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The genus of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ not contained in quadrics

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Abstract
A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree and the dimension of the space where they are embedded. Fixed integers r, d, s, Castelnuovo-Halphen's theory states a sharp upper bound for the genus of a non-degenerate, reduced and irreducible curve of degree d in $\mathbb{P}^r$, under the condition of being not contained in a surface of degree $< s$. This theory can be generalized in several ways. For instance, fixed integers r, d, k, one may ask for the maximal genus of a curve of degree d in $\mathbb{P}^r$, not contained in a hypersurface of degree $< k$. In the present paper we examine the genus of curves C of degree d in $\mathbb{P}^r$ not contained in quadrics (i.e. $h^0(\mathbb{P}^r, \mathcal{I}_C(2)) = 0$). When $r = 4$ and $r = 5$, and $d \gg 0$, we exhibit a sharp upper bound for the genus. For certain values of $r \geq 7$, we are able to determine a sharp bound except for a constant term, and the argument applies also to curves not contained in cubics.

Keywords Projective curve · Castelnuovo-Halphen theory · Quadric and cubic hypersurfaces · Veronese surface · Projection of a rational normal scroll surface · Maximal rank.

Mathematics Subject Classification Primary 14N15 · Secondary 14N25 · 14M05 · 14M06 · 14M10 · 14J26 · 14J70

1 Introduction

A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree and the dimension of the space where they are embedded [3, 9, 11]. Fixed integers r, d, s, Castelnuovo-Halphen's theory states a sharp upper bound for the genus of a non-degenerate, reduced and irreducible curve of degree d in $\mathbb{P}^r$, under the condition of being not contained in a surface of degree $< s$ [3, 9, 10]. This theory can be generalized imposing flag conditions [4]. For instance, fixed integers r, d, k, one may ask for the maximal genus of a curve of degree d in $\mathbb{P}^r$, not contained in a hypersurface of degree $< k$. As far as we know, in this case there are only rough

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estimates [13, 4, p. 726, (2.10)]. In the present paper we examine the case of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ not contained in quadrics. Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1 *Let $C \subset \mathbb{P}^4$ be a reduced and irreducible complex curve, of degree d and arithmetic genus $p_a(C)$, not contained in quadrics (i.e. $h^0(\mathbb{P}^4, \mathcal{I}_C(2)) = 0$)*

- If $d > 16$, then
$$p_a(C) \leq \frac{1}{8}d(d-6)+1. \tag{1}$$
The bound is sharp, and $p_a(C) = \frac{1}{8}d(d-6)+1$ if and only if d is even, and C is contained in the isomorphic projection in $\mathbb{P}^4$ of the Veronese surface.
- If $d > 143$, and either d is odd or C is arithmetically Cohen-Macaulay (a.C.M. for short), then
$$p_a(C) \leq \frac{d^2}{10} - \frac{d}{2} - \frac{1}{10}(\epsilon-4)(\epsilon+1) + \binom{\epsilon}{4} + 1, \tag{2}$$

where ϵ is defined by dividing $d-1 = 5m+\epsilon$, $0 \leq \epsilon \leq 4$. The bound is sharp. Every extremal curve C is a.C.M., and it is contained in a flag $S \subset T \subset \mathbb{P}^4$, where S is a surface uniquely determined by C, of degree 5, and T is a cubic threefold. Moreover, the surface S has sectional genus $\pi = 1$.

When d is even, in contrast with the classical case, the extremal curves are never a.C.M. (at least for $d > 16$). Moreover, the asymptotic behaviour of the bound is different, depending on whether d is even or odd. As for the proof of Theorem 1.1, the first part, when $d \gg 0$, easily follows combining the geometry of the Veronese surface (see Lemma 3.1 below), with [9, p. 117, Theorem (3.22)]. In order to establish the bound under the hypothesis $d > 16$, we need to refine the analysis of the Hilbert function of a general hyperplane section of C, similarly as in [8, p. 74–75]. It is a rather long numerical argument, relying on [3]. We relegate it to an appendix at the end of the paper (Section 7). We will not insist on this analysis in the other cases (i.e. in second part of Theorem 1.1, and in next Theorem 1.2, Propositions 1.3, and 1.4). We will be content with coarser hypotheses on d, which can be obtained without too much effort, simply using [9, loc. cit.], [7, Corollary 3.11], and [3, Main Theorem]. However, we think our assumptions on d, in the second part of Theorems 1.1 and 1.2, can be significantly improved. We hope to return on this question in a forthcoming paper. When d is odd, or C is a.C.M., the proof of Theorem 1.1 is more involved, and we need [14, Theorem 1] in order to extend certain results of [7] in the range $r = 4$ and $s = 5$. We point out that, in this case, the extremal curves we are able to produce are singular. We do not know whether there are nonsingular extremal curves.

Theorem 1.2 *Let $C \subset \mathbb{P}^5$ be a reduced and irreducible complex curve, of degree d and arithmetic genus $p_a(C)$, not contained in quadrics. Assume $d > 215$, and divide $d-1 = 6m+\epsilon, 0 \leq \epsilon \leq 5$. Then:*
$$p_a(C) \leq 6\binom{m}{2} + m\epsilon.$$

The bound is sharp [1]. *Every extremal curve C is not a.C.M., and it is contained in a flag $S \subset T \subset \mathbb{P}^5$, where S is a surface uniquely determined by C, of degree 6, and T is a cubic hypersurface of $\mathbb{P}^5$. Moreover, S has sectional genus $\pi = 0$, and arithmetic genus $p_a(S) = 0$*

.

The analysis of the arithmetic genus of a surface of degree 6 in $\mathbb{P}^5$ appearing in [7, Corollary 3.6], combined with [9, loc. cit.], enables us to state the bound in Theorem 1.2. As for the sharpness, we simply project a general Castelnuovo's curve in $\mathbb{P}^7$, which is not contained in quadrics because so is a general projection of a smooth rational normal scroll surface [1, Theorem 2, Lemma 3.1].

In the case of curves contained in $\mathbb{P}^r$, for certain values of $r \geq 7$, we are able to compute the sharp bound for $p_a(C)$, except for a constant term. We obtain similar results also in the case of cubics. In fact, we prove the following Propositions 1.3 and 1.4.

Proposition 1.3 *Fix an integer $r \geq 7$, not divisible by 3, and denote by s the integer such that*[2]

$$\binom{r+2}{2} = 3(s+1).$$

Let $C \subset \mathbb{P}^r$ be a reduced and irreducible complex curve, of degree d and maximal arithmetic genus $p_a(C)$ with respect to the conditions of being of degree d and not contained in a quadric hypersurface. Assume $d > d_1(r,s)$ (where $d_1(r,s)$ is defined in (14) and (16) below), and divide $d-1 = ms + \epsilon$, $0 \leq \epsilon \leq s-1$. Then:

$$p_a(C) = \frac{d^2}{2s} - \frac{d}{2s}(s+2) + O(1), \quad \text{with } 0 < O(1) \leq \frac{s^3}{r-2}. \tag{3}$$

Moreover, C is not a.C.M., and it is contained in a flag $S \subset T \subset \mathbb{P}^r$, where S is a surface uniquely determined by C, of degree s, and T is a cubic hypersurface of $\mathbb{P}^r$. The surface S has sectional genus $\pi = 0$.

Proposition 1.4 *Fix an integer $r \geq 9$, and assume that the number*

$$s := \frac{1}{6}\left[\binom{r+3}{3} - 4\right]$$

is an integer.[3] *Let $C \subset \mathbb{P}^r$ be a reduced and irreducible complex curve, of degree d and maximal arithmetic genus $p_a(C)$ with respect to the conditions of being of degree d and not contained in a cubic hypersurface. Assume $d > d_1(r,s)$ (where $d_1(r,s)$ is defined in (14) and (16) below), and divide $d-1 = ms + \epsilon$, $0 \leq \epsilon \leq s-1$. Then:*

$$p_a(C) = \frac{d^2}{2s} - \frac{d}{2s}(s+2) + O(1), \quad \text{with } 0 < O(1) \leq \frac{s^3}{r-2}. \tag{4}$$

1 This is the Castelnuovo's bound for curves of degree d in $\chi(\mathcal{O}_S)$ (see Section 2, (v), below).

2 If r is not divisible by 3, then $\chi(\mathcal{O}_S)$ is divisible by 3. For instance, we have K_S^2.

3 This is equivalent to say that the class $[r]$ of r modulo 36 is one of the following classes: [1], [2], [9], [10], [11], [18], [19], [27], [29] (e.g. K_S^2).

Moreover, C is not a.C.M., and it is contained in a flag $S \subset T \subset \mathbb{P}^r$, where S is a surface uniquely determined by C, of degree s, and T is a quartic hypersurface of $\mathbb{P}^r$. The surface S has sectional genus $\pi = 0$.

The proof follows the same line of Theorem 1.2. However, this analysis, when $r \geq 7$, leads to examine surfaces whose degree is out of the range considered in [7, Theorem 2.2]. We partly overcome this difficulty, using an estimate appearing in [5, p. 792, Lemma]. This explains why we are not able to determine the sharp bound (i.e. the constant $O(1)$).

We observe that the bounds appearing in [4, 5, 7] do not apply to our setting, except the estimate

$$p_a(C) \leq \frac{r!d^2}{4t^{r-2}} + O(d)$$

for an integral non-degenerate curve $C \subset \mathbb{P}^r$ of degree d not contained in a hypersurface of degree $< t$ [4, p. 726, (2.10)]. In the cases $t = 2$ and $t = 3$, this bound is coarse compared with the bounds of the present paper. Moreover, observe that the number $\frac{1}{8}d(d-6)+1$, which appears in (1), is the genus of a plane curve of degree $d/2$. In [6, 8] there are sharp lower bounds $K_S^2 \geq -d(d-6)$ and $\chi(\mathcal{O}_S) \geq -\frac{1}{8}d(d-6)$ for a smooth projective surface S of degree $d \gg 0$. Extremal surfaces are contained in $\mathbb{P}^5$, and have sectional genus given by the same number $\frac{1}{8}d(d-6)+1$. However, unlike what happens in the setting of Theorem 1.1, general hyperplane sections of such S come from curves in the Veronese surface $V \subset \mathbb{P}^5$ via projection from a point $x \in V$ [6, p. 226, Corollary 1.2] (it would remain to interpret the last case, i.e. projections of curves of V from a point $x \in \mathrm{Sec}(V)\backslash V$).

2 Notations and preliminary remarks

- (i) For a projective subscheme $V \subseteq \mathbb{P}^N$ we will denote by $\mathcal{I}_V = \mathcal{I}_{V,\mathbb{P}^N}$ its ideal sheaf in $\mathbb{P}^N$, and by $M(V) := \oplus_{i\in\mathbb{Z}} H^1(V, \mathcal{I}_V(i))$ the Hartshorne-Rao module of V. We will denote by h_V the Hilbert function of V, and by Δh_V the first difference of h_V, i.e. $\Delta h_V(i) = h_V(i) - h_V(i-1)$. Observe that

$$h_V(i) = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(i)) - h^0(\mathbb{P}^N, \mathcal{I}_V(i)) \leq h^0(V, \mathcal{O}_V(i)). \tag{5}$$

We denote by $p_a(V)$ the arithmetic genus of V. We say that V is *arithmetically Cohen-Macaulay* (shortly a.C.M.) if all the restriction maps $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(i)) \to H^0(V, \mathcal{O}_V(i))$ ($i \in \mathbb{Z}$) are surjective, and $H^j(V, \mathcal{O}_V(i)) = 0$ for all $i \in \mathbb{Z}$ and $1 \leq j \leq \dim V - 1$. If V is one-dimensional of degree d, and V' denotes a general hyperplane section of V, then [9, p. 83–84]:

$$\Delta h_V(i) \geq h_{V'}(i) \text{ for every } i, \tag{6}$$

and

$$p_a(V) \leq \sum_{i=1}^{+\infty} d - h_{V'}(i). \tag{7}$$

Moreover, the equality occurs in (6) (resp. in (7)) if and only if V is a.C.M.. If V is integral (i.e. reduced and irreducible) and one-dimensional, we also have [9, p. 86–87, Corollary (3.5) and (3.6)]:

$$h_{V'}(i+j) \geq \min\{d,\ h_{V'}(i) + h_{V'}(j) - 1\} \text{ for every } i,j, \tag{8}$$

and

$$h_{V'}(i) \geq \min\{d,\ i(r-1)+1\} \text{ for every } i. \tag{9}$$

If V is integral, and $\dim V \geq 2$, then V is a.C.M. if and only if a general hyperplane section of V is.

- (ii) Let $\Sigma \subset \mathbb{P}^{r-1}$ be an integral curve of degree s in $\mathbb{P}^{r-1}$, $r \geq 4$, and $\Sigma' \subset \mathbb{P}^{r-2}$ a general hyperplane section of Σ. For every i we have: $h^0(\Sigma, \mathcal{O}_\Sigma(i)) = 1 - \pi + is + h^1(\Sigma, \mathcal{O}_\Sigma(i))$, where π denotes the arithmetic genus of Σ. From the exact sequence $0 \to \mathcal{O}_\Sigma(i-1) \to \mathcal{O}_\Sigma(i) \to \mathcal{O}_{\Sigma'}(i) \to 0$ we get the exact sequence $0 \to H^0(\Sigma, \mathcal{O}_\Sigma(i-1)) \to H^0(\Sigma, \mathcal{O}_\Sigma(i)) \to H^0(\Sigma', \mathcal{O}_{\Sigma'}(i)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i-1)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i)) \to 0.$
$0 \to H^0(\Sigma, \mathcal{O}_\Sigma(i-1)) \to H^0(\Sigma, \mathcal{O}_\Sigma(i)) \to H^0(\Sigma', \mathcal{O}_{\Sigma'}(i)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i-1)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i)) \to 0.$
We deduce that $h^1(\Sigma, \mathcal{O}_\Sigma(i)) \leq h^1(\Sigma, \mathcal{O}_\Sigma(i-1))$. In particular, when $i \geq 1$, we have $h^1(\Sigma, \mathcal{O}_\Sigma(i)) \leq \pi = h^1(\Sigma, \mathcal{O}_\Sigma)$. Observe that if, for some $i \geq 1$, one has $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = \pi$, then $\pi = 0$. In fact, in this case, we have $h^1(\Sigma, \mathcal{O}_\Sigma(1)) = \pi$, hence $h^0(\Sigma, \mathcal{O}_\Sigma(1)) = 1 + s$. This means that Σ is contained in $\mathbb{P}^s$ as a non-degenerate curve of degree s, so $\pi = 0$. Similarly, if for some $i \geq 1$, one has $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = \pi - 1$, then $\pi = 1$. In fact, in this case, we have $h^1(\Sigma, \mathcal{O}_\Sigma(1)) = \pi - 1$ (it cannot be $h^1(\Sigma, \mathcal{O}_\Sigma(1)) = \pi$, otherwise, arguing as above, we obtain $\pi = 0$, in contrast with the assumption $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = \pi - 1$), hence $h^0(\Sigma, \mathcal{O}_\Sigma(1)) = s$. This means that Σ is contained in $\mathbb{P}^{s-1}$ as a non-degenerate curve of degree s. By Castelnuovo's bound, it follows that $\pi = 1$. In conclusion, since $h^0(\Sigma, \mathcal{O}_\Sigma(i)) = 1 - \pi + is + h^1(\Sigma, \mathcal{O}_\Sigma(i))$, for every $i \geq 1$ we have:

$$h^0(\Sigma, \mathcal{O}_\Sigma(i)) \leq \begin{cases} 1 + is & \text{if } \pi \geq 0, \\ is & \text{if } \pi \geq 1, \\ -1 + is & \text{if } \pi \geq 2. \end{cases} \tag{10}$$

- (iii) Let $S \subset \mathbb{P}^r$ $(r \geq 4)$ be an integral surface of degree s, and Σ a general hyperplane section of S. From the exact sequence $0 \to \mathcal{O}_S(i-1) \to \mathcal{O}_S(i) \to \mathcal{O}_\Sigma(i) \to 0$ we get, for every $i \geq 1$, $h^0(S, \mathcal{O}_S(i)) \leq \sum_{j=0}^{i} h^0(\Sigma, \mathcal{O}_\Sigma(j))$. Hence, for every $i \geq 1$, we have (compare with (5)) $h^0(\mathbb{P}^r, \mathcal{I}_S(i)) \geq \binom{r+i}{i} - \sum_{j=0}^{i} h^0(\Sigma, \mathcal{O}_\Sigma(j))$. Combining with (10), it follows that:

$$h^0(\mathbb{P}^r, \mathcal{I}_S(i)) \geq \begin{cases} \binom{r+i}{i} - \left[i + 1 + \binom{i+1}{2} s\right] & \text{if } \pi \geq 0, \\ \binom{r+i}{i} - \left[1 + \binom{i+1}{2} s\right] & \text{if } \pi \geq 1, \\ \binom{r+i}{i} - \left[1 - i + \binom{i+1}{2} s\right] & \text{if } \pi \geq 2. \end{cases} \tag{11}$$

- (iv) Let $S \subset \mathbb{P}^{s+1}$ be a non-degenerate smooth rational normal scroll surface, of minimal degree s. Fix an integer $k \geq 1$. Since S is a.C.M., we have $h^1(S, \mathcal{O}_S(k)) = 0$. Moreover, one has $K_S = -2\Sigma + (s-2)W$, where Σ is a general hyperplane section, and W the ruling. Therefore, one has $h^2(S, \mathcal{O}_S(k)) = h^0(S, K_S - k\Sigma) = 0$. By Riemann-Roch Theorem it follows that: $h^0(S, \mathcal{O}_S(k)) = 1 + \frac{1}{2}k\Sigma \cdot (k\Sigma - K_S) = k + 1 + \binom{k+1}{2}s$. In particular:

$$h^0(S, \mathcal{O}_S(2)) = 3(1+s), \quad \text{and} \quad h^0(S, \mathcal{O}_S(3)) = 4 + 6s. \tag{12}$$

- (v) Fix integers $s \geq 2$ and $d \geq s+1$, and divide $d - 1 = ms + \epsilon$, $0 \leq \epsilon \leq s-1$. The number

$$G(s+1;d) := \binom{m}{2}s + m\epsilon$$

is the celebrated Castelnuovo's bound for the genus of a non-degenerate integral curve of degree d in $\mathbb{P}^{s+1}$ [9, p. 87, Theorem (3.7)]. Observe that

$$G(s+1;d) = \frac{d^2}{2s} + \frac{d}{2s}(-s-2) + \frac{1+\epsilon}{2s}(s+1-\epsilon) \leq \frac{d^2}{2s}. \tag{13}$$

- (vi) Fix integers r, d and s, with $s \geq r-1 \geq 2$. Denote by $G(r;d,s)$ the maximal arithmetic genus for an integral non-degenerate projective curve $C \subset \mathbb{P}^r$ of degree d, not contained in any surface of degree $< s$ [3]. Put

$$d_0(r,s) := \frac{2s}{r-2}\prod_{i=1}^{r-2}[(r-1)!s]^{\frac{1}{r-1-i}}. \tag{14}$$

By [3, Main Theorem] and [5, p. 792, Lemma], one knows that, for $d > d_0(r,s)$, the number $G(r;d,s)$ has the following form:

$$G(r;d,s) = \frac{d^2}{2s} + \frac{d}{2s}[2G(r-1;s) - 2 - s] + R, \quad \text{with} \quad |R| \leq \frac{s^3}{r-2}. \tag{15}$$

When $r = 4$ and $s = 6$, this holds true also for $d > 143$, instead than simply for $d > d_0(4,6)$ [9, Theorem (3.22), p. 117]. And, when $r = 5$ and $s = 7$, this holds true also for $d > 179$, instead than simply for $d > d_0(5,7)$ [9, loc. cit.]. Moreover, taking into account (15), and that $G(r-1;s+1) \leq \frac{(s+1)^2}{2(r-2)}$ (compare with (13)), an elementary computation, which we omit, shows that, for (compare with (14))

$$d > d_1(r,s) := \max\left\{d_0(r,s), \frac{4s}{r-2}(s+1)^3\right\}, \tag{16}$$

one has

$$G(r;d,s+1) < G(s+1;d). \tag{17}$$

- (vii) Let $S \subset \mathbb{P}^r$ be an integral non-degenerate projective surface of degree $s \geq r-1 \geq 2$. Denote by σ the integer part of the number

$$(s-r+2)\left(\frac{s^2}{2(r-2)} + 1\right) + 1.$$

By [7, Lemma 3.8] we know that

$$h^1(\mathbb{P}^r, \mathcal{I}_S(i)) = 0 \quad \text{for every } i \geq \sigma. \tag{18}$$

Now, suppose there exists an a.C.M. curve C on S of degree d, with $d-1 = ms + \epsilon$, $0 \leq \epsilon \leq s-1$, and $m \geq \sigma$. Let Σ and Γ be general hyperplane sections of S and C. By Bezout's theorem we have $h_S(i) = h_C(i)$ and $h_\Sigma(i) = h_\Gamma(i)$ for every $i \leq m$. Since C is a.C.M., we have $\Delta h_C(i) = h_\Gamma(i)$ for every i (compare with (6) and three lines below). It follows that $\Delta h_S(i) = h_\Sigma(i)$ for every $i \leq m$. Therefore, since

$$\mu_i := \Delta h_S(i) - h_\Sigma(i) = \dim_{\mathbb{C}} \left[\ker \left(H^1(\mathbb{P}^r, \mathcal{I}_S(i-1)) \to H^1(\mathbb{P}^r, \mathcal{I}_S(i))\right)\right] \tag{19}$$

(see [7, Lemma 3.4 and Remark 3.5]), we deduce that $H^1(\mathbb{P}^r, \mathcal{I}_S(i-1))$ maps injectively to $H^1(\mathbb{P}^r, \mathcal{I}_S(i))$ for every $i \leq m$. Combining with (18), it follows that the Hartshorne-Rao module $M(S)$ vanishes. Summing up: *if a surface S contains an a.C.M. curve of degree d with $m \geq \sigma$,then $H^1(\mathbb{P}^r, \mathcal{I}_S(i)) = 0$ for every $i \in \mathbb{Z}$* (compare with [7, Remark 3.10, (i)]). Notice that the condition $m \geq \sigma$ is satisfied when $r = 5$ and $s = 6$ and $d > 179$, or when $d > d_1(r, s)$.

- ($viii$) Let $S \subset \mathbb{P}^5$ be an integral non-degenerate projective surface of degree 6 and sectional genus $\pi = 0$. Denote by $p_a(S)$ its arithmetic genus. By [7, Corollary 3.6 and Remark 3.7] (here one has to put $\pi_0 = 2$) we know that

$$-3 \leq p_a(S) \leq 0,$$

and that if $p_a(S) = -3$, then $M(S) = 0$. By [7, Lemma 3.4] we have

$$p_a(S) = -\dim_{\mathbb{C}} M(\Sigma) + \sum_{i=1}^{+\infty} \mu_i,$$

where $\Sigma \subset \mathbb{P}^4$ denotes a general hyperplane section of S, and the numbers μ_i are defined as in (19). Moreover, by [7, Proposition 3.1], we also know that

$$h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(1)) = 2 \quad \text{and} \quad h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(i)) \leq \max(0, 3-i) \quad \text{for every } i \geq 1.$$

3 The proof of theorem 1.1.

We start with the proof of the first part of Theorem 1.1. We need the following Lemma 3.1, which is certainly well known. We prove it for lack of a suitable reference.

Lemma 3.1 *Let $V \subset \mathbb{P}^5$ be the Veronese surface. Let* $\mathrm{Sec}(V)$ *be the secant variety of V, $x \in \mathbb{P}^5 \backslash \mathrm{Sec}(V)$ be a point. Then the projection in $\mathbb{P}^4$ of V from x is a surface of degree* 4, *not contained in quadrics. Conversely, every integral surface of degree* 4 *in $\mathbb{P}^4$ not contained in quadrics is the projection of V from a point $x \in \mathbb{P}^5 \backslash \mathrm{Sec}(V)$.*

Proof Let V' be the projection in $\mathbb{P}^4$ of V from a point $x \in \mathbb{P}^5 \backslash \mathrm{Sec}(V)$. Suppose there exists a quadric Q in $\mathbb{P}^4$ containing V'. If Q were smooth, then, by Severi's theorem, V would be a complete intersection of two quadrics. This implies that the sectional genus of V is 1, in contrast with the fact that it is 0. If Q were singular, then Q would be a cone. Let $H \subset \mathbb{P}^4$ be a general hyperplane, passing from the vertex of the cone (if it is a point). Then $H \cap V$ is an integral curve of degree 4 of a quadric cone in $\mathbb{P}^3$. Therefore, the genus of $H \cap V$ would be 1, a contradiction, because it is 0.

Conversely, let S be a surface of degree 4 in $\mathbb{P}^4$ not contained in quadrics. Let π be the sectional genus of S. By Castelnuovo's bound, we have either $\pi = 0$ or $\pi = 1$. If $\pi = 1$, then a general hyperplane section $H \cap S$ of S is a Castelnuovo's curve in $\mathbb{P}^3$. This curve is contained in a quadric, which lifts to a quadric containing S because $H \cap S$ is a.C.M., hence also S is. Therefore, $\pi = 0$. In this case, by [15, Lemma 7, p. 411], we know that S is the projection of the Veronese surface V from some point $x \in \mathbb{P}^5 \backslash V$. Now we observe that $x \notin \mathrm{Sec}(V) \backslash V$, otherwise S is contained in a quadric [12, Remark 2.1, p. 60–61]. Hence, S is the projection of the Veronese surface V from a point $x \in \mathbb{P}^5 \backslash \mathrm{Sec}(V)$. □

We are in position to prove Theorem 1.1, first part.

Let $C \subset \mathbb{P}^4$ be a curve of degree d not contained in quadrics. Since every surface of degree 3 in $\mathbb{P}^4$ is contained in a quadric (Section 2, (11)), in view of Lemma 3.1, it suffices to prove that *if C is not contained in a surface of degree < 5,then $p_a(C) < \frac{1}{8}d(d-6)+1$.* To this purpose, set

$$G := \frac{1}{10}d^2 - \frac{3}{10}d + \frac{1}{5} + \frac{1}{10}v - \frac{1}{10}v^2 + w,$$

where v is defined by dividing $d-1 = 5n + v$, $0 \le v \le 4$, and $w := \max\{0, [\frac{v}{2}]\}$. By [9, Theorem (3.22), p. 117] (with the notation of [9] one has $G = \pi_2(d,4)$), we know that, if C is not contained in a surface of degree < 5, and $d > 143$, then $p_a(C) \le G$. An elementary computation shows that $G < \frac{1}{8}d(d-6)+1$ for $d > 18$. This concludes the proof of the first part of Theorem 1.1, when $d > 143$. We may examine the remaining cases $16 < d \le 143$ in a similar manner as in [8, p. 74–75]. For details, we refer to the Appendix (see Section 7 below).

Now we are going to prove Theorem 1.1, second part.

First, we notice that if d is odd, or C is a.C.M., and C is not contained in quadrics, then C is not contained in surfaces of degree < 5. In fact, by (11), every surface in $\mathbb{P}^4$ of degree 3 is contained in a quadric. This is true also for surfaces of degree 4, except for an isomorphic projection S of the Veronese surface (Lemma 3.1). But on the Veronese surface every curve has even degree. And every curve on S cannot be a.C.M. (otherwise, from the natural sequence: $0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-C) \to 0$, we would get the exact sequence: $H^0(S, \mathcal{O}_S(H_S - C)) \to H^1(\mathbb{P}^4, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^4, \mathcal{I}_C(1))$; this is impossible, because $H^0(S, \mathcal{O}_S(H_S - C)) = 0$, $H^1(\mathbb{P}^4, \mathcal{I}_C(1)) = 0$, and $H^1(\mathbb{P}^4, \mathcal{I}_S(1)) \neq 0$).

On the other hand, if C is not contained in a surface of degree < 6, and $d > 143$, then (compare with Section 2, (*vi*), and (15))

$$p_a(C) \le G(4;d,6) \le \frac{d^2}{12} + 108.$$

By elementary arithmetics, it follows that, for $d > 143$, $p_a(C)$ is strictly less than the bound (2) appearing in our claim. Therefore, in order to prove the second part of Theorem 1.1, we may assume that C is contained in an integral surface $S \subset \mathbb{P}^4$ of degree 5, not contained in quadrics. Observe that, by Bezout's theorem, such a surface S is unique, and is contained in a cubic hypersurface of degree 3 by (11).

Let $\Sigma \subset \mathbb{P}^3$ be a general hyperplane section of S. Since $\deg \Sigma = 5$, by Castelnuovo's bound, the arithmetic genus π of Σ satisfies the condition $0 \le \pi \le 2$. The sectional genus π cannot be 2, otherwise Σ should be a Castelnuovo's curve, hence a.C.M., and contained in a quadric. This would imply that also S is a.C.M., and contained in a quadric (since $h^1(\mathbb{P}^4, \mathcal{I}_S(1)) = 0$, the restriction map $H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^3, \mathcal{I}_{\Sigma,\mathbb{P}^3}(2))$, induced by the exact sequence $0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{\Sigma,\mathbb{P}^3}(2) \to 0$, is onto). Therefore, we have $0 \le \pi \le 1$. Since Σ is non-degenerate, from the exact sequence $0 \to \mathcal{I}_{\Sigma,\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_\Sigma(1) \to 0$ it follows that $h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma,\mathbb{P}^3}(1)) = 2 - \pi > 0$ (observe that $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = 0$ for every $i \ge 1$, because $0 \le \pi \le 1$). Moreover, by [14, Theorem 1], we have

$$h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma,\mathbb{P}^3}(i-\pi)) = 0 \quad \text{for every } i \ge 3. \tag{20}$$

Let $\Sigma' \subset \mathbb{P}^2$ be a general plane section of Σ. By (9) we have $h_{\Sigma'}(i) = 5$ for every $i \ge 2$. Hence, $h^1(\mathbb{P}^2, \mathcal{I}_{\Sigma',\mathbb{P}^2}(i)) = 0$ for every $i \ge 2$, and from the exact sequence $0 \to \mathcal{I}_{\Sigma,\mathbb{P}^3}(i-1) \to \mathcal{I}_{\Sigma,\mathbb{P}^3}(i) \to \mathcal{I}_{\Sigma',\mathbb{P}^2}(i) \to 0$ it follows that $h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma,\mathbb{P}^3}(2)) \le h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma,\mathbb{P}^3}(1))$. Summing up, for every $i \ge 1$, we have:

$$h^1(\mathbb{P}^3, \mathcal{I}_\Sigma(i)) \le \max\{0,\ 3 - \pi - i\} + \mu(i),$$

with $\mu(i) = 1$ if $\pi = 0$ and $i = 2$, and $\mu(i) = 0$ otherwise. Now, divide $d - 1 = 5m + \epsilon$, $0 \le \epsilon \le 4$, and set (recall that $0 \le \pi \le 1$):

$$h_{d,\pi}(i) = 1 - \pi + 5i - \max\{0,\ 3 - \pi - i\} - \mu(i)$$

for $1 \le i \le m$, and, for $i \ge m+1$, $h_{d,\pi}(i) = d$, except for the case $\pi = 1$, $\epsilon = 4$, $i = m+1$, in which we set $h_{d,\pi}(m+1) = d - 1$. If we denote by Γ a general hyperplane section of C, the same analysis appearing in the proof of [7, Lemma 3.3], shows that $h_\Gamma(i) \ge h_{d,\pi}(i)$ for every $i \ge 1$. By (7) it follows that

$$p_a(C) \le \sum_{i=1}^{+\infty} d - h_\Gamma(i) \le \sum_{i=1}^{+\infty} d - h_{d,\pi}(i) =: G_{d,\pi}. \tag{21}$$

Now, observe that

$$G_{d,0} = 5\binom{m}{2} + m\epsilon + 4, \quad \text{and} \quad G_{d,1} = 5\binom{m}{2} + m(\epsilon+1) + 1 + \binom{\epsilon}{4}.$$

Therefore, since $G_{d,0} < G_{d,1}$, and taking into account that $G_{d,1}$ is exactly the bound appearing in (2), in order to complete the proof we only have to exhibit examples with $p_a(C) = G_{d,1}$ (in this case, by (21) and Section 2, (i), we know that C is a.C.M.).

To this purpose, fix an elliptic curve $\Sigma \subset \mathbb{P}^3$ of degree 5 (compare with [7, p. 101–103]). By (20) we know that $h^1(\mathbb{P}^3, \mathcal{I}_\Sigma(2)) = 0$. It follows that Σ is not contained in quadrics, because $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ and $H^0(\Sigma, \mathcal{O}_\Sigma(2))$ have the same dimension, 10. Let $S = C(\Sigma) \subset \mathbb{P}^4$ be the cone over Σ. Let $D \subset \Sigma$ be a subset formed by $4 - \epsilon$ points. Let $C(D) \subset C(\Sigma)$ be the cone over D. Let $\mu \ge 3$ be an integer. Let $F \subset \mathbb{P}^4$ be a hypersurface of degree $\mu + 1$ containing $C(D)$, consisting of $\mu + 1$ sufficiently general hyperplanes passing through the vertex. Let R be the residual curve to $C(D)$ in the complete intersection of F with S. Equipped with the reduced structure, R is a cone over k distinct points of Σ, with $k - 1 = 5\mu + \epsilon$. In particular, R is a (reducible) a.C.M. curve of degree k on S, and, if we denote by R' the hyperplane section of R with the hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing Σ, we have $p_a(R) = \sum_{i=1}^{+\infty} k - h_{R'}(i)$ (compare with (7) and the first line below). On the other hand, since $R' \subset \Sigma$, it is clear that $h_{R'}(i) = h_{k,1}(i)$ for every $i \ge 1$. Hence, we have:

$$p_a(R) = G_{k,1} = 5\binom{\mu}{2} + \mu(1+\epsilon) + 1 + \binom{\epsilon}{4}.$$

Now, let $m \gg 0$, and let $G \subset \mathbb{P}^4$ be a hypersurface of degree $m+1$ containing $C(D)$ such that the residual curve C in the complete intersection of G with S, equipped with the reduced structure, is an integral curve of degree $d = 5m + \epsilon + 1$, with a singular point of multiplicity k at the vertex p of S, and tangent cone at p equal to R. We are going to prove that C is the curve we are looking for, i.e.

$$p_a(C) = G_{d,1}.$$

To this aim, let $\widetilde{S}$ be the blowing-up of S at the vertex. By [11, p. 374], we know that $\widetilde{S}$ is the ruled surface $\mathbb{P}(\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-1)) \to \Sigma$. Denote by E the exceptional divisor, by f the line of the ruling, and by L the pull-back of the hyperplane section. We have $L^2 = 5$, $L \cdot f = 1$, $f^2 = 0$, $L \equiv E + 5f$ and $K_{\widetilde{S}} \equiv -2L + 5f$. Let $\widetilde{C} \subset \widetilde{S}$ be the blowing-up of C at p, which is nothing but the normalization of C. Since C has degree d, $\widetilde{C}$ belongs to the numerical class of $(m+1+a)L + (1+\epsilon-5(a+1))f$ for some integer a. Moreover $E \cdot \widetilde{C} = 1 + \epsilon - 5(a+1) = k$, so

$$a = -\frac{k+4-\epsilon}{5} = -\mu - 1.$$

By the adjunction formula we get

$$p_a(\widetilde{C}) = 5\binom{m}{2} + m(\epsilon+1) + 1 - \frac{5}{2}a^2 + a\left(\epsilon - \frac{3}{2}\right).$$

On the other hand, we have $p_a(C) = p_a(\widetilde{C}) + \delta_p$, where δ_p is the delta invariant of the singularity (C, p) [11, p. 298, Exercise 1.8]. Since the tangent cone of C at p is R, the delta invariant is equal to the difference between the arithmetic genus of R and the arithmetic genus of k disjoint lines in the projective space, i.e.

$$\delta_p = p_a(R) - (1-k) = 5\binom{\mu}{2} + \mu(1+\epsilon) + 1 + \binom{\epsilon}{4} - (1-k).$$

It follows that

$$\begin{aligned} p_a(C) = 5\binom{m}{2} + m(\epsilon+1) + 1 - \frac{5}{2}a^2 + a\left(\epsilon - \frac{3}{2}\right) \\ +5\binom{\mu}{2} + \mu(1+\epsilon) + 1 + \binom{\epsilon}{4} - (1-k). \end{aligned}$$

Taking into account that $a = -\mu - 1$, a direct computation proves that this number is exactly $G_{d,1}$. This concludes the proof of Theorem 1.1.

4 The proof of theorem 1.2

By (11), every surface of degree ≤ 5 in $\mathbb{P}^5$ is contained in a quadric. Moreover, if C is not contained in a surface of degree < 7, and $d > 179$, then (see Section 2, (*vi*), and (15)):

$$p_a(C) \leq G(5;d,7) \leq \frac{d^2}{14} - \frac{3}{14}d + 115.$$

An elementary comparison, relying on (13), shows that this number is strictly less than $G(7;d) = 6\binom{m}{2} + m\epsilon$ for $d > 179$. Therefore, in order to prove Theorem 1.2, we may assume that C is contained in a surface S of degree 6. Such a surface is unique by Bezout's theorem, and is contained in a hypersurface of degree 3 by (11). If π denotes the sectional genus of S, by Castelnuovo's bound we have $0 \leq \pi \leq 2$, and π cannot be equal to 1 or 2, otherwise, by (11), S is contained in a quadric. It follows that $\pi = 0$.

Let $S \subset \mathbb{P}^5$ be a surface of degree 6 and sectional genus $\pi = 0$. By [7, Corollary 3.6] we know that $-3 \leq p_a(S) \leq 0$. Now we are going to prove that if $-3 \leq p_a(S) \leq -1$, then S is contained in a quadric. The proof relies on certain results appearing in [7], that we have collected in Section 2, (*viii*).

- If $p_a(S) = -3$, by [7, Remark 3.7] we get $h^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0$. Therefore, the map $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2))$ is onto. This implies that S is contained in a quadric, because it is so for every curve of degree 6 in $\mathbb{P}^4$ by (5) and (10).
- Assume $p_a(S) = -2$. Since Σ has degree 6 and arithmetic genus 0, by [7, Proposition 3.1] we know that $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(1)) = 2$, $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) \leq 1$, and $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(i)) = 0$ for $i \geq 3$.

In the case $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$, by [7, Lemma 3.4] we get (compare with (19), and recall (*viii*), Section 2): $-2 = p_a(S) = -\dim_{\mathbb{C}} M(\Sigma) + \sum_{i=1}^{+\infty} \mu_i = -2 + \sum_{i=1}^{+\infty} \mu_i$. Therefore, $\sum_{i=1}^{+\infty} \mu_i = 0$, and so $h^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0$. As before, we deduce that S lies on a quadric.

In the case $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 1$, we have $h^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 3$. In view of the exact sequence $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) \to H^1(\mathbb{P}^5, \mathcal{I}_S(1))$, in order to prove that S lies on a quadric, it suffices to prove that

$$h^1(\mathbb{P}^5, \mathcal{I}_S(1)) \leq 2. \tag{22}$$

This follows from the exact sequence $0 \to \mathcal{I}_S \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_S \to 0$, taking into account that $h^0(\mathcal{O}_S(1)) \leq 8$, in view of the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(1) \to \mathcal{O}_\Sigma(1) \to 0$.

- Assume $p_a(S) = -1$. As before, we have two cases, $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$, or $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 1$. If $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0$, then (recall (*viii*), Section 2) $-1 = p_a(S) = -\dim_{\mathbb{C}} M(\Sigma) + \sum_{i=1}^{+\infty} \mu_i = -2 + \sum_{i=1}^{+\infty} \mu_i$. Hence $\sum_{i=1}^{+\infty} \mu_i = 1$. Combining with the exact sequence $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) \to H^1(\mathbb{P}^5, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^5, \mathcal{I}_S(2))$, we get $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \neq 0$, otherwise the map on the right should have a kernel of dimension $\geq 2 = h^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2))$, in contrast with the fact that $\mu_2 \leq 1$. If $h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 1$, then $h^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 3$. Hence, the map $H^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) \to H^1(\mathbb{P}^5, \mathcal{I}_S(1))$ has non trivial kernel, because $h^1(\mathbb{P}^5, \mathcal{I}_S(1)) \leq 2$ (see (22)). This implies $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \neq 0$. Summing up, in order to prove Theorem 1.2, we may assume that C is contained in a surface S of degree 6, sectional genus $\pi = 0$, and arithmetic genus $p_a(S) = 0$. On such a surface, the genus of C satisfies the bound in view of [7, Corollary 3.11] (here we need $d > 215$). Moreover, C cannot be a.C.M.. Otherwise, $M(S) = 0$ (see Section 2, (*vii*)). In particular, $H^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0$, and so the restriction map $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_{\mathbb{P}^4,\Sigma}(2))$ is onto. This is impossible, because Σ is contained in a quadric, and S is not. It remains to prove that the bound is sharp. To this purpose, let $S' \subset \mathbb{P}^7$ be a smooth rational normal scroll surface, of degree 6. Let C' be a Castelnuovo's curve on S' of degree d. Let S be a general projection of S' in $\mathbb{P}^5$ (observe that the degree of S is 6, the sectional genus is $\pi = 0$, and that $p_a(S) = 0$ because S is isomorphic to S' via projection). We claim that the image C of C' is an extremal curve. To prove this, first we notice that S is not contained in quadrics. In fact, S is of maximal rank [1, Theorem 2, Lemma 3.1]. In particular, the map $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)) \to H^0(S, \mathcal{O}_S(2))$ is injective or surjective. Since both spaces have the same dimension (see (12)), it follows that $h^0(\mathbb{P}^5, \mathcal{I}_S(2)) = 0$. Since S is not contained in quadrics, and $d > 12$, by Bezout's theorem it follows that also C is not contained in quadrics. Furthermore, since C is isomorphic to C', and C' is a Castelnuovo's curve, we get

$$p_a(C) = p_a(C') = 6\binom{m}{2} + m\epsilon.$$

This concludes the proof of Theorem 1.2.

5 The proof of proposition 1.3

By the definition of s, we have $\binom{r+2}{2} - (3 + 3(s-1)) > 0$. Hence, by (11), every surface of degree $< s$ is contained in a quadric. In particular, the extremal curve C cannot be contained in a surface of degree $< s$. Moreover, if C is not contained in a surface of degree $< s+1$, then (compare with (16), (17), and (13))

$$p_a(C) \leq G(r;d, s+1) < G(s+1;d) = \frac{d^2}{2s} - \frac{d}{2s}(s+2) + \frac{1+\epsilon}{2s}(s+1-\epsilon). \tag{23}$$

On the other hand, let $S' \subset \mathbb{P}^{s+1}$ be a smooth rational normal scroll surface, of degree s. Let D' be a Castelnuovo's curve on S' of degree d. Let $S\,(\cong S')$ be a general projection of S' in $\mathbb{P}^r$. By [1, loc. cit.], S is of maximal rank. Since

$$h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) = \binom{r+2}{2} = 3 + 3s = h^0(S, \mathcal{O}_S(2))$$

(compare with the definition of s and (12)), it follows that the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(S, \mathcal{O}_S(2))$ is injective. Hence, S, and so the image D of D', are not contained in quadrics. Moreover, since D is a Castelnuovo's curve, we have (compare with (13))

$$p_a(D) = p_a(D') = G(s+1;d) > \frac{d^2}{2s} - \frac{d}{2s}(s+2). \tag{24}$$

In view of (23) and (24), in order to prove (3), we may assume that C is contained in a surface S of degree s. By (11), the sectional genus of S should be $\pi = 0$, otherwise S is contained in a quadric (here we have to assume that r is not divisible by 3). Now, if C is contained in a surface S of degree s and sectional genus $\pi = 0$, then by [5, p. 792, Lemma] we know that

$$p_a(C) \leq \frac{d^2}{2s} - \frac{d}{2s}(s+2) + O(1), \quad \text{with} \quad O(1) \leq \frac{s^3}{r-2}.$$

Taking into account (24), we deduce (3).

Since $d > d_1(r,s)$, by Bezout's theorem the surface S containing C is unique. By (11) and the definition of s, it follows that S is contained in a cubic hypersurface. Moreover, C cannot be a.C.M.. Otherwise, $M(S) = 0$ (see Section 2, (*vii*)). In particular, $H^1(\mathbb{P}^r, \mathcal{I}_S(1)) = 0$, and so the restriction map $H^0(\mathbb{P}^r, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^{r-1}, \mathcal{I}_{\mathbb{P}^{r-1},\Sigma}(2))$ is onto (Σ = a general hyperplane section of S). This is impossible, because, in view of (5) and (10), Σ is contained in a quadric, and S is not. This concludes the proof of Proposition 1.3.

Remark 5.1 When 3 divides r, previous argument does not work. In fact, in order to preserve the inequality in the first line of this section, one is lead to define s as the minimal integer σ such that $\binom{r+2}{2} - 3(\sigma+1)$ is ≤ 0. It turns out, when 3 divides r, that $3s+1 = \binom{r+2}{2}$. But, unlike what happens when r is not divisible by 3, in this case (at least when $r=6$) there are surfaces of degree s with $\pi > 0$ and not contained in quadrics (this can happen only if $\pi = 1$, by (11)). For instance, consider a general projection S in $\mathbb{P}^6$ of a 3-ple Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$ [1, Theorem 3]. In this case, repeating the same argument as before (compare with [1, loc. cit.] and [5, loc. cit.]), one may prove that the sharp bound (for curves in $\mathbb{P}^6$ not contained in quadrics, and of degree $d = 0$ mod. 3, $d > d_1(r,s)$, $r = 6$) is

$$p_a(C) = \frac{d^2}{2s} - \frac{d}{2} + O(1), \tag{25}$$

with $s = 9$, and $0 < O(1) \leq 182$. The formula (25) remains true (with $0 < O(1) \leq \frac{s^3}{r-2}$) if there exists a surface of degree s in $\mathbb{P}^r$, with sectional genus $\pi = 1$, and not contained in quadrics (at least when $\epsilon = s-1$, i.e. when d is a multiple of s).

6 The proof of proposition 1.4.

The proof is quite similar to the proof of previous Proposition 1.3. Hence, we omit some details.

By the definition of s and by (11), every surface of degree $< s$ is contained in a cubic (here we need that $6s+4 \leq \binom{r+3}{3}$). In particular, the extremal curve C cannot be contained in a surface of degree $< s$. Moreover, if C is not contained in a surface of degree $< s+1$, then (compare with (16), (17), and (13))

$$p_a(C) \leq G(r;d,s+1) < G(s+1;d) = \frac{d^2}{2s} - \frac{d}{2s}(s+2) + \frac{1+\epsilon}{2s}(s+1-\epsilon). \tag{26}$$

On the other hand, let $S' \subset \mathbb{P}^{s+1}$ be a smooth rational normal scroll surface, of degree s. Let D' be a Castelnuovo's curve on S' of degree d. Let $S\,(\cong S')$ be a general projection of S' in $\mathbb{P}^r$. By [1, loc. cit.], S is of maximal rank. Since

$$h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(3)) = \binom{r+3}{3} = 6s+4 = h^0(S, \mathcal{O}_S(3))$$

(compare with the definition of s and (12)), it follows that the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(3)) \to H^0(S, \mathcal{O}_S(3))$ is injective (actually, here we only need that

$6s+4 \geq \binom{r+3}{3}$). Hence, S, and so the image D of D', are not contained in quadrics. Moreover, since D is a Castelnuovo's curve, we have (compare with (13))

$$p_a(D) = p_a(D') = G(s+1;d) > \frac{d^2}{2s} - \frac{d}{2s}(s+2). \tag{27}$$

In view of (26) and (27), in order to prove (4), we may assume that C is contained in a surface S of degree s. By (11), the sectional genus of S should be $\pi = 0$, otherwise S is contained in a cubic (again, here we need that $6s+4 \leq \binom{r+3}{3}$). Now, if C is contained in a surface S of degree s and sectional genus $\pi = 0$, then by [5, p. 792, Lemma] we know that

$$p_a(C) \leq \frac{d^2}{2s} - \frac{d}{2s}(s+2) + O(1), \quad \text{with} \quad O(1) \leq \frac{s^3}{r-2}.$$

Taking into account (27), we deduce (4). One may prove the remaining properties in a similar way as in Proposition 1.3. This concludes the proof of Proposition 1.4.

Appendix.

In order to conclude the proof of Theorem 1.1, first part, it remains to prove that *if C is not contained in a surface of degree <5, then $p_a(C) < \frac{1}{8}d(d-6)+1$*, when $16 < d \leq 143$. We are going to do an analysis similar to the one that appears in [8, pp. 74–75]. The same calculation in [8, loc. cit.] proves that $p_a(C) < \frac{1}{8}d(d-6)+1$ for $d > 30$. We need to refine the argument to deal with the case $16 < d \leq 30$. Let $\Gamma \subset \mathbb{P}^3$ be a general hyperplane section of C. In the sequel, we will apply (6), (7), (8), and (9) (see Section 2, (*i*)).

- Case I: $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) \geq 2$. This case can't happen. Otherwise, since $d > 4$, by monodromy [3, Proposition 2.1], Γ would be contained in an integral curve of $\mathbb{P}^3$ of degree ≤ 4. Since $d > 16$, from [2, Theorem (0.2)] we would deduce that C is contained in a surface of degree ≤ 4, against our hypothesis.
- Case II: $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ and $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) > 4$. This is the most complicated case. Since $d > 6$, by monodromy [3, Proposition 2.1], in this case Γ is contained in an integral curve X of $\mathbb{P}^3$ of degree ≤ 6. Based on what was said in the previous case, we may suppose the degree of X is either 5 or 6.

First assume $\deg X = 5$.

If $d \geq 21$, by Bezout's theorem we have $h_\Gamma(i) = h_X(i)$ for all $i \leq 4$. Let X' be a general hyperplane section of X. By (6) we get $h_X(i) \geq \sum_{j=0}^{i} h_{X'}(j)$. By (9) it follows that $h_X(3) \geq 14$ and $h_X(4) \geq 19$. Therefore, if $d \geq 21$, taking into account (8), we get:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) \geq 14,\ h_\Gamma(4) \geq 19,$$
$$h_\Gamma(5) \geq \min\{d,\ 22\},\ h_\Gamma(6) \geq \min\{d,\ 27\},\ h_\Gamma(7) = d.$$

If $28 \leq d \leq 30$, using (7) we deduce:

$$p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d-4) + (d-9) + (d-14) + (d-19) + 8 + 3 = 4d - 35,$$

which is $< \frac{1}{8}d(d-6)+1$.

If $24 \le d \le 27$, we have:

$$p_a(C) \le \sum_{i=1}^{+\infty} d - h_\Gamma(i) \le (d-4) + (d-9) + (d-14) + (d-19) + 5 = 4d - 41,$$

which is $< \frac{1}{8}d(d-6)+1$, except for $d = 24$ for which we have $4d - 41 = \frac{1}{8}d(d-6)+1$. However, $p_a(C)$ should be strictly less than such number, otherwise $p_a(C) = \sum_{i=1}^{+\infty} d - h_\Gamma(i)$ and C would be a.C.M. (Section 2, (i)). This is impossible, because C is not contained in quadrics, while $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ (we will use this argument later as well).

If $21 \le d \le 23$, we have:

$$p_a(C) \le \sum_{i=1}^{+\infty} d - h_\Gamma(i) \le (d-4) + (d-9) + (d-14) + (d-19) + 1 = 4d - 45,$$

which is $< \frac{1}{8}d(d-6)+1$.

Now assume $17 \le d \le 20$. By Bezout's theorem we have $h_\Gamma(3) = h_X(3)$. Hence we get:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) \ge 14,\ h_\Gamma(4) \ge 17,\ h_\Gamma(7) = d.$$

And we may conclude with a calculation similar to the previous one (when $18 \le d \le 20$, we use again the fact that C cannot be a.C.M.).

Assume $\deg X = 6$.

By [3, Proposition 2.1], X is a single component of the intersection of the quadric containing Γ, with a general cubic containing Γ. Since $\deg X = 6$, it follows that X is a complete intersection of bi-degree (2, 3).

If $d \ge 25$, by Bezout's theorem we have $h_\Gamma(i) = h_X(i)$ for every $i \le 4$. Let X' be a general plane section of X. Similarly as before, since $\deg X = 6$, we have $h_X(3) \ge 15$ e $h_X(4) \ge 21$. Therefore, if $d \ge 25$, we have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) \ge 15,\ h_\Gamma(4) \ge 21,$$
$$h_\Gamma(5) \ge \min\{d, 23\},\ h_\Gamma(6) \ge \min\{d, 29\},\ h_\Gamma(7) = d.$$

It follows that:

$$p_a(C) \le \sum_{i=1}^{+\infty} d - h_\Gamma(i) \le (d-4) + (d-9) + (d-15) + (d-21) + 7 = 4d - 42,$$

which is $< \frac{1}{8}d(d-6)+1$.

If $19 \le d \le 24$ we have $h_\Gamma(3) = h_X(3)$, and so:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) \ge 15,\ h_\Gamma(4) \ge 18, h_\Gamma(5) \ge \min\{d, 23\},\ h_\Gamma(6) = d.$$

Hence:

$$p_a(C) \le \sum_{i=1}^{+\infty} d - h_\Gamma(i) \le (d-4) + (d-9) + (d-15) + (d-18) + 1 = 4d - 45,$$

which is $< \frac{1}{8}d(d-6)+1$. It remains to examine the cases $d = 18$ e $d = 19$.

If $d = 18$ and $h_\Gamma(3) = h_X(3)$, then $h_\Gamma(3) \geq 15$. Otherwise, Γ is a complete intersection of type (2, 3, 3), and from Koszul's complex we get $h_\Gamma(3) \geq 14$. Therefore, in every case, we have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) \geq 14,\ h_\Gamma(4) \geq 17,\ h_\Gamma(5) = 18.$$

Hence $p_a(C) < \sum_{i=1}^{+\infty} 18 - h_\Gamma(i) \leq 28 = \frac{1}{8}18(18-6)+1$.

When $d = 17$, previous argument does not work. We may argue as follows. If $h_\Gamma(3) = h_X(3)$ we may repeat previous computation (in this case $h_X(3) \geq 15$). Otherwise, $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) > h^0(\mathbb{P}^3, \mathcal{I}_X(3))$. On the other hand, since X is a complete intersection of bidegree (2, 3), we have $h^1(\mathbb{P}^3, \mathcal{I}_X(2+i)) = h^2(\mathbb{P}^3, \mathcal{I}_X(2+i)) = 0$ for all $i \geq 0$. It follows from [3, Corollary 1.2] that

$$\Delta h_\Gamma(4) \leq \max\left\{0,\ \Delta h_\Gamma(3) - 2\right\}.$$

If $\Delta h_\Gamma(4) \leq 0$ then $h_\Gamma(3) = 17$. Otherwise, $\Delta h_\Gamma(4) \leq \Delta h_\Gamma(3) - 2$, hence $h_\Gamma(3) \geq 14$, and we may conclude as in previous computation.

- Case III: $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ e $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) = 4$. We have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 9,\ h_\Gamma(3) = 16,\ h_\Gamma(4) \geq \min\{d,\ 19\},$$
$$h_\Gamma(5) \geq \min\{d,\ 24\},\ h_\Gamma(6) = d.$$

  If $26 \leq d \leq 30$, we have:

$$p_a(C) < \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d-4) + (d-9) + (d-16) + 11 + 6 = 3d - 12,$$

  which is $\leq \frac{1}{8}d(d-6)+1$. If $21 \leq d \leq 25$, we have:

$$p_a(C) < \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d-4) + (d-9) + (d-16) + 6 + 1 = 3d - 22,$$

  which is $\leq \frac{1}{8}d(d-6)+1$, except the case $d = 21$. In this case $3d - 22 = 41$, hence $p_a(C) \leq 40 < \frac{1}{8}21(21--6)+1 = 40 + \frac{3}{8}$. If $17 \leq d \leq 20$, we have:

$$p_a(C) < \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d-4) + (d-9) + (d-16) + 1 = 3d - 28,$$

  which is $\leq \frac{1}{8}d(d-6)+1$.
- Case IV: $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0$. We have:

$$h_\Gamma(1) = 4,\ h_\Gamma(2) = 10,\ h_\Gamma(3) \geq 13,\ h_\Gamma(4) \geq \min\{d,\ 19\},$$
$$h_\Gamma(5) \geq \min\{d,\ 22\},\ h_\Gamma(6) \geq \min\{d,\ 28\},\ h_\Gamma(6) = d.$$

  So, we may conclude with a similar computation as in the previous case.

Remark 7.1 From previous analysis it follows that *if $C \subset \mathbb{P}^4$ is not contained in quadrics, and $d > 16$, then $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) \leq 1$*. In fact, otherwise C should be contained in an isomorphic projection of the Veronese surface. Therefore, Γ should be contained in an irreducible rational quartic X in $\mathbb{P}^3$. On the other hand, if Γ lies in two independent quadrics, by monodromy it lies in an irreducible elliptic quartic Y. Since $\Gamma \subseteq X \cap Y$, by Bezout's theorem we get $d \leq 16$, in contrast with the assumption $d > 16$.

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