbb{A}$-manifold and θ is a bivariant class of degree one for f, then θ is a strong orientation for f.*

Proof. First assume that θ is a strong orientation for f.

Let $U \subset Y$ be a Zariski non-empty open subset of Y such that $f^{-1}(U) \cong U$ via f. Product by θ gives an isomorphism:

$$H^0(f^{-1}(U) \to X) \xrightarrow{\bullet\theta} H^0(U \to Y).$$

On the other hand, by Verdier Duality [4, p. 803], and [8, Corollary 3.2.12., p. 65], we have:

$$H^0(f^{-1}(U) \to X) \cong H^0(f^{-1}(U)), \quad \text{and} \quad H^0(U \to Y) \cong H^0(U).$$

Therefore, θ induces an isomorphism $H^0(f^{-1}(U)) \to H^0(U)$. It follows that, up to multiplication by a unit, θ is a bivariant class of degree one.

Conversely, assume X is an $\mathbb{A}$-manifold, and θ is a bivariant class of degree one for f.

In this case, by Theorem 6.1, we know that also Y is an $\mathbb{A}$-homology manifold, and that θ corresponds to 1_X in the isomorphism $H^0(X \xrightarrow{f} Y) \cong H^0(X)$. Since X and Y are $\mathbb{A}$-manifolds, we get:

$$f^!(\mathbb{A}_Y) = D(f^*(D(\mathbb{A}_Y))) = D(f^*(\mathbb{A}_Y[2n])) = D(\mathbb{A}_X[2n]) = \mathbb{A}_X.$$

Therefore, θ corresponds to an isomorphism in

$$Hom_{D^b_c(X)}(\mathbb{A}_X, f^!\mathbb{A}_Y) \cong Hom_{D^b_c(X)}(\mathbb{A}_X, \mathbb{A}_X) \cong H^0(X).$$

By [10, 7.3.2, proof of Proposition, p. 85], we deduce that θ is a strong orientation for f. $\square$

Proposition 6.4. *Let $f : X \to Y$ be a projective birational morphism between irreducible and quasi-projective complex varieties of the same complex dimension n. Let $\theta \in H^0(X \xrightarrow{f} Y)$ be a bivariant class. If θ is a strong orientation for f, and Y is an $\mathbb{A}$-homology manifold, then also X is so.*

Proof. Since Y is an $\mathbb{A}$-homology manifold, we have:

$$f^!(\mathbb{A}_Y) = D(f^*(D(\mathbb{A}_Y))) = D(f^*(\mathbb{A}_Y[2n])) = D(\mathbb{A}_X[2n]).$$

On the other hand, if θ is a strong orientation, then [10, loc. cit.]

$$f^!(\mathbb{A}_Y) \cong \mathbb{A}_X.$$

Therefore, we get $D(\mathbb{A}_X[2n]) \cong \mathbb{A}_X$. This means that $\mathbb{A}_X[n]$ is self-dual, i.e. X is an $\mathbb{A}$-homology manifold [4, proof of Theorem 3.7]. $\square$

Remark 6.5. Let $f : X \to Y$ be a birational, projective local complete intersection morphism between complex irreducible quasi-projective algebraic varieties. Let $\theta \in H^0(X \xrightarrow{f} Y)$ be the orientation class of f. Then θ has degree one (Remark 2.1, (*iii*)). But, in general, in view of previous Proposition 6.4, θ cannot be a strong orientation.

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