INTERSECTING JONES PROJECTIONS

SEBASTIANO CARPI Dipartimento di Scienze

Università "G. d'Annunzio" di Chieti-Pescara Viale Pindaro 87, I-65127 Pescara, Italy E-mail: carpi@sci.unich.it

Abstract

Let M be a von Neumann algebra on a Hilbert space \mathcal{H} with a cyclic and separating unit vector Ω and let ω be the faithful normal state on M given by $\omega(\cdot) = (\Omega, \cdot \Omega)$. Moreover, let $\{N_i : i \in I\}$ be a family of von Neumann subalgebras of M with faithful normal conditional expectations E_i of M onto N_i satisfying $\omega = \omega \circ E_i$ for all $i \in I$ and let $N = \bigcap_{i \in I} N_i$. We show that the projections e_i , e of \mathcal{H} onto the closed subspaces $\overline{N_i\Omega}$ and $\overline{N\Omega}$ respectively satisfy $e = \bigwedge_{i \in I} e_i$. This proves a conjecture of V.F.R. Jones and F. Xu in [1].

1 Introduction

Let M be a von Neumann algebra on a Hilbert space \mathcal{H} and let $\Omega \in \mathcal{H}$ be a vector of norm 1 which is cyclic and separating for M. Given a family $\{N_i : i \in I\}$ of von Neumann subalgebras of M it is often useful to consider the closed subspaces $\overline{N_i\Omega}$, $i \in I$ and the corresponding projections $e_i \in \mathbf{B}(\mathcal{H})$. If N denotes the intersection $\bigcap_{i \in I} N_i$ and e is the projection of \mathcal{H} onto $\overline{N\Omega}$ one always have $e \leq \bigwedge_{i \in I} e_i$ namely $e\mathcal{H} \subset \bigcap_{i \in I} e_i\mathcal{H}$. However in general the equality does not hold and in fact it is not hard to give examples with $N = \mathbb{C}1$ but $e_i = 1$ for all $i \in I$ even when the set I contains just two elements.

In this paper we prove (see Corollary 3.4) that if for every $i \in I$ there is a faithful normal conditional expectation of M onto N_i with $\omega \circ E_i = \omega$, where ω denotes the faithful normal state on M given by $\omega(\cdot) = (\Omega, \cdot \Omega)$, then $e = \bigwedge_{i \in I} e_i$.

In a recent paper V.F.R. Jones and F. Xu gave a proof of this equality for a relevant class of examples associated with inclusions of loop groups models of conformal nets on S^1 [1, Lemma 4.14]. Moreover, they conjectured that this conclusion is not restricted to specific models but holds in general for inclusions of of completely rational conformal nets [1, Remark 4.15]. In these examples M is a local algebra of the larger conformal net (a type III₁ factor) and Ω is the vacuum vector (if M is a Type II₁ factor and the vector state ω is the trace on M the equality holds by [2], cf. [1, Remark 4.16]).

Our result proves the conjecture of Jones and Xu in [1] and actually shows that assumption (2) in [1, Corollary 4.9] is not needed.

The proof we shall give is partially inspired by the one given in [1, Lemma 4.14]. The main new idea is to replace the smeared vertex operators used in [1] by suitable closed operators provided by the Tomita-Takesaki modular theory. In fact we shall prove a more general result (Theorem 3.3) where a normal semifinite normal weight on the von Neumann algebra M is given instead of the vector state ω .

2 Preliminaries and notations

Let M be a von Neumann algebra and let φ be a normal semifinite faithful (n.s.f.) weight on M. Then the set

$$\mathfrak{N}_{\varphi} = \{ x \in M : \varphi(x^*x) < \infty \}.$$
(1)

is σ -weakly dense left ideal of M. \mathfrak{N}_{φ} with the inner product $(x, y) = \varphi(x^*y)$ can be completed to a complex Hilbert space \mathcal{H}_{φ} . Accordingly $\mathfrak{N}_{\varphi} \subset M$ will be considered as a dense subspace of \mathcal{H}_{φ} via the mapping $\mathfrak{N}_{\varphi} \ni x \mapsto x_{\varphi} \in \mathcal{H}_{\varphi}$ so that for $x, y \in \mathfrak{N}_{\varphi}$ we have $(x_{\varphi}, y_{\varphi}) = \varphi(x^*y)$. The GNS representation π_{φ} of M on \mathcal{H}_{φ} is determined by $\pi_{\varphi}(x)y_{\varphi} = (xy)_{\varphi}$ for all $x \in M$ and $y \in \mathfrak{N}_{\varphi}$. Then π_{φ} is normal and faithful i.e. a *-isomorphism of M onto the von Neumann algebra $\pi_{\varphi}(M) \subset \mathbf{B}(\mathcal{H}_{\varphi})$ (see [3, §2]).

The set $\mathfrak{N}_{\varphi} \cap \mathfrak{N}_{\varphi}^*$ is a σ -weakly dense self-adjoint subalgebra \mathfrak{A}_{φ} of M which can also be considered as a dense subspace of \mathcal{H}_{φ} . The antilinear operator S_{φ}^0 on \mathcal{H}_{φ} with domain \mathfrak{A}_{φ} , defined by

$$S^0_{\varphi} x_{\varphi} = (x^*)_{\varphi}, \quad x \in \mathfrak{A}_{\varphi}, \tag{2}$$

is preclosed and we denote its closure by S_{φ} . The modular operator $\Delta_{\varphi} = S_{\varphi}^* S_{\varphi}$ and the modular conjugation J_{φ} associated with M and φ are obtained from the polar decomposition $S_{\varphi} = J_{\varphi} \Delta_{\varphi}^{1/2}$ of S_{φ} . Moreover the following fundamental relations hold

$$J_{\varphi}\pi_{\varphi}(M)J_{\varphi} = \pi_{\varphi}(M)', \ \Delta_{\varphi}^{it}\pi_{\varphi}(M)\Delta_{\varphi}^{-it} = \pi_{\varphi}(M), \ t \in \mathbb{R}$$
(3)

and the modular automorphism group $\{\sigma_t^{\varphi}\}_{t\in\mathbb{R}}$ of M associated with φ is defined by

$$\Delta_{\varphi}^{it}\pi_{\varphi}(x)\Delta_{\varphi}^{-it} = \pi_{\varphi}(\sigma_t^{\varphi}(x)), \ x \in M, t \in \mathbb{R}.$$
(4)

For every $\eta \in \mathcal{H}_{\varphi}$ one defines a linear operator R_{η}^{0} on \mathcal{H}_{φ} with domain \mathfrak{A}_{φ} by

$$R^0_\eta x_\varphi = \pi_\varphi(x)\eta, \ x \in \mathfrak{A}_\varphi.$$
(5)

If η is in the domain $D(S_{\varphi}^*)$ of S_{φ}^* then R_{η}^0 is preclosed and its closure R_{η} is affiliated with $\pi_{\varphi}(M)'$, see [3, Chapter I, §2]. The subset $\mathfrak{A}'_{\varphi} \subset \mathcal{H}_{\varphi}$ defined by

$$\mathfrak{A}'_{\varphi} = \{\eta \in D(S^*_{\varphi}) : R_{\eta} \in \mathbf{B}(\mathcal{H}_{\varphi})\}$$
(6)

is a dense subspace of \mathcal{H}_{φ} and the set $\{R_{\eta} : \eta \in \mathfrak{A}_{\varphi}'\}$ is a σ -weakly dense self-adjoint subalgebra of $\pi_{\varphi}(M)'$.

Similarly, for every $\xi \in \mathcal{H}_{\varphi}$ one defines a linear operator L^0_{ξ} on \mathcal{H}_{φ} with domain \mathfrak{A}'_{φ} by

$$L^0_{\xi}\eta = R_{\eta}\xi, \ \eta \in \mathfrak{A}'_{\varphi}.$$
(7)

If ξ is in the domain $D(S_{\varphi})$ of S_{φ} then L_{ξ}^{0} is preclosed and its closure L_{ξ} is affiliated with $\pi_{\varphi}(M)$.

The operators L_{ξ} , $\xi \in D(S_{\varphi})$, which in general can be unbounded, will play a crucial role in the proof of our main result.

We conclude this section with a proposition (cf. [2]). which we shall need later.

Proposition 2.1. Let $\{R_i : i \in I\}$ be a family of von Neumann algebras on a Hilbert space \mathcal{H} and let T be a closed linear operator on \mathcal{H} . If T is affiliated with R_i for every $i \in I$ then T is also affiliated with $R = \bigcap_{i \in I} R_i$.

Proof. Let A be the unital self-adjoint subalgebra of $\mathbf{B}(\mathcal{H})$ generated by the union of the algebras $R'_i, i \in I$. If $x \in A$ then $xT \subset Tx$. Now A'' = R' and thus A is strong-operator dense in R'. For $x \in R'$ let x_{λ} be a net in A converging to x in the strong-operator topology. If ξ is in the domain D(T)

of T then $x_{\lambda}\xi \in D(T)$ and $Tx_{\lambda}\xi = x_{\lambda}T\xi$ for each λ . Hence $\lim x_{\lambda}\xi = x\xi$ and $\lim Tx_{\lambda}\xi = xT\xi$. Since T is closed it follows that $x\xi \in D(T)$ and $Tx\xi = xT\xi$. Hence $xT \subset Tx$ for every $x \in R'$ namely T is affiliated with R.

3 Results

Let M be a von Neumann algebra and let φ be a n.s.f. weight on M. If N is a von Neumann subalgebra of M we can define a closed subspace \mathcal{H}_N of \mathcal{H}_{φ} by

$$\mathcal{H}_N = \overline{\{x_{\varphi} : x \in \mathfrak{N}_{\varphi} \cap N\}}.$$
(8)

For all $x \in N$ we have $x(\mathfrak{N}_{\varphi} \cap N) \subset \mathfrak{N}_{\varphi} \cap N$ and hence \mathcal{H}_N is invariant for $\pi_{\varphi}(N)$.

If $\sigma_t^{\varphi}(N) = N$ for all $t \in \mathbb{R}$ and the restriction ψ of φ to N is semifinite we say that N is a *modular covariant* von Neumann subalgebra of M relatively to φ or simply a modular covariant subalgebra if the corresponding weight on M is unambiguously defined from the context. Note that if φ is a state, i.e. $\varphi(1) = 1$, then $N \subset M$ is modular covariant iff $\sigma_t^{\varphi}(N) = N$ for all $t \in \mathbb{R}$.

A von Neumann subalgebra $N \subset M$ is modular covariant if and only if there exists a faithful normal conditional expectation E of M onto N such that $\varphi = \varphi \circ E$, namely $\varphi(x) = \varphi(E(x))$ for each positive element x of M [4] (see also [3, §10]). In this case the conditional expectation E is completely determined by

$$\pi_{\varphi}(E(x))e = e\pi_{\varphi}(x)e, \ x \in M, \tag{9}$$

where e denotes the projection of \mathcal{H}_{φ} onto \mathcal{H}_{N} (the Jones projection), and the fact that \mathcal{H}_{N} is separating for $\pi_{\varphi}(M)$, being $N \cap \mathfrak{N}_{\varphi}$ σ -weakly dense in N.

Lemma 3.1. Let φ be a n.s.f. weight on the von Neumann algebra M. If $N \subset M$ is a modular covariant von Neumann subalgebra and e is the corresponding Jones projection then

$$\pi_{\varphi}(N) = \pi_{\varphi}(M) \cap \{e\}'.$$

Proof. From the fact that \mathcal{H}_N is invariant for $\pi_{\varphi}(N)$ it follows that $e \in \pi_{\varphi}(N)'$. Assume now that $x \in M$ and that $\pi_{\varphi}(x)$ commutes with e. Then it follows from Eq. (9) that $\pi_{\varphi}(E(x) - x)e = 0$ and hence, being \mathcal{H}_N separating for $\pi_{\varphi}(M)$ and π_{φ} faithful, that $x = E(x) \in N$.

Proposition 3.2. Let M, φ and N be as in the previous lemma and for $\xi \in D(S_{\varphi})$ let L_{ξ} be the closed operator affiliated with $\pi_{\varphi}(M)$ defined after Eq. (7). Then L_{ξ} is affiliated with $\pi_{\varphi}(N)$ for all $\xi \in D(S_{\varphi}) \cap \mathcal{H}_N$.

Proof. Let $\xi \in D(S_{\varphi}) \cap \mathcal{H}_N$. Since we know that L_{ξ} is a closed operator affiliated with $\pi_{\varphi}(M)$ and by Lemma 3.1 $\pi_{\varphi}(N) = \pi_{\varphi}(M) \cap \{e\}'$ it follows from Proposition 2.1 that it is enough to show that L_{ξ} is affiliated with $\{e\}'$ namely that $eL_{\xi} \subset L_{\xi}e$.

From $eJ_{\varphi} = J_{\varphi}e$ (see pag. 131 of [3]) and $\mathfrak{A}'_{\varphi} = J_{\varphi}\mathfrak{A}_{\varphi}$ [3, 2.12] it follows that $e\mathfrak{A}'_{\varphi} = J_{\varphi}e\mathfrak{A}_{\varphi}$. Now, for every $x \in \mathfrak{N}_{\varphi}$ we have $E(x) \in \mathfrak{N}_{\varphi}$ and $ex_{\varphi} = E(x)_{\varphi}$, see [3, 10.3]. Since E is a self-adjoint map it follows that $e\mathfrak{A}_{\varphi} \subset \mathfrak{A}_{\varphi}$ and hence that $e\mathfrak{A}'_{\varphi} \subset \mathfrak{A}'_{\varphi}$. Given $\eta \in \mathfrak{A}'_{\varphi}$, $x \in \mathfrak{A}_{\varphi}$ we have

$$eR_{\eta}ex_{\varphi} = eR_{\eta}E(x)_{\varphi} = e\pi_{\varphi}(E(x))\eta = \pi_{\varphi}(E(x))e\eta$$
$$= R_{e\eta}E(x)_{\varphi} = R_{e\eta}ex_{\varphi}.$$

Since R_{η} and $R_{e\eta}$ are bounded and \mathfrak{A}_{φ} is dense in \mathcal{H}_{φ} it follows that $eR_{\eta}e = R_{e\eta}e$ for every $\eta \in \mathfrak{A}'_{\varphi}$. Hence, using the assumption that $\xi = e\xi$, for every $\eta \in \mathfrak{A}'_{\varphi}$ we find

$$L_{\xi}e\eta = R_{e\eta}\xi = R_{e\eta}e\xi = eR_{\eta}e\xi$$
$$= eR_{\eta}\xi = eL_{\xi}\eta$$

and the conclusion follows from the fact that \mathfrak{A}'_{φ} is a core for L_{ξ} .

We are now ready prove the main result of this paper.

Theorem 3.3. Let M be a von Neumann algebra with a n.s.f. weight φ and let $\{N_i : i \in I\}$ be a family of modular covariant von Neumann subalgebras of M with Jones projections $\{e_i : i \in I\}$. Assume that the restriction of φ to $N = \bigcap_{i \in I} N_i$ is semifinite. Then N is a modular covariant subalgebra of Mwith Jones projection e satisfying $e = \bigwedge_{i \in I} e_i$.

Proof. We have to show that $\bigcap_{i \in I} \mathcal{H}_{N_i} = \mathcal{H}_N$. For all $i \in I$ we have $\mathfrak{N}_{\varphi} \cap N \subset \mathfrak{N}_{\varphi} \cap N_i$ and hence $\mathcal{H}_N \subset \bigcap_{i \in I} \mathcal{H}_{N_i}$. To prove the other inclusion let us consider the projection f of \mathcal{H}_{φ} onto $\bigcap_{i \in I} \mathcal{H}_{N_i}$. Since $e_i \Delta_{\varphi} \subset \Delta_{\varphi} e_i$ (see pag. 131 of [3]), Δ_{φ} is affiliated with $\{e_i\}'$ for each $i \in I$ and hence, by Proposition 2.1 it is affiliated with $\bigcap_{i \in I} \{e_i\}' \subset \{f\}'$. It follows that $f \Delta_{\varphi}^{1/2} \subset \Delta_{\varphi}^{1/2} f$ and thus that $D(S_{\varphi}) \cap f \mathcal{H}_{\varphi} = f D(S_{\varphi})$.

Now let $\xi \in D(S_{\varphi})$. Then $f\xi \in \bigcap_{i \in I} (\mathcal{H}_{N_i} \cap D(S_{\varphi}))$ and by Propositions 2.1 and 3.2 $L_{f\xi}$ is affiliated with $\bigcap_{i \in I} N_i = N$. It follows that $eL_{f\xi} \subset L_{f\xi}e$ and hence that $eR_{\eta}f\xi = R_{e\eta}f\xi$ for every $\eta \in \mathfrak{A}'_{\varphi}$. Thus, using the fact that $\mathfrak{A}'_{\varphi} = J_{\varphi}\mathfrak{A}_{\varphi}$, we find $R_{eJ_{\varphi}x_{\varphi}}f\xi \in e\mathcal{H}_{\varphi}$ for all $x \in \mathfrak{A}_{\varphi}$. From the equalities $R_{J_{\varphi}y_{\varphi}} = J_{\varphi}\pi_{\varphi}(y)J_{\varphi}, y \in \mathfrak{A}_{\varphi}$ (see pag. 26 of [3]) and $J_{\varphi}e = eJ_{\varphi}$ it follows that $J_{\varphi}\pi_{\varphi}(E(x))J_{\varphi}f\xi \in e\mathcal{H}_{\varphi}$ for every $x \in \mathfrak{A}_{\varphi}$, where E is the faithful normal conditional expectation of M onto N satisfying $\varphi \circ E = \varphi$. Hence, being $\mathfrak{A}_{\varphi} \sigma$ -weakly dense in M and E normal, we find $f\xi \in e\mathcal{H}_{\varphi} = \mathcal{H}_N$. Since $\xi \in D(S_{\varphi})$ was arbitrary we can conclude that $f\mathcal{H}_{\varphi} \subset \mathcal{H}_N$.

Corollary 3.4. Let M be a von Neumann algebra on a Hilbert space \mathfrak{H} with a cyclic and separating unit vector Ω and let ω be the faithful normal state on M given by $\omega(\cdot) = (\Omega, \cdot \Omega)$. Assume that for a given family $\{N_i : i \in I\}$ of von Neumann subalgebras of M there exist faithful normal conditional expectations E_i of M onto N_i satisfying $\omega = \omega \circ E_i$ for all $i \in I$ and let $N = \bigcap_{i \in I} N_i$. Then the Jones projections e_i , e of \mathfrak{H} onto the closed subspaces $\overline{N_i\Omega}$ and $\overline{N\Omega}$ respectively satisfy $e = \bigwedge_{i \in I} e_i$.

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References

- [1] V.F.R. Jones and F. Xu: Intersections of finite families of finite index subfactors. *Internat. J. Math.* **15** (2004) 717-733.
- [2] C. F. Skau: Finite subalgebras of a von Neumann algebra. J. Funct. Anal. 25 (1977) 211-235.
- [3] S. Strătilă: Modular theory in operator algebras. Abacus Press, Tunbridge Wells, Kent (1981).
- [4] Takesaki M.: Conditional expectations in von Neumann algebras. J. Funct. Anal. 9 (1972) 306–321.