What is refutation?

Gabriele Pulcini* and Tomasz Skura

Abstract The paper offers a logical analysis of the concept of refutation and illustrates some possible directions of research in the field of philosophical logic as well as in the methodology of propositional calculi.

Key words: refutation rules, axiomatic refutation systems, methodology of propositional logic, philosophy of logic, meaning theory

1 Introduction

We analyze the concept of refutation in both philosophical and mathematical logic. First, we deal with axiomatic refutation systems. Two types of such systems are considered. Systems with *reverse substitution* (having theoretical applications) and systems without this rule (applicable in counter-model constructions and decision procedures). Then, we consider a list of topics, most of them in philosophical logic, which could benefit from the specific technical and conceptual tools offered by the refutational approach.

Gabriele Pulcini
Institute for Logic, Language and Computation
University of Amsterdam
e-mail: `g.pulcini@uva.nl`

Tomasz Skura
Institute of Philosophy
University of Zielona Góra
e-mail: `t.skura@ifil.uz.zgora.pl`

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2 Axiomatic Refutation Systems

In formal logic, one can find two approaches to refutation: an indirect one and a direct one. In the indirect approach, you refute a formula by failing to prove it. For example, you search for a proof of A, and if all the possibilities of finding a proof for A have been exhausted, you say that A is refuted. In the direct approach, a single refutation of A, which is a derivation, justifies refuting A.

Although the indirect approach is standard, we believe that combining two direct approaches (one looking for a proof and the other for a refutation) is more attractive and can yield new results that are both interesting and useful. Note that if A is non-valid, then finding a refutation for A may be simpler than producing all possibilities of proving it.

2.1 Basic Concepts

An axiomatic refutation system is just like a traditional axiomatic system, but it is applied to non-valid formulas rather than valid ones.

Let $\mathbf{L}$ be a logic, that is, a set of formulas closed under *substitution*, *modus ponens*, and possibly some other rules (e.g. *necessitation*). We say that a rule

$$\frac{A_1,\ldots,A_n}{B}$$

is a *refutation rule for* $\mathbf{L}$ iff $B \notin \mathbf{L}$ whenever every $A_i \notin \mathbf{L}$. A *refutation system for* $\mathbf{L}$ is a pair $\mathbf{S} = (AX, RU)$, where AX is a set of formulas that are not in $\mathbf{L}$ (called refutation axioms for $\mathbf{L}$) and RU is a set of refutation rules for $\mathbf{L}$. A formula A is $\mathbf{S}$-*refutable* (in symbols $\dashv_{\mathbf{S}} A$, or just $\dashv A$) iff A has a derivation in $\mathbf{S}$. Moreover, we say that a refutation system $\mathbf{S}$ is *characteristic for* $\mathbf{L}$ iff, for every formula A, we have:

$$A \notin \mathbf{L} \text{ iff } \dashv_{\mathbf{S}} A.$$

In the literature, various kinds of refutation rules can be found. Here, we focus on refutation rules preserving non-validity (for another approach, see e.g. Fiorentini and Ferrari 2017). Let us start with the following refutation rules introduced by (Łukasiewicz, 1951).

- *Reverse substitution (RS): B/A where B is a substitution instance of A.*
- *Reverse modus ponens (RMP): B/A where $\vdash A \to B$.*

Remark 1. In this section, by a rule we mean a set of pairs Γ/A, where $\Gamma \cup \{A\}$ is a set of formulas. So, our description of RMP fits into the above definition of a refutation rule.

Remark 2. We sometimes present refutation rules bottom-up:

$$\frac{B}{A_1 \mid \ldots \mid A_n}$$

as multiple-conclusion rules preserving validity. (If $B \in \mathbf{L}$ *then some* $A_i \in \mathbf{L}$*).*

2.2 Systems with RS

CL (Classical Propositional Logic) can be characterized by the following simple refutation system.

Refutation axiom: $\bot$ (the false).
Refutation rules: *RS*, *RMP*.

Refutation systems with *RS* (involving certain characteristic formulas of finite algebras as refutation axioms) characterize every intermediate logic (and every normal modal logic) with the *FMP* (finite model property) (Skura, 1992, 1994, 2013; Citkin, 2013). However, there are logics without the *FMP* (that is, they cannot be characterized by any class of finite models) that do have finite refutation systems (Skura, 1992, 1994, 2013).

Furthermore, there are logics with problematic (or unknown) proof theories, but having neat syntactic descriptions of their non-validities. For example, Medvedev's logic is characterized by the following refutation system (Skura, 1992).

Refutation axiom: $\bot$ (the false).
Refutation rules: *RS*, RMP_{KP}, *RD*, where

(RMP_{KP}) B/A where $\vdash_{KP} A \to B$.

Here $\vdash_{KP}$ means provability in the Kreisel-Putnam logic, which is the extension of Intuitionistic Logic (**Int**) by the axiom

$$(\neg A \to B \vee C) \to (\neg A \to B) \vee (\neg A \to C).$$

(*RD*) $A, B/A \vee B$ (This rule reverses the disjunction property.)

Also, refutation systems with *RS* are useful for establishing certain facts about the lattice of extensions of a given logic, especially, concerning maximality and minimality (Skura, 2004, 2009).

Another duality concerns the way you create a non-classical logic:

- (Positive approach) You reject some unacceptable classical law (for example, the law of explosion or the positive paradox) and derive provable formulas from the acceptable axioms getting **P** (Paraconsistent Logic or Implicational Relevance Logic).
- (Negative approach) You keep the rejected law as a refutation axiom and you declare a formula *refutable* iff the refutation axiom is derivable from it. The set of refutable formulas is thus defined. If the complement **N** of this set is closed under the inference rules, then **N** is our new logic (Skura, 2004, 2017a).

Humorously speaking, in the positive approach, we want what is good, and in the negative approach, we prevent what is bad. **P** and **N** are the two extremes of possible solutions.

2.3 Systems without RS

Constructing Counter-Models

Of course, *reverse substitution* is not good for constructing counter-models. However, *reverse modus ponens* is okay, but in Johansson's logic and extensions (including **Int** and intermediate logics), it must have the following form.

$(RMP')\ A \to C/B \to C$ where $\vdash A \to B$.

Roughly speaking, in Johansson's logic and extensions as well as in normal modal logics, finite countermodels can be constructed from syntactic refutations (which can be presented as finite trees consisting of formulas) as follows.

- For every formula A, we construct its Mints-normal form F_A such that $\vdash A$ iff $\vdash F_A$ (Mints, 1990). Every normal form has its natural number called its rank (Skura, 2011a, 2013).
- We give a Scott-style refutation rule involving normal forms.

  $(R) \qquad \dfrac{F_1, \ldots, F_n}{F}$

  where F is a normal form of rank $n > 0$ and each F_i is (after simple *modus ponens* transformations) a normal form of rank smaller than n (Scott, 1957; Skura, 2011a, 2013).
- Our refutation system consits of refutation axioms (which are normal forms of rank 0 that are non-valid) and refutation rules: R and RMP (or RMP').
- We prove, by a simple inductive argument, that every normal form F is either provable or refutable (Scott, 1957; Skura, 2011a, 2013). So, if F is not provable then F has a refutation tree.
- We transform every syntactic refutation tree into a Kripke frame by removing the nodes obtained by RMP and by defining a suitable accessibility relation (Skura, 2002, 2013, 2017a). From the normal forms, we extract a valuation falsifying the refutable formulas. So, if F is a node in a syntactic refutation tree, then it is false at some point in the corresponding model built from this tree. (We remark that the corresponding frames need not be trees.) Hence if F is refutable then F has a countermodel, so F is not provable.
- As a result we get both syntactic completeness (F is not provable iff F is refutable) and semantic completeness (F is provable iff F is valid in all finite tree-type frames).

Decision Procedures

Of course, *RMP* is not good for refutation search procedures. However, we do not need the whole *RMP* in our syntactic completeness proof (Skura, 2011a). Just a few simple auxiliary rules are enough. The completeness proof provides a refutation search procedure that is a finite tree consisting of finite sets of formulas and having the following property: the origin is non-valid iff some end node is non-valid (Skura, 2017b). (Here we say that a set of formulas is non-valid iff every member of it is non-valid.) Note that it is in fact a decision procedure.

Refutation Procedures and Tableau Procedures

We focus on Modal Logic and follow Goré in our account of tableau systems (Goré, 1999). (We assume the reader to be familiar with basic concepts concerning tableaux.)

Tableau procedures are viewed as refutation procedures in the sense that in order to show that a formula A is valid we assume that A is not valid (Fitting, 1999). Then we apply tableau rules

$$\frac{\Gamma}{\Gamma_1 \mid ... \mid \Gamma_k}$$

where $\Gamma, \Gamma_1, .., \Gamma_k$ are finite sets of formulas. The interpretation is that if Γ is satisfiable, then so is some Γ_i. As a result, we get a tableau for $\{\neg A\}$, which is a finite tree with origin $\{\neg A\}$. A formula A is *tableau-provable* iff there is a closed tableau for $\{\neg A\}$. Tableau procedures are a generalization of disjunctive normal form procedures.

It is easier to compare refutation procedures with tableau procedures when refutation rules are presented bottom-up (see Remark 2). In a bottom-up refutation procedure, we assume that A is valid. Then we apply bottom-up refutation rules generating finite trees consisting of formulas. A formula A is refutable iff there is a finite refutation tree with origin A and refutation axioms as end-nodes. Refutation procedures are a generalization of conjunctive normal form procedures. Thus, as syntactic procedures, tableau procedures and refutation procedures are complementary.

Tableau procedures also provide counter-model constructions. Assume that there is no closed tableau for $\{\neg A\}$. Then a finite tree-type counter-model for A can be constructed from various open tableaux for $\{\neg A\}$. Note that in transitive logics (**K**4 and extensions) such constructions involve cycles.

On the other hand, we get a finite tree-type counter-model for A from a refutation tree for A *directly*, by deleting the nodes obtained by *RMP* and transforming the resulting tree into a frame (Skura, 2002, 2013). We remark that our constructions are cycle-free. It turns out that, at least in some cases, our counter-model constructions are simpler than the tableau constructions (Skura, 2013, p.125).

3 Concluding remarks and research perspectives

In short, philosophical logic is intended to provide the interface through which philosophy and logic interact. On the one hand, philosophical logic concerns the study of philosophical problems — especially problems in epistemology, (analytic) metaphysics and the philosophy of language — by means of the specific mathematical tools offered by logic (Grayling, 1982). On the other hand, once logically addressed, meaningful philosophical settings never fail to produce challenging new logical problems to be solved mathematically.

In what follows we provide a list of topics in philosophical logic and, more in general, in the methodology of propositional logic which could benefit from the specific formal and conceptual tools made available by refutation systems.

3.1 General proof-theory program

In 1974, Prawitz proposed to emancipate proof-theory from classical foundational studies by means of a new research program that he termed *general proof-theory*. The core of this program consists in the fact that "proofs are studied in their own right where one is interested in general questions about the nature and structure of proofs [...]" (Prawitz, 1974, p. 66).

The introduction of refutation calculi can be seen as a way to 'maximize' Prawitz's program inasmuch as these systems allow us to widen the space of proofs by including derivations ending with *invalid* formulas/sequents (Varzi, 1990, 1992; Goranko, 1994; Skura, 2009, 2011b; Piazza and Pulcini, 2016; Carnielli and Pulcini, 2017; Piazza and Pulcini, 2019). Here the basic idea is that one can have a better understanding of the structure of proofs once the 'affirmative' and the 'refutational' parts are considered together as two alternative and complementary ways to syntactically characterize a given (decidable) logic.

On the one hand, this idea echoes what happens in Girard's *Ludics*, where proofs and para-proofs peacefully co-exist and interact (Girard, 2001). On the other, it can be also seen as a way to further extend Wansing's proposal of accommodating dual proofs logically (Wansing, 2017). Indeed, from an epistemic point of view, ontological parsimony is more an obstacle to the real understanding of the structure of proofs than a virtue.

3.2 Meaning theory and proof-theoretic semantics

As a byproduct of the ontological extension posited in the previous point, refutation calculi might provide new conceptual tools in the fields of anti-realist meaning theory and proof-theoretic semantics (Dummett, 1975; Schroeder-Heister, 2018).

Hardcore proof-theorists tend to believe that the meaning of logical operators is primarily conveyed by their rules in a suitable proof-system. Model-theoretic semantics comes into play at a later moment to provide a mathematical account of these very epistemic insights (Sundholm, 1986; Schroeder-Heister, 2018; Kremer, 1988). Yet this view has been discussed and criticized on several occasions; in this regard, the logico-philosophical debate about the *tonk* connective is very well-known (Prior, 1960; Belnap, 1962; Avron, 2010).

It is our conviction that several of the problems arising in this field might be fruitfully addressed by considering the meaning of logical operators as given by their rules in *both* the affirmative and the refutational parts. Take, for instance, the classical conjunction operator ($\wedge$). According to this view, its meaning should be considered as being conveyed not only through its rules in the affirmative part LK:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\vdash \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \vdash\wedge,$$

but also by its rules in the refutational part $\overline{\mathsf{LK}}$ (Goranko, 1994):

$$\frac{\Gamma, A, B \dashv \Delta}{\Gamma, A \wedge B \dashv \Delta} \wedge\dashv \quad \frac{\Gamma \dashv A, \Delta}{\Gamma \dashv A \wedge B, \Delta} \dashv\wedge(1) \quad \frac{\Gamma \dashv B, \Delta}{\Gamma \dashv A \wedge B, \Delta} \dashv\wedge(2).$$

It turns out that negation ($\neg$) is the only classical connective which allows for the very same (left and right) introduction rules in both parts, affirmative and negative:

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta} \neg\vdash \quad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \vdash\neg$$

$$\frac{\Gamma \dashv \Delta, A}{\Gamma, \neg A \dashv \Delta} \neg\dashv \quad \frac{\Gamma, A \dashv \Delta}{\Gamma \vdash \Delta, \neg A} \dashv\neg$$

3.3 Comparative theory of formalisms

Formalized proofs and the study of their structural properties always relate to a specific proof-system of reference, and new formalisms are expected to improve the already known deductive engines in some respects (proof-search effectiveness, naturalness, perspicuity, identity of proofs, etc.). Put thus, besides being a theory of proofs, proof-theory can be also conceived of as a comparative meta-theory of formalisms. In this regard, refutation calculi might suggest new strategies for improving our proof-systems and challenging new problems. To take an example, the problem of designing a satisfactory proof-net theory for classical logic still remains, in many respects, an open problem (Girard, 1987). Surprisingly enough, once refutationally addressed, classical logic comes with a very simple and efficient proof-net theory which may, in turn, provide new information about the deeper geometrical structure of proof-nets for the affirmative counterpart (Pulcini and Varzi, 2019).

3.4 Philosophy of logic

According to (Carnielli and Pulcini, 2017), we indicate with $\overline{\mathsf{LK}}$ the sequent system sound and complete with respect to the set of classically invalid sequents (Goranko, 1994). It seems quite reasonable to assert that $\overline{\mathsf{LK}}$ is actually a *logic*, to the extent that it provides an alternative syntactic characterization of propositional classical logic (Casati and Varzi, 2000; Pulcini and Varzi, 2018).

It is worth observing that, unlike LK, $\overline{\mathsf{LK}}$ is *paraconsistent*. In other words, classical logic can be syntactically grasped, albeit in the negative, by means of a paraconsistent sequent system. In general, any decidable logic whose semantics circumscribes a set of contingent formulas allows for a refutational characterization which is paraconsistent (Pulcini and Varzi, 2018). This kind of observation offers new insights into the logical nature of paraconsistency, which turns out to be sensitive to the specific syntactic formulation of our logic. The system $\overline{\mathsf{LK}}$ does not need to resort to structural rules either, therefore similar considerations also apply to the notion of *substructurality* (Restall, 2018).

3.5 Rejection / assertion debate

Is the negation of a proposition A the same as the *rejection* of A? Frege famously maintained that rejection and assertion do not have to be treated separately, since the act of rejecting a proposition A is nothing but the act of affirming its negation (Frege, 1919). The opposite view is called *bilateralism* and finds in Smiley one of its more tenacious proponents (Smiley, 1996).

This debate has remained lively and intense over the years, involving both philosophers of language, linguists, and logicians interested in investigating the nature of the negation operator (Rumfitt, 2000; Incurvati and Smith, 2010; Ripley, 2011; Incurvati and Schlöder, 2017). It would be interesting to understand how technical advances in the proof-theory of rejection calculi might contribute to the elucidation of this rejection/negation rapport.

3.6 Metaphysical grounding

In metaphysics, the ground relation is meant to connect the truth of some set of facts $A_1,\ldots,A_n$ to the truth of some other fact B. This relation has to be something conceptually deeper than mere logical implication: it has to explain *why* B is true, by virtue of the truth of each one of the As. Put differently, when we are in presence of a ground relation, it is *metaphysically necessary* that the truth of the fact B came from the truth of the premisses $A_1,\ldots,A_n$ (Fine, 2012).

Logic enters the debate at the moment we want to grasp methaphysical necessitation by means of a suitable set of formal constraints regulating the transition from

$A_1, \ldots, A_n$ to B. Needless to say, the resort to formal logic has been usually intended as the resort to the affirmative part of logical systems (Fine, 2012). We guess that refutation systems could provide new tools to fine-tune the formal grasp of grounding by allowing us to introduce complementary considerations about the deductive transmission of *un*grounding.

3.7 Methodology of propositional logics

Our analysis shows that the concept of refutation provides new tools having the following interesting applications in the methodology of propositional logics.

- Specific/generic descriptions of the non-validities of logics.
- Establishing maximality/minimality in the lattices of logics.
- Non-classical logics *via* refutability.
- Constructive completeness proofs that are simple.
- Refined semantic characterizations of logics by finite tree-type frames.
- Cycle-free constructions of counter-models.
- Refutation search procedures that, at least in some cases, are simpler than those provided by standard methods.

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