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Open sets of exponentially mixing Anosov flows

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Abstract. We prove that an Anosov flow with $\mathcal{C}^1$ stable bundle mixes exponentially whenever the stable and unstable bundles are not jointly integrable. This allows us to show that if a flow is sufficiently close to a volume-preserving Anosov flow and $\dim \mathbb{E}_s = 1$, $\dim \mathbb{E}_u \geq 2$ then the flow mixes exponentially whenever the stable and unstable bundles are not jointly integrable. This implies the existence of non-empty open sets of exponentially mixing Anosov flows. As part of the proof of this result we show that $\mathcal{C}^{1+}$ uniformly expanding suspension semiflows (in any dimension) mix exponentially when the return time is not cohomologous to a piecewise constant.

Keywords. Anosov flows, robust, exponential mixing

1. Introduction & results

Anosov flows [1], which have been studied extensively since the 1960s, are arguably the canonical examples of chaotic dynamical systems and the rate of mixing (decay of correlation) is one of the most important statistical properties. Nevertheless our knowledge of the rate of mixing of Anosov flows remains unsatisfactory. The study of the rate of mixing for hyperbolic systems goes back to the work of Sinai [38] and Ruelle [36] in the 1970s and plenty of results were obtained for maps during the subsequent years. However various results for flows have been established only recently and several basic questions remain open. Exponential mixing is interesting in its own right, it is an intrinsic property of a dynamical system which describes the rate at which initial information is lost, but also it is crucial for establishing other quantitative statistical properties and when working on more intricate models (prominently in nonequilibrium statistical mechanics, e.g., in questions of energy transport [22]).

Let $\phi^t : \mathcal{M} \to \mathcal{M}$ be an Anosov flow on $\mathcal{M}$, a smooth compact connected Riemannian manifold. That ϕ^t is Anosov means that there exists a ϕ^t-invariant continuous splitting of the tangent space, $T\mathcal{M} = \mathbb{E}_s \oplus \mathbb{E}_0 \oplus \mathbb{E}_u$ where $\mathbb{E}_0$ is the line bundle tangent

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to the flow, $\mathbb{E}_s$ is the stable bundle in which there is exponential contraction and $\mathbb{E}_u$ is the unstable bundle in which there is exponential expansion. It is known that each transitive Anosov flow admits a unique SRB measure which will be denoted μ (see [43] for extensive information on SRB measures). This invariant measure is most relevant from the physical point of view.

The focus of this text is to prove exponential mixing with respect to the SRB measure. By *exponential mixing* we mean the existence of $C, \gamma > 0$ such that

$$\left| \int_{\mathcal{M}} f \cdot g \circ \phi^t \, d\mu - \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu \right| \\ \leq C \|f\|_{\mathcal{C}^1} \|g\|_{\mathcal{C}^1} e^{-\gamma t} \quad \text{for all } f, g \in \mathcal{C}^1(\mathcal{M}, \mathbb{R}) \text{ and all } t \geq 0.$$

(An approximation argument shows that exponential mixing for $\mathcal{C}^1$ observables implies exponential mixing for Hölder observables [21, proof of Corollary 1].) In the following we will use the expression *mixes exponentially* to mean with respect to the unique SRB measure for the flow, often without explicit mention of the measure.

Not all Anosov flows mix exponentially, indeed those which are constant time suspensions over Anosov maps are not mixing.[1] One wonders if this degenerate case is the only way that Anosov flows can fail to mix exponentially or if other slower rates are possible. Taking a suspension over an Anosov diffeomorphism is one way to construct Anosov flows but not all Anosov flows are of this type. The geodesic flow of any compact Riemannian manifold of strictly negative curvature is an Anosov flow, and these were a major motivation at the beginning of the study of Anosov flows. Some initial progress was made by proving exponential mixing for geodesic flows in the case of constant curvature and low dimension (see the introduction of [32] for details and further references), but these methods, which are group-theoretical in nature, were not suitable for adaption to the general case of variable curvature, let alone for Anosov flows which are not geodesic flows.

In the late 1990s a major advance was made by Dolgopyat [21] who, building on the dynamical argument introduced by Chernov [18], showed that transitive Anosov flows with $\mathcal{C}^1$ stable and unstable bundles mix exponentially whenever the stable and unstable bundles are not jointly integrable.[2] In particular this means that geodesic flows on surfaces of negative curvature mix exponentially (in this special case the regularity of the bundle is a consequence of the low dimension and the preserved contact structure which exists naturally for geodesic flows). However, a question of foremost importance is to show that statistical properties hold for an open and dense set of systems and the problem here is that the requirement of regularity for both bundles simultaneously is not typically satisfied for Anosov flows [28]. Both stable and unstable foliations are always Hölder but

[1] Suspensions over Anosov diffeomorphisms by a return time that is cohomologous to a constant are not mixing either but these can always be written as constant time suspensions.

[2] A subbundle $\mathbb{E} \subset T\mathcal{M}$ is said to be *integrable* if, for each point $p \in \mathcal{M}$, there exists an immersed submanifold $S \subset \mathcal{M}$ which contains p and such that $\mathbb{E}(q) = T_q S$ for all $q \in S$.

the regularity cannot in general be expected to be better than Hölder: a generic smooth perturbation[3] will destroy the Lipschitz regularity of at least one of the foliations.[4]

If a flow preserves a contact form then it is said to be a *contact flow*. Liverani [32] showed that all contact (with $\mathcal{C}^2$ contact form) Anosov flows mix exponentially with no requirement on the regularity of the stable and unstable bundles. This provides a complete answer for geodesic flows on manifolds of negative curvature since all such geodesic flows are contact Anosov flows with smooth contact form.[5] Liverani's requirement of a $\mathcal{C}^2$ contact form has two important consequences: firstly it guarantees that $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable and this is a property which is robust under perturbation; secondly, the smoothness of the contact form guarantees the smoothness of the subbundle $\mathbb{E}_s \oplus \mathbb{E}_u$ and the smoothness of the temporal function [32, Figure 2]. This smoothness is essential to Liverani's argument. Unfortunately the existence of a $\mathcal{C}^2$ contact form cannot be expected to be preserved by perturbations of the Anosov flow (the consequences of the existence of a smooth contact structure would contradict the prevalence of foliations with bad regularity which was mentioned above).

In the case of Axiom A flows[6] there exist flows which are mixing but mix arbitrarily slowly [37]. These are constructed as suspensions over Axiom A maps with piecewise constant (but not constant) return time and consequently are not Anosov flows. It would be interesting to understand if this phenomenon can only exist in the Axiom A case and not for Anosov flows.

The Bowen–Ruelle conjecture states that every mixing Anosov flow mixes exponentially. At the present moment this conjecture remains wide open: there is a substantial distance between the above discussed results and the statement of the conjecture. One obvious possibility of proceeding is to separate this conjecture into two separate conjectures: (A) if an Anosov flow is mixing then $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable; (B) a transitive Anosov flow mixes exponentially whenever $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable. A related, but seemingly slightly easier problem is to understand whether exponential mixing is an open and dense property for Anosov flows. Statement (A) was proved by Plante [34, Theorem 3.7] under the additional assumption that the Anosov flow is codimension one[7] but the general statement remains an open conjecture. Our main aim is to show statement (B) in the greatest generality possible, i.e., to show exponential mixing under the assumption that $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable. The question of exponential mixing continues to be of significant importance, beyond the (rather special) setting of Anosov flows. In particular it would be

[3] Here and in the following, by perturbation of the flow we mean a $\mathcal{C}^r$ $(r \geq 1)$ perturbation of the vector field associated to the flow. The structural stability of Anosov flows means that such a perturbed vector field (under a small perturbation) also defines an Anosov flow.

[4] Stoyanov [39] obtained results similar to Dolgopyat [21] for Axiom A flows but, among other assumptions, required that local stable and unstable laminations are Lipschitz.

[5] Not every contact Anosov flow is a geodesic flow on a Riemannian manifold, for example the flows constructed by Foulon & Hasselblatt [26].

[6] Axiom A flows are a generalization of Anosov flows, they are uniformly hyperbolic but the maximal invariant set is permitted to be a proper subset of the underlying manifold (for further details see e.g., [10]).

[7] An Anosov flow is said to be *codimension one* if $\dim \mathbb{E}_s = 1$ or $\dim \mathbb{E}_u = 1$.

easily argued that, from the physical point of view (e.g. for the multitudes of uniformly hyperbolic billiard flows [19]), discontinuities are natural. In such situations part (B) of the conjecture is the important part.[8] Given the Axiom A examples mentioned above, it would be surmised that part (A) is a peculiarity of the special properties of Anosov flows. The main advance to date for flows with discontinuities is the work of Baladi, Demers & Liverani [7] which proves exponential mixing for Sinai billiard flows (three-dimensional) and, as in the work mentioned above, their argument uses crucially the contact structure which is present in such billiard flows.

Major progress on exponential mixing for flows was made recently by Tsujii [41] who demonstrated the existence of a $\mathcal{C}^3$-open and $\mathcal{C}^r$-dense subset of volume-preserving three-dimensional Anosov flows which mix exponentially. Interestingly the set Tsujii constructs does not contain flows which have $\mathcal{C}^1$ stable and unstable bundles (and consequently does not contain flows which preserve a $\mathcal{C}^2$ contact form). In some sense the new ideas introduced in his work are the main recent advance towards settling the Bowen–Ruelle conjecture. One of the consequences of the present text is that in certain higher-dimensional settings the result analogous to Tsujii's can, to some extent, be proved rather more easily.

It is enlightening to consider the three-dimensional case in more detail. As mentioned above, it is known [21, 32] that any contact Anosov flow (and hence any geodesic flow of a negatively curved surface) mixes exponentially. Tsujii [41] uses the expression "twist of the stable subbundle along pieces of unstable manifolds" to describe the geometric mechanism which produces exponential mixing for flows. For contact Anosov flows a key part of the argument, and one which is clear in the work of Liverani [32], is to use the contact structure to guarantee that (in the language of Tsujii) moving along the unstable manifold a prescribed distance guarantees a uniform amount of twist of the stable subbundle. On the other hand, Tsujii uses the fact that the twist "will be 'random' and 'rough' in generic cases". The core of our work in this paper will be to study the flows by quotienting along stable manifolds. We will then take advantage of a twist in the sense discussed above, but since we have already quotiented, we will not distinguish between the two different cases.[9]

Given the evidence currently available it is reasonable to conjecture that (B) is true, i.e., transitive Anosov flows mix exponentially whenever $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable. However, a complete solution of this problems appears to be difficult and the path in this direction is not clear. It is also reasonable to hope that the above holds more generally and that uniformly hyperbolic flows (with discontinuities permitted) mix exponentially whenever $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable (assuming sufficient structure such that integrability of this bundle has a meaning). One of the motivations behind the present work is to bet-

[8] In some settings (e.g., symbolic systems) it is not clear that the notion of integrability (or non-integrability) of $\mathbb{E}_s \oplus \mathbb{E}_u$ always makes sense. However, for Axiom A attractors, since unstable disks are contained within the maximal invariant set, the notion is fine and corresponds to the existence of a foliation of a neighbourhood of the attractor [2, §3]. Another relevant direction is to consider dispersing billiard flows in the presence of a small external field.

[9] In practice we will consider the picture with stable and unstable exchanged but this seems to be merely a preference and not significant when studying Anosov flows.

ter understand and enlarge the set of Anosov flows which are known to be exponentially mixing in order to eventually improve our understanding of the general case.

At this point it is worth noting that the mechanism behind the exponential mixing of Anosov flows is also important in some partially hyperbolic maps (see e.g. [20, Appendix C]) and is essential in semiclassical analysis (see e.g. [24]).

Our first result concerns exponential mixing under relatively weak regularity assumptions.

Theorem 1. *Suppose that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a transitive $\mathcal{C}^{1+}$ Anosov flow*[10] *and that the stable bundle is $\mathcal{C}^{1+}$. If the stable and unstable bundles are not jointly integrable, then ϕ^t mixes exponentially with respect to the unique SRB measure.*

This result improves the result of Dolgopyat [21] since regularity is only required for the stable bundle whereas in the cited work it was required for both bundles. Although this change is small when measured in terms of the number of characters altered in the statement, we have to redo the proof in a somewhat different fashion (even though the essential ideas are the same). More to the point, the improvement over Dolgopyat's previous result is substantial in terms of the advantage it gives in finding open sets of exponentially mixing flows. This is illustrated by the following theorem.

Theorem 2. *Suppose that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}^{2+}$ volume-preserving Anosov flow and that $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \geq 2$. There exists a $\mathcal{C}^1$-neighbourhood of this flow such that for all $\mathcal{C}^{2+}$ Anosov flows in the neighbourhood, if the stable and unstable bundles are not jointly integrable, then the flow mixes exponentially with respect to the unique SRB measure.*

Since the set of Anosov flows where the stable and unstable bundles are not jointly integrable is $\mathcal{C}^1$-open and $\mathcal{C}^r$-dense in the set of all Anosov flows (see [25] and references therein concerning the prior work of Brin), the above theorem implies a wealth of open sets of exponentially mixing Anosov flows. To the best of our knowledge, this is the first proof of the existence of open sets of Anosov flows which mix exponentially (observe that the neighbourhood in the statement of the theorem, although centred on a volume-preserving flow, is a neighbourhood in the set of all Anosov flows). Similarly the set of Anosov flows where the stable and unstable bundles are not jointly integrable is $\mathcal{C}^1$-open and $\mathcal{C}^r$-dense in the set of volume-preserving Anosov flows.[11] This means that an open and dense subset of the volume-preserving Anosov flows such that $\dim \mathbb{E}_s = 1$ and

[10] For any $k \in \mathbb{N}$, the notation $\mathcal{C}^{k+}$ means $\mathcal{C}^{k+\alpha}$ for some $\alpha \in (0, 1]$. That a flow is $\mathcal{C}^{k+}$ is shorthand for requiring that the map $\mathcal{M} \times \mathbb{R} \to \mathcal{M}$; $(x, t) \mapsto \phi^t x$ is $\mathcal{C}^{k+}$.

[11] Consider a volume-preserving Anosov flow and assume that $\mathbb{E}_s \oplus \mathbb{E}_u$ is integrable. There exists a section such that the flow can be described as a suspension with constant return time. We will perturb the flow by smoothly modifying the magnitude of the associated vector field in a small ball. Following [25] we can do this so as to guarantee that, for the perturbed flow, $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable. Note that the perturbed system is still Anosov and as smooth as before. Since we only changed the magnitude of the vector field, the cross-section remains a cross-section and the return map also remains unchanged. Consequently, we ensure that the perturbed flow also preserves a smooth volume.

$\dim \mathbb{E}_u \geq 2$ mix exponentially. The ideas used here and the application of Theorem 1 actually show exponential mixing for an even larger set of Anosov flows than stated in the above theorem, but further details concerning this are postponed to the remarks in Section 2.2 (in particular we can prove the same conclusions in many cases where $\dim \mathbb{E}_s > 1$).

Let us consider the particular case of four-dimensional volume-preserving flows $\phi^t : \mathcal{M} \to \mathcal{M}$. Since the flow is Anosov and four-dimensional, either $\dim \mathbb{E}_s = 1$ or $\dim \mathbb{E}_u = 1$. In the first case Theorem 2 applies directly. For the other case observe that the SRB measure for a volume-preserving Anosov flow is the preserved volume and consequently the SRB measure for the time reversed flow ϕ^{-t} is equal to the SRB measure for ϕ^t. Since $\int_{\mathcal{M}} f \cdot g \circ \phi^t \, d\mu = \int_{\mathcal{M}} f \circ \phi^{-t} \cdot g \, d\mu$ and since stable and unstable are swapped for the time reversed flow, we can again apply Theorem 2. Consequently, the above result implies the following statement: Suppose that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}^{2+}$ four-dimensional volume-preserving Anosov flow. Then, if the stable and unstable bundles are not jointly integrable, the flow mixes exponentially with respect to the volume. In particular a $\mathcal{C}^1$-open and $\mathcal{C}^r$-dense subset of four-dimensional volume-preserving flows mix exponentially. This means that Tsujii's result holds in four dimensions. As discussed above, Plante demonstrated that mixing implies that $\mathbb{E}_s \oplus \mathbb{E}_u$ is not integrable in the codimension-one case. Consequently, the results of this paper provide a complete resolution of the Bowen–Ruelle conjecture in the volume-preserving four-dimensional case.

Remark. The proof of Theorem 2 requires the flow to be transitive in order to apply Theorem 1. However, due to Verjovsky [42], codimension-one Anosov flows on higher-dimensional manifolds ($\dim \mathcal{M} > 3$) are transitive and so transitivity is automatic[12] in the case of Theorem 2.

Section 2 contains the proof of Theorem 1 and the details of how Theorem 2 is derived from it. The proof of the first result rests heavily on a result (Theorem 3 below) concerning exponential mixing for $\mathcal{C}^{1+}$ expanding semiflows. Our motivation for proving Theorem 3 was proving Theorem 1, but Theorem 3 is also of interest in its own right. Details concerning past work on similar questions follow after we precisely introduce the setting.

We observe that the ideas in this text are very much limited to the argument presented here and will not suffice to fully answer the question of when in general Anosov flows mix exponentially. For this, we hope that the work of Dolgopyat [21], Liverani [32], Baladi & Vallée [8] and Tsujii [41] (among others) can eventually be extended and improved.

We proceed to define the class of $\mathcal{C}^{1+}$ expanding semiflows. Firstly we require two pieces of information concerning the geometry of the set in question. Let X be the disjoint union of a finite number of connected bounded open subsets of $\mathbb{R}^d$ (we use the convention that the distance between two points in different connected components is infinite).

[12] In the case where both the stable and unstable bundles are at least two-dimensional there are examples of non-transitive Anosov flows [27]. Also, as remarked in [27], in the three-dimensional case, where Verjovsky's proof does not work, this question of transitivity of Anosov flows remains open.

Definition 1.1. We say that $X \subset \mathbb{R}^d$ is *almost John* if there exist constants $C, \epsilon_0 > 0$ and $s \geq 1$ such that, for all $\epsilon \in (0, \epsilon_0)$ and all $x \in X$, there exists $y \in X$ such that $\mathrm{d}(x, y) \leq \epsilon$ and the ball centred at y of radius $C\epsilon^s$ is contained in X.[13]

We will always assume that X is almost John and that the boundary of X has upper box dimension strictly less than d. Let $T : X \to X$ denote a *uniformly expanding* $\mathcal{C}^{1+}$ *Markov map*, by which we mean that there exists a finite partition $\mathcal{P}$ of a full measure subset of X into connected open sets such that, for each $\omega \in \mathcal{P}$, T is a $\mathcal{C}^1$ diffeomorphism from ω to $T\omega$ and $T\omega$ is a full measure subset of one of the connected components of X.[14]

Remark. The conditions on X would be satisfied if the boundary of X were a finite union of $\mathcal{C}^1$ submanifolds. However, in view of the intended application, we must allow lower regularity of the boundary since such low regularity is the unfortunate reality for Markov partitions [12].

We require that there exist $C_1, \lambda > 0$ such that

$$\|(DT^n(x))^{-1}\| \leq C_1 e^{-\lambda n} \qquad \text{for all } x \in X \text{ and } n \in \mathbb{N}, \tag{1}$$

and there exist $C_2 > 0$ and $\alpha \in (0, 1)$ such that

$$\left| \ln \frac{\det(DT(x))}{\det(DT(y))} \right| \leq C_2\, \mathrm{d}(Tx, Ty)^\alpha \qquad \text{for all } x, y \in \omega \text{ and } \omega \in \mathcal{P}. \tag{2}$$

We also require T to be covering in the sense that for every open ball $B \subset X$ there exists $n \in \mathbb{N}$ such that $T^n B = X$ (modulo a zero measure set). For such maps it is known that there exists a unique T-invariant probability measure absolutely continuous with respect to Lebesgue measure. We denote this measure by ν. The density of ν is Hölder (on each partition element) and bounded away from zero. Let $\tau : X \to \mathbb{R}_+$ denote the *return time function*. We require that τ is $\mathcal{C}^{1+\alpha}$, that there exists $C_3 > 0$ such that[15]

$$\|D\tau(x) DT(x)^{-1}\| \leq C_3 \qquad \text{for all } x \in \omega \text{ and } \omega \in \mathcal{P}, \tag{3}$$

and that there exists $C_4 > 0$ such that

$$\tau(x) \leq C_4 \qquad \text{for all } x \in \omega \text{ and } \omega \in \mathcal{P}. \tag{4}$$

The suspension semiflow $T_t : X_\tau \to X_\tau$ is defined as usual, $X_\tau := \{(x, u) : x \in X, 0 \leq u < \tau(x)\}$ and $T_t : (x, u) \mapsto (x, u + t)$ modulo the identifications $(x, \tau(x)) \sim (Tx, 0)$. The unique absolutely continuous T_t-invariant probability measure[16] is denoted by ν_τ.

[13] This condition on X is similar in spirit to the requirement of a John domain as used in [6]. However, they are not equivalent; in our case we need only weaker properties and so we can make do with weaker assumptions. See the discussion in the Appendix for further details.

[14] That is, the map is required to be Markov but it is not necessarily full-branch.

[15] In our setting, (3) could be simplified but we choose to write it like this because it corresponds to $D(\tau \circ \ell)$ where ℓ is an inverse branch of T.

[16] $\nu_\tau(f) = \frac{1}{\nu(\tau)} \int_X \int_0^{\tau(x)} f(x, u)\, du\, d\nu(x)$.

Baladi and Vallée [8] showed that semiflows similar to the above, but with the $\mathcal{C}^2$ version of assumptions, typically mix exponentially when X is one-dimensional. The same argument was shown to hold by Avila, Gouëzel & Yoccoz [6], again in the $\mathcal{C}^2$ case, irrespective of the dimension of X. Recently Araújo & Melbourne [3] showed that the argument still holds in the $\mathcal{C}^{1+}$case when X is one-dimensional. This evidence means that the following result is not unexpected.

Theorem 3. *Suppose that $T_t : X_\tau \to X_\tau$ is a uniformly expanding $\mathcal{C}^{1+}$ suspension semi-flow as above. Then either τ is cohomologous to a piecewise constant function or there exist $C, \gamma > 0$ such that, for all $f, g \in \mathcal{C}^1(X_\tau, \mathbb{R})$ and $t \geq 0$,*

$$\left| \int_{X_\tau} f \cdot g \circ T_t \, d\nu_\tau - \int_{X_\tau} f \, d\nu_\tau \int_{X_\tau} g \, d\nu_\tau \right| \leq C \|f\|_{\mathcal{C}^1} \|g\|_{\mathcal{C}^1} e^{-\gamma t}.$$

The proof of the above is the content of Section 3. The estimate for exponential mixing relies on estimates of the norm of the twisted transfer operator given in Proposition 3.16. In some sense Proposition 3.16 is the main result of this part of the paper and the exponential mixing which we use here is merely one consequence of it. For many other applications, for example, to other statistical properties or to the study of perturbations, the extra information contained in the functional-analytic result is key. However, we avoid giving the statement here because it relies on a significant amount of notation which is yet to be introduced.

The argument of Araújo & Melbourne [4] follows closely the argument of Avila, Gouëzel & Yoccoz [6], which in turn follows closely the argument of Baladi & Vallée [8]. Everything suggests that exactly this argument could be used with minor modifications in order to prove Theorem 3. That the structure of the proof contained in Section 3 is superficially rather different is merely due to the aesthetic opinion of the present authors.

2. Anosov flows

This section is devoted to the proof of Theorems 1 and 2. The proof of Theorem 1 relies crucially on Theorem 3. The proof of Theorem 2 relies crucially on Theorem 1.

2.1. *Proof of Theorem 1*

Suppose that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}^{1+\alpha}$ Anosov flow and that the stable bundle is $\mathcal{C}^{1+\alpha}$ for some $\alpha > 0$. The proof is based (as per [5, 2, 3]) on quotienting along local stable manifolds and reducing the problem to the study of the corresponding expanding suspension semiflow. We then use the estimate which is given by Theorem 3.

The argument is the same idea as used previously [2] for Axiom A flows,[17] the only difference being that some of the estimates are now Hölder and not $\mathcal{C}^1$ since here we have

[17] Lemma 5 in [2] contains an inaccuracy: it is claimed that the domain of the uniformly expanding map is a $\mathcal{C}^2$ disk whereas in reality it is a subset of such a disk but with a boundary of poor smoothness.

a merely $\mathcal{C}^{1+\alpha}$ stable bundle whereas in [2] the bundle is $\mathcal{C}^2$. One important element in this argument is the regularity of the boundary of the elements of the Markov partition. The Appendix is devoted to further details concerning the construction and various important estimates which will be required, in particular estimates concerning the boundary of elements of the partition.

We recall that Bowen [9] constructed Markov partitions for Axiom A diffeomorphisms and then extended this construction to Axiom A flows, in particular Anosov flows [10]. Ratner [35] also constructed Markov partitions for Anosov flows, again based on Bowen's work. We will take Ratner's description of the construction as our primary reference since several parts of that presentation are more amenable to our present purposes.

The main idea is that we can find a section which consists of a family of local sections which are $\mathcal{C}^{1+\alpha}$ and foliated by local stable manifolds. The return map is a uniformly hyperbolic Markov map on the family of local sections [10]. Let Y denote the union of the local sections and let $S : Y \to Y$ and $\tau : Y \to \mathbb{R}_+$ denote the return map and return time for ϕ^t to this section. Let η denote the unique SRB measure for $S : Y \to Y$. Note that τ is constant along the local stable manifolds [2, §3].

We now quotient along the local stable manifolds (within the local sections) letting $\pi : Y \to X$ denote the quotient map. Consequently, we obtain a map $T : X \to X$ such that $T \circ \pi = \pi \circ S$. Since the original flow is Anosov (in particular has an attractor), the set X is the finite union of connected components. Each connected component is a subset of a $\mathcal{C}^{1+\alpha}$ submanifold of the same dimension as the unstable bundle. However, the boundary of these components, viewed as a subset of this submanifold, cannot be expected to be smooth [12].

That the assumptions on X which are required by Theorem 3 are satisfied is shown in Section A.1 and Lemma A.3. Because of the properties of S (in particular due to the use of the Markov partition in the above construction), the map T is a uniformly expanding Markov map and satisfies conditions (1)–(4). Therefore, by Theorem 3, either the suspension semiflow T_t mixes exponentially or τ is cohomologous to a constant function. In the latter case the stable and unstable bundles are jointly integrable [2, Lemma 12], so for the rest of the proof we suppose that τ is not cohomologous to a constant function and hence T_t mixes exponentially.

Let ν denote the unique SRB measure for T ($\nu = \pi_* \eta$). To proceed we observe that ν admits a disintegration into conditional measures along local stable manifolds. We observe using [17] that there exists a family $\{\nu_x\}_{x \in X}$ of conditional measures (with ν_x supported on $\pi^{-1}x$) such that

$$\eta(v) = \int_X \nu_x(v)\, d\eta(x)$$

for all continuous functions $v : Y \to \mathbb{R}$. We also know that this disintegration has good regularity in the sense that $x \mapsto \nu_x(v)$ is Hölder on each partition element and has uniformly bounded Hölder norm for any Hölder $v : Y \to \mathbb{R}$ [17, Proposition 6].

Let Y_τ, S_t, η_τ be defined analogously to X_τ, T_t, ν_τ. Suppose $u, v : Y_\tau \to \mathbb{R}$ are Hölder continuous functions. Points in Y are denoted by (x, a) according to the product representation of Y as X times the local stable manifolds. To prove that S mixes exponentially, it is convenient to write

$$\int_{Y_\tau} u \cdot v \circ S_{2t} \, d\eta_\tau = \int_{Y_\tau} u \cdot (v \circ S_t - v_t \circ \pi_\tau) \circ S_t \, d\eta_\tau + \int_{X_\tau} \tilde{u} \cdot v_t \circ T_t \, d\nu_\tau \tag{5}$$

where $\tilde{u}, v_t : X_\tau \to \mathbb{R}$ are defined as

$$\tilde{u}(x,a) := \int_{\pi^{-1}x} u(y,a) \, d\nu_x(y), \qquad v_t(x,a) := \int_{\pi^{-1}x} v \circ S_t(y,a) \, d\nu_x(y).$$

The new observables $\tilde{u}$ and v_t are $\mathcal{C}^\alpha$ on each partition element as observed above. To estimate the first term on the right hand side of (5) we observe that

$$(v \circ S_t - v_t \circ \pi_\tau)(y,u) = \int_{\pi^{-1}(\pi y)} [v \circ S_t(y,u) - v \circ S_t(z,u)] \, d\nu_{\pi y}(z).$$

Consequently, the function v_t is exponentially close to $v \circ S_t$ on each local stable manifold and so

$$\left| \int_{Y_\tau} u \cdot (v \circ S_t - v_t \circ \pi_\tau) \circ S_t \, d\eta_\tau \right| \le C \|u\|_{\mathcal{C}^\alpha} \|v\|_{\mathcal{C}^\alpha} e^{-\tilde{\gamma} t} \tag{6}$$

where $\tilde{\gamma} > 0$ depends on the contraction rate on the stable bundle.

The second term on the right hand side of (5) is estimated using Theorem 3 which says that T_t mixes exponentially since τ is not cohomologous to a piecewise constant. We have

$$\left| \int_{X_\tau} \tilde{u} \cdot v_t \circ T_t \, d\nu_\tau - \int_{X_\tau} \tilde{u} \, d\nu_\tau \cdot \int_{X_\tau} v_t \, d\nu_\tau \right| \le C \|\tilde{u}\|_{\mathcal{C}^\alpha} \|v_t\|_{\mathcal{C}^\alpha} e^{-\gamma t}. \tag{7}$$

Using estimates (6) and (7) in (5) shows that the flow $S_t : Y_\tau \to Y_\tau$ mixes exponentially. This in turn implies that the flow ϕ^t is exponentially mixing.

2.2. *Proof of Theorem 2*

The proof consists in showing that if ϕ^t is $\mathcal{C}^1$-close to a volume-preserving flow and if $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \ge 2$ then the stable bundle is $\mathcal{C}^{1+}$. We then apply Theorem 1.

We recall that the regularity of the invariant bundle of an Anosov flow is given by Hirsch, Pugh & Shub [29] (see also [4, Theorem 4.12]) under the following *bunching* condition. Suppose that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a $\mathcal{C}^{2+}$ Anosov flow.[18] If there exist $t, \alpha > 0$ such that

$$\sup_{x \in \mathcal{M}} \|D\phi^t|_{\mathbb{E}_s}(x)\| \, \|D\phi^t|_{\mathbb{E}_{cu}}^{-1}(x)\| \, \|D\phi^t|_{\mathbb{E}_{cu}}(x)\|^{1+\alpha} < 1, \tag{8}$$

then the stable bundle is $\mathcal{C}^{1+\alpha}$ ($\mathbb{E}_{cu} = \mathbb{E}_u \oplus \mathbb{E}_0$ is called the *central unstable subbundle*).

Following Plante [34, Remark 1], we observe that if the Anosov flow is volume-preserving, and $\dim \mathbb{E}_s = 1$ and $\dim \mathbb{E}_u \ge 2$, then the above bunching condition holds

[18] This is the only place where the flow is required to be $\mathcal{C}^{2+}$, everywhere else $\mathcal{C}^{1+}$ suffices.

true and consequently the stable bundle is $\mathcal{C}^{1+\alpha}$ for some $\alpha > 0$. This is because volume-preserving means that the contraction in $\mathbb{E}_s$ must equal the volume expansion in $\mathbb{E}_u$. Since $\dim \mathbb{E}_u \geq 2$ the maximum expansion in any given direction must be dominated by the contraction. Consequently, the stable bundle is $\mathcal{C}^{1+\alpha}$. From its definition the bunching condition (8) is robust under $\mathcal{C}^1$ perturbations of the Anosov flow.

Remark. This argument for the robust regularity of the stable bundle crucially uses the fact that the unstable bundle has dimension at least 2 whilst the stable bundle has dimension 1. Such an argument is therefore impossible if the Anosov flow is three-dimensional (see [34] for a counterexample). Of course regular bundles are possible in the three-dimensional case but not in a robust way.

Remark. In general, when $\dim \mathbb{E}_s < \dim \mathbb{E}_u$ it is again possible to find open sets such that the bunching condition is satisfied although this will not be possible for all such flows. A natural assumption to add would be isotropy of the hyperbolicity, i.e., the expansion is of equal strength in all directions and similarly for the contraction. In this case we can again obtain (8) robustly and prove the analog of Theorem 2.

Remark. In higher dimensions, with a large difference between the dimensions of the stable and unstable bundles, it is sometimes possible to obtain stronger bunching and therefore to guarantee that the stable bundle is $\mathcal{C}^2$ in a robust way. In this case results for $\mathcal{C}^2$ expanding semiflows [6] can be applied with the same argument as in this paper and exponential mixing proved for the flow [2]. A substantial part of this paper is to prove Theorem 3 which generalizes prior work to the higher-dimensional $\mathcal{C}^{1+}$ case. This is required to be able to handle a significantly larger set of Anosov flows, in particular to handle flows in dimension 4 and higher when $\dim \mathbb{E}_s = 1$.

3. Expanding semiflows

This section is devoted to the proof of Theorem 3. Throughout the section we suppose the setting of the theorem. Recall that the semiflow is a combination of a uniformly expanding map $T : X \to X$ and return time $\tau : X \to \mathbb{R}_+$. Let m denote Lebesgue measure on X. We will assume, by scaling if required, that the diameter of X is not greater than 1 and that $m(X) \leq 1$. We will also assume that $C_1 = 1$ in assumption (1). Suppose that this is not the case originally; then there exists some iterate such that $C_1 e^{-\lambda n} < 1$. We choose some partition element such that returning to this element takes at least n iterates. We take $\tilde{X}$ (which will replace X) to be equal to this partition element and choose for $\tilde{T}$ the first return map to $\tilde{X}$. The new return time τ is given by the corresponding sum of the return time. There is then a one-to-one correspondence between the new suspension semiflow and the original. It is simply a different choice of coordinates for the flow which has the effect that the expansion per iterate is increased and the return time increases correspondingly. This is not essential but it is convenient because below we can choose a constant cone field which is invariant. We will also assume for notational simplicity that $C_4 \leq 1$, i.e., $\tau(x) \leq 1$ for all x. This can be done without loss of generality, simply by scaling uniformly in the flow direction. Let $\Lambda > 0$ be such that $\|DT(x)\| \leq e^{\Lambda}$ for all x. This relates to the maximum possible expansion, whereas $\lambda > 0$ relates to the minimum expansion. In view of this discussion the suspension semiflow is controlled by the constants $\alpha \in (0, 1)$, $\Lambda \geq \lambda > 0$ and $C_2, C_3 > 0$.

Central to the argument of this section are Propositions 3.6, 3.9 and 3.16. The first describes how we see, in an exponential way, a key geometric property. The second proposition uses this geometric property and the idea of oscillatory integrals in order to see cancellations on average. The third proposition is the combination of the previous estimates to produce the key estimate on the norm of the twisted operators.

3.1. *Basic estimates*

Let $C_5 := 2C_3/(1 - e^{-\lambda})$, $\tau_n := \sum_{j=0}^{n-1} \tau \circ T^j$ and let $\mathcal{P}_n$ denote the n^{th} refinement of the partition. For convenience we will systematically use the notation $\ell_\omega := (T^n|_\omega)^{-1}$ for any $n \in \mathbb{N}$ and $\omega \in \mathcal{P}_n$. Let $J_n(x) = 1/\det(DT^n(x))$.

Lemma 3.1. *$\|D(\tau_n \circ \ell_\omega)(x)\| \leq \frac{1}{2} C_5$ for all $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$ and $x \in T^n\omega$.*

Proof. Let $y = \ell_\omega(x)$ and observe that

$$D(\tau_n \circ \ell_\omega)(x) = \sum_{k=0}^{n-1} D\tau(T^k y) D(T^k \circ \ell_\omega)(T^k y)$$

Consequently, using also (1) and (3), we see that $\|D(\tau_n \circ \ell_\omega)\| \leq C_3 \sum_{k=0}^{n-1} e^{-\lambda(n-k)}$. As $\sum_{k=0}^{\infty} e^{-\lambda k} = (1 - e^{-\lambda})^{-1}$, the required estimate holds. □

Lemma 3.2. *There exists $C_6 > 0$ such that, for all $n \in \mathbb{N}$ and $\omega \in \mathcal{P}_n$,*

$$\left| \ln \frac{\det(D\ell_\omega(x))}{\det(D\ell_\omega(y))} \right| \leq C_6 \, \mathrm{d}(x, y)^\alpha \quad \text{for all } x, y \in T^n\omega.$$

Proof. We write $\ell_\omega = g_1 \circ \cdots \circ g_n$ where each g_k is the inverse of T restricted to the relevant domain. Let $x_k = T^k \ell_\omega x$ and $y_k = T^k \ell_\omega y$. Consequently, $\det(D\ell_\omega(x)) = \prod_{k=1}^{n} \det(Dg_k(x_k))$ and so

$$\left| \ln \frac{\det(D\ell_\omega(x))}{\det(D\ell_\omega(y))} \right| \leq \sum_{k=1}^{n} \left| \ln \frac{\det(Dg_k(x_k))}{\det(Dg_k(y_k))} \right|.$$

Assumption (2) implies that $\left| \ln \frac{\det(Dg_k(x_k))}{\det(Dg_k(y_k))} \right| \leq C_2 \, \mathrm{d}(x_k, y_k)^\alpha$. Using also assumption (1) we obtain a bound $\sum_{k=1}^{n} C_2 (e^{-\lambda(n-k)})^\alpha \, \mathrm{d}(x, y)^\alpha$. To finish the proof let $C_6 := C_2 \sum_{j=0}^{\infty} e^{-\lambda \alpha j}$. □

Lemma 3.3. *There exists $C_7 > 0$ such that*

$$\sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \leq C_7 \quad \text{for all } n \in \mathbb{N}.$$

Proof. For each $\omega \in \mathcal{P}_n$ there exists some $x_\omega \in \omega$ such that $m(\omega) = J_n(x_\omega)m(T^n\omega)$. This means that $\sum_{\omega\in\mathcal{P}_n} J_n(x_\omega) \le m(x)(\inf_\omega m(T^n\omega))^{-1}$. By Lemma 3.2,

$$\|J_n\|_{L^\infty(\omega)}/J_n(x_\omega) \le e^{C_6}.$$

Consequently, $\sum_{\omega\in\mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \le C_7$ where $C_7 := e^{C_6}/\inf_\omega m(T^n\omega)$. □

3.2. *Twisted transfer operators*

For $z \in \mathbb{C}$, the *twisted transfer operator* $\mathcal{L}_z : L^\infty(X) \to L^\infty(X)$ is defined as

$$\mathcal{L}_z^n f = \sum_{\omega\in\mathcal{P}_n} (e^{-z\tau_n} \cdot f \cdot J_n) \circ \ell_\omega \cdot \mathbf{1}_{T^n\omega}.$$

We use the standard notation for the *Hölder seminorm* $|f|_{\mathcal{C}^\alpha(J)}$ where J is any metric space: $|f|_{\mathcal{C}^\alpha(J)}$ is the supremum of $C \ge 0$ such that $|f(x) - f(y)| \le C\,\mathrm{d}(x, y)^\alpha$ for all $x, y \in J$, $x \ne y$. The *Hölder norm* is defined to be $\|f\|_{\mathcal{C}^\alpha(J)} := |f|_{\mathcal{C}^\alpha(J)} + \|f\|_{L^\infty(J)}$. Recall that X is the disjoint union of a finite number of connected subsets of $\mathbb{R}^d$. In this case

$$|f|_{\mathcal{C}^\alpha(X)} := \sup_{x,y} \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)^\alpha}$$

where the supremum is taken over all distinct $x, y \in X$ which are in the same connected component. As before let $\|f\|_{\mathcal{C}^\alpha(X)} := |f|_{\mathcal{C}^\alpha(X)} + \|f\|_{L^\infty(X)}$. Let $\mathcal{C}^\alpha(X) := \{f : X \to \mathbb{R} : |f|_{\mathcal{C}^\alpha(X)} < \infty\}$. This is a Banach space when equipped with the norm $\|\cdot\|_{\mathcal{C}^\alpha(X)}$. Define, for all $b \in \mathbb{R}$, the equivalent norm

$$\|f\|_{(b)} := \frac{1}{1 + |b|^\alpha}|f|_{\mathcal{C}^\alpha(X)} + \|f\|_{L^\infty(X)}.$$

Observe that, by Lemma 3.3, $\|\mathcal{L}_z^n f\|_{L^\infty(X)} \le C_7 e^{-\Re(z)n}\|f\|_{L^\infty(X)}$ for all $n \in \mathbb{N}$ and $f \in L^\infty(X)$.

The argument of this section depends on choosing $\sigma > 0$ sufficiently small in a way which depends only on the system (X, T, τ). We suppose from now on that such a $\sigma > 0$ is fixed (sufficiently small) and the precise constraints on σ will appear at the relevant places in the following paragraphs.

Lemma 3.4. *There exists $C_8 > 0$ such that, for all $z = a + ib$ with $a > -\sigma$, and all $f \in \mathcal{C}^\alpha(X)$ and $n \in \mathbb{N}$,*

$$\|\mathcal{L}_z^n f\|_{\mathcal{C}^\alpha(X)} \le C_8 e^{-(\alpha\lambda-\sigma)n}|f|_{\mathcal{C}^\alpha(X)} + C_8 e^{\sigma n}(1 + |b|^\alpha)\|f\|_{L^\infty(X)}.$$

Proof. Suppose that $\omega \in \mathcal{P}_n$, $f \in \mathcal{C}^\alpha(X)$ and $x, y \in T^n\omega$, $x \ne y$. Then

$$(e^{-z\tau_n} \cdot f \cdot J_n)(\ell_\omega x) - (e^{-z\tau_n} \cdot f \cdot J_n)(\ell_\omega y) = A_1 + A_2 + A_3 + A_4$$

where

$$\begin{aligned}A_1 &= (e^{-ib\tau_n(\ell_\omega x)} - e^{-ib\tau_n(\ell_\omega y)})(e^{-a\tau_n} \cdot f \cdot J_n)(\ell_\omega x),\\ A_2 &= e^{-ib\tau_n(\ell_\omega y)}(e^{-a\tau_n(\ell_\omega x)} - e^{-a\tau_n(\ell_\omega y)})(f \cdot J_n)(\ell_\omega x),\\ A_3 &= e^{-z\tau_n(\ell_\omega y)}(f(\ell_\omega x) - f(\ell_\omega y)) \cdot J_n(\ell_\omega x),\\ A_4 &= e^{-z\tau_n(\ell_\omega y)} f(\ell_\omega y)(J_n(\ell_\omega x) - J_n(\ell_\omega y)).\end{aligned}$$

By Lemma 3.1, $|A_1| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x) 2 \min(|b|(C_5/2)\, \mathrm{d}(x, y), 1)$. Since $\min(u, 1) \leq u^\alpha$ for all $u \geq 0$, we have $|A_1| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x) 2|b|^\alpha ((C_5/2))^\alpha\, \mathrm{d}(x, y)^\alpha$. Again, by Lemma 3.1,

$$\begin{aligned}|A_2| &\leq e^{-a\tau_n(\ell_\omega x)} |1 - e^{-a(\tau_n(\ell_\omega y) - \tau_n(\ell_\omega x))}| (|f| \cdot J_n)(\ell_\omega x)\\ &\leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega x)|a|(C_5/2)\, \mathrm{d}(x, y).\end{aligned}$$

Using assumption (1) we find that $|A_3| \leq (e^{-a\tau_n} \cdot J_n)(\ell_\omega y) e^{-\alpha\lambda n}\, \mathrm{d}(x, y)^\alpha |f|_{\mathcal{C}^\alpha(\omega)}$. Finally, by Lemma 3.2, $|A_4| \leq (e^{-a\tau_n} \cdot |f| \cdot J_n)(\ell_\omega y) C_6\, \mathrm{d}(x, y)^\alpha$. Summing over $\omega \in \mathcal{P}_n$ we obtain

$$\begin{aligned}&\frac{|\mathcal{L}_z^n f(x) - \mathcal{L}_z^n f(y)|}{\mathrm{d}(x, y)^\alpha}\\ &\quad \leq \|\mathcal{L}_a^n 1\|_{L^\infty(X)} \big[\big((2|b|^\alpha + |a|) C_5/2 + C_6\big) \|f\|_{L^\infty(X)} + C_7 e^{-\lambda n} |f|_{\mathcal{C}^\alpha(X)} \big] \qquad (9)\end{aligned}$$

To finish the proof we observe that $\|\mathcal{L}_z^n f\|_{L^\infty(X)} \leq \|\mathcal{L}_\sigma^n 1\|_{L^\infty(X)} \|f\|_{L^\infty(X)}$ and $\|\mathcal{L}_\sigma^n 1\|_{L^\infty(X)} \leq C_7 e^{\sigma n}$ and choose C_8 according to (9). □

In view of the definition of the $\|\cdot\|_{(b)}$ norm, Lemma 3.4 implies the following uniform estimate.

Lemma 3.5. *For all $z = a + ib$ with $a > -\sigma$,*

$$\|\mathcal{L}_z^n f\|_{(b)} \leq C_8 e^{\sigma n} (e^{-\lambda n} \|f\|_{(b)} + \|f\|_{L^\infty(X)}) \quad \text{for all } f \in \mathcal{C}^\alpha(X) \text{ and } n \in \mathbb{N}.$$

3.3. Exponential transversality

The goal of this subsection is to prove Proposition 3.6 below. This is an extension of Tsujii's [40, Theorem 1.4] to the present higher-dimensional situation. Much of the argument follows the reasoning of the above mentioned reference with some changes due to the more general setting.

Define the $(d + 1)$-dimensional square matrix $\mathcal{D}^n(x) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ by

$$\mathcal{D}^n(x) = \begin{pmatrix} DT^n(x) & 0 \\ D\tau_n(x) & 1 \end{pmatrix}.$$

This is notationally convenient since $DT_t(x, s) = \mathcal{D}^n(x)$ whenever $\tau_n(x) \leq s + t < \tau_{n+1}(x)$.[19] To proceed it is convenient to introduce the notion of an invariant unstable

[19] If one wished to study the skew product $G : (x, u) \mapsto (Tx, u - \tau(x))$, this is also the relevant object to study since $\mathcal{D}^n = DG^n$.

cone field. Recall that $C_5 = 2C_3/(1 - e^{-\lambda})$. We define $\mathcal{K} \subset \mathbb{R}^{d+1}$ as

$$\mathcal{K} = \left\{ \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) : a \in \mathbb{R}^d,\ b \in \mathbb{R},\ |b| \leq C_5|a| \right\}.$$

We refer to $\mathcal{K}$ as a *cone*. We will see now that the cone has been chosen sufficiently wide to guarantee invariance. Note that

$$\begin{pmatrix} DT(x) & 0 \\ D\tau(x) & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} DT(x)a \\ D\tau(x)a + b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix}.$$

Let $\omega \in \mathcal{P}$ be such that $a = D\ell_\omega(Tx)a'$. Using conditions (1) and (3), we have

$$\begin{aligned} |b'| &= |D\tau(x)\, a + b| = |D(\tau \circ \ell_\omega)(Tx)\, a' + b| \\ &\leq C_3|a'| + C_5 e^{-\lambda}|a'| \leq \tfrac{1}{2}C_5|a'|. \end{aligned} \tag{10}$$

Suppose that $x_1, x_2 \in X$ and $n \in \mathbb{N}$ are such that $T^n x_1 = T^n x_2$. We write

$$\mathcal{D}^n(x_1)\mathcal{K} \pitchfork \mathcal{D}^n(x_2)\mathcal{K}$$

if $\mathcal{D}^n(x_1)\mathcal{K} \cap \mathcal{D}^n(x_2)\mathcal{K}$ does not contain a d-dimensional linear subspace. In such a case we say that the image cones are *transversal*.

Proposition 3.6. *Let $T : X \to X$ be a $\mathcal{C}^{1+}$ uniformly expanding Markov map and $\tau : X \to \mathbb{R}_+$ as above. Further suppose that there is no $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element. Then there exist $C_9, \gamma > 0$ such that, for all $y \in X$ and $x_0 \in T^{-n}y$,*

$$\sum_{\substack{x \in T^{-n}y \\ \mathcal{D}^n(x)\mathcal{K} \not\pitchfork \mathcal{D}^n(x_0)\mathcal{K}}} J_n(x) \leq C_9 e^{-\gamma n}. \tag{11}$$

The major part of the remainder of this subsection is devoted to the proof of this proposition, but first we record a consequence of transversality.

Lemma 3.7. *Suppose that $\omega, \varpi \in \mathcal{P}_n$, $y \in X$ and $\mathcal{D}^n(\ell_\omega y)\mathcal{K} \pitchfork \mathcal{D}^n(\ell_\varpi y)\mathcal{K}$. Then there exists a 1-dimensional linear subspace $L \subset \mathbb{R}^d$ such that, for all $v \in L$,*

$$|D(\tau_n \circ \ell_\omega)(y)v - D(\tau_n \circ \ell_\varpi)(y)v| > C_5(|D\ell_\omega(y)v| + |D\ell_\varpi(y)v|).$$

Proof. Let $x_1 = \ell_\omega y$ and $x_2 = \ell_\varpi y$. That $\mathcal{D}^n(x_1)\mathcal{K} \pitchfork \mathcal{D}^n(x_2)\mathcal{K}$ means there exists a line $L \subset \mathbb{R}^d$ which passes through the origin and such that, when restricted to the two-dimensional subspace $L \times \mathbb{R} \subset \mathbb{R}^{d+1}$, the image cones $\mathcal{D}^n(x_1)\mathcal{K}$ and $\mathcal{D}^n(x_2)\mathcal{K}$ fail to intersect, except at the origin. To see this, suppose this were false. Then, for all L, the cones restricted to $L \times \mathbb{R}$ intersect and the intersection contains a 1D subspace. We can do this for $\{L_k\}_{k=1}^d$ which are pairwise orthogonal. This yields a d-dimensional subspace in the intersection of the images of the cones, contradicting the assumed transversality.

Observe that

$$\mathcal{D}^n(x)\mathcal{K} \cap L \times \mathbb{R} = \left\{ \begin{pmatrix} DT^n(x)a \\ D\tau_n(x)a + b \end{pmatrix} : |b| \leq C_5|a|,\ DT^n(x)a \in L \right\}$$
$$= \left\{ \begin{pmatrix} v \\ D\tau_n(x)DT^{-n}(x)v + b \end{pmatrix} : v \in L,\ |b| \leq C_5|DT^{-n}(x)v| \right\}.$$

Consequently, $\mathcal{D}^n(x_1)\mathcal{K} \cap \mathcal{D}^n(x_2)\mathcal{K} \cap L \times \mathbb{R} = \{0\}$ implies

$$|[(D\tau_n(x_1)DT^{-1}(x_1) - (D\tau_n(x_2)DT^{-1}(x_2)]v| > C_5|DT^{-n}(x_1)v| + C_5|DT^{-n}(x_2)v|. \qquad \square$$

For all $n \in \mathbb{N}$, let

$$\phi(n) := \sup_{y \in X} \sup_{x_0 \in T^{-n}y} \sum_{\substack{x \in T^{-n}y \\ \mathcal{D}^n(x)\mathcal{K} \pitchfork \mathcal{D}^n(x_0)\mathcal{K}}} J_n(x).$$

Let h_ν denote the density of ν (the T-invariant probability measure). It is convenient to introduce the quantity

$$\varphi(n, P, y) := \sum_{\substack{x \in T^{-n}(y) \\ \mathcal{D}^n(x)\mathcal{K} \supset P}} J_n(x) \cdot \frac{h_\nu(x)}{h_\nu(y)}, \tag{12}$$

where $P \subset \mathbb{R}^{d+1}$ is a d-dimensional linear subspace. Let

$$\varphi(n) := \sup_y \sup_P \varphi(n, P, y).$$

The benefit of this definition is that $\varphi(n)$ is submultiplicative, i.e., $\varphi(n+m) \leq \varphi(n)\varphi(m)$ for all $n, m \in \mathbb{N}$; and $\varphi(n) \leq 1$ for all $n \in \mathbb{N}$. In order to prove Proposition 3.6 it suffices to prove the following lemma.

Lemma 3.8. *The following statements are equivalent:*

(i) $\liminf_{n\to\infty} \phi(n)^{1/n} = 1$*;*
(ii) $\lim_{n\to\infty} \varphi(n)^{1/n} = 1$*;*
(iii) *for all $n \in \mathbb{N}$ and $y \in X$ there exists a d-dimensional linear subspace $Q_n(y) \subset \mathcal{K}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n(y)$ for all y and all $x \in T^{-n}y$;*
(iv) *there exists $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element.*

Proof. (i)⇒(ii). Let $m_2 \in \mathbb{N}$ and $n = \lceil 2(\Lambda/\lambda)m_2 \rceil$. Since $\Lambda \geq \lambda$, we have $n > m_2$. Let $m_1 \in \mathbb{N}_+$ be such that $n = m_1 + m_2$. Let $P_n(x_1) := \mathcal{D}^n(x_1)(\mathbb{R}^d \times \{0\})$. We will first show that $\mathcal{D}^n(x_1)\mathcal{K} \pitchfork \mathcal{D}^n(x_2)\mathcal{K}$ implies $\mathcal{D}^{m_2}(T^{m_1}x_2)\mathcal{K} \supset P_n(x_1)$. Observe that

$$\mathcal{D}^n(x)\mathcal{K} = \left\{ \begin{pmatrix} a \\ D\tau_n(x)DT^{-n}(x)a + b \end{pmatrix} : a \in \mathbb{R}^d,\ b \in \mathbb{R},\ |b| \leq C_5|DT^{-n}(x)a| \right\}.$$

That transversality fails means that $P_n(x_1)$ (being contained in $\mathcal{D}^n(x_1)\mathcal{K}$) is close to the image cone $\mathcal{D}^n(x_2)\mathcal{K}$ by a factor of $C_5 e^{-\lambda n}$. We also know that $\mathcal{D}^{m_2}(T^{m_1}x_2)$ is sufficiently bigger than $\mathcal{D}^n(x_2)\mathcal{K}$ in the sense that

$$\mathcal{D}^{m_2}(T^{m_1}x_2) \supset \left\{ \begin{pmatrix} a \\ b_1 + b_2 \end{pmatrix} : \begin{pmatrix} a \\ b_1 \end{pmatrix} \in \mathcal{D}^n(x_2)\mathcal{K},\ |b_2| \le C_5 e^{-\lambda n}|a| \right\}.$$

To prove this let $a \in \mathbb{R}^d$ and $b_0, b_1, b_2 \in \mathbb{R}$ be such that $|b_0| \le C_5 |DT^{-n}(x_2)a|$, $b_1 = D\tau_n(x_2)DT^{-n}(x_2)a + b_0$ and $|b_2| \le C_5 e^{-\lambda n}|a|$. It will suffice to prove that

$$|(b_1 + b_2 - D\tau_{m_2}(T^{m_1}x_2)DT^{-m_2}(T^{m_1}x_2))a| \le C_5 |DT^{-m_2}(T^{m_1}x_2))a|.$$

We estimate

$$\begin{aligned}
&|(b_1 + b_2 - D\tau_{m_2}(T^{m_1}x_2)DT^{-m_2}(T^{m_1}x_2))a| \\
&\quad = |(b_0 + b_1 + D\tau_{m_1}(x_2)DT^{-m_1}(x_2))DT^{-m_2}(T^{m_1}x_2)a| \\
&\quad \le C_5\big(\tfrac{1}{2}|DT^{-m_2}(T^{m_1}x_2))a| + 2e^{-\lambda n}|a|\big) \\
&\quad \le C_5\big(2e^{-\lambda n}|a| - \tfrac{1}{2}|DT^{-m_2}(T^{m_1}x_2))a|\big) + C_5|DT^{-m_2}(T^{m_1}x_2))a| \qquad (13)
\end{aligned}$$

That $|DT^{-m_2}(T^{m_1}x_2))a| \ge e^{-\Lambda m_2} \ge e^{-\lambda n/2}$ means $\frac{1}{2}|DT^{-m_2}(T^{m_1}x_2))a| \ge 2e^{-\lambda n}|a|$ for n sufficiently large (depending only on λ and Λ). We therefore conclude that $P_n(x_1) \subset \mathcal{D}^{m_2}(T^{m_1}x_2)$. Suppose that $x_1 \in T^{-n}y$. Then

$$\begin{aligned}
\sum_{\substack{x_2 \in T^{-n}y \\ \mathcal{D}^n(x_2)\mathcal{K} \not\pitchfork \mathcal{D}^n(x_1)\mathcal{K}}} J_n(x_2) &\le \sum_{\substack{x_2 \in T^{-n}y \\ \mathcal{D}^{m_2}(T^{m_1}x_2)\mathcal{K} \supset P_n(x_1)}} J_{m_2}(T^{m_1}x_2) J_{m_1}(x_2) \\
&\le \sum_{\substack{x_3 \in T^{-m_2}y \\ \mathcal{D}^{m_2}(x_3)\mathcal{K} \supset P_n(x_1)}} J_{m_2}(x_3) \sum_{x_2 \in T^{-m_1}x_3} J_{m_1}(x_2).
\end{aligned}$$

Consequently, $\varphi(n) \le C\phi(m_2(n))$ where $m_2(n) = \left\lfloor \frac{n\lambda}{2\Lambda} \right\rfloor$ and $C = \sup_{x,y} f_\nu(x)/f_\nu(y)$.

(ii)⇒(iii). First observe that $\lim_{n\to\infty} \varphi(n)^{1/n} = 1$ implies $\varphi(n) = 1$ for all n since $\varphi(n)$ is submultiplicative and bounded by 1. Consequently, the following statement holds:

(iii′) For each n there exists $y_n \in X$ and a d-dimensional linear subspace $Q_n \subset \mathbb{R}^{d+1}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n$ for every $x \in T^{-n}(y_n)$.

It remains to prove that the above statement implies the following:

(iii) For all $n \in \mathbb{N}$ and $y \in X$ there exists a d-dimensional linear subspace $Q_n(y) \subset \mathcal{K}$ such that $\mathcal{D}^n(x)\mathcal{K} \supset Q_n(y)$ for all y and all $x \in T^{-n}y$.

We will prove the contrapositive. Suppose the negation of (iii), i.e., there exist $n_0 \in \mathbb{N}$, $y_0 \in X$ and $x_1, x_2 \in T^{-n_0}(y_0)$ such that $\mathcal{D}^{n_0}(x_1)\mathcal{K} \cap \mathcal{D}^{n_0}(x_2)\mathcal{K}$ does not contain a d-dimensional linear subspace. Let $\omega_1, \omega_2 \in \mathcal{P}_{n_0}$ be such that $x_1 = \ell_{\omega_1}y_0$ and $x_2 = \ell_{\omega_2}y_0$. These inverses are defined on some neighbourhood Δ containing y_0 and due to the openness related to the cones not intersecting we can assume that

$\mathcal{D}^{n_0}(\ell_{\omega_1}(y_0))\mathcal{K} \cap \mathcal{D}^{n_0}(\ell_{\omega_2}(y_0))\mathcal{K}$ does not contain a d-dimensional linear subspace for all $y \in \Delta$ (after shrinking Δ if required).

There exist $m_0 \in \mathbb{N}$ and $\varpi \in \mathcal{P}_{m_0}$ such that $\ell_\varpi X \subset \Delta$ (by the covering property of T). Observe that, for all $z \in X$,

$$\mathcal{D}^{n_0+m_0}(\ell_{\omega_1}(\ell_\varpi z))\mathcal{K} \subset \mathcal{D}^{m_0}(\ell_\varpi z)\mathcal{D}^{n_0}(\ell_{\omega_1} y)\mathcal{K}$$

where $y = \ell_\varpi z$ (and similarly for ω_2). This means that for all $z \in X$ there exist $x_1, x_2 \in T^{-(m_0+n_0)}(z)$ such that $\mathcal{D}^{m_0+n_0}(x_1)\mathcal{K} \cap \mathcal{D}^{m_0+n_0}(x_2)\mathcal{K}$ fails to contain a d-dimensional linear subspace and consequently contradicts (iii′).

(iii)⇒(iv). Let $(\omega_1, \omega_2, \ldots)$ be a sequence of elements of the partition $\mathcal{P}$. For each $n \in \mathbb{N}$ let $G_n := \ell_{\omega_n} \circ \cdots \circ \ell_{\omega_1}$. Consider

$$D(\tau_n \circ G_n)(x) = \sum_{k=1}^{n} D(\tau \circ \ell_{\omega_k})(G_{k-1}x)DG_{k-1}(x) \tag{14}$$

and observe that, by (3) and (1), this series converges uniformly. Moreover this limit is independent of the choice of a sequence of inverse branches. This is a consequence of (iii). Observe that

$$\mathcal{D}^n(x)\mathcal{K} = \left\{\begin{pmatrix} v \\ D\tau_n(x)DT^{-n}(x)v+b \end{pmatrix} : v \in \mathbb{R}^d,\ |b| \le C_5|DT^{-n}(x)v|\right\}.$$

Therefore, for all n, $y \in X$,

$$\|D\tau_n(x_1)DT^{-n}(x_1)v - D\tau_n(x_2)DT^{-n}(x_2)v\| \le 2C_5\|v\|\lambda^{-n}$$

for all $x_1, x_2 \in T^{-n}y$.

Consequently, we can denote by $\Omega(x)$ the limit of (14). Then, for all $\omega \in \mathcal{P}$,

$$\Omega(x) = D(\tau \circ \ell_\omega)(x) + \Omega(\ell_\omega x)D\ell_\omega(x).$$

Fix $x_0 \in X$. The function series $\sum_{n=1}^{\infty}(\tau \circ G_n - \tau \circ G_n(x_0))$ is summable in $\mathcal{C}^1$. Denote its sum by θ. By construction $\Omega(x) = D\theta(x)$. Consequently, $D(\tau + \theta - \theta \circ T) = 0$.

(iv)⇒(i). Let

$$Q(x) := \left\{\begin{pmatrix} a \\ D\theta(x)a \end{pmatrix} : a \in \mathbb{R}^d\right\}.$$

Observe that $Q(x) \subset \mathcal{K}$. Since $D\tau_n(x) = D\theta(T^n x)DT^n(x) - D\theta$,

$$\begin{aligned}\mathcal{D}^n(x)Q(x) &= \left\{\begin{pmatrix} DT^n(x) & 0 \\ D\tau_n(x) & 1 \end{pmatrix}\begin{pmatrix} a \\ D\theta(x)a \end{pmatrix} : a \in \mathbb{R}^d\right\} \\ &= \left\{\begin{pmatrix} DT^n(x)a \\ D\theta(T^n x)DT^n(x)a \end{pmatrix} : a \in \mathbb{R}^d\right\} = Q(T^n x).\end{aligned}$$

This means that for all $y \in X$ we have $\mathcal{D}^n(x)\mathcal{K} \supset Q(y)$ for all $x \in T^{-n}y$. Consequently, $\mathcal{D}^n(x)\mathcal{K} \not\pitchfork \mathcal{D}^n(x_0)\mathcal{K}$ whenever $x, x_0 \in T^{-n}y$. The required conclusion then follows from the definition of $\phi(n)$. □

3.4. Oscillatory cancellation

In this subsection we take advantage of the geometric property established above and estimate the resulting cancellations. The following estimate concerns the case when f is more or less constant on a scale of $|b|^{-1}$. The argument will depend on the following choice of constants (convenient but not optimal)

$$\beta_1 := \frac{2}{\lambda}, \quad \beta_2 := \frac{\alpha}{8\Lambda}, \quad q := \frac{\alpha\lambda}{2}.$$

Let $n_1 = \lfloor \beta_1 \ln |b| \rfloor$, $n_2 = \lfloor \beta_2 \ln |b| \rfloor$ and $n := n_1 + n_2$, $\beta := \beta_1 + \beta_2$. The first n_1 iterates will be so that the dynamics evenly spreads the function f across the space X. Then n_2 iterates will be to see the oscillatory cancellations. The assumptions of the following proposition are identical to the assumptions of Proposition 3.6.

Proposition 3.9. *Let $T : X \to X$ be a $\mathcal{C}^{1+}$ uniformly expanding Markov map and $\tau : X \to \mathbb{R}_+$ as above. Further suppose that there is no $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element.*

Then there exist $\xi > 0$, $b_0 > 1$ and $\sigma > 0$ such that, for all $z = a + ib$ with $a \in (-\sigma, \sigma)$ and $|b| > b_0$, for $n = \lfloor \beta \ln |b| \rfloor$, and for all $f \in \mathcal{C}^\alpha(X)$ satisfying $|f|_{\mathcal{C}^\alpha(X)} \le e^{qn} |b|^\alpha \|f\|_{L^\infty(X)}$, we have

$$\|\mathcal{L}_z^n f\|_{L^1(X)} \le e^{-\xi n} \|f\|_{L^\infty(X)}.$$

The proof is via several lemmas.

It is convenient to localize in space using a partition of unity. Using the assumption that the box-counting dimension of the boundary is strictly smaller than the ambient dimension we have the following partition of unity.

Lemma 3.10. *There exist $C_{10}, C_{11}, r_0 > 0$ and $d_1 \in [0, d)$ such that for all $r \in (0, r_0)$ there exists a set $\{x_p\}_{p=1}^{N_r}$ of points and a $\mathcal{C}^1$ partition of unity $\{\rho_p\}_{p=1}^{N_r}$ on X (i.e., $\sum_p \rho_p(x) = 1$ for all $x \in X$, and $\rho_p \in \mathcal{C}^1(X, [0, 1])$) with the following properties:*

- $N_r \le C_{10} r^{-d}$;

for each p,

- $\rho_p(x) = 1$ *for all* $x \in \mathrm{B}(x_p, r)$;
- $\mathrm{supp}(\rho_p) \subset \mathrm{B}(x_p, C_{10} r)$;
- $\|\rho_p\|_{\mathcal{C}^1} \le C_{10} r^{-1}$;

and, with $\mathcal{R}_\partial := \{p : \mathrm{B}(x_p, C_{10} r) \cap \partial\omega \ne \emptyset$ for some $\omega \in \mathcal{P}\}$,

- $\#\mathcal{R}_\partial \le C_{11} r^{-d_1}$.

The construction of such a partition of unity and the proof of the above estimates are given in Subsection A.1 of the Appendix.

At each point of X we have a direction in which we see cancellations. One major use of the partition of unity is to consider the direction as locally constant. We choose $r = r(b) = |b|^{-1/2}$. Take $f \in \mathcal{C}^\alpha(X)$. Using Jensen's inequality we get

$$\begin{aligned}
\|\mathcal{L}_z^n f\|_{L^1(X)} &= \int_X \Big| \sum_{\omega \in \mathcal{P}_n} (J_n \cdot f \cdot e^{-z\tau_n}) \circ \ell_\omega(x) \cdot \mathbf{1}_{T^n\omega}(x) \Big| \, dx \\
&= \sum_{p=1}^{N_r} \int \Big| \sum_{\omega \in \mathcal{P}_n} \rho_p \cdot (J_n \cdot f \cdot e^{-z\tau_n}) \circ \ell_\omega(x) \cdot \mathbf{1}_{T^n\omega}(x) \Big| \, dx \\
&\le \Bigg(\sum_{p \notin \mathcal{R}_\partial} \sum_{\omega, \varpi \in \mathcal{P}_n} \Big| \int_{T^n\omega \cap T^n\varpi} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_\varpi \cdot e^{ib\theta_{\omega,\varpi}})(x) \, dx \Big| \Bigg)^{1/2} \\
&\quad + e^{\sigma n} \|f\|_{L^\infty} \sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \sum_{p \in \mathcal{R}_\partial} \int_{T^n\omega} \rho_p(x) \, dx
\end{aligned}$$

where $K := J_n \cdot f \cdot e^{-a\tau_n}$ and $\theta_{\omega,\varpi} := \tau_n \circ \ell_\omega - \tau_n \circ \ell_\varpi$. Using Lemmas 3.3 and 3.10, we can bound the final term above by

$$C_{11} 2^d r^{d-d_1} C_7 e^{\sigma n} \|f\|_{L^\infty(X)} \le C_{11} 2^d C_7 e^{-(\frac{d-d_1}{2\beta} - \sigma)n} \|f\|_{L^\infty(X)}. \tag{15}$$

It remains to estimate the other term. We consider separately the set

$$\mathcal{Q}_{n,p,\omega} := \{\varpi \in \mathcal{P}_n : \mathcal{D}^{n_2}(T^{n_1}\ell_\varpi x_p) \not\pitchfork \mathcal{D}^{n_2}(T^{n_1}\ell_\omega x_p)\}$$

and the set of ϖ where this is not the case. In the second case we see oscillatory cancellations.

Lemma 3.11. *There exists $C_{12} > 0$ such that*

$$|K \circ \ell_\omega(x) - K \circ \ell_\omega(y)| \le e^{\sigma n} \|J_n\|_{L^\infty(\omega)} (C_{12} \|f\|_{L^\infty(\omega)} + |f|_{\mathcal{C}^\alpha(\omega)} e^{-\alpha\lambda n}) \, \mathrm{d}(x, y)^\alpha$$

for all $n \in \mathbb{N}$, $\omega \in \mathcal{P}_n$ and $x, y \in T^n\omega$.

Proof. Since $K \circ \ell_\omega(x) = (J_n \cdot f \cdot e^{-a\tau_n}) \circ \ell_\omega(x)$, for all $x, y \in T^n\omega$ we have

$$\begin{aligned}
K \circ \ell_\omega(x) - K \circ \ell_\omega(y) &= (e^{-a\tau_n(\ell_\omega x)} - e^{-a\tau_n(\ell_\omega y)}) f(\ell_\omega x) \cdot J_n(\ell_\omega x) \\
&\quad + e^{-a\tau_n(\ell_\omega y)} f(\ell_\omega y)(J_n(\ell_\omega x) - J_n(\ell_\omega y)) \\
&\quad + e^{-a\tau_n(\ell_\omega y)} (f(\ell_\omega x) - f(\ell_\omega y)) \cdot J_n(\ell_\omega x).
\end{aligned}$$

Using the estimates of Lemmas 3.1 and 3.2, and (1), we obtain

$$\begin{aligned}
&|K \circ \ell_\omega(x) - K \circ \ell_\omega(y)| \\
&\qquad \le e^{\sigma n} \|J_n\|_{L^\infty(\omega)} \big((\sigma C_5/2 + C_6) \|f\|_{L^\infty(\omega)} + |f|_{\mathcal{C}^\alpha(\omega)} e^{-\alpha\lambda n} \big) \, \mathrm{d}(x, y)^\alpha
\end{aligned}$$

The lemma follows by choosing $C_{12} := C_6 + \sigma C_5/2$. □

Lemma 3.12. *There exists $C_{13} > 0$ such that, for all $n \in \mathbb{N}$ and $\omega, \varpi \in \mathcal{P}_n$,*

$$\|D\theta_{\omega,\varpi}\|_{\mathcal{C}^\alpha} \leq C_{13}.$$

Proof. Since $D(\tau_n \circ \ell_\omega)(x) = \sum_{k=0}^{n-1} D\tau(h_k x) Dh_k(x)$ where $h_k := T^k \circ \ell_\omega$, we have

$$\begin{aligned}\|D(\tau_n \circ \ell_\omega)(x) - D(\tau_n \circ \ell_\omega)(y)\| &\leq \sum_{k=0}^{n-1} \|D\tau\|_{\mathcal{C}^\alpha} \, \mathrm{d}(h_k x, h_k y)^\alpha \\ &\leq \|D\tau\|_{\mathcal{C}^\alpha} \sum_{k=0}^{n-1} e^{-\lambda n \alpha} \, \mathrm{d}(x, y)^\alpha,\end{aligned}$$

and so $\|D\theta_{\omega,\varpi}\|_{\mathcal{C}^\alpha} \leq 2\|D\tau\|_{\mathcal{C}^\alpha} \sum_{k=0}^{\infty} e^{-\lambda n \alpha}$. □

Lemma 3.13. *In the setting of Proposition 3.9, there exists $C_{14} > 0$ such that*

$$\left(\sum_{p \notin \mathcal{R}_\partial} \sum_{\omega \in \mathcal{P}_n} \sum_{\varpi \in \mathcal{Q}_{n,p,\omega}} \left| \int (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_\varpi \cdot e^{ib\theta_{\omega,\varpi}})(x)\, dx \right| \right)^{1/2} \leq C_{14} e^{-(\frac{\gamma\beta_2}{2\beta} - \sigma)n} \|f\|_{L^\infty}. \quad (16)$$

Proof. Fixing for the moment $p \notin \mathcal{R}_\partial$ and $\omega \in \mathcal{P}_n$ we want to perform the sum over ϖ:

$$\sum_{\varpi \in \mathcal{Q}_{n,p,\omega}} \left| \int_{T^n\omega \cap T^n\varpi} (\rho_p \cdot K \circ \ell_\varpi \cdot K \circ \ell_\omega \cdot e^{ib\theta_{\omega,\varpi}})(x)\, dx \right| \leq \Big(\sum_{\varpi \in \mathcal{Q}_{n,p,\omega}} \|J_n\|_{L^\infty(\varpi)} \Big) e^{2\sigma n} \|J_n\|_{L^\infty(\omega)} \|f\|_{L^\infty}^2 \|\rho_p\|_{L^1}. \quad (17)$$

Observe that

$$\sum_{\varpi \in \mathcal{Q}_{n,p,\omega}} \|J_n\|_{L^\infty(\varpi)} \leq \Big(\sum_{\varpi_1 \in \mathcal{P}_{n_1}} \|J_n\|_{L^\infty(\varpi_1)} \Big) \Big(\sum_{\varpi_2} \|J_n\|_{L^\infty(\varpi_2)} \Big)$$

where the second sum is over the set of $\varpi_2 \in \mathcal{P}_{n_2}$ which satisfy

$$\mathcal{D}^{n_2}(T^{n_1}\ell_{\varpi_2} x_p) \pitchfork \mathcal{D}^{n_2}(T^{n_1}\ell_\omega x_p).$$

Consequently, the estimate of Proposition 3.6 shows that the term in (17) is bounded by

$$C_9 C_7 e^{-\gamma n_2} e^{2\sigma n} \|J_n\|_{L^\infty(\omega)} \|f\|_{L^\infty}^2 \|\rho_p\|_{L^1}.$$

Using again Lemma 3.3, we see that $\sum_{\omega \in \mathcal{P}_n} \|J_n\|_{L^\infty(\omega)} \leq C_7$ and we sum over p. □

Now we turn our attention to the $\varpi \in \mathcal{P}_n$ where we observe oscillatory cancellations. The crucial technical part of the estimate is the following oscillatory integral bound.

Lemma 3.14. *Suppose that $J \subset [0, 1]$ is an interval, $k \in \mathcal{C}^\alpha(J)$, $\theta \in \mathcal{C}^{1+\alpha}(J)$, $|\theta'| \geq \kappa > 0$, $|b| > 1$ and $k \in \mathcal{C}^\alpha(J)$. Then*

$$\left| \int_J e^{ib\theta(x)} k(x)\, dx \right| \leq \frac{C}{\kappa^2 |b|^\alpha} \|k\|_{\mathcal{C}^\alpha(J)}. \quad \textit{where} \quad C = (\|\theta'\|_{L^\infty(X)} + 6)(1 + |\theta'|_{\mathcal{C}^\alpha(X)}).$$

Proof. We assume that $b > 1$, the other case being identical. We also assume without loss of generality that $\theta' \geq \kappa$, otherwise we can exchange $-\theta$ for θ. Since k/θ' is α-Hölder, there exists[20] $g_b \in \mathcal{C}^1(J, \mathbb{R})$ such that

$$\|g_b - k/\theta'\|_{L^\infty} \leq b^{-\alpha}|k/\theta'|_{\mathcal{C}^\alpha}, \quad \|g_b'\|_{L^\infty} \leq 2b^{1-\alpha}|k/\theta'|_{\mathcal{C}^\alpha}.$$

Changing variables via $y = \theta(x)$ yields

$$\begin{aligned}\int_J k(x) \cdot e^{ib\theta(x)}\,dx &= \int_{\theta(J)} \frac{k}{\theta'} \circ \theta^{-1}(y) e^{iby}\,dy \\ &= \int_{\theta(J)} g_b \circ \theta^{-1}(y) e^{iby}\,dy + \int_{\theta(J)} \left(\frac{k}{\theta'} - g_b\right) \circ \theta^{-1}(y) e^{iby}\,dy.\end{aligned}$$

Observe that the final term is equal to $\int_J (k/\theta' - g_b)(x) e^{ib\theta(x)}\theta'(x)\,dx$. Integrating by parts the penultimate term we get

$$\begin{aligned}\int_{\theta(J)} g_b \circ \theta^{-1}(y) e^{iby}\,dy &= -\frac{i}{b}[g_b \circ \theta^{-1}(y) e^{iby}]_{\theta(J)} + \frac{i}{b}\int_{\theta(J)} \frac{g_b'}{\theta'} \circ \theta^{-1}(y) e^{iby}\,dy \\ &= -\frac{i}{b}[g_b e^{ib\theta}]_J + \frac{i}{b}\int_J g_b'(x) e^{ib\theta(x)}\,dx.\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}\left|\int_J e^{ib\theta(x)} k(x)\,dx\right| &\leq \left|\int_J \left(\frac{k}{\theta'} - g_b\right)(x) e^{ib\theta(x)}\theta'(x)\,dx\right| + \left|\frac{1}{b}[g_b e^{ib\theta}]_J\right| \\ &\quad + \left|\frac{1}{b}\int_J g_b'(x) e^{ib\theta(x)}\,dx\right| \\ &\leq \left(\frac{\|\theta'\|_\infty |J|}{b^\alpha} + \frac{2}{b^{1+\alpha}} + \frac{2|J|}{b^\alpha}\right)\left|\frac{k}{\theta'}\right|_\alpha + \frac{2\|k\|_\infty}{b\kappa}.\end{aligned}$$

To finish we observe that

$$\begin{aligned}\left|\frac{k}{\theta'}(x) - \frac{k}{\theta'}(y)\right| &= \left|\frac{k(x) - k(y)}{\theta'(x)} + \frac{k(y)(\theta'(y) - \theta'(x))}{\theta'(x)\theta'(y)}\right| \\ &\leq \left(\frac{|k|_\alpha}{\kappa} + \frac{\|k\|_\infty |\theta'|_\alpha}{\kappa^2}\right)|x - y|^\alpha.\end{aligned}$$

□

Lemma 3.15. *In the setting of Proposition 3.9, there exists $C_{15} > 0$ such that*

$$\left(\sum_{p \notin \mathcal{R}_\partial} \sum_{\omega \in \mathcal{P}_n} \sum_{\varpi \in \mathcal{P}_n \setminus \mathcal{Q}_{n,p,\omega}} \left|\int (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_\varpi \cdot e^{ib\theta_{\omega,\varpi}})(x)\,dx\right|\right)^{1/2} \leq C_{15}|b|^{-\alpha/4} e^{(\Lambda\beta_2/\beta+\sigma)n}\|f\|_{L^\infty} \leq C_{15} e^{-(\frac{\alpha}{8\beta}-\sigma)n}\|f\|_{L^\infty}. \tag{18}$$

[20] Take a molifier $\rho \in \mathcal{C}^1(\mathbb{R}, [0, 1])$ such that $\operatorname{supp}(\rho) \subset (-1, 1)$ and $\int \rho = 1$, $\int |\rho'| \leq 2$. Define $g_b(x) := \int \rho_b(x - y)\frac{k}{\theta'}(y)\,dy$ where $\rho_b(z) := b\rho(bz)$. Observe that $g_b(x) - \frac{k}{\theta'}(x) = \int \rho_b(x-y)\left[\frac{k}{\theta'}(y) - \frac{k}{\theta'}(x)\right]dy$, $g_b'(x) = \int \rho_b'(x-y)\left[\frac{k}{\theta'}(y) - \frac{k}{\theta'}(x)\right]dy$, $\int |\rho_b| = 1$ and $\int |\rho_b'| \leq 2b$.

Proof. Fixing p and ω for the moment we want to perform the sum over ϖ, i.e., we estimate

$$\sum_{\varpi\in\mathcal{P}_n\setminus\mathcal{Q}_{n,p,\omega}}\left|\int_{T^n\omega\cap T^n\varpi}(\rho_p\cdot K\circ\ell_\omega\cdot K\circ\ell_\varpi\cdot e^{ib\theta_{\omega,\varpi}})(x)\,dx\right|.$$

Since $\mathcal{D}^{n_2}(T^{n_1}\ell_\varpi x_p)\mathcal{K}\pitchfork\mathcal{D}^{n_2}(T^{n_1}\ell_\omega x_p)\mathcal{K}$ there exists (Lemma 3.7) a 1-dimensional linear subspace $L\subset\mathbb{R}^d$ (which depends on ϖ and ω) such that, for all $v\in L$,

$$|D(\tau_{n_2}\circ T^{n_1}\circ\ell_\varpi)(x_p)v-D(\tau_{n_2}\circ T^{n_1}\circ\ell_\omega)(x_p)v|>C_5|D(T^{n_1}\circ\ell_\varpi)(x_p)v|.$$

(We could also write another term on the right hand side of the above but this worse estimate suffices for what follows.) By Lemma 3.1,

$$|D(\tau_{n_1}\circ\ell_\varpi)(x_p)v|\le\tfrac{1}{2}C_5|D(T^{n_1}\circ\ell_\varpi)(x_p)v|.$$

Consequently,

$$\begin{aligned}&|D(\tau_n\circ\ell_\varpi)(x_p)v-D(\tau_n\circ\ell_\omega)(x_p)v|\\&\qquad>\tfrac{1}{2}C_5(|D(T^{n_1}\circ\ell_\varpi)(x_p)v|+|D(T^{n_1}\circ\ell_\varpi)(x_p)v|).\end{aligned}$$

Rotating and translating, we choose an orthogonal coordinate system $(y_1,\dots,y_d)$ such that y_1 corresponds to L and $x_p=(0,\dots,0)$. We have

$$\left|\frac{\partial\theta_{\omega,\varpi}}{\partial y_1}\right|(0,\dots,0)=\left|\frac{\partial(\tau_n\circ\ell_\varpi)}{\partial y_1}-\frac{\partial(\tau_n\circ\ell_\omega)}{\partial y_1}\right|(0,\dots,0)\ge C_5e^{-\Lambda n_2}.$$

Since $r>0$ is sufficiently small, the transversality holds along this direction for the entire ball (by Lemma 3.12). In order to show this we will show that $C_{13}r(b)^\alpha\le\frac{C_5}{2}e^{-\Lambda n_2}$ since $\|D\theta_{\omega,\varpi}\|_{\mathcal{C}^\alpha}\le C_{13}$. This is equivalent to $\exp\bigl(-\bigl[\frac{\alpha}{2\beta_2}-\Lambda\bigr]n_2\bigr)\le\frac{C_5}{2C_{13}}$, which holds for $|b|$ sufficiently large since β_2 was chosen such that $\beta_2\le\frac{\alpha}{2\Lambda}$. Here b_0 is chosen sufficiently large to guarantee that $|b|$ is large enough to satisfy the above condition. We have

$$\left|\frac{\partial\theta_{\omega,\varpi}}{\partial y_1}\right|(y_1,\dots,y_d)=\left|\frac{\partial(\tau_n\circ\ell_\varpi)}{\partial y_1}-\frac{\partial(\tau_n\circ\ell_\omega)}{\partial y_1}\right|(y_1,\dots,y_d)\ge\frac{1}{2}C_5e^{-\Lambda n_2}$$

for all $(y_1,\dots,y_d)\in B_{r_b}(0)$. To proceed we must estimate the Hölder norm of $\rho_p\cdot K\circ\ell_\omega\cdot K\circ\ell_\varpi$. By Lemma 3.11, since we assume that $|f|_{\mathcal{C}^\alpha(\omega)}\le e^{(q+\alpha/\beta)n}\|f\|_{L^\infty(\omega)}$ in Proposition 3.9 and $q+\alpha/\beta\le\alpha(\lambda/2+1/\beta_1)=\alpha\lambda$ (for some $C>0$), we have

$$|K\circ\ell_\omega(x)-K\circ\ell_\omega(y)|\le Ce^{\sigma n}\|J_n\|_{L^\infty(\omega)}\|f\|_{L^\infty(\omega)}\,\mathrm{d}(x,y)^\alpha.$$

Consequently, using Lemmas 3.11 and 3.10 we get

$$|\rho_p\cdot K\circ\ell_\omega\cdot K\circ\ell_\varpi|_{\mathcal{C}^\alpha(T^n\omega)}\le C(1+r^{-\alpha})e^{\sigma n}\|J_n\|_{L^\infty(\omega)}\|f\|_{L^\infty(\omega)}.$$

Using the estimate of Lemma 3.14, for $(y_2, \ldots, y_d)$ fixed we obtain

$$\left| \int_{-r}^{r} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_\varpi \cdot e^{ib\theta_{\omega,\varpi}})(y_1, \ldots, y_d)\, dy_1 \right| \\ \leq C r^{-\alpha} e^{2\Lambda n_2} |b|^{-\alpha} e^{2\sigma n} \|J_n\|_{L^\infty(\omega)} \|J_n\|_{L^\infty(\varpi)} \|f\|_{L^\infty}^2.$$

If $d = 1$ we are done, otherwise we integrate over the other directions. We also recall that $r = |b|^{-1/2}$. Thus

$$\left| \int_{T^n\omega \cap T^n\varpi} (\rho_p \cdot K \circ \ell_\omega \cdot K \circ \ell_\varpi \cdot e^{ib\theta_{\omega,\varpi}})(x)\, dx \right| \\ \leq C |b|^{-\alpha/2} e^{2\Lambda n_2 + 2\sigma n} \|J_n\|_{L^\infty(\omega)} \|J_n\|_{L^\infty(\varpi)} \|f\|_{L^\infty}^2.$$

Using Lemma 3.3 we sum over ω and ϖ to obtain the estimate. □

Proof of Proposition 3.9. The estimates from (15) and Lemmas 3.15 and 3.13 imply that, for some $C > 0$,

$$\|\mathcal{L}_z^n f\|_{L^1(X)} \leq C\left(e^{-(\frac{\gamma\beta_2}{2\beta} - \sigma)n} + e^{-(\frac{\alpha}{8\beta} - \sigma)n} + e^{-(\frac{d-d_1}{2\beta} - \sigma)n}\right) \|f\|_{L^\infty(X)}.$$

Here we take $\sigma > 0$ sufficiently small, depending only on the system. □

Proposition 3.16. *Let $T : X \to X$ be a $\mathcal{C}^{1+}$ uniformly expanding Markov map and $\tau : X \to \mathbb{R}_+$ as above. Further suppose that there is no $\theta \in \mathcal{C}^1(X, \mathbb{R})$ such that $\tau = \theta \circ T - \theta + \chi$ where χ is constant on each partition element.*

Then there exist $\zeta, b_0, B > 0$ such that, for all $z = a + ib$ with $a \geq -\sigma$ and $|b| \geq b_0$, and all $n \geq B \ln |b|$,

$$\|\mathcal{L}_z^n\|_{(b)} \leq e^{-\zeta n}.$$

Proof. We start by estimating $\|\mathcal{L}_z^n\|_{(b)}$ for $n = \beta \ln |b|$. First we consider the case when

$$\|f\|_{L^\infty(X)} \leq e^{-qn} \|f\|_{(b)}.$$

We apply Lemma 3.5:

$$\|\mathcal{L}_z^n f\|_{(b)} \leq C_8 e^{\sigma n}(e^{-\lambda n}\|f\|_{(b)} + \|f\|_{L^\infty(X)}) \leq C e^{\sigma n}(e^{-\lambda n} + e^{-qn})\|f\|_{(b)}.$$

It remains to consider the case when $\|f\|_{L^\infty(X)} \geq e^{-qn}\|f\|_{(b)}$. This means that $|f|_{\mathcal{C}^\alpha(X)} \leq e^{qn}(1+|b|^\alpha)\|f\|_{L^\infty(X)}$. Recall that $s \geq 1$ is the exponent associated to the almost-John property. The interpolation result of Lemma A.4 means that there exist $C, \epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$,

$$\|f\|_{L^\infty(X)} \leq C\epsilon^{-d}\|f\|_{L^1(X)} + \epsilon^{\alpha/s}|f|_{\mathcal{C}^\alpha(X)}. \tag{19}$$

Here we choose $\epsilon = e^{-\frac{\xi}{2d}n}$. Applying Lemma 3.5 twice yields

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq C_8 e^{2\sigma n} e^{-\lambda n}\|f\|_{(b)} + C_8 e^{\sigma n}\|\mathcal{L}_z^n f\|_{L^\infty(X)}.$$

Using also the above estimate (19) we get

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq \big(C_8 e^{-(\lambda-2\sigma)n} + e^{-\frac{\alpha\xi}{2ds}n}\big)\|f\|_{(b)} + C_8 e^{(\sigma+\xi/2)n}\|\mathcal{L}_z^n f\|_{L^1(X)}.$$

The estimate of Proposition 3.9 means that

$$\|\mathcal{L}_z^{2n} f\|_{(b)} \leq \big(C_8 e^{-(\lambda-2\sigma)n} + e^{-\frac{\alpha\xi}{2ds}n}\big)\|f\|_{(b)} + C_8 e^{-(\xi/2-\sigma)n}\|f\|_{(b)}.$$

Again we take $\sigma > 0$ sufficiently small. We have obtained the estimate $\|\mathcal{L}_z^n\|_{(b)} \leq e^{-\zeta n}$ when $n = \lfloor \beta \ln |b| \rfloor$. Iterating this estimate and choosing $B > 0$ sufficiently large concludes the proof. □

3.5. *Rate of mixing*

It remains to complete the proof of Theorem 3. In the present setting, in particular that the twisted transfer operators satisfy a Lasota–Yorke style estimate (Lemma 3.5), the required conclusion of exponential mixing follows in an established fashion (for example [3, §2.7] or [6, §7.5]) from the estimate of Proposition 3.16. In the first cited reference the $\mathcal{C}^1$ norm is used whilst we using the $\mathcal{C}^\alpha$ norm, but the same argument holds since it depends on the spectral properties of the twisted transfer operator and the norm estimate (Proposition 3.16), and these are identical in the present case. In the second cited reference the $\mathcal{C}^\alpha$ norm is used but for functions on the interval and not in the higher-dimensional situation of the present work. Again the argument presented there depends only on the spectral properties of the operator and so holds also in this setting.

For the convenience of the reader we here summarize the general argument which was cited above; at each stage the relevant paragraph in one of the references is detailed. The main part of the argument is to observe that the Laplace transform of the correlation function can be written in terms of a sum of twisted transfer operators [3, Proposition A.3]. The Laplace transform of the correlation is then shown to admit an analytic extension to a neighbourhood of each point $z = ib$. For $b \neq 0$ this is because the existence of poles on the imaginary axis would contradict mixing since they form groups and for $z = 0$ this uses the fact that the problem reduces to the case when one of the observables has zero average [3, Lemma 2.22]. This part of the argument uses the quasi-compactness of the twisted transfer operators. When $|b|$ is large the main functional-analytic estimate (Proposition 3.16) is used to imply an analytic extension of uniform size [3, Lemma 2.23]. The above is done in a way which is independent of the choice of observables. Combining the above gives an analytic extension of the correlation function to a strip about the imaginary axis. The result of exponential mixing then follows from a Paley–Wiener type estimate [3, §2.7].

Appendix. The boundary of Markov partitions

In the early 1970s, Bowen [10] and Ratner [35] showed that it is possible to construct Markov partitions for Anosov flows. However, it is known [12] that the regularity of the boundary of these partitions is normally rather poor. This is unfortunate for our present purposes since we need some degree of regularity of the unstable part of the Markov construction in order to complete our argument. Ratner [35] showed that the boundary of the Markov partition has Lebesgue measure zero but this is not quite sufficient for our purposes. Fortunately, as shown by Horita & Viana [30, Proposition 3.5], we also have estimates for the box-counting dimension[21] of the boundary. Section A.1 is devoted to reviewing this topic and the information on the dimension of the boundary is a key point in constructing the partition of unity of Lemma 3.10. Section A.2 is devoted to showing a different control on the geometry of the Markov partition, namely that the set satisfies a generalization of the notion of a John domain. This piece of information is used in order to have a convenient interpolation result (Lemma A.4). Note that the constructions of Bowen [10] and Ratner [35] are very similar but that Bowen's latter description [11] of the construction of Markov partitions is described rather differently. The later method of construction is based on shadowing in a way that works elegantly for all Axiom A systems. However, the geometry is rather lost in the construction and a clear hold of the geometry is precisely what we require for our present purposes. In this appendix we will follow the construction of Ratner [35] and for clarity use, whenever possible, identical notation as used in that reference.

Throughout this section we assume that $\phi^t : \mathcal{M} \to \mathcal{M}$ is a transitive Anosov flow. First we recall the notation and the general idea behind the construction of the Markov partition. For any x let $W^s_\epsilon(x)$ (resp. $W^{cs}_\epsilon(x)$, $W^u_\epsilon(x)$, $W^{cu}_\epsilon(x)$) denote the ϵ-sized local stable (resp. centre-stable, unstable, centre-unstable) manifold of x. As usual, we know that there exist $\epsilon_0, \gamma > 0$ such that for all x, all $y \in W^s_\gamma(x)$ and all $z \in W^{cu}_\gamma(x)$ the sets $W^s_{\epsilon_0}(y)$ and $W^{cu}_{\epsilon_0}(z)$ intersect in exactly one point which we denote by $[y, x]$; this defines the *canonical local product structure*. From now on we suppose that such a choice of $\epsilon_0, \gamma > 0$ is fixed. Let $\mathbf{C} \subset W^u_\gamma(x)$ and $\mathbf{D} \subset W^s_\gamma(x)$. A *parallelogram* is a set $\mathbf{A} = [\mathbf{C}, \mathbf{D}]$ defined as all the points $[y, z]$ such that $y \in \mathbf{C}$ and $z \in \mathbf{D}$. Observe that the set $\mathbf{A}$ is foliated by stable manifolds but not, in general, by unstable manifolds. Let $\mathfrak{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_k\}$, $\mathbf{A}_i = [\mathbf{C}_i, \mathbf{D}_i]$, $\mathbf{A}_i \cap \mathbf{A}_j = \emptyset$ for $i \neq j$, be a finite complete system of parallelograms. (Here *complete* means that for every point in $\mathcal{M}$ there is an interval on the trajectory of the point whose end points each lie in one of the parallelograms.) Let $\mathbf{M}_\mathfrak{A}$ be the set-theoretic union of the parallelograms $\{\mathbf{A}_i\}_i$ with the induced topology. For any $x \in \mathbf{M}_\mathfrak{A}$, consider the trajectory of the flow ϕ^t extending from x to its first intersection x' with $\mathbf{M}_\mathfrak{A}$. Let T denote the one-to-one mapping of $\mathbf{M}_\mathfrak{A}$ onto itself which maps x to x'. A system $\mathfrak{A}$ is said to be *Markovian* for the flow ϕ^t if, whenever $x \in \operatorname{Int} \mathbf{A}_i \cap T^{-1}(\operatorname{Int} \mathbf{A}_j)$,[22]

$$T(\operatorname{Int} \mathbf{D}_i(x)) \subset \mathbf{D}_j(T(x)) \quad \text{and} \quad T(\mathbf{C}_i(x)) \supset \operatorname{Int} \mathbf{C}_j(T(x)). \tag{20}$$

As mentioned previously, we rely on the following result.

[21] In general the upper box-counting dimension may differ from the lower box-counting dimension. Throughout this text our only interest is in an upper bound for the upper box-counting dimension and for conciseness we omit explicit mention of this detail. Note that in [30] the term *limit capacity* is used for the same concept.

[22] Here we use the notation $\mathbf{D}_i(x) = [x, \mathbf{D}_i]$ and $\mathbf{C}_i(x) = [\mathbf{C}_i, x]$.

Theorem A.1 ([10, Theorem 2.5] or [35, Theorem 2.1]). *For every $\epsilon > 0$ the transitive Anosov flow $\phi^t : \mathcal{M} \to \mathcal{M}$ has a Markov partition with all elements of size at most ϵ.*

Since we will need more details of the construction of the Markov partition, particularly some information on the geometry of the partition elements, we recall the most relevant details of the construction. During the construction, $\alpha, \delta > 0$ are chosen to satisfy, amongst other conditions, the requirement $0 < \alpha < \delta < \min(\epsilon, \gamma, \epsilon_0)$. To start the construction we fix $\mathfrak{A}^0 = \{\mathbf{A}_1^0, \ldots, \mathbf{A}_k^0\}$, a complete finite system of parallelograms $\mathbf{A}_i^0 = [\mathbf{C}_i^0, \mathbf{D}_i^0]$ with $\mathbf{C}_i^0 = W_u^\alpha(x_i)$ and $\mathbf{D}_i^0 = W_s^\alpha(x_i)$. By a recursive procedure[23] [35, §2] we define the sets $\mathbf{C}_i^n \subset W_u^\delta(x_i)$ and $\mathbf{D}_i^n \subset W_s^\delta(x_i)$. The procedure involves starting with the system of parallelograms $\{\mathbf{A}_i^0\}_i$ and using the dynamics, both in forward and backward time, to add the appropriate images of the already defined sets in order to become closer to the required Markov property. For the sets $\mathbf{C}_i$ this means applying a strong contraction to the sets already defined in order to add small additional sets to the sets already defined. At the beginning some m is chosen sufficiently large. For each i, j we consider if $\phi^{-m}\mathbf{C}_j^n$ contributes a part which should be added to the set $\mathbf{C}_i$. The successive approximation means that these leaves converge to the Markov property. The unstable part of the partition element is defined by a countable union

$$\mathbf{C}_i = \overline{\bigcup_{n \geq 1} \mathbf{C}_i^n} \subset W_\delta^u(x_i).$$

The stable part, $\mathbf{D}_i$, is defined similarly but using ϕ^m in place of ϕ^{-m}.

A.1. Box-counting dimension of the boundary

The structure of the constructed Markov partition leads to the following result.

Proposition A.2 ([30, Proposition 3.5]). *The box-counting dimension of the union of the unstable boundaries of the elements of the Markov partition of an Anosov map is strictly smaller than the dimension of the unstable bundle.*

The proof of the above is based on estimates available in Bowen [11] and on a standard relation [23] which connects the measure of a neighbourhood of a set to the box-counting dimension of that set. Although the result is stated for Anosov diffeomorphisms, the same result holds for the Markov structure of an Anosov flow as described above.

We will use this information about the box-counting dimension of the boundary to prove the previously stated Lemma 3.10 which concerns the existence of a partition of unity. This construction is essentially standard but since the details are crucial and the estimates concerning the boundary of the set are less common, we give the details of the construction and the proof of the required estimates.

Fix a function $\Phi \in \mathcal{C}^1(\mathbb{R}, [0, 1])$ such that $\Phi(u) = 1$ whenever $|u| \leq 1/4$, $\Phi(u) = 0$ whenever $|u| \geq 3/4$ and $\sum_{k=-\infty}^{\infty} \Phi(u - k) = 1$ for all $u \in \mathbb{R}$. For any $x \in \mathbb{R}^d$ and $r > 0$

[23] The reference [35, §2] contains a clear description of this recursive construction. See also the pertinent details reproduced in §A.2.

we denote by $\mathrm{B}(x,r)$ the open ball centred at x and of radius $r > 0$. For each $\epsilon > 0$, $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d$ define $\Phi_\ell^{(\epsilon)} \in \mathcal{C}^1(\mathbb{R}^d, [0,1])$ by

$$\Phi_\ell^{(\epsilon)}(x_1, \ldots, x_d) := \prod_{k=1}^{d} \Phi(\epsilon^{-1}(x_k - \epsilon \ell_k)).$$

Such a function is "centred" at the point $\epsilon\ell = (\epsilon\ell_1, \ldots, \epsilon\ell_d) \in \mathbb{R}^d$. Observe that

- the support of $\Phi_\ell^{(\epsilon)}$ is contained within $\mathrm{B}(\epsilon\ell, 3\epsilon/4)$,
- for all $x \in \mathrm{B}(\epsilon\ell, \epsilon/4)$,
$$\Phi_\ell^{(\epsilon)}(x) = 1,$$
- for each $x \in \mathbb{R}^d$,
$$\sum_{\ell \in \mathbb{Z}^d} \Phi_\ell^{(\epsilon)}(x) = 1,$$
- There exists $K > 0$ such that $\|\Phi_\ell^{(\epsilon)}\|_{\mathcal{C}^1} \leq K\epsilon^{-1}$ for all $\epsilon > 0$ and $\ell \in \mathbb{Z}^d$.

We suppose that $X \subset \mathbb{R}^d$ is bounded and that ∂X has box-counting dimension strictly less than d. That the set is bounded implies there exists $K > 0$ such that the cardinality of the set $\{\ell \in \mathbb{Z}^d : \mathrm{B}(\epsilon\ell, 3\epsilon/4) \cap X \neq \emptyset\}$ is bounded from above by $K\epsilon^{-d}$.

Consider the ϵ-mesh where the cubes of the mesh are centred on the points $\{\epsilon\ell : \ell \in \mathbb{Z}^d\}$. For any set $E \subset \mathbb{R}^d$ let $N_\epsilon(E)$ denote the number of cubes in the ϵ-mesh which intersect E. (There are several equivalent definitions of box-counting dimension [23, §3.1].) Since the boundary ∂X has box-counting dimension strictly less than d, there exist $K > 0$ and $d_1 \in [0, d)$ such that, for all $\epsilon > 0$,

$$N_\epsilon(\partial X) \leq K\epsilon^{-d_1}.$$

Consequently, the cardinality of the set $\{\ell \in \mathbb{Z}^d : \mathrm{B}(\epsilon\ell, 3\epsilon/4) \cap \partial X \neq \emptyset\}$ is bounded from above by $K\epsilon^{-d_1}$ (increasing $K > 0$ if required, independently of ϵ). This completes the proof of Lemma 3.10.

A.2. *Markov partitions are almost John*

The construction of the unstable part of the Markov partition can be conveniently rephrased[24] as follows (for full details consult [35]). There is a collection $\{\mathbf{C}_i\}_{i=1}^N$ of bounded subsets of $\mathbb{R}^d$. (These sets are the unstable parts of the complete finite system of parallelograms $\mathbf{A}_i^0 = [\mathbf{C}_i^0, \mathbf{D}_i^0]$ which are the starting point of the construction of the Markov structure.) For each set there is a subset $\mathbf{C}_i^0 \subset \mathbf{C}_i$ which has nice geometry in the sense that the boundary of $\mathbf{C}_i^0$ is $\mathcal{C}^1$. Let $\mathfrak{C}$ denote the disjoint union $\bigsqcup_i \mathbf{C}_i$. Again with a slight abuse of notation, there is a map $T : \mathfrak{C} \to \mathfrak{C}$ which corresponds to the Anosov flow

[24] Strictly speaking, the objects here are $\mathbf{C}_i$ and T as above combined with an appropriate choice of d_u-dimensional chart. Abusing notation we suppress this detail and use the same symbols.

for some large time (this is the return map associated to the family of parallelograms after projecting along local stable manifolds). There is an index set $\mathcal{A} \subset \{1, \ldots, N\}^2$ and, for each $(j,k) \in \mathcal{A}$, a map $h_{j,k} : \mathbf{C}_j \to \mathbf{C}_k$ such that $T \circ h_{j,k} = \mathrm{id}$. Moreover these maps are strong contractions in the sense that there exist $0 < \lambda_2 \leq \lambda_1 < 1$ such that, for all $(j,k) \in \mathcal{A}$ and $x, y \in \mathbf{C}_j$,

$$\lambda_2 \, \mathrm{d}(x,y) \leq \mathrm{d}(h_{j,k}(x), h_{j,k}(y)) \leq \lambda_1 \, \mathrm{d}(x,y).$$

Define

$$\mathbf{C}_j^n = \bigcup_{k:\,(j,k)\in\mathcal{A}} h_{j,k}(\mathbf{C}_k^{n-1}), \qquad \mathbf{C}_i = \overline{\bigcup_{n\geq 1} \mathbf{C}_i^n}.$$

Note that $\mathbf{C}_i^n \supset \mathbf{C}_i^{n-1}$ for all n. The above structure suffices to show some modest control on the geometry.

Since $0 < \lambda_2 \leq \lambda_1 < 1$ there exists $s \geq 1$ such that

$$\lambda_2 = \lambda_1^s. \tag{21}$$

Observe that $s \geq 1$ can be taken to be equal to 1 in the special case when the expansion is isotropic. In particular this is the situation when the unstable bundle is one-dimensional. Recall (Definition 1.1) that a set $\Omega \subset \mathbb{R}^d$ is *almost John* with exponent $s \geq 1$ if there exist $K_2, \epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$ and all $x \in \Omega$, there exists $y \in \Omega$ such that $\mathrm{d}(x,y) \leq \epsilon$ and $\mathrm{B}(y, K_2\epsilon^s) \subset \Omega$.

Lemma A.3. *Each set $\mathbf{C}_i$ is almost John. The exponent $s \geq 1$ is that given by (21).*

Proof. Let $\delta > 0$ be such that $\mathrm{diam}(\mathbf{C}_i^0) \leq \delta$ for each i. Since the set $\mathbf{C}_i^0$ has smooth boundary, there exist $K_1 > 0$ and $\epsilon_1 > 0$ such that for all i, all $x \in \mathbf{C}_i^0$ and all $\epsilon \in (0, \epsilon_1)$ there exists $y \in \mathbf{C}_i^0$ such that

$$\mathrm{B}(y, K_1\epsilon) \subset \mathbf{C}_i^0 \quad \text{and} \quad \mathrm{d}(x,y) \leq \epsilon.$$

Fix the constants

$$K_3 = \frac{2\delta}{\lambda_1(1-\lambda_1)}, \qquad K_4 = \min(1/2, \epsilon_0 K_3^{-1}).$$

Let $\epsilon \in (0, \epsilon_0)$. Define $N_\epsilon \in \mathbb{N}$ by the requirement that $K_3\lambda_1^{N_\epsilon+1} \leq \epsilon \leq K_3\lambda_1^{N_\epsilon}$. For $x \in \mathbf{C}_i$, we will consider two cases.

Case 1 ($x \in \mathbf{C}_i^{N_\epsilon}$)**:** Let j be such that $T^n x \in \mathbf{C}_j$. We choose $\epsilon' = \epsilon K_4 \lambda_1^{-N_\epsilon} \in (0, \epsilon_0)$. We know that there exists $y \in \mathbf{C}_i^{N_\epsilon}$ such that $T^{N_\epsilon} y \in \mathbf{C}_j^0$, $\mathrm{d}(T^{N_\epsilon}x, T^{N_\epsilon}y) \leq \epsilon'$ and $\mathrm{B}(T^{N_\epsilon}y, K_1\epsilon') \subset \mathbf{C}_i^0$. Consequently, $\mathrm{d}(x,y) \leq \epsilon'\lambda_1^{N_\epsilon}$ and $T^{N_\epsilon}\mathrm{B}(y, K_1\epsilon'\lambda_2^{N_\epsilon}) \subset \mathrm{B}(T^{N_\epsilon}y, K_1\epsilon') \subset \mathbf{C}_i^0$. This means that

$$\mathrm{d}(x,y) \leq \epsilon K_4 \leq \epsilon$$

as required. Since the definition of $s > 1$ implies $\lambda_2/\lambda_1 = \lambda_1^{s-1}$, we see that

$$K_1\epsilon'\lambda_2^{N_\epsilon} = K_1K_4\left(\frac{\lambda_2}{\lambda_1}\right)^{N_\epsilon}\epsilon = K_1K_4\lambda_1^{N_\epsilon(s-1)}\epsilon \geq K_1K_4\left(\frac{\epsilon}{K_3}\right)^{s-1}\epsilon = \frac{K_1K_4}{K_3^{s-1}}\epsilon^s.$$

This means that we have shown that $B(y, K_1'\epsilon^s) \subset \mathbf{C}_i^{N_\epsilon}$ where $K_1' = K_1K_4/K_3^{s-1}$.

Case 2 ($x \in \mathbf{C}_i \setminus \mathbf{C}_i^{N_\epsilon}$)**:** In this case we know that there exists $z \in \mathbf{C}_i^{N_\epsilon}$ such that $d(x, z) \leq \delta\frac{\lambda_1^{N_\epsilon}}{1-\lambda_1}$. This is because, from the construction, the diameter of every component of $\mathbf{C}_i^n$ is not greater than $\delta\lambda_1^n$ and must intersect some previously defined set. Using now what we demonstrated in the other case we know that there exists $y \in \mathbf{C}_i^{N_\epsilon}$ which satisfies $d(z, y) \leq K_4\epsilon$ and $B(y, K_1'\epsilon^s) \subset \mathbf{C}_i^{N_\epsilon}$. Observe that

$$d(x, y) \leq \delta\frac{\lambda_1^{N_\epsilon}}{1-\lambda_1} + K_4\epsilon \leq \left(\frac{\delta}{\lambda_1(1-\lambda_1)K_3} + \frac{1}{2}\right)\epsilon \leq \epsilon$$

as required. □

Remark. The work of Avila, Gouëzel & Yoccoz [6] required the domain of the expanding Markov map to be a John domain in a sense which corresponds to our definition if $s = 1$. However, when the expansion is not the same in all directions, it seems unlikely that a condition better than the one we use here could be satisfied. A weakening of the definition of a John domain similar to ours has been studied in other contexts (see, e.g., [33] and references therein). In the case $s = 1$ the John domain property implies the estimate on the box-counting dimension of the boundary [33, Corollary 6.2]. However, in general when $s > 1$, this is not sufficient for a useful estimate of the dimension [33, §7.3]. We therefore show independently the two properties we require.

In our application we use the above lemma for the following key interpolation result.

Lemma A.4. *Let $\Omega \subset \mathbb{R}^d$ be almost John with exponent $s \geq 1$. Let $\gamma = 1/s \in (0, 1]$. There exist $K_5 > 0$ and $\epsilon_1 > 0$ such that, for all $\epsilon \in (0, \epsilon_1)$ and $f \in \mathcal{C}^\alpha(\Omega)$,*

$$\|f\|_{L^\infty(\Omega)} \leq K_5\epsilon^{-d}\|f\|_{L^1(\Omega)} + \epsilon^{\gamma\alpha}|f|_{\mathcal{C}^\alpha(\Omega)}.$$

Proof. Let $x \in \Omega$, $\epsilon \in (0, \epsilon_1)$ and $f \in \mathcal{C}^\alpha(\Omega)$. Since Ω is almost John, there exists $y \in \Omega$ such that $B(y, K_2\epsilon) \subset \Omega$ and $d(x, y) \leq \epsilon^\gamma$. Let V_d denote the appropriate constant such that the volume of the d-ball of radius ϵ is equal to $V_d\epsilon^d$. There must exist $z \in B(y, \epsilon)$ such that $|f(z)| \leq V_d^{-1}\epsilon^{-d}\|f\|_{L^1(\Omega)}$ because otherwise there would be a contradiction for the L^1 norm (if the statement were false then $|f(z)| > V_d^{-1}\epsilon^{-d}\|f\|_{L^1(\Omega)}$ for all $z \in B(y, \epsilon)$ and consequently $\|f\|_{L^1(B(y,\epsilon))} > \|f\|_{L^1(\Omega)}$). This means that

$$\begin{aligned}|f(x)| &\leq |f(z)| + |f(x) - f(z)| \\ &\leq V_d^{-1}\epsilon^{-d}\|f\|_{L^1(\Omega)} + |f|_{\mathcal{C}^\alpha(\Omega)}\, d(x, z)^\alpha \\ &\leq K_5\epsilon^{-d}\|f\|_{L^1(\Omega)} + \epsilon^{\gamma\alpha}|f|_{\mathcal{C}^\alpha(\Omega)}.\end{aligned}$$

This estimate holds for all $x \in \Omega$, $\epsilon \in (0, \epsilon_1)$ and $f \in \mathcal{C}^\alpha(\Omega)$. □

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