Corvinus University of Budapest
Doctoral School of Economics, Business and Informatics

THESIS
on

Price effects and profitability in heterogeneous agent models

Attila A. Víg
Ph.D. dissertation

Supervisor:
Zsolt Bihary, Ph.D.
associate professor

Budapest, 2022

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Chapter 1

Introduction

According to the traditional paradigm of financial economics, participants of capital markets are rational, and the markets are efficient (Malkiel and Fama, 1970). Although the rationality of market participants and the efficiency of markets are not considered realistic by many critics, these ideas are deeply rooted in financial economics. The heterogeneous agent literature of financial markets breaks with efficient markets and rational investors and imagines the market as a multidimensional dynamic system where the portfolio of traders, asset prices, and market weights evolve simultaneously. This thesis aims to join this literature. In such models, market participants are boundedly rational – they follow heuristic investment strategies and make their decisions based only on information available to them.

The thesis has a theoretical and a technical goal. The the theoretical goal is to examine the impact of investor heterogeneity and heuristic distortions on asset prices and investor profitability using agent-based modeling. The main idea of the thesis is that supply-demand disequilibrium in the market is reflected in the asset price dynamics. Through this, we demonstrate phenomena such as the separation of the asset price from the fundamental value, the sudden rise and fall of the asset price, the positive or negative autocorrelation and the heavy-tailed distribution of the returns. In addition to these phenomena, we will examine the profitability and price impact of canonical investment strategies: fundamental, technical and index trading.

The technical goal is to produce a realistic stochastic financial model in which the stochastic behavior of the phenomena mentioned above can be analyzed analytically. A shortcoming of the agent based literature of financial markets is that it does not examine the stochastic nature of the models in sufficient depth. In almost all cases, the analytical results are only about the deterministic skeleton of the models, while the stochastic model is only examined through simulation. In this paper, we try to break with this tradition and provide (quasi)analytical results in stochastic cases as well.

Structure of the thesis

Chapter 2 serves two functions. On the one hand, it is a literature review of agent-based financial modeling, with focus on its branch in which a few canonical investment strategies compete, and the mathematical model is usually a few-dimensional stochastic dynamic system. On the other hand, in the chapter we present – through simple dynamic equations – the most important phenomena and effects modeled in the later parts of the thesis; the agents, the evolution realized through the development of their wealth and the market price mechanism.

We discuss two continuous-time models in the thesis. Chapter 3 presents the author's study published in a Hungarian economics journal Bihary and Víg (2020). In this study, the volatility is constant as usual in the literature, and interesting nonlinear effects appear through the traders' position function. We examine the long-term stability of the stochastic model through the invariant distribution of certain relevant variables. Due to nonlinear effects, multimodal behavior appears in the case of distributions.

In Chapter 4, we break with the usual constant volatility assumption and assume instead that the supply-demand disequilibrium appears directly in the volatility as well. This endogenous volatility effect solves the problem of the previous chapter: none of the traders can ultimately dominate the other in the long term. Due to stochastic volatility, the distribution of returns has a heavy-tail, which is a well-known and welcomed feature of returns in a stochastic financial model.

Chapter 2

Agent-based modelling

The most important elements of agent-based models to be clarified are the agents themselves, whom we call traders or investors. Traders make portfolio decisions based on information available to them, typically the asset price and its history, dividend data, but other macro indicators are also conceivable. In the thesis, we follow the classic path where the market is populated by a finite number of traders following a well-defined trading strategy. In the following, the most frequently modeled canonical investment strategies (agent types) are presented.

2.1 Fundamental trader

The fundamental trader can be considered the classical, fully rational agent of financial markets. The fundamental trader attempts to determine the fundamental (intrinsic) value of an asset and then takes a position depending on whether they deem the asset over- or undervalued. The fundamental trader can be thought of as an investment fund with significant resources and infrastructure that conducts its analysis effectively and makes investment decisions based on this analysis. In the financial heterogeneous agent literature, the fundamental value often appears explicitly, and we will follow this modelling practice in the thesis as well. In the next section, we present a simple model of the fundamental value.

2.1.1 Fundamental value

The fundamental value can appear in models in two ways. On the one hand, the fundamental value itself can be an exogenously evolving stochastic process, such as an exponential random walk. On the other hand, the future cash flow (dividend) of the asset can be described by an exogenously developing stochastic process, and then the fundamental value is defined as the discounted present value of the expected value of this cash flow. In the following, we present a simple model in which the exogenously defined stochastic dividend process leads to a fundamental value – as a discounted present value – with the same dynamics as the dividend process itself.

Now let the process of dividends $D(t)$ be a geometric Brownian motion with drift μ. The dividend process is interpreted such that the asset is paying the amount $D(t)dt$ at time t. The fundamental trader calculates the fundamental value $F(t)$ as the discounted present value of future dividends at a subjective discount rate $\delta > \mu$:

$$F(t) = E\left[\int_t^\infty e^{-\delta(s-t)} D(s)\, ds \middle| \mathcal{F}(t)\right] = \frac{D(t)}{\delta - \mu}, \tag{2.1}$$

thus the current fundamental value is a constant multiple of the current dividend rate, which means that it is also a geometric Brownian motion. In light of this result, in the later parts of the thesis, instead of introducing dividends, we will assume directly of the fundamental value that it is an exogenous geometric Brownian motion.

2.2 Technical trader

Technical traders believe that changes in supply and demand can be read from the price, so they try to make predictions about the future price using patterns of price movements of the past. In heterogeneous agent-based models, technical traders are most often identified with trend-following (and contrarian) strategies. Traders following a trend-following strategy believe in the inertia of price movements: if they identify a trend, they trade on the assumption that this trend will continue in the near future. Traders following a contrarian strategy believe in the “return to the average”, and therefore trade against the trend. The most important effect of technical traders is that they generate bubbles and large collapses. In the following, we present the dynamics of the trend indicator followed by technical traders, which indicator will be one of the focal objects of the thesis.

2.2.1 Trend indicator

The trend indicator is a stochastic process that is meant to show whether the price of an asset is currently in an increasing or decreasing trend. In the thesis we present two equivalent definitions of the trend indicator, of which only one is discussed here. In this case, the trend indicator $x(t)$ is defined directly as an exponentially weighted moving average of (log)returns:

$$x(t) = \int_{-\infty}^{t} e^{-\frac{1}{\tau}(t-u)}\, ds(u), \tag{2.2}$$

The intuition behind the (2.2) integral formula is that we interpret it as a positive (negative) trend if positive (negative) returns were more typical in the recent past. The dynamics of (2.2) trend indicator process results from the application of the Leibniz integral rule:

$$dx(t) = -\frac{1}{\tau}x(t)\, dt + ds(t), \tag{2.3}$$

which is a continuous generalization of the trend indicator of Chiarella et al. (2006).

Since technical traders base their strategy on the trend indicator in the thesis, it can ultimately be considered one of the most important state variables, the dynamics and stochastic properties of which are examined thoroughly in the thesis. Furthermore, in addition to being a strategic state variable, the trend indicator can also be considered a continuously calculated τ-time return. This is different from the usual τ-time returns, since they are usually calculated from increments between discrete time points, but nevertheless statements about the distribution of the trend indicator can be interpreted as statements about the returns in the usual sense. If, for example, the invariant distribution of the trend indicator defined according to the equation (2.3) turns out to be heavy-tailed, then this property carries over to returns as well.

2.3 Evolution, wealth dynamics and equilibrium

Evolution is often at the heart of agent models: it is the main source of dynamics between agents in both practical and philosophical sense. The profitability of individual agents (traders or investment funds) depends on the one hand on their own strategy, on the other hand on the strategy of the others, as they all have an impact on market prices. In this sense, the competition of different strategies can be placed in a game theoretic context. However, instead of the standard game theory setup, in this thesis we examine the strategies through the lens of evolutionary game theory, which identifies the success and failure of the participants with their population ratio.

In the thesis, we present a simple two-player example for the population dynamics and examine the relationship of this dynamic system with the usual static game theoretic equilibrium concept. Two different sources of wealth development are considered:

1. In the first case, the two traders achieve different returns, and this return is naturally into account in their portfolio value. The trader who achieves a higher return will eventually have a larger market weight, which we consider a measure of success. LeBaron (2011) calls this slow evolution passive learning, and following his terminology we will use the terms passive wealth development or passive evolution.

2. In the second case, we think of individual traders as investment funds that manage other people's invested money. In this case as well, the investment funds achieve different returns, but we do not take this into account in the value of their managed portfolio. However, investors notice the difference in returns and allocate part of their wealth to the better performing fund. LeBaron (2011) calls this form of evolution active learning, and following his terminology we will use the terms active wealth reallocation or active evolution.

The example we discuss here is the classic hawk-dove game interpreted for capital markets. Let's say that there is a *professional* (P) and an *amateur* (A) strategy to choose from in the market. The trader who follows the professional strategy is considered to be the informed trader. They can be identified as the previously discussed fundamental or even the technical trader. The trader who follows the amateur strategy is considered to be the uninformed trader, who can be identified as, for example, a noise trader. They essentially take random positions and provide liquidity to the market. In the thesis, the game is described as the sum of three subgames, which capture three market phenomena. Here we skip this breakdown, let us look at the resulting game instead:

	P	A
P	$-2, -2$	$5, -1$
A	$-1, 5$	$3, 3$

Table 2.1: The game of the professional and the amateur trader.

A cell of the matrix should be interpreted as the first number being the row player's payoff if they trade "against" the corresponding column player, while the second number is the column player's payoff if they trade against the corresponding row player.

The game is famous for having only one stable equilibrium. According to the standard game theory approach, each player uses a mixed strategy: they specify a probability distribution over the possible actions. Let $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$ be the probabilities that the row and column player chooses the professional (P) action respectively. A probability pair (p_1, p_2) from which neither player will deviate is called a Nash equilibrium (Nash, 1951), which in the case of our game are the following:

$$E_{\text{Nash}} = \left\{ \left(\frac{2}{3}, \frac{2}{3} \right), (1, 0), (0, 1) \right\} \tag{2.4}$$

Among the three equilibrium solutions, the first represents true mixed strategies, while the other two represent pure strategies. In the following two sections, we will present two evolutionary game theoretic interpretations and solutions of the same game, which are strongly related to the standard equilibrium solutions just discussed.

2.3.1 Passive wealth development

Passive wealth development is parallel with Friedman's market selection hypothesis (Friedman, 1953), according to which good strategies will be successful and will account for an ever increasing share of the market, while the weight of bad strategies will decrease over time.

Suppose there is a continuum of traders in the market who follow one (P) or the other (A) trading strategy. Let $W_P(t)$ and $W_A(t)$ denote the funds of each strategy at a time t. The market weight of each strategy is identified with its market share. Let the wealth ratio of those following the professional strategy at a time t be $p(t)$:

$$p(t) = \frac{W_P(t)}{W_P(t) + W_A(t)},$$

and thus the wealth ratio of those following the amateur strategy is $1 - p(t)$. In this case, we do not assume a probability distribution over the individual actions, but instead we evolve the population ratio (wealth ratio) of those following each strategy over time. We assume that a given trader trades randomly "against" another trader. The probability of what type of trader they will encounter depends on the market weight – the wealth ratio – of the opposing strategy. Since the number of traders is a continuum, the change of wealth managed by each strategy is governed by their average profit, which in turn depends on their market wealth ratios. The dynamics of the market share is

$$dp(t) = p(t)\big(1 - p(t)\big)\big(2 - 3p(t)\big)\, dt \tag{2.5}$$

The dynamics (2.5) is the so-called *replicator equation*, which has a wide literature, see for example the book of Hofbauer and Sigmund (1998). The resting points of the replicator equation are the same as the Nash equilibria discussed in the previous section. This is easy to see in this simple example, since a resting point is defined by $dp(t) = 0$:

$$E_{\text{passive}} = \Big\{p \in [0,1] : p(1-p)(2-3p) = 0\Big\} = \left\{\frac{2}{3}, 0, 1\right\},$$

which resting points are exactly the same as the previous E_{Nash} equilibrium points.

2.3.2 Active wealth reallocation

The active asset reallocation process is presumably closer to the evolutionary process that a layman first thinks of in relation to financial markets. Agents actively choose between different strategies according to a well-defined objective function and cluster behind the strategy that performs better than the other. This selection process can also be seen in real markets.

Similarly as before, let $W_p(t)$ and $W_A(t)$ denote the funds managed by the two trading strategies, while $p(t)$ is the market share of the professional strategy. Let us think of the players in this case as investment funds that manage the wealth of a continuum of investors. The idea behind the wealth dynamics is that a part of the managed funds flows into a common pool with a certain intensity, and then the investment funds receive their share from this pool in accordance with a selection rule. A discrete time version of this type of wealth dynamics can be seen in the flow of funds model of Palczewski et al. (2016). The corresponding continuous-time dynamics of the market share is:

$$dp(t) = \kappa \left(\frac{1}{1 + e^{\gamma(3p(t)-2)}} - p(t) \right) dt,$$

where $\kappa \geq 0$ controls the intensity of wealth reallocation, while $\gamma \geq 0$ is controls the intensity of choice. The resting point of this dynamics is given by the value $p(t)$ where $dp(t) = 0$:

$$E_{\text{active}}(\gamma) = \left\{ p \in [0, 1] : \frac{1}{1 + e^{\gamma(3p-2)}} = p \right\}$$

Although the equation defining the resting point does not have an analytical solution, in the thesis we consider that $\lim_{\gamma\to\infty} E_{\text{active}}(\gamma) = \frac{2}{3}$. This means that the resting point of pure active wealth reallocation dynamics coincides with both the standard game-theoretic equilibrium and the resting point of the passive wealth development dynamics if the underlying investors react very sensitively to the difference in returns of the investment funds.

2.4 The traded instruments and the market price mechanism

The traded instruments are the basic building blocks of market models. The type and number of instruments included in the model must be clarified. Since the emphasis in heterogeneous agent market models is on the agents and their competition, the number of traded risky assets is often only one. In this case, this single risky asset usually represents an index: the traders hold a portfolio of the index and a risk-free asset, which is in accordance with the classical CAPM (Sharpe, 1964).

The market price mechanism serves to establish the asset as a result of market demand and supply. LeBaron (2001) distinguishes three types of market mechanism: micro-level order book-driven market, equilibrium market, and permanent disequilibrium market. In the thesis, we use the permanent disequilibrium approach.

The price mechanism of permanent disequilibrium models is simpler than that of micro-level and equilibrium models, so its use in continuous-time models is more prevalent. In such models, the market is never in equilibrium, but an external force is constantly trying to steer it towards equilibrium. This external force is usually identified with a usually latent agent, the market maker. The market maker aggregates the current positions of investors, determines the level of current disequilibrium, and then takes an opposite position to satisfy excess demand or supply. Finally, it sets a price for the next period in such a way that the absolute value of excess demand – and thus the level of disequilibrium in the market – becomes lower. Following these ideas, the continuous-time price dynamics of a risky asset in the most general sense takes the following form:

$$\frac{dS(t)}{S(t)} = \mu\big(\mathcal{D}(t)\big)\,dt + \sigma\big(\mathcal{D}(t)\big)\,dB(t),$$

where $\mathcal{D}(t)$ is the measure of current disequilibrium, $dB(t)$ is the change of a Brownian motion. In the case of the $\mu : \mathbb{R} \mapsto \mathbb{R}$ function, it is usually assumed that it is monotone increasing, since if the level of disequilibrium is high, the market maker adjusts the price upwards, whereas if the level of disequilibrium is low (even negative), they adjust the price downwards. The function $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is often the constant function, because drift is usually the focus of these type of models. However, in the thesis we will also examine a case where volatility also depends on the current level of disequilibrium in such a way that volatility is high when the absolute value of disequilibrium is high. The current level of disequilibrium $\mathcal{D}(t)$ is usually defined as the difference between the average position of traders and the supply:

$$\mathcal{D}(t) = \sum_{i=1}^{n} p_i(t) Z_i(t) - \mathcal{S}$$

Permanent disequilibrium models provide a lot of freedom for the modeler through the μ and σ functions, although this freedom comes with the price of being heuristic in nature. Interesting and realistic price dynamics usually emerge in the case of nonlinear assumptions. A first example of permanent disequilibrium models similar to the one described above is the article of Beja and Goldman (1980). Another early example is the model of Jarrow (1992), who assume that sufficiently large players move prices by trading. The articles of He et al. (2019) and He et al. (2018) can be considered as the closest continuous-time antecedents of this thesis, while examples of similar ideas in discrete-time are Chiarella et al. (2009) and Chiarella et al. (2013).

Chapter 3

Fundamental and technical traders in a constant volatility model

Chapter 3 of the dissertation is a slightly edited version of the author's article (Bihary and Víg, 2020) published in a Hungarian economics journal. In this chapter, we examine a market whose active players are fundamental and trend-following traders, and index-following traders appear as passive characters. The model in this chapter relies heavily on the continuous-time model of He and Li (2015).

The behavior of market participants is controlled by market indicators, which indicators are also state variables of the model. The model postulates a fundamental value that differs from the market price. One market indicator is price dislocation, which is the difference between the market price and the fundamental value. This is followed by fundamental traders. The other is the trend indicator, which is the difference between the current price and its moving average. Trend followers base their strategy on this indicator. The joint dynamics of the two indicators can be derived from the dynamic equation of the market price. Unlike the usual approach of the literature, we do not ignore the stochastic nature of the dynamics in our calculations. On the contrary, our discussion focuses on the long-term statistical behavior of the market. As the market price is akin to a random walk, we cannot talk about its invariant distribution. However, the market indicators, as quantities determining the state of the market, can behave in a stationary manner. In our stochastic description, we define the stability of the market precisely through this;

we say that the market is stable when the indicators possess an invariant distribution. We examine the stochastic behavior of a main model and two of its simplified versions.

- The two central quantities of the main model are the price dislocation followed by fundamental traders and the trend indicator followed by trend followers. In this model, the position function of trend followers is nonlinear, as usual in the literature (see for example He and Li (2012)). Therefore the two-dimensional stochastic dynamics is also nonlinear. The invariant distribution of this stochastic dynamic system is investigated numerically using the forward Kolmogorov method.

- The first simplified model is obtained by linearization. We linearize the position function of trend followers around zero. Then the price dislocation and trend indicator two-dimensional stochastic dynamic system to be investigated becomes linear, which we are able to examine analytically.

- The second simplified model is obtained by dimension reduction. Here we omit the fundamental value and assume that the fundamental trader also trades based on the trend indicator, but in the opposite direction to the trend-following trader. Then the fundamental trader actually becomes a contrarian trader and the dynamic system becomes one-dimensional, but the interesting nonlinearity remains.

3.1 Illustrations of the price dynamics

Figure 3.1 illustrates the role and price impact of fundamental and trend-following traders in the case of the main – two-dimensional, nonlinear – specification using generated trajectories.

The fundamental value (green) is a driftless, exogenous Brownian motion with relatively low volatility. The market price (yellow) fluctuates around the fundamental value. We also show the moving average (red) that lags behind the market price. We can visualize the two market indicators used by the respective traders. Price dislocation (green) is the difference between the market price and the fundamental

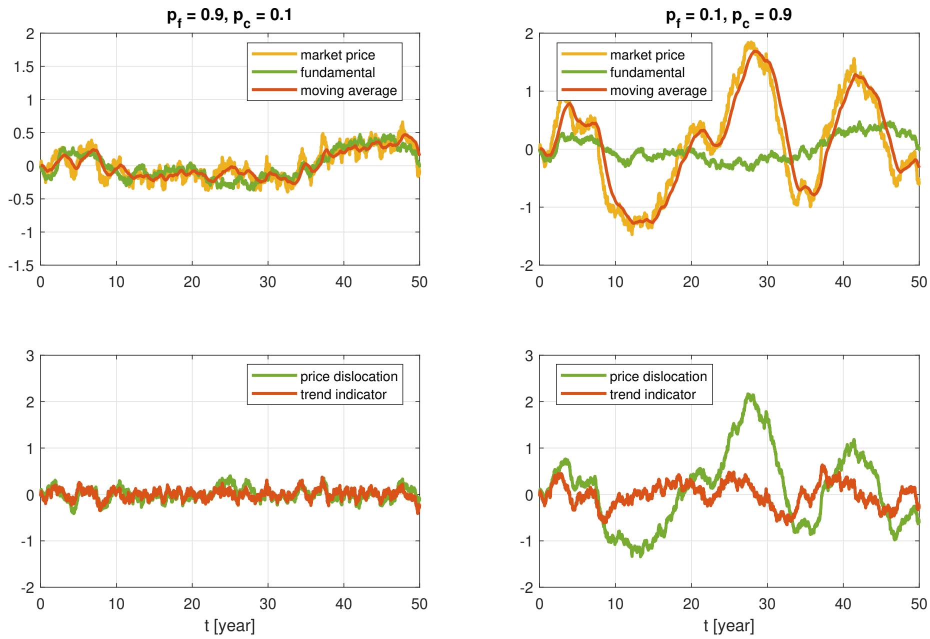

Figure 3.1: Sample trajectories for the main – two-dimensional, nonlinear – specification.

value. Trend indicator (red) is the difference between the market price and the moving average.

On the left panels we illustrate a mature market where fundamental traders are more numerous ($p_f = 0.9$), while on the right the market is dominated by trading-following chartists $p_c = 0.9$). The market influence of the traders is evident on the charts. On the left the market price closely follows the fundamental value as fundamental traders' activity pulls the market price towards the fundamental value. Therefore fluctuations of the price dislocation is relatively low. On the right we see wild excursions in the price. These are caused by trend following chartists, who build up and burst bubbles periodically. However, due to our nonlinear specification, trend followers can only build a position of bounded size, so sooner or later the price effect of the relatively small number of fundamental traders will pull the price back to the fundamental. The market price fluctuates strongly, but does not explode ultimately.

3.2 Invariant distributions

Based on the stochastic differential equation system of the two indicators of the main model (price dislocation and trend indicator), we calculate the invariant joint probability density which characterizes the long-term state of the system. We calculate the invariant density function by solving the Kolmogorov forward equation of the system on a grid (see Appendix A.1). Figure 3.2 shows the invariant density function of the same two characteristically different markets as the previous section: one is dominated by fundamental traders, while the other is dominated by trend-following chartists.

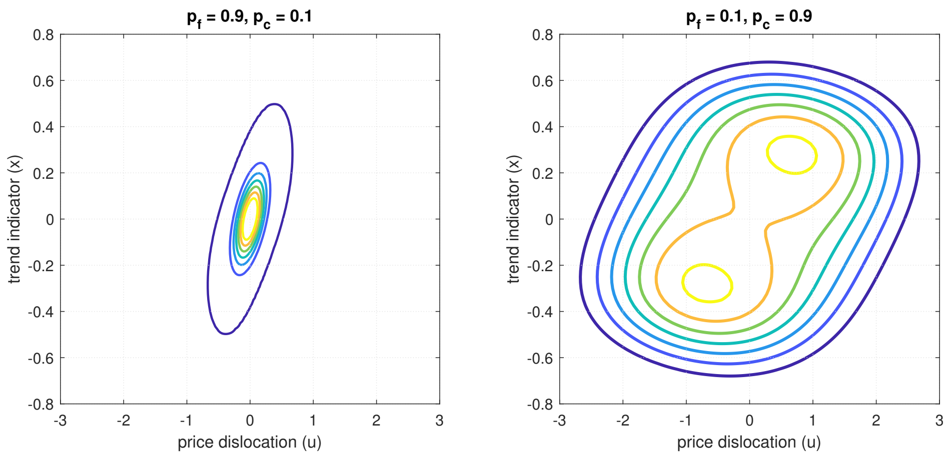

Figure 3.2: Contour plot of the invariant density functions of the main (non-linear, two-dimensional) specification. The two dimensions are price distribution and trend indicator. The parameters coincide with those in Figure 3.1.

On the left plot (market dominated by fundamental traders) we see a density function that is strongly localized at the $(x, u) = (0, 0)$ origin. This corresponds to the left side of Figure 3.1 where price dislocations and trend indicator values are small throughout the trajectory. On the right plot (market dominated by trend-following chartists) the density is more spread out and is bimodal. This corresponds to the periodically increasing and decreasing market price.

We can see such effects in the simplified models as well. In the two dimensional linear specification, we can make the following

Proposition 1. *In the linear specification the joint dynamics of price dislocation* *and trend indicator* $\boldsymbol{Y}(t) = \big(u(t), x(t)\big)^\top$ *has a limiting distribution (which we regard as "stable") if the conditions*

$$\beta \left(p_c \alpha_c - p_f \alpha_f\right) < \frac{1}{\tau} \tag{C1}$$

$$\beta p_f \alpha_f > 0 \tag{C2}$$

both hold. Furthermore, if (C1) *and* (C2) *both hold, the limiting distribution is multivariate normal with*

$$\begin{aligned}
E\left[u_\infty\right] &= E\left[x_\infty\right] = 0\\
D^2\left[u_\infty\right] &= \frac{\left(\frac{1}{\tau}\boldsymbol{\sigma}_u + q_c\boldsymbol{\sigma}_f\right)^\top \left(\frac{1}{\tau}\boldsymbol{\sigma}_u + q_c\boldsymbol{\sigma}_f\right) + q_f\frac{1}{\tau}\boldsymbol{\sigma}_u^\top\boldsymbol{\sigma}_u}{2q_f\frac{1}{\tau}\left(q_f + \frac{1}{\tau} - q_c\right)},\\
COV\left[u_\infty, x_\infty\right] &= \frac{\left(q_c - \frac{1}{\tau}\right)\boldsymbol{\sigma}_f^\top\boldsymbol{\sigma}_f + \frac{1}{\tau}\boldsymbol{\sigma}_s^\top\boldsymbol{\sigma}_s}{2\frac{1}{\tau}\left(q_f + \frac{1}{\tau} - q_c\right)},\\
D^2\left[x_\infty\right] &= \frac{q_f\boldsymbol{\sigma}_f^\top\boldsymbol{\sigma}_f + \frac{1}{\tau}\boldsymbol{\sigma}_s^\top\boldsymbol{\sigma}_s}{2\frac{1}{\tau}\left(q_f + \frac{1}{\tau} - q_c\right)},
\end{aligned}$$

where $\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_s - \boldsymbol{\sigma}_f$*,* $q_j = \beta p_j \alpha_j$*,* $j \in \{f, c\}$*. We use the notation* $\boldsymbol{Y}_\infty = (u_\infty, x_\infty)^\top$ *for the Gaussian random variable which the process* $\boldsymbol{Y}(t)$ *converges in distribution to.*

Proof. Appendix and Bihary and Víg (2020). □

The first part of Proposition 1 says that if chartists are strong enough ($p_c\alpha_c$ is large enough), they can destabilize the market to such an extent, that the price dislocation – trend indicator system does not even have an invariant distribution. If (C1) does not hold, chartists can blow a bubble which never bursts. The second part of Proposition 1 says that if the market is stable in the above sense, then the limiting distribution is multivariate normal. The relevant support of the ellipsoid density of (u_∞, x_∞) gets larger as fundamental traders get weaker and/or chartists get stronger, as can be seen at the denominator of $D\left[u_\infty^2\right]$ for instance.

In the main model the trend-following chartists' effect has a floor and a ceiling due to the nonlinearity of their position function. In the one dimensional model this nonlinearity remains, and we can make the following

Proposition 2. *In the one dimensional (nonlinear) model, the dynamics of the trend indicator defined by equation* (A.6) *is always stable and its limiting probability density function is*

$$f_\infty(x) = C\left(\cosh\left(\frac{x}{x^\star}\right)\right)^{\frac{2\beta p_c \alpha_c (x^\star)^2}{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}} \exp\left(-\frac{\frac{1}{\tau} + \beta p_f \alpha_f}{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s} x^2\right), \tag{3.1}$$

where $C > 0$ is a suitable normalizing constant.

Proof. Appendix and Bihary and Víg (2020). □

Remark. Note that if the exponent of the cosh factor in equation (3.1) is one, then the density function describes a symmetric mixture of two normal distributions. This results from $\cosh(x) \cdot e^{-\frac{x^2}{2}} = \frac{e^x + e^{-x}}{2} e^{-\frac{x^2}{2}} = e^{\frac{1}{2}}\left(\frac{1}{2}e^{-\frac{(x-1)^2}{2}} + \frac{1}{2}e^{-\frac{(x+1)^2}{2}}\right)$.

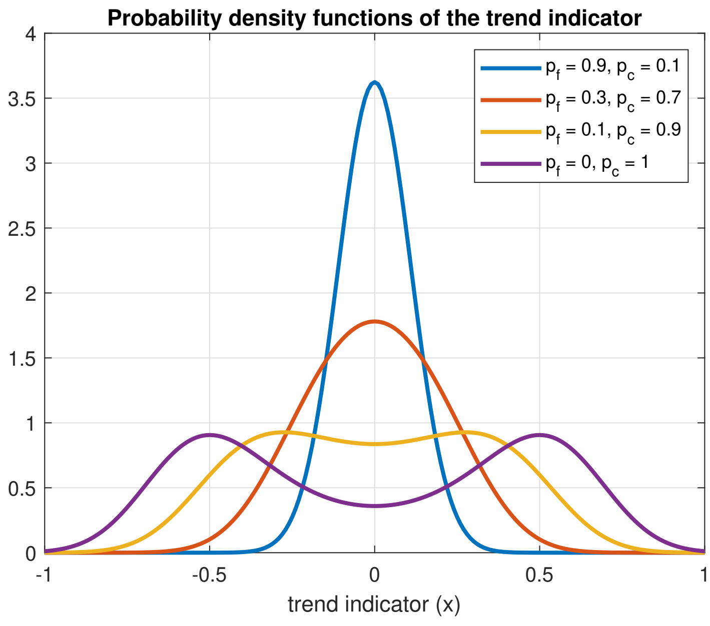

Figure 3.3: Density functions of equation (3.1) with different population ratios. The parameters used are the same as in Figure 3.1.

Figure 3.3 shows representative density curves with the trend-following chartists' population gradually replacing the fundamental traders'. We used the same exogenous parameters as in Figure 3.2. The blue curve corresponds to a mature market with the same population ratios as the one in the left panel of Figure 3.2 and Figure 3.1, dominated by fundamental traders. The density in this case shows little variance, and is unimodal. The yellow curve corresponds to a market with the same population ratios as the one in the right panel of Figure 3.2 and Figure 3.1, dominated by chartists. The density in this case shows greater variance and is bimodal. The red curve is an intermediary case, and the purple curve shows an extreme market populated by only chartists.

In this simple tractable model, we can investigate the appearance of bimodality analytically. Bimodality appears when the second derivative of the density function (3.1) at zero changes from negative to positive:

Proposition 3. *The condition of unimodality of trend indicator in the one-dimensional nonlinear model described by the limiting density* (3.1) *is*

$$\beta \left(p_c \alpha_c - p_f \alpha_f\right) < \frac{1}{\tau}. \tag{C3}$$

Proof. Appendix and Bihary and Víg (2020). □

Condition (C3) shows that bimodality tends to appear when trend-following chartists dominate the market over fundamental traders. This result is equivalent to the stability condition (C1) of Proposition 1. This formal equivalence suggests that the bimodal invariant distribution appearing in the nonlinear model can be interpreted as a milder form of instability.

As long as fundamental traders dominate the market, they stabilize it with their mean-reverting effect, strong trends do not develop. If the market power of trend followers exceeds a critical value, self-inflicted bubble cycles appear. With a linear specification, there is nothing to prevent the trend followers from generating never-ending bubbles, the market explodes. In this case, an invariant distribution does not exist. The nonlinear specification solves this problem by establishing bounds for the demand of the trend-following chartists, and ultimate bubbles do not develop.

In this case, the market price will be a noisy, quasi-periodic series of moderately strong rising and falling trends.

3.3 Profitability

In this section, we examine the long-term profitability of traders, and through this we draw attention to a weakness of the model of this chapter. We suppose that the traders invest in the risky asset and a risk-free money market account according to their position function, and we evolve their wealth $W_j(t)$ in a self-financing way. This idea is parallel to the passive wealth development of section 2.3.1. Then we define the profitability of the traders with their long-term expected log return:

$$\Pi_j = \lim_{T\to\infty} E\left[\frac{\log W_j(T) - \log W_j(0)}{T}\right], \tag{3.2}$$

In the case of the two-dimensional linear specification, we provide analytical results for profitability:

Proposition 4. *Assume that conditions* (C1) *and* (C2) *both hold in the case of the linear model. Then the average logarithmic growth* (3.2) *of the traders' wealth is:*

$$\Pi_f = \beta p_f \alpha_f^2 E\left[u_\infty^2\right] - \beta p_c \alpha_c \alpha_f E\left[u_\infty x_\infty\right] - \alpha_f^2 \frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{2} E\left[u_\infty^2\right]$$
$$\Pi_c = \beta p_c \alpha_c^2 E\left[x_\infty^2\right] - \beta p_f \alpha_f \alpha_c E\left[u_\infty x_\infty\right] - \alpha_c^2 \frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{2} E\left[x_\infty^2\right]$$

Proof. Appendix and Bihary and Víg (2020). □

With these long-term profits in our hands, we can examine how the difference of profits depends on the market shares of the traders. It is easy to show that

$$\frac{\partial\left(\Pi_f - \Pi_c\right)}{\partial\left(p_f - p_c\right)} > 0$$

which means that the trader with a larger market share in the market will have a larger profit in the long run. This is unfortunate because if we were to evolve the market share of the traders, then we would get a model in which, in the long run, one trader would dominate the other ultimately. In other words, in the long term, not even some type of statistical equilibrium can arise between the different traders. In the next chapter of the thesis, we outline a model in which the market shares are endogenized, and thanks to a new effect in the volatility, a long-term equilibrium arises between traders.

Chapter 4

Market heterogeneity and endogenous volatility

The purpose of Chapter 4 of the dissertation is to outline a model that provides a solution to the problem related to long-term profitability discussed at the end of the previous chapter. We introduce a new, realistic effect which "punishes" the trader that becomes dominant in the market. In this chapter, we break with the usual constant volatility of the heterogeneous agent literature, and assume that the market demand-supply disequilibrium arising from active trading appears directly in the volatility as well; the greater the current disequilibrium, the greater the volatility. Our chosen σ function is a parameterized version of $\sqrt{1+x^2}$, so the dynamics of the asset price is

$$\frac{dS(t)}{S(t)} = \beta \cdot \mathcal{D}(t)\, dt + \sigma \cdot \sqrt{1+\gamma^2 \mathcal{D}^2(t)}\, dB(t),$$

where $\mathcal{D}(t)$ is the measure of current disequilibrium, which depends on the difference of the weighted average of the traders' positions and the supply. This endogenous volatility effect solves the problem that appeared in the previous chapter: none of the traders can ultimately dominate the other in the long term. Furthermore, due to endogenous volatility, the distribution of returns will have a heavy tail, which is a welcomed feature in the a stochastic financial model.

4.1 Analysis of the trend indicator with endogenous volatility

One of the main results of the chapter is the characterization of the stochastic behavior of the trend indicator process. The stochastic differential equation of the trend indicator is as follows:

$$dx(t) = -Kx(t)\,dt + \sigma\sqrt{1+\Gamma^2 x^2(t)}\,dB(t), \tag{4.1}$$

where K, σ and Γ are constants. The square root factor appearing in the stochastic term is the central innovation of the chapter compared to the literature. This factor (or this effect) is called endogenous volatility, which arises from the current level of disequilibrium: the greater the disequilibrium in absolute value, the greater the volatility in the market.

The invariant distribution of the trend indicator process (4.1) based on Cherny (2004) is a scaled version Student's t-distribution (see Appendix A.3):

$$x_\infty \overset{d}{=} \frac{T}{\Gamma\sqrt{1+\frac{2K}{\Gamma^2\sigma^2}}} \quad \text{where} \quad T \sim \mathrm{t}\left(1+\frac{2K}{\Gamma^2\sigma^2}\right)$$

Since the degrees of freedom of the t-distribution must be positive, the invariant distribution exists only if

$$1+\frac{2K}{\Gamma^2\sigma^2} > 0 \quad \Longrightarrow \quad \frac{2K}{\Gamma^2\sigma^2} > -1$$

Note that the invariant distribution may exist even for negative K, since for K the condition is $K > -\frac{\Gamma^2\sigma^2}{2}$. This is an interesting observation, since intuition would say that a process can only have an invariant distribution if it possesses some type of mean-reverting behavior in the drift. On the other hand, the trend indicator process (4.1) with endogenous volatility may have an invariant distribution even if an – although weak – repulsive effect appears in the drift. This suggests that endogenous volatility stabilizes the process in terms of its distribution.

In order to gain a deeper understanding of the behavior of the trend indicator process (4.1), let us examine its Lamperti-transform (see Møller and Madsen (2010));

we are searching for the function $g : \mathbb{R} \mapsto \mathbb{R}$ under which the process $\xi(t) = g\big(x(t)\big)$ has constant volatility. Since according to Itô's lemma

$$d\xi(t) = (\dots)\ dt + g'\big(x(t)\big)\sigma\sqrt{1+\Gamma^2 x^2(t)}\, dB(t),$$

the suitable transformation function g can be calculated by simple integration:

$$g'(x)\,\sigma\sqrt{1+\Gamma^2 x^2} = \sigma \qquad \Longrightarrow \qquad g(x) = \int^x \frac{1}{\sqrt{1+\Gamma^2 u^2}}\, du = \frac{\operatorname{arsinh}(\Gamma x)}{\Gamma},$$

Thus, if we transform the process with the inverse of the hyperbolic sine function[1], then the resulting process has constant volatility, the dynamics of which is

$$d\xi(t) = -\frac{2K + \Gamma^2\sigma^2}{2\Gamma}\tanh\big(\Gamma\xi(t)\big)\, dt + \sigma\, dB(t) \tag{4.2}$$

In light of the transformed process (4.2), the heavy tail of the invariant distribution of the original process (4.1) is not surprising. In the process (4.2), due to the hyperbolic tangent function, mean-reversion appears very weakly. Moreover, the original $x(t)$ process would be the result of taking the sine hyperbolic[2] transform of $\xi(t)$, which is a rapidly increasing odd function. That is, we would spread out a weakly mean-reverting process with a rapidly increasing function, so it seems reasonable to obtain a process whose invariant distribution has a heavy tail, in our case a t-distribution specifically.

As explained in section 2.2.1, the trend indicator $x(t)$ can also be interpreted as the τ-yield in the model. Therefore, the heavy tail of the invariant distribution of the trend indicator also means that the returns in the usual sense also have a heavy tail due to the trading activity of the trend following traders as a result of our endogenous volatility assumption. The non-normality of returns is one of the classic and most fundamental empirical observations of the financial literature (see, for example Officer (1972)), which our theoretical model is able to reproduce.

[1] $\operatorname{arsinh}(x) = \log\left(x + \sqrt{x^2+1}\right)$

[2] $\sinh(x) = \frac{e^x - e^{-x}}{2}$

4.2 Competition of the trend follower and the index trader

At the end of chapter 4 of the thesis, we examine a simplified characteristic case of the model where a trend follower and an index trader compete with each other. The purpose of the study is to explore a realistic model in which neither trader dominates the other ultimately, and a quasi-cyclical, long-term equilibrium emerges between them.

At the heart of the model is a two-dimensional stochastic dynamic system, in which one dimension is the trend indicator $x(t)$ and the other is the logit-transform of the market weight ratio $y(t) = \log\left(\frac{p(t)}{1-p(t)}\right)$ of the trend follower. Volatility depends endogenously on current market disequilibrium. For a precise understanding of the dynamics of the two-dimensional $\big(x(t), y(t)\big)$ system driven by a single Brownian motion, we examine its deterministic skeleton, where the $dB(t)$ terms are omitted.

The characteristic behavior of the deterministic skeleton is shown in Figure 4.1. The most interesting result is the swirling behavior in the the upper left phase diagram of Figure 4.1. The two diagrams on the left characterize the drift of the system, while the two diagrams on the right characterize its volatility. All four diagrams are located on the (x, y) plane, where x is the trend indicator, y is the logit transform of the market weight of the trend-following chartist. In the upper two diagrams, the arrows are normalized to equal length in order to illustrate the direction of the change of the system. In the case of drift (top left diagram), the arrows mean the direction in which the system is expected to move. This figure shows a swirling behavior around the $(1, 0)$ and $(-1, 0)$ points. In the case of volatility (upper right diagram), arrowheads are not shown, since volatility has no expected direction. The bottom two diagrams illustrate the rate of change of the system using heat maps. In the case of drift (bottom left diagram), three "valleys" can be identified: one at the origin and one each at the centers of the vortices. The valley at the origin represents a trivial stationary state, which can be easily verified by examining the differential equations. The two additional valleys can also represent two non-trivial stationary states, but analytical verification of this is not trivial. Volatility (lower right diagram) increases in the "north-east" and "north-west" directions, which means that volatility is high when trend followers are dominant and the market is in a large trend.

A financial description of the swirling behavior is as follows. Assume that the trend follower chartists and index-traders have weight in the market, and the trend-followers identify a small – say, positive – trend. In response to the small trend, the trend followers take a small positive position, with which they drive the price up – thanks to their price effect. Thus, a larger and larger trend begins to form, which the trend followers react with larger and larger positions to. Moreover, in the meantime, their market weight is also increasing: thanks to their increasing position, they gain more and more from price increases. However, in this euphoric state of the market, market disequilibrium begins reach levels such that it already greatly increases volatility. Volatility becomes so great that trend followers' expected profit becomes negative. When they start losing, their market share is still quite large (although it is decreasing), so the bubble is growing even further for a while, albeit at a slower pace. However, due to endogenous volatility, trend followers suddenly suffer enormous losses, and they also miss the trend reversal due to the delay of the trend indicator compared to their market share. The trend followers have thus become victims of their own success. If market disequilibrium appears in the price dynamics both at the level of expected returns and at the level of volatility, then the trend followers are able to blow bubbles, during which they are able to take profits for a while. Eventually, however, they lose a lot due to self-inflicted high volatility.

The behavior of the model is illustrated in Figure 4.2. with simulated trajectories. The diagrams show the history of a 50-year trajectory along several dimensions. It is worth treating these diagrams as if they were a kind of caricature of real market phenomena. The top left diagram shows the dollar price of the single risky asset, $S(t)$. Several large trends can be identified in the diagram, and the change in volatility is also noticeable. The most striking event is the huge spike around the 15th year, followed by a sudden decline. In the top right diagram, the trend indicator $x(t)$ can be seen, which stays within the $(-1, 1)$ interval for most of the 50 years, but it spikes up to 2 around the 15th year. Since $\tau = 1$, a value of 1 in this figure corresponds roughly to a 100% annual return.

The bottom left diagram shows the evolution of the market share $p(t)$ of the trend followers. It is clear that at the same time as they induce large trends – positive or negative – their market share also increases. However, the correction also happens quickly: their market share increases sharply, but it can fall back just as quickly due to sudden losses. Stressed periods are followed by calm periods.

The lower right diagram shows the volatility calculated directly from the model. This diagram resembles the empirical volatility process calculated from real-world asset price data, where highly volatile and calm periods alternate. Of course, the 200% volatility seen around year 15 may seem unrealistic to a practicing trader, so we emphasize the caricaturistic nature of the model again. However, barring the largest spike, volatility remains roughly in the 20% to 60% range, which no longer seems unrealistic.

The effect of endogenous, stochastic volatility is also illustrated by Figure 4.3, where the distribution of log returns can be seen for different time intervals, calculated from a 2000-year simulated trajectory. In addition to the histograms of the daily, monthly and annual returns, the density function of the corresponding normal distribution was also plotted for comparison. Since the asset price defined by the model is an Itô process, the conditional distribution of short-term returns is by definition normal. However, due to stochastic volatility, the unconditional distribution of returns is as a mixture of normal distributions with different standard deviations. As a result, the returns of all three time intervals show excess kurtosis and a heavier tail compared to normal.

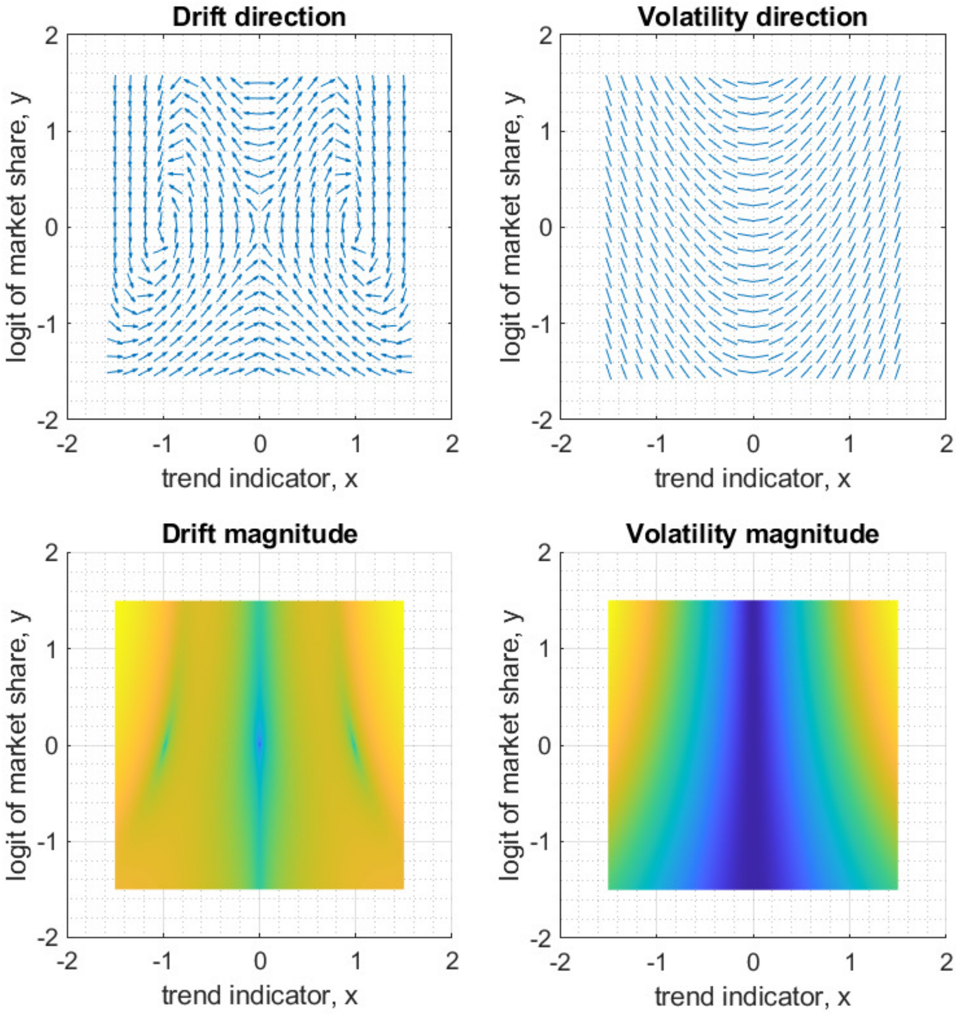

Figure 4.1: Phase diagrams of the two-dimensional system of the trend indicator $x(t)$ and the logit of the trend follower's market share $y(t)$. The two diagrams on the left describe the drift of the system, while the two diagrams on the right describe the volatility of the system. The top two diagrams show the direction of the change of the system with normalized arrows of equal length, while the bottom two diagrams show the speed of the change with a heat map. Blue color means a slow change, while yellow means fast change.

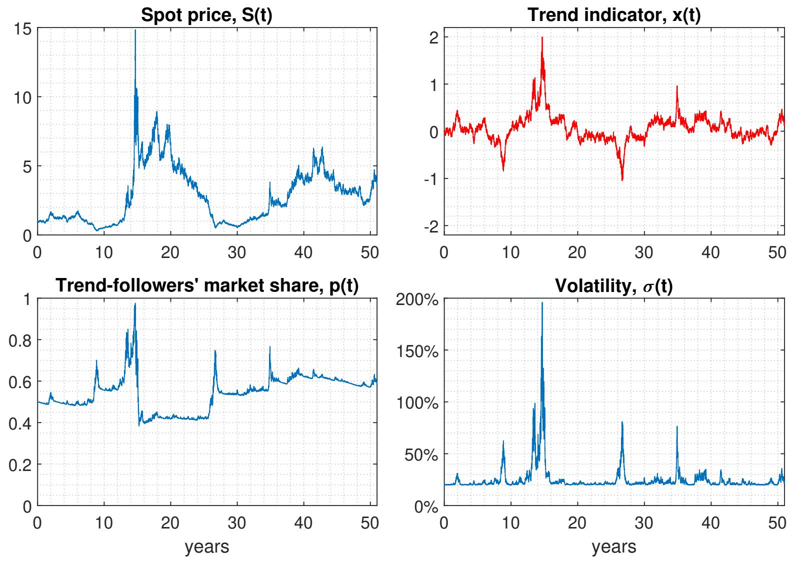

Figure 4.2: Simulated trajectory of the model.

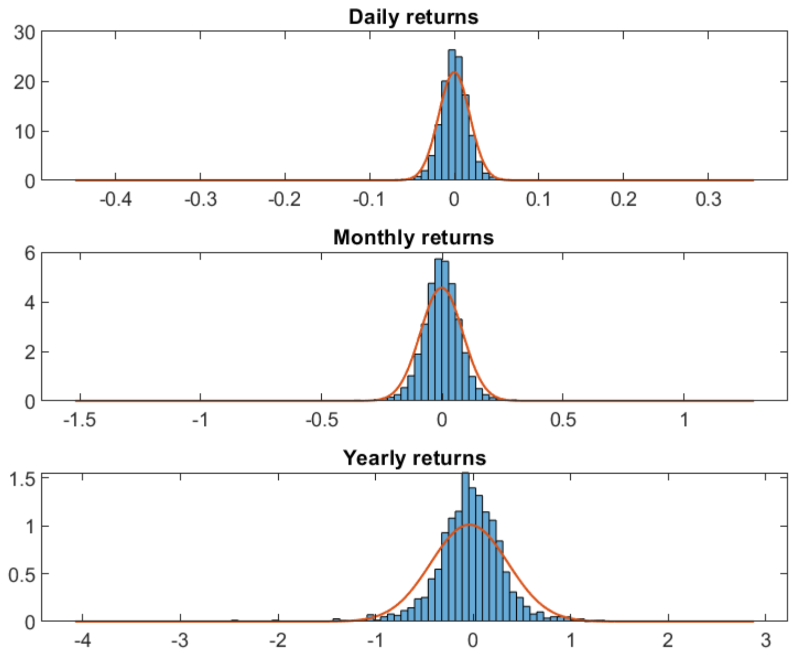

Figure 4.3: Histograms (and the corresponding normal densities) of the returns of the model. Calculated from a simulated 2000 year trajectory.

Chapter 5

The author's publications

Bajai, M., Hortay, O. and Víg, A. A. (2022). Electricity Market Liquidity and Price Spikes. *Periodica Polytechnica Social and Management Sciences*, 30(1), pp. 49–56.

Havran, D., Kerényi, P. and Víg, A. A. (2021). Social Finance and Agricultural Funding. In *Innovations in Social Finance.* pp. 269–290.

Bihary, Z. and Víg, A. A. (2020). Heterogén kereskedési stratégiák hatása a piaci árfolyamokra. *Közgazdasági Szemle*, 67(7–8), pp. 688–707.

Hortay, O. and Víg, A. A. (2020). Potential effects of market power in Hungarian solar boom. *Energy*, 213, p. 118857.

Bihary, Z. and Víg, A. A. (2018). Portfólióallokáció csődveszély esetén, korlátolt felelősség mellett. *Közgazdasági Szemle*, 65(7–8), pp. 711–725.

Vidovics-Dancs, Á. and Víg, A. A. (2017). Indexed bonds with mean-reverting risk factors. In *Proceedings 31st European Conference on Modelling and Simulation.* pp. 81–86.

Havran, D., Kerényi, P. and Víg, A. A. (2017). Szállítói finanszírozás vagy bankhitelek? - A magyar vállalatok 2010 és 2015 közötti tanulságai. *Hitelintézeti Szemle*, 16(4), pp. 86–121.

Appendix A

Appendix

A.1 Numeric solution of the Kolmogorov forward equation

The two-dimensional (price dislocation and trend indicator) SDE of the main specification is the following:

$$
\begin{aligned}
du(t) &= \beta \left(p_c \alpha_c x^* \tanh\left(\frac{x(t)}{x^*}\right) - p_f \alpha_f u(t) \right) dt + (\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_f)^\top \, d\boldsymbol{B}(t) \\
dx(t) &= \left[\beta \left(p_c \alpha_c x^* \tanh\left(\frac{x(t)}{x^*}\right) - p_f \alpha_f u(t) \right) - \frac{1}{\tau} x(t) \right] dt + \boldsymbol{\sigma}_s^\top \, d\boldsymbol{B}(t).
\end{aligned}
\tag{A.1}
$$

Assume that the dynamics of the N-dimensional stochastic process $\boldsymbol{Y}(t)$ is given by the SDE

$$
d\boldsymbol{Y}(t) = \boldsymbol{a}\big(\boldsymbol{Y}(t)\big)\, dt + \boldsymbol{\Sigma}\big(\boldsymbol{Y}(t)\big)\, d\boldsymbol{B}(t)
$$

where $\boldsymbol{a} : \mathbb{R}^N \mapsto \mathbb{R}^N$, $\boldsymbol{\Sigma} : \mathbb{R}^N \mapsto \mathbb{R}^{N \times M}$ and $\boldsymbol{B}(t)$ is an M-dimensional Wiener-process with independent components. Let $\boldsymbol{D}(\boldsymbol{y}) = \frac{1}{2}\boldsymbol{\Sigma}(\boldsymbol{y})\boldsymbol{\Sigma}^\top(\boldsymbol{y})$. Then the dynamics of the underlying density $f_t(\boldsymbol{y})$ of the system is governed by the so-called forward Kolmogorov equation.

$$
\frac{\partial f_t(\boldsymbol{y})}{\partial t} = -\sum_{j=1}^{N} \frac{\partial a_i(\boldsymbol{y}) f_t(\boldsymbol{y})}{\partial y_i} + \sum_{k=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 D_{j,k}(\boldsymbol{y})\, f_t(\boldsymbol{y})}{\partial y_j \partial y_k}
\tag{A.2}
$$

The invariant density $f_\infty(\boldsymbol{y})$ is arrived at when $\frac{\partial f_\infty(\boldsymbol{y})}{\partial t} = 0$. We calculate $f_\infty(u,x)$ of equation (A.1) by solving (A.2) numerically on the grid $\mathbb{T} \times \mathbb{U} \times \mathbb{X} = \{0, 0 + \Delta t, \ldots, T\} \times \{\underline{u}, \underline{u} + \Delta u, \ldots, \overline{u}\} \times \{\underline{x}, \underline{x} + \Delta x, \ldots, \overline{x}\}$. Our main assumption throughout the numerical calculations is that the relevant support of the density $f_t(u,x)$ is finite, i.e. $f_t(u,x) \approx 0$ if the point (u,x) is far enough from the origin:

$$\lim_{(u^2+x^2)\to\infty} f_t(u,x) = 0 \quad \forall t \in [0,\infty),$$

During computations this assumption means that at the edge of the grid $\mathbb{U} \times \mathbb{X}$ we set the density to zero:

$$\begin{aligned} f_t(u,\underline{x}) &= f_t(u,\overline{x}) = 0 \quad \forall u \in \mathbb{U} \text{ and } \forall t \in \mathbb{T}, \\ f_t(\underline{u},x) &= f_t(\underline{u},x) = 0 \quad \forall x \in \mathbb{X} \text{ and } \forall t \in \mathbb{T}. \end{aligned}$$

Assume now that for $t \in \mathbb{T}$ we have the density matrix $f_t(\mathbb{U},\mathbb{X}) \in \mathbb{R}^{|\mathbb{U}|\times|\mathbb{X}|}$. In a single iteration of our algorithm we solve (A.2) explicitly by approximating the derivatives with their finite differences on the inner points of the grid $\mathbb{U} \times \mathbb{X}$:

$$\begin{aligned} f^*_{t+\Delta t}(u,x) = f_t(u,x) &- \frac{(a_u \cdot f_t)(u+\Delta u,x) - (a_u \cdot f_t)(u-\Delta u,x)}{2\Delta u}\Delta t \\ &- \frac{(a_x \cdot f_t)(u,x+\Delta x) - (a_x \cdot f_t)(u,x-\Delta x)}{2\Delta x}\Delta t \\ &+ D_{u,u}\frac{f_t(u+\Delta u,x) - 2f_t(u,x) + f_t(u-\Delta u,x)}{(\Delta u)^2}\Delta t \\ &+ D_{x,x}\frac{f_t(u,x-\Delta x) - 2f_t(u,x) + f_t(u,x-\Delta x)}{(\Delta x)^2}\Delta t \\ &+ D_{u,x}\frac{f_t(u+\Delta u,x+\Delta x) + f_t(u-\Delta u,x-\Delta x)}{2(\Delta u)(\Delta x)}\Delta t \\ &- D_{u,x}\frac{f_t(u+\Delta u,x-\Delta x) + f_t(u-\Delta u,x+\Delta x)}{2(\Delta u)(\Delta x)}\Delta t \end{aligned}$$

$\forall (u,x) \in \overline{\underline{\mathbb{U}}} \times \overline{\underline{\mathbb{X}}} = \{\underline{u} + \Delta u, \ldots, \overline{u} - \Delta u\} \times \{\underline{x} + \Delta x, \ldots, \overline{x} - \Delta x\}$.

$f^*_{t+\Delta t}(\mathbb{U},\mathbb{X})$ is not necessarily a probability density matrix in the sense that it may have negative components, and $\sum_{u\in\mathbb{U}}\sum_{x\in\mathbb{X}} f^*_{t+\Delta t}(u,x)\Delta u \Delta x = 1$ might not hold. To this end, we make the following corrections in each iteration:

$$
\begin{aligned}
f^{**}_{t+\Delta t}(u,x) &= \max\left(f^{*}_{t+\Delta t}(u,x),0\right) \quad \forall (u,x)\in \mathbb{U}\times\mathbb{X} \quad \text{and} \\
f_{t+\Delta t}(u,x) &= \frac{f^{**}_{t+\Delta t}(u,x)}{\sum_{u\in\mathbb{U}}\sum_{x\in\mathbb{X}} f^{**}_{t+\Delta t}(u,x)\Delta u\Delta x} \quad \forall (u,x)\in \mathbb{U}\times\mathbb{X}.
\end{aligned}
$$

The initial density we use in our calculations is the uniform distribution:

$$
f_0(u,x) = \frac{1}{|\underline{\mathbb{U}}|\cdot|\underline{\mathbb{U}}|\Delta u\Delta x} \quad \forall (u,x)\in \underline{\mathbb{U}}\times\underline{\mathbb{X}}
$$

A.2 Proofs

After linearization of (A.1) the resulting SDE is

$$
\begin{aligned}
du(t) &= \beta\big(p_c\alpha_c x(t) - p_f\alpha_f u(t)\big)\,dt + (\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_f)^\top \, d\boldsymbol{B}(t) \\
dx(t) &= \beta\big(p_c\alpha_c x(t) - p_f\alpha_f u(t)\big)\,dt - \frac{1}{\tau}x(t)\,dt + \boldsymbol{\sigma}_s^\top\, d\boldsymbol{B}(t).
\end{aligned}
$$

$Y(t) = \big(u(t),x(t)\big)^\top$ is a two dimensional Ornstein-Uhlenbeck process, i.e. is of the form

$$
d\boldsymbol{Y}(t) = -\boldsymbol{\Theta}\boldsymbol{Y}(t)\,dt + \boldsymbol{\Sigma}\,d\boldsymbol{B}(t) \tag{A.3}
$$

where

$$
\boldsymbol{\Theta} = \begin{pmatrix} \beta p_f\alpha_f & -\beta p_c\alpha_c \\ \beta p_f\alpha_f & \frac{1}{\tau} - \beta p_c\alpha_c \end{pmatrix} \in \mathbb{R}^{2\times 2} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma}_s^\top - \boldsymbol{\sigma}_f^\top \\ \boldsymbol{\sigma}_s^\top \end{pmatrix} \in \mathbb{R}^{2\times 2}
$$

Proof of Proposition 1. $Y(t)$ defined by (A.3) has the solution[1]

$$
\boldsymbol{Y}(t) = \boldsymbol{e}^{-\boldsymbol{\Theta}t}\boldsymbol{Y}(0) + \int_0^t \boldsymbol{e}^{-\boldsymbol{\Theta}(t-v)}\boldsymbol{\Sigma}\,d\boldsymbol{B}(v), \tag{A.4}
$$

which is asymptotically stationary if and only if – see Shreve (2004) – the ordinary differential equation system $\dot{\boldsymbol{y}} = -\boldsymbol{\Theta}\boldsymbol{y}$ is strongly stable, which requires that the real part of all the eigenvalues of $\boldsymbol{\Theta}$ are strictly positive. This holds if $\beta\,(p_c\alpha_c - p_f\alpha_f) < \frac{1}{\tau}$

[1] Here $e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!}$, and it holds that $\left(e^{At}\right)' = Ae^{At}$.

and $\beta p_f \alpha_f > 0$ both hold.

Furthermore, (A.4) is a simple stochastic integral, thus it is normally distributed for all t. The expectation of the stochastic integral component is 0 for all t, thus

$$\lim_{t\to\infty} E\left[\boldsymbol{Y}(t)\right] = \lim_{t\to\infty} \boldsymbol{e}^{-\boldsymbol{\Theta}t}\boldsymbol{Y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{A.5}$$

if the eigenvalues of $\boldsymbol{\Theta}$ are positive, which holds if $\beta\left(p_c\alpha_c - p_f\alpha_f\right) < \frac{1}{\tau}$ and $\beta p_f \alpha_f > 0$.

Now let $\boldsymbol{\rho}(t) = COV\left[\boldsymbol{Y}(t)\right] = E\left[\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right] - E\left[\boldsymbol{Y}(t)\right]E\left[\boldsymbol{Y}^\top(t)\right]$ and let t be large enough so that $\boldsymbol{\rho}(t) \approx E\left[\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right]$. Then

$$\begin{aligned}
d\boldsymbol{\rho}(t) &= dE\left[\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right] = E\left[d\left(\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right)\right] = \\
&= E\left[\left(d\boldsymbol{Y}(t)\right)\boldsymbol{Y}^\top(t)\right] + E\left[\boldsymbol{Y}(t)\left(d\boldsymbol{Y}^\top(t)\right)\right] + E\left[d\boldsymbol{Y}(t)d\boldsymbol{Y}^\top(t)\right] = \\
&= -\boldsymbol{\Theta}E\left[\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right]dt - E\left[\boldsymbol{Y}(t)\boldsymbol{Y}^\top(t)\right]\boldsymbol{\Theta}^\top dt + \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top dt \Longrightarrow \\
\Longrightarrow \dot{\boldsymbol{\rho}} &= -\boldsymbol{\Theta}\boldsymbol{\rho} - \boldsymbol{\rho}\boldsymbol{\Theta}^\top + \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top
\end{aligned}$$

The covariance matrix of the asymptotic distribution is got by setting $\dot{\boldsymbol{\rho}} = 0$, and solving the resulting system of algebraic equations. □

The dynamics of the trend indicator $x(t)$ in the one-dimensional specification is

$$dx(t) = -\left(\beta p_f\alpha_f + \frac{1}{\tau}\right)x(t)\,dt + \beta p_c\alpha_c x^\star \tanh\left(\frac{x(t)}{x^\star}\right)dt + \boldsymbol{\sigma}_s^\top\,d\boldsymbol{B}(t) \tag{A.6}$$

Proof of Proposition 2. The process (A.6) is a time-homogeneous Ito-process of the form

$$dx(t) = b\big(x(t)\big)\,dt + \boldsymbol{\sigma}^\top\big(x(t)\big)\,d\boldsymbol{B}(t)$$

with $\boldsymbol{\sigma}\big(x(t)\big) = \boldsymbol{\sigma}_s > \boldsymbol{0}\ \forall x_t \in \mathbb{R}$. The main theorem of Cherny (2004) applies, and the unique limiting (invariant) distribution is given by

$$f_\infty\left(x\right) = C\frac{\exp\left(\int^x \frac{2b(y)}{\boldsymbol{\sigma}^\top(y)\boldsymbol{\sigma}(y)}\,dy\right)}{\boldsymbol{\sigma}^\top(x)\boldsymbol{\sigma}(x)}$$

where $\int^x$ denotes the indefinite integral and $C > 0$ is a suitable normalizing constant.

Substituting in

$$b(y) = -\left(\beta p_f \alpha_f + \frac{1}{\tau}\right) y + \beta p_c \alpha_c x^\star \tanh\left(\frac{y}{x^\star}\right)$$
$$\text{és} \quad \boldsymbol{\sigma}(x) = \boldsymbol{\sigma}_s,$$

and we arrive at the invariant density function:

$$f_\infty(x) = C \left(\cosh\left(\frac{x}{x^\star}\right)\right)^{\frac{2\beta p_c \alpha_c (x^\star)^2}{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}} \exp\left(-\frac{\frac{1}{\tau} + \beta p_f \alpha_f}{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s} x^2\right)$$

□

Proof of Proposition 3. The function $f_\infty(x)$ of equation (3.1) is unimodal if $f''_\infty(0) < 0$, and bimodal if $f''_\infty(0) > 0$:

$$f''_\infty(0) < 0$$
$$\frac{\frac{2\beta p_c \alpha_c (x^\star)^2}{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}}{(x^\star)^2} - \frac{2}{\frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{\frac{1}{\tau} + \beta p_f \alpha_f}} < 0$$
$$\beta (p_c \alpha_c - p_f \alpha_f) < \frac{1}{\tau}$$

□

Proof of Proposition 4. Megmutatjuk a számításokat Π_f esetén. Π_c esetén minden hasonlóan számolható. We show the calculations for Π_f, which go similarly for Π_c. First, dynamics of $S_t = e^{s_t}$ is

$$\frac{dS(t)}{S(t)} = \left(\frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{2} + \beta \sum_{j \in \{f,c,i\}} p_j Z_j(t)\right) dt + \boldsymbol{\sigma}_s^\top \, d\boldsymbol{B}(t)$$

With the self-financing assumption, the dynamics of log-wealth $\log W_f(t)$ is

$$d(\log W_f)(t) = Z_f(t) \frac{dS(t)}{S(t)} - \frac{Z_f^2(t)}{2} \frac{d\langle S \rangle(t)}{S^2(t)}$$

In the linear specification, the position functions are $Z_f(t) = \alpha_f u(t)$, $Z_c(t) = \alpha_c x(t)$

and $Z_i(t) = 0$, so the dynamics of log-wealth can be written in terms of $u(t)$ and $x(t)$:

$$d\left(\log W_f\right)(t) = \left(\alpha_f \frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{2} u(t) + \beta p_c \alpha_c \alpha_f u(t) x(t) + \alpha_f^2 \left(\beta p_f - \frac{\boldsymbol{\sigma}_s^\top \boldsymbol{\sigma}_s}{2}\right) u^2(t)\right) dt + \\ + \alpha_f u(t) \boldsymbol{\sigma}_s^\top \, d\boldsymbol{B}(t)$$

$$d\left(\log W_f\right)(t) = g\left(u(t), u(t)x(t), u^2(t)\right) \, dt + \alpha_f u(t) \boldsymbol{\sigma}_s^\top \, d\boldsymbol{B}_t$$

where $g : \mathbb{R}^3 \mapsto \mathbb{R}$ is an affine function. From here, by using the law of iterated expectations[2] and the fact that we know the asymptotic distribution of $(u_t, m_t)^\top$ we get that

$$\begin{aligned}
\Pi_f = \lim_{T\to\infty} E\left[\frac{\log W_f(T) - \log W_f(0)}{T}\right] = \\
&= \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T d\left(\log W_f\right)(t)\right] = \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T E_t\big[d\left(\log W_f\right)(t)\big]\right] = \\
&= \lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T g\left(u(t), u(t)x(t), u^2(t)\right) \, dt\right] = \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T g\left(E[u(t)], E[u(t)x(t)], E[u^2(t)]\right) \, dt = \\
&= \lim_{T\to\infty} \frac{1}{T} \int_0^T g\left(E[u_\infty], E[u_\infty x_\infty], E[u_\infty^2]\right) \, dt = \\
&= g\left(0, E[u_\infty x_\infty], E[u_\infty^2]\right)
\end{aligned}$$

□

A.3 Invariant distribution of the trend indicator

The invariant density by Cherny (2004) is

$$x_\infty \sim f(x) = \frac{C}{\sigma^2(x)} \exp\left(\int^x \frac{2\mu(u)}{\sigma^2(u)} \, du\right) \tag{A.7}$$

[2]Here we denote the conditional expectation by E_t, i.e. $E\left[X_T | \mathcal{F}_t\right] \doteq E_t\left[X_T\right]$.

where $\mu(x)$ and $\sigma(x)$ are the drift and volatility functions respectively, $C > 0$ is a suitable normalization constant. Then

$$\int \frac{2\mu(x)}{\sigma^2(x)}\,dx = \int \frac{-2Kx}{\sigma^2\left(1+\Gamma^2x^2\right)}\,dx =$$
$$= -\frac{K}{\Gamma^2\sigma^2}\int \frac{2\Gamma^2 x}{\left(1+\Gamma^2x^2\right)}\,dx = -\frac{K}{\Gamma^2\sigma^2}\log\left(1+\Gamma^2x^2\right)$$

and then the invariant density is

$$f(x) = \frac{C}{\sigma^2\left(1+\Gamma^2x^2\right)}\left(1+\Gamma^2x^2\right)^{-\frac{K}{\Gamma^2\sigma^2}} = C\left(1+\Gamma^2x^2\right)^{-\left(1+\frac{K}{\Gamma^2\sigma^2}\right)} \tag{A.8}$$

As the the probability density function of the standard t-distribution with degrees of freedom ν is $g(x) = C\cdot\left(1+\frac{x^2}{\nu}\right)^{-\frac{1+\nu}{2}}$, it is easy to check that $f(x)$ is the density of a scaled t-distribution with degrees of freedom $\nu = 1+\frac{2K}{\Gamma^2\sigma^2}$:

$$x_\infty \overset{d}{=} \frac{T}{\Gamma\sqrt{1+\frac{2K}{\Gamma^2\sigma^2}}} \quad \text{where} \quad T \sim \mathrm{t}\left(1+\frac{2K}{\Gamma^2\sigma^2}\right)$$

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