Linearly ordered sets with only one operator have the amalgamation property

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Abstract. The class of linearly ordered sets with one order preserving unary operation has the Strong Amalgamation Property (SAP). The class of linearly ordered sets with one strict order preserving unary operation has AP but not SAP. The class of linearly ordered sets with two order preserving unary operations has not AP. For every set F, the class of linearly ordered sets with an F-indexed family of automorphisms has SAP. Corresponding results are proved in the case of order reversing operations. Various subclasses of the above classes are considered and some model-theoretical consequences are presented.

1. Introduction

The amalgamation property (AP) has found deep applications in algebra and logic, and is nontrivially linked to categorical notions. In the special case of groups the amalgamation property has been considered in Schreier [S]. Then Fraïssé [F1, F2] and Jónsson [Jón] introduced the abstract general definitions and initiated a flourishing line of research with applications in model theory. Subsequently, another line of research connected the amalgamation property with algebraic logic. See [Ev, GM, H, KMPT, MMT] for more details and further references.

If one starts with some theory having AP and adds a set of operators with suitable properties, sometimes the resulting theory has still AP. Many results of this kind are known for fields with operators, e. g., [W, Z]. See [BHKK, GP] for more recent results and further references. A similar preservation phenomenon sometimes occurs for ordered structures [LP]. In particular, many kinds of Boolean algebras with operators have the strong amalgamation property (SAP). See [Joh, EC, MS, N][1]. The case of partially ordered sets with any number of order preserving unary operations is probably folklore; anyway, see Corollary 2.4 below.

In Section 3 we prove the quite curious fact that, on the other hand, for *linearly* ordered sets, adding a single order preserving unary operation maintains SAP, while AP fails when two operations are present. Linearly ordered sets

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[1]Results in [EC, Theorem 4] are stated for just one operator, but the proof works for an arbitrary set of operators.

with one strict order preserving operation have AP but not SAP. Corresponding results are proved in Section 4 for linearly ordered sets with order reversing operations, but in this case strong amalgamation generally fails. On the positive side, any class of linearly ordered sets with families of automorphisms and antiautomorphisms with a common fixed point has SAP.

An appealing aspect of our proofs is that we always construct the amalgamating structure on the set-theoretical union of the domains of the structures to be amalgamated. Henceforth we need no further effort in order to get AP for various subclasses of the above classes. The existence of Fraïssé limits follows in most cases, and sometimes we even get model completions for appropriate theories. This aspect is discussed in Section 5.

1.1. Outline of the proofs. Our main techniques are summarized as follows. Given linearly ordered sets $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ to be amalgamated, with ground $\mathbf{C}$, we first use Fraïssé [F2] and Jónsson's [Jón] method in order to embed $\mathbf{A}$ and $\mathbf{B}$ into a partially ordered set $\mathbf{E}$ over $A \cup B$. In the absence of operations, it is enough to extend the partial order on $\mathbf{E}$ to some arbitrary linear order, but a cleaner (classical) way to do this is to consider some element of A to be always smaller than some element of B, provided there is no other relation to be satisfied and which implies the converse.

Given a triple to be amalgamated, the above idea provides a rather uniform method to extend a partial order, and the method works even in the presence of one operation f. Of course, we are not allowed to always set $a < b$, whenever $a \in A$ and $b \in B$ are not comparable in $\mathbf{E}$. Indeed it may happen that $f(a)$ and $f(b)$ are comparable in $\mathbf{E}$ and in this case the relative position of a and b should be set accordingly. We check that all the conditions arising in a similar way can be consistently put together, hence we succeed in getting a linear order.

The most delicate case is when the operation is supposed to be strict order preserving. In this case, some elements of A and B possibly need to be identified: this means that *strong* amalgamation fails. However, the relations involved in the identifications exactly determine the structure relative to such particular elements. In other words, the ground structure can be extended to some model $\mathbf{C}_1$ which then becomes a strong amalgamation base. Needless to say, details are delicate in each case, since the method works only for one operation, not for a pair of operations, the counterexamples being quite easy. In contrast, and to make the situation even more involved, SAP holds for any number of operations, under the assumption that the operations are automorphisms.

The arguments need to be modified when dealing with order reversing operations. In this case, we cannot always put $a < b$, when $a \in A$, $b \in B$ and the ordering relation between a and b is not determined by other conditions. Indeed, if g is order reversing, then $a < b$ implies $g(b) \leq g(a)$, but still $g(a) \in A$ and $g(b) \in B$. However, in the presence of an order reversing operation, the elements of a linearly ordered set can be obviously divided into "lower" and "upper" elements: we set $a < b$ for lower elements and $b < a$ for upper elements, again, when there is no other condition to be satisfied.

2. Preliminaries

In this section we recall the basic definitions and some classical constructions which show amalgamation for partial and linear orders.

Definition 2.1. If $\mathcal{K}$ is a class of structures of the same type, then $\mathcal{K}$ is said to have the *amalgamation property* (AP) if, whenever $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\iota_{\mathbf{C},\mathbf{A}} \colon \mathbf{C} \rightarrowtail \mathbf{A}$ and $\iota_{\mathbf{C},\mathbf{B}} \colon \mathbf{C} \rightarrowtail \mathbf{B}$ are embeddings, then there is a structure $\mathbf{D} \in \mathcal{K}$ and embeddings $\iota_{\mathbf{A},\mathbf{D}} \colon \mathbf{A} \rightarrowtail \mathbf{D}$ and $\iota_{\mathbf{B},\mathbf{D}} \colon \mathbf{B} \rightarrowtail \mathbf{D}$ such that $\iota_{\mathbf{C},\mathbf{A}} \circ \iota_{\mathbf{A},\mathbf{D}} = \iota_{\mathbf{C},\mathbf{B}} \circ \iota_{\mathbf{B},\mathbf{D}}$. Namely, the following diagram can be commutatively completed as requested.

$$\begin{array}{ccc} \mathbf{A} & & \mathbf{B} \\ & \nwarrow \quad \nearrow & \\ & \mathbf{C} & \end{array} \qquad \text{completes to} \qquad \begin{array}{ccc} & \mathbf{D} & \\ & \nearrow \quad \nwarrow & \\ \mathbf{A} & & \mathbf{B} \\ & \nwarrow \quad \nearrow & \\ & \mathbf{C} & \end{array}$$

If, in addition, the above model and embeddings can be always chosen in such a way that the intersection of the images of $\iota_{\mathbf{A},\mathbf{D}}$ and $\iota_{\mathbf{B},\mathbf{D}}$ is equal to the image of $\iota_{\mathbf{C},\mathbf{A}} \circ \iota_{\mathbf{A},\mathbf{D}}$, then $\mathcal{K}$ is said to have the *strong amalgamation property* (SAP).

The latter condition can be simplified under the assumption that $\mathcal{K}$ is closed under isomorphism. Under this assumption, $\mathcal{K}$ has SAP if and only if, whenever $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{C} \subseteq \mathbf{A}$, $\mathbf{C} \subseteq \mathbf{B}$ and $C = A \cap B$, then there is a structure $\mathbf{D} \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{D}$ and $\mathbf{B} \subseteq \mathbf{D}$. Here, say, $\mathbf{C} \subseteq \mathbf{A}$ means that $C \subseteq A$ as sets and that the inclusion is an embedding from $\mathbf{C}$ to $\mathbf{A}$.

A triple of models $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as above shall be called an *amalgamation triple*, or a *triple to be amalgamated*. The structure $\mathbf{D}$ shall be called a *(strong) amalgam*, or a *(strong) amalgamating structure*. Whenever possible, we shall consider the simplified setting described in the previous paragraph, namely, we shall deal with inclusions rather than with arbitrary embeddings. The setting in which we work shall always be clear from the context.

Even if some class $\mathcal{K}$ has not AP, it is anyway interesting to ask when a diagram as above can be completed. In particular, it is interesting to consider those specific $\mathbf{C}$ for which the diagram can be always completed. In detail, a structure $\mathbf{C}$ is said to be a *(strong) amalgamation base for a class* $\mathcal{K}$ if every amalgamation triple with $\mathbf{C}$ at the bottom and $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ has some (strong) amalgamating structure $\mathbf{D} \in \mathcal{K}$.

Notice that here we are always dealing with *embeddings*, not *homomorphisms*. E. g., in the case of ordered sets, $\iota \colon \mathbf{C} \rightarrowtail \mathbf{A}$ is an embedding if, for every $c, d \in C$, it happens that $c \leq_{\mathbf{C}} d$ if and only if $\iota(c) \leq_{\mathbf{A}} \iota(d)$. In the definition of an homomorphism the sole "only if" implication is required. Homomorphisms of partially ordered sets are frequently called *ordermorphisms*.

In the above inequalities we have written $\leq_{\mathbf{C}}$ and $\leq_{\mathbf{A}}$ to distinguish the order relation considered on $\mathbf{C}$ from the order relation considered on $\mathbf{A}$. We shall use a similar convention when dealing with unary operations. As customary, we shall drop the subscripts when there is no risk of confusion.

As a final detail, slightly distinct notions arise if in (S)AP one allows or does not allow $\mathbf{C}$ to be an empty structure. The results here shall not be affected by the distinction, hence the reader might use her or his favorite version of the definition.

The following classical construction will be the starting point for our proofs. *Poset* is an abbreviation for *partially ordered set*.

Theorem 2.2. *(Fraïssé* [F2, 9.3]*, Jónsson* [Jón, Lemma 3.3]*) The class of posets has the strong amalgamation property.*

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are posets,* $\mathbf{C} \subseteq \mathbf{A}$*,* $\mathbf{C} \subseteq \mathbf{B}$ *and* $C = A \cap B$*, then an amalgamating structure* $\mathbf{E}$ *is obtained as follows. The domain of* $\mathbf{E}$ *is* $A \cup B$ *and, for* $d, e \in A \cup B$*, let* $d \leq_{\mathbf{E}} e$ *if either*

$$\begin{aligned} &d, e \in A \text{ and } d \leq_{\mathbf{A}} e, \text{ or} \\ &d, e \in B \text{ and } d \leq_{\mathbf{B}} e, \text{ or} \\ &d \in A, e \in B \text{ and there is } c \in C \text{ such that } d \leq_{\mathbf{A}} c \text{ and } c \leq_{\mathbf{B}} e, \text{ or} \\ &d \in B, e \in A \text{ and there is } c \in C \text{ such that } d \leq_{\mathbf{B}} c \text{ and } c \leq_{\mathbf{A}} e. \end{aligned} \tag{2.1}$$

For short, $\leq_{\mathbf{E}} = \leq_{\mathbf{A}} \cup \leq_{\mathbf{B}} \cup (\leq_{\mathbf{A}} \circ \leq_{\mathbf{B}}) \cup (\leq_{\mathbf{B}} \circ \leq_{\mathbf{A}})$.

See [F2, Jón] for a proof and [Li] for various generalizations.

The order $\leq_{\mathbf{E}}$ as defined above is the finest, i.e., smallest order on $D = A \cup B$ which makes D an amalgamating structure. However, $\leq_{\mathbf{E}}$ is not the unique such order. For example, if $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are linear orders, then, for any extension $\leq_{\mathbf{D}}$ of $\leq_{\mathbf{E}}$ to a linear order, the inclusions from $\mathbf{A}$ and $\mathbf{B}$ to $(A \cup B, \leq_{\mathbf{D}})$ still remain embeddings. This is enough to show that the class of linearly ordered sets has SAP, but a cleaner method is to extend $\leq_{\mathbf{E}}$ in such a way that $a <_{\mathbf{D}} b$, whenever $a \in A$, $b \in B$ and the relative order between a and b is not decided by $\leq_{\mathbf{E}}$. This is a classical argument, see, e. g., [F2, 9.2], [Ev, Example 2.2.1]. We present full details in the next corollary, since similar methods will prove useful in the following sections.

Corollary 2.3. *The class of linearly ordered sets has SAP.*

Proof. Let $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ be a triple of linear orders to be amalgamated. In particular, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are partial orders, hence Theorem 2.2 provides an amalgamating *partial* order $\mathbf{E} = (D, \leq_{\mathbf{E}})$ with $D = A \cup B$. If $d, e \in D$ are $\leq_{\mathbf{E}}$-incomparable, then necessarily $d \in A$ and $e \in B$, or conversely, since $\mathbf{A}$ and $\mathbf{B}$ are linearly ordered and $D = A \cup B$. Extend $\leq_{\mathbf{E}}$ to a relation $\leq_{\mathbf{D}}$ by always letting the element in A be $<_{\mathbf{D}}$ than the element in B, for every pair of $\leq_{\mathbf{E}}$-incomparable elements. In view of (2.1), it is easy to check transitivity and antisymmetry of $\leq_{\mathbf{D}}$. The resulting order is linear by construction.

Letting $\mathbf{D} = (D, \leq_{\mathbf{D}})$, the identity map ι is an ordermorphism from $\mathbf{E}$ to $\mathbf{D}$, though not necessarily an embedding. However, the composition $\iota_{\mathbf{A},\mathbf{E}} \circ \iota$ is indeed an embedding from $\mathbf{A}$ to $\mathbf{D}$, and similarly for $\mathbf{B}$, hence $\mathbf{D}$ is an amalgamating structure in the class of *linear* orders.

As we mentioned, we could have done by simply extending the partial order $\leq_{\mathbf{E}}$ to some arbitrary linear order $\leq_{\mathbf{F}}$; however, the assumption that every partial order can be extended to a linear order is a weak form of the axiom of choice [HR, Form 49]; on the other hand, the argument we have recalled seems to need no form of choice. Moreover, the above method, with variations, shall be used in order to prove Theorems 3.1 and 4.3 below. □

As another immediate consequence of Theorem 2.2, if we add monotone unary operations to partial orders, then SAP is maintained. See the next corollary. We shall see in the following sections that this is not always the case, when dealing with linear orders.

If $\mathbf{A}$ is a poset, a unary operation $f\colon A \to A$ is *order preserving*, resp., *order reversing* if, for every $a, b \in A$, $a \leq b$ implies $f(a) \leq f(b)$, resp., $f(a) \geq f(b)$. We say that f is *strict order preserving*, resp., *strict order reversing* if, for every $a, b \in A$, $a < b$ implies $f(a) < f(b)$, resp., $f(a) > f(b)$.

Corollary 2.4. *The class of posets with any (fixed in advance) number of order preserving, order reversing, strict order preserving and strict order reversing unary operations has SAP, actually, the superamalgamation property; see* [GM, p. 173].

More formally, Corollary 2.4 asserts that, for every set $F = F_1 \cup F_2 \cup F_3 \cup F_4$ of unary function symbols, if $\mathcal{PO}_F$ is the class of posets with additional functions such that all the symbols in F_1 are interpreted as order preserving functions, all the symbols in F_2 are interpreted as order reversing functions, etc., then $\mathcal{PO}_F$ has SAP.

Proof. Given three structures $\mathbf{A}, \mathbf{B}, \mathbf{C}$ for the appropriate language and to be amalgamated, first construct a partial order $\leq_{\mathbf{E}}$ on $D = A \cup B$ as in Theorem 2.2. Since $D = A \cup B$ and the operations under consideration are unary and agree on $C = A \cap B$, then each operation can be uniquely extended over D. Notice that here it is fundamental to have the strong version of the amalgamation property for posets. It is immediate from (2.1) that if, say, f is interpreted by an order preserving operation on $\mathbf{A}, \mathbf{B}, \mathbf{C}$, then the extension of $f_{\mathbf{A}}$ and $f_{\mathbf{B}}$ to $\mathbf{E}$ is still order preserving. For example, if $a \leq_{\mathbf{E}} b$ is given by $a \leq_{\mathbf{A}} c \leq_{\mathbf{B}} b$, for some $c \in C$, then $f_{\mathbf{A}}(a) \leq_{\mathbf{A}} f_{\mathbf{A}}(c)$ and $f_{\mathbf{B}}(c) \leq_{\mathbf{B}} f_{\mathbf{B}}(b)$, since $f_{\mathbf{A}}$ and $f_{\mathbf{B}}$ are order preserving in $\mathbf{A}$ and $\mathbf{B}$, respectively. Hence $f(a) \leq_{\mathbf{E}} f(b)$. □

3. Linearly ordered sets with operators

We have seen in the previous section that posets, possibly with operators, always share SAP. The situation with linearly ordered sets is much more delicate. Everything runs smoothly when at most one order preserving operation is added, but even AP fails when more than one operation are present. One strict order preserving operation prevents SAP, but AP is still satisfied. In contrast, SAP holds when an arbitrary family of automorphisms is added.

The method recalled in the proof of Corollary 2.3 needs to be modified, since it is possible that, say, a and b are not comparable in $\mathbf{E}$, while $f(a)$ and $f(b)$ turn out to be comparable. We check that all the conditions arising in similar ways can be consistently put together in the case of just an operation, while this is not possible for two or more operations.

As usual, if f is a unary operation, f^n is defined inductively as follows: $f^0(a) = a$ and $f^{n+1}(a) = f(f^n(a))$.

Theorem 3.1. *(a) The class $\mathcal{LO}_p$ of linearly ordered sets with an order preserving unary operation has SAP.*

(b) The class $\mathcal{LO}_{sp}$ of linearly ordered sets with a strict order preserving unary operation has AP but not SAP.

(c) The classes of linearly ordered sets with two order preserving, resp., two strict order preserving unary operations have not AP.

(d) For every set F, the class $\mathcal{LO}_{Fa}$ of linearly ordered sets with an F-indexed family of order automorphisms has SAP.

Proof. (a) Fix a triple $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ to be amalgamated. The proofs of Theorem 2.2 and Corollary 2.4 furnish an amalgamating *partial* order $\mathbf{E} = (D, \leq_{\mathbf{E}}, f)$ on $D = A \cup B$ such that f is an order preserving operation with respect to $\leq_{\mathbf{E}}$. Szigeti and Nagy [SN] provided a condition under which a partial order D with an order preserving unary operation f can be linearized in such a way that the operation is still order preserving with respect to the new linear order. This holds if and only if f is *acyclic*, namely, whenever $d \in D$ and $f^{m+1}(d) \in \{d, f(d), f^2(d), \dots, f^m(d)\}$, for some $m \in \mathbb{N}$, then $f^{m+1}(d) = f^m(d)$. In the case at hand, f is surely acyclic, since $D = A \cup B$ and f is trivially acyclic on both A and B. Relying on [SN] is thus sufficient in order to prove (a). To make the paper self-contained, we shall also present a more direct proof of (a) which has the advantage of making no use of the axiom of choice.

So let again $\mathbf{E} = (D, \leq_{\mathbf{E}}, f)$ be given by Theorem 2.2 and Corollary 2.4, with $D = A \cup B$ and $\leq_{\mathbf{E}}$ only a partial order. As in the proof of Corollary 2.3, we shall extend $\leq_{\mathbf{E}}$ to some linear order $\leq$ on D, but in the present case the behavior of the operation f should be taken into account. The values of f shall not be modified.

We first explicitly describe the way the linear order on $\mathbf{C}$ forms the "backbone" of the partial order $\leq_{\mathbf{E}}$. This description does not involve f and in principle is not strictly necessary, but it will greatly simplify the subsequent arguments. Recall that if $\mathbf{C}$ is a linearly ordered set, a *cut* of $\mathbf{C}$ is a pair (C_1, C_2) such that $C_1 \cup C_2 = C$ and $c_1 < c_2$, for every $c_1 \in C_1$ and $c_2 \in C_2$, in particular, $C_1 \cap C_2 = \emptyset$. We allow C_1 or C_2 to be empty. If $\mathcal{G} = (C_1, C_2)$ is a cut of $\mathbf{C}$, the *component (of D) associated to $\mathcal{G}$* is the set $O_\mathcal{G} = \{ d \in D \mid c_1 < d < c_2, \text{ for all } c_1 \in C_1 \text{ and } c_2 \in C_2 \}$. Thus $O_\mathcal{G} \subseteq D \setminus C$, since $C = C_1 \cup C_2$. Actually, every element d of $D \setminus C$ belongs to some component: if $d \in D \setminus C$, then d *determines* a cut $\mathcal{G}$ of $\mathbf{C}$ by setting $C_1 = \{ c \in C \mid c <_\mathbf{E} d \}$ and $C_2 = \{ c \in C \mid d <_\mathbf{E} c \}$. Then $\mathcal{G} = (C_1, C_2)$ is a cut, since $d \notin C$, $\mathbf{C}$ embeds in $\mathbf{E}$ and $\mathbf{A}$, $\mathbf{B}$ are linearly ordered. If d determines $\mathcal{G}$, then d belongs to the component associated to $\mathcal{G}$; actually, this is the only component to which d belongs. In other words, the nonempty components partition $D \setminus C$.

If d and d' lie in two distinct components, then either $d \leq_\mathbf{E} d'$ or $d' \leq_\mathbf{E} d$. Indeed, let (C_1, C_2) and (C'_1, C'_2) be the cuts determined by d and d', thus $C_1 \neq C'_1$. Since $C_1 \cup C_2 = C'_1 \cup C'_2 = C$, then either $C_1 \cap C'_2 \neq \emptyset$ or $C'_1 \cap C_2 \neq \emptyset$. If $c \in C_1 \cap C'_2$, then $d' \leq_\mathbf{E} c \leq_\mathbf{E} d$. Similarly, if $c \in C'_1 \cap C_2$, then $d \leq_\mathbf{E} c \leq_\mathbf{E} d'$.

Moreover, if $c \in C$ and $d \in D$, then either $c \leq_\mathbf{E} d$ or $d \leq_\mathbf{E} c$, since $\leq_\mathbf{E}$ extends the linear orders $\leq_\mathbf{A}$ and $\leq_\mathbf{B}$, $D = A \cup B$, $c \in C = A \cap B$, hence either $c, d \in A$ or $c, d \in B$, thus the relative position of c and d is already determined by either $\leq_\mathbf{A}$ or $\leq_\mathbf{B}$. It follows that

(*) in order to extend $\leq_\mathbf{E}$ to a linear order on D it is enough to extend to a linear order each restriction of $\leq_\mathbf{E}$ to each component.

So let O be the component associated to some cut. Recall that $O \subseteq D \setminus C$. If $a, a' \in O \cap A$, then either $a \leq_\mathbf{E} a'$ or $a' \leq_\mathbf{E} a$, by (2.1), since $\leq_\mathbf{A}$ is a linear order, $\leq_\mathbf{E}$ coincides with $\leq_\mathbf{A}$ on A and a, a' lie in the same component. The situation is similar if $b, b' \in O \cap B$. If $a \in O \cap A$ and $b \in O \cap B$, then a and b are $\leq_\mathbf{E}$-incomparable, by the last two lines in condition (2.1) and since a and b lie in the same component. Henceforth it is enough to set the relative order for each pair $a \in A \cap O$ and $b \in B \cap O$.

Let $<_O$ extend $<_\mathbf{E}$ on O by setting

$$a <_O a_1, \qquad \text{if } a, a_1 \in O \cap A \text{ and } a <_\mathbf{A} a_1 \tag{3.1}$$

$$b <_O b_1, \qquad \text{if } b, b_1 \in O \cap B \text{ and } b <_\mathbf{B} b_1 \tag{3.2}$$

$$\left.\begin{array}{ll} b <_O a, & \text{if } f^n(b) <_\mathbf{E} f^n(a), \\ & \text{for some } n \geq 1, \text{ or} \\ a <_O b, & \text{otherwise,} \end{array}\right\} \quad \text{if } a \in A \cap O \text{ and } b \in B \cap O \tag{3.3}$$

We now show that $<_O$ is a linear order on O. As above, all pairs of distinct elements in $A \cap O$ are $<_\mathbf{A}$ comparable, hence $<_O$ comparable. A similar remark holds for $B \cap O$. By construction, all pairs $a \in A \cap O$ and $b \in B \cap O$ are $<_O$ comparable and we cannot have both $a <_O b$ and $b <_O a$. Notice that $A \cap B \cap O = \emptyset$, since $A \cap B = C$ and O is a component, hence $O \subseteq D \setminus C$. Moreover, $d <_O d$ is impossible, if $d \in O$.

It remains to show that $<_O$ is transitive on O. The proof goes by considering all possible cases. Notice that Clause (3.3) is not symmetrical. Let $a, a_1 \in A \cap O$ and $b, b_1 \in B \cap O$.

(i) If $a <_O a_1 <_O b$, then for no n $f^n(b) <_{\mathbf{E}} f^n(a_1)$, a fortiori for no n $f^n(b) <_{\mathbf{E}} f^n(a)$. Indeed, $a <_O a_1$ means $a <_{\mathbf{A}} a_1$ hence $f^n(a) \leq_{\mathbf{A}} f^n(a_1)$, since f is order preserving on $\mathbf{A}$, thus $f^n(a) \leq_{\mathbf{E}} f^n(a_1)$. If by contradiction $f^n(b) <_{\mathbf{E}} f^n(a)$, then $f^n(b) <_{\mathbf{E}} f^n(a) \leq_{\mathbf{E}} f^n(a_1)$, a contradiction, since $\leq_{\mathbf{E}}$ is a partial order. Hence $a <_O b$.

(ii) If $a <_O b <_O a_1$, then $f^n(b) <_{\mathbf{E}} f^n(a_1)$, for some n. Were $a_1 \leq_O a$, then $f^n(a_1) \leq_{\mathbf{A}} f^n(a)$, since f is order preserving on $\mathbf{A}$, thus $f^n(b) <_{\mathbf{E}} f^n(a_1) \leq_{\mathbf{E}} f^n(a)$, contradicting $a <_O b$. Hence $a <_O a_1$. Here we have used the assumption that $\leq_{\mathbf{A}}$ is a linear order on A.

(iii) If $a <_O b <_O b_1$, then for no n $f^n(b) <_{\mathbf{E}} f^n(a)$, a fortiori for no n $f^n(b_1) <_{\mathbf{E}} f^n(a)$, since $f^n(b) \leq_{\mathbf{E}} f^n(b_1)$. Hence $a <_O b_1$.

(iv) If $b <_O a <_O a_1$, then $f^n(b) <_{\mathbf{E}} f^n(a)$, for some n, hence $f^n(b) <_{\mathbf{E}} f^n(a) \leq_{\mathbf{E}} f^n(a_1)$, thus $b <_O a_1$.

(v) If $b <_O a <_O b_1$, then $f^n(b) <_{\mathbf{E}} f^n(a)$, for some n, hence we cannot have $b_1 \leq_O b$, since otherwise $f^n(b_1) \leq_{\mathbf{E}} f^n(b) <_{\mathbf{E}} f^n(a)$, contradicting $a <_O b_1$. Hence $b <_O b_1$. Here we have used the assumption that $\leq_{\mathbf{B}}$ is a linear order on B.

(vi) If $b <_O b_1 <_O a$, then $f^n(b_1) <_{\mathbf{E}} f^n(a)$, for some n, hence $f^n(b) \leq_{\mathbf{E}} f^n(b_1) <_{\mathbf{E}} f^n(a)$, thus $b <_O a$.

The remaining cases (three elements in A or three elements in B) are trivial.

We have showed that, for each component O, the relation $<_O$ given by (3.1) - (3.3) linearly (strict) orders O. By the considerations at the beginning, in particular, by (*), if, for $d, e \in D$, we let

$$d \leq e \quad \text{if} \quad \text{either } d \leq_{\mathbf{E}} e, \text{ or } d <_O e, \text{ for some component } O, \tag{3.4}$$

then we get a linear order on D which extends $\leq_{\mathbf{E}}$, thus $(D, \leq)$ amalgamates $(A, \leq_{\mathbf{A}})$ and $(B, \leq_{\mathbf{B}})$ over $(C, \leq_{\mathbf{C}})$ in the class of linear orders. It remains to show that f is order preserving on D with respect to $\leq$.

As in the proof of Corollary 2.4, from (2.1) and from the assumption that f is order preserving on both $\mathbf{A}$ and $\mathbf{B}$ it follows that if $d \leq_{\mathbf{E}} e$, then $f(d) \leq_{\mathbf{E}} f(e)$, hence $f(d) \leq f(e)$. This case covers also (3.1) and (3.2), hence we only need to consider the case in (3.3). Suppose that $a \in A$ and $b \in B$ belong to the same component O. There are two cases with various subcases.

Case $b <_O a$. If $b <_O a$, then $f^n(b) <_{\mathbf{E}} f^n(a)$, for some $n \geq 1$, by (3.3). (i) If $n = 1$, then $f(b) <_{\mathbf{E}} f(a)$, hence $f(b) < f(a)$ and we are done. (iia) If $n > 1$ and $f(b)$, $f(a)$ lie in the same component O', then we get $f(b) < f(a)$ applying (3.3) to $<_{O'}$ with $n-1$ in place of n. (iib) If $n > 1$ and $f(b)$ and $f(a)$ lie in distinct components, then either $f(b) \leq_{\mathbf{E}} c \leq_{\mathbf{E}} f(a)$, or $f(a) \leq_{\mathbf{E}} c \leq_{\mathbf{E}} f(b)$, for some $c \in C$. But the latter eventuality cannot occur, since it implies $f^n(a) \leq_{\mathbf{E}} f^{n-1}(c) \leq_{\mathbf{E}} f^n(b)$, contradicting $f^n(b) <_{\mathbf{E}} f^n(a)$. Hence $f(b) \leq f(a)$ in this case, as well.

Case $a <_O b$. If $a <_O b$, then for no n $f^n(b) <_{\mathbf{E}} f^n(a)$, hence for no m $f^m(f(b)) <_{\mathbf{E}} f^m(f(a))$. (i) If $f(b)$ and $f(a)$ lie in the same component, we are done, by applying (3.3) with $f(a)$ and $f(b)$ in place of a and b. (ii) If $f(b)$ and $f(a)$ lie in distinct components, then, as above, either $f(b) \leq_{\mathbf{E}} f(a)$, or $f(a) \leq_{\mathbf{E}} f(b)$. Now observe that if $a <_O b$, then $f(b) <_{\mathbf{E}} f(a)$ does not occur, otherwise the first clause in (3.3) should have been applied. Thus $f(a) \leq_{\mathbf{E}} f(b)$.

We have proved that f is $\leq$-preserving, hence $\mathbf{D} = (D, \leq, f)$ is linearly ordered and amalgamates $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$.

(b) We first show that SAP fails. Let $\mathbf{C}$ be $\mathbb{N}$ with the usual order and with f interpreted as the successor function. Let $A = \{a\} \cup \mathbb{N}$ and let $\mathbf{A}$ extend $\mathbf{C}$ by setting $a < 0$ and $f(a) = 0$. Similarly, let $B = \{b\} \cup \mathbb{N}$ with $a \neq b$ and let $\mathbf{B}$ extend $\mathbf{C}$ by setting $b < 0$ and $f(b) = 0$. If an amalgamating algebra is a linear order and f is still to be strict order preserving, then a and b should be identified, since $f(a) = f(b)$ and then both $a < b$ and $b < a$ contradict the assumption that f is strict order preserving. Hence SAP fails.

In order to prove AP, we shall show that the situation in the above counterexample essentially provides all kinds of failures of SAP. In summary, if $a \in A$, $b \in B$ and $f^n(a) = f^n(b) \in C$, then a and b should be identified. After the identification is made, we are left with a triple which can be amalgamated using the techniques of part (a) and then it is quite easy to see that f is strict order preserving in the amalgamating algebra.

We now proceed wih the details. Suppose that $\iota_{\mathbf{C},\mathbf{A}} \colon \mathbf{C} \rightarrowtail \mathbf{A}$ and $\iota_{\mathbf{C},\mathbf{B}} \colon \mathbf{C} \rightarrowtail \mathbf{B}$ are embeddings. Let

$$\begin{aligned} C_A = \{\, a \in A \mid {} & \text{there are } n \in \mathbb{N},\, c \in C \text{ and } b \in B \text{ such that} \\ & f^n_{\mathbf{A}}(a) = \iota_{\mathbf{C},\mathbf{A}}(c) \text{ and } f^n_{\mathbf{B}}(b) = \iota_{\mathbf{C},\mathbf{B}}(c) \,\} \text{ and, symmetrically,} \\ C_B = \{\, b \in B \mid {} & \text{there are } n \in \mathbb{N},\, c \in C \text{ and } a \in A \text{ such that} \\ & f^n_{\mathbf{A}}(a) = \iota_{\mathbf{C},\mathbf{A}}(c) \text{ and } f^n_{\mathbf{B}}(b) = \iota_{\mathbf{C},\mathbf{B}}(c) \,\}. \end{aligned}$$

Notice that C_A and C_B are closed under applications of f, hence they are domains for substructures $\mathbf{C}_A$ and $\mathbf{C}_B$ of $\mathbf{A}$ and $\mathbf{B}$, respectively. Moreover, $n = 0$ is allowed, hence $\mathbf{C}$ embeds both in $\mathbf{C}_A$ and $\mathbf{C}_B$. Now observe that every strict order preserving unary operation on some linearly ordered set is injective. Hence if $a \in C_A$, then the $b \in B$ witnessing that a satisfies the defining condition for C_A is unique. Moreover, such b clearly belongs to C_B, as witnessed, in turn, by a. Thus if we define $\varphi \colon C_A \to C_B$ by setting $\varphi(a) = b$, for a, b as above, we get a bijective correspondence from C_A onto C_B. Surjectivity of φ is given by the symmetrical argument.

The correspondence φ is a homomorphism with respect to f, since if $\varphi(a) = b$, then $\varphi(f(a)) = f(b)$. This is obvious if $a \in C$; otherwise consider $n - 1$ in place of n in the definitions of C_A and C_B. We are going to show that φ is also an order isomorphism, hence $\mathbf{C}_A$ and $\mathbf{C}_B$ are isomorphic structures. Indeed, suppose that $a, a_1 \in C_A$ with $f^n_{\mathbf{A}}(a) = \iota_{\mathbf{C},\mathbf{A}}(c)$ and $f^m_{\mathbf{A}}(a_1) = \iota_{\mathbf{C},\mathbf{A}}(c_1)$ with, say, $m > n$. If $r = m - n$, then $f^m_{\mathbf{A}}(a) = \iota_{\mathbf{C},\mathbf{A}}(f^r_{\mathbf{C}}(c))$ and, since $\iota_{\mathbf{C},\mathbf{A}}$ is an embedding and f is strict order preserving, we get that $a < a_1$ if and only if $f^m_{\mathbf{A}}(a) < f^m_{\mathbf{A}}(a_1)$, if and only if $f^r_{\mathbf{C}}(c) < c_1$. The last inequality is computed in $\mathbf{C}$, hence it is also equivalent to $\varphi(a) < \varphi(a_1)$.

Since $\mathbf{C}_A$ and $\mathbf{C}_B$ are isomorphic, then, by replacing $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ with suitable isomorphic structures, it is no loss of generality to assume that $\mathbf{C}_1 \subseteq \mathbf{A}, \mathbf{B}$, where $\mathbf{C}_1$ is isomorphic to $\mathbf{C}_A$. Since $\mathbf{C}$ embeds into $\mathbf{C}_1$, if we can amalgamate the copies of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}_1$, then we have embeddings amalgamating the original structures. The relevant property of $\mathbf{C}_1$ that we have obtained is that

$$\text{if } a \in A,\ b \in B,\ c \in C_1 \text{ and } f_{\mathbf{A}}(a) = c = f_{\mathbf{B}}(b), \text{ then } a = b \in C_1. \tag{3.5}$$

Now apply the construction in (a) with $\mathbf{C}_1$ in place of $\mathbf{C}$. Since f is strict order preserving on $\mathbf{A}$ and $\mathbf{B}$, it is in particular order preserving, hence (a) can be applied, obtaining some structure $\mathbf{D}$ with an order preserving f. It remains to show that f is strict order preserving, and we shall show that this follows from (3.5). So let $d < e$. We know from (a) that $f(d) \leq f(e)$; it remains to show that $f(d) \neq f(e)$. If either $d, e \in A$ or $d, e \in B$, this is immediate from the assumption that f is strict order preserving on $\mathbf{A}$, respectively, $\mathbf{B}$. Otherwise, say, $d \in A$ and $e \in B$, thus $f(d) \in A$ and $f(e) \in B$. If $f(d) = f(e)$, then $f(d), f(e) \in C_1 = A \cap B$, hence $d = e$, by (3.5). This contradicts $d < e$.

(c) We shall present two counterexamples, since they have quite distinct features.

(c)(i) Let $C = \{0\}$ with the only possible interpretations, $A = \{a, 0\}$, $B = \{b, 0\}$, $A \cap B = C$ with

$$\begin{aligned} &a <_{\mathbf{A}} 0,\ f_{\mathbf{A}}(a) = 0,\ h_{\mathbf{A}}(a) = a, \text{ and} \\ &b <_{\mathbf{B}} 0,\ f_{\mathbf{B}}(b) = b,\ h_{\mathbf{A}}(b) = 0. \end{aligned} \tag{3.6}$$

If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ can be amalgamated to a *linear* order, then in the amalgamating algebra we have either $a \leq b$ or $b \leq a$. If f is required to be order preserving, the first eventuality cannot occur, since then $0 = f(a) \leq f(b) = b$ and similarly the second eventuality cannot occur, if h is required to be order preserving.

(c)(ii) If we want f and h to be strict order preserving, consider $\mathbb{N}$, as $\mathbf{C}$, with the standard order and interpret both f and h in $\mathbf{C}$ as the successor function. Then extend $\mathbf{C}$ with one more element in two possible ways with $A = \{a\} \cup \mathbb{N}$, $B = \{b\} \cup \mathbb{N}$ and the same new relations (3.6) as above. Then repeat the same argument.

(d) In this case the order used in the proof of Corollary 2.3 works. In detail, given $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ to be amalgamated, let $D = A \cup B$ and, for each $f \in F$, define f on D in the unique compatible way. Thus each f is bijective, since f is bijective both on A and B and the values of f agree on $C = A \cap B$. Then extend the order $\leq_{\mathbf{E}}$ on D from Theorem 2.2 by setting $d \leq e$ if

$$\begin{aligned} &\text{either } d \leq_{\mathbf{E}} e, \text{ or} \\ &\text{not } d \leq_{\mathbf{E}} e, \text{ not } e \leq_{\mathbf{E}} d \text{ and } d \in A,\ e \in B. \end{aligned} \tag{3.7}$$

By the arguments in (a), if the second alternative in equation (3.7) holds, then d and e belong to the same component. Now notice that if f is bijective and order preserving on C and (C_1, C_2) is a cut of $\mathbf{C}$, then $(f(C_1), f(C_2))$ is a cut. The assumption that f is surjective is used in order to get $f(C_1) \cup f(C_2) = C$. Hence if d and e belong to the component O associated to (C_1, C_2), then $f(d)$ and $f(e)$ belong to the component O' associated to $(f(C_1), f(C_2))$, since f is strict order preserving on both $\mathbf{A}$ and $\mathbf{B}$.

Now we can prove that each f is order preserving with respect to $\leq$. If $d <_{\mathbf{E}} e$, then we get $f(d) \leq_{\mathbf{E}} f(e)$. On the other hand, if $d < e$ is given by the second alternative in (3.7), then $d \in A$, $e \in B$ and d, e belong to the same component, call it O. But also $f(d) \in A$, $f(e) \in B$ and $f(d), f(e)$ belong to the same component O' as described above, hence (3.7) gives $f(d) < f(e)$. Put in another way, the first alternative in (3.3) never occurs when dealing with bijective functions.

The conclusion follows from the fact that a bijective order preserving function on a linearly ordered set is necessarily an order automorphism. □

In the special case of a single strictly increasing automorphism AP in Theorem 3.1 is a consequence of [LP, Theorem 2.2]. Recall that a theory with model completion has AP.

Remarks 3.2. The counterexample in (c)(i) in the proof of Theorem 3.1 shows that the class of *finite* linearly ordered sets with two order preserving operations fails to have AP. Indeed, the counterexample shows a bit more.

Recall that an *(order-theoretical) closure operation* on some poset P is an order preserving unary operation f such that $f(f(x)) = f(x) \geq x$ holds for every $x \in P$. See [Er] for information about closure operations, pictures and for the interest of the notion in the general order-theoretical setting. The counterexample in (c)(i) shows that the class of (finite) linearly ordered sets with two closure operations fails to have AP.

The counterexample (c)(ii) works both for the strict and the nonstrict case, but in the former situation the counterexample should necessarily be infinite, since a strict order preserving operation is the identity on a finite linearly ordered set.

Actually, the example shows that it is not always the case that a triple of linearly ordered set with two strict order preserving operations can be amalgamated into a linearly ordered set with two order preserving operations, namely, without requiring in the amalgamating structure that the operations are strict order preserving.

The proof of Theorem 3.1(d) shows that if $\mathbf{C}$ is a linearly ordered set with an F-indexed set of automorphisms, then $\mathbf{C}$ is a strong amalgamation base for the class of linearly ordered sets with an F-indexed set of strict order preserving unary operations.

4. Order reversing operations

If we consider order reversing operations, the arguments of the previous section generally carry over. However, a linearly ordered set with an order reversing operation has at most one element c such that $g(c) = c$, and obviously embeddings must preserve such "centers", if they exist. This fact prevents strong amalgamation. Moreover, elements greater than the center should be treated in a different—but symmetrical—way in comparison with elements smaller than the center. We now give precise definitions and collect some trivial facts about these notions.

Definition 4.1. If $\mathbf{C}$ is a linearly ordered set with a unary operation g, an element c of C is said to be a *center*, or a *fixed point* of g if $g(c) = c$.

An element d of C is an *upper* (resp., *lower*) element if $g(d) < d$ (resp., $g(d) > d$).

Lemma 4.2. *Suppose that $\mathbf{C}$ is a linearly ordered set with one order reversing unary operation g.*

(a) Every element of C is either upper, lower, or a center. The alternatives are mutually exclusive and $\mathbf{C}$ has at most one center.

(b) Suppose that $\mathbf{C}$ has a center c. Then, for every $d \in C$, d is upper if and only if $c < d$ and d is lower if and only if $d < c$.

(c) If d is upper, then $g(d)$ is either lower or the center, and symmetrically. All the upper elements are greater than all the lower elements,

(d) If $\mathbf{C}$ has no center, then $\mathbf{C}$ can be extended by adding just one element (in a unique way modulo isomorphisms preserving $\mathbf{C}$) to a linearly ordered set $\mathbf{C}^*$ with an order reversing unary operation and a center. If g is strict order reversing in $\mathbf{C}$, then the operation in $\mathbf{C}^*$ is strict order reversing, too.

(e) Suppose that $\iota\colon \mathbf{C} \rightarrowtail \mathbf{A}$ is an embedding.

If $\mathbf{C}$ has a center c, then $\mathbf{A}$ has a center and $\iota(c)$ is the center of $\mathbf{A}$.

If $\mathbf{C}$ has not a center and $\mathbf{A}$ has a center, then ι extends uniquely to an embedding from $\mathbf{C}^*$ to $\mathbf{A}$, where $\mathbf{C}^*$ is defined as in (d).

If neither $\mathbf{C}$ nor $\mathbf{A}$ have a center, then ι extends uniquely to an embedding from $\mathbf{C}^*$ to $\mathbf{A}^*$.

Proof. (a) - (c) are immediate from the assumptions that $\mathbf{C}$ is linearly ordered and g is order reversing. For example, to prove (c), observe that if $g(d) < d$, then $g(g(d)) \geq g(d)$, hence either $g(g(d)) > g(d)$, thus $g(d)$ is lower, or $g(g(d)) = g(d)$, thus $g(d)$ is the center. Suppose that d is upper and e is lower. If $d < e$, then $g(d) < d < e < g(e)$ contradicts the assumption that g is order reversing. Hence $e < d$, since the order is linear and d and e are necessarily distinct.

(d) By (b), the new element supposed to be a center, call it c, should be greater than all the lower elements and smaller then all the upper elements, hence the position of c in the order is fully determined. Setting $g(c) = c$, clause (c) implies that, endowed with the above structure, $C \cup \{c\}$ is linearly ordered and g is order reversing.

(e) It follows from the definitions that an embedding (actually, just a morphism) sends a center to a center. All the rest follows from (a) - (d). □

It follows from Lemma 4.2 that embeddings preserve upper and lower elements, as well as centers, if they exist. In particular, given a triple to be amalgamated, there is no need to mention some specific structure **A**, **B** or **C**, when referring to the center.

Theorem 4.3. *(a) The classes $\mathcal{LO}_r$, resp., $\mathcal{LO}_{sr}$ of linearly ordered sets with one order reversing, resp., one strict order reversing unary operation have AP but not SAP.*

(b) The classes of linearly ordered sets with two order reversing, resp., two strict order reversing unary operations have not AP. Similarly for the case of an order preserving and an order reversing operation. AP fails even if we assume that all the operations have a common center.

Proof. (a) To prove that SAP fails, just let **C** have no center and **A**, **B** have a center. In any amalgamating structure the centers of **A** and **B** must be identified, by Lemma 4.2, hence SAP fails. The simplest concrete example is when **C** is an empty model and **A** and **B** have only one element, necessarily, the center. However, we want to prove that also the weaker version of SAP fails when **C** is required to be nonempty. Cf. the final comment in Definition 2.1.

So let **C** be the model with domain $C = \{-\infty, \infty\}$ and such that $-\infty < \infty$, $g(-\infty) = \infty$ and $g(\infty) = -\infty$. Extend **C** to **A** by letting $A = \{-\infty, a, \infty\}$, with $-\infty < a < \infty$ and $g(a) = a$. Similarly, extend **C** to **B** by letting $B = \{-\infty, b, \infty\}$ with $b \neq a$, $-\infty < b < \infty$ and $g(b) = b$ (in fact, **A** and **B** are just two isomorphic copies of the structure $\mathbf{C}^*$ constructed in Lemma 4.2(d)). By Lemma 4.2, in any amalgamating structure we must have $a = b$, hence SAP fails. Notice that g is strict order reversing on **A**, **B** and **C**, but $a = b$ in any amalgamating structure **D**, even if g is assumed to be (possibly, not necessarily strict) order reversing in **D**.

The proof of AP is similar to the proof of Theorem 3.1(a)(b), except that the possible overlapping of centers should be fixed (in the case of order reversing operations this is the only obstacle to strong amalgamation) and that the actual definition of the linear order involves still another division into cases.

Let $\mathbf{C} \subseteq \mathbf{A}, \mathbf{B}$ be a triple to be amalgamated. If some algebra above has no center, add a center to it according to Lemma 4.2(d). Possibly, replace **A** and **B** with isomorphic copies, so that their centers are identified (this is necessary exactly in case the original **A** and **B** have some center and **C** has not a center). Because of Lemma 4.2, there is just one way to add the centers and the original embeddings can be extended in a unique way.

Hence we can suppose that **A**, **B** and **C** all have a center. As in the proof of Theorem 3.1, Theorem 2.2 and Corollary 2.4 furnish an amalgamating *partial* order $\mathbf{E} = (D, \leq_{\mathbf{E}}, g)$ with an order reversing operation. Lengvárszky [Le] showed that a poset with a unary order reversing function g can be linearized in such a way that g is still order reversing if and only if g^2 is acyclic and g has at most one fixed point (in the original poset). Hence we can apply [Le] in order to get a proof of the positive part of (a). As in the case of Theorem 3.1, we shall present a direct and more explicit construction.

To simplify the notation, let $<^n_{\mathbf{E}}$ be $<_{\mathbf{E}}$ for n even, and $<^n_{\mathbf{E}}$ be $>_{\mathbf{E}}$ for n odd. Apply a similar convention for $\leq^n_{\mathbf{E}}$. Recall from the proof of Theorem 3.1 the definition of a component, and recall that the relative order between two elements lying in distinct components is completely determined by $\leq_{\mathbf{E}}$. Notice also that, as we mentioned, the definition of the components does not rely on the operations, it depends only on the orderings.

As in the proof of 3.1(a), we need to linearize each component. Let us call a component *lower* if its elements are $\leq_{\mathbf{E}} c$ and *upper* otherwise, where c is the center of **C**. The distinction makes sense, since each component is convex and contained in $D \setminus C$. If O is a lower component, linearize O according to the conditions (3.1) - (3.3), but replacing $f^n(b) <_{\mathbf{E}} f^n(a)$ in (3.3) by $g^n(b) <^n_{\mathbf{E}} g^n(a)$. The proof that $<_O$ is a linear order on O carries over just by replacing $<_{\mathbf{E}}$ and $\leq_{\mathbf{E}}$, respectively, by $<^n_{\mathbf{E}}$ and $\leq^n_{\mathbf{E}}$ in all the expressions involving f^n (here, g^n) and with no further modification.

As far as upper components are concerned, we have to exchange the role of A and B in (3.3), since we want g to be order reversing. In detail, replace (3.3) by

$$\left.\begin{array}{ll} a <_O b, & \text{if } g^n(a) <^n_{\mathbf{E}} g^n(b), \\ & \text{for some } n \geq 1, \text{ or} \\ b <_O a, & \text{otherwise,} \end{array}\right\} \quad \text{if } a \in A \cap O \text{ and } b \in B \cap O \tag{4.1}$$

for upper components

By symmetry, $<_O$ is a linear order on O in this case, too.

We now can define $\leq$ on D as in equation (3.4). The proof that g is order reversing with respect to $\leq$ is similar to 3.1(a), just considering separately the cases when a and b belong to the same lower or upper component. For example, we shall treat the case when $a \in A$ and $b \in B$ belong to the same upper component O.

Case $a <_O b$. If $a <_O b$, then $g^n(a) <^n_{\mathbf{E}} g^n(b)$, for some $n \geq 1$, by (4.1). (i) If $n = 1$, then $g(a) <^1_{\mathbf{E}} g(b)$, namely, $g(b) <_{\mathbf{E}} g(a)$, by the definition of $<^n_{\mathbf{E}}$, hence $g(b) < g(a)$ and we are done. (ii) If $n > 1$, then $g^{n-1}(g(a)) <^n_{\mathbf{E}} g^{n-1}(g(b))$, that is, $g^{n-1}(g(b)) <^{n-1}_{\mathbf{E}} g^{n-1}(g(a))$. (iia) First suppose that $g(b)$ and $g(a)$ lie in the same component P. Then P is a lower component, by Lemma 4.2(c) applied to **A** and **B**. Notice that, by construction, each component has empty intersection with C, hence $g(b)$ and $g(a)$ are not centers, since C is assumed to have a center and the center is unique. Thus we get $g(b) <_P g(a)$ applying the modified version of (3.3) with $n-1$ in place of n. (iib) If $g(b)$ and $g(a)$ lie in distinct components, then either $g(b) \leq_{\mathbf{E}} c \leq_{\mathbf{E}} g(a)$, or $g(a) \leq_{\mathbf{E}} c \leq_{\mathbf{E}} g(b)$, for some $c \in C$. The latter eventuality cannot occur, since it implies $g^n(a) \leq_{\mathbf{E}}^{n-1} g^{n-1}(c) \leq_{\mathbf{E}}^{n-1} g^n(b)$, hence $g^n(b) \leq_{\mathbf{E}}^n g^n(a)$ contradicting $g^n(a) <_{\mathbf{E}}^n g^n(b)$. Hence $g(b) \leq g(a)$ in this case, as well.

Case $b <_O a$. If $b <_O a$, then for no n $g^n(a) <_{\mathbf{E}}^n g^n(b)$, hence for no m $g^m(g(a)) <_{\mathbf{E}}^{m+1} g^m(g(b))$, that is, for no m $g^m(g(b)) <_{\mathbf{E}}^m g^m(g(a))$. (i) If $g(b)$ and $g(a)$ lie in the same component P, necessarily, as we mentioned, a lower component, then we get $g(a) <_P g(b)$, by applying the variant of (3.3) with $g(a)$ and $g(b)$ in place of a and b. Hence $g(a) < g(b)$. (ii) If $g(b)$ and $g(a)$ lie in distinct components, then, as above, either $g(b) \leq_{\mathbf{E}} g(a)$, or $g(a) \leq_{\mathbf{E}} g(b)$. If $b <_O a$, then $g(a) <_{\mathbf{E}}^1 g(b)$, equivalently, $g(b) <_{\mathbf{E}} g(a)$, does not occur, otherwise the first clause in (4.1) should have been applied. Thus $g(a) \leq_{\mathbf{E}} g(b)$, hence $g(a) \leq g(b)$.

The case when g is assumed to be strict order reversing presents no essential difference with respect to the proof of Theorem 3.1(b). The only minor detail is in the proof that φ, as defined in the proof of 3.1(b), is an order isomorphism. In the present case, assume that $a, a_1 \in C_A$, $g_{\mathbf{A}}^n(a) = \iota_{\mathbf{C},\mathbf{A}}(c)$ and $g_{\mathbf{A}}^m(a_1) = \iota_{\mathbf{C},\mathbf{A}}(c_1)$ with, say, $m > n$. If $r = m - n$, then $g_{\mathbf{A}}^m(a) = \iota_{\mathbf{C},\mathbf{A}}(g_{\mathbf{C}}^r(c))$. Since g is strict order reversing, then $a < a_1$ if and only if $g_{\mathbf{A}}^m(a) <^m g_{\mathbf{A}}^m(a_1)$, if and only if $g_{\mathbf{C}}^r(c) <^m c_1$. As in 3.1(b), the last inequality is computed in $\mathbf{C}$, hence it is also equivalent to $\varphi(a) < \varphi(a_1)$. All the rest goes as in 3.1(b)

(b)(i) We provide the example of three finite nonamalgamable algebras with two order reversing operations g and k. Let $C = \{c\}$; $A = \{a, c, d\}$, with $a <_{\mathbf{A}} c <_{\mathbf{A}} d$, $g_{\mathbf{A}}(a) = g_{\mathbf{A}}(d) = c$, $k_{\mathbf{A}}(a) = d$, $k_{\mathbf{A}}(d) = a$, and $B = \{b, c, e\}$, with $A \cap B = \{c\}$, $b <_{\mathbf{B}} c <_{\mathbf{B}} e$, $g_{\mathbf{B}}(b) = e$, $g_{\mathbf{B}}(e) = b$, $k_{\mathbf{B}}(b) = k_{\mathbf{B}}(e) = c$, Then argue as in 3.1(c)(i).

(b)(ii) In this example we construct three nonamalgamable finite algebras with an order preserving operation f and an order reversing operations g. Let the domains A, B, C, the orderings and the operations g be as in (b)(i). Let $f_{\mathbf{A}}$ be the identity and $f_{\mathbf{B}}(b) = f_{\mathbf{B}}(e) = c$. If $a \leq b$ in some amalgamating structure with g order reversing, then $e = g(b) \leq g(a) = c$, contradicting $c < e$. If $b \leq a$ in some amalgamating structure with f order preserving, then $c = f(b) \leq f(a) = a$, again a contradiction.

(b)(iii) Now we present three nonamalgamable algebras with two strict order reversing bijective operations g and k and which cannot be amalgamated into a linear order on which g is order reversing. Let $C = \{-\infty, \infty\}$ with $-\infty < \infty$, $g_{\mathbf{C}}(-\infty) = k_{\mathbf{C}}(-\infty) = \infty$ and $g_{\mathbf{C}}(\infty) = k_{\mathbf{C}}(\infty) = -\infty$. Extend $\mathbf{C}$ to $\mathbf{A}$ by letting $A = C \cup \mathbb{Z}$, with $-\infty < z < \infty$ and $g_{\mathbf{A}}(z) = k_{\mathbf{A}}(z) = -z$, for every $z \in \mathbb{Z}$. Let $\mathbb{Z}' = \{\dots, -2', -1', 0', 1', 2', \dots\}$ be a disjoint copy of $\mathbb{Z}$ and extend $\mathbf{C}$ to $\mathbf{B}$ by letting $B = C \cup \mathbb{Z}'$, with $-\infty < z' < \infty$, $g_{\mathbf{B}}(z') = -z'$ and $k_{\mathbf{B}}(z') = -z' + 2'$, for every $z' \in \mathbb{Z}'$.

In view of Lemma 4.2, in any amalgamating structure with g (not necessarily strict) order reversing, the centers 0 and $0'$ with respect to g should be identified, but this is incompatible with $k_{\mathbf{A}}(0) = 0$ and $k_{\mathbf{B}}(0') = 2'$.

Notice that $k_{\mathbf{A}}(0) = 0$ and $k_{\mathbf{B}}(0') \neq 0'$ are the only properties of k needed in the above argument, hence, by changing the other values of k, the counterexample can be modified in order to take care of the case of a (strict) order reversing together with a (strict) order preserving operation, possibly both bijective. Actually, there are plenty of further similar possibilities.

(b)(iv) The main point in (b)(iii) above is that the operations in $\mathbf{C}$ have no center and then centers are added in different ways to $\mathbf{A}$ and $\mathbf{B}$. On the other hand, we can merge the ideas in (b)(i) and 3.1(c)(ii) in order to get failure of AP even under the assumption that the two operations are strict order reversing with a common center.

So let $\mathbf{C}$ be $\mathbb{Z} \setminus \{-1, 1\}$ with the standard order, $g_{\mathbf{C}}(0) = k_{\mathbf{C}}(0) = 0$, $g_{\mathbf{C}}(n) = k_{\mathbf{C}}(n) = -n$ and $g_{\mathbf{C}}(-n) = k_{\mathbf{C}}(-n) = n+1$, for $n \in \mathbb{N} \setminus \{0, 1\}$. Extend $\mathbf{C}$ to $\mathbf{A}$ with $A = \mathbb{Z}$, $g_{\mathbf{A}}(1) = k_{\mathbf{A}}(1) = -1$ and $g_{\mathbf{A}}(-1) = 1$, $k_{\mathbf{A}}(-1) = 2$. Let $B = C \cup \{1', -1'\}$ with $-2 < -1' < 0 < 1' < 2$, $g_{\mathbf{B}}(1') = k_{\mathbf{B}}(1') = -1'$ and $g_{\mathbf{B}}(-1') = 2$, $k_{\mathbf{B}}(-1') = 1'$. In any amalgamating structure with a linear order, either $1 \leq 1'$ or $1' \leq 1$. If $1 \leq 1'$, then $2 = k^2(1) \leq k^2(1') = 1'$, a contradiction. If $1' \leq 1$, then $2 = g^2(1') \leq g^2(1) = 1$, still a contradiction.

If we want a counterexample with a strict order preserving operation f and a strict order reversing operation g, again, with a common center, just take $f = k^2$ in the above counterexample. □

Remark 4.4. The proof of Theorem 4.3(a) shows that the class of linearly ordered sets with an order reversing unary operation with a center has SAP.

Actually, in the class of linearly ordered sets with an order reversing unary operation, a structure $\mathbf{C}$ is a strong amalgamation base if and only if $\mathbf{C}$ has a center.

The counterexample (b)(iii) in the proof of Theorem 4.3 shows that Theorem 3.1(d), as it stands, does not generalize to order reversing bijective operations, *antiautomorphisms*, for short.

The counterexamples (b)(i) and (b)(iv) show that the class of linearly ordered sets with two order reversing operations with the same center fails to have AP.

However, Theorem 3.1(d) does generalize if we put together the two assumptions. Moreover, we can deal with automorphisms and antiautomorphisms at the same time, provided they all respect the same center.

Theorem 4.5. *For every pair F and G of sets, let $\mathcal{LO}_{FGac}$ be the class of linear orders with an F-indexed family of automorphisms and a G-indexed family of antiautomorphisms such that all the operations in F and in G have a common center. Then $\mathcal{LO}_{FGac}$ has SAP.*

Proof. If $G = \emptyset$, this is Theorem 3.1(d); actually, no assumption on centers is needed. So let us assume that $G \neq \emptyset$, hence the center is unique and is preserved by embeddings, by Lemma 4.2.

Given a triple **A**, **B**, **C** to be amalgamated and with center c, extend the order $\leq_{\mathbf{E}}$ given by Theorem 2.2 on $A \cup B$ by setting $d \leq e$ if either

$$d \leq_{\mathbf{E}} e, \text{ or} \tag{4.2}$$

$$\text{not } d \leq_{\mathbf{E}} e, \text{ not } e \leq_{\mathbf{E}} d, \text{ and } d \in A,\ e \in B,\ d, e <_{\mathbf{E}} c, \text{ or} \tag{4.3}$$

$$\text{not } d \leq_{\mathbf{E}} e, \text{ not } e \leq_{\mathbf{E}} d, \text{ and } d \in B,\ e \in A,\ c <_{\mathbf{E}} d, e. \tag{4.4}$$

The definition provides a linear order since if, say, $d <_{\mathbf{E}} c <_{\mathbf{E}} e$, then $d <_{\mathbf{E}} e$, hence (4.2) applies. If $f_{\mathbf{A}}$ and $f_{\mathbf{B}}$ are automorphisms of **A** and **B**, define f on $A \cup B$ in the unique compatible way as in Corollary 2.4. If $d \leq_{\mathbf{E}} e$, then $f(d) \leq_{\mathbf{E}} f(e)$ as in 2.4. Since $f_{\mathbf{A}}$ and $f_{\mathbf{B}}$ have center $c \in C$, then $d, e <_{\mathbf{E}} c$ implies $f(d), f(e) <_{\mathbf{E}} f(c) = c$, hence the arguments in the proof of Theorem 3.1(d) show that if $d < e$ and clause (4.2) does not apply, then $f(d) < f(e)$. If $c <_{\mathbf{E}} d, e$, the symmetrical arguments apply when A and B are exchanged. Thus f is order preserving on D. Since f is bijective, it is an automorphism.

Now suppose that g is interpreted as an antiautomorphism on A and B and, again, define g on $A \cup B$ in the unique compatible way. If $d, e <_{\mathbf{E}} c$, then $c = g(c) <_{\mathbf{E}} g(d), g(e)$. Recalling the definition of a component from the proof of Theorem 3.1(a), and by the comments before (*) there, if either (4.3) or (4.4) applies, then d and e lie in the same component. Arguing in a way similar to 3.1(d), the assumption that g is bijective implies that if d and e lie in the same component, say, the component associated to the cut (C_1, C_2), then $g(d)$ and $g(e)$ lie in the component associated to $(g(C_2), g(C_1))$. Thus if (4.3) applies to d, e, then (4.4) applies to $g(e)$, $g(d)$ and conversely. This implies that g is order reversing on D, hence an antiautomorphism, since g is bijective. □

5. Further remarks

In this section we present a few model-theoretical consequences of the above results. It is almost immediate from Theorems 3.1 and 4.3 that the classes of finite linearly ordered sets with one order preserving, resp., one order reversing unary operation have a Fraïssé limit. Moreover, say, if T is the theory of linearly ordered sets with one order preserving unary operation f satisfying $f^{m+1}(x) = f^m(x)$, for some fixed m, and **M** is the Fraïssé limit of the class of finite models of T, then $Th(\mathbf{M})$ is ω-categorical and is the model completion of T.

With a bit more notation, we can prove AP, JEP and the existence of Fraïssé limits for many more classes. The relevant aspect in the following considerations is that in all the previous constructions the amalgamating model has been always constructed on the set theoretical union of A and B. We shall elaborate further on this aspect in [Li].

Recall the definitions of the classes $\mathcal{PO}_F$, $\mathcal{LO}_p$, $\mathcal{LO}_{sp}$, $\mathcal{LO}_{Fa}$, $\mathcal{LO}_r$, $\mathcal{LO}_{sr}$ and $\mathcal{LO}_{FGac}$ from Proposition 2.4 and Theorems 3.1, 4.3, 4.5. Recall the definition of a closure operation from Example 3.2.

For every class $\mathcal{K}$ of structures and every set H of appropriate conditions, let $\mathcal{K}^H$ denote the subclass of $\mathcal{K}$ consisting of those structures in $\mathcal{K}$ satisfying all the conditions in H. We allow H to be the empty set of conditions; in this case $\mathcal{K}^H = \mathcal{K}$. In a few cases, for certain combinations of $\mathcal{K}$ and H, the class $\mathcal{K}^H$ will turn out to be an empty class; formally, the results remain true in this trivial situation.

For each class we have considered in this note, AP and SAP are preserved by adding various kinds of conditions. In some cases, the classes we have considered have the Joint Embedding Property (JEP), even when AP fails.

Lemma 5.1. *The classes $\mathcal{PO}_F^H$, $\mathcal{LO}_p^H$, $\mathcal{LO}_{Fa}^H$ and $\mathcal{LO}_{FGac}^H$ have SAP and JEP, for any pair of sets F and G and for any set H of conditions chosen among the following ones.*

The ordered set has no maximum (minimum); is finite; finitely generated; countable; of cardinality $< \lambda$, for λ an infinite cardinal; is well-ordered; some operation f (or some iteration f^ℓ, $\ell \in \mathbb{N}$) has some (no) fixed point; is surjective; is (strictly) increasing (decreasing); is a closure operation; for some $m, n \in \mathbb{N}$ satisfies $f^{m+1}(x) = f^n(x)$ for every (some) x; some given pair of operations commute. In general, we can allow any condition which can be expressed by a universal-existential first-order sentence such that only one variable is bounded by the universal quantifier.

The classes $\mathcal{LO}_{sp}^H$, $\mathcal{LO}_r^H$, $\mathcal{LO}_{sr}^H$ have AP and JEP, for any set H of conditions as above.

The class of linearly ordered sets with any (fixed in advance) number of order preserving and strict order preserving unary operations has JEP. Each subclass determined by any set of conditions as above has JEP .

Proof. If some property from H holds in $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, then the property holds in the amalgamating structure $\mathbf{D}$, since in each case we have constructed $\mathbf{D}$ on $A \cup B$. Hence (S)AP holds in all the classes under consideration.

All the classes for which we have proved AP have also JEP, since, for languages without constants, JEP means exactly that the empty structure is an amalgamation base. Formally, the class $\mathcal{LO}_{FGac}$ has not an empty model; however, modulo isomorphism, there is a unique "initial" 1-element model, hence JEP follows from AP. Notice that it is not necessary to assume that the 1-element model, call it $\mathbf{C}$, belongs to $\mathcal{LO}_{FGac}^H$ in order to prove JEP, it is enough to observe that $\mathbf{C} \in \mathcal{LO}_{FGac}$.

To prove the last statement, given $\mathbf{A}$ and $\mathbf{B}$, set all the elements from A to be $<$ than all the elements from B in $A \cup B$ and define the operations on $A \cup B$ in the unique compatible way. $\square$

Even more general conditions under which AP and SAP are preserved are presented in [Li].

Recall that if $\mathcal{F}$ is a class of finitely generated structures in a countable language, a Fraïssé limit of $\mathcal{F}$ is a countable ultrahomogeneous structure of age $\mathcal{F}$. See [H, Section 7.1] for further details.

We say that a first-order sentence σ is 1-*universal* if σ is universal and only one variable appears in σ. Examples of 1-universal sentences are sentences asserting that some unary operation is increasing, decreasing, strictly increasing, strictly decreasing, idempotent, has no fixed point; that some pair of unary operations commute, etc. In particular, there is a 1-universal sentence asserting that an order preserving unary operation is a closure operation.

Theorem 5.2. *(1) Let $\mathcal{K}$ be either $\mathcal{LO}_p^H$, $\mathcal{LO}_r^H$ or $\mathcal{PO}_F^H$ for F finite, where H is any, possibly empty, set of conditions expressible by a* 1-*universal sentence.*

If $\mathcal{K}^{fin}$ is the the class of finite members of $\mathcal{K}$ and $\mathcal{K}^{fin}$ is not empty, then $\mathcal{K}^{fin}$ has a Fraïssé limit $\mathbf{M}$ in $\mathcal{K}$.

If $\mathcal{K}$ is either $\mathcal{LO}_p^H$ or $\mathcal{LO}_r^H$ and H includes the condition $f^{m+2}(x) = f^m(x)$, for some m, then the first-order theory $Th(\mathbf{M})$ of $\mathbf{M}$ is ω-categorical and has quantifier elimination. Moreover, $Th(\mathbf{M})$ is the model-completion of $Th(\mathcal{K})$.

(2) Let $\mathcal{K}$ be either $\mathcal{LO}_p^H$, $\mathcal{LO}_{sp}^H$, $\mathcal{LO}_r^H$ or $\mathcal{LO}_{sr}^H$, where H is any, possibly empty, set of conditions expressible by a 1-*universal sentence.*

If $\mathcal{K}$ is nonempty and $\mathcal{K}^{fg}$ is the class of all finitely generated members of $\mathcal{K}$, then $\mathcal{K}^{fg}$ has a Fraïssé limit in $\mathcal{K}$.

Proof. (1) In each case $\mathcal{K}^{fin}$ has AP and JEP, by Lemma 5.1. Obviously $\mathcal{K}^{fin}$ is closed under taking substructures, hence the Fraïssé limit of $\mathcal{K}^{fin}$ exists by Fraïssé's Theorem. See, e. g., [H, Theorem 7.1.2]. The finiteness of F in $\mathcal{PO}_F$ is necessary in order to have only a countable number of nonisomorphic structures in $\mathcal{K}^{fin}$.

The Fraïssé limit belongs to $\mathcal{K}$ since the limit is constructed as the union of a chain of structures in $\mathcal{K}^{fin}$ and $\mathcal{K}$ is closed under unions of chains. To prove the last statement, use [H, Theorem 7.4.1], noticing that if $f^{m+2}(x) = f^m(x)$ holds for some m, then any member of $\mathcal{K}$ generated by n elements has cardinality $\leq (m+1)n$. Finally, $Th(\mathbf{M})$ is model-complete and $\mathbf{M}$ is existentially closed in $\mathcal{K}$; moreover, $Th(\mathcal{K})$ and $Th(\mathbf{M})$ have the same universal consequences.

(2) is proved in a similar way. Just check that in each case $\mathcal{K}$ has only a countable number of nonisomorphic finitely generated members. □

Remark 5.3. Fraïssé method does not apply to the classes $\mathcal{LO}_{Fa}$ and $\mathcal{LO}_{FGac}$, since such classes are generally not closed under taking substructures. The problem can be circumvented, since operations in F and in G are assumed to be bijective, hence we get an inessential expansion of the language if we assume that, for every $f \in F$, there is another operation symbol in F interpreted as the inverse of f, and similarly for each $g \in G$. Thus Theorem 5.2(2) holds for

$\mathcal{LO}^H_{Fa}$ when F has two function symbols, assumed to be one the inverse of the other. A similar result holds for linearly ordered set with an antiautomorphism together with its inverse.

However, we face another problem when two or more (anti)automorphisms are considered, together with their inverses. Consider $\mathbb{Z} \times \mathbb{Z}$ with the lexicographic order, let f and h be defined by $f(z, w) = (z + 1, w)$ and $h(z, w) = (z, w + n(z))$, where n is an arbitrary function from $\mathbb{Z}$ to $\{1, -1\}$. If we add to the language operations representing the inverses of f and g, then $(0, 0)$ generates the whole of $\mathbb{Z} \times \mathbb{Z}$. Letting the function n vary, we get continuum many nonisomorphic 1-generated structures, hence the method in Fraïssé construction, as it stands, cannot be applied.

Of course, for certain sets H of conditions, it is possible that $\mathcal{LO}^H_{Fa}$ and $\mathcal{LO}^H_{FGac}$ have only countably many models modulo isomorphism, in which case a result analogous to Theorem 5.2(2) holds, provided inverses are present in the language, as specified above. In the case of $\mathcal{LO}^H_{FGac}$ we also need to dispense for a constant interpreted as the center.

Problem 5.4. Lemma 5.1 and the proof of Theorem 5.2(1) imply that many *locally finite* theories of *partially* ordered sets with further operations have model completion, the simplest case being posets with a finite number of pairwise commuting closure operations. In view of the counterexamples in the proofs of Theorems 3.1(c) and 4.3(b), theories of *linearly* ordered sets with many operations generally have not model completion. Recall that some theory has model completion if and only if it has both AP and model companion.

However, it is partially an open problem to characterize companionable theories of linear orders with further operations.

Remark 5.5. Theorem 2.2 and Corollary 2.4 can be strengthened further. We can consider many order relations at the same time, and add conditions asserting that some order is coarser than another order. Again, conditions involving the operations can be added, for example, conditions asserting that some operation is increasing, or that it is idempotent. In many cases, Theorem 2.2 and Corollary 2.4 apply also to binary relations which are not necessarily orders. See [Li] for more details. Moreover, Corollary 2.4 holds for any number of n-ary operations, with n varying.

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References

[BHKK] Ö. Beyarslan, D. M. Hoffmann, M. Kamensky, P. Kowalski, *Model theory of fields with free operators in positive characteristic*, Trans. Amer. Math. Soc. **372**, 5991–6016 (2019).

[Er] M. Erné, *Closure*, in *Beyond topology*, 163–238, Contemp. Math. **486**, Amer. Math. Soc., Providence, RI, 2009.

[Ev] D. M. Evans, *Examples of $\aleph_0$-categorical structures*, in *Automorphisms of first-order structures*, R. Kaye, D. Macpherson (eds.), Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 33–72 (1994).

[F1] R. Fraïssé, *Sur certaines relations qui généralisent l'ordre des nombres rationnels*, C. R. Acad. Sci. Paris, **237**, 540–542 (1953).

[F2] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3), **71**, 363–388 (1954).

[GM] D. M. Gabbay, L. Maksimova, *Interpolation and definability. Modal and intuitionistic logics*, Oxford Logic Guides **46**, The Clarendon Press, Oxford University Press, Oxford (2005).

[GP] N. Guzy, F. Point, *Topological differential fields*, Ann. Pure Appl. Logic **161**, 570–598 (2010).

[H] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications **42**, Cambridge University Press, Cambridge, 1993.

[HR] P. Howard, J. E. Rubin, *Consequences of the axiom of choice*, Mathematical Surveys and Monographs **59**, American Mathematical Society, Providence, 1998.

[Joh] J. S. Johnson, *Amalgamation of polyadic algebras*, Trans. Amer. Math. Soc. **149**, 627–652 (1970).

[Jón] B. Jónsson, *Universal relational systems*, Math. Scand. **4**, 193–208 (1956).

[KMPT] E. W. Kiss, L. Márki, P. Pröhle, W. Tholen, *Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness, and injectivity*, Studia Sci. Math. Hungar. **18** (1982), 79–140.

[LP] M. C. Laskowski, K, Pal, *Model companion of ordered theories with an automorphism*, Trans. Amer. Math. Soc. **367**, 6877–6902 (2015).

[Le] Z. Lengvárszky, *Linear extensions of partial orders preserving antimonotonicity*, Publ. Math. Debrecen **38**, 279–285 (1991),

[EC] P. Lipparini, *Existentially complete closure algebras*, Boll. Un. Mat. Ital. D (6) **1**, 13–19 (1982).

[Li] P. Lipparini, *The strong amalgamation property into union*, arXiv:2103.00563

[MS] J. Madárasz, T. Sayed-Ahmed, *Amalgamation, interpolation and epimorphisms in algebraic logic*, Algebra Universalis **56**, 179–210 (2007).

[MMT] G. Metcalfe, F. Montagna, C. Tsinakis, *Amalgamation and interpolation in ordered algebras*, J. Algebra **402**, 21–82 (2014).

[N] I. Németi, *Cylindric-relativised set algebras have strong amalgamation*, J. Symbolic Logic **50**, 689–700 (1985).

[S] O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandlungen Hamburg **5**, 161–183 (1927).

[SN] J. Szigeti, B. Nagy, *Linear Extensions of Partial Orders Preserving Monotonicity*, Order **4**, 31–35 (1987).

[W] C. Wood, *The model theory of differential fields of characteristic $p \neq 0$*, Proc. Amer. Math. Soc. **40**, 577–584 (1973).

[Z] M. Ziegler, *Separably closed fields with Hasse derivations*, J. Symbolic Logic, **68**, 311–318 (2003).

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