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# On the existence of global-in-time weak solutions and scaling laws for Kolmogorov's two-equation model for turbulence

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This paper is concerned with Kolmogorov's two-equation model for turbulence in $\mathbb{R}^3$ involving the mean velocity $\boldsymbol{u}$, the pressure $p$, an average frequency $\omega > 0$, and a mean turbulent kinetic energy $k$. We consider the system with space-periodic boundary conditions in a cube $\Omega = (]0, a[)^3$, which is a good choice for studying the decay of free turbulent motion sufficiently far away from boundaries. In particular, this choice is compatible with the rich set of similarity transformations for turbulence. The main part of this work consists in proving existence of global weak solutions of this model. For this we approximate the system by adding a suitable regularizing $r$-Laplacian and invoke existence result for evolutionary equations with pseudo-monotone operators. An important point constitutes the derivation of pointwise a priori estimates for $\omega$ (upper and lower) and $k$ (only lower) that are independent of the box size $a$, thus allow us to control the parabolicity of the diffusion operators.

## 1 | INTRODUCTION

In 1942, A. N. Kolmogorov (see Kolmogorov [1] and pp. 214–216 in Spalding [2] for an English translation) postulated the following system of PDEs as a model for the isotropic homogeneous turbulent motion of an incompressible fluid $(x, t) \in \mathbb{R}^3 \times ]0, \infty[$:

$$\operatorname{div} \boldsymbol{u} = 0\,, \tag{1.1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu_0 \operatorname{div}\left(\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u})\right) - \nabla p + \boldsymbol{f}, \tag{1.1b}$$

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla\omega = \nu_1 \operatorname{div}\left(\frac{k}{\omega}\nabla\omega\right) - \alpha_1\omega^2, \tag{1.1c}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u} \cdot \nabla k = \nu_2 \operatorname{div}\left(\frac{k}{\omega}\nabla k\right) + \nu_0\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - \alpha_2 k\omega. \tag{1.1d}$$

Throughout the paper, bold letters denote functions with values in $\mathbb{R}^3$ or $\mathbb{R}^9$ as well as normed spaces of such functions. Here, the unknowns have the following physical meaning:

$\boldsymbol{u}$ is the velocity of the mean flow, $p$ is the average of the pressure, $k$ is the mean turbulent kinetic energy,

$\omega$ is the average of the frequency associated with the turbulent kinetic energy. (1.2)

The velocity field $\boldsymbol{v}$ of the fluid motion is given by $\boldsymbol{v} = \boldsymbol{u} + \tilde{\boldsymbol{u}}$, where $\tilde{\boldsymbol{u}}$ denotes the turbulent fluctuation velocity, such that the scalar $k$ is the time average $\overline{\frac{1}{2}|\tilde{\boldsymbol{u}}|^2}$. Further,

$$\begin{aligned}
&\nu_0,\ \nu_1,\ \nu_2 > 0 \text{ and } \alpha_2,\ \alpha_1 > 0 \text{ are dimensionless constant;}\\
&\boldsymbol{f} \text{ is a given averaged external force,}\\
&\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\big) \text{ is the mean strain-rate tensor.}
\end{aligned} \tag{1.3}$$

The function $\nu_0\frac{k}{\omega}$ denotes the kinematic eddy viscosity, while $\nu_1\frac{k}{\omega}$ and $\nu_2\frac{k}{\omega}$ denote the corresponding diffusion constants for the scalars $\omega$ and $k$. The constants $\nu_0,\ \nu_1,\ \nu_2 > 0$ and $\alpha_2,\ \alpha_1 > 0$ in Equation (1.1) are related to the constants $A,\ A',\ A''$ [1] (cf. also p. 213 in Spalding [2] where $b = \frac{2}{3}k$) as follows:

$$\nu_0 = \frac{4}{3}A, \quad \nu_1 = \frac{2}{3}A', \quad \nu_2 = \frac{2}{3}A'', \quad \alpha_1 = \frac{7}{11}, \quad \alpha_2 = 1. \tag{1.4}$$

In Section 2, we discuss the scaling properties of the two-equation model (1.1) with the special viscosities "$\nu_j\, k/\omega$" and loss terms "$\alpha_1\omega^2$" and "$\alpha_2\, k\omega$." These specific choices of power-law nonlinearities relate to specific scaling laws in free turbulence. In Kolmogorov [1], there is no indication why the particular values of $\alpha_1$ and $\alpha_2$ were chosen.

Since the numerical values of $\nu_1$ and $\nu_2$ are not relevant for the existence theory of weak solutions for Equation (1.1) we are going to develop below, we assume them to be equal to 1. A detailed discussion of the numerical values of closure coefficients and their role in turbulence modeling can be found, for example, in Baumert [3] and Chap. 4.3.1 in Wilcox [4]. However, we keep the coefficient $\nu_0$ to emphasize that the viscous dissipation generated by the viscous term in Equation (1.1a) is feeding into the mean turbulent kinetic energy, see the second last term in Equation (1.1d). Hence, for sufficiently smooth solutions, we have the formal energy relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(\frac{1}{2}|\boldsymbol{u}|^2 + k\right)\mathrm{d}x = \int_{\mathbb{R}^3}(\boldsymbol{f}\cdot\boldsymbol{u} - \alpha_2\omega k)\,\mathrm{d}x, \tag{1.5}$$

where the first term on the right-hand side gives the power of the external forces, while the second term is Kolmogorov's way of modeling dissipative losses, for example, through thermal radiation. We refer to Refs. [5, 6] for general issues in turbulent modeling, in particular to Chap. 7+8 in Chacón Rebello and Lewandowski [6] for the mathematical analysis of the NS-TKE model (Navier–Stokes equation with Turbulent Kinetic Energy), where Equation (1.1c) for $\omega$ is absent and the energetic losses in Equation (1.1d) are modeled via $k^{3/2}/\ell$ with a suitable mixing length $\ell$ instead of $\alpha_2 k\omega$ (see e.g., Equation (4.137) in Chacón Rebello and Lewandowski [6].

System (1.1) is an outgrowth of A. N. Kolmogorov's theory of turbulence published in a series of papers in 1941. Comprehensive presentations of this theory can be found, for example, in Frisch [7] and Vol. I, Chap. 6.1, 6.2; Vol. II, Chap. 8 in Monin [8] (see also the article pp. 488–503 in Tikhomirov [9]). The function $L = \frac{k^{1/2}}{\omega}$ ("external length scale" or "size of largest eddies") plays an important role for the study of the energy spectrum of the turbulence (see Chap. 33 in Landau and Lifschitz [10], Chap. 8.1 in Wilcox [4]). A review of the work of A. N. Kolmogorov and the Russian school of turbulence can be found in Yaglom [11]. This paper contains also some remarks about a possibly "missing source term" in Equation (1.1c) (cf. p. 212 in Spalding [2]).

A profound discussion of the mathematical background of Obukhoff–Kolmogorov's spectral theory of turbulence (K41-functions, bounds for the energy spectrum for low and high frequencies) is given in Vigneron [12].

In Bulíček and Málek [13], the authors study system (1.1) in $\Omega\times]0,T[$, where $\Omega\subset\mathbb{R}^3$ is a bounded $C^{1,1}$ domain, with mixed boundary conditions for $\omega$ and $k$, the condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$ and a condition for the normal traction of the tensor $-p\boldsymbol{I}+\nu_0\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u})$ on $\partial\Omega\times]0,T[$. Under these boundary conditions, system (1.1) characterizes a wall-bounded turbulent motion, that is, turbulence is generated at the Dirichlet part of the boundary. The authors complete this boundary value problem by the initial conditions (1.10b) and prove the existence of a weak solution by combining a truncation method and the Galerkin approximation. Wall-generated turbulence is an important topic in engineering applications where two-equation models, including the $k$,- $\varepsilon$ model, are heavily used, see Chacón Rebello and Lewandowski [6] and the references there.

The emphasis of this paper is quite different as we are interested in free turbulence (also called isotropic or homogeneous turbulence) that develops far away of the boundary and is rather governed by suitable scaling symmetries in the sense of Oberlack [14] and Klingenberg et al. [15]. In Ref. [1] Kolmogorov writes about the derivation of his model: "*We may submit to a rather less complete mathematical investigation the turbulent motion which is homogeneous and isotropic (in all scales), and from which mean flow is absent; such a flow decays continuously with time. … Starting from the above local properties of turbulence (and with the help of some more coarsely approximate assumptions), we may construct the following complete system of equations to describe turbulent motion*:" and then he states his two-equation model (cited from English translation in Spalding [2]).

To preserve these similarity transforms, we avoid boundaries and use periodic boundary conditions and on a cube size with side length $a$, that can be chosen much larger than the structures under consideration. A bonus of the scaling invariance of Equation (1.1) for $\boldsymbol{f}\equiv 0$ is the existence of a rich class of similarity solutions. Compatible with the periodic boundary conditions, we have the following explicit spatially constant solutions:

$$\boldsymbol{u}\equiv\boldsymbol{u}_\circ,\qquad p\equiv 0,\qquad \omega(t)=\frac{\omega_\circ}{1{+}\alpha_1\omega_\circ t},\qquad k(t)=\frac{k_\circ}{(1{+}\alpha_1\omega_\circ t)^{\alpha_2/\alpha_1}},\tag{1.6}$$

that is, the mean turbulent kinetic energy decays like $t^{-\alpha_2/\alpha_1}$, if there is no feeding through macroscopic viscous dissipation. Indeed, independent of $\boldsymbol{u}$ and $k$, Equation (1.1c) for $\omega$ can always be solved by the spatially constant solution $\omega(x,t)=\omega_\circ/(1{+}\alpha_1\omega_\circ t)$. The occurrence of asymptotically self-similar behavior for $\Omega=\mathbb{R}^d$ for a closely related, but much simpler coupled system (obtained by replacing the Navier–Stokes equation by a scalar equation for shear flows and neglecting lower order terms) is discussed in Mielke [16].

To show the effect of energy feeding from viscous dissipation into the turbulent kinetic energy $k$ via the source term $\nu_0\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2$, we can look at the following family of exact shear flow solutions:

$$\boldsymbol{u}(x,t)=\frac{U}{1{+}\alpha_1\omega_\circ t}\begin{pmatrix}\sin(\lambda x_3)\\ \cos(\lambda x_3)\\ 0\end{pmatrix},\qquad \omega(x,t)=\frac{\omega_\circ}{1{+}\alpha_1\omega_\circ t},\qquad k(x,t)=\frac{k_\circ}{(1{+}\alpha_1\omega_\circ t)^2}.\tag{1.7}$$

with $p\equiv 0$, where the positive constant parameters $\omega_\circ$, $k_\circ$, $\lambda$, and $U$ are related by

$$U^2=\frac{\alpha_2-2\alpha_1}{\alpha_1}k_\circ\quad\text{ and }\quad\lambda^2=\frac{2\alpha_1}{\nu_0}\frac{\omega_\circ^2}{k_\circ}.\tag{1.8}$$

These solutions only exist for the case $\alpha_2/\alpha_1>2$, and thus the decay of $k$ like $1/t^2$ is slower than $1/t^{\alpha_2/\alpha_1}$ in Equation (1.6), because of the spatially constant source term $\nu_0\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2=\alpha_1\omega_\circ U^2(1{+}\alpha_1\omega_\circ t)^{-3}$. As in Oberlack [14], these invariant solutions exist because of the scaling symmetries, and moreover they are indeed compatible with period boundary conditions if $\lambda a\in 2\pi\mathbb{N}$. For a given $a$, we find infinitely many solutions by choosing $\lambda_n=2\pi n/a$ and suitable $k_\circ$ and $\omega_\circ$. This also highlights the fact that there are no uniform compactness properties unless we prescribe a lower bound for $k$.

In place of $\mathbb{R}^3\times]0,\infty[$, in the present paper, we study system (1.1) in the space–time cylinder $Q=\Omega\times]0,T[$, where $\Omega=(]0,a[)^3$ with $T,a>0$ arbitrary but fixed. To implement periodic boundary conditions, we interpret $\Omega$ as a torus by identifying the opposite sides. If $\partial\Omega$ denotes the boundary of the cube $\Omega\subset\mathbb{R}^3$ we set

$$\Gamma_i=\partial\Omega\cap\{x_i=0\},\quad \Gamma_{i+3}=\partial\Omega\cap\{x_i=a\}\quad\text{ for } i=1,2,3,\tag{1.9}$$

and complement (1.1) with periodic boundary conditions and initial conditions as follows:

$$\left.\begin{aligned} \boldsymbol{u}|_{\Gamma_i\times\,]0,T[} = \boldsymbol{u}|_{\Gamma_{i+3}\times\,]0,T[} \quad & \text{and analogously for } p,\omega,k, \\ \boldsymbol{D}(\boldsymbol{u})|_{\Gamma_i\times\,]0,T[} = \boldsymbol{D}(\boldsymbol{u})|_{\Gamma_{i+3}\times\,]0,T[} \quad & \text{and analogously for } \nabla\omega,\nabla k \end{aligned}\right\} \quad \text{for } i=1,2,3; \tag{1.10a}$$

$$\boldsymbol{u}=\boldsymbol{u}_0,\ \ \omega=\omega_0,\ \ k=k_0 \ \text{ in } \ \Omega\times\{0\}. \tag{1.10b}$$

Initial/boundary-value problem (1.1) and (1.10) characterizes a turbulent motion of an incompressible fluid in $Q$ that evolves from $\{\boldsymbol{u}_0,\omega_0,k_0\}$ at time $t=0$. We assume the pressure to be periodic thus avoiding additional pressure gradients that might occur when assuming that $\nabla p$ is periodic only. As a consequence the mean flow $a^{-3}\int_\Omega \boldsymbol{u}(x,t)\,\mathrm{d}x$ is constant, when assuming $\boldsymbol{f}\equiv 0$, cf. [17, 18]. The usage of periodic boundary conditions is common in theoretical investigations of the Navier–Stokes equations and modeling of free turbulence, see, for example, Refs. [7, 12, 19–22].

On physical grounds, the size $a$ of the underlying cube $\Omega$ should be greater than certain quantities of the turbulent motion. A detailed discussion of this aspect is given on pp. 25–26 and 424–435 in Davidson [23] (cf. also item 2° below). This is one of the main reasons why we consider a cube $\Omega$ of side length $a$ and periodic boundary conditions, which provides an analysis that is completely independent of $a$. In particular, we can choose $a$ much bigger than the "external length scale" $L(x,t) := k(x,t)^{1/2}/\omega(x,t)$.

Our proof of the existence of weak solutions of Equations (1.1) and (1.10), which has been already sketched in Mielke and Naumann [24], is entirely independent of the discussion in Bulíček and Málek [13]. More specifically, the basic aspects of our paper are:

1° In Section 3, we introduce the notion of weak solution $\{\boldsymbol{u},\omega,k\}$ with defect measure $\mu$ for Equations (1.1) and (1.10). This notion leads to a balance law for $\int_\Omega k(x,\cdot)\,\mathrm{d}x$ and gives a connection between the energy equality for $\frac{1}{2}\int_\Omega |\boldsymbol{u}(x,\cdot)|^2\,\mathrm{d}x$ and the vanishing of $\mu$, cf. Proposition 3.7, which states that Equation (1.5) holds if $\mu=0$.

2° In Section 4, we present our existence theorem for weak solutions $\{\boldsymbol{u},\omega,k\}$ with defect measure $\mu$. Based on comparison arguments with the explicit solution in Equation (1.6), our solutions $\{\boldsymbol{u},\omega,k\}$ satisfy, for a.a. $(x,t)\in\Omega\times\,]0,T[$,

$$\frac{\omega^*}{1{+}\alpha_1\omega^* t} \geq \omega(x,t) \geq \frac{\omega_*}{1{+}\alpha_1\omega_* t} \quad \text{and} \quad k(x,t) \geq \frac{k_*}{(1{+}\alpha_1\omega_* t)^{\alpha_2/\alpha_1}}\,, \tag{1.11}$$

if the initial conditions in Equation (1.10b) satisfy the corresponding estimates at $t=0$. It is important to preserve these estimates even through the necessary approximations, since that provide a lower bound for the diffusion coefficients $k/w$ in the three evolution equations.

3° Moreover, the bounds in Equation (1.11) provide a physically relevant lower bound for Kolmogorov's external length scale $L=k^{1/2}/\omega$, namely

$$L(x,t) = \frac{k(x,t)^{1/2}}{\omega(x,t)} \geq c\,(1{+}t)^{1-\alpha_2/(2\alpha_1)} \quad \text{for all } t\in[0,T\,], \tag{1.12}$$

where $\alpha_2$ and $\alpha_1$ are from Equations (1.1c) and (1.1d), and where $c=\text{const}>0$ neither depends on $a$ nor on $T$ (cf. Corollary 4.3 in Section 4). Using A. N. Kolmogorov's values from Equation (1.4), we have $\alpha_2/\alpha_1 = 11/7$ and $L$ *grows at least as* $t^{3/14}$, which compares well to $t^{2/7}$ mentioned in Kolmogorov [1].

4° The proof of our existence theorem is given in Section 5. It is based on the existence of an approximate solution $\{\boldsymbol{u}_\varepsilon,\omega_\varepsilon,k_\varepsilon\}$ (without defect measure) of Equations (1.1) and (1.10), establishing a priori estimates independently of $\varepsilon$ and then carrying out the limit passage $\varepsilon\to 0$. The existence of the approximate solutions is obtained by applying an abstract existence results for evolutionary equations with pseudo-monotone operators from Thm. 8.9 in Roubíček [25], see Appendix A for the details.

5° Our approach is easily adaptable to more general domains with suitable boundary conditions, and to the full-space $\mathbb{R}^d$ with general $d\in\mathbb{N}$. However, for notational convenience and physical relevance, we restrict ourselves to $d=3$ and the spatially periodic case.

6° In Lewandowski [26], a simplified one-equation model of turbulence is studied, where a defect measure appears as well (see the pages 397 and 416 there). Weak solutions for the full one-equation model were obtained in Bulíček et al. [27].

The parallel work in Bulíček and Málek [13] developed completely independently to the present work, which had its origin in Mielke and Naumann [24]. The former work is based on an intricate Galerkin approximation with several regularization parameters and is devoted to the case of bounded domains with nontrivial (even non-smooth) boundary conditions that can trigger the generation of turbulence. For the initial condition $k_0 := k(\cdot, 0)$, we rely on the stronger assumption $k_0(x) \geq k_* > 0$ to obtain the very explicit lower bound for $k(x,t)$ in Equation (1.11) that is independent of the domain size $a$. In Bulíček and Málek [13], it is sufficient to assume the much weaker condition $\min\{0, \log k_0\} \in L^1(\Omega)$, but estimates are given in terms of domain-dependent constants. Moreover, Bulíček and Málek [13] has a *stronger notion* of solution that additionally guarantees the validity of a local balance equation for the total energy density $E(x,t) = k(x,t) + \frac{1}{2}|\boldsymbol{u}(x,t)|^2$, see Remark 3.6 and relation (3.19) there.

In subsequent work, we will investigate similarity solutions that are induced by the scaling laws discussed in Section 2. The most challenging question will be the derivation of suitable solution concepts that allow the turbulent kinetic energy $k$ to vanish on parts of the domain. This would allow us to study the predictions of the Kolmogorov model (1.1) in which way turbulent regions invade nonturbulent regions.

## 2 | SCALING LAWS AND SIMILARITY

We consider the free turbulent motion of an incompressible fluid in $\mathbb{R}^3 \times ]0, \infty[$, which is governed by the following system of PDEs (note that $\boldsymbol{f} \equiv 0$):

$$\operatorname{div} \boldsymbol{u} = 0, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \operatorname{div}\left(d_1(\omega, k)\boldsymbol{D}(\boldsymbol{u})\right) - \nabla p, \tag{2.1b}$$

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla\omega = \operatorname{div}\left(d_2(\omega, k)\nabla\omega\right) - g_2(\omega, k)\omega, \tag{2.1c}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u} \cdot \nabla k = \operatorname{div}\left(d_3(\omega, k)\nabla k\right) + d_1(\omega, k)\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - g_3(\omega, k)k, \tag{2.1d}$$

where $\boldsymbol{u}$, $p$, $\omega$ and $k$ are the unknowns, and

$$d_i : (]0, \infty[)^2 \longrightarrow \ ]0, \infty[ \quad (i = 1, 2, 3) \qquad \text{and} \qquad g_m : (]0, \infty[)^2 \longrightarrow \ ]0, \infty[ \quad (m = 2, 3) \tag{2.2}$$

are given coefficients. The coefficient $d_1(\omega, k)$ represents a "generalized" viscosity of the fluid. System (2.1) obviously includes Kolmogorov's two-equation model (1.1) with

$$d_1(\omega, k) = \nu_0 \frac{k}{\omega}, \quad d_2(\omega, k) = \nu_1 \frac{k}{\omega}, \quad d_3(\omega, k) = \nu_2 \frac{k}{\omega}, \quad g_2(\omega, k) = \alpha_1 \omega, \quad g_3(\omega, k) = \alpha_2 \omega. \tag{2.3}$$

We want to show that these choices are special, because they give a richer structure of scaling invariances than arbitrary nonlinear functions. In particular, they respect the classical Reynolds symmetry (see Sec. 3.3 in Chacón Rebello and Lewandowski [6]), but go one step beyond because the viscosities $d_j(\omega, k)$ also have scaling properties. We refer to Refs. [14, 28, 29] where the importance of scaling symmetries for the modeling of free turbulence is discussed.

Let $\{\boldsymbol{u}, \omega, k\}$ be a classical solution of Equation (2.1) that has a suitable decay for $|x| \to \infty$ such that the following integrals over $\mathbb{R}^3$ exist. We multiply Equation (2.1b) by $\boldsymbol{u}$, integrate by parts over $\mathbb{R}^3$, integrate Equation (2.1d) over $\mathbb{R}^3$, and add the equations obtained. This gives the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\left(\frac{1}{2}|\boldsymbol{u}|^2+k\right)\mathrm{d}x=-\int_{\mathbb{R}^3}g_3(\omega,k)k\,\mathrm{d}x,\quad t\in\,]0,\infty[,\tag{2.4}$$

cf. Proposition 3.7 in Section 4.

We are now studying the invariance of $\{\boldsymbol{u},\omega,k\}$ under the scaling

$$\partial_t\mapsto\alpha\partial_t,\quad \partial_{x_j}\mapsto\beta\partial_{x_j},\quad \boldsymbol{u}\mapsto\gamma\boldsymbol{u},\quad \omega\mapsto\rho\omega,\quad k\mapsto\sigma k,\tag{2.5}$$

where $(\alpha,\beta,\gamma,\rho,\sigma)\in(\,]0,+\infty\,[\,)^5$. Here, the pressure $p$ is omitted, for it can be always suitably scaled. In addition to the well-known scaling laws for the Navier–Stokes equations, the scaling (2.5) have to leave invariant the coefficients $d_i(\omega,k)$ and $g_m(\omega,t)$ for $i=1,2,3$ and $m=2,3$, too.

To this end, we consider the following conditions for the family of parameters $(\alpha,\beta,\gamma,\rho,\sigma)$ and the coefficients $d_i$ and $g_m$:

$$\alpha=\beta\gamma,\qquad \sigma=\gamma^2,\tag{2.6}$$

$$\forall\,\omega,k>0:\ \begin{cases}\beta^2 d_i(\rho\omega,\sigma k)=\alpha d_i(\omega,k), & i=1,2,3,\\ g_m(\rho\omega,\sigma k)=\alpha g_m(\omega,k), & m=2,3.\end{cases}\tag{2.7}$$

The first condition in Equation (2.6) implies the invariance of the convective derivative $\partial_t+\boldsymbol{u}\cdot\nabla$ under Equation (2.5), while the second condition implies that $|\boldsymbol{u}|^2$ and $k$ have the same scaling property which is necessary for the conservation law (2.4) to hold. It is now easy to see that system (2.1) is invariant under the scaling laws (2.5) if the conditions (2.6) and (2.7) hold.

In order to relate the present discussion to Kolmogorov's two-equation model (1.1) we make an "ansatz" for the parameter $\beta$ as well as for the coefficients $d_i$ and $g_m$. For $(\gamma,\rho),(\omega,k)\in(]0,\infty[)^2$ define

$$\beta=\rho^A\gamma^{1-2B}\tag{2.8}$$

$$d_i(\omega,k)=D_i\omega^{-A}k^B,\quad g_m(\omega,k)=G_m\omega^A k^{1-B},\tag{2.9}$$

where $D_i,G_m$ $(i=1,2,3;\,m=2,3)$ and $A,B$ are arbitrary positive constants. Condition (2.8) is equivalent to

$$\frac{\beta}{\gamma}\rho^{-A}\gamma^{2B}=1\quad\text{resp.}\quad\frac{1}{\beta\gamma}\rho^A\gamma^{2(1-B)}=1.\tag{2.10}$$

Observing Equation (2.6), it is readily seen that $d_i$ and $g_m$ as in Equation(2.9) obey the scaling conditions (2.7) for all choices of $D_i,G_m,A$, and $B$.

Finally, let $A=B=1$ in Equations (2.8) and (2.9), that is, $g_m$ does not depend on $k$. Then we obtain

$$d_i(\omega,k)=D_i\frac{k}{\omega},\quad g_m(\omega,k)=G_m\omega\quad(i=1,2,3;\ m=2,3).\tag{2.11}$$

Hence, Kolmogorov's two-equation model of turbulence, which is obtained for $D_i=\nu_{i-1}$, $G_2=\alpha_1$, and $G_3=\alpha_2$, is invariant under the scaling (2.5) with the two-parameter family

$$(\rho,\gamma)\mapsto(\alpha,\beta,\gamma,\rho,\sigma)=\left(\rho,\frac{\rho}{\gamma},\gamma,\rho,\gamma^2\right).\tag{2.12}$$

## 3 | DEFINITION OF WEAK SOLUTIONS

We begin with introducing notations that will be used throughout the paper.

Let $X$ denote any real normed space with norm $|\cdot|_X$, and let $\langle x^*, x\rangle_X$ denote the dual pairing of $x^* \in X^*$ and $x \in X$. By $L^p(0,T;X)$ $(1 \le p \le +\infty)$ we denote the vector space of all equivalence classes of Bochner measurable mappings $u : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T |u(t)|_X^p \, dt\right)^{1/p} & \text{if } 1 \le p < +\infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} |u(t)|_X & \text{if } p = +\infty \end{cases} \tag{3.1}$$

is finite (see e.g., Chap. III, §3, Chap. IV, §3 in Bourbaki [30], App. in Brézis [31], and Droniou [32] for details). Let $\Omega \subseteq \mathbb{R}^N$ $(N \ge 2)$ be any open set, and let $Q = \Omega \times ]0,T[$ for $T > 0$. For $1 \le p < \infty$ and $u \in L^p(Q)$ define

$$[u](t)(\cdot) = u(\cdot, t) \quad \text{for a.a. } t \in [0,T]. \tag{3.2}$$

By Fubini's theorem, the function $t \mapsto \int_\Omega |u(x,t)|^p dx$ is in $L^1(0,T)$ and there holds

$$\int_0^T \|[u](t)\|_{L^p(\Omega)}^p \, dt = \int_Q |u(x,t)|^p \, dx\, dt. \tag{3.3}$$

An elementary argument shows that the mapping $u \mapsto [u]$ is a linear isometry of $L^p(Q)$ onto $L^p(0,T;L^p(\Omega))$. Therefore, these spaces will be identified in what follows. By $W^{1,p}(\Omega)$, we denote the usual Sobolev space, and we set $\boldsymbol{W}^{1,p}(\Omega) = (W^{1,p}(\Omega))^N$.

Unless otherwise stated, from now on let $\Omega = (]0,a[)^3$ denote the cube introduced in Section 1. We define

$$\begin{aligned}
&W^{1,p}_{\text{per}}(\Omega) = \left\{ u \in W^{1,p}(\Omega);\ u|_{\Gamma_i} = u|_{\Gamma_{i+3}} \text{ for } i = 1,2,3 \right\}, \\
&\boldsymbol{W}^{1,p}_{\text{per,div}}(\Omega) = \left\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}_{\text{per}}(\Omega);\ \operatorname{div} \boldsymbol{u} = 0 \text{ a.e. in } \Omega \right\}, \\
&C^1_{\text{per},T}(\overline{Q}) = \left\{ \varphi \in C^1(\overline{Q});\ \varphi|_{\Gamma_i\times]0,T[} = \varphi|_{\Gamma_{i+3}\times]0,T[},\ \nabla\varphi|_{\Gamma_i\times]0,T[} = \nabla\varphi|_{\Gamma_{i+3}\times]0,T[} \text{ for } i = 1,2,3,\ \varphi(\cdot,T) = 0 \text{ on } \Omega \right\}, \\
&\boldsymbol{C}^1_{\text{per},T,\text{div}}(\overline{Q}) = \left\{ \boldsymbol{v} \in \boldsymbol{C}^1_{\text{per},T}(\overline{Q});\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } Q \right\}.
\end{aligned} \tag{3.4}$$

We emphasize that the test functions in $C^1_{\text{per},T}(\overline{Q})$ vanish at $t = T$. Finally, by $\mathcal{M}_{\ge}(\overline{Q})$, we denote the set of all non-negative, bounded Radon measures on the $\sigma$-algebra of Borel sets $\subseteq \overline{Q}$, which is a closed cone in the vector space $\mathcal{M}(\overline{Q}) \simeq C(\overline{Q})^*$ of all (signed) Radon measures.

To simplify the notation, we subsequently set $\alpha_1 = 1$ and $\nu_2 = 1$, which can always be achieved by exploiting the scaling (2.12). We further set $\nu_1 = 1$, but keep the constant $\nu_0 > 0$ to emphasize that the source term in Equation (1.1d) for the turbulent energy $k$ arises from the dissipation in the momentum equation (1.1b) for $\boldsymbol{u}$.

**Definition 3.1.** Let $\boldsymbol{f} \in \boldsymbol{L}^1(Q)$, $\boldsymbol{u}_0 \in \boldsymbol{L}^1(\Omega)$ and $\omega_0, k_0 \in L^1(\Omega)$ such that $\omega_0, k_0 \ge 0$ a.e. in $\Omega$. A triple of measurable functions $\{\boldsymbol{u}, \omega, k\}$ in $Q$ is called *weak solution of Equations* (1.1) *and* (1.10) *with a non-negative defect measure* $\mu \in \mathcal{M}_{\ge}(\overline{Q})$, if

$$\omega > 0, \quad \frac{k}{\omega} \ge \text{const} > 0 \ \text{ a.e. in } Q, \tag{3.5}$$

$$\left.\begin{aligned}&\boldsymbol{u}\in L^{\infty}\big(0,T;\boldsymbol{L}^2(\Omega)\big)\cap L^2\Big(0,T;\boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)\Big),\\&\omega\in L^{\infty}\big(0,T;L^2(\Omega)\big)\cap L^2\Big(0,T;W^{1,2}_{\mathrm{per}}(\Omega)\Big),\\&k\in L^{\infty}(0,T;L^1(\Omega))\cap L^{15/14}\Big(0,T;W^{1,15/14}_{\mathrm{per}}(\Omega)\Big),\end{aligned}\right\}\tag{3.6}$$

$$\int_Q\frac{k}{\omega}\big((1+|\boldsymbol{D}(\boldsymbol{u})|)|\boldsymbol{D}(\boldsymbol{u})|+|\nabla\omega|+|\nabla k|\big)\,\mathrm{d}x\,\mathrm{d}t<\infty,\tag{3.7}$$

the following weak equations hold

$$\left.\begin{aligned}&-\int_Q\boldsymbol{u}\cdot\frac{\partial\boldsymbol{v}}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_Q(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t+\nu_0\int_Q\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t\\&=\int_\Omega\boldsymbol{u}_0(x)\cdot\boldsymbol{v}(x,0)\,\mathrm{d}x+\int_Q\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\quad\text{for all }\boldsymbol{v}\in\boldsymbol{C}^1_{\mathrm{per},T,\mathrm{div}}(\overline{Q}),\end{aligned}\right\}\tag{3.8}$$

$$\left.\begin{aligned}&-\int_Q\omega\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_Q\omega\boldsymbol{u}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\frac{k}{\omega}\nabla\omega\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\\&=\int_\Omega\omega_0(x)\varphi(x,0)\,\mathrm{d}x-\int_Q\omega^2\varphi\,\mathrm{d}x\,\mathrm{d}t\quad\text{for all }\varphi\in C^1_{\mathrm{per},T}(\overline{Q}),\end{aligned}\right\}\tag{3.9}$$

$$\left.\begin{aligned}&-\int_Q k\frac{\partial z}{\partial t}\,\mathrm{d}x\,\mathrm{d}t-\int_Q k\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t+\int_Q\frac{k}{\omega}\nabla k\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t\\&=\int_\Omega k_0(x)z(x,0)\,\mathrm{d}x+\int_Q\Big(\nu_0\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2-\alpha_2k\omega\Big)z\,\mathrm{d}x\,\mathrm{d}t+\int_{\overline{Q}}z\,\mathrm{d}\mu\quad\text{for all }z\in C^1_{\mathrm{per},T}(\overline{Q}),\end{aligned}\right\}\tag{3.10}$$

the Leray–Hopf type energy bound for the Navier–Stokes equation

$$\int_\Omega\frac12|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x+\int_0^t\int_\Omega\nu_0\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2\,\mathrm{d}x\,\mathrm{d}s\le\int_\Omega\frac12|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x+\int_0^t\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s\ \text{ for a.a. }t\in[0,T],\tag{3.11}$$

and the total energy satisfies the estimate

$$\int_\Omega\Big(\frac12|\boldsymbol{u}(x,t)|^2+k(x,t)\Big)\,\mathrm{d}x+\int_0^t\int_\Omega\alpha_2k\omega\,\mathrm{d}x\,\mathrm{d}s\le\int_\Omega\Big(\frac12|\boldsymbol{u}_0(x)|^2+k_0(x)\Big)\,\mathrm{d}x+\int_0^t\int_\Omega\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s\ \text{ for a.a. }t\in[0,T].\tag{3.12}$$

It is easy to see that all integrals in Equations (3.8)–(3.10) are well-defined. It suffices to consider the integrals with integrands $k\boldsymbol{u}\cdot\nabla z$ and $\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2z$ in Equation (3.10). First, it is well-known that condition (3.6) on $\boldsymbol{u}$ implies $\boldsymbol{u}\in\boldsymbol{L}^{10/3}(Q)$ (combine Hölder's inequality and Sobolev's embedding theorem). Analogously, the condition (3.6) on $k$ implies $k\in L^{10/7}(Q)$ (take $N=3$, $\theta=3/4$, $(p_1,p_2)=(1,\frac{15}{14})$, and $(s_1,s_2)=(\infty,\frac{15}{14})$ in Lemma 4.1(B) below). Hence, $k\boldsymbol{u}\in\boldsymbol{L}^1(Q)$. Second, $\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2\in L^1(Q)$ by virtue of Equation (3.7).

*Remark* 3.2. *The condition $k/\omega\ge$ const $>0$ is crucial for our existence theory, in particular for obtaining the regularities for $\{\boldsymbol{u},\omega,k\}$ stated in Equation (3.6). It would be desirable to develop an existence theory without this condition, because this would allow us to study how the support of $k$, which may be called the "turbulent region", invades the "non-turbulent region" where $k\equiv0$.*

*Remark* 3.3 (Classical solutions are weak solutions). *Every sufficiently regular classical solution* $\{\boldsymbol{u}, \omega, k\}$ *of Equations* (1.1) *and* (1.10) *satisfies the variational identities* (3.8)–(3.10) *with defect measure* $\mu = 0$. *To verify this, we multiply* (1.1b)–(1.1d) *by the test functions* $\boldsymbol{v}$, $\varphi$, *and* $z$, *respectively, and integrate by parts over the cube* $\Omega$ *and then over the interval* $[0, T]$. *Moreover, it is easy to see that the energy inequalities* (3.11) *and* (3.12) *hold as equalities.*

Of course, the important implication to be shown is that smooth weak solutions are indeed classical solutions. In order to establish this, we crucially use that the inequality (3.12) for the total energy $\int_\Omega (\frac{1}{2}|\boldsymbol{u}|^2 + k)\,\mathrm{d}x$ and combine it with the upper estimate (3.11) for the macroscopic kinetic energy $\int_\Omega \frac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}x$ and a lower energy estimate for the turbulent kinetic energy $\int_\Omega k\,\mathrm{d}x$, which will be derived next.

**Lemma 3.4.** *Let* $\{\boldsymbol{u}, \omega, k\}$ *be a weak solution of Equations* (1.1) *and* (1.10) *with defect measure* $\mu$. *Then, we have the integral relations*

$$\int_\Omega \omega(x,t)\,\mathrm{d}x + \int_0^t \int_\Omega \omega^2\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \omega_0(x)\,\mathrm{d}x \quad \textit{for all } t \in [0,T], \tag{3.13a}$$

$$\int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k_0(x)\,\mathrm{d}x + \int_0^t \int_\Omega \left(\nu_0 \frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - \alpha_2 k\omega\right)\mathrm{d}x\,\mathrm{d}s + \mu\left(\overline{\Omega}\times[0,t]\right) \quad \textit{for a.a. } t \in [0,T], \tag{3.13b}$$

$$\lim_{t\to 0} \int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k_0(x)\,\mathrm{d}x + \mu\left(\overline{\Omega}\times\{0\}\right), \tag{3.13c}$$

$$\int_\Omega k(x,t)\,\mathrm{d}x = \int_\Omega k(x,s)\,\mathrm{d}x + \int_s^t \int_\Omega \left(\nu_0 \frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - \alpha_2 k\omega\right)\mathrm{d}x\,\mathrm{d}\tau + \mu\left(\overline{\Omega}\times\,]s,t]\right) \quad \textit{for a.a. } s,t \textit{ with } s<t. \tag{3.13d}$$

*Proof.* It suffices to prove Equation (3.13b). The same reasoning gives Equation (3.13a), and the relations (3.13c) and (3.13d) follow from Equation (3.13b). For $t \in\, ]0,T[$, and $m > \frac{1}{T-t}$ with $m \in \mathbb{N}$ we define

$$\eta_m(s) = \begin{cases} 1 & \text{if } \ 0 \le s \le t, \\ m(t-s)+1 & \text{if } \ t \le s \le t + \frac{1}{m}, \\ 0 & \text{if } \ s \ge t + \frac{1}{m}. \end{cases} \tag{3.14}$$

Then, $\eta_m \in C([0,\infty[)$ and $\dot{\eta}_m = m\mathbb{1}_{]t,t+1/m[}$. For the Steklov average $\eta_{m,\lambda}(s) = \frac{1}{\lambda}\int_s^{s+\lambda} \eta_m(\tau)\,\mathrm{d}\tau$ with $s \ge 0$, $\lambda > 0$, we find

$$\eta_{m,\lambda} \in C^1([0,T]), \quad \eta_{m,\lambda} \overset{\lambda\to 0^+}{\longrightarrow} \eta_m \text{ in } C^0([0,T]), \quad \eta_{m,\lambda}(0) = 1 \text{ and } \eta_{m,\lambda}(T) = 0 \tag{3.15}$$

for $\lambda \in\, ]0,t[$. Moreover, we have $\dot{\eta}_{m,\lambda}(s) \to \dot{\eta}_m(s)$ for all $s \in [0,T] \setminus \{t, t+\frac{1}{m}\}$, and for all $f \in L^1(Q_T)$ we find

$$\int_{Q_T} f(x,s)\dot{\eta}_{m,\lambda}(s)\,\mathrm{d}x\,\mathrm{d}s \longrightarrow -m\int_t^{t+1/m} \int_\Omega f(x,s)\,\mathrm{d}x\,\mathrm{d}s \quad \text{as } \lambda \to 0^+. \tag{3.16}$$

Inserting the function $z(x,s) = \mathbb{1}_\Omega(x)\eta_{m,\lambda}(s)$ for $(x,s) \in Q$ into Equation (3.10) with $s$ in place of $t$, the limit $\lambda \to 0^+$ gives the relation

$$m\int_t^{t+1/m}\int_\Omega k(x,s)\,\mathrm{d}x\,\mathrm{d}s = \int_\Omega k_0(x)\,\mathrm{d}x + \int_0^{t+1/m}\int_\Omega \left(\nu_0\frac{k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2 - \alpha_2 k\omega\right)\eta_m\,\mathrm{d}x\,\mathrm{d}\tau + \mu\left(\overline{\Omega}\times[0,t]\right) + \int_{\overline{\Omega}\times]t,t+\frac{1}{m}[} \eta_m(\tau)\,\mathrm{d}\mu. \tag{3.17}$$

Because of $\eta_m(\tau) \in [0,1]$ for all $s \in [0,T]$, we have $\int_{\overline{\Omega}\times]t,t+\frac{1}{m}[} \eta_m(\tau)\,\mathrm{d}\mu \le \mu(\overline{\Omega}\times]t,t+\frac{1}{m}[) \to 0$ as $m \to \infty$. The limit passage $m \to \infty$ in Equation (3.17) gives Equation (3.13b) for every Lebesgue point $t \in [0,T]$ of the function $t \mapsto \int_\Omega k(x,t)\,\mathrm{d}x$. □

We are now ready to show that smooth enough weak solutions are indeed classical solutions and that the associated defect measure has to vanish.

**Proposition 3.5** (Smooth weak solutions are classical). *If $\{\boldsymbol{u},\omega,k\}$ is a weak solution of Equations (1.1) and (1.10) with defect measure $\mu$ (in the sense of Definition 3.1) such that $\boldsymbol{u}$, $\omega$, and $k$ are sufficiently smooth (e.g., twice continuously differentiable in $x$ and once in $t$), then $\{\boldsymbol{u},\omega,k\}$ is a classical solution of Equations (1.1) and (1.10).*

*Proof.* By definition, weak solutions lie in $\boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)$, which implies Equation (1.1a). Similarly, the periodic boundary conditions (1.10a) follow from the choice of spaces for the weak solution.

Using the smoothness of $\{\boldsymbol{u},\omega,k\}$, we can integrate by parts in the weak equations (3.8) and (3.9). From this, we obtain the validity of the classical equations (1.1b) and (1.1c) for $\boldsymbol{u}$ and $\omega$, respectively, and the initial conditions $\boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0$ and $\omega(0,\cdot) = \omega_0$.

Since the Navier–Stokes equation is classically satisfied, the kinetic energy satisfies Equation (3.11) with equality. Adding this equality to relation (3.13b) for the turbulent energy, the term $\nu_0\frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2$ exactly cancels; and we obtain

$$\int_\Omega\Big(\frac{1}{2}|\boldsymbol{u}(x,t)|^2+k(x,t)\Big)\,\mathrm{d}x = \int_\Omega\Big(\frac{1}{2}|\boldsymbol{u}_0|^2+k_0\Big)\,\mathrm{d}x + \int_0^t\int_\Omega(\boldsymbol{f}\cdot\boldsymbol{u}-\alpha_2 k\omega)\,\mathrm{d}x\,\mathrm{d}s + \mu(\overline{\Omega}\times[0,t]) \tag{3.18}$$

for a.a. $t \in [0,T]$. Comparing this to the total energy inequality (3.12) and using $\mu \ge 0$, we conclude $\mu(\overline{\Omega}\times[0,t]) = 0$ for a.a. $t \in [0,T]$. Thus, we find $\mu(\overline{\Omega}\times[0,T[) = 0$, which gives $\int_{\overline{Q}} z\,\mathrm{d}\mu = \int_{\overline{\Omega}} z(x,T)\,\mathrm{d}\mu(T,x) = 0$ in Equation (3.10). For the last identity we exploit that $z \in C^1_{\mathrm{per},T}(\overline{Q})$ implies $z(x,T) = 0$ on $\overline{\Omega}$.

Again, using the smoothness of $\{\boldsymbol{u},\omega,k\}$, we can integrate by parts in the weak equations (3.10) and obtain the validity of the classical equations (1.1d) and the initial conditions $k(0,\cdot) = k_0$. □

We note that by Equation (3.13b) the defect measure $\mu \ge 0$ contributes positively to the integrated turbulent energy $\int_\Omega k(x,t)\,\mathrm{d}x$. In contrast, the energy inequality (3.11) for weak solutions of the Navier–Stokes equations provides an upper bound for the integrated kinetic energy $\int_\Omega \frac{1}{2}|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x$ in terms of possibly different defect measure $\mu_{\mathrm{NS}}$. The expectation is that these two measures exactly cancel each other when considering the total kinetic energy $\int_\Omega(\frac{1}{2}|\boldsymbol{u}(x,t)|^2 + k(x,t))\,\mathrm{d}x$, and then Equation (3.12) holds as an equality. Our methods will not be strong enough to show this cancellation but we establish the corresponding upper bound stated in Equation (3.12), which may be interpreted as $\mu \le \mu_{\mathrm{NS}}$. In the related work Bulíček and Málek [13], the desired cancellation is derived by completely different methods.

*Remark* 3.6 (Conservation law for the energy density $E$). *For fluid models involving an additional energy equation, it is natural to derive equations for the total energy density, which in our case reads $E(x,t) = k(x,t) + \frac{1}{2}|\boldsymbol{u}(x,t)|^2$. This idea goes back to Feireisl and Málek in Refs. [33, 34] and provides a local balance law for the total energy density $E$. We expect that the result of Thm. 1.1, Eqn. (1.50) in Bulíček and Málek [13] also holds in our case and conjecture that there exist weak solutions as stated in Theorem 4.1 that additionally satisfy the distributional form of the local balance equation*

$$\frac{\partial}{\partial t}E + \mathrm{div}\,((E+p)\boldsymbol{u}) = \mathrm{div}\left(\frac{k}{\omega}\nabla k + \nu_0\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u})\,\boldsymbol{u}\right) + \boldsymbol{f}\cdot\boldsymbol{u} - \alpha_2 k\omega, \tag{3.19}$$

*A close inspection of our estimates shows that all terms in this equation can be defined as distributions, if the pressure $p$ is recovered from Equation (1.1b) in the standard way. However, at present, it remains unclear how this relation can be derived using our approach based on pseudo-monotone operators.*

Clearly, integrating the local balance law (3.19) over $\Omega$ and using the periodic boundary condition implies that the total-energy inequality (3.12) holds as equality for all $t \in [0,T]$:

$$\int_\Omega \left(\frac{1}{2}|\boldsymbol{u}(x,t)|^2 + k(x,t)\right) \mathrm{d}x + \alpha_2 \int_0^t \int_\Omega k\omega \,\mathrm{d}x\,\mathrm{d}s = \int_\Omega \left(\frac{1}{2}|\boldsymbol{u}_0(x)|^2 + k_0(x)\right) \mathrm{d}x + \int_0^t \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s. \tag{3.20}$$

The following result shows that in this case, the defect measure $\mu$ in Equation (3.10) is closely related to the defect measure associated with the weak solution of the Navier–Stokes equation. The result follows simply by subtracting Equation (3.13b) from Equation (3.20).

**Proposition 3.7** (Energy equalities and defect measure). *Let $\{\boldsymbol{u}, \omega, k\}$ and $\mu$ be a weak solution as in Definition 3.1. If additionally the energy equality (3.20) holds, then the following two statements are equivalent:*

*(i)* $\mu = 0$;
*(ii)* $\int_\Omega \frac{1}{2}|\boldsymbol{u}(x,t)|^2\,\mathrm{d}x + \nu_0 \int_0^t\int_\Omega \frac{k}{\omega}|\boldsymbol{D}(\boldsymbol{u})|^2\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega |\boldsymbol{u}_0(x)|^2\,\mathrm{d}x + \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}s$ *for a.a.* $t \in [0,T]$.

This result shows that the two energy inequalities (3.11), (3.12) and the defect measure $\mu$ in Equation (3.10) are related to the classical problem of proving an energy equality for weak solutions of the Navier–Stokes equations. A similar result for the case of Navier–Stokes equations with temperature dependent viscosities has been obtained in Naumann [35]. Defect measures also appear in a natural way in the context of weak solutions of other types of nonlinear PDEs (see, e.g., Refs. [36–38]).

## 4 | AN EXISTENCE THEOREM FOR WEAK SOLUTIONS

We define the function spaces

$$\begin{aligned} C^\infty_{\mathrm{per}}(\Omega) &= \{u|_\Omega\ ;\ u \in C^\infty(\mathbb{R}^3), u \text{ is } a\text{-periodic in the directions } \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}, \\ \boldsymbol{C}^\infty_{\mathrm{per,div}}(\Omega) &= \{\boldsymbol{u} \in \boldsymbol{C}^\infty_{\mathrm{per}}(\Omega)\ ;\ \mathrm{div}\,\boldsymbol{u} = 0 \text{ in } \Omega\}. \end{aligned} \tag{4.1}$$

We impose the following conditions upon the right-hand side in Equation (1.1b) and the initial data in Equation (1.10b):

$$\left.\begin{aligned} &\boldsymbol{f} \in \boldsymbol{L}^2(Q);\ \boldsymbol{u}_0 \in \boldsymbol{L}^2_{\mathrm{div}}(\Omega) := \overline{\boldsymbol{C}^\infty_{\mathrm{per,div}}(\Omega)}^{\|\cdot\|_{L^2(\Omega)}},\ \omega_0 \in L^\infty(\Omega),\ k_0 \in L^1(\Omega), \\ &\textit{there exist positive}\omega_*,\ \omega^* \textit{ such that } \omega_* \le \omega_0(x) \le \omega^* \textit{ for a.a. } x \in \Omega, \\ &\textit{there exist positive} k_* \textit{ such that } k_0(x) \ge k_* \textit{ for a.a. } x \in \Omega. \end{aligned}\right\} \tag{4.2}$$

The following theorem is the main result of our paper.

**Theorem 4.1** (Main existence result). *Assume Equation (4.2) and $\alpha_2 = \mathrm{const} > 0$ (cf. (1.1d)). Then there exists a triple of measurable functions $\{\boldsymbol{u}, \omega, k\}$ in $Q$ and a non-negative defect measure $\mu \in \mathcal{M}_\ge(\overline{Q})$ such that*

$$\frac{\omega_*}{1+t\omega_*} \le \omega(x,t) \le \frac{\omega^*}{1+t\omega^*} \quad \textit{and} \quad \frac{k_*}{(1+t\omega^*)^{\alpha_2}} \le k(x,t) \quad \textit{for a.a. } (x,t) \in Q; \tag{4.3}$$

$$\left.\begin{aligned} &\boldsymbol{u} \in C_{\mathrm{w}}\left([0,T]; \boldsymbol{L}^2(\Omega)\right) \cap L^2(0,T; \boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)), \\ &\omega \in C_{\mathrm{w}}\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; W^{1,2}_{\mathrm{per}}(\Omega)\right), \\ &k \in L^\infty\left(0,T; L^1(\Omega)\right) \cap \bigcap_{1\le p<2} L^p\left(0,T; W^{1,p}_{\mathrm{per}}(\Omega)\right); \end{aligned}\right\} \tag{4.4}$$

$$\int_Q k\left(\left|\boldsymbol{D}(\boldsymbol{u})\right|^2+|\nabla\omega|^2\right)\mathrm{d}x\,\mathrm{d}t<\infty, \tag{4.5}$$

$$\boldsymbol{u}'\in\bigcap\nolimits_{\sigma>16/5}L^{4/3}\left(0,T;\left(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)\right)^*\right),\quad \textit{and}\quad \omega'\in\bigcap\nolimits_{\sigma>16/5}L^{4/3}\left(0,T;\left(W^{1,\sigma}_{\mathrm{per}}(\Omega)\right)^*\right). \tag{4.6}$$

*The triple $\{\boldsymbol{u},k,\omega\}$ is a weak solution of Equations (1.1) and (1.10) in the sense of Definition 3.1 with*

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \textit{ in } \boldsymbol{L}^2(\Omega)\ \textit{ and }\ \omega(0)=\omega_0 \textit{ in } L^2(\Omega); \tag{4.7}$$

*In particular, Equation (3.10) holds and for all $\sigma>16/5$ we have*

$$\int_0^T\left\langle\boldsymbol{u}'(t),\boldsymbol{v}(t)\right\rangle_{W^{1,\sigma}_{\mathrm{per,div}}}\mathrm{d}t+\int_Q\left(-(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{v}+\nu_0\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})\right)\mathrm{d}x\,\mathrm{d}t=\int_Q\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t\ \ \textit{for all } \boldsymbol{v}\in L^\sigma\left(0,T;\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)\right), \tag{4.8}$$

$$\int_0^T\left\langle\omega'(t),\varphi(t)\right\rangle_{W^{1,\sigma}_{\mathrm{per}}}\mathrm{d}t-\int_Q\omega\boldsymbol{u}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_Q\frac{k}{\omega}\nabla\omega\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=-\int_Q\omega^2\varphi\,\mathrm{d}x\,\mathrm{d}t\ \ \textit{for all } \varphi\in L^\sigma\left(0,T;W^{1,\sigma}_{\mathrm{per}}(\Omega)\right). \tag{4.9}$$

Of course, in Equations (4.8) and (4.9), it suffices to consider $\sigma=\frac{16}{5}+\eta$ for an arbitrarily small $\eta>0$. The derivatives $\boldsymbol{u}'$ and $\omega'$ in Equation (4.6) are understood in the sense of distributions from $]0,T[$, into $(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega))^*$ and $(W^{1,\sigma}_{\mathrm{per}}(\Omega))^*$, respectively (see, e.g., App. in Brézis [31] or pp. 54–56 in Droniou [32] for details). Here we have used the continuous and dense embeddings

$$W^{1,2}_{\mathrm{per}}(\Omega)\subset L^2(\Omega)\subset\left(W^{1,\sigma}_{\mathrm{per}}(\Omega)\right)^*\quad \text{for } \sigma\geq\frac{6}{5}. \tag{4.10}$$

To see that $\{\boldsymbol{u},\omega,k\}$ together with the measure $\mu$ in the above theorem are a weak solution of Equations (1.1) and (1.10) in the sense of the Definition 3.1, it suffices to note that Equations (3.8) and (3.9) follow from Equations (4.8) and (4.9), respectively, by integration by parts of the first integrals on the left-hand sides.

Before starting the proof, it is instructive to check that the above estimates (4.3)–(4.6) are enough to show that all terms in (4.8)–(3.10) are well-defined. For this, we first recall the classical Gagliardo-Nirenberg estimate and then provide an anisotropic version that is adjusted to the parabolic problems on $Q=[0,T]\times\Omega$, we use the short-hand notations

$$L^s(L^p):=L^s(0,T;L^p(\Omega))\quad\text{and}\quad J_\theta(a,b):=a^{1-\theta}(a+b)^\theta. \tag{4.11}$$

**Lemma 4.1** (Gagliardo-Nirenberg estimates). *For $N\in\mathbb{N}$ consider a bounded Lipschitz domain $\Omega\subset\mathbb{R}^N$. (A) (Classical isotropic version) Assume $1\leq p_1<p<\infty$, $p_2\in\,]1,N[$ and $\theta\in\,]0,1[$ such that*

$$\frac{1}{p}=(1-\theta)\frac{1}{p_1}+\theta\left(\frac{1}{p_2}-\frac{1}{N}\right). \tag{4.12}$$

*Then, there exists a constant $C>0$ such that for all $\psi\in W^{1,p_2}(\Omega)$ we have*

$$\|\psi\|_{L^p(\Omega)}\leq C\,J_\theta\left(\|\psi\|_{L^{p_1}(\Omega)},\|\nabla\psi\|_{L^{p_2}(\Omega)}\right). \tag{4.13}$$

*(B) (Anisotropic version) Consider $p,\ p_1,\ p_2$, and $\theta$ as in (A) and $s,\ s_1$, and $s_2$ satisfying*

$$1\leq s_2\leq s\leq s_1\quad \textit{and}\ \frac{1}{s}=(1-\theta)\frac{1}{s_1}+\theta\frac{1}{s_2}. \tag{4.14}$$

*Then, there exists* $C^* > 0$ *such that for all* $\varphi \in L^{s_2}(0,T;W^{1,p_2}(\Omega))$ *we have*

$$\|\varphi\|_{L^s(L^p)} \leq C^* J_\theta\big(\|\varphi\|_{L^{s_1}(L^{p_1})}, \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\big). \tag{4.15}$$

*Proof.* Part (A) is well-known, see, for example, Thm. 1.24 in Roubíček [25].

To establish Part (B) we apply Part (A) for $\psi = \varphi(t)$ a.a. $t \in [0,T]$. Thus, we obtain (abbreviating $\|\psi\|_p := \|\psi\|_{L^p(\Omega)}$)

$$\|\varphi\|^s_{L^s(L^p)} = \int_0^T \|\varphi(t)\|^s_p \,\mathrm{d}t \overset{4.13)}{\leq} C_1 \int_0^T \|\varphi(t)\|^{(1-\theta)s}_{p_1}\big(\|\varphi(t)\|_{p_1} + \|\nabla\varphi(t)\|_{p_2}\big)^{\theta s}\,\mathrm{d}t \tag{4.16}$$

$$\overset{\text{Hölder}+(4.14)}{\leq} C_1 \big\| \|\varphi\|_{p_1}\big\|^{(1-\theta)s}_{L^{s_1}(0,T)} \Big\| \|\varphi\|_{p_1} + \|\nabla\varphi\|_{p_2}\Big\|^{\theta s}_{L^{s_2}(0,T)} \tag{4.17}$$

$$\overset{s_1\geq s_2}{\leq} C_1 \|\varphi\|^{(1-\theta)s}_{L^{s_1}(L^{p_1})}\big(T^{1/s_2-1/s_1}\|\varphi\|_{L^{s_1}(L^{p_1})} + \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\big)^{\theta s} \leq C_2\big(J_\theta\big(\|\varphi\|_{L^{s_1}(L^{p_1})}, \|\nabla\varphi\|_{L^{s_2}(L^{p_2})}\big)\big)^s, \tag{4.18}$$

which is the desired estimate. □

Remark 4.2 (Well-definedness of nonlinear terms). *We first show that the second integral on the left-hand side of the variational identity in Equation (4.8) is well-defined. For the integral of* $(\boldsymbol{u}\otimes\boldsymbol{u}):\nabla\boldsymbol{v}$ *we see that Equation (4.4) allows us to use Lemma 4.1 with* $N = 3$, $(s_1,p_1) = (\infty,2)$ *and* $(s_2,p_2) = (2,2)$. *With* $\theta = 3/4$ *part (A) gives*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)} \leq C\Big(\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}\|^{1/4}_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{u}\|^{3/4}_{\boldsymbol{L}^2(\Omega)}\Big), \tag{4.19}$$

*whereas part (B) leads to* $\boldsymbol{u} \in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega))$, *which implies*

$$\boldsymbol{u}\otimes\boldsymbol{u} \in L^{4/3}(0,T;L^2(\Omega)). \tag{4.20}$$

*With* $\sigma > 16/5 > 2$, *we have* $\nabla\boldsymbol{v} \in L^2(0,T;L^2(\Omega))$ *and* $\int_Q(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t$ *is well-defined. Using* $\theta = 3/5$ *in Lemma 4.1(B) we obtain* $s = p = 10/3$ *and hence conclude*

$$\|\boldsymbol{u}\|_{L^{10/3}(Q)} \leq C_2 J_{3/5}\big(\|\boldsymbol{u}\|_{L^\infty(\boldsymbol{L}^2)}, \|\nabla\boldsymbol{u}\|_{L^2(L^2)}\big). \tag{4.21}$$

*For the integral of* $\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})$ *we use* $\omega \geq \omega_*/(1+T\omega_*) > 0$ *from Equation (4.3),* $k^{1/2}\boldsymbol{D}(\boldsymbol{u}) \in L^2(Q)$ *from Equation (4.5). Using Equation (4.4), we can apply Lemma 4.1(B) to* $k$ *with* $N = 3$, $(s_1,p_1) = (\infty,1)$, *and* $s_2 = p_2 \in [1,2[$. *Choosing* $\theta = 3/4$, *we obtain* $s = p = 4p_2/3$, *such that* $k$ *lies in* $L^{4p_2/3}(0,T;L^{4p_2/3}(\Omega)) = L^{4p_2/3}(Q)$. *As* $p_2 \in [1,2[$ *is arbitrary, we have* $k^{1/2} \in L^q(Q)$ *for all* $q \in [1,16/3[$. *By Hölder's inequality, we arrive at*

$$k\boldsymbol{D}(\boldsymbol{u}) = k^{1/2}\,k^{1/2}\boldsymbol{D}(\boldsymbol{u}) \in L^{\overline{p}}(Q) \ \text{ for all } \overline{p} \in [1,16/11[. \tag{4.22}$$

*Using* $\boldsymbol{D}(\boldsymbol{v}) \in L^\sigma(0,T;L^\sigma(\Omega)) = L^\sigma(Q)$ *with* $\sigma > 16/5$, *we see that there is always a* $\overline{p} \in [1,16/11[$ *such that* $\frac{1}{\sigma} + \frac{1}{\overline{p}} \leq 1$. *Hence, we conclude*

$$\int_Q \Big|\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v})\Big|\,\mathrm{d}x\,\mathrm{d}t \leq C\|k\boldsymbol{D}(\boldsymbol{u})\|_{L^{\overline{p}}(Q)}\|\boldsymbol{D}(\boldsymbol{v})\|_{L^\sigma(Q)} < \infty. \tag{4.23}$$

*Thus, by a routine argument, Equations (4.20) and (4.22) lead to the existence of the distributional derivative* $\boldsymbol{u}'$ *as in Equation (4.6), see also Sections 5.4–5.6.*

*An analogous reasoning applies to the second and the third integral on the left-hand side of the variational identity in Equation (4.9).*

*Finally, combining* $\boldsymbol{u} \in \boldsymbol{L}^2(Q)$ *and* $\nabla k \in L^p(Q)$ *for all* $p \in [1,2[$ *(see Equation (4.4)) and* $k \in L^{4p/3}(Q)$ *from above, Hölder's inequality gives*

$$k\boldsymbol{u} \in \boldsymbol{L}^q(Q) \text{ and } k\nabla k \in \boldsymbol{L}^q(Q) \quad \text{for all } q \in [1, 8/7[, \tag{4.24}$$

*that is, the second and third integral on the left-hand side in Equation (3.10) are well-defined.*

The estimates (4.3), which will be derived by using suitable comparison arguments, allow us to deduce the following result (based on the choice $\alpha_1 = 1$).

**Corollary 4.3.** *For a.a.* $(x,t) \in Q$*, we have the following estimates:*

$$L(x,t) := \frac{k(x,t)^{1/2}}{\omega(x,t)} \geq \frac{k_*^{1/2}}{\omega^*}(1 + t\omega^*)^{1-\alpha_2/2}, \tag{4.25}$$

$$\frac{1}{\omega^*} + t \leq \frac{1}{\omega(x,t)} \leq \frac{1}{\omega_*} + t. \tag{4.26}$$

Kolmogorov claimed in Ref. [1] that $L = L(x,t)$ "… grows in proportion of $t^{2/7}$ …" (see also p. 215 in Spalding [2] or p. 329 in Tikhomirov [9]). Clearly, from Equation (4.25) with $\alpha_2 = 10/7$, it follows

$$L(x,t) \geq \frac{k_*^{1/2}}{\omega^*}(1 + t\omega^*)^{2/7} \quad \text{for a.a. } (x,t) \in \Omega \times ]\, t_0, T\, [\,. \tag{4.27}$$

Of course, Kolmogorov's claim is compatible with our lower estimate for any choice $\alpha_2 \geq 10/7$ (and in Kolmogorov [1] $\alpha_2 = 11/7$ was chosen). However, it cannot be true for $\alpha_2 \in ]0, 10/7[$.

## 5 | PROOF OF THE EXISTENCE THEOREM

The proof of the main Theorem 4.1 proceeds in several steps. First, we regularize the problem by adding small higher-order dissipation terms of $r$-Laplacian type and small coercivity-generating lower-order terms. A general result for pseudo-monotone operators, which is detailed in Appendix A, then provides approximate solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$. In Section 5.2, we provide $\varepsilon$-independent upper and lower bounds for $\omega_\varepsilon$ and $k_\varepsilon$ by comparison arguments. In Section 5.3, we complement the standard energy estimates by improved integral estimates for $k_\varepsilon$ that allow us to pass to the limit $\varepsilon \searrow 0$ in Section 5.5.

### 5.1 | Defining suitable approximate solutions $\{\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, \boldsymbol{k}_\varepsilon\}$

Let be $\omega_*$, $\omega^*$, and $k_*$ as in Equation (4.2). We introduce the comparison functions

$$\underline{\omega}(t) = \frac{\omega_*}{1 + t\omega_*}, \quad \overline{\omega}(t) = \frac{\omega^*}{1 + t\omega^*}, \quad \varkappa(t) = \frac{k_*}{(1 + t\omega^*)^{\alpha_2}} \quad \text{for } t \in [0,T], \tag{5.1}$$

which will be the desired bounds for $\omega_\varepsilon$ and $k_\varepsilon$ in $Q$. Subsequently, we will use the notion

$$\xi^+ := \max\{\xi, 0\} \geq 0 \quad \text{and} \quad \xi^- = \min\{\xi, 0\} \leq 0 \tag{5.2}$$

for the positive and negative parts of real numbers or real-valued functions.

We choose a fixed number $r \in ]3, \infty[$ and consider for all small $\varepsilon > 0$ the following *r-Laplacian approximation* of Equation (1.1), where we add the coercivity-generating terms $\varepsilon|\boldsymbol{u}|^{r-1}\boldsymbol{u}$, $\varepsilon|\omega|^{r-2}\omega$, and $\varepsilon|k|^{r-2}k$ to the right-hand sides of Equations (1.1b)–(1.1d), respectively:

$$\operatorname{div} \boldsymbol{u} = 0, \tag{5.3a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu_0 \operatorname{div}\left(\frac{k^+}{\varepsilon+\omega^+}\boldsymbol{D}(\boldsymbol{u})\right) - \nabla p + \boldsymbol{f} + \varepsilon\big(\operatorname{div}\left(|\boldsymbol{D}(\boldsymbol{u})|^{r-2}\boldsymbol{D}(\boldsymbol{u})\right) - |\boldsymbol{u}|^{r-2}\boldsymbol{u}\big), \tag{5.3b}$$

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega = \operatorname{div}\left(\frac{k^+}{\varepsilon+\omega^+}\nabla\omega\right) - \omega^+\omega + \varepsilon\big(\operatorname{div}\left(|\nabla\omega|^{r-2}\nabla\omega\right) - |\omega|^{r-2}\omega\big) + \varepsilon\big(\underline{\omega}(t)\big)^{r-1}, \tag{5.3c}$$

$$\frac{\partial k}{\partial t} + \boldsymbol{u}\cdot\nabla k = \operatorname{div}\left(\frac{k^+}{\varepsilon+\omega^+}\nabla k\right) + \nu_0\frac{k^+\,|\boldsymbol{D}(\boldsymbol{u})|^2}{\varepsilon+\omega^+ + \varepsilon k^+} - \alpha_2 k\omega^+ + \varepsilon\big(\operatorname{div}\left(|\nabla k|^{r-2}\nabla k\right) - |k|^{r-2}k\big) + \varepsilon(\varkappa(t))^{r-1}. \tag{5.3d}$$

The additional terms $\varepsilon(\underline{\omega}(t))^{r-1}$ and $\varepsilon(\varkappa(t))^{r-1}$ are added in Equations (5.3b) and (5.3c), respectively, to make the comparison principle work again. In principle, it would be possible to use different exponents $r_{\boldsymbol{u}}$, $r_\omega$, and $r_k$ in Equations (5.3a)–(5.3c), because they need to satisfy different restrictions. In our case, $r = r_{\boldsymbol{u}} = r_\omega = r_k$ is sufficient and fits exactly with the assumptions in Equation (A.1) with $p = r$ for the abstract existence Theorem A.1.

We consider system (5.3) with initial data $\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\}$ satisfying

$$\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\} \in \boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\times W^{1,r}_{\mathrm{per}}(\Omega)\times W^{1,r}_{\mathrm{per}}(\Omega), \tag{5.4a}$$

$$\omega_* \le \omega_{0,\varepsilon}(x) \le \omega^* \ \text{ and } \ k_{0,\varepsilon}(x) \ge k_* \quad \text{a.e. in } \Omega, \tag{5.4b}$$

$$\boldsymbol{u}_{0,\varepsilon} \longrightarrow \boldsymbol{u}_0 \text{ in } \boldsymbol{L}^2(\Omega), \quad \omega_{0,\varepsilon} \longrightarrow \omega_0 \text{ a.e. in } \Omega, \quad k_{0,\varepsilon} \longrightarrow k_0 \ \text{ in } \ L^1(\Omega) \ \text{ for } \varepsilon \to 0. \tag{5.4c}$$

The existence of a sequence $\{\boldsymbol{u}_{0,\varepsilon}\}_{\varepsilon>0}$, which satisfies Equation (5.4a) follows immediately from the condition on $\boldsymbol{u}_0$ in Equation (4.2), whereas the existence of sequences $\{\omega_{0,\varepsilon}\}_{\varepsilon>0}$ and $\{k_{0,\varepsilon}\}_{\varepsilon>0}$ satisfying Equation (5.4) can be derived by routine argument from the conditions on $\omega_0$ and $k_0$ in Equation (4.2).

The following lemma states the existence of weak solutions of Equation (5.3) under the periodic boundary conditions (1.10a) and initial data (5.4). This result, which we derive in Appendix A by a direct application of existence results for pseudo-monotone evolutionary problems (see Theorem A.1), forms the starting point for our discussion in Sections 5.2–5.6.

**Proposition 5.1** (Existence of approximate solutions). *Let $\{\boldsymbol{u}_{0,\varepsilon}, \omega_{0,\varepsilon}, k_{0,\varepsilon}\}_{\varepsilon>0}$ be as in Equation (5.4), $r > 3$, and $\boldsymbol{f} \in \boldsymbol{L}^2(Q)$. Then, for every $\varepsilon > 0$, there exists a triple $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ such that*

$$\boldsymbol{u}_\varepsilon \in C\big([\,0,T\,];\boldsymbol{L}^2(\Omega)\big)\cap L^r\left(0,T;\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\right), \tag{5.5a}$$

$$\omega_\varepsilon, k_\varepsilon \in C\big([\,0,T\,];L^2(\Omega)\big)\cap L^r\left(0,T;W^{1,r}_{\mathrm{per}}(\Omega)\right), \tag{5.5b}$$

$$\boldsymbol{u}_\varepsilon' \in L^{r'}\left(0,T;\left(\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\right)^*\right), \quad \omega_\varepsilon', k_\varepsilon' \in L^{r'}\left(0,T;\left(W^{1,r}_{\mathrm{per}}(\Omega)\right)^*\right), \tag{5.5c}$$

*and*

$$\begin{aligned}&\int_0^T \left\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{v}(t)\right\rangle_{W^{1,r}_{\mathrm{per,div}}} \mathrm{d}t + \int_Q \sum_{i=1}^3 u_{\varepsilon,i}(\partial_i \boldsymbol{u}_\varepsilon)\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t + \nu_0\int_Q \frac{k_\varepsilon^+}{\varepsilon+\omega_\varepsilon^+}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v})\,\mathrm{d}x\,\mathrm{d}t \\ &+ \varepsilon\int_Q \left(\left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^{r-2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v}) + |\boldsymbol{u}_\varepsilon|^{r-2}\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}\right)\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \boldsymbol{v} \in L^r\left(0,T;\boldsymbol{W}^{1,r}_{\mathrm{per,div}}(\Omega)\right),\end{aligned} \tag{5.6a}$$

$$\int_0^T \big\langle \omega_\varepsilon'(t), \varphi(t) \big\rangle_{W^{1,r}_{\rm per}} \,\mathrm{d}t + \int_Q \varphi \boldsymbol{u}_\varepsilon \cdot \nabla\omega_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{k_\varepsilon^+}{\varepsilon + \omega_\varepsilon^+} \nabla\omega_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_Q \omega_\varepsilon^+ \omega_\varepsilon \varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \varepsilon \int_Q \left( |\nabla\omega_\varepsilon|^{r-2} \nabla\omega_\varepsilon \cdot \nabla\varphi + |\omega_\varepsilon|^{r-2}\omega_\varepsilon\varphi \right) \mathrm{d}x\,\mathrm{d}t = \varepsilon \int_Q \big(\underline{\omega}(t)\big)^{r-1} \varphi \,\mathrm{d}x\,\mathrm{d}t \quad \textit{for all } \varphi \in L^r\Big(0,T; W^{1,r}_{\rm per}(\Omega)\Big), \tag{5.6b}$$

$$\int_0^T \big\langle k_\varepsilon'(t), z(t) \big\rangle_{W^{1,r}_{\rm per}} \,\mathrm{d}t + \int_Q z \boldsymbol{u}_\varepsilon \cdot \nabla k_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_Q \frac{k_\varepsilon^+}{\varepsilon + \omega_\varepsilon^+} \nabla k_\varepsilon \cdot \nabla z \,\mathrm{d}x\,\mathrm{d}t - \nu_0 \int_Q \frac{k_\varepsilon^+}{\varepsilon + \omega_\varepsilon^+ + \varepsilon k_\varepsilon^+} \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 z \,\mathrm{d}x\,\mathrm{d}t \tag{5.6c}$$
$$+ \int_Q \alpha_2 k_\varepsilon \omega_\varepsilon^+ z \,\mathrm{d}x\,\mathrm{d}t + \int_Q \varepsilon\Big( |\nabla k_\varepsilon|^{r-2} \nabla k_\varepsilon \cdot \nabla z + |k_\varepsilon|^{r-2} k_\varepsilon z \Big) \mathrm{d}x\,\mathrm{d}t = \int_Q \varepsilon(\varkappa(t))^{r-1} z \,\mathrm{d}x\,\mathrm{d}t \ \textit{ for all } z \in L^r\Big(0,T; W^{1,r}_{\rm per}(\Omega)\Big),$$

$$\boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_{0,\varepsilon}, \quad \omega_\varepsilon(0) = \omega_{0,\varepsilon}, \quad k_\varepsilon(0) = k_{0,\varepsilon}. \tag{5.7}$$

The proof of Proposition 5.1 is the content of Appendix A. Observing the separability of $\boldsymbol{W}^{1,r}_{\rm per,div}(\Omega)$ and $W^{1,r}_{\rm per}(\Omega)$ and using Equation (5.5), a routine argument yields that the system (5.6) is equivalent to the following conditions for a.a. $t \in [0,T]$:

$$\big\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{w} \big\rangle_{\boldsymbol{W}^{1,r}_{\rm per,div}} + \int_\Omega \left( (\boldsymbol{u}_\varepsilon(t)\cdot\nabla\boldsymbol{u}_\varepsilon(t))\cdot\boldsymbol{w} + \nu_0 \frac{k_\varepsilon^+(t)}{\varepsilon + \omega_\varepsilon^+(t)} \boldsymbol{D}(\boldsymbol{u}_\varepsilon(t)) : \boldsymbol{D}(\boldsymbol{w}) \right) \mathrm{d}x - \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x$$
$$+ \varepsilon \int_\Omega \left( \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t))\big|^{r-2} \boldsymbol{D}(\boldsymbol{u}_\varepsilon(t)) : \boldsymbol{D}(\boldsymbol{w}) + \big|\boldsymbol{u}_\varepsilon(t)\big|^{r-2} \boldsymbol{u}_\varepsilon(t)\cdot\boldsymbol{w} \right) \mathrm{d}x = 0 \ \text{ for all } \boldsymbol{w} \in \boldsymbol{W}^{1,r}_{\rm per,div}(\Omega), \tag{5.8a}$$

$$\big\langle \omega_\varepsilon'(t), \psi \big\rangle_{W^{1,r}_{\rm per}} q + \int_\Omega \left( \psi \boldsymbol{u}_\varepsilon(t)\cdot\nabla\omega_\varepsilon(t) + \frac{k_\varepsilon^+(t)}{\varepsilon + \omega_\varepsilon^+(t)} \nabla\omega_\varepsilon(t)\cdot\nabla\psi + \omega_\varepsilon^+(t)\omega_\varepsilon(t)\psi \right) \mathrm{d}x$$
$$+ \int_\Omega \Big( +\varepsilon\Big( \big|\nabla\omega_\varepsilon(t)\big|^{r-2} \nabla\omega_\varepsilon(t)\cdot\nabla\psi + \big|\omega_\varepsilon(t)\big|^{r-2}\omega_\varepsilon(t)\psi \Big)\Big) \mathrm{d}x = \varepsilon\big(\underline{\omega}(t)\big)^{r-1} \int_\Omega \psi \,\mathrm{d}x \ \text{ for all } \psi \in W^{1,r}_{\rm per}(\Omega), \tag{5.8b}$$

$$\big\langle k_\varepsilon'(t), z \big\rangle_{W^{1,r}_{\rm per}} + \int_\Omega \left( z\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t) + \frac{k_\varepsilon^+(t)}{\varepsilon + \omega_\varepsilon^+(t)} \nabla k_\varepsilon(t)\cdot\nabla z \right) \mathrm{d}x - \int_\Omega \frac{\nu_0 k_\varepsilon^+(t)\, |\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t))|^2}{\varepsilon + \omega_\varepsilon^+(t) + \varepsilon k_\varepsilon^+(t)} z\,\mathrm{d}x + \int_\Omega \alpha_2 k_\varepsilon(t)\omega_\varepsilon^+(t) z\,\mathrm{d}x$$
$$+ \varepsilon \int_\Omega \Big( \big|\nabla k_\varepsilon(t)\big|^{r-2} \nabla k_\varepsilon(t)\cdot\nabla z + \big|k_\varepsilon(t)\big|^{r-2} k_\varepsilon(t) z \Big) \mathrm{d}x = \varepsilon(\varkappa(t))^{r-1} \int_\Omega z\,\mathrm{d}x \quad \text{for all } z \in W^{1,r}_{\rm per}(\Omega) \tag{5.8c}$$

We notice that the set $\mathcal{N} \subset [\,0,T\,]$ of measure zero of those $t$ where Equation (5.8) fails, does not depend on $(\boldsymbol{w},\psi,z)$. More specifically, if $\varepsilon = \varepsilon_m > 0$ with $\lim_{m\to\infty} \varepsilon_m = 0$, then $\mathcal{N}$ can be chosen independently of $m$.

The variational identities in Equation (5.8) are the point of departure for the proof of a series of the a priori estimates for $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ we are going to derive in Sections 5.2–5.4.

## 5.2 | Upper and lower bounds for $\{\omega_\varepsilon, k_\varepsilon\}$

Let $\underline{\omega}$, $\overline{\omega}$, and $\varkappa$ be as in Equation (5.1) and $r > 3$ as chosen in Section 5.1. The following result provides pointwise upper and lower bounds that are obtained via classical comparison arguments for weak solutions of the scalar parabolic equations for $\omega$ and $k$, cf. Equations (1.1c) and (1.1d), respectively.

**Lemma 5.2.** *Let be $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ a triple according to Proposition 5.1 with $r > 3$. Then,*

$$\underline{\omega}(t) \le \omega_\varepsilon(x,t) \le \overline{\omega}(t) \quad \text{and} \quad \varkappa(t) \le k_\varepsilon(x,t) \tag{5.9}$$

*for a.a. $(x,t) \in Q$ and for all $\varepsilon > 0$.*

*Proof.* For notational simplicity, we set $\boldsymbol{u} \equiv \boldsymbol{u}_\varepsilon$, $\omega \equiv \omega_\varepsilon$ and $k \equiv k_\varepsilon$ within this proof.

Step 1: $\omega \ge \underline{\omega}$. The function $\psi = (\omega(\cdot,t) - \underline{\omega}(t))^-$ is an admissible test function for Equation (5.8b). Since $\underline{\omega}(t)$ does not depend on $x$, we have $\frac{1}{2}\nabla(\psi^2) = \psi\nabla\omega$ and $\nabla\omega \cdot \nabla\psi = |\nabla\psi|^2 \ge 0$. Using $\underline{\omega} > 0$ and the monotonicity of $\omega \mapsto |\omega|^{r-2}\omega$ we arrive at

$$\left\langle \omega'(t), \left(\omega(t) - \underline{\omega}(t)\right)^- \right\rangle_{W^{1,r}_{\text{per}}} + \int_\Omega \omega^2 \left(\omega - \underline{\omega}(t)\right)^- \mathrm{d}x \le \varepsilon \int_\Omega \left( \left(\underline{\omega}(t)\right)^{r-1} - |\omega|^{r-2}\omega \right) \left(\omega - \underline{\omega}(t)\right)^- \mathrm{d}x \le 0 \tag{5.10}$$

for a.a. $t \in [0,T]$. By construction, we have $\underline{\omega}'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega}(t) = -(\underline{\omega}(t))^2$. Identifying $\underline{\omega}$ with a function in $C^1([0,T]; W^{1,r}_{\text{per}}(\Omega))$, the estimate (5.10) leads to

$$\left\langle \omega'(t) - \underline{\omega}'(t), \left(\omega(t) - \underline{\omega}(t)\right)^- \right\rangle_{W^{1,r}_{\text{per}}} \le - \int_\Omega \left( \omega^2 - \left(\underline{\omega}(t)\right)^2 \right) \left(\omega - \underline{\omega}(t)\right)^- \mathrm{d}x \le 0. \tag{5.11}$$

By Equations (5.1) and (5.4b), we have $\omega(x,0) - \underline{\omega}(0) \ge 0$, which means $\psi(x,0) = 0$ for a.a. $x \in \Omega$. Using a slight modification of pp. 290–291 in Lions [39], we find

$$\int_\Omega \frac{1}{2}\psi(t)^2 \,\mathrm{d}x = \int_\Omega \frac{1}{2}\psi(0)^2 \,\mathrm{d}x + \int_0^t \langle \psi', \psi \rangle_{W^{1,r}_{\text{per}}} \,\mathrm{d}t = 0 + \int_0^t \langle \omega' - \underline{\omega}', (\omega - \underline{\omega})^- \rangle_{W^{1,r}_{\text{per}}} \,\mathrm{d}t \le 0. \tag{5.12}$$

Hence, we conclude $\psi(t) = 0$ for all $t$, which means that

$$\omega(x,t) \ge \underline{\omega}(t) \quad \text{for a.a. } (x,t) \in Q. \tag{5.13}$$

Step 2: $\omega \le \overline{\omega}$. Next, we insert $\psi = (\omega(\cdot,t) - \overline{\omega}(t))^+$ in Equation (5.8b) and argue as in Step 1:

$$\left\langle \omega', (\omega - \overline{\omega})^+ \right\rangle_{W^{1,r}_{\text{per}}} + \int_\Omega \omega^2 \left(\omega - \overline{\omega}\right)^+ \mathrm{d}x \le \varepsilon \int_\Omega \left( (\underline{\omega})^{r-1} - \omega^{r-1} \right) \left(\omega - \overline{\omega}\right)^+ \mathrm{d}x \le 0. \tag{5.14}$$

For the last estimate, we used $\omega \ge \underline{\omega}$, which was obtained in Step 1. Hence, as above

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2}\psi(t)^2 \,\mathrm{d}x = \left\langle \omega'(t) - \dot{\overline{\omega}}(t), \left(\omega(t) - \overline{\omega}(t)\right)^+ \right\rangle_{W^{1,r}_{\text{per}}} \le - \int_\Omega \left( \omega^2 - \overline{\omega}^2 \right) \left(\omega - \overline{\omega}\right)^+ \mathrm{d}x \le 0 \tag{5.15}$$

for a.a. $t \in [0,T]$. Again by Equations (5.1) and (5.4b), we have $\psi(0) = 0$ a.e. in $\Omega$ and conclude

$$\omega(x,t) \le \overline{\omega}(t) \quad \text{for a.a. } (x,t) \in Q. \tag{5.16}$$

Step 3: $k \ge \varkappa$. We first insert $z = k^-(\cdot,t)$ into Equation (5.8c) and find $k \ge 0$ a.e. in $Q$. Next, we insert the test function $z(x,t) = (k(x,t) - \varkappa(t))^-$ and obtain as above

$$\left\langle k'(t), (k(t) - \varkappa(t))^- \right\rangle_{W^{1,r}_{\text{per}}} + \alpha_2 \int_\Omega k(t)\omega(t)(k - \varkappa(t))^- \,\mathrm{d}x \le 0 \tag{5.17}$$

for a.a. $t \in [0,T]$. By construction $\varkappa$ satisfies $\varkappa'(t) = -\alpha_2\varkappa(t)\overline{\omega}(t)$ for all $t \in [0,T]$. It follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\big((k(t){-}\varkappa(t))^-\big)^2\,\mathrm{d}x = \big\langle k'(t){-}\dot\varkappa(t), (k(t){-}\varkappa(t))^-\big\rangle_{W^{1,r}_{\mathrm{per}}} \le -\int_\Omega \alpha_2\big(k(t)\omega(t) - \varkappa(t)\overline{\omega}(t)\big)(k(t){-}\varkappa(t))^-\,\mathrm{d}x \;\le\; 0. \tag{5.18}$$

To see the last inequality, we use $\omega \le \overline{\omega}$ a.e. in $Q$ from Step 2, which gives $k(x,t)\omega(x,t) \le \varkappa(t)\overline{\omega}(t)$ for a.a. $x$ of the set $\{x \in \Omega\,;\ k(x,t) \le \varkappa(t)\}$. Since $k(x,0) \ge \varkappa(0)$ for a.a. $x \in \Omega$ by Equations (5.1) and (5.4b) we obtain, as above, $k(x,t) \ge \varkappa(t)$ for a.a. $(x,t) \in Q$. Altogether the upper and lower bounds in Equation (5.9) are established. □

## 5.3 | Energy estimates for $(\boldsymbol{u}_\varepsilon, \omega_\varepsilon)$ and improved estimates for $k_\varepsilon$

For the subsequent estimates, we fix the data

$$\mathfrak{D} = \{T,\ \boldsymbol{f},\ \omega_*,\ \omega^*,\ k_*,\ r\} \tag{5.19}$$

and will indicate constants that only depend on $\mathfrak{D}$ by $C_\mathfrak{D}$. However, depending on the context, the constants $C_\mathfrak{D}$ may be different. We also define the constant

$$\beta_* = \frac{k_*}{(1+\omega^*)(1+T\omega^*)^{\alpha_2}}, \tag{5.20}$$

which according to Lemma 5.2 is a lower bound for $k_\varepsilon/(\varepsilon{+}\omega_\varepsilon)$. This allows us to derive the standard estimates for $\boldsymbol{u}_\varepsilon$ and $\omega_\varepsilon$.

**Lemma 5.3.** *There exists a constant $C_\mathfrak{D} > 0$ such for all $\varepsilon \in \,]0,1]$ and all solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ as in Proposition 5.1 we have the estimates*

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(\boldsymbol{L}^2)} + \int_Q\Big(\beta_* + \frac{k_\varepsilon}{\varepsilon{+}\omega_\varepsilon}\Big)\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_Q\Big(\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^r{+}|\boldsymbol{u}_\varepsilon|^r\Big)\,\mathrm{d}x\,\mathrm{d}t \le C_\mathfrak{D}\Big(\|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2} + \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2}\Big), \tag{5.21a}$$

$$\|\omega_\varepsilon\|^2_{L^\infty(L^2)} + \int_Q\Big(\beta_* + \frac{k_\varepsilon}{\varepsilon{+}\omega_\varepsilon}\Big)|\nabla\omega_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_Q(|\nabla\omega_\varepsilon|^r + \omega_\varepsilon^r)\,\mathrm{d}x\,\mathrm{d}t \le C_\mathfrak{D}\Big(1 + \|\omega_{0,\varepsilon}\|^2_{L^2}\Big). \tag{5.21b}$$

*Proof.* We insert the test functions $\boldsymbol{w} = \boldsymbol{u}_\varepsilon$ and $\psi = \omega_\varepsilon$ in Equations (5.8a) and (5.8b), respectively. Integrating over $[0,t]$ and using $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon} \ge \beta_*$ a.e. in $Q$ (cf. Equation (5.9)), the desired estimates (5.21) are readily obtained by the aid of Gronwall's lemma. □

By Equation (5.4), the approximative initial conditions satisfy $\sup_{0<\varepsilon\le 1}(\|\boldsymbol{u}_{0,\varepsilon}\|_{\boldsymbol{L}^2}{+}\|\omega_{0,\varepsilon}\|_{L^2}) < \infty$. Therefore, all terms on the left-hand sides of Equation (5.21) are bounded independently of $\varepsilon \in \,]0,1]$.

Of course, one obtains a trivial bound for $k_\varepsilon$ in $L^\infty(0,T;L^1(\Omega))$ by testing Equation (5.8c) with $z \equiv 1$. We include this result in the following nontrivial estimate that implies uniform higher integrability of $k_\varepsilon$ as well as suitable bounds for $\nabla k_\varepsilon$. For this, we test Equation (5.8c) by $z = 1 - (1{+}k_\varepsilon)^{-\delta}$ for $\delta \in \,]0,1[$, which is a well-known technique for treating diffusion equations with an $L^1$ right-hand side, see, for example, Refs. [40–42].

**Proposition 5.4.** *For given data $\mathfrak{D}$, $p \in [1,2[$, and $\delta \in \,]0,1[$, there exists $C^{p,\delta}_\mathfrak{D} > 0$ such that for all $\varepsilon \in \,]0,1]$ and all $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ as in Proposition 5.1, we have the estimate*

$$\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \int_Q\Big(k_\varepsilon^{4p/3} + |\nabla k_\varepsilon|^p + \frac{|\nabla k_\varepsilon|^2}{(1{+}k_\varepsilon)^\delta} + \frac{\varepsilon|\nabla k_\varepsilon|^r}{(1{+}k_\varepsilon)^{1+\delta}} + \varepsilon k_\varepsilon^{r-1}\Big)\,\mathrm{d}x\,\mathrm{d}t \le C^{p,\delta}_\mathfrak{D}\Big(1{+}\|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2(\Omega)}{+}\|k_{0,\varepsilon}\|_{L^1(\Omega)}\Big). \tag{5.22}$$

*Proof.* Step 1: For $0 < \delta < 1$, we define $\Phi : [0,\infty[ \to [0,\infty[$ via

$$\Phi(\tau) = \tau + \frac{1}{1-\delta}\big(1-(1+\tau)^{1-\delta}\big), \quad 0 \le \tau < \infty. \tag{5.23}$$

Hence, $\Phi$ is convex and satisfies, for all $\tau \ge 0$, the estimates

$$\frac{\tau}{2} - \frac{2}{1-\delta} \le \Phi(\tau) \le \tau, \quad \Phi'(\tau) = 1 - \frac{1}{(1+\tau)^{\delta}} \in [0,1], \quad \Phi''(\tau) = \frac{\delta}{(1+\tau)^{1+\delta}}. \tag{5.24}$$

From pp. 360–361 and 365–366 in Rakotoson [43] (with $W^{1,p}_{\mathrm{per}}(\Omega)$ in place of $W^{1,p}_0(\Omega)$) we have the chain rule

$$\int_0^t \big\langle k_\varepsilon'(s), \Phi'(k_\varepsilon(s))\big\rangle_{W^{1,r}_{\mathrm{per}}} \,\mathrm{d}s = \int_\Omega \Phi(k_\varepsilon(x,t))\,\mathrm{d}x - \int_\Omega \Phi\big(k_{0,\varepsilon}(x)\big)\,\mathrm{d}x \tag{5.25}$$

for all $t \in [\,0,T\,]$. Using $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$ we obtain

$$\int_\Omega \Phi'(k_\varepsilon(\cdot,t))\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_\varepsilon(t)\cdot\nabla(\Phi(k_\varepsilon(\cdot,t))\,\mathrm{d}x = 0 \quad \text{for a.a. } t\in[0,T]. \tag{5.26}$$

Inserting $z = \Phi'(k_\varepsilon(\cdot,t))$ into Equation (5.8c) and using the last relation, we find (recall $\nu_0 = 1 = \alpha_2$)

$$\int_\Omega \Phi(k_\varepsilon(x,t))\,\mathrm{d}x + \delta\int_0^t\int_\Omega \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\,\frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{1+\delta}}\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_0^t\int_\Omega\left(\delta\,\frac{|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{1+\delta}} + k_\varepsilon^{r-1}\left(1-\frac{1}{(1+k_\varepsilon)^{\delta}}\right)\right)\mathrm{d}x\,\mathrm{d}s \tag{5.27}$$

$$= \int_\Omega \Phi\big(k_{0,\varepsilon}(x)\big)\,\mathrm{d}x + + \int_0^t\int_\Omega\left(\varepsilon(\varkappa(s))^{r-1} + \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\,\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 - k_\varepsilon\omega_\varepsilon\right)\left(1-\frac{1}{(1+k_\varepsilon)^{\delta}}\right)\mathrm{d}x\,\mathrm{d}s \tag{5.28}$$

for all $t\in[\,0,T\,]$. By Equations (5.21a), (5.24), and $k_\varepsilon/((\varepsilon+\omega_\varepsilon)(1+k_\varepsilon)) \ge 1/(1+\overline{\omega}(T)) > 0$, we find

$$\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \delta\int_Q \frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{\delta}}\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\delta\int_Q\frac{|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{1+\delta}}\,\mathrm{d}x\,\mathrm{d}s + \varepsilon\int_Q k_\varepsilon^{r-1}\,\mathrm{d}x\,\mathrm{d}t$$

$$\le c\left(\frac{1}{1-\delta} + \|\boldsymbol{u}_{0,\varepsilon}\|^2_{\boldsymbol{L}^2} + \|k_{0,\varepsilon}\|_{L^1} + \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2} + k_*^{r-1}\right), \tag{5.29}$$

where the constant $c$ is independent of $\delta$ and $\varepsilon$. Thus, we have estimated all the terms on the left-hand side of Equation (5.22) except for the second and third.

Step 2: To estimate $\nabla k_\varepsilon$, we choose $p\in\,]1,2[$ and $\delta = (2-p)/p \in\,]0,1[$. With Hölder's inequality, we find

$$\int_Q |\nabla k_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{|\nabla k_\varepsilon|^p}{(1+k_\varepsilon)^{p\delta/2}}(1+k_\varepsilon)^{p\delta/2}\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_Q\frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{\delta}}\,\mathrm{d}x\,\mathrm{d}t\right)^{p/2}\left(\int_Q (1+k_\varepsilon)^{\delta p/(2-p)}\,\mathrm{d}x\,\mathrm{d}t\right)^{(2-p)/2} \tag{5.30}$$

$$\le \frac{1}{\delta^{p/2}}\left(\delta\int_Q\frac{|\nabla k_\varepsilon|^2}{(1+k_\varepsilon)^{\delta}}\,\mathrm{d}x\,\mathrm{d}t\right)^{p/2} T\big(|\Omega|+\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}\big). \tag{5.31}$$

Using Equation (5.29), this provides the estimate for the third term on the left-hand side of Equation (5.22).

Step 3: To show higher integrability of $k_\varepsilon$, we simply use the Gagliardo–Nirenberg interpolation from Lemma 4.1 for $z\in W^{1,p}(\Omega)$ with $\Omega\subset\mathbb{R}^3$ where $p\in[1,2[$ as in Step 2:

$$\|z\|_{L^{4p/3}(\Omega)} \le C_{\mathrm{GN}}\|z\|^{1/4}_{L^1(\Omega)}\big(\|z\|_{L^1(\Omega)} + \|z\|_{L^p(\Omega)}\big)^{3/4}, \tag{5.32}$$

Applying this to $z = k_\varepsilon(t)$, taking the power $4p/3$, and integrating $t \in [0,T]$ we obtain

$$\int_Q |k_\varepsilon|^{4p/3}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \|k_\varepsilon(t)\|^{4p/3}_{L^{4p/3}(\Omega)}\,\mathrm{d}t \le C_{\mathrm{GN}}^{4p/3}\int_0^T K_\varepsilon^{p/3}\big(K_\varepsilon + \|\nabla k_\varepsilon(t)\|_{L^p(\Omega)}\big)^p\,\mathrm{d}t, \tag{5.33}$$

where $K_\varepsilon := \|k_\varepsilon(\cdot)\|_{L^\infty(L^1(\Omega))} \le C < \infty$ by Step 1. Hence, together with Step 2, the second term on the left-hand side of Equation (5.22) is uniformly bounded by the right-hand side of Equation (5.22).

In summary, the desired a priori estimate (5.22) is established. □

## 5.4 | Estimates for $\{\boldsymbol{u}_\varepsilon', \omega_\varepsilon', k_\varepsilon'\}$

We now provide a priori estimates on the time derivative. To obtain estimates that are independent of $\varepsilon \in ]0,1]$, we recall $r \ge 3$ and will use $\sigma > r$ and estimate in the dual space of $W^{1,\sigma}(\Omega)$. While for $\boldsymbol{u}_\varepsilon'$ and $\omega_\varepsilon'$, we obtain estimates in spaces $L^q(0,T;((W^{1,\sigma}(\Omega))^*)$ with $q > 1$, the time derivative $k_\varepsilon'$ can be estimated only for $q = 1$, because of the source term $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2$, for which the only $\varepsilon$-independent a priori estimate is in $L^1(Q) = L^1(0,T;L^1(\Omega))$. This problem will result in the occurrence of the defect measure $\mu$. The estimates for $\boldsymbol{u}_\varepsilon'$ and $\omega_\varepsilon'$ will work for arbitrary $r \ge 3$, however, for the estimate of $k_\varepsilon'$, we need to restrict $r$ to the small interval $[3, 11/3[$. Here the upper bound $r < 11/3$ seems to be critical for $N = 3$, while $2 < r < 3$ might still be considered.

**Proposition 5.5.** *Let $\mathfrak{D}$ be fixed.*

*(A) For all $r \ge 3$ (implying $r' = r/(r-1) \le 3/2$) and $\sigma > r$, there exists a constant $C_1$ such that for all $0 < \varepsilon \le 1$ the solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ of Proposition 5.1 satisfy the estimates*

$$\|\boldsymbol{u}_\varepsilon'\|_{L^{r'}(0,T;(\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega))^*)} + \|\omega_\varepsilon'\|_{L^{r'}(0,T;(W^{1,\sigma}_{\mathrm{per}}(\Omega))^*)} \le C_1. \tag{5.34}$$

*(B) For all $r \in [3, 11/3[$ and $\sigma > 8r/(11-3r)$, there exists a constant $C_2$ such that for all $0 < \varepsilon \le 1$, the solutions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ of Proposition 5.1 satisfy*

$$\|k_\varepsilon'\|_{L^1(0,T;(W^{1,\sigma}_{\mathrm{per}})^*)} \le C_2. \tag{5.35}$$

*Proof.* Step 1. Estimate for $\boldsymbol{u}_\varepsilon'$: For $\boldsymbol{w} \in \boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)$, we write Equation (5.8a) in the form

$$\big\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{w}\big\rangle_{\boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}} = \big\langle \boldsymbol{u}_\varepsilon'(t), \boldsymbol{w}\big\rangle_{\boldsymbol{W}^{1,r}_{\mathrm{per,div}}} = \int_\Omega (\boldsymbol{u}_\varepsilon(t)\otimes\boldsymbol{u}_\varepsilon(t)) : \nabla\boldsymbol{w}\,\mathrm{d}x - \nu_0\int_\Omega \frac{k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)}\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t)) : \boldsymbol{D}(\boldsymbol{w})\,\mathrm{d}x \tag{5.36}$$

$$- \varepsilon\int_\Omega \Big(\big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t))\big|^{r-2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t)) : \boldsymbol{D}(\boldsymbol{w}) + \big|\boldsymbol{u}_\varepsilon(t)\big|^{r-2}\boldsymbol{u}_\varepsilon(t)\cdot\boldsymbol{w}\Big)\,\mathrm{d}x + \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{w}\,\mathrm{d}x =: \sum_{m=1}^4 I_{\varepsilon,m}(t)$$

for a.a. $t \in [0,T]$. The aim is to show $|I_{\varepsilon,m}(t)| \le f_{\varepsilon,m}(t)\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}(\Omega)}$ with $f_{\varepsilon,m}$ bounded in $L^{\overline{q}_m}(0,T)$ for some $\overline{q}_m \ge r/(r-1)$. For this, we proceed as in Remark 4.2, but use now that $\boldsymbol{w} \in \boldsymbol{W}^{1,\sigma}_{\mathrm{per,div}}(\Omega)$ is fixed.

For $I_{\varepsilon,1}$, we use $\nabla\boldsymbol{w} \in \boldsymbol{L}^\sigma(\Omega)$ and need to bound $|\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon| \le |\boldsymbol{u}_\varepsilon|^2$ in $L^{\sigma'}(\Omega)$, which means $\boldsymbol{u}_\varepsilon \in \boldsymbol{L}^p(\Omega)$ with $p = 2\sigma/(\sigma-1)$. For this, we use the bounds (5.21a) for $\boldsymbol{u}_\varepsilon$, which allow us to apply Lemma 4.1(B) with $(s_1, p_1) = (\infty, 2)$, $(s_2, p_2) = (2, 2)$, $N = 3$, and $\theta = 3/(2\sigma) < 1/2$. This provides the desired $p = 2\sigma/(\sigma-1)$ and $\overline{q}_1 = s = 4\sigma/3$.

To estimate $I_{\varepsilon,2}$, we use $\varepsilon + \omega_\varepsilon(x,t) \ge \underline{\omega}(T) > 0$ and need to bound

$$|k_\varepsilon\boldsymbol{D}(\boldsymbol{u}_\varepsilon)| = k_\varepsilon^{1/2}\;|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)| \quad \text{in } L^{\overline{q}_2}(0,T;L^{\sigma'}(\Omega)). \tag{5.37}$$

By Equation (5.21a), we have a uniform bound for $|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|$ in $L^2(Q) = L^2(0,T;L^2(\Omega))$. Moreover, Equation (5.22) provides uniform bounds for $\|k_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))}$ and for $\|\nabla k_\varepsilon\|_{L^p(Q)}$ with $p \in [1,2[$. Hence, restricting to $\overline{q}_2 \in [1,2]$, we proceed as follows:

$$\|k_\varepsilon \boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|^{\overline{q}_2}_{L^{\overline{q}_2}(0,T;L^{\sigma'}(\Omega))} \le \int_0^T \Big( \|k_\varepsilon^{1/2}\|_{L^{2\sigma/(\sigma-2)}} \|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|_{L^2}\Big)^{\overline{q}_2} \,\mathrm{d}t$$

$$\le \int_0^T \|k_\varepsilon\|^{\overline{q}_2/2}_{L^{\sigma/(\sigma-2)}} \|k_\varepsilon^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\|^{\overline{q}_2}_{L^2}\,\mathrm{d}t \overset{\text{Hölder}}{\le} \left(\int_0^T \|k_\varepsilon\|^{\overline{q}_2/(2-\overline{q}_2)}_{L^{\sigma/(\sigma-2)}}\,\mathrm{d}t\right)^{(2-\overline{q}_2)/2} \left(\int_Q k_\varepsilon |\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}t\right)^{\overline{q}_2/2}. \tag{5.38}$$

The second term in the last product is already uniformly bounded. To estimate the first term, we apply Lemma 4.1(B) with $(s_1,p_1)=(\infty,1)$, $s_2=p_2\in[1,2[$, $N=3$, and $\theta=6p_2/((4p_2-3)\sigma)\in\,]0,1[$, where we use $\sigma>r\ge 3$ such that $p_2$ can be chosen close to 2. From the interpolation condition (4.14), we obtain the range of possible $\overline{q}_2$ via

$$\frac{2}{\overline{q}_2}-1=\frac{2-\overline{q}_2}{\overline{q}_2}=\frac{1}{s}=(1-\theta)\frac{1}{s_1}+\theta\frac{1}{s_2}=0+\theta\frac{1}{p_2}=\frac{6}{(4p_2-3)\sigma}. \tag{5.39}$$

Thus, we are able to choose all $\overline{q}_2\in[1,10\sigma/(5\sigma+6)[$ by adjusting $p_2$ suitably. As $\sigma>r\ge 3$ we see that $\overline{q}_2=3/2$ is always admissible.

Using $\sigma\ge r\ge 3$ and Hölder's inequality, we obtain

$$\big|I_{\varepsilon,3}(t)\big| \le f_{\varepsilon,3}(t)\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}} \quad \text{with } f_{\varepsilon,3}(t)=C\varepsilon\big\|\boldsymbol{u}_\varepsilon(t)\big\|^{r-1}_{\boldsymbol{W}^{1,r}}. \tag{5.40}$$

By the uniform bound (5.21a), we obtain $\|f_{\varepsilon,3}\|_{L^{r'}(0,T)}\le C_*\varepsilon^{1/(r-1)}$ with a constant $C_*$ independent of $\varepsilon$. Thus, we can choose $\overline{q}_3=r'=r/(r-1)\le 3/2$.

With $|I_{\varepsilon,4}(t)|\le\|\boldsymbol{f}(t)\|_{L^2}\|\boldsymbol{w}(t)\|_{\boldsymbol{L}^2}\le C\|\boldsymbol{f}(t)\|_{L^2}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\sigma}}$ and $\boldsymbol{f}\in\boldsymbol{L}^2(Q)=L^2(0,T;\boldsymbol{L}^2(\Omega))$, we obtain $\overline{q}_4=2$, and conclude that in all cases, we have $\overline{q}_m\ge r'=r/(r-1)$ and the first part of Equation (5.34) is established.

<u>Step 2. Estimate for $\omega_\varepsilon'$:</u> We proceed as in Step 1 by writing Equation (5.8b) in the form

$$\big\langle \omega_\varepsilon'(t),\psi\big\rangle_{W^{1,\sigma}}=\sum_{m=1}^{5} J_{\varepsilon,m}(t) \quad \text{with } |J_{\varepsilon,m}(t)|\le g_{\varepsilon,m}(t)\|\psi\|_{W^{1,\sigma}}, \tag{5.41}$$

where $g_{\varepsilon,m}$ has to be bounded in $L^{\widetilde{q}_m}(0,T)$ for suitable $\widetilde{q}_m\ge r'=r/(r-1)$. Exploiting Lemma 5.2, namely $0<\underline{\omega}(T)\le\omega_\varepsilon(x,t)\le\overline{\omega}(0)=\omega^*$ and Equation (5.21b) and proceeding as in Step 1 we easily find $\widetilde{q}_1=\widetilde{q}_3=\widetilde{q}_5=\infty$, $\widetilde{q}_2=10\sigma/(5\sigma+6)\ge 3/2$, and $\widetilde{q}_4=r'\le 3/2$. Thus, the second part of Equation (5.34), and hence all of Equation (5.34), is established.

<u>Step 3. Estimate for $k_\varepsilon'$:</u> We again write

$$\big\langle k_\varepsilon'(t),z\big\rangle=-\int_\Omega\left(z\boldsymbol{u}_\varepsilon(t)\cdot\nabla k_\varepsilon(t)-\frac{k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)}\nabla k_\varepsilon(t)\cdot\nabla z+\frac{\nu_0 k_\varepsilon(t)}{\varepsilon+\omega_\varepsilon(t)+\varepsilon k_\varepsilon(t)}|\boldsymbol{D}(\boldsymbol{u}_\varepsilon(t))|^2 z-\alpha_2 k_\varepsilon(t)\omega_\varepsilon(t)z\right)\mathrm{d}x$$

$$-\int_\Omega\varepsilon\big(|\nabla k_\varepsilon(t)|^{r-2}\nabla k_\varepsilon(t)\cdot\nabla z+|k_\varepsilon(t)|^{r-2}k_\varepsilon(t)z\big)\,\mathrm{d}x+\varepsilon(\varkappa(t))^{r-1}\int_\Omega z\,\mathrm{d}x=:\sum_{m=1}^{7}K_{\varepsilon,m}(t)$$

and have to show that $K_{\varepsilon,m}(t)\le h_{\varepsilon,m}(t)\|z\|_{W^{1,\sigma}}$, where each $h_{\varepsilon,m}$ is bounded in $L^1(0,T)$ independently of $\varepsilon\in\,]0,1[$.

Before starting the estimates, we note that the condition $r\in[3,11/3[$ and $\sigma>8r/(11-3r)$ implies $\sigma>12$, which will be useful below.

For $m=1$, we integrate by parts using $\operatorname{div}\boldsymbol{u}_\varepsilon=0$ and obtain

$$|K_{\varepsilon,1}(t)|=\left|\int_\Omega k_\varepsilon\boldsymbol{u}_\varepsilon\cdot\nabla z\,\mathrm{d}x\right|\le h_{\varepsilon,1}(t)\|z\|_{W^{1,\sigma}} \quad \text{with } h_{\varepsilon,1}(t)=\|k_\varepsilon\boldsymbol{u}_\varepsilon\|_{L^{\sigma'}}. \tag{5.42}$$

Using Equation (5.21a) for $\boldsymbol{u}_\varepsilon$ and applying Lemma 4.1 with $(s_1,p_1)=(\infty,2)$, $(s_2,p_2)=(2,2)$, $N=3$, and $\theta=3/5$, we find $(s,p)=(10/3,10/3)$, which means that $\boldsymbol{u}_\varepsilon$ is uniformly bounded in $L^{10/3}(Q)$. Using the uniform bound (5.22) for $k_\varepsilon$ in

$L^q(Q)$ for all $q \in [1, 8/3[$, we can choose $q$ such that $\frac{1}{q} + \frac{3}{10} \leq 1/\sigma' < 1$ as $\sigma > 40/13$ and obtain

$$\int_0^T h_{\varepsilon,1}(t)\,\mathrm{d}t \leq \int_0^T C\|k_\varepsilon(t)\|_{L^q(\Omega)}\|\boldsymbol{u}_\varepsilon(t)\|_{L^{10/3}(\Omega)}\,\mathrm{d}t \leq C_T\|k_\varepsilon\|_{L^q(Q)}\|\boldsymbol{u}_\varepsilon\|_{L^{10/3}(Q)} \leq C_{T,1}. \tag{5.43}$$

For $m = 2$, we again use Equation (5.22) and $\sigma > 8$. Choosing $p \in [1, 2[$ with $3/(4p) + 1/p + 1/\sigma \leq 1$, Hölder's inequality gives

$$\int_0^T |K_{\varepsilon,2}(t)|\,\mathrm{d}t \leq \int_0^T \|k_\varepsilon\|_{L^{4p/3}}\|\nabla k_\varepsilon\|_{L^p}\|\nabla z\|_{L^\sigma}\,\mathrm{d}t \leq C_{T,2}\|k_\varepsilon\|_{L^{4p/3}(Q)}\|\nabla k_\varepsilon\|_{L^p(Q)}\|z\|_{W^{1,\sigma}}. \tag{5.44}$$

The case $m = 3$ follows easily as $\|z\|_{L^\infty(\Omega)} \leq C\|z\|_{W^{1,\sigma}}$ because $\sigma > N$. Together with the simple energy estimate (5.21a) (uniform boundedness of the dissipation), we obtain

$$\int_0^T |K_{\varepsilon,3}(t)|\,\mathrm{d}t \leq C\int_Q \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2\,\mathrm{d}x\,\mathrm{d}t\,\|z\|_{L^\infty} \leq C_3\|z\|_{W^{1,\sigma}}. \tag{5.45}$$

The case $m = 4$ is also trivial, since $|K_{\varepsilon,4}(t)| \leq C\|k_\varepsilon(t)\|\omega^*\|z\|_{L^\infty}$.

The most difficult term is $K_{\varepsilon,5}$ because we do not have an a priori bound on $\varepsilon|\nabla k_\varepsilon|^r$. We adapt the method developed in Step 2 of the proof of Proposition 5.4. Using

$$|K_{\varepsilon,5}(t)| \leq h_{\varepsilon,5}(t)\|z\|_{W^{1,\sigma}} \quad \text{with } h_{\varepsilon,5}(t) = \varepsilon\big\||\nabla k_\varepsilon(t)|^{r-1}\big\|_{L^{\sigma'}} \tag{5.46}$$

we proceed as follows:

$$\int_0^T h_{\varepsilon,5}\,\mathrm{d}t = \varepsilon\int_0^T \|\nabla k_\varepsilon(t)\|^{r-1}_{L^{(r-1)\sigma'}}\,\mathrm{d}t \leq \varepsilon T^{1/\sigma}\|\nabla k_\varepsilon\|^{r-1}_{L^{(r-1)\sigma'}(Q)} \leq \varepsilon T^{1/\sigma}\bigg(\int_Q \frac{|\nabla k_\varepsilon|^{(r-1)\sigma'}}{(1+k_\varepsilon)^\rho}(1+k_\varepsilon)^\rho\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/\sigma'} \tag{5.47}$$

for a $\rho > 0$ to be chosen appropriately. Applying Hölder's inequality with $p = r'/\sigma' > 1$ and using $\varepsilon = \varepsilon^{1/r}\varepsilon^{1/(p\sigma')}$, we continue

$$\leq \varepsilon^{1/r}T^{1/\sigma}\bigg(\int_Q \frac{\varepsilon|\nabla k_\varepsilon|^r}{(1+k_\varepsilon)^{p\rho}}\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/(p\sigma')}\bigg(\int_Q (1+k_\varepsilon)^{p'\rho}\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/(p'\sigma')}. \tag{5.48}$$

According to Equation (5.22), both integral terms are uniformly bounded if we can choose $\rho$ such that $p\rho \in ]1, 2]$ and $p'\rho < 8/3$. Writing $\varkappa = 1/p$, this means $\varkappa < \rho < \min\{2\varkappa, 8(1-\varkappa)/3\}$, which has solutions $\rho$ if and only if $\varkappa \in ]0, 8/11[$, that is, we need $p = r'/\sigma' > 11/8$, which in term can only be possible if $r' > 11/8$ or $r < 11/3$. Then, $p = r'/\sigma' > 11/8$ is equivalent to $\sigma > 8r/(11-3r)$. This explains the restriction for $r$ and $\sigma$ in Equation (5.35) and provides the $L^1$ bound $\int_0^T |K_{\varepsilon,5}(t)|\,\mathrm{d}t \leq \varepsilon^{1/r}C_{r,\sigma}\|z\|_{W^{1,\sigma}}$.

The estimate of $K_{\varepsilon,6}$ follows easily from Equation (5.22) using $r-1 \in [2, 8/3[$, which implies $\|k_\varepsilon\|_{L^{r-1}(Q)} \leq C$ and thus

$$\int_0^T |K_{\varepsilon,6}(t)|\,\mathrm{d}t \leq \int_0^T \varepsilon\|k_\varepsilon\|^{r-1}_{L^{r-1}}\,\mathrm{d}t\,\|z\|_{L^\infty} \leq \varepsilon C\|z\|_{W^{1,\sigma}}. \tag{5.49}$$

The case of $K_{\varepsilon,7}$ is trivial.

For later use in the limit passage $\varepsilon \to 0$, we note that

$$\int_0^T \big(|K_{\varepsilon,5}(t)| + |K_{\varepsilon,6}(t)| + |K_{\varepsilon,7}(t)|\big)\,\mathrm{d}t \;\leq\; \varepsilon^{1/r}C_{r,\sigma}\|z\|_{W^{1,\sigma}}. \tag{5.50}$$

Hence, the a priori estimate (5.35) for $k_\varepsilon'$ is established. □

## 5.5 Convergent subsequences

After having derived a series of a priori estimates, we are now able to choose weakly converging subsequences for $\varepsilon \to 0$. Of course, the major step is to identify the limits of the nonlinear terms. For simplicity, we now choose one fixed $r_* \in [3, 11/3[$ and a $\sigma_* > 12$, which implies that Parts (A) and (B) of Proposition 5.5 can be applied. From Equations (5.9), (5.21), (5.22), (5.34), and (5.35), we obtain a limit triple $\{\boldsymbol{u}, \omega, k\}$ with the properties

$$\left.\begin{aligned}
&\underline{\omega} \le \omega \le \overline{\omega} \text{ a.e. on } Q,\\
&\boldsymbol{u} \in L^2(0,T;\boldsymbol{W}^{1,2}(\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega)) \cap W^{1,r_*'}\Big(0,T;(\boldsymbol{W}^{1,\sigma_*}_{\text{per,div}}(\Omega))^*\Big),\\
&\omega \in L^\infty(Q) \cap L^2(0,T;W^{1,2}(\Omega)) \cap W^{1,r_*'}\Big(0,T;(W^{1,\sigma_*}_{\text{per}}(\Omega))^*\Big),\\
&k \in L^\infty(0,T;L^1(\Omega)) \cap L^{4p/3}(Q) \cap L^p(0,T;W^{1,p}_{\text{per}}(\Omega)) \cap \text{BV}(0,T;(W^{1,\sigma_*}_{\text{per}}(\Omega))^*\Big)
\end{aligned}\right\} \tag{5.51}$$

for all $p \in [1, 2[$, such that along a suitable subsequence (not relabeled) we have

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \ \text{ in } L^2\Big(0,T;\boldsymbol{W}^{1,2}_{\text{per,div}}(\Omega)\Big) \ \text{ and weakly* in } L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big), \tag{5.52a}$$

$$\boldsymbol{u}_\varepsilon' \rightharpoonup \boldsymbol{u}' \ \text{ in } L^{r_*'}\Big(0,T;(W^{1,\sigma_*}_{\text{per,div}}(\Omega))^*\Big), \tag{5.52b}$$

$$\omega_\varepsilon \rightharpoonup \omega \ \text{ in } L^2\Big(0,T;W^{1,2}_{\text{per}}(\Omega)\Big) \ \text{ and weakly* in } L^\infty(Q), \tag{5.52c}$$

$$\omega_\varepsilon' \rightharpoonup \omega' \ \text{ in } L^{r_*'}\Big(0,T;(W^{1,\sigma_*}_{\text{per}}(\Omega))^*\Big), \tag{5.52d}$$

$$k_\varepsilon \rightharpoonup k \ \text{ in } L^p\Big(0,T;W^{1,p}_{\text{per}}(\Omega)\Big) \text{ and in } L^{4p/3}(Q) \ \text{ for all } p \in [1,2[. \tag{5.52e}$$

These weak convergences imply the corresponding properties of the limits $\boldsymbol{u}$ and $\omega$ in Equation (5.51). Moreover, $\|k\|_{L^\infty(0,T;L^1(\Omega))} \le C < \infty$ follows from Equations (5.22) and (5.52e) by a routine argument. As in Sec. 1.3.2 in Barbu and Precupanu [44], the space $BV(0,T;X)$, where $X$ is a Banach space, denotes all functions $g : [0,T] \to X$ such that $\text{Var}_X(g,[a,b]) := \sup \sum_{i=1}^N \|g(t_i) - g(t_{i-1})\|_X < \infty$ where the supremum is taken over all finite partitions $a \le t_0 < t_1 < \cdots < t_N \le b$. Clearly, Equation (5.35) implies $\text{Var}_{(W^{1,\sigma}_{\text{per}})^*}(k_\varepsilon,[0,T]) = \|k_\varepsilon'\|_{L^1(0,T;(W^{1,\sigma}_{\text{per}})^*)} \le C_2$. Since for all partitions, we have

$$\sum_{i=1}^N \|k(t_i) - k(t_{i-1})\|_{(W^{1,\sigma}_{\text{per}})^*} \le \liminf_{\varepsilon\to 0} \sum_{i=1}^N \|k_\varepsilon(t_i) - k_\varepsilon(t_{i-1})\|_{(W^{1,\sigma}_{\text{per}})^*} \le C_2, \tag{5.53}$$

which provides $\|k\|_{\text{BV}(0,T;(W^{1,\sigma_*}_{\text{per}}(\Omega))^*)} \le C_2 < \infty$ as stated at the end of Equation (5.51).

We next apply the Aubin–Lions–Simon lemma (see Cor. 4, p. 85 in Simon [45], Thm. 5.1, p. 58 in Lions [39], or Lem. 7.7 in Roubíček [25]) to obtain strong convergence. By taking a further subsequence (not relabeled) Vitali's theorem implies the pointwise convergence almost everywhere.

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \ \text{ in } \boldsymbol{L}^s(Q) \ \text{ for all } s \in [1, 10/3[ \ \text{ and a.e. in } Q, \tag{5.54a}$$

$$\omega_\varepsilon \to \omega \quad \text{in } L^p(Q) \text{ for all } p > 1 \text{ and a.e. in } Q, \tag{5.54b}$$

$$k_\varepsilon \to k \quad \text{in } L^q(Q) \text{ for all } q \in [1, 8/3[ \text{ and a.e. in } Q, \tag{5.54c}$$

To obtain the results in Equations (5.54b) and (5.54c), we first derive strong convergence for $s = p = q = 2$ and then use the boundedness of the sequence for higher $s$, $p$, and $q$ to obtain strong convergence for intermediate values by Riesz interpolation (use Equation (4.21) for $\boldsymbol{u}_\varepsilon$).

We are now ready to consider also the limits of the nonlinear terms. We first treat the diffusive terms.

**Lemma 5.6.** *Along the chosen subsequences for $\varepsilon \to 0$, we have the convergences*

$$\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) \ \textit{and} \ \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\nabla\omega_\varepsilon \rightharpoonup \frac{k}{\omega}\nabla\omega \ \textit{in } \boldsymbol{L}^s(Q) \textit{ for all } s \in [1, 16/11[, \tag{5.55a}$$

$$\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\nabla k_\varepsilon \rightharpoonup \frac{k}{\omega}\nabla k \qquad \textit{in } \boldsymbol{L}^\sigma(Q) \textit{ for all } \sigma \in [1, 8/7[. \tag{5.55b}$$

*Proof.* We first recall the weak convergences of the gradients $\boldsymbol{D}(\boldsymbol{u}_\varepsilon)$, $\nabla\omega_\varepsilon$, and $\nabla k_\varepsilon$ in $L^p(Q)$ for all $p \in [1, 2[$, see Equation (5.52). Next, we establish the strong convergence

$$\left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2} \to \left(\frac{k}{\omega}\right)^{1/2} \quad \text{in } L^q(Q) \text{ for all } q \in [1, 16/3[. \tag{5.56}$$

To see this, we use the explicit estimate

$$\begin{aligned}\left\|\left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2} - \left(\frac{k}{\omega}\right)^{1/2}\right\|_{L^q(Q)} &\le \left\|\left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2} - \left(\frac{k}{\varepsilon+\omega_\varepsilon}\right)^{1/2}\right\|_{L^q(Q)} + \left\|\left(\frac{k}{\varepsilon+\omega_\varepsilon}\right)^{1/2} - \left(\frac{k}{\omega}\right)^{1/2}\right\|_{L^q(Q)} \\ &\le \frac{\|k_\varepsilon - k\|^{1/2}_{L^{q/2}(Q)}}{(1+\underline{\omega}(T))^{1/2}} + \frac{\|(\varepsilon+\omega_\varepsilon - \omega)\, k^{1/2}\|_{L^q(Q)}}{2(1+\underline{\omega}(T))^{3/2}}.\end{aligned} \tag{5.57}$$

Clearly, the first term on the right-hand side tends to 0 using Equation (5.54c) and $q/2 < 8/3$. For the second term, we can still choose $\widetilde{q} \in ]q, 16/3[$ and $\widetilde{p} \gg 1$ such that $1/q = 1/\widetilde{q} + 1/\widetilde{p}$. Then, Hölder's inequality, $k^{1/2} \in L^{\widetilde{q}}(Q)$, and Equation (5.54b) for $p = \widetilde{p}$ yield the convergence to 0. Hence, the convergence (5.56) is established.

Now using the weak convergences $\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \boldsymbol{D}(\boldsymbol{u})$ and $\nabla\omega_\varepsilon \rightharpoonup \nabla\omega$, and $\nabla k_\varepsilon \rightharpoonup \nabla k$ in $L^p(Q)$ for $p \in [1, 2[$ and Equation (5.56), we obtain the weak convergences

$$\left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon) \rightharpoonup \left(\frac{k}{\omega}\right)^{1/2}\boldsymbol{D}(\boldsymbol{u}), \quad \left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2}\nabla\omega_\varepsilon \rightharpoonup \left(\frac{k}{\omega}\right)^{1/2}\nabla\omega, \quad \left(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\right)^{1/2}\nabla k_\varepsilon \rightharpoonup \left(\frac{k}{\omega}\right)^{1/2}\nabla k \tag{5.58}$$

in $L^q(Q)$ for all $q \in [1, 16/11[$.

However, by the standard a priori estimates (5.21), we see that the first two sequences are bounded in $L^2(Q)$ and hence converge weakly in $L^2(Q)$ as well. The convergence of the third term cannot be improved, because we do not have appropriate a priori bounds.

Multiplying once again by $(k_\varepsilon/(\varepsilon+\omega_\varepsilon))^{1/2}$, which converges strongly according to Equation (5.56), we obtain the results in Equation (5.55). □

## 5.6 | Limit passage $\varepsilon \to 0$ and appearance of the defect measure

In this subsection, we finalize the proof of Theorem 4.1.

Using the convergences derived above, it is now straight forward to perform the limit passage $\varepsilon \to 0$ in the equation for $\boldsymbol{u}_\varepsilon$ and $\omega_\varepsilon$. In the energy equation for $k_\varepsilon$, we have to be a little more careful to show the occurrence of the defect measure $\mu$.

In the Steps 1–3, the limit $\varepsilon \to 0$ will be done with test functions with high integrability $\overline{s}$ in $t \in [0,T]$ taking values in the Sobolev $W^{1,\overline{\tau}}(\Omega)$ with large $\overline{\tau}$. This choice will be independent of the chosen $r_*$ in the regularization terms. After the artificial $r_*$ has disappeared in the limit, in Step 4, we discuss which minimal $\overline{s}$ and $\overline{\tau}$ can be chosen in the weak form.

Step 1. Limit in the momentum balance for $\boldsymbol{u}_\varepsilon$, from Equations (5.6a)–(4.8): We consider a fixed test function $\boldsymbol{v} \in L^{\overline{s}}(0,T;\boldsymbol{W}^{1,\overline{\tau}}_{\mathrm{per,div}}(\Omega))^*)$ with $\overline{s} = 4$ and $\overline{\tau} \geq s_* > 12$ and discuss the convergence of the five terms on the left-hand side of Equation (5.6a) individually.

The first term is linear in $\boldsymbol{u}_\varepsilon'$ and converges because of Equation (5.52b). The second term can be rewritten as $\int_\Omega (\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla \boldsymbol{v} \,\mathrm{d}x\,\mathrm{d}t$ and converges by Equation (5.54a).

For the third term, we use the nonlinear convergences from Lemma 5.6, cf. the first in Equation (5.55a). The fourth and fifth terms converge to 0 by the estimate $\int_0^T |I_{\varepsilon,3}(t)|\,\mathrm{d}t \leq C_* \varepsilon^{1/(r_*-1)} \|\boldsymbol{D}(\boldsymbol{v})\|_{L^{r_*}(\boldsymbol{L}^{\sigma_*})} \leq C\varepsilon^{1/(r_*-1)}\|\boldsymbol{v}\|_{L^{\overline{s}}(\boldsymbol{W}^{1,\overline{\tau}})}$, see Step 1 of the proof of Proposition 5.5.

Thus, Equation (4.8) is established for test functions $v \in L^{\overline{s}}(0,T;\boldsymbol{W}^{1,\overline{\tau}}_{\mathrm{per,div}}(\Omega))^*)$.

Step 2. Limit for $\omega_\varepsilon$, from Equation (5.6b) to Equation (4.9): This case works similar as Step 1.

Step 3. Limit in the energy equation for $k_\varepsilon$, from Equation (5.6c) to Equation (3.10): For this limit passage, we choose a test function $z \in C^1_{\mathrm{per},T}(\overline{Q})$,

because we want to take the limit of the dissipation, which is bounded only in $L^1(Q)$.

The first term of the left-hand side in Equation (5.6c) is integrated by parts in time to obtain

$$\int_0^T \left\langle k_\varepsilon'(t), z(t)\right\rangle_{W^{1,r}_{\mathrm{per}}} \mathrm{d}t = \int_\Omega k_{0,\varepsilon} z(\cdot,0)\,\mathrm{d}x - \int_Q k_\varepsilon z'\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_\Omega k_0 z(\cdot,0)\,\mathrm{d}x - \int_Q kz'\,\mathrm{d}x\,\mathrm{d}t \tag{5.59}$$

by Equations (5.4c) and (5.52e). For the second term, we use Equation (5.54) and conclude

$$\int_Q z\boldsymbol{u}_\varepsilon\cdot\nabla k_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = -\int_Q k_\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \nabla z\,\mathrm{d}x\,\mathrm{d}t \;\to\; -\int_Q k\boldsymbol{u}\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t. \tag{5.60}$$

For the third term, Lemma 5.6 can be exploited (cf. Equation (5.55a)) to find

$$\int_Q \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\nabla k_\varepsilon\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_Q \frac{k}{\omega}\nabla k\cdot\nabla z\,\mathrm{d}x\,\mathrm{d}t. \tag{5.61}$$

We return to the fourth term at the end and continue with the fifth term. Using Equation (5.54) and $\omega_\varepsilon^+ = \omega_\varepsilon \geq \underline{\omega}(\cdot) > 0$, we easily find $\int_Q k_\varepsilon \omega_\varepsilon^+ z\,\mathrm{d}x\,\mathrm{d}t \to \int_Q k\omega z\,\mathrm{d}x\,\mathrm{d}t$.

The sixth and seventh terms on the left-hand side and the single term on the right-hand side converge to 0, which was established in Step 3 of the proof of Proposition 5.5, see Equation (5.50).

For the fourth term, it remains to prove the *appearance of the non-negative defect measure* $\mu \in \mathcal{M}_{\geq}(\overline{Q})$ such that

$$\int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^2 \phi\,\mathrm{d}x\,\mathrm{d}t \;\longrightarrow\; \int_Q \frac{\nu_0 k}{\omega}\left|\boldsymbol{D}(\boldsymbol{u})\right|^2\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{\overline{Q}} \phi\,\mathrm{d}\mu \quad \text{for all } \phi \in C(\overline{Q}). \tag{5.62}$$

Indeed, by the positivity of the integrand and the a priori estimate (5.21a), we can apply Riesz' Representation Theorem for linear continuous functionals on $C(\overline{Q})$. Hence, there exist $\widehat{\mu} \in \mathcal{M}_{\geq}(\overline{Q})$ such that

$$\int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon}\left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^2\phi\,\mathrm{d}x\,\mathrm{d}t \;\to\; \int_{\overline{Q}}\phi\,\mathrm{d}\widehat{\mu} \quad \text{for all } \phi \in C(\overline{Q}). \tag{5.63}$$

As in Lemma 5.6 ,we can show that $(\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon})^{1/2}\boldsymbol{D}(\boldsymbol{u}_\varepsilon)$ converges weakly to $(k/\omega)^{1/2}\boldsymbol{D}(\boldsymbol{u})$ in $\boldsymbol{L}^2(Q)$. Of course, this weak convergence remains true if we multiply by a continuous function $\psi \in C(\overline{Q})$. Thus, the lower semi-continuity of the $L^2$

norm yields

$$\int_Q \psi^2 \,\mathrm{d}\hat{\mu} = \lim_{\varepsilon\to 0} \int_Q \frac{\nu_0 k_\varepsilon}{\varepsilon + \omega_\varepsilon + \varepsilon k_\varepsilon} \left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t \geq \int_Q \frac{\nu_0 k}{\omega} \left|\boldsymbol{D}(\boldsymbol{u})\right|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t \tag{5.64}$$

for all $\psi \in C(\overline{Q})$. Thus, the linear functional $\phi \mapsto \int_Q \phi \,\mathrm{d}\hat{\mu} - \int_Q \frac{\nu_0 k}{\omega} |\boldsymbol{D}(\boldsymbol{u})|^2 \phi \,\mathrm{d}x\,\mathrm{d}t$ is non-negative and defines the desired defect measure $\mu \in \mathcal{M}_{\geq}(\overline{Q})$, and

$$\int_{\overline{Q}} \phi \,\mathrm{d}\hat{\mu} = \int_Q \frac{\nu_0 k}{\omega} \left|\boldsymbol{D}(\boldsymbol{u})\right|^2 \phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\overline{Q}} \phi \,\mathrm{d}\mu \quad \text{for all } \phi \in C(\overline{Q}), \tag{5.65}$$

which gives the desired convergence (5.62).

Step 4. More test functions: After having passed to the limit $\varepsilon \to 0$, the regularization terms involving the exponent $r$ have disappeared. From the a priori estimates (5.51) for $\{\boldsymbol{u}, \omega, k\}$, we know that $\boldsymbol{u}\otimes\boldsymbol{u} \in L^{5/3}(Q)$ and $\frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) \in L^q(Q)$ for all $q \in [1, 16/11[$. Thus, by density, we can extend the set of test function $\boldsymbol{v}$ in Equation (4.6) can be chosen in $L^{\overline{s}}(0,T; W^{1,\overline{\tau}}_{\mathrm{per,div}}(\Omega))$ for any $\overline{s} > 16/5$ and $\overline{\tau} > 16/5$. This proves Equations (4.8) and (4.9) for the full set of test functions.

Moreover, we find $\boldsymbol{u}' \in L^q((W^{1,q'}_{\mathrm{per,div}}(\Omega))^*)$ for all $q \in [1, 16/11[$, which proves Equation (4.6).

Step 5. Several further statements: To derive Equation (4.5), we define the functional $\mathcal{J} : (k, \boldsymbol{u}, \omega) \mapsto \int_Q k\,(|\boldsymbol{D}(\boldsymbol{u})|^2 + |\nabla\omega|^2)\,\mathrm{d}x\,\mathrm{d}t$ and use the a priori estimate $\mathcal{J}(k_\varepsilon, \boldsymbol{u}_\varepsilon, \omega_\varepsilon) \leq C$, which follows from Equation (5.21) since $\omega_\varepsilon \geq \underline{\omega}(T) > 0$. The functional is convex in $\boldsymbol{u}$ and $\omega$, hence, it is lower semicontinuous with respect to strong convergence in $k$ (see Equation (5.54c)) and weak convergence for $(\boldsymbol{u}, \omega)$ (see Equations (5.52a) and (5.52c)), so that

$$\mathcal{J}(k, \boldsymbol{u}, \omega) \leq \liminf_{\varepsilon\to 0} \mathcal{J}(k_\varepsilon, \boldsymbol{u}_\varepsilon, \omega_\varepsilon) \leq C, \tag{5.66}$$

which is the desired estimate (4.5). The limit passage $\varepsilon \to 0$ in the pointwise a priori estimates (5.9) leads immediately to the pointwise estimates (4.3) for $\omega$ and $k$.

By Equations (5.52b) and (5.52d), the functions $\boldsymbol{u}_\varepsilon(\cdot)$ and $\omega_\varepsilon$ are uniformly bounded with respect to $\varepsilon \in ]0,1]$ in $W^{1,r_*}(0,T; (W^{1,\sigma_*}(\Omega))^*) \subset C^{1/r_*}([0,T]; (W^{1,\sigma_*}(\Omega))^*)$. Thus, we have uniform convergence and obtain $(\boldsymbol{u}, \omega) \in C^{1/r_*}([0,T]; (\boldsymbol{W}^{1,\sigma_*}(\Omega))^* \times (W^{1,\sigma_*}(\Omega))^*)$. Together with the essential boundedness of $(\boldsymbol{u}, \omega)$ in $L^2(\Omega)\times L^2(\Omega)$, this implies

$$(\boldsymbol{u}, \omega) \in C_{\mathrm{w}}([0,T]; \boldsymbol{L}^2(\Omega)\times L^2(\Omega)). \tag{5.67}$$

Hence, Equation (4.4) is established. Moreover, with Equation (5.4c) and the uniform convergence, we deduce the initial conditions (4.7), that is, $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0$ and $\omega(\cdot,0) = \omega_0$.

Step 6. Energy estimates: To obtain the energy-dissipation inequality (3.11) for the Navier–Stokes equation, we insert $\boldsymbol{w} = \boldsymbol{u}_\varepsilon(t)$ into Equation (5.8a), integrate over the interval $[0,t]$, drop the non-negative term $\int_0^t \int_\Omega \varepsilon |\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^r \,\mathrm{d}x\,\mathrm{d}t$, and take the limit $\varepsilon \to 0$.

Finally, we insert $z \equiv 1$ into Equation (5.8c), integrate over $[0,t]$ and add this identity to the one just obtained for $\boldsymbol{u}_\varepsilon$. Using $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon} - \frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon+\varepsilon k_\varepsilon} \geq 0$, we can drop the two dissipation terms involving $|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)|^2$. Moreover, the regularization term $\int_\Omega \varepsilon |\nabla k_\varepsilon|^{r-2} \nabla k_\varepsilon \cdot \nabla z \,\mathrm{d}x$ with $z \equiv 1$ gives 0. Hence, taking the limit $\varepsilon \to 0$ yields inequality (3.12) for the total energy.

With this, the proof of our main existence result in Theorem 4.1 is complete.

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## CONFLICT OF INTEREST

The authors have declared no conflict of interest.

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## APPENDIX A: EXISTENCE OF APPROXIMATE SOLUTIONS

We now provide the proof of Proposition 5.1, which will be obtained as an application of a general existence result of evolutionary equations of pseudo-monotone type.

We consider a separable reflexive Banach space $\boldsymbol{V}$ that is continuously and densely embedded in a Hilbert space $\boldsymbol{H}$ such that $\boldsymbol{V} \subset \boldsymbol{H} \approx \boldsymbol{H}^* \subset \boldsymbol{V}^*$. For $U \in \boldsymbol{V}$ and $\Xi \in \boldsymbol{V}^*$, we denote the dual pairing by $\langle \Xi, U\rangle$. Our operator $A : \boldsymbol{V} \to \boldsymbol{V}^*$ is assumed to satisfy the following conditions depending on $p > 1$:

$$p\text{-boundedness:} \qquad \exists\, C_1 > 0 : \ \|A(U)\|_{\boldsymbol{V}^*} \le C_1\Big(1+\|U\|_{\boldsymbol{V}}^{p-1}\Big) \ \text{ for all } U \in \boldsymbol{V}; \tag{A.1a}$$

$$p\text{-coercivity:} \qquad \exists\, C_2 > 0 : \ \langle A(U), U\rangle \ge \frac{1}{C_2}\|U\|_{\boldsymbol{V}}^p - C_2 \quad \text{ for all } U \in \boldsymbol{V}; \tag{A.1b}$$

$$\text{pseudo-monotonicity:} \qquad \left\{ \begin{array}{l} \text{if } U_m \rightharpoonup U \text{ in } \boldsymbol{V} \text{ and } \limsup\limits_{m\to\infty}\langle A(U_m), U_m - U\rangle \le 0, \text{ then} \\ \langle A(U), U-V\rangle \le \liminf\limits_{m\to\infty}\langle A(U_m), U_m - V\rangle \text{ for all } V \in \boldsymbol{V}. \end{array} \right\} \tag{A.1c}$$

Under these conditions, the following existence result is available.

**Theorem A.1** see, e.g., Thm. 8.9 in Roubíček [25]. *Let $\boldsymbol{V}$ and $\boldsymbol{H}$ be as above and let the operator $A : \boldsymbol{V} \to \boldsymbol{V}^*$ satisfy the assumptions (A.1) with $p > 1$. Then, for all $T > 0$, all $u_0 \in \boldsymbol{H}$, and all $f \in L^{p'}([0,T]; \boldsymbol{V}^*)$, there exists a solution $u \in L^p(0,T;\boldsymbol{V}) \cap C([0,T];\boldsymbol{H}) \cap W^{1,p'}(0,T;\boldsymbol{V}^*)$ of the Cauchy problem*

$$u'(t) + A(u(t)) = f(t) \ \text{ in } \boldsymbol{V}^* \text{ for a.a. } t \in [0,T] \qquad \text{and} \qquad u(0) = u_0. \tag{A.2}$$

To apply this result, we choose $p = r > 3$, $U = (\boldsymbol{u}, \omega, k)$,

$$\boldsymbol{H} = \boldsymbol{L}^2_{\text{div}}(\Omega) \times L^2(\Omega) \times L^2(\Omega), \quad \text{and} \quad \boldsymbol{V} = \boldsymbol{W}^{1,r}_{\text{per,div}}(\Omega) \times W^{1,r}_{\text{per}}(\Omega) \times W^{1,r}_{\text{per}}(\Omega). \tag{A.3}$$

The operator $A$ is defined to make the approximate system (5.6) equivalent to the abstract Cauchy problem (A.2). We recall that $\varepsilon > 0$ is fixed in Proposition 5.1, so we do not keep track of the dependence on $\varepsilon$. With $V = (\boldsymbol{v}, \varphi, w)$, we define $A : V \to V^*$ by

$$\langle A(U), V\rangle = I(U,V)$$
$$:= \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + \int_\Omega \varphi\boldsymbol{u}\cdot\nabla\omega + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\nabla\omega\cdot\nabla\varphi \tag{A.4}$$

$$+ \int_\Omega \omega^+\omega\varphi + \int_\Omega w\boldsymbol{u}\cdot\nabla k + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\nabla k\cdot\nabla w - \int_\Omega \frac{k^+}{\varepsilon+\omega^++\varepsilon k^+}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 w + \int_\Omega k^+\omega^+ w$$

$$+ \varepsilon\int_\Omega \Big(\big|\boldsymbol{D}(\boldsymbol{u})\big|^{r-2}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v}) + |\boldsymbol{u}|^{r-2}\boldsymbol{u}\cdot\boldsymbol{v} + |\nabla\omega|^{r-2}\nabla\omega\cdot\nabla\varphi + |\omega|^{r-2}\omega\varphi + |\nabla k|^{r-2}\nabla k\cdot\nabla w + |k|^{r-2}kw\Big). \tag{A.5}$$

For the rest of this appendix, we continue to omit the measure symbol "$\mathrm{d}x$" for integration over $\Omega$. Moreover, we have set $\alpha_2 = \nu_0 = 1$ for notational simplicity, because these numerical constant have no influence on the analysis.

*Proof of Proposition* 5.1. It remains to establish the conditions (A.1) on the operator $A$.

Step 1. $r$-boundedness (A.1a): Using $r > 3$ and Hölder's inequality, it is easily seen that all integrals in the definition of $I(U,V)$ are well-defined. In particular, we find a constant $c_1 > 0$ such that

$$\big|I(U,V)\big| \le c_1\big(\|U\|_{\boldsymbol{V}}^2 + \|U\|_{\boldsymbol{V}}^{r-1}\big)\|V\|_{\boldsymbol{V}} \ \text{ for all } U,V\in\boldsymbol{V}. \tag{A.6}$$

But this implies Equation (A.1a) because of $r \ge 3$.

Step 2. $r$-coercivity (A.1b): For estimating $\langle A(U), U\rangle = I(U,U)$ from below, we see that all convective terms disappear because of $\operatorname{div}\boldsymbol{u} = 0$. After dropping the three non-negative terms arising from the dissipation terms involving $k^+/(\varepsilon+\omega^+)$, we find

$$\langle A(U), U\rangle = I(U,U) \ge \varepsilon\big\|(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{u}, \nabla\omega, \omega, \nabla k, k)\big\|_{L^r(\Omega)}^r - \int_\Omega \frac{k^+}{\varepsilon+\omega^++\varepsilon k^+}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k \tag{A.7}$$

for all $U \in \boldsymbol{V}$. We now use $k^+/(\varepsilon+\omega^++\varepsilon k^+) \le 1/\varepsilon$ and $r \gneq 3$. By Hölder's and Young's inequality, we find $c_2 > 0$ such that

$$\int_\Omega \frac{k^+}{\varepsilon+\omega^++\varepsilon k^+}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k \le \frac{1}{\varepsilon}\int_\Omega \big|\boldsymbol{D}(\boldsymbol{u})\big|^2 k \le \frac{\varepsilon}{2}\int_\Omega \big|\boldsymbol{D}(\boldsymbol{u})\big|^r + \frac{\varepsilon}{2}\int_\Omega |k|^r + c_2, \tag{A.8}$$

where the constant $c_2$ depends on $\varepsilon > 0$, $r > 3$, and $\mathrm{vol}(\Omega)$. Inserting this into Equation (A.7) and using Korn's inequality in $\boldsymbol{W}^{1,r}(\Omega)$, we have established Equation (A.1b) for $p = r$.

Step 3. Strong convergence: In the remaining two steps, we consider a sequence $U_m = (\boldsymbol{u}_m, \omega_m, k_m)$ satisfying the assumptions in condition (A.1c), namely

$$\text{(a)}\ \ U_m \rightharpoonup U \ \text{ in } \boldsymbol{V} \qquad\qquad \text{(b)}\ \limsup_{m\to\infty}\,\langle A(U_m), U_m - U\rangle \le 0. \tag{A.9}$$

In this step, we first show that this implies the strong convergence $U_m \to U$ in $\boldsymbol{V}$, and in Step 4, we deduce the liminf estimate for Equation (A.1c).

Combining parts (a) and (b) of Equation (A.9), we immediately obtain

$$\limsup_{m\to\infty}\,\big\langle A(U_m) - A(U)\,,\, U_m - U\big\rangle \le 0. \tag{A.10}$$

We decompose these duality products into ten separate integrals, namely

$$\langle A(U_m) - A(U), U_m - U \rangle = \sum_{j=1}^{10} K_{j,m} \tag{A.11}$$

$$\begin{aligned} := &\int_\Omega [\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m - \boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot(\boldsymbol{u}_m-\boldsymbol{u}) + \int_\Omega \left[\frac{k_m^+}{\varepsilon+\omega_m^+}\boldsymbol{D}(\boldsymbol{u}_m) - \frac{k^+}{\varepsilon+\omega^+}\boldsymbol{D}(\boldsymbol{u})\right]:\boldsymbol{D}(\boldsymbol{u}_m-\boldsymbol{u}) + \int_\Omega (\boldsymbol{u}_m\cdot\nabla\omega_m - \boldsymbol{u}\cdot\nabla\omega)(\omega_m-\omega) \\ &+ \int_\Omega \left[\frac{k_m^+}{\varepsilon+\omega_m^+}\nabla\omega_m - \frac{k^+}{\varepsilon+\omega^+}\nabla\omega\right]\cdot\nabla(\omega_m-\omega) + \int_\Omega (\omega_m^+\omega_m - \omega^+\omega)(\omega_m-\omega) + \int_\Omega (\boldsymbol{u}_m\cdot\nabla k_m - \boldsymbol{u}\cdot\nabla k)(k_m-k) \\ &+ \int_\Omega \left[\frac{k_m^+}{\varepsilon+\omega_m^+}\nabla k_m - \frac{k^+}{\varepsilon+\omega^+}\nabla k\right]\cdot\nabla(k_m-k) + \int_\Omega (k_m\omega_m^+ - k\omega^+)(k_m-k) - \int_\Omega \left(\frac{k_m^+|\boldsymbol{D}(\boldsymbol{u}_m)|^2}{\varepsilon+\omega_m^++\varepsilon k_m^+} - \frac{k^+|\boldsymbol{D}(\boldsymbol{u})|^2}{\varepsilon+\omega^++\varepsilon k^+}\right)(k_m-k) \\ &+ \int_\Omega \varepsilon\,[(\Phi_r(\boldsymbol{D}(\boldsymbol{u}_m)) - \Phi_r(\boldsymbol{D}(\boldsymbol{u}))):\boldsymbol{D}(u_m-\boldsymbol{u}) + (\Phi_r(\boldsymbol{u}_m)-\Phi_r(\boldsymbol{u}))\cdot(\boldsymbol{u}_m-\boldsymbol{u}) + (\Phi_r(\nabla\omega_m)-\Phi_r(\nabla\omega))\cdot\nabla(\omega_m-\omega) \\ &+ (\Phi_r(\omega_m)-\Phi_r(\omega))(\omega_m-\omega) + (\Phi_r(\nabla k_m)-\Phi_r(\nabla k))\cdot\nabla(k_m-k) + (\Phi_r(k_m)-\Phi_r(k))(k_m-k)\,], \end{aligned} \tag{A.12}$$

where $\Phi_r(\boldsymbol{\xi}) := |\boldsymbol{\xi}|^{r-2}\boldsymbol{\xi}$. The last term $K_{10,m}$ can be used to control $U_m - U$ in the norm of $\boldsymbol{V}$ by using the estimate

$$\big(\Phi_r(\boldsymbol{\xi}) - \Phi_r(\boldsymbol{\eta})\big)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) \geq 2^{2-r}|\boldsymbol{\xi}-\boldsymbol{\eta}|^r \quad \text{for all } \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N, \tag{A.13}$$

see Lindqvist [46] for the derivation of the exact constant. In particular, we find

$$K_{10,m} \geq \varepsilon 2^{2-r}\|U_m - U\|_V^r, \tag{A.14}$$

and the strong convergence $U_m \to U$ follows if we show $\limsup_{m\to\infty} K_{10,m} \leq 0$.

By Equation (A.10), we control the limsup of $\sum_1^{10} K_{j,m}$ and hence obtain

$$\limsup_{m\to\infty} K_{10,m} = \limsup_{m\to\infty}\left(\sum_{j=1}^{10}K_{j,m} - \sum_{l=1}^{9}K_{l,m}\right) \leq \limsup_{m\to\infty}\sum_{j=1}^{10}K_{j,m} - \liminf_{m\to\infty}\sum_{l=1}^{9}K_{l,m} \overset{(A.10)}{\leq} 0 - \sum_{l=1}^{9}\liminf_{m\to\infty}K_{l,m}. \tag{A.15}$$

Thus, it suffices to show $\liminf_{m\to\infty} K_{l,m} \geq 0$ for $l \in \{1,\dots,9\}$. To do so, we use $U_m \rightharpoonup U$ (i.e., Equation (A.9 a)), which by $r > 3$ and the compact embedding $W^{1,r}(\Omega) \Subset C^0(\overline{\Omega})$ implies

$$\boldsymbol{u}_m \to \boldsymbol{u}, \quad \omega_m \to \omega, \quad k_m \to k \quad \text{uniformly in } \overline{\Omega}. \tag{A.16}$$

For treating $K_{1,m}$, we use integration by parts and $\operatorname{div}\boldsymbol{u}_m = \operatorname{div}\boldsymbol{u} = 0$ to find

$$K_{1,m} = \int_\Omega (\operatorname{div}(\boldsymbol{u}_m\otimes\boldsymbol{u}_m) : \nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_m) \to \int_\Omega (\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}) = 0, \tag{A.17}$$

because of the uniform convergence $\boldsymbol{u}_m \to \boldsymbol{u}$.

Similarly, the other convective terms $K_{3,m}$ and $K_{6,m}$ converge to 0, since $\omega_m \to \omega$ and $k_m \to k$ converge uniformly.

For the second term $K_{2,m}$, we again use the uniform convergence in the decomposition

$$K_{2,m} = \int_\Omega \left(\frac{k_m^+}{\varepsilon+\omega_m^+} - \frac{k^+}{\varepsilon+\omega^+}\right)\boldsymbol{D}(\boldsymbol{u}_m) : \boldsymbol{D}(\boldsymbol{u}_m-\boldsymbol{u}) + \int_\Omega \frac{k^+}{\varepsilon+\omega^+}\boldsymbol{D}(\boldsymbol{u}_m-\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{u}_m-\boldsymbol{u}). \tag{A.18}$$

The first integral converges to 0 as the two terms involving $\boldsymbol{D}$ are bounded in $\boldsymbol{L}^r(\Omega) \subset \boldsymbol{L}^2(\Omega)$ while the prefactor converges to 0 uniformly. The second integral is non-negative, hence $\liminf_{m\to\infty} K_{2,m} \geq 0$ follows. Analogously, the $\liminf_{m\to\infty}$ of $K_{4,m}$ and $K_{7,m}$ is non-negative.

By uniform convergence of the integrands, we easily obtain $K_{5,m} \to 0$ and $K_{8,m} \to 0$.

In $K_{9,m}$, the integrand is a product of a function bounded uniformly in $L^{r/2}(\Omega)$ and $k_m - k$, which converges uniformly to 0; hence $K_{9,m} \to 0$ as well.

This finishes the proof of Step 3 guaranteeing $U_m \to U$ in $\boldsymbol{V}$.

Step 4. $A$ is pseudo-monotone: For the sequence $U_m$ satisfying Equation (A.9) we have to show

$$\langle A(U), U - V\rangle \le \liminf_{m\to\infty} \langle A(U_m), U_m - V\rangle \text{ for all } V = (\boldsymbol{v}, \varphi, w) \in \boldsymbol{V} \tag{A.19}$$

By Step 3, we are now able to use the strong convergence $U_m \to U$. Again we split the duality-product term into 10 parts and treat the parts separately:

$$\big\langle A(U_m), U_m - V \big\rangle = \sum_{j=1}^{10} G_{j,m} \tag{A.20}$$

$$\begin{aligned}
:= &\int_\Omega \boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m \cdot (\boldsymbol{u}_m - \boldsymbol{v}) + \int_\Omega \frac{k_m^+}{\varepsilon + \omega_m^+} \boldsymbol{D}(\boldsymbol{u}_m) : \boldsymbol{D}(\boldsymbol{u}_m - \boldsymbol{v}) + \int_\Omega \boldsymbol{u}_m \cdot \nabla \omega_m \, (\omega_m - \varphi) \\
&+ \int_\Omega \frac{k_m^+}{\varepsilon + \omega_m^+} \nabla \omega_m \cdot \nabla(\omega_m - \varphi) + \int_\Omega \omega_m^+ \omega_m (\omega_m - \varphi) + \int_\Omega \boldsymbol{u}_m \cdot \nabla k_m \, (k_m - w) \\
&+ \int_\Omega \frac{k_m^+}{\varepsilon + \omega_m^+} \nabla k_m \cdot \nabla(k_m - w) + \int_\Omega k_m \omega_m^+ (k_m - w) - \int_\Omega \frac{k_m^+}{\varepsilon + \omega_m^+ + \varepsilon k_m^+} \left|\boldsymbol{D}(\boldsymbol{u}_m)\right|^2 (k_m - w) \\
&+ \int_\Omega \varepsilon \big(\Phi_r(\boldsymbol{D}(\boldsymbol{u}_m)) : \boldsymbol{D}(\boldsymbol{u}_m - \boldsymbol{v}) + \Phi_r(\boldsymbol{u}_m) \cdot (\boldsymbol{u}_m - \boldsymbol{v}) + \Phi_r(\nabla \omega_m) \cdot \nabla(\omega_m - \varphi) \\
&+ \Phi_r(\omega_m)(\omega_m - \varphi) + \Phi_r(\nabla k_m) \cdot \nabla(k_m - w) + \Phi_r(k_m)(k_m - w)\big).
\end{aligned} \tag{A.21}$$

Using the uniform convergence of $U_m$ (see Equation (A.16)) and the strong convergence in $L^r(\Omega)$ of the derivatives $\nabla U_m$, it is straight forward to see that the integrals $G_{j,m}$ for $j \in \{1, \dots, 9\}$ converge to their respective limits. For $G_{10,m}$, we can use the estimate

$$\left|\Phi_r(\boldsymbol{\xi}) - \Phi_r(\boldsymbol{\eta})\right| \le 3r\big(|\boldsymbol{\xi}| + |\boldsymbol{\eta}|\big)^{r-2} \left|\boldsymbol{\xi} - \boldsymbol{\eta}\right| \quad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N, \tag{A.22}$$

see “exerc. 10.a”, p. 257 in Bourbaki [30]. Thus, we conclude that Equation (A.19) holds, even with equality.

Hence, all the assumptions in Equation (A.1) are established, Theorem A.1 is applicable, and the proof of Proposition 5.1 is complete. □

*Remark* A.2. An alternative proof for Proposition 5.1 is given in the first draft [47] of the present work. That proof is based on the method of elliptic regularization of abstract evolution equations, cf. Ch. 3, Thm. 1.2 [39].