DOMAIN INVARIANCE FOR LOCAL SOLUTIONS OF SEMILINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

P. CANNARSA, G. DA PRATO, AND H. FRANKOWSKA

ABSTRACT. A closed set K of a Hilbert space H is said to be invariant under the evolution equation

$$X'(t) = AX(t) + f\big(t, X(t)\big) \qquad (t > 0)$$

whenever all solutions starting from a point of K, at any time $t_0 \geqslant 0$, remain in K as long as they exist.

For a self-adjoint strictly dissipative operator A, perturbed by a (possibly unbounded) nonlinear term f, we give necessary and sufficient conditions for the invariance of K, formulated in terms of A, f, and the distance function from K. Then, we also give sufficient conditions for the viability of K for the control system

$$X'(t) = AX(t) + f\big(t, X(t), u(t)\big) \qquad (t > 0,\ u(t) \in U).$$

Finally, we apply the above theory to a bilinear control problem for the heat equation in a bounded domain of $\mathbb{R}^N$, where one is interested in keeping solutions in one fixed level set of a smooth integral functional.

Date: January 7, 2019.

2010 *Mathematics Subject Classification.* 58D25, 47H06, 37L25.

Key words and phrases. Evolution equations, invariance, viability, dissipative operators, distance function.

The first author wishes to express his gratitude to the Institut de Mathématiques de Jussieu - Paris Rive Gauche for supporting his visit through a CNRS Poste Rouge. He also thankfully acknowledges support from the following institutions/research programs:

- MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP: E83C18000100006;
- Mission Sustainability 2017 Program of the University of Rome Tor Vergata, CUP: E86C18000550005;
- Istituto Nazionale di Alta Matematica, within the LIA COPDESC network and EuLaMS Project.

The second author was partially supported by AFOSR, grant no. FA 9550-18-1-0254, and by the "FMJH Program Gaspard Monge in optimization and operation research" under grant PGMO 2018-0047H.

Corresponding author: Piermarco Cannarsa.

Declarations of interest: none.

1. Introduction

In a real separable Hilbert space H, with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$, we consider the evolution equation

$$\begin{cases} X'(t) = AX(t) + f\big(t, X(t)\big), \quad t > t_0 \\ X(t_0) = x_0 \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a densely defined negative self-adjoint linear operator on H and $f : [0,\infty[\times D((-A)^{\theta/2}) \to H$, with $(-A)^{\theta/2}$ standing for the fractional power of $-A$ for any $\theta \in [0,1]$, is assumed to be locally square summable in t and locally Lipschitz in x. Then, it is well-known that, for any $(t_0, x_0) \in [0,\infty[\times D((-A)^{\theta/2})$, problem (1.1) has a unique maximal solution $X(\cdot; t_0, x_0)$, defined on some interval $[t_0, T(t_0, x_0)[$ which can be proved to equal $[t_0,\infty[$ under additional assumptions. In the latter case, we say that the maximal solution is global.

We say that a nonempty closed set $K \subset H$ is invariant under (1.1) if, for all $t_0 \geqslant 0$ and $x_0 \in K$, we have that $X(t; t_0, x_0) \in K$ for all $t \in [t_0, T(t_0, x_0)[$.

More generally, given a complete separable metric space U—called the control space—and a Lebesgue measurable map $u : [t_0,\infty[\to U$, one can also consider the control system

$$\begin{cases} X'(t) &= AX(t) + f(t, X(t), u(t)), \quad u(t) \in U \\ X(t_0) &= x_0. \end{cases} \tag{1.2}$$

We denote by $X(\cdot; t_0, x_0, u)$ the maximal solution of (1.2) and by $T(t_0, x_0, u)$ the right end-point of the interval on which it is defined. Then, K is called *viable* under (1.2) if for every initial condition $(t_0, x_0) \in \mathbb{R}_+ \times K$ there exists a control function $u : [t_0,\infty) \to U$ such that $X(t; t_0, x_0, u) \in K$ for all $t \in [t_0, T(t_0, x_0, u)[$. Notice that the notion of invariance can be introduced even for control systems by requiring that, for every $(t_0, x_0) \in \mathbb{R}_+ \times K$ and every $u : [t_0,\infty) \to U$, $X(t; t_0, x_0, u) \in K$ for all $t \in [t_0, T(t_0, x, u)[$. Such a property is clearly more restrictive than viability.

The analysis of control systems is one motivation for assuming just measurability in time for f in (1.1). Indeed, such settings allow to regard (1.2) as a special case of (1.1) not only as far as well-posedness is concerned but for some invariance issues as well. On the other hand, viability is a different notion which requires a specific treatment, as we explain below.

There is an extensive literature addressing domain invariance issues in infinite dimensional spaces, although only part of it can be applied to partial differential equations. For instance, Martin [11] studied the invariance of K under equation (1.1) in Banach spaces, in the special case of $A = 0$, extending the classical condition introduced by Nagumo [12]. Then, Pavel [13] established necessary and sufficient conditions for the invariance of K under (1.1) in Banach spaces, assuming the semigroup generated by A, e^{tA}, to be compact, f continuous in both variables, and $\theta = 0$. In [14], the same author removed the compactness hypothesis on e^{tA} replacing it with the dissipativity of f. In both papers [13] and [14], the condition for invariance takes the form

$$\lim_{\lambda\downarrow 0} \frac{d_K\big(e^{\lambda A}x + \lambda f(t,x)\big)}{\lambda} = 0 \quad \forall t \in [0,\infty[\,,\ \forall x \in K. \tag{1.3}$$

Later on, in [17], Shi studied an analogous viability problem for a differential inclusion, that is, for a set-valued map f. Assuming both K and e^{tA} to be compact, he derived a necessary and sufficient condition, analogous to (1.3), with $\lim_{\lambda\downarrow 0}$ replaced by $\liminf_{\lambda\downarrow 0}$ (lower Dini derivative). A detailed exposition of this theory, and more, can be found in the monograph [8] by O. Carja, M. Necula and K. Vrabie. We observe that a common feature of the above results is that conditions for invariance are:

- imposed at (boundary) points of K, and
- expressed in terms of the semigroup e^{tA}.

In [7], assuming $f = f(x)$ in (1.1) to be continuous and quasi-dissipative, we showed that a necessary and sufficient condition for the invariance of K is that all points $x \in D(A)\setminus K$, sufficiently close to K, satisfy the inequality

$$D^- d_K(x)\,(Ax + f(x)) \leqslant C\, d_K(x) \tag{1.4}$$

for some constant $C \geqslant 0$. Notice that (1.4) is formulated just in terms of the generator A, which is simpler to use in applications.

In order to treat more general control systems, it is convenient to allow the nonlinear term f to be defined on suitable subspaces of H, such as $D((-A)^{\theta/2})$. For instance, consider the controlled heat equation

$$\begin{cases} \frac{\partial X}{\partial t}\,(t,\xi) = \Delta X(t,\xi) + g(t)X(t,\xi) & (t>0, \xi \in \mathcal{O}) \\ X = 0, & \text{on } (0,\infty)\times\partial\mathcal{O} \\ X(0,\xi) = x_0(\xi), & \xi \in \mathcal{O}, \end{cases} \tag{1.5}$$

where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary and $g \in L^\infty(0,T)$ is called a bilinear control. For a given smooth convex function $\phi : \mathbb{R} \to \mathbb{R}$, an interesting viability problem is to find conditions to ensure the existence of a control g such that the corresponding maximal solution X_g of (1.5) remains in the ϕ-energy level of x_0, that is

$$\int_{\mathcal{O}} \phi\big(X_g(t,\xi)\big)d\xi = \int_{\mathcal{O}} \phi\big(x_0(\xi)\big)d\xi \qquad \forall t \in [0, T(0,x_0)[. \tag{1.6}$$

We note that a special case of the above problem, for $\phi(s) = s^2$, is considered by Caffarelli and Lin in [5]. As we show in section 4 of this paper, the problem of determining g to satisfy (1.6) can be reduced to the invariance, under (1.1), of a suitable level set of the integral functional associated with ϕ. In this case, f in (1.1) turns out to be defined on the domain of $(-A)^{1/2}$.

More generally, in section 3, we study the invariance of $K \cap D((-A)^{\theta/2})$ under (1.1). First, we give sufficient conditions for invariance that are formulated—like (1.4)—in terms of the lower Dini derivatives of the distance function d_K at points of $D(A)$ which are exterior to K (Theorem 3.2

and Corollary 3.3). Then, under a further assumption connecting K with $(-A)^{\theta/2}$, we show that the above conditions are also necessary (Theorem 3.7 and Corollary 3.9). These results become particularly simple when K is proximally smooth since, in this case, Dini derivatives are replaced by scalar products with proximal normals (Theorem 3.4 and Theorem 3.10). Notably, in the latter case, one only needs to impose conditions on $\partial K \cap D(A)$.

Since f in (1.1) or (1.2) is just measurable in time, an essential technical tool of our approach is a Scorza-Dragoni type theorem, which is the object of Proposition 2.9. Such a result ensures the existence of a negligible subset of times $\mathcal{N} \subset \mathbb{R}_+$ such that, for every $(t_0, x_0) \in \big(\mathbb{R}_+ \setminus \mathcal{N}\big) \times D(A)$, the maximal solution $X(\cdot; t_0, x_0)$ of problem (1.1) is differentiable at t_0.

As for viability of the control system (1.2), when K is invariant under the action of e^{tA}, we can provide a sufficient condition under an additional compactness assumption for e^{tA}. Such a condition is given in terms of Clarke's derivatives of the distance to K (Theorem 5.2).

Although both invariance and viability are of great interest in their own right, these properties also have applications to other important issues in dynamical systems. For example, they can be used to derive lower bounds for the blow-up time of solutions, possibly yielding that solutions are global. The semilinear problem we discuss in section 4.3 below is a case in point.

Finally, we would like to stress the fact that this paper is restricted to Hilbert space settings for two main reasons: the use of maximal L^2-regularity for solutions of linear evolution equations and the convenience of formulating our condition for invariance in terms of proximal normals. Part of our results could certainly be extended to suitable classes of Banach spaces, and also to set-valued operators associated with maximal dissipative graphs.

This paper is organized as follows. In section 2, we discuss assumptions and well-posedness for problems (1.1) and (1.2). In section 3, we derive our conditions for invariance. In section 4, we study a bilinear control problem for system (1.5), transforming it into an invariance problem, and use invariance to extract nontrivial information on the maximal time of existence. In section 5, we analyse viability under the control system (1.2). In section 6, we prove proximal smoothness for level sets of certain integral functionals.

2. Preliminaries

Let H be a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. For any $x \in H$ and $r > 0$ we set

$$B_r(x) = \big\{ y \in H \ : \ |y - x| < r \big\}, \quad B_r = B_r(0).$$

We denote by $\mathcal{L}_1$ the Lebesgue measure on $\mathbb{R}$. We set

$$\mathbb{R}_+ = [0, \infty[, \quad \mathbb{R}_+^* =]0, \infty[, \quad \text{and} \quad \lfloor r \rfloor = \min \big\{ n \in \mathbb{N} \ : \ r \leqslant n \big\} \ \ \forall r \in \mathbb{R}_+.$$

We say that a measurable set $F \subset \mathbb{R}_+$ is of full measure if $\mathcal{L}_1\big(\mathbb{R}_+ \setminus F\big) = 0$.

For any $(t_0, x_0) \in \mathbb{R}_+ \times H$, consider the Cauchy problem

$$\begin{cases} X'(t) = AX(t) + f\big(t, X(t)\big), & t > t_0 \\ X(t_0) = x_0 \end{cases} \tag{2.1}$$

where A is a linear operator on H and f is a nonlinear term. In this paper, both A and f are allowed to be unbounded.

2.1. **Assumptions.** In this section, we introduce the assumptions that will be enforced in the rest of the paper. We begin with operator A which is hereafter supposed to satisfy the following.

(H_A)**:** *$A : D(A) \subset H \to H$ is a densely defined self-adjoint linear operator such that $A \leqslant -\omega I$ for some $\omega > 0$.*

Assumption (H_A) ensures that A is the infinitesimal generator of a strongly continuous semigroup of contractions on H, which we denote by e^{tA} $(t \geqslant 0)$. Moreover, e^{tA} is analytic and $e^{tA}x \in D(A)$ for all $t > 0$ and $x \in H$.

For any $\theta \in \mathbb{R}$ we denote by $(-A)^\theta$ the *fractional powers* of $-A$ (see, for instance, [15, Section 2.6]) and we set

$$H_\theta = D\big((-A)^{\theta/2}\big) \quad \text{with} \quad |x|_\theta = |(-A)^{\theta/2}x|, \;\; \forall x \in H_\theta \quad (\theta \geqslant 0)$$

We recall that, for all $\theta \geqslant 0$,

$$\omega^{\theta/2}|(-A)^{-\theta/2}x| \leqslant |x| \qquad \forall x \in H, \tag{2.2}$$

$$\omega^{\theta/2}|x| \leqslant |x|_\theta \qquad \forall x \in H_\theta, \tag{2.3}$$

and, for all $0 \leqslant \theta \leqslant 1$,

$$\omega^{(1-\theta)/2}|x|_\theta \leqslant |x|_1 \qquad \forall x \in H_1, \tag{2.4}$$

where $\omega > 0$ is the constant given by (H_A).

We now give the assumptions on the nonlinear term f.

(H_f)**:** *$f : \mathbb{R}_+ \times H_\theta \to H$ for some $\theta \in [\,0, 1\,]$ and*

(a) *for all $x \in H_\theta$, $t \mapsto f(t, x)$ is Lebesgue measurable on $[\,0, \infty\,[$;*

(b) *there exists a function $L : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, with*

$$\begin{cases} r \mapsto L(t, r) \text{ nondecreasing for a.e. } t \geqslant 0 \\ t \mapsto L(t, r) \text{ locally square-summable for all } r \geqslant 0, \end{cases} \tag{2.5}$$

such that, for every $r \geqslant 0$ and a.e. $t \geqslant 0$,

$$\big|f(t, x)\big| \leqslant L(t, r) \tag{2.6}$$

and

$$\big|f(t, x) - f(t, y)\big| \leqslant L(t, r)|x - y|_\theta \tag{2.7}$$

for all $x, y \in H_\theta$ with $|x|_\theta, |y|_\theta \leqslant r$.

Remark 2.1. Observe that, for a.e. $t \geqslant 0$ and every $r \geqslant 0$, (2.7) yields

$$\begin{aligned}\langle f(t,x)-f(t,y),x-y\rangle &\leqslant \big|f(t,x)-f(t,y)\big|\,|x-y|\\ &\leqslant L(t,r)\,|x-y|_\theta|x-y| \end{aligned}\tag{2.8}$$

for all $x,y \in H_\theta$ with $|x|_\theta, |y|_\theta \leqslant r$. Consequently, for every $\epsilon > 0$, every $r \geqslant 0$, and every x, y as above we have that

$$\langle f(t,x)-f(t,y),x-y\rangle \leqslant \frac{\epsilon}{2}\,|x-y|_\theta^2 + \frac{L(t,r)^2}{2\epsilon}\,|x-y|^2. \tag{2.9}$$

Typical examples of partial differential equations which can be recast in the form (2.1), with A and f satisfying assumptions (H_A) and (H_f) respectively, are semilinear parabolic equations.

Example 2.2. Let $\mathcal{O} \subset \mathbb{R}^n$ $(n \geqslant 3)$ [(1)] be a bounded domain with boundary of class C^2. Consider the semilinear initial-boundary value problem

$$\begin{cases} \frac{\partial X}{\partial t} = \Delta X + F(t,\xi,X) & \text{in }]0,\infty[\times\mathcal{O}\\ X = 0 & \text{on }]0,\infty[\times\partial\mathcal{O}\\ X(0,\xi) = x_0(\xi) & \xi\in\mathcal{O} \text{ a.e.}\end{cases}\tag{2.10}$$

where $F : \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is such that

$$\begin{cases} v \mapsto F(t,\xi,v) \text{ is of class } C^1(\mathbb{R}) \text{ for a.e. } (t,\xi)\in\mathbb{R}_+\times\mathcal{O},\\ (t,\xi)\mapsto F(t,\xi,v) \text{ is Lebesgue measurable for all } v\in\mathbb{R},\end{cases}\tag{2.11}$$

and satisfies, for some given

$$p \in \Big[1, \frac{n}{n-2}\Big],\ \ \varphi_0 \in L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{O})),\ \text{ and }\ \varphi_1 \in L^2_{loc}(\mathbb{R}_+; L^n(\mathcal{O})),\tag{2.12}$$

the growth conditions

$$|F(t,\xi,v)| \leqslant C_0\big(\varphi_0(t,\xi) + |v|^p\big)\tag{2.13}$$

$$\Big|\frac{\partial F}{\partial v}(t,\xi,v)\Big| \leqslant C_1\big(\varphi_1(t,\xi) + |v|^{p-1}\big)\tag{2.14}$$

for a.e. $(t,\xi) \in \mathbb{R}_+ \times \mathcal{O}$, every $v \in \mathbb{R}$, and some constants $C_0, C_1 \geqslant 0$.

We will now show that the above problem can be recast as an evolution equation like (2.1), for some operator A and nonlinear map f satisfying assumptions (H_A) and (H_f), respectively. In the Hilbert space $H = L^2(\mathcal{O})$, with norm

$$|x| = \Big(\int_{\mathcal{O}} |x(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \qquad \forall x \in L^2(\mathcal{O}),$$

define the linear operator A by

$$\begin{cases} D(A) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O})\\ Ax = \Delta x & \forall x\in D(A).\end{cases}\tag{2.15}$$

[(1)]We assume $n \geqslant 3$ for simplicity. The analysis of this example is even simpler for $n = 1, 2$.

It is well known that A satisfies (H_A) and $H_1 = H_0^1(\mathcal{O})$. Moreover, in view of *Poincaré's inequality*

$$\int_{\mathcal{O}} |x(\xi)|^2 d\xi \leqslant C_{\mathcal{O}}^2 \int_{\mathcal{O}} |\nabla x(\xi)|^2 d\xi \qquad \forall x \in H_0^1(\mathcal{O}), \tag{2.16}$$

an equivalent *norm* in H_1 is the following

$$|x|_1 = \Big(\int_{\mathcal{O}} |\nabla x(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \qquad \forall x \in H_0^1(\mathcal{O}),$$

with the associated scalar product

$$\langle x, y \rangle_1 = \int_{\mathcal{O}} \nabla x(\xi) \cdot \nabla y(\xi)\, d\xi \qquad \forall x, y \in H_0^1(\mathcal{O}).$$

Moreover, appealing to the Sobolev embedding theorem, we have that

$$H_0^1(\mathcal{O}) \subset L^p(\mathcal{O}) \quad \forall p \in \Big[1, \frac{2n}{n-2}\Big]$$

and

$$\Big(\int_{\mathcal{O}} |x(\xi)|^p d\xi \Big)^{\frac{1}{p}} \leqslant C_p(\mathcal{O}) \Big(\int_{\mathcal{O}} |\nabla x(\xi)|^2 d\xi \Big)^{\frac{1}{2}} \quad \forall x \in H_0^1(\mathcal{O}). \tag{2.17}$$

Therefore, (2.11), (2.13), and (2.17) ensure that the map $f : \mathbb{R}_+ \times H_1 \to H$, defined by

$$f(t,x)(\xi) = F\big(t, \xi, x(\xi)\big) \quad \forall (t,x) \in \mathbb{R}_+ \times H_1,\ \xi \in \mathcal{O} \text{ a.e.}, \tag{2.18}$$

satisfies (H_f)-(a), (2.5), and (2.6). Moreover, by (2.14) we have that

$$\begin{aligned} |f(t,x) - f(t,y)|^2 &= \int_{\mathcal{O}} \big| F\big(t,\xi,x(\xi)\big) - F\big(t,\xi,y(\xi)\big)\big|^2 d\xi \\ &\leqslant C \int_{\mathcal{O}} \Big[\varphi_1(t,\xi)^2 + \big(|x(\xi)| + |y(\xi)|\big)^{2(p-1)} \Big] \big| x(\xi) - y(\xi) \big|^2 d\xi \\ &\leqslant C \Big\{ \int_{\mathcal{O}} \Big[\varphi_1(t,\xi)^n + \big(|x| + |y|\big)^{n(p-1)} \Big] d\xi \Big\}^{\frac{2}{n}} \Big\{ \int_{\mathcal{O}} \big| x - y \big|^{\frac{2n}{n-2}} d\xi \Big\}^{\frac{n-2}{n}} \end{aligned}$$

which gives (2.7). So, f satisfies (H_f) with $\theta = 1$ (see also [10, Example 3.6]).

As we show next, the abstract model (2.1) also allows to treat equations with nonlocal terms.

Example 2.3. In $\mathcal{O} \subset \mathbb{R}^n$ as in the above example, consider the initial-boundary value problem for the heat operator with a nonlocal source term

$$\begin{cases} \frac{\partial X}{\partial t} = \Delta X + \Big(\int_{\mathcal{O}} |\nabla X|^2 d\xi \Big) X & \text{in }]0,\infty[\times \mathcal{O} \\ X = 0 & \text{on }]0,\infty[\times \partial\mathcal{O} \\ X(0,\xi) = x_0(\xi) & \xi \in \mathcal{O} \text{ a.e.} \end{cases} \tag{2.19}$$

In the Hilbert space $H = L^2(\mathcal{O})$, define A as in (2.15) and take

$$f(t,x)(\xi) = |x|_1^2\, x(\xi) \qquad \forall (t,x) \in \mathbb{R}_+ \times H_0^1(\mathcal{O}),\ \xi \in \mathcal{O} \text{ a.e.}$$

Then, one immediately has that f satisfies (H_f)-(a), (2.5), and (2.6) with $\theta = 1$. We proceed to check that (2.7) also holds true. Fix $R > 0$ and let $x, y \in H_0^1(\mathcal{O})$ be such that $|x|_1, |y|_1 \leqslant R$. Then, by (2.16),

$$\begin{aligned}\big|f(t,x) - f(t,y)\big| &= \big||x|_1^2\, x - |y|_1^2\, y\big| \\ &\leqslant \; |x|_1^2\, |x-y| + |y|\big||x|_1^2 - |y|_1^2\big| \\ &\leqslant \; R^2\, |x-y| + 2C_{\mathcal{O}} R^2\, |x-y|_1\end{aligned}$$

which yields (2.7) with $L(t,R) = 3C_{\mathcal{O}}R^2$. □

2.2. **Well-posedness.** In this section, we discuss well-posedness for problem (2.1). These results will be needed in the next sections. Let us consider the linear problem

$$\begin{cases} X'(t) = AX(t) + g(t), \quad t \in]0,T[\\ X(0) = x_0, \end{cases} \tag{2.20}$$

where $g \in L^1(0,T;H)$ and $x_0 \in H$. We recall that $X \in C([0,T];H)$ given by

$$X(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} g(s) ds \qquad \forall t \in [0,T] \tag{2.21}$$

is called the *mild solution* of (2.20). For more regular data, such a solution has additional regularity properties, some of which are summarized below.

Proposition 2.4. *Let $x_0 \in H$. If $g \in L^2(0,T;H)$ then X, given by (2.21), satisfies the equation in (2.20) for a.e. $t \in [0,T]$. If, in addition, $x_0 \in H_\theta$ for some $\theta \in [\,0,1\,]$, then*

$$X \in C([0,T]; H_\theta). \tag{2.22}$$

Furthermore, for $\theta = 1$ we have that

$$X \in H^1(0,T;H) \cap L^2(0,T;D(A)). \tag{2.23}$$

Proof. We begin by observing that property (2.23) is the well-known maximal L^2-regularity of the solution of (2.20), which holds true in Hilbert spaces and in suitable classes of Banach spaces (see, for instance, [3]). This maximal regularity result also ensures that X satisfies the equation in (2.20) for a.e. $t \in [0,T]$.

At this point, we note that the above applies, in particular, to the mild solution

$$G(t) = \int_0^t e^{(t-s)A} g(s) ds \qquad \forall t \in [0,T]$$

of (2.20) with $x_0 = 0$. So, G satisfies

$$G'(t) = AG(t) + g(t) \qquad (t \in]0,T[\text{ a.e.}) \tag{2.24}$$

and, since $H^1(0,T;H) \cap L^2(0,T;D(A)) \subset C([0,T];H_1)$, we also have that $G \in C([0,T];H_1)$. Consequently, $G \in C([0,T];H_\theta)$ by (2.4).

Next, for any $x_0 \in H$ by (2.21) we have that $X(t) = e^{tA}x_0 + G(t)$, where $\frac{d}{dt}\, e^{tA}x_0$ exists and equals $Ae^{tA}x_0$ for every $t > 0$. This fact and (2.24) imply that X satisfies the equation in (2.20) for a.e. $t \in [0, T]$.

Finally, (2.22) holds true if $x_0 \in H_\theta$: indeed

$$t \mapsto (-A)^{\theta/2} e^{tA} x_0 = e^{tA}(-A)^{\theta/2} x_0 \quad (t \geqslant 0)$$

is continuous and $G \in C([0, T]; H_1)$. Inequality (2.4) ends the proof. □

We now turn to the nonlinear problem (2.1) under assumptions $(H_A), (H_f)$. Let $(t_0, x_0) \in \mathbb{R}_+ \times H_\theta$.

Definition 2.5. *A mild solution of* (2.1) *on the time interval* $[t_0, T]$ *is a vector-valued function* $X \in C\big([t_0, T]; H_\theta\big)$ *such that*

$$X(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A} f\big(s, X(s)\big)ds \qquad \forall t \in [t_0, T]. \tag{2.25}$$

Notice that, for $X \in C\big([t_0, T]; H_\theta\big)$, the function $g(s) = f\big(s, X(s)\big)$ belongs to $L^2(0, T; H)$ by (2.6). So, the integral in (2.25) makes sense.

Our next result establishes the local well-posedness of problem (2.1).

Proposition 2.6. *Assume* (H_A) *and* (H_f). *Then, for any* $T, R > 0$ *there exists* $\tau = \tau(T, R) > 0$ *such that, for every* $(t_0, x_0) \in \mathbb{R}_+ \times H_\theta$ *with* $t_0 < T$ *and* $|x_0|_\theta \leqslant R$, *system* (2.1) *has a unique mild solution on* $[t_0, t_0 + \tau]$. *Such a solution satisfies equation* (2.1) *for a.e.* $t \in [t_0, t_0 + \tau]$ *as well as the bound*

$$|X(t)|_\theta \leqslant 2R \qquad \forall t \in [t_0, t_0 + \tau]. \tag{2.26}$$

Moreover, there exists a function $\ell : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *with*

$$\begin{cases} r \mapsto \ell(t, r) & \text{nondecreasing for a.e. } t \geqslant 0 \\ t \mapsto \ell(t, r) & \text{locally summable for all } r \geqslant 0, \end{cases}$$

such that, for any $y_0 \in H_\theta$ *with* $|y_0|_\theta \leq R$, *the corresponding solution* Y *of* (2.1) *with initial condition* y_0 *satisfies*

$$|X(t) - Y(t)| \leqslant e^{\int_{t_0}^{t} \ell(s,R)ds} |x_0 - y_0| \quad \forall\, t \in [t_0, t_0 + \tau]. \tag{2.27}$$

and

$$|X(t) - Y(t)|_\theta \leqslant e^{\int_{t_0}^{t} \ell(s,R)ds} |x_0 - y_0|_\theta \quad \forall\, t \in [t_0, t_0 + \tau]. \tag{2.28}$$

Furthermore, for $\theta = 1$ *we have that*

$$X \in H^1(t_0, t_0 + \tau; H) \cap L^2(t_0, t_0 + \tau; D(A)). \tag{2.29}$$

Proof. Let T and R be fixed positive numbers. The existence and uniqueness of the mild solution to (2.1)—as well as estimate (2.26)—can be obtained by a standard fixed-point argument for the map $\Phi : \mathcal{X}_\tau \to \mathcal{X}_\tau$ defined by

$$\Phi(X)(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A} f\big(s, X(s)\big)ds \quad \big(X \in \mathcal{X}_\tau,\, t \in [t_0, t_0 + \tau]\big),$$

where $0 < \tau \leqslant T - t_0$ has to be properly chosen and

$$\mathcal{X}_\tau = \Big\{ X \in C\big([t_0, t_0+\tau]; H_\theta\big) \ : \ \sup_{t_0 \leqslant t \leqslant t_0+\tau} |X(t)|_\theta \leqslant 2R \Big\}.$$

Once the mild solution X of (2.1), satisfying (2.26), has been constructed, one can appeal to Proposition 2.4 with

$$g(t) = f\big(t, X(t)\big) \quad (t \in [t_0, t_0+\tau]),$$

which belongs to $L^2(t_0, t_0+\tau; D(A))$, to deduce the fact that

$$X'(t) = AX(t) + f\big(t, X(t)\big)$$

for a.e. $t \in [t_0, , t_0+\tau]$. For $\theta = 1$, (2.29) follows from the same proposition.

We now proceed to justify (2.27) and (2.28). Let Y be the mild solution of (2.1) with any initial condition $y_0 \in H_\theta$ such that $|y_0|_\theta \leq R$. Then,

$$(X-Y)'(t) = A(X-Y)(t) + f\big(t, X(t)\big) - f\big(t, Y(t)\big) \quad (t \in [t_0, t_0+\tau] \text{ a.e.})$$

So, taking the scalar product of both sides of the above identity by $X - Y$, in view of (2.9) with $\epsilon = 2\omega^{1-\theta}$ and (2.4) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|X-Y|^2 \\ &\quad\leqslant -|X-Y|_1^2 + \omega^{1-\theta}|X-Y|_\theta^2 + \frac{L(t,2R)^2}{4\omega^{1-\theta}}|X-Y|^2 \\ &\quad\leqslant \frac{L(t,2R)^2}{4\omega^{1-\theta}}|X-Y|^2. \end{aligned} \tag{2.30}$$

Thus, (2.27) follows by Gronwall's lemma with $\ell(t,R) = \frac{L(t,2R)^2}{4\omega^{1-\theta}}$.

Estimate (2.28) can be deduced in a similar way. This time, taking the scalar product with $(-A)^\theta(X-Y)$ and recalling (2.7), we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|X-Y|_\theta^2 \\ &\quad= -|X-Y|_{1+\theta}^2 + \langle f(t,X) - f(t,Y), (-A)^\theta(X-Y)\rangle \\ &\quad\leqslant -|X-Y|_{1+\theta}^2 + L(t,2R)\,|X-Y|_\theta\,|X-Y|_{2\theta}. \end{aligned}$$

Thus, instead of (2.30) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|X-Y|_\theta^2 \\ &\quad\leqslant -|X-Y|_{1+\theta}^2 + \omega^{1-\theta}|X-Y|_{2\theta}^2 + \frac{L(t,2R)^2}{4\omega^{1-\theta}}|X-Y|_\theta^2. \end{aligned} \tag{2.31}$$

Since (2.4) yields

$$\begin{aligned} \omega^{1-\theta}|X-Y|_{2\theta}^2 &= \omega^{1-\theta}|(-A)^\theta(X-Y)|^2 \\ &\leqslant |(-A)^{\theta/2}(X-Y)|_1^2 = |X-Y|_{1+\theta}^2, \end{aligned}$$

we derive (2.28) by Gronwall's lemma, with the same function ℓ as above. $\square$

Once the existence and uniqueness of the local mild solution has been proved, one constructs the *maximal solution* of problem (2.1), $X(\cdot;t_0,x_0)$, which exists on the maximal interval $[t_0,T(t_0,x_0)[$ defined as follows:

$$T(t_0,x_0)=\sup\big\{T\geqslant t_0\ :\ \exists\,X(\cdot)\text{ mild solution of (2.1) on }[t_0,T]\big\}. \quad (2.32)$$

Then, for any fixed $T\in[t_0,T(t_0,x_0)[$, thanks to the uniqueness property of Proposition 2.6 one has that $X(t;t_0,x_0)=X(t)$ for all $t\in[t_0,T]$ where X is the mild solution of (2.1) on $[t_0,T]$. Moreover, $X(\cdot;t_0,x_0)$ satisfies (2.29), (2.27), and (2.28) on all compact subintervals of $[t_0,T(t_0,x_0)[$.

2.3. **Control systems.** We now want to extend the above well-posedness result to the semilinear control system

$$\begin{cases} X'(t) &=& AX(t)+f(t,X(t),u(t)), \quad u(t)\in U\\ X(t_0) &=& x_0, \end{cases} \quad (2.33)$$

where $u:[t_0,\infty[\to U$ is Lebesgue measurable and U is a complete separable metric space. This goal is obtained at essentially no cost, because in the previous section f has been assumed to be measurable in time. We just need to adapt assumption (H_f) to control systems, as we do next.

(H_f'): *$f:\mathbb{R}_+\times H_\theta\times U\to H$ for some $\theta\in[0,1]$ and satisfies the following:*

(a) *for all $(x,u)\in H_\theta\times U$, the map $t\mapsto f(t,x,u)$ is Lebsesgue measurable on $\mathbb{R}_+$;*
(b) *for a.e. $t\geqslant 0$, the map $(x,u)\mapsto f(t,x,u)$ is continuous;*
(c) *$f_u(t,x):=f(t,x,u)$ satisfies (H_f) uniformly in $u\in U$.*

Directly from Proposition 2.6 we deduce the well-posedness of (2.33) for any $(t_0,x_0)\in\mathbb{R}_+\times H_\theta$ and any measurable control $u:[t_0,\infty[\to U$. We denote by $X(\cdot;t_0,x_0,u)$ the maximal solution of such a problem.

As is well-known, $X(\cdot;t_0,x_0,u)$ is global under an additional assumption.

Proposition 2.7. *Assume $(H_A),(H_f')$, and suppose there exists a nonnegative function $c\in L^2_{loc}(\mathbb{R}_+)$ such that*

$$\big|f(t,x,u)\big|\leqslant c(t)(1+|x|_\theta)\quad\forall\,x\in H_\theta, \quad (2.34)$$

for a.e. $t\geqslant 0$ and every $u\in U$. Then, for any $(t_0,x_0)\in\mathbb{R}_+\times H_\theta$, the maximal solution of (2.33) is global, that is, $T(t_0,x_0)=\infty$ and for every $T>t_0$ there exists a constant $C_T>0$ such that

$$|X(t;t_0,x_0,u)|_\theta\leqslant C_T(1+|x_0|_\theta)\quad\forall t\in[t_0,T]. \quad (2.35)$$

Proof. Fix any $(t_0,x_0)\in\mathbb{R}_+\times H_\theta$ and let $T\in]t_0,T(t_0,x_0)[$. Since

$$X'(t)=AX(t)+f\big(t,X(t),u(t)\big)\quad(t\in[t_0,T]\text{ a.e.}),$$

where $X(\cdot) = X(\cdot; t_0, x_0, u)$, taking the scalar product of both sides of the above identity by $(-A)^\theta X$ and arguing as in the proof of (2.31) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|X(t)|_\theta^2 &\leqslant -|X(t)|_{1+\theta}^2 + \omega^{1-\theta}|X(t)|_{2\theta}^2 + \frac{c(t)^2}{4\omega^{1-\theta}}\left(1+|X(t)|_\theta\right)^2 \\ &\leqslant \frac{c(t)^2}{4\omega^{1-\theta}}\left(1+|X(t)|_\theta\right)^2 \quad (t \in [t_0,T] \text{ a.e.})\end{aligned}$$

Then, by Gronwall's lemma we conclude that

$$|X(t)|_\theta \leqslant C_T(1+|x_0|_\theta) \quad \forall t \in [t_0,T]$$

for some constant $C_T > 0$. This implies that the maximal solution of (2.33) is global and satisfies (2.35). □

Next, we state a Scorza-Dragoni type theorem that can be easily deduced from [4, Theorem 1].

Theorem 2.8. *Let $T > 0$, Y and Z be separable metric spaces and consider a Carathéodory map $F : [0,T] \times Y \to Z$. Then, for any $\varepsilon > 0$ there exists a compact set $\mathcal{T}_\varepsilon \subset [0,T]$ with $\mathcal{L}_1([0,T] \backslash \mathcal{T}_\varepsilon) < \varepsilon$ such that the restriction of F to $\mathcal{T}_\varepsilon \times Y$ is continuous.*

Recall that $F : [0,T] \times Y \to Z$ is called a *Carathéodory map* if

$$\begin{cases} t \mapsto F(t,y) \ \text{ is Lebesgue measurable for all } y \in Y, \\ y \mapsto F(t,y) \ \text{ is continuous for a.e. } t \in [0,T]. \end{cases}$$

From the above theorem we deduce the very useful result below.

Proposition 2.9. *Under assumptions (H_A) and (H_f') there exists a set $\mathcal{N} \subset \mathbb{R}_+$, of Lebesgue measure zero, such that for every*

$$(t_0, x_0, u_0) \in \big(\mathbb{R}_+ \setminus \mathcal{N}\big) \times H_\theta \times U,$$

the maximal solution $X(\cdot) = X(\cdot; t_0, x_0, u_0)$ of (2.33), with the constant control $u(\cdot) = u_0$, satisfies

$$\lim_{h \to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} e^{A(t_0+h-s)} f(s, X(s), u_0)ds = f(t_0, x_0, u_0).$$

Proof. Having fixed any $T \geqslant 0$, we will construct a set (of full measure) $\mathcal{M}_T \subset [0,T]$ such that the conclusion holds on $\mathcal{M}_T \times H_\theta \times U$. Then, to obtain our result, it suffices to consider the union, say $\mathcal{M}$, of such sets for a sequence $T_i \uparrow \infty$ and take $\mathcal{N} = \mathbb{R}_+ \setminus \mathcal{M}$.

Define

$$\varepsilon = \frac{1}{i}, \quad \mathcal{T} = [0,T], \quad Y = H_\theta \times U, \quad Z = H.$$

Let $\mathcal{M}_i := \mathcal{T}_{1/i}$ be as in Theorem 2.8 and set $L_j(\cdot) = L(\cdot, j)$ for all $j \geqslant 1$, where the latter is the function in (H_f')-(b). Consider the subset $\widetilde{\mathcal{M}}_i \subset \mathcal{M}_i$ of all points which are both Lebesgue density point for $\mathcal{M}_i$ and Lebesgue points for $L_j\chi_{[0,T]\setminus\mathcal{M}_i}$ and L_j for every integer $j \geqslant 1$. Then $\mathcal{L}_1(\widetilde{\mathcal{M}}_i) = \mathcal{L}_1(\mathcal{M}_i)$. Set $\mathcal{M}_T = \cup_{i\geqslant 1}\widetilde{\mathcal{M}}_i$. Then $\mathcal{M}_T$ is of full measure in $[0, T]$.

Fix any $(t_0, x_0, u_0) \in \mathcal{M}_T \times H_\theta \times U$ and consider the solution $X(\cdot)$ of (2.33) with $u(\cdot) \equiv u_0$ defined on some interval $[t_0, t_0 + \delta]$, where $\delta > 0$. Setting $R := \max_{s\in[t_0,t_0+\delta]} |X(s)|_\theta$, for any integer $j > R$ we have that

$$\left|\int_{t_0}^{t_0+h} e^{(t_0+h-s)A}(f(s, X(s), u_0) - f(s, x_0, u_0))ds\right| \leqslant \int_{t_0}^{t_0+h} L_j(s)|X(s) - x_0|_\theta ds.$$

Furthermore,

$$\frac{1}{h}\int_{t_0}^{t_0+h} L_j(s)|X(s) - x_0|_\theta ds \leqslant \max_{s\in[t_0,t_0+h]} |X(s) - x_0|_\theta \ \frac{1}{h}\int_{t_0}^{t_0+h} L_j(s)ds.$$

Hence

$$\lim_{h\to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} e^{(t_0+h-s)A}\big[f(s, X(s), u_0) - f(s, x_0, u_0)\big]ds = 0.$$

Let $i \geqslant 1$ be such that $t_0 \in \widetilde{\mathcal{M}}_i$. Then for all $j \geqslant 1$

$$\lim_{h\to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} L_j(s)\chi_{[0,T]\setminus\mathcal{M}_i}(s)ds = 0$$

and therefore

$$\lim_{h\to 0+} \frac{1}{h}\int_{t_0}^{t_0+h} e^{(t_0+h-s)A}f(s, x_0, u_0)ds = \lim_{h\to 0+} \int_{[t_0,t_0+h]\cap\mathcal{M}_i} e^{(t_0+h-s)A}f(s, x_0, u_0)ds.$$

By the continuity of $f(\cdot, x_0, u_0)$ on $[t_0, t_0 + h] \cap \mathcal{M}_i$, we have that

$$\lim_{h\to 0+} \frac{1}{h}\int_{[t_0,t_0+h]\cap\mathcal{M}_i} e^{(t_0+h-s)A}f(s, x_0, u_0)ds = \lim_{h\to 0+} \frac{1}{h}\int_{[t_0,t_0+h]\cap\mathcal{M}_i} e^{(t_0+h-s)A}f(t_0, x_0, u_0)ds = \lim_{h\to 0+} \frac{1}{h}\int_{[t_0,t_0+h]\cap\mathcal{M}_i} f(t_0, x_0, u_0)ds.$$

Next, notice that

$$\frac{1}{h}\int_{[t_0,t_0+h]\cap\mathcal{M}_i} f(t_0, x_0, u_0)ds = \frac{1}{h}\mathcal{L}_1([t_0, t_0 + h] \cap \mathcal{M}_i)\, f(t_0, x_0, u_0)$$

which implies, by the choice of t_0 and i, that

$$\lim_{h\to 0+} \frac{1}{h}\int_{[t_0,t_0+h]\cap\mathcal{M}_i} f(t_0, x_0, u_0)ds = f(t_0, x_0, u_0).$$

Hence, we deduce that

$$\lim_{h\to 0+} \frac{1}{h} \int_{[t_0,t_0+h]\cap \mathcal{M}_i} e^{(t_0+h-s)A} f(s,x_0,u_0)ds = f(t_0,x_0,u_0)$$

completing the proof. □

We would like to underline the importance of Proposition 2.9, which lies in the fact that the negligible set $\mathcal{N}$ is independent of the choice of x_0 and u_0.

By recalling (2.25) and appealing to Proposition 2.9 in the special case of $f(t,x,u) = f(t,x)$, one deduces the following differentiability result.

Corollary 2.10. *Under assumptions* (H_A) *and* (H_f) *there exists* $\mathcal{N} \subset [\,0,\infty[$, *of Lebesgue measure zero, such that for every* $(t_0,x_0) \in \big(\mathbb{R}_+ \setminus \mathcal{N}\big) \times D(A)$ *the maximal solution* $X(\cdot;t_0,x_0)$ *of problem* (2.1) *is differentiable at* t_0 *and*

$$\frac{d}{dt} X(t;t_0,x_0)_{|t=t_0} = Ax_0 + f(t_0,x_0). \tag{2.36}$$

3. Invariance

In all the results of this section, we assume without further notice that:

- (H_A) and (H_f) are satisfied, and
- $K \subset H$ is a nonempty closed set.

We denote by $\theta \in [0,1]$ the number given by (H_f). We know that, for any $(t_0,x_0) \in \mathbb{R}_+ \times H_\theta$, problem (2.1) has a unique solution $X(\cdot;t_0,x_0)$ and we denote by $[t_0, T(t_0,x_0)[$ its interval of existence.

We begin by defining invariance under (2.1).

Definition 3.1. *We say that* $K \cap H_\theta$ *is* invariant *under* (2.1) *if, for all* $t_0 \geqslant 0$ *and* $x_0 \in K \cap H_\theta$, *we have that*

$$X(t;t_0,x_0) \in K \cap H_\theta \quad \forall t \in [t_0, T(t_0,x)[.$$

Denote by $d_K(x)$ the distance of x from K, that is,

$$d_K(x) = \inf_{y\in K} |x-y|, \quad \forall\, x \in H. \tag{3.1}$$

We recall that d_K is Lipschitz continuous (with constant 1) on H and the lower Dini derivative of d_K at $x \in H$ in the direction $v \in H$ is given by

$$D^- d_K(x)\, v = \liminf_{\lambda \downarrow 0} \frac{d_K(x+\lambda v) - d_K(x)}{\lambda}.$$

For any $\delta > 0$ we set

$$K_\delta = \big\{ x \in H \setminus K \;:\; d_K(x) < \delta \big\}. \tag{3.2}$$

3.1. **Sufficient Conditions for Invariance.** We first provide sufficient conditions for the invariance of K, which apply to rather general settings.

Theorem 3.2. *Suppose there exists a set $\mathcal{T} \subset \mathbb{R}_+$, of Lebesgue measure zero, and a number $\delta > 0$ such that, for all $t \in \mathbb{R}_+ \setminus \mathcal{T}$ and all $x \in D(A) \cap K_\delta$,*

$$D^- d_K(x)\,(Ax + f(t,x)) \leqslant C(t, |x|_\theta)\, d_K(x), \tag{3.3}$$

where $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\begin{cases} r \mapsto C(t,r) & \text{nondecreasing for a.e. } t \geqslant 0 \\ t \mapsto C(t,r) & \text{locally summable for all } r \geqslant 0. \end{cases} \tag{3.4}$$

Then $K \cap H_\theta$ is invariant under (2.1).

Proof. Fix any $(t_0, x_0) \in \mathbb{R}_+ \times (K \cap H_\theta)$ and denote by $X(\cdot)$ the maximal solution $X(\cdot; t_0, x_0)$ of (2.1). Let us argue by contradiction assuming there is $T \in]t_0, T(t_0, x_0)[$ such that $X(T) \notin K$. Set

$$\bar{t} = \max\big\{t \in [t_0, T] \ : \ X(t) \in K\big\} \quad \text{and} \quad \bar{x} = X(\bar{t}).$$

We will show that $\bar{t} = T$, which contradicts the assumption $X(T) \notin K$.

Suppose $\bar{t} < T$ and observe that

$$X(t) \notin K \qquad \forall t \in]\bar{t}, T]. \tag{3.5}$$

In light of Proposition 2.6, there exists $T' \in]\bar{t}, T]$ such that $|X(t)|_\theta \leqslant 2|\bar{x}|_\theta$ for every $t \in [\bar{t}, T']$ and equation (2.1) holds true for a.e. $t \in [\bar{t}, T']$. Denote by F the set, of full measure in $[\bar{t}, T']$, of all times t such that condition (3.3) is fulfilled for all $x \in D(A) \cap K_\delta$. Define

$$E = \big\{t \in [\bar{t}, T'] \ : \ X'(t) = AX(t) + f\big(t, X(t)\big)\big\}.$$

Owing to (3.3), for all $t \in E \cap F \setminus \{\bar{t}\}$ we have that

$$D^- d_K(X(t))\,(AX(t) + f(t, X(t))) \leqslant C(t, 2|\bar{x}|_\theta)\, d_K(X(t)). \tag{3.6}$$

Now, since d_K is Lipschitz, the function

$$\phi(t) := d_K\big(X(t)\big) \qquad (t \in [\bar{t}, T'])$$

is absolutely continuous, hence differentiable on a set D of full measure in $[\bar{t}, T']$. Then, in view of (3.6), for all $t \in D \cap E \cap F \setminus \{\bar{t}, T'\}$ we have that

$$\begin{aligned} \phi'(t) &= \lim_{h \downarrow 0} \frac{1}{h}\big[d_K\big(X(t+h)\big) - d_K\big(X(t)\big)\big] \\ &= \lim_{h \downarrow 0} \frac{1}{h}\big\{d_K\big(X(t) + h\big[AX(t) + f(t, X(t))\big]\big) - d_K\big(X(t)\big)\big\} \\ &\leqslant C(t, 2|\bar{x}|_\theta)\phi(t). \end{aligned}$$

Since $D \cap E \cap F$ has full measure in $[\bar{t}, T']$ and $\phi(\bar{t}) = 0$, Gronwall's lemma ensures that $\phi(t) \equiv 0$, or $X(t) \in K$ for all $t \in [\bar{t}, T']$, in contrast with (3.5). So, $\bar{t} = T$, as claimed, and we have reached the announced contradiction. □

When K has further geometric properties, the above sufficient condition can be given in different forms that are easier to handle in specific situations. For any $x \in H$ we denote by $\Pi_K(x)$ the set of *projections* of x onto K, that is, the (possibly empty) subset of K at which the infimum in (3.1) is attained. We recall that K is said to be *proximally smooth*, if $\Pi_K(x)$ is a singleton for all $x \in K_\delta$ and some $\delta > 0$ or, equivalently, if d_k is continuously Frechét differentiable on K_δ. In this case, we have that

$$Dd_K(x) = \frac{x - \Pi_K(x)}{d_K(x)} \qquad \forall x \in K_\delta.$$

Consequently, if K is proximally smooth, then

$$D^- d_K(x)v = \langle Dd_K(x), v\rangle \qquad \forall x \in K_\delta\,,\ \forall v \in H.$$

So, Theorem 3.2 yields the following.

Corollary 3.3. *Let K be proximally smooth and fix any $\delta > 0$ such that Π_K is single-valued on K_δ. Furthermore, suppose that, for a.e. $t \in \mathbb{R}_+$ and every $x \in D(A) \cap K_\delta$,*

$$\langle x - \Pi_K(x), Ax + f(t,x)\rangle \leqslant C(t, |x|_\theta)\, d_K^2(x), \tag{3.7}$$

where $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (3.4).
Then $K \cap H_\theta$ is invariant under (2.1).

In our next result, we propose a condition that needs to be satisfied only at boundary points. For any $x \in K$ we denote by $N_K^P(x)$ the *proximal normal cone* to K at x. We recall that a vector $p \in H$ belongs to $N_K^P(x)$ if and only if, for some $\lambda > 0$, we have that

$$\langle p, y - x\rangle \leqslant \frac{|p|}{2\lambda}\,|y - x|^2 \qquad \forall y \in K. \tag{3.8}$$

Observe that $0 \in N_K^P(x)$ for all $x \in K$. Whenever (3.8) holds for some $p \neq 0$, we say that p is realized by a ball of radius λ. Indeed, in this case, one has that $B_\lambda(x + \lambda p/|p|) \subset H \setminus K$.

Theorem 3.4. *Let K be proximally smooth and fix any $\delta > 0$ such that Π_K is single-valued on K_δ. Suppose $\Pi_K\big(D(A) \cap K_\delta\big) \subset D(A)$ and*

$$\langle p, Ax + f(t,x)\rangle \leqslant 0 \quad \forall\, p \in N_K^P(x) \cap D(A), \tag{3.9}$$

for a.e. $t \in \mathbb{R}_+$ and every $x \in \partial K \cap D(A)$.
Then $K \cap H_\theta$ is invariant under (2.1).

Proof. Let $x \in D(A) \cap K_\delta$ and set $\overline{x} = \Pi_K(x)$. Since $x - \bar{x} \in N_K^P(\bar{x}) \cap D(A)$, (3.9) yields $\langle x - \overline{x}, A\overline{x} + f(t, \overline{x})\rangle \leqslant 0$ for a.e. $t \in \mathbb{R}_+$. Recalling (2.4), by

(2.9) with $\epsilon = 2\omega^{1-\theta}$ we then obtain

$$\begin{aligned}
&\langle x-\overline{x}, Ax+f(t,x)\rangle \\
&\quad = \langle x-\overline{x}, A\overline{x}+f(t,\overline{x})\rangle + \langle x-\overline{x}, A(x-\overline{x})\rangle + \langle x-\overline{x}, f(t,x)-f(t,\overline{x})\rangle \\
&\quad \leqslant -|x-\overline{x}|_1^2 + \omega^{1-\theta}|x-\overline{x}|_\theta^2 + \frac{L\big(t,M(|x|_\theta)\big)^2}{4\omega^{1-\theta}}\,|x-\overline{x}|^2 \\
&\quad \leqslant \frac{L\big(t,M(|x|_\theta)\big)^2}{4\omega^{1-\theta}}\,d_K^2(x).
\end{aligned}$$

Hence (3.7) is satisfied. So, $K\cap H_\theta$ is invariant in view iof Corollary 3.3. □

Example 3.5. In the Hilbert space $H = L^2(\mathcal{O})$, consider the closed convex cone

$$H^+ = \big\{x\in H \ : \ x(\xi)\geqslant 0,\ \xi\in\mathcal{O} \text{ a.e.}\big\}.$$

Denoting by x^+ and x^- the positive and negative parts of $x\in H$, respectively, we have that any $x\in H$ can be represented as $x = x^+ - x^-$. So,

$$\Pi_{H^+}(x) = x^+ \quad \text{and} \quad d_{H^+}(x) = |x^-| \quad \forall x\in H.$$

Consequently, H^+ is proximally smooth and we can study its invariance under the flow associated with (2.10) using Corollary 3.3.

Suppose assumptions (2.11), (2.12), (2.13), and (2.14) are satisfied with $\varphi_0\equiv 0$ and define A and f as in (2.15) and (2.18), respectively. Then, appealing to Corollary 3.3 we conclude that (3.7) is a sufficient condition for the invariance of $H^+\cap H_1$ under (2.10). In order to check (3.7), observe that, for all $x\in D(A)\setminus H^+$, integrating by parts we have that

$$\begin{aligned}
&\langle x-\Pi_{H^+}(x), Ax+f(t,x)\rangle \qquad (3.10)\\
&\quad = -\int_{\mathcal{O}} x^-(\xi)\Big(\Delta x(\xi) + F\big(t,\xi,x(\xi)\big)\Big)d\xi \\
&\quad = -\int_{\mathcal{O}} |\nabla x^-(\xi)|^2 d\xi - \int_{\mathcal{O}} x^-(\xi)F\big(t,\xi,-x^-(\xi)\big)d\xi
\end{aligned}$$

because x^+ and x^- vanish on the intersection of their supports. Now, use (2.13) (with $\varphi_0\equiv 0$) and the Sobolev inequality (2.17) to derive

$$\begin{aligned}
&-\int_{\mathcal{O}} x^-(\xi)F\big(t,\xi,-x^-(\xi)\big)d\xi \leqslant C_0\int_{\mathcal{O}} |x^-(\xi)|\,|x^-(\xi)|^p d\xi \\
&\quad \leqslant C_0\,|x^-|\Big(\int_{\mathcal{O}} |x^-(\xi)|^{2p}d\xi\Big)^{\frac{1}{2}} \leqslant C_0C_{2p}(\mathcal{O})^p\,|x^-|\,|x^-|_1^p \\
&\quad \leqslant |x^-|_1^2 + \frac{C_0^2C_{2p}(\mathcal{O})^{2p}}{4}\,|x^-|_1^{2(p-1)}\,|x^-|^2.
\end{aligned}$$

Finally, combine the last inequality with (3.10), to obtain (3.7) with

$$C(t,r) = \frac{C_0^2C_{2p}(\mathcal{O})^{2p}}{4}\,r^{2(p-1)} \quad (t,r\geqslant 0),$$

thus yielding the claimed invariance of $H^+\cap H_1$. □

Since, in the above example, H^+ is invariant under the semigroup e^{tA}, the conclusion could be interpreted saying that invariance is preserved for a perturbation F which satisfies (2.13) and (2.14) with $\varphi_0 \equiv 0$.

Remark 3.6. The analysis of Example 3.5 shows that, without assuming $\varphi_0 \equiv 0$ in (2.13), a sufficient condition for the invariance of $H^+ \cap H_1$ under (2.10) is that, for a.e. $(t, \xi) \in \mathbb{R}_+ \times \mathcal{O}$,

$$F(t, \xi, v) \geqslant Cv \qquad \forall v \leqslant 0 \tag{3.11}$$

for some constant $C \geqslant 0$.

3.2. **Necessary Conditions for Invariance.** In this section, we show that the above sufficient conditions become also necessary for the invariance of $K \cap H_\theta$ under a further assumption, (H_ρ), which connects $D(A)$ with K.

Theorem 3.7. *Suppose there exists a number $\rho > 0$ and a nondecreasing function $M : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $M(s) \geqslant s$ for all $s \in \mathbb{R}_+$, such that:*

(H_ρ) *for all $x \in D(A) \cap K_\rho$ and all $h > 0$ one can find $x_h \in K \cap H_\theta$ with*

$$|x_h - x| < d_K(x) + h \quad \textit{and} \quad |x_h|_\theta \leqslant M(|x|_\theta).$$

If $K \cap H_\theta$ is invariant under (2.1)*, then there exists a set $\mathcal{T} \subset \mathbb{R}_+$, of Lebesgue measure zero, and a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (3.4) *such that* (3.3) *holds true for all $t \in \mathbb{R}_+ \setminus \mathcal{T}$ and all $x \in D(A) \cap K_\rho$.*

Proof. Let ℓ be the function given by Proposition 2.6 and observe that

$$\ell(t, r) \leqslant \ell(t, \lfloor r \rfloor) \quad \text{for a.e. } t \in \mathbb{R}_+\,,\ \forall r \in \mathbb{R}_+. \tag{3.12}$$

For every integer $n \geqslant 1$, denote by $\mathcal{P}_n$ the set of Lebesgue points of the function $\ell(\cdot, n)$ and set $\mathcal{Q}_n = \mathbb{R}_+ \setminus \mathcal{P}_n$ and $\mathcal{Q} = \cup_{n \geqslant 1} \mathcal{Q}_n$. Since $\mathcal{P}_n$ is a set of full measure, $\mathcal{Q}_n$ is a negligible set and so is $\mathcal{Q}$. Then, $\mathcal{T} := \mathcal{N} \cup \mathcal{Q},$, where $\mathcal{N}$ is given by Proposition 2.9, is negligible too.

Suppose $K \cap H_\theta$ is invariant under (2.1). Fix $t_0 \in \mathbb{R}_+ \setminus \mathcal{T}$, $x_0 \in D(A) \cap K_\rho$, and $h > 0$. Invoke assumption (H_ρ) to construct $x_h \in K \cap H_\theta$ such that

$$|x_h - x_0| \leqslant (1 + h^2) d_K(x_0) \quad \text{and} \quad |x_h|_\theta \leqslant M(|x_0|_\theta). \tag{3.13}$$

Owing to Proposition 2.6 the maximal solutions $X(\cdot) := X(\cdot; t_0, x_0)$ and $X_h(\cdot) := X(\cdot; t_0, x_h)$ of (2.1) are defined on some common interval $[t_0, T]$, on which they satisfy (2.26). Since $X_h(t_0 + h) \in K \cap H_\theta$ for all $h \in [0, T - t_0]$

because we assume invariance, by (2.27), (3.13), and (3.12) we get

$$\begin{aligned}
&\frac{1}{h}\big[d_K(X(t_0+h)) - d_K(x)\big]\\
&= \frac{1}{h}\big[d_K(X(t_0+h)) - d_K(X_h(t_0+h)) - d_K(x)\big]\\
&\leqslant \frac{1}{h}\Big(|X(t_0+h)) - X_h(t_0+h)| - \frac{|x-x_h|}{1+h^2}\Big)\\
&\leqslant \frac{1}{h}\Big(e^{\int_{t_0}^{t_0+h}\ell(s,M(|x_0|_\theta))ds} - \frac{1}{1+h^2}\Big)|x-x_h|\\
&\leqslant \frac{1}{h}\Big(e^{\int_{t_0}^{t_0+h}\ell(s,\lfloor M(|x_0|_\theta)\rfloor)ds} - \frac{1}{1+h^2}\Big)|x-x_h|
\end{aligned}$$

for all $0 < h \leqslant T - t_0$. Hence, again by (3.13), we obtain

$$\begin{aligned}
&\frac{1}{h}\big[d_K(X(t_0+h)) - d_K(x)\big] \qquad (3.14)\\
&\leqslant \Big(\frac{e^{\int_{t_0}^{t_0+h}\ell(s,\lfloor M(|x_0|_\theta)\rfloor)ds} - 1}{h} + \frac{h}{1+h^2}\Big)(1+h^2)d_K(x)
\end{aligned}$$

for all $0 < h \leqslant T - t_0$. Since d_K is Lipschitz and, in view of the choice of t_0, X satisfies (2.36), $D^- d_K(x_0)(Ax + f(t_0,x_0))$ coincides with the lower limit as $h \downarrow 0$ of the left side of (3.14). Moreover,

$$\lim_{h\downarrow 0}\frac{1}{h}\int_{t_0}^{t_0+h}\ell(s,\lfloor M(|x_0|_\theta)\rfloor)ds = \ell(t_0,\lfloor M(|x_0|_\theta)\rfloor)$$

because t_0 is a Lebesgue point of $\ell(\cdot,\lfloor r\rfloor)$ for any $r > 0$. We have thus obtained (3.3) with $C(t,r) = \ell(\cdot,\lfloor M(r)\rfloor)$. □

Remark 3.8. Whenever assumption (H_ρ) of Theorem 3.7 is satisfied, condition (3.3) is necessary and sufficient for the invariance of $K \cap H_\theta$.

Theorems 3.7 and 3.2 yield the following.

Corollary 3.9. *Assume that:*

(a) *K is proximally smooth and let $\delta > 0$ be such that Π_K is single-valued on K_δ;*

(b) *$\Pi_K\big(D(A) \cap K_\delta\big) \subset H_\theta$ and*

$$|\Pi_K(x)|_\theta \leqslant M(|x|_\theta) \quad \forall x \in D(A) \cap K_\delta. \qquad (3.15)$$

where $M : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $M(s) \geqslant s$ for all $s \in \mathbb{R}_+$.

Then $K \cap H_\theta$ is invariant under (2.1) *if and only if there exists a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, satisfying* (3.4)*, such that* (3.7) *holds true for a.e. $t \in \mathbb{R}_+$ and every $x \in D(A) \cap K_\delta$.*

Finally, we show the necessity of condition (3.9) for boundary points.

Theorem 3.10. *In addition to assumptions (a) and (b) of Corollary 3.9, suppose that $\Pi_K\big(D(A)\cap K_\delta\big) \subset D(A)$. Then $K\cap H_\theta$ is invariant under* (2.1) *if and only if* (3.9) *holds true for a.e. $t \in \mathbb{R}_+$ and every $x \in \partial K \cap D(A)$.*

Proof. Sufficiency follows from Theorem 3.4. In order to prove necessity, observe that if $K \cap H_\theta$ is invariant under (2.1), then (3.7) holds true by Corollary 3.9. Let $x \in \partial K \cap D(A)$ and $p \in N_K^P(x) \cap D(A)$. Then $x_\lambda := x + \lambda p \in D(A) \setminus K$ and $\Pi_K(x_\lambda) = x$ for $\lambda > 0$ sufficiently small. Thus (3.7) yields

$$\langle x_\lambda - x, Ax_\lambda + f(t, x_\lambda)\rangle \leqslant C(t, |x|_\theta)\, \lambda^2 |p|^2$$

for a.e. $t \in \mathbb{R}_+$ and every $x \in \partial K \cap D(A)$. So, dividing by λ,

$$\langle p, Ax + \lambda Ap + f(t, x_\lambda)\rangle \leqslant C(t, |x|_\theta)\, \lambda |p|^2.$$

Passing to the limit as $\lambda \downarrow 0$ we obtain (3.9). □

If, in Example 3.5, we have used the sufficient condition given by Corollary 3.3 to obtain invariance, the fact that (3.7) is also a necessary condition for invariance (Corollary 3.9) can help to reconstruct structural properties of the data from the behaviour of solutions, as we show in our next example.

Example 3.11. With the notation of Example 3.5, let us study the invariance of the closed convex cone H^+ under the flow associated with the semilinear initial-boundary value problem

$$\begin{cases} \frac{\partial X}{\partial t} = \Delta X + F(X) & \text{in }]0, \infty[\times \mathcal{O} \\ \frac{\partial X}{\partial \nu} = 0 & \text{on }]0, \infty[\times \partial\mathcal{O} \\ X(0, \xi) = x_0(\xi) & \xi \in \mathcal{O} \text{ a.e.} \end{cases} \tag{3.16}$$

Here, $F \in C^1(\mathbb{R})$ is supposed to satisfy assumptions (2.13) and (2.14) with $\varphi_0 \equiv 1 \equiv \varphi_1$. Defining

$$\begin{cases} D(A) = \left\{ x \in H^2(\mathcal{O}) \ : \ \frac{\partial x}{\partial \nu}|_{\partial\mathcal{O}} = 0 \right\} \\ Ax = \Delta x - x & \forall x \in D(A) \end{cases} \tag{3.17}$$

we have that (H_A) is satisfied and $H_1 = H^1(\mathcal{O})$. Moreover,

$$f(x)(\xi) := x(\xi) + F\big(x(\xi)\big) \quad \forall x \in H_1,\ \xi \in \mathcal{O} \text{ a.e.} \tag{3.18}$$

satisfies in turn hypothesis (H_f). Furthermore, assumptions (a) and (b) of Corollary 3.9 hold true with $\delta = \infty$. So, we have that (3.7) is necessary (and sufficient) for the invariance of $H^+ \cap H_1$ under (3.16). By the same computations as in Example 3.5, one has that (3.7) can be rewritten as

$$-\int_{\mathcal{O}} x^-(\xi) F\big(-x^-(\xi)\big) d\xi \leqslant \int_{\mathcal{O}} |\nabla x^-(\xi)|^2 d\xi + C(|x|_1) \int_{\mathcal{O}} |x^-|^2 d\xi$$

for all $x \in D(A) \setminus H^+$. Now, restricting the above inequality to the constant functions $x(\xi) \equiv v \leqslant 0$ gives $vF(v) \leqslant C(0)v^2$. Thus, we deduce that, if $H^+ \cap H_1$ is invariant under (3.16), then, for some constant $C \geqslant 0$,

$$F(v) \geqslant Cv \quad \forall v \leqslant 0. \tag{3.19}$$

Moreover, recalling Remark 3.6, we conclude that (3.19) is a necessary and sufficient condition for the invariance of $H^+ \cap H_1$. □

3.3. **The case of $\theta = 1$.** In this section we show that, when (H_f) is satisfied with $\theta = 1$, the invariance of $D(A)$ under Π_K can be dropped from the assumptions of Theorem 3.10. We begin by adapting Corollary 3.9.

Proposition 3.12. *Assume that:*

(a) *K is proximally smooth and let $\delta > 0$ be such that Π_K is single-valued on K_δ;*

(b$_1$) *$\Pi_K\big(H_1 \cap K_\delta\big) \subset H_1$, $\Pi_K : H_1 \cap K_\delta \to H_1$ is continuous, and*

$$|\Pi_K(x)|_1 \leqslant M(|x|_1) \quad \forall x \in H_1 \cap K_\delta, \tag{3.20}$$

where $M : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $M(s) \geqslant s$ for all $s \in \mathbb{R}_+$.

Then $K \cap H_1$ is invariant under (2.1) if and only if there exists a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, satisfying (3.4), such that

$$\langle x - \Pi_K(x), f(t,x)\rangle - \langle (-A)^{1/2}(x - \Pi_K(x)), (-A)^{1/2}x)\rangle \leqslant C(t, |x|_1)\, d_K^2(x), \tag{3.21}$$

for a.e. $t \in \mathbb{R}_+$ and every $x \in H_1 \cap K_\delta$.

Proof. Sufficiency follows directly from Corollary 3.9 because (3.21) implies (3.7) with $\theta = 1$. In order to prove necessity, fix any $x \in H_1 \cap K_\delta$ and let $x_j = e^{A/j}x$ for all integers $j \geqslant 1$. Then $x_j \in D(A)$, $|x_j|_1 \leqslant |x|_1$ for all $j \geqslant 1$, and $x_j \to x$ in H_1 as $j \to \infty$. Therefore, $x_j \in D(A) \cap K_\delta$ for j sufficiently large and so, in view of (3.7), we have that for a.e. $t \in \mathbb{R}_+$

$$\begin{aligned} C(t, |x|_1)\, d_K^2(x_j) &\geqslant C(t, |x_j|_1)\, d_K^2(x_j) \geqslant \langle x_j - \Pi_K(x_j), Ax_j + f(t, x_j)\rangle \\ &= \langle x_j - \Pi_K(x_j), f(t, x_j)\rangle - \langle (-A)^{1/2}(x_j - \Pi_K(x_j)), (-A)^{1/2}x_j\rangle \end{aligned}$$

where $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (3.4). By taking the limit as $j \to \infty$, one recovers (3.21). □

We now turn to the analogue of Theorem 3.10.

Proposition 3.13. *Let assumptions* (a) *and* (b$_1$) *of Proposition 3.12 be satisfied. Then $K \cap H_1$ is invariant under* (2.1) *if and only if*

$$\langle p, f(t,x)\rangle - \langle (-A)^{1/2}p, (-A)^{1/2}x)\rangle \leqslant 0 \quad \forall p \in N_K^P(x) \cap H_1 \tag{3.22}$$

for a.e. $t \in \mathbb{R}_+$ and every $x \in H_1 \cap \partial K$.

We omit the proof of the above result that can be reconstructed from the one of Theorem 3.10, by replacing Corollary 3.9 with Proposition 3.12.

4. Level set preserving bilinear controls

In this section, we apply our invariance result to a class of nonlinear parabolic equations that generalize the one considered in Example 2.3. In order to motivate our analysis, we begin with a viability problem for the heat equation under the action of a bilinear control.

4.1. **A bilinear viability problem.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a nonnegative function of class $C^2(\mathbb{R})$ such that

$$\begin{array}{ll} (i) & \phi(0) = \phi'(0) = 0 \\ (ii) & 0 < \lambda_- \leqslant \phi''(s) \leqslant \lambda_+ \quad \forall s \in \mathbb{R} \end{array} \tag{4.1}$$

for some constants $\lambda_-, \lambda_+ > 0$. Observe that, owing to the above assumptions, the following estimates are true:

$$\frac{\lambda_+}{2}\, s^2 \geqslant \phi(s) \geqslant \frac{\lambda_-}{2}\, s^2 \qquad \forall s \in \mathbb{R} \tag{4.2}$$

and

$$s\phi'(s) \geqslant \phi(s) + \frac{\lambda_-}{2}\, s^2 \geqslant \lambda_-\, s^2 \qquad \forall s \in \mathbb{R}. \tag{4.3}$$

Moreover, (4.1) and (4.3) yield

$$\lambda_-^2 s^2 \leqslant |\phi'(s)|^2 \leqslant \lambda_+^2 s^2 \qquad \forall s \in \mathbb{R}. \tag{4.4}$$

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2. In view of (4.2), for any $c > 0$ the level set

$$\Lambda_\phi(c) = \Big\{ x \in L^2(\mathcal{O}) \ : \ \int_{\mathcal{O}} \phi\big(x(\xi)\big) d\xi = c \Big\} \tag{4.5}$$

is nonempty and closed in the Hilbert space $H = L^2(\mathcal{O})$.

For a fixed $c > 0$ and a given $x_0 \in \Lambda_\phi(c) \cap H_0^1(\mathcal{O})$ we seek a time $T > 0$ and a control $g \in L^\infty(0, T)$ such that the solution, say X_g, of the initial-boundary value problem

$$\begin{cases} \frac{\partial X}{\partial t}\,(t,\xi) = \Delta X(t,\xi) + g(t) X(t,\xi) & (t > 0, \xi \in \mathcal{O}) \\ X = 0, & \text{on } (0,\infty) \times \partial\mathcal{O} \\ X(0,\xi) = x_0(\xi), & \xi \in \mathcal{O}. \end{cases} \tag{4.6}$$

satisfies

$$X_g(t,\cdot) \in \Lambda_\phi(c) \qquad \forall t \in [0, T].$$

Now, let $g \in L^\infty(0, T)$ be any control with such a property. Since

$$\int_{\mathcal{O}} \phi'(X_g)\, \frac{\partial X_g}{\partial t}\, d\xi = \frac{d}{dt} \int_{\mathcal{O}} \phi(X_g)\, d\xi = 0,$$

multiplying the equation in (4.6) by $\phi'(X_g)$ and integrating by parts we find that $g = G(X_g)$ where

$$G(x) = \frac{\int_{\mathcal{O}} \phi''(x(\xi)) |\nabla x(\xi)|^2 d\xi}{\int_{\mathcal{O}} x(\xi) \phi'(x(\xi)) d\xi} \quad \forall x \in H_0^1(\mathcal{O}) \setminus \{0\}. \tag{4.7}$$

Notice that the denominator in the above quotient is strictly positive in view of (4.3) and the fact that $x \neq 0$.

4.2. **Invariance of level sets.** We now show how to apply the abstract results of this paper to study the invariance of all level sets $\Lambda_\phi(c)\cap H^1_0(\mathcal{O})$, with $c>0$, under the flow associated with the initial-boundary value problem

$$\begin{cases} \frac{\partial X}{\partial t} = \Delta X + G(X)X & \text{in } (0,\infty)\times\mathcal{O} \\ X = 0 & \text{on } (0,\infty)\times\partial\mathcal{O} \\ X(0,\xi) = x_0(\xi), & \xi\in\mathcal{O} \end{cases} \tag{4.8}$$

where G is given by (4.7). For this purpose, we recast the problem in abstract form as in Example 2.3, defining operator A as in (2.15). In this case, it is well-known that $H_1 = H^1_0(\mathcal{O})$ and we define $f: H_1 \to H$ by

$$f(x)(\xi) = \begin{cases} G(x)\,x(\xi) & \forall x\in H^1_0(\mathcal{O})\setminus\{0\} \\ 0 & \text{if } x = 0 \end{cases} \qquad (\xi\in\mathcal{O} \text{ a.e.})$$

Then, one can check that f satisfies assumption (H_f)-(b) with $\theta = 1$. So, for any $x_0\in H_1$, problem (4.8) has a unique maximal solution.

Now, applying Proposition 3.13 we obtain the following.

Proposition 4.1. *Assume* (4.1)*. Then, for any $c>0$, the set $\Lambda_\phi(c)\cap H_1$ is invariant under* (4.8)*.*

Proof. Consider the functional $\Phi : H\to\mathbb{R}$ defined by

$$\Phi(x) = \int_{\mathcal{O}} \phi\big(x(\xi)\big)d\xi \qquad \forall x\in H.$$

Since $\phi\in C^2(\mathbb{R})$, we have that Φ is Fréchet differentiable and

$$D\Phi(x)(\xi) = \phi'\big(x(\xi)\big) \quad \forall x\in H,\ \xi\in\mathcal{O} \text{ a.e.}$$

Moreover, $\Phi\in C^{1,1}(H)$ because, by (4.1),

$$|D\Phi(x) - D\Phi(y)| \leqslant \lambda_+|x-y| \quad \forall x,y\in H.$$

Furthermore, for every $x\in\Lambda_\phi(c)$ with $c>0$, (4.4) and (4.3) ensure that

$$\begin{aligned} |D\Phi(x)|^2 &= \int_{\mathcal{O}} \big|\phi'\big(x(\xi)\big)\big|^2 d\xi \geqslant \lambda_-^2 \int_{\mathcal{O}} \big|x(\xi)\big|^2 d\xi \\ &\geqslant \frac{2\lambda_-^2}{\lambda_+}\int_{\mathcal{O}}\phi\big(x(\xi)\big)d\xi = \frac{2\lambda_-^2 c}{\lambda_+}. \end{aligned}$$

This shows that $D\Phi$ satisfies the lower bound in (6.2) of Proposition 6.1 below. The upper bound in (6.2) follows by a similar argument. Indeed, by (4.2) and (4.4) we have that

$$|D\Phi(x)|^2 \leqslant \lambda_+^2\int_{\mathcal{O}}\big|x(\xi)\big|^2 d\xi \leqslant \frac{2\lambda_+^2}{\lambda_-}\int_{\mathcal{O}}\phi\big(x(\xi)\big)d\xi = \frac{2\lambda_+^2 c}{\lambda_-}.$$

Therefore, owing to Proposition 6.1 we conclude that $\Lambda_\phi(c)$ is proximally smooth for any $c>0$. So, condition (a) of Proposition 3.13 holds true for

$K = \Lambda_\phi(c)$. We now check that condition (b_1) of the same proposition is also satisfied. Still denoting $\Lambda_\phi(c)$ by K, observe that

$$N_K^P(x) \subset \{\mu D\Phi(x) \ : \ \mu \in \mathbb{R}\} \qquad \forall x \in K. \tag{4.9}$$

Suppose $\delta > 0$ is such that any point $x \in K_\delta$ admits a unique projection onto K and call $\overline{x}$ such a projection. Then, for any $x \in H_1 \cap K_\delta$ we have that $x - \overline{x} \in N_K^P(\overline{x})$ and so, by (4.9),

$$x = \overline{x} + \mu D\Phi(\overline{x}) \quad \text{with} \quad |\mu| = \frac{d_K(x)}{|D\Phi(\overline{x})|}\,. \tag{4.10}$$

Hence, Lemma 4.2 below guarantees that $\overline{x} \in H_1$ and $\Pi_K : H_1 \cap K_\delta \to H_1$ is continuous whenever $\delta > 0$ is sufficiently small. Furthermore, (3.20) follows from (4.12).

Therefore, by Proposition 3.13 we conclude that condition (3.22) is necessary and sufficient for the invariance of $K \cap H_1$. Since

$$\begin{aligned}\langle D\Phi(x), G(x)x\rangle &- \langle (-A)^{1/2} D\Phi(x), (-A)^{1/2} x\rangle \\ &= \frac{\int_{\mathcal{O}} \phi''(x)|\nabla x|^2\, d\xi}{\int_{\mathcal{O}} x\,\phi'(x)\, d\xi} \int_{\mathcal{O}} x\,\phi'(x)\, d\xi - \int_{\mathcal{O}} \nabla\phi'(x)\cdot\nabla x\, d\xi = 0\end{aligned}$$

for all $x \in K \cap H_1$, (3.22) is satisfied and $K \cap H_1$ is invariant. □

Lemma 4.2. *Assume* (4.1) *and let* $\mu \in \mathbb{R}$ *be such that* $|\mu| \leqslant 1/(2\lambda_+)$. *Then, for any* $x \in H = L^2(\mathcal{O})$ *the equation*

$$x(\xi) = y(\xi) + \mu\,\phi'\big(y(\xi)\big) \quad (\xi \in \mathcal{O} \ a.e.) \tag{4.11}$$

has a unique solution $y_x \in H$. *Moreover,* $y_x \in H_1$ *if* $x \in H_1$, *the map* $\Psi : H_1 \to H_1$ *defined by* $\Psi(x) = y_x$ *is continuous, and*

$$|\Psi(x)|_1 \leqslant M|x|_1 \tag{4.12}$$

for some constant $M > 0$ *depending only on* ϕ *and* $\mathcal{O}$.

Proof. Define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(s) = s + \mu\phi'(s)$ for all $s \in \mathbb{R}$. Observe that ψ is of class $C^2(\mathbb{R})$ and surjective. Since $|\mu| \leqslant 1/(2\lambda_+)$, $\psi'(s) = 1 + \mu\phi''(s) \geqslant 1/2$ for all $s \in \mathbb{R}$. Therefore, for all such μ's, ψ is invertible, $\psi^{-1} \in C^2(\mathbb{R})$, and $(\psi^{-1})'$ is bounded independently of μ. Consequently, for any $x \in H$ the unique solution of (4.11) is given by $y_x = \psi^{-1}(x)$, which belongs to H and, for $x \in H_1$, satisfies

$$\nabla y_x(\xi) = (\psi^{-1})'\big(x(\xi)\big)\nabla x(\xi) \quad (\xi \in \mathcal{O} \text{ a.e.})$$

Moreover, since both ψ and ψ^{-1} vanish at zero, we have that y_x has null trace on $\partial\mathcal{O}$. This shows that $y_x \in H_1$ for $x \in H_1$ and (4.12) holds true.

As for the continuity of the map $x \mapsto y_x$ in the H_1 norm, fix any sequence $x_j \to x$ in H_1. Without loss of generality, we can assume that $x_j \to x$ a.e. in $\mathcal{O}$. Then

$$\nabla y_{x_j} - \nabla y_x = (\psi^{-1})'(x_j)\big(\nabla x_j - \nabla x\big) + \big((\psi^{-1})'(x_j) - (\psi^{-1})'(x)\big)\nabla x,$$

where we realize that

$$\lim_{j\to\infty}(\psi^{-1})'(x_j)\big(\nabla x_j - \nabla x\big) = 0 \quad \text{in} \quad L^2(\mathcal{O};\mathbb{R}^n),$$

because $(\psi^{-1})'$ is bounded and $x_j \to x$ in H_1, and

$$\lim_{j\to\infty}\big((\psi^{-1})'(x_j) - (\psi^{-1})'(x)\big)\nabla x = 0 \quad \text{in} \quad L^2(\mathcal{O};\mathbb{R}^n)$$

by the dominated convergence theorem. Therefore, $y_{x_j} \to y_x$ in H_1. □

Remark 4.3. We observe that an *ad hoc* computation can be used to prove invariance under (4.8) for the level set $\Lambda_\phi(c)\cap H_1$ $(c>0)$. Indeed, it suffices to check that the time derivative of Φ along the maximal solution of (4.8) vanishes for a. e. t. Nevertheless, the goal of this section is to illustrate the interplay between our abstract conditions for invariance and the specific features of a concrete example.

Example 4.4. Consider the heat equation with a nonlocal term we presented in Example 2.3. One can easily check that, taking $\phi(s) = s^2$, such an equation reduces to (4.8) provided that $X(t,\cdot) \in \Lambda_\phi(1)$. Therefore, Proposition 4.1 ensures that the unit sphere of $L^2(\mathcal{O})$ is invariant under the equation

$$\frac{\partial X}{\partial t}(t,\xi) = \Delta X(t,\xi) + \Big(\int_{\mathcal{O}} |\nabla X(t,\xi')|^2 d\xi'\Big) X(t,\xi) \qquad (t>0, \xi\in\mathcal{O})$$

with homogeneous Dirichlet boundary conditions. The question remains whether the maximal solution of the above equation, with initial condition $X(0,\cdot) = x_0 \in \Lambda_\phi(1)\cap H_0^1(\mathcal{O})$, is global (that is, $T(0,x_0)=\infty$) or not. We return to this problem in the next section.

4.3. **Maximal interval of existence.** Finally, we apply the above invariance results to give a lower bound for the maximal interval of existence of the solution of (4.8). For simplicity, we suppose $t_0 = 0$ but the same result can be obtained—by exactly the same reasoning—for any $t_0 \geqslant 0$. Also, we assume $x_0 \neq 0$ since, when $x_0 = 0$, the null solution is trivially global.

Theorem 4.5. *Assume* (4.1)*. Then, for any* $x_0 \in H_0^1(\mathcal{O})\setminus\{0\}$ *we have that*

$$T(0,x_0)\begin{cases} \geqslant \dfrac{c_0\,\lambda_-}{2\lambda_+(\lambda_+-\lambda_-)|x_0|_1^2} & \text{if } \lambda_- < \lambda_+ \\ = \infty & \text{if } \lambda_- = \lambda_+, \end{cases} \tag{4.13}$$

where

$$c_0 = \int_{\mathcal{O}} \phi\big(x_0(\xi)\big)d\xi. \tag{4.14}$$

Proof. First, observe that $c_0 > 0$ and, obviously, $x_0 \in \Lambda_\phi(c_0)\cap H_1$. Therefore, abbreviating $T(0,x_0)$ to T_0, Theorem 4.1 ensures that

$$X(t,\cdot)\in\Lambda_\phi(c_0) \qquad \forall t\in[0,T_0[\tag{4.15}$$

where X denotes the maximal solution of (4.8). Fix any $T \in [0, T_0[$, multiply by X both sides of the equation in (4.8), and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi = -\int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi + G(X(t,\cdot))\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi \quad (4.16)$$

for all $t \in [0,T]$. Similarly, multiplying the equation by $\frac{\partial X}{\partial t}$ we have that

$$0 \leqslant \int_{\mathcal{O}} \Big|\frac{\partial X}{\partial t}(t,x)\Big|^2 d\xi = -\frac{1}{2}\,\frac{d}{dt}\int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi + G(X(t,\cdot))\,\frac{1}{2}\,\frac{d}{dt}\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi. \quad (4.17)$$

By combining (4.16) and (4.17) we conclude that

$$\frac{d}{dt}\int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi \leqslant 2\,G(X(t,\cdot))\Big[G(X(t,\cdot))\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi - \int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi\Big]. \quad (4.18)$$

Now, in view of (4.2), (4.3), and (4.15) we have the estimates

$$G(X(t,\cdot)) \leqslant \frac{\lambda_+ \int_{\mathcal{O}} |\nabla X(t,\xi)|^2\, d\xi}{\int_{\mathcal{O}} \phi(X(t,\xi))\, d\xi} = \frac{\lambda_+}{c_0}\int_{\mathcal{O}} |\nabla X(t,\xi)|^2\, d\xi$$

and

$$\begin{aligned} G(X(t,\cdot))\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi &\leqslant \frac{\lambda_+ \int_{\mathcal{O}} |\nabla X(t,\xi)|^2\, d\xi}{\lambda_- \int_{\mathcal{O}} |X(t,\xi)|^2 d\xi}\int_{\mathcal{O}} |X(t,\xi)|^2 d\xi \\ &= \frac{\lambda_+}{\lambda_-}\int_{\mathcal{O}} |\nabla X(t,\xi)|^2\, d\xi. \end{aligned}$$

So, returning to (4.18) we obtain

$$\frac{d}{dt}\int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi \leqslant 2\,\frac{\lambda_+}{c_0}\Big(\frac{\lambda_+}{\lambda_-} - 1\Big)\Big(\int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi\Big)^2$$

for all $t \in [0,T]$. Setting

$$E(t) = \int_{\mathcal{O}} |\nabla X(t,\xi)|^2 d\xi,$$

we can recast the above inequality as $E'(t) \leqslant \kappa E^2(t)$ with

$$\kappa = 2\,\frac{\lambda_+}{c_0}\Big(\frac{\lambda_+}{\lambda_-} - 1\Big).$$

Then, the comparison principle for ordinary differential equation yields

$$E(t) \leqslant \frac{E(0)}{1 - t\,\kappa\,E(0)} \qquad \forall t \in \Big[0, \frac{1}{\kappa E(0)}\Big[.$$

This shows that the maximal solution of (4.8) is bounded in $H_0^1(\mathcal{O})$ on every interval $[0,T]$ such that

$$2\,T\,\frac{\lambda_+}{c_0}\Big(\frac{\lambda_+}{\lambda_-}-1\Big)\int_{\mathcal{O}}|\nabla x_0(\xi)|^2 d\xi<1.$$

Therefore, T_0 satisfies (4.13). □

Example 4.6. We return to Example 2.3 with $\phi(s)=s^2$ and $K=\Lambda_\phi(1)$. Since $\lambda_+=\lambda_-$, the maximal solution of (2.19) is global by Theorem 4.5. In other terms, the flow associated with (2.19) preserves the L^2 energy: this result was first observed in [5], where it was applied to study the convergence of a family of singularly perturbed systems of nonlocal parabolic equations.

5. Viability of a Semilinear Control System

Given a nonempty closed set $K\subset H$, we consider the semilinear control system (2.33) under the state constraint $X(s)\in K$ $(s\geqslant t_0)$.

We recall that $K_\delta\,(\delta>0)$ is defined in (3.2) and we list below the assumptions that will be imposed in this section.

- (H_A') $A:D(A)\subset H\to H$ satisfies (H_A) and e^{tA} is a compact linear operator on H for all $t>0$.
- (H_f'') $f:\mathbb{R}_+\times H_\theta\times U\to H$ satisfies assumptions (H_f') for some $\theta\in[\,0,1\,]$, with (2.6) replaced by (2.34) for some function $c\in L^2_{loc}(\mathbb{R}_+)$, and $f(t,x,U)$ is closed and convex for a.e. $t\geqslant 0$ and every $x\in H_\theta$.
- (H_ρ') There exists $\rho>0$ and a nondecreasing function $M:\mathbb{R}_+\to\mathbb{R}_+$ such that, for all $x\in K_\rho\cap H_\theta$ and all $h>0$, one can find $x_h\in K\cap H_\theta$ satisfying

$$|x_h-x|<d_K(x)+h\quad\text{and}\quad|x_h|_\theta\leqslant M(|x|_\theta).$$

Under assumptions (H_A') and (H_f''), in light of Proposition 2.7, for every initial condition $(t_0,x_0)\in\mathbb{R}_+\times H_\theta$ and control $u:[t_0,\infty)\to U$, (2.33) has a unique global solution $X(\cdot;t_0,x_0,u)$. Moreover, arguing exactly as in [6], one can prove the following compactness result.

Lemma 5.1. *Under assumptions (H_A') and (H_f''), for all $(t_0,x_0)\in\mathbb{R}_+\times H_\theta$ and all $T>t_0$ the set*

$$S_{[t_0,T]}(x_0):=\big\{X(\cdot;t_0,x_0,u)\ :\ u:[t_0,T]\to U\ \text{ measurable}\big\}$$

is compact in $C([t_0,T];H_\theta)$.

The set $K\cap H_\theta$ is called *viable* under the control system (2.33) if for every initial condition $(t_0,x_0)\in\mathbb{R}_+\times(K\cap H_\theta)$ there exists a control function $u:[t_0,\infty)\to U$ such that $X(t;t_0,x_0,u)\in K\cap H_\theta$ for all $t\geqslant t_0$.

In what follows, we provide sufficient conditions for the viability of $K\cap H_\theta$ in terms of Clarke's derivative of d_K which, for any point $x\in H$ and any direction $v\in H$, is given by

$$d_K^0(x)v=\limsup_{y\to x,\ \lambda\downarrow 0}\frac{d_k(y+\lambda v)-d_k(y)}{\lambda}\,.$$

Notice that, if K is proximally smooth, then for all $x \in H \setminus K$, sufficiently close to K, one has that

$$d_K^0(x)v = \langle Dd_K(x), v\rangle \quad \forall v \in H.$$

Theorem 5.2. *Assume (H_A'), (H_f''), and (H_δ'). Suppose that*

$$e^{tA}K \subset K \qquad \forall t \geqslant 0 \tag{5.1}$$

and there exists a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+^*$, satisfying (3.4), such that

$$\inf_{u\in U} d_K^0(x) f(t,x,u) \leqslant C(t, |x|_\theta)\, d_K(x) \tag{5.2}$$

for a.e. $t \in \mathbb{R}_+$ and every $x \in K_\delta \cap H_\theta$. Then $K \cap H_\theta$ is viable under (2.33).

Proof. Fix any $t_0 \in \mathbb{R}_+$, $x_0 \in K \cap H_\theta$.

Step 1. Define, for all $t \geqslant t_0$,

$$R(t) := \big\{X(t; t_0, x_0, u) \ : \ X \in S_{[t_0,t]}(x_0)\big\}\big(\subset H_\theta\big),$$

$$g(t) := \inf\big\{|x_t - y| \ : \ x_t \in R(t),\ y \in K\big\}.$$

Claim 1. *For every $t > t_0$ there exists $x_t \in R(t)$ such that*

$$g(t) = d_K(x_t). \tag{5.3}$$

Indeed, it is clear that

$$g(t) = \inf_{z\in R(t)} d_K(z).$$

Consider $y_i \in R(t)$ such that $g(t) = \lim_{i\to\infty} d_K(y_i)$. Since $R(t)$ is compact we may assume that the sequence y_i converges to some $x_t \in R(t)$. The continuity of $d_K(\cdot)$ yields (5.3). For $t = t_0$ we take $x_{t_0} = x_0$.

Claim 2. *g is continuous.*

Indeed, by our assumptions, this is obvious at t_0. We first show that g is lower semicontinuous on $]t_0, \infty[$. Consider any $t > t_0$, and a sequence t_i converging to t such that $\liminf_{s\to t} g(s) = \lim_{i\to\infty} g(t_i)$. For each i pick $x_{t_i} \in R(t_i)$ such that $g(t_i) = d_K(x_{t_i})$. Fix $T > t$ and let $X_i \in S_{[t_0,T]}(x_0)$ be such that $X_i(t_i) = x_{t_i}$. By Lemma 5.1, there exists a subsequence X_{i_j} converging uniformly to some $X \in S_{[t_0,T]}(x_0)$. Therefore $\lim_{j\to\infty} X_{i_j}(t_{i_j}) = X(t) \in R(t)$. From the continuity of $d_K(\cdot)$ we deduce that $g(t) \leqslant d_K(X(t)) = \lim_{j\to\infty} g(t_{i_j})$.

We show next that g is upper semicontinuous on $]t_0, \infty[$. Fix any $t > t_0$ and let $x_t \in R(t)$ be such that $g(t) = d_K(x_t)$. For any fixed $u \in U$ consider the solution $X(\cdot) = X(\cdot; t, x_t, u)$. It is not difficult to realize that for every $\varepsilon > 0$, there exists $\rho > 0$ such that for all $h \in [0, \rho]$,

$$|e^{hA}x_t - x_t| \leqslant \frac{\varepsilon}{2}, \qquad \left|\int_t^{t+h} e^{(t+h-s)A} f(s, X(s), u)ds\right| \leqslant \frac{\varepsilon}{2}.$$

Hence $|X(t+h) - x_t| \leqslant \varepsilon$ and therefore

$$g(t+h) \leqslant d_K(X(t+h)) \leqslant d_K(x_t) + \varepsilon.$$

Consequently, $g(\cdot)$ is upper semicontinuous from the right at t. In order to prove that g is upper semicontinuous from the left at t, let $X \in S_{[t_0,t]}(x_0)$ be such that $X(t) = x_t$. Then $\lim_{h\to 0+} X(t-h) = X(t)$. Since

$$g(t-h) \leqslant d_K(X(t-h)) \leqslant d_K(x_t) + |X(t-h) - x_t| = g(t) + |X(t-h) - x_t|,$$

we conclude that g is upper semicontinuous from the left at t.

Claim 3. $g \equiv 0$.

Observe first that, for any $y_0 \in H\backslash K$,

$$\limsup_{h\to 0+} \frac{1}{h}\Big(d_K(e^{hA}y_0) - d_K(y_0)\Big) \leqslant 0. \tag{5.4}$$

Indeed, let $y_0 \notin K$. Fix $h > 0$ and let $y_h \in K$ be such that

$$|y_0 - y_h| \leqslant d_K(y_0) + h^2.$$

Since e^{tA} is contractive, we get $|e^{hA}y_0 - e^{hA}y_h| \leqslant |y_0 - y_h|$. Recalling that $e^{hA}y_h \in K$ for all $h \geqslant 0$, we obtain

$$d_K(e^{hA}y_0) - d_K(y_0) \leqslant |e^{hA}y_0 - e^{hA}y_h| - d_K(y_0) \leqslant d_K(y_0) + h^2 - d_K(y_0) = h^2.$$

Dividing by $h > 0$ the above inequality and taking the limit yields (5.4).

Next, observe that, if the set $\{t > t_0 \ : \ g(t) > 0\}$ is empty, then $g \equiv 0$. Otherwise, let

$$\bar t := \inf\big\{t > t_0 \ : \ g(t) > 0\big\}.$$

Since g is continuous, $g(\bar t) = 0$ and there exists $\varepsilon > 0$ such that for any $t \in]\bar t, \bar t + \varepsilon]$ with $g(t) > 0$ and x_t as in (5.3), we have $x_t \in K_\delta$. Let $t'' \in]\bar t, \bar t + \varepsilon]$ be such that $g(t'') > 0$ and $t' = \sup\{t \in [\bar t, t''] \ : \ g(t) = 0\}$. Then, $g > 0$ on $]t', t'']$. Fix $T > t''$ and let $\mathcal{M} = [0,T]\setminus\mathcal{N}$, with $\mathcal{N}$ as in Proposition 2.9. Pick any $t \in]t', t''[\cap\mathcal{M}$ such that (5.2) holds true, and let $\bar u \in U$ be such that

$$d_K^0(x_t)f(t, x_t, \bar u) \leqslant 2C(t, |x_t|_\theta)d_K(x_t).$$

Observe further that

$$|x_t|_\theta \leqslant C_T(1 + |x_0|_\theta) \quad \forall t \in [t', t''],$$

where C_T is the positive constant given by (2.35), to conclude that

$$d_K^0(x_t)f(t, x_t, \bar u) \leqslant \widetilde C(t)d_K(x_t). \tag{5.5}$$

where we have set $\widetilde C(t) = 2C(t, C_T(1 + |x_0|_\theta))$. Then we have that

$$\begin{aligned} D_\uparrow g(t) &:= \liminf_{h\downarrow 0} \frac{g(t+h) - g(t)}{h} \\ &\leqslant \limsup_{h\downarrow 0} \frac{d_K(X(t+h)) - d_K(e^{hA}x_t)}{h} + \limsup_{h\downarrow 0} \frac{d_K(e^{hA}x_t) - d_K(x_t)}{h}. \end{aligned}$$

where $X(\cdot) = X(\cdot; t, x_t, \bar u)$. On the other hand, by our assumptions and the choice of t,

$$X(t+h) = e^{hA}x_t + \int_t^{t+h} e^{(t+h-s)A}f(s, X(s), \bar u)ds = e^{hA}x_t + hf(t, x_t, \bar u) + o(h)$$

and $\lim_{h\to 0+} e^{hA}x_t = X(t)$. Hence, from (5.4), (5.5), and the growth bound (2.35) we deduce that

$$D_\uparrow g(t) \leqslant d_K^0(x_t)f(t,x_t,\bar u) \leqslant \widetilde{C}(t)d_K(x_t) = \widetilde{C}(t)g(t), \tag{5.6}$$

Let us introduce the set valued map $[t',t''] \rightsquigarrow P(t) = \{g(t)\} + \mathbb{R}_+$, which has closed values. We claim that $P(\cdot)$ is locally left absolutely continuous on $[t',t'']$. This means that, for every $\varepsilon > 0$ and every compact set $Q \subset \mathbb{R}$, there exists $\delta > 0$ such that, for every partition

$$t' \leqslant t_1 < \tau_1 \leqslant ... \leqslant t_j < \tau_j < .... \leqslant t''$$

satisfying $\Sigma(\tau_i - t_i) \leqslant \delta$, we have that $\Sigma e(P(t_i)\cap Q, P(\tau_i)) \leqslant \varepsilon$, where $e(C,D) = \inf\{\lambda > 0 \ : \ C \subset D + \lambda[-1,1]\}$ is the excess of $C \subset \mathbb{R}$ with respect to $D \subset \mathbb{R}$. To prove this claim fix $t \in]t',t'']$ and $h > 0$ with $t-h \geqslant t'$. Consider $x_{t-h} \in R(t-h)$ such that $g(t-h) = d_K(x_{t-h})$ and let $y_h \in K \cap H_\theta$ be such that

$$|y_h - x_{t-h}| \leqslant g(t-h) + h \quad \text{and} \quad |y_h|_\theta \leqslant M(|x_{t-h}|_\theta).$$

Fix any $\bar u \in U$ and let $X_h(\cdot) = X(\cdot; t-h, x_{t-h}, \bar u)$. Then

$$\begin{aligned} g(t) &\leqslant d_K(X_h(t)) \leqslant \left|X_h(t) - e^{hA}y_h\right| \\ &= \left|e^{hA}x_{t-h} + \int_{t-h}^t e^{(t-s)A}f(s,X_h(s),\bar u)ds - e^{hA}y_h\right| \\ &\leqslant |x_{t-h} - y_h| + \int_{t-h}^t \left|e^{(t-s)A}f(s,X_h(s),\bar u)\right| ds \\ &\leq g(t-h) + \int_{t-h}^t \left(1 + \left|e^{(t-s)A}f(s,X_h(s),\bar u)\right|\right) ds. \end{aligned}$$

Since $t \in [t',t'']$ is arbitrary, this inequality and our assumptions on f imply that $P(\cdot)$ is left absolutely continuous on $[t',t'']$.

On the other hand, recalling that

$$\text{Graph}(P) = \{(t,z) \ : \ t \in [t',t''], \ z \in P(t)\},$$

we realize that (5.6) says that, for a.e. $t \in [t',t'']$, $(1, \widetilde{C}(t)g(t))$ belongs to the contingent cone (see, e.g. [1]) to Graph(P) at $(t,g(t))$, which is denoted by $T_{\text{Graph}(P)}(t,g(t))$. This fact and the continuity of g imply that, for a.e. $t \in [t',t'']$ and every $y \in P(t)$,

$$(1, \widetilde{C}(t)y) \in T_{\text{Graph}(P)}(t,y).$$

By [9, Theorem 4.2] the solution of the differential equation

$$z'(t) = \widetilde{C}(t)z(t), \quad z(t') = 0$$

satisfies $z(t) \in P(t)$ for all $t \in [t',t'']$. But $z \equiv 0$ and therefore $g(t) \leqslant 0$ for all $t \in [t',t'']$. The derived contradiction yields our claim.

Step 2. To simplify notations we consider only the case $t_0 = 0$. Let $x_0 \in K \cap H_\theta$. Fix an integer $j \geqslant 1$. By Claim 3, using the fact that $R(\frac{1}{2^j})$ is compact we can find a solution of (2.33), X_j, such that $X_j(\frac{1}{2^j}) \in K$. We now proceed by induction: suppose that, for some $p \in \{1,\dots,2^j-1\}$, we have already constructed a solution to (2.33), corresponding to some control $u(\cdot)$, such that $X_j(\frac{k}{2^j}) \in K$ for every integer $0 \leqslant k \leqslant p$. By Step 1 we can extend

X_j to a solution of (2.33) with $t_0 = \frac{p}{2^j}$ satisfying $X_j(\frac{p+1}{2^j}) \in K$. In this way we construct a solution $X_j(\cdot)$ of (2.33) on $[0,1]$ such that $X_j(\frac{k}{2^j}) \in K$ for every $k \in \{0, \dots, 2^j\}$.

By Lemma 5.1 the sequence X_j has a subsequence X_{j_i} which converges uniformly to a solution $X(\cdot)$ of (2.33) defined on $[0,1]$. Fix an integer $n \geqslant 1$. Then for all large i, $X_{j_i}(\frac{k}{2^n}) \in K$ for $k \in \{0, ..., 2^n\}$. Therefore $X(\frac{k}{2^n}) \in K$ for $k \in \{0, ..., 2^n\}$. Since this holds true for every n and $X(\cdot)$ is continuous, we deduce that $X(t) \in K$ for all $t \in [0,1]$.

Step 3. Assume that for some integer $i \geqslant 1$ and every integer $1 \leqslant k \leqslant i$ we have constructed solutions $X_k(\cdot)$ of (2.33) defined on $[0,k]$ such that the restriction of X_i to $[0,k]$ is equal to X_k and $X_i([0,i]) \subset K$.

Consider the interval $[i, i+1]$. Applying the same arguments as in Step 2, with $t_0 = 0$ replaced by $t_0 = i$ and x_0 by $X_i(i)$, we extend X_i on the time interval $[i, i+1]$ as a solution of (2.33) satisfying $X([0, i+1]) \subset K$. Then, using an induction argument we complete the proof. □

Example 5.3. Let us apply Theorem 5.2 to study a viability problem for the following parabolic system

$$\begin{cases} \frac{\partial y_1}{\partial t} = \Delta y_1 + F_1(t,\xi,y,u_1) & \text{in }]0,\infty[\times \mathcal{O} \\ \frac{\partial y_2}{\partial t} = \Delta y_2 + F_2(t,\xi,y,u_2) & \text{in }]0,\infty[\times \mathcal{O} \\ y_1 = 0 = y_2 & \text{on }]0,\infty[\times \partial\mathcal{O} \\ y_1(0,\xi) = x_1(\xi)\,,\ y_2(0,\xi) = x_2(\xi) & \xi \in \mathcal{O} \text{ a.e.} \end{cases} \tag{5.7}$$

where $\mathcal{O} \subset \mathbb{R}^n\,(n \geqslant 3)$ is a bounded domain with boundary of class C^2. Here, we denote by $y(t,\xi) = (y_1(t,\xi), y_2(t,\xi))$ the vector-valued solution of system (5.7) subject to control $u(t,\xi) = (u_1(t,\xi), u_2(t,\xi))$, that is, a measurable map $u : \mathbb{R}_+ \times \mathcal{O} \to V \times V$, where V is a bounded closed subset of a separable Banach space. We assume that $F_i : \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}^2 \times V \to \mathbb{R}^2$ satisfies the following for $i = 1, 2$:

$$\begin{cases} z \mapsto F_i(t,\xi,z,v) \text{ is } C^1(\mathbb{R}^2) \text{ for a.e. } (t,\xi) \in \mathbb{R}_+ \times \mathcal{O} \text{ and every } v \in V, \\ v \mapsto F_i(t,\xi,z,v) \text{ is continuous for a.e. } (t,\xi) \in \mathbb{R}_+ \times \mathcal{O} \text{ and every } z \in \mathbb{R}^2, \\ (t,\xi) \mapsto F_i(t,\xi,z,v) \text{ is Lebesgue measurable for all } v \in V \text{ and } z \in \mathbb{R}^2. \end{cases}$$

Moreover, we suppose that, for some given functions

$$\varphi_0 \in L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{O})),\ \varphi_1 \in L^2_{loc}(\mathbb{R}_+; L^\infty(\mathcal{O})),$$

the growth conditions $(i = 1, 2)$

$$|F_i(t,\xi,z,v)| \leqslant C_0\big(\varphi_0(t,\xi) + |z|\big)\,, \quad \left|\frac{\partial F_i}{\partial z}(t,\xi,z,v)\right| \leqslant \varphi_1(t,\xi)$$

are satisfied for a.e. $(t,\xi) \in \mathbb{R}_+ \times \mathcal{O}$, every $v \in V$, every $z \in \mathbb{R}^2$, and some constant $C_0 \geqslant 0$. Finally, we impose that, for a.e. $(t,\xi) \in \mathbb{R}_+ \times \mathcal{O}$ and every

$z \in \mathbb{R}^2$, the following is a closed convex set

$$\Big\{\big(F_1(t,\xi,z,v_1), F_1(t,\xi,z,v_2)\big) \;:\; v_1, v_2 \in V\Big\} \subset \mathbb{R}^2.$$

Arguing as in Example 2.2, one can show that the above problem can be recast as a semilinear control system like (2.33). For this purpose, one just needs to take $H = L^2(\mathcal{O};\mathbb{R}^2)$ and

$$\begin{cases} D(A) = H^2(\mathcal{O};\mathbb{R}^2) \cap H^1_0(\mathcal{O};\mathbb{R}^2) \\ Ax = (\Delta x_1, \Delta x_2) \qquad\qquad\qquad \forall x = (x_1,x_2) \in D(A). \end{cases}$$

Then, assumption (H'_A) follows from the positivity and compactness of the heat semigroup. Next, let us denote by U the family of all measurable maps $u : \mathcal{O} \to V \times V$. Defining $f : \mathbb{R}_+ \times H \times U \to H$ by

$$f(t,x,u)(\xi) = \Big(F_1\big(t,\xi,x(\xi),u_1(\xi)\big), F_2\big(t,\xi,x(\xi),u_2(\xi)\big)\Big), \tag{5.8}$$

for a.e. $(t,\xi) \in \mathbb{R}_+ \times \mathcal{O}$, every $x \in H$, and every $u = (u_1,u_2) \in U$, we have that assumptions (H''_f) hold true with $\theta = 0$.

So, by applying Theorem 5.2 we conclude that the closed convex cone

$$H^+ = \big\{x \in H \;:\; x_i(\xi) \geqslant 0,\ \xi \in \mathcal{O} \text{ a.e. } (i=1,2)\big\}.$$

is viable under the flow associated with (5.7) provided that (5.2) is satisfied. Now, observe that H^+ is proximally smooth and

$$\Pi_{H^+}(x) = x^+ \quad \text{and} \quad d_{H^+}(x) = |x^-| \quad \forall x \in H,$$

where we have set $x^\pm = (x_1^\pm, x_2^\pm)$. for all $x \in D(A) \setminus H^+$. So, condition (5.2) reduces to

$$\inf_{u\in U} \langle x - \Pi_{H^+}(x), f(t,x,u)\rangle \leqslant C(t,|x|)\,|x^-|^2$$

for a.e. $t \in \mathbb{R}_+$, every $x \in H$, and some function $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^*_+$ satisfying (3.4). In view of (5.8), the above condition reads as

$$\inf_{u\in U} \int_{\mathcal{O}} \Big\{ - x_1^-\, F_1\big(t,\xi,(-x_1^-, x_2), u_1\big) - x_2^-\, F_2\big(t,\xi,(x_1, -x_2^-)), u_2\big)\Big\} d\xi$$
$$\leqslant C(t,|x|) \int_{\mathcal{O}} \big(|x_1^-|^2 + |x_2^-|^2\big) d\xi,$$

which can in turn be reduced to

$$\inf_{u\in U} \int_{\mathcal{O}} \Big\{ - x_1^-\, F_1\big(t,\xi,(-x_1^-, x_2^+), u_1\big) - x_2^-\, F_2\big(t,\xi,(x_1^+, -x_2^-)), u_2\big)\Big\} d\xi$$
$$\leqslant C(t,|x|) \int_{\mathcal{O}} \big(|x_1^-|^2 + |x_2^-|^2\big) d\xi \tag{5.9}$$

by appealing to the Lipschitz continuity of $z \mapsto F_i(t,\xi,z,v)$.

It is easy to give pointwise conditions, like the one in Remark 3.6, which ensure the validity of (5.9). For instance, we can assume that, for some constant $C > 0$ and a.e. $(t,\xi) \in \mathbb{R}_+ \times \mathcal{O}$ the following holds true:

$$\begin{cases} \forall z \in \mathbb{R}_- \times \mathbb{R}_+ \ \exists v_1 \in V \text{ such that} & z_1 F_1\big(t,\xi,z,v_1\big) \leqslant C|z|^2 \\ \forall w \in \mathbb{R}_+ \times \mathbb{R}_- \ \exists v_2 \in V \text{ such that} & w_2 F_2\big(t,\xi,w,v_2\big) \leqslant C|w|^2. \end{cases} \tag{5.10}$$

Indeed, owing to well-known results in set-valued analysis (see, for instance, [2, Theorem 8.2.9]), from (5.10) we deduce that for a.e. $t \in \mathbb{R}_+$ and every $x \in H$ there exist measurable maps $u_i : \mathcal{O} \to V$ $(i = 1, 2)$, such that

$$\begin{aligned} -x_1^-(\xi)\, F_1\big(t,\xi,(-x_1^-(\xi), x_2^+(\xi)), u_1(\xi)\big) &\leqslant C|x(\xi)|^2 \\ -x_2^-(\xi)\, F_2\big(t,\xi,(x_1^+(\xi), -x_2^-(\xi)), u_2(\xi)\big) &\leqslant C|x(\xi)|^2 \end{aligned}$$

for a.e. $\xi \in \mathcal{O}$. So, taking $u = (u_1, u_2)$ we have that (5.9) is satisfied. □

6. Appendix

In this Appendix, we prove the proximal smoothness of level sets of smooth functionals on a real Hilbert space H. We denote by $C^{1,1}(H)$ the Banach space of all maps $\Phi : H \to \mathbb{R}$ that are continuously Fréchet differentiable at every point of H with a Lipschitz continuous gradient $D\Phi : H \to H$, and we set

$$Lip(D\Phi) = \sup_{x \neq y} \frac{|D\Phi(x) - D\Phi(y)|}{|x - y|} \qquad \forall \Phi \in C^{1,1}(H).$$

Finally, for any $c \in \mathbb{R}$, we denote by $\Lambda(c)$ the level set

$$\Lambda(c) := \big\{x \in H \ : \ \Phi(x) = c\big\}. \tag{6.1}$$

The following result establishes the proximal smoothness of "nondegenerate" level sets.

Proposition 6.1. *Let $\Phi \in C^{1,1}(H)$ and let $c_0 \in \mathbb{R}$ be such that $\Lambda(c_0) \neq \varnothing$. Suppose that there exist positive constants σ and S such that*

$$\sigma \leqslant |D\Phi(x)| \leqslant S \quad \text{for every } x \in \Lambda(c_0). \tag{6.2}$$

Then $\Lambda(c_0)$ is proximally smooth.

Proof. Let us abbreviate $\Lambda(c_0)$ to Λ_0. We want to prove that there exists $\delta > 0$ such that any $x \in H$ with $d_{\Lambda_0}(x) < \delta$ has a unique projection onto Λ_0. We prove the existence of the projection first, then its uniqueness.

Existence. Fix any $x \in H \setminus \Lambda_0$ and suppose that $\Phi(x) < c_0$. Since d_{Λ_0} is Fréchet differentiable on a dense subset of H by a well-known result due to Preiss [16], we can find a sequence $\{x_n\}_n$, converging to x, such that $\Pi_{\Lambda_0}(x_n) = \{y_n\}$. Moreover, we have that

$$x_n = y_n - d_{\Lambda_0}(x_n)\, \frac{D\Phi(y_n)}{|D\Phi(y_n)|} \qquad (n \geqslant 1). \tag{6.3}$$

Set $\mu_n = d_{\Lambda_0}(x_n)/|D\Phi(y_n)|$. In view of (6.2), $\{\mu_n\}$ is bounded in $\mathbb{R}$. So, we can extract a convergent subsequence—still denoted by $\{\mu_n\}$. Since, for all sufficiently large integers m and n, we have that

$$y_n - y_m = x_n - x_m + d_{\Lambda_0}(x_n)\,\frac{D\Phi(y_n)}{|D\Phi(y_n)|} - d_{\Lambda_0}(x_m)\,\frac{D\Phi(y_m)}{|D\Phi(y_m)|}\,,$$

we deduce that

$$\begin{aligned} |y_n - y_m|^2 &= \langle x_n - x_m, y_n - y_m\rangle \\ &+ d_{\Lambda_0}(x_n)\Big\langle \frac{D\Phi(y_n)}{|D\Phi(y_n)|} - \frac{D\Phi(y_m)}{|D\Phi(y_m)|}, y_n - y_m\Big\rangle \\ &+ \big(d_{\Lambda_0}(x_n) - d_{\Lambda_0}(x_m)\big)\Big\langle \frac{D\Phi(y_m)}{|D\Phi(y_m)|}, y_n - y_m\Big\rangle. \end{aligned}$$

Notice that (6.2) yields the following inequality

$$\Big|\,\frac{D\Phi(x)}{|D\Phi(x)|} - \frac{D\Phi(y)}{|D\Phi(y)|}\,\Big| \leqslant \frac{2\,Lip(D\Phi)}{\sigma}\,|x-y| \tag{6.4}$$

for all $x, y \in \Lambda_0$. Hence, owing to (6.4),

$$|y_n - y_m|^2 \leqslant 2|x_n - x_m|\,|y_n - y_m| + d_{\Lambda_0}(x_n)\,\frac{2\,Lip(D\Phi)}{\sigma}\,|y_n - y_m|^2.$$

Since $d_{\Lambda_0}(x_n) \to d_{\Lambda_0}(x)$, by taking $\delta > 0$ such that

$$\delta\, Lip(D\Phi) < \frac{\sigma}{2}\,, \tag{6.5}$$

for $d_{\Lambda_0}(x) < \delta$ we conclude that $\{y_n\}$ is a Cauchy sequence in H. Consequently, its limit $y \in \Lambda_0$ satisfies

$$|x - y| = \lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} d_{\Lambda_0}(x_n) = d_{\Lambda_0}(x),$$

thus qualifying as a projection of x onto Λ_0.

Finally, for $\Phi(x) > c_0$ the reasoning is exactly the same as above, with the only difference that, in this case, (6.3) is replaced by

$$x_n = y_n + d_{\Lambda_0}(x_n)\,\frac{D\Phi(y_n)}{|D\Phi(y_n)|} \qquad (n \geqslant 1).$$

Uniqueness. Let $x \in H$ be such that $0 < d_{\Lambda_0}(x) < \delta$, with δ as in (6.5), and suppose that $x', x'' \in \Pi_{\Lambda_0}(x)$. Then

$$x' + \mu' D\Phi(x') = x = x'' + \mu'' D\Phi(x'')$$

for real numbers μ' and μ'' such that $\mu'\mu'' > 0$ and

$$|\mu'|\,|D\Phi(x')| = d_{\Lambda_0}(x) = |\mu''|\,|D\Phi(x'')|\,.$$

Therefore, on account of (6.4), we conclude that

$$\frac{|x' - x''|}{d_{\Lambda_0}(x)} = \Big|\,\frac{D\Phi(x')}{|D\Phi(x')|} - \frac{D\Phi(x'')}{|D\Phi(x'')|}\,\Big| \leqslant \frac{2\,Lip(D\Phi)}{\sigma}\,|x' - x''|,$$

which in turn yields $x' = x''$ because $2d_{\Lambda_0}(x)Lip(D\Phi)/\sigma < 1$ by (6.5). $\square$

References

[1] J.-P. Aubin. *Viability theory.* Birkhäuser Boston, Inc., Boston, MA, 1991.
[2] Jean-Pierre Aubin and Hélène Frankowska. *Set-valued analysis*, volume 2 of *Systems & Control: Foundations & Applications.* Birkhäuser Boston, Inc., Boston, MA, 1990.
[3] A. Bensoussan, G. Da Prato, M. Delfour, and S. K. Mitter. *Representation and control of infinite-dimensional systems. Vol. 1.* Birkhäuser Boston, Inc., Boston, MA, 1992.
[4] G. Bonanno. Two theorems on the Scorza Dragoni property for multifunctions. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 83:51–56, 1989.
[5] L. Caffarelli and F. Lin. Nonlocal heat flows preserving the L^2 energy. *Discrete Contin. Dyn. Syst.*, 23(1-2):49–64, 2009.
[6] P. Cannarsa and H. Frankowska. Value function and optimality conditions for semilinear control problems. *Appl. Math. Optim.*, 26(2):139–169, 1992.
[7] P. Cannarsa, G. Da Prato, and H. Frankowska. Invariance for quasi-dissipative systems in Banach spaces. *J. Math. Anal. Appl.*, 457(2):1173–1187, 2018.
[8] O. Cârjă, M. Necula, and I. I. Vrabie. *Viability, invariance and applications.* Amsterdam: Elsevier, 2007.
[9] H. Frankowska, S. Plaskacz, and T. Rzeżuchowski. Measurable viability theorems and the Hamilton-Jacobi-Bellman equation. *J. Differential Equations*, 116(2):265–305, 1995.
[10] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1981.
[11] R. H. Martin. Differential equations on closed subsets of a Banach space. *Trans. Am. Math. Soc.*, 179:399–414, 1973.
[12] M. Nagumo. Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen. *Proc. Phys.-Math. Soc. Japan, III. Ser.*, 24:551–559, 1942.
[13] N. H. Pavel. Invariant sets for a class of semi-linear equations of evolution. *Nonlinear Anal., Theory Methods Appl.*, 1:187–196, 1977.
[14] N. H. Pavel. Semilinear equations with dissipative time-dependent domain perturbations. *Isr. J. Math.*, 46:103–122, 1983.
[15] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, New York etc.: Springer-Verlag., 1983.
[16] D. Preiss. Differentiability of Lipschitz functions on Banach spaces. *J. Funct. Anal.*, 91(2):312–345, 1990.
[17] S. Shi. Viability theorems for a class of differential-operator inclusions. *J. Differ. Equations*, 79(2):232–257, 1989.

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133 Roma, Italy
E-mail address: cannarsa@mat.uniroma2.it

Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 7, 56125 Pisa, Italy
E-mail address: daprato@sns.it

CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche,
Sorbonne Université, Case 247, 4 Place Jussieu, 75252 Paris, France
E-mail address: helene.frankowska@imj-prg.fr