On Asymptotic Stability of Prime Ideals in Noncommutative Rings

Nicholas Richard Collier

Thesis submitted for the degree of Doctor of Philosophy
University of Warwick Mathematics Institute

September 2003

Contents

Acknowledgements

I would like to thank gratefully my supervisor, Dr. C.R. Hajarnavis, for his advice, suggestions and support during the time I have spent writing this thesis.

I also give acknowledgement to the Engineering and Physical Sciences Research Council (EPSRC) for the funding provided to me during the first three years of my research, from 1999 until 2002. Further I thank the University of Warwick Mathematics Institute, for providing teaching and supervising opportunities which were a valuable source of additional funding.

Declaration

To the best of my knowledge I can state that all results in this thesis are, unless otherwise acknowledged, original and my own work.

This thesis has not been submitted for a degree at any other university.

List of Frequently Used Symbols

Let R be a ring. Let I be an ideal of R. Let P and Q be prime ideals of R. Let X be a subset of R. Let M and N be right R-modules.

Symbol	*Meaning*
M_R	M is a right R-module
${}_RM$	M is a left R-module
$A \subseteq B$	the set A is a subset of the set B
$A \subsetneqq B$	the set A is a subset of the set B and $A \neq B$
$A \nsubseteq B$	the set A is not a subset of the set B
$\mathbb{Z}$	the set of integers
$N(R)$	the prime radical of R
$\mathcal{C}(I)$	the set of all elements of R regular modulo I
$\mathcal{C}'(I)$	the set of all elements of R right regular modulo I
${}'\mathcal{C}(I)$	the set of all elements of R left regular modulo I
$r_R(Y)$ or $r(Y)$	the right annihilator in R of a subset $Y \subseteq M$
$l_M(X)$	the left annihilator in M of a subset $X \subseteq R$
$Ass(M)_R$	the set of all associated prime ideals of M_R
$Aff(M)_R$	the set of all affiliated prime ideals of M_R
$Q \rightsquigarrow P$	Q is linked to P
$Cl(P)$	the clique containing P
$\cong$	an isomorphism (of rings or modules)
$M_n(R)$	the set of all $n \times n$ matrices over R
RX^{-1}	the right ring of fractions of R with respect X
$X^{-1}R$	the left ring of fractions of R with respect X
MX^{-1}	the module of fractions of M with respect X
I^e	the extension of I into RX^{-1}
J^c	the contraction of an ideal J of RX^{-1} into R

Symbol	*Meaning*
$End_R(M)$	the set of all R-endomorphisms of M
1_R	the multiplicative identity element of R
$M \otimes_R N$	the tensor product of M and N
$Z(M)$	the singular submodule of M
$t_X(M)$	the X-torsion submodule of M
$R[t^{-1}, It]$	the Rees ring of R with respect to I
$gr_I(R)$	the associated graded ring of R at I
S_n	the nth symmetric group
s_n	the nth standard polynomial
$PIdeg(R)$	the PI degree of R
R^{op}	the opposite ring of R
$Brod(R, I)$	the Brodmann stabilizing point of I

Summary

This thesis is concerned with the asymptotic behaviour of the prime divisors of powers of an ideal I in a ring R. The main focus is on the case when R is a prime Noetherian polynomial identity ring.

Our starting point is a theorem, first proved by M. Brodmann ([2]), which shows that in a commutative Noetherian ring, the prime divisors of powers of an ideal turn out to be asymptotically stable. We first generalize this result to Azumaya algebras, and then attempt to find an overring which satisfies the following two criteria:

1. prime divisors 'go up and down' between this overring and the original ring;

2. prime divisors 'go up and down' between this overring and its centre.

After finding conditions under which such an overring exists, it is possible to formulate a theorem concerning asymptotic stability of prime divisors in the original ring, using the fact that prime divisors are asymptotically stable in the centre of the overring.

In chapter 1 we outline the basic definitions, results and theory required for this thesis. In chapter 2 we first establish some properties concerning associated primes of a module, and in Proposition 2.1.3 we note that a result connecting two sequences, due to M. Brodmann in the commutative case, is equally valid in the noncommutative setting. Next we outline a proof of the definitive asymptotic stability result in the commutative theory, due to M. Brodmann ([2]). The proof followed in this thesis is that given by S. McAdam and P. Eakin in [10]. In the final section of this chapter we discuss some results concerning invertible ideals, due to A.J. Gray, and their applications to aspects of the proof of M. Brodmann's result. We note difficulties with this approach and give reasons why this does not lead to a generalization of M. Brodmann's theorem.

In chapter 3 we provide the generalization of M. Brodmann's theorem to Azumaya algebras. Affiliated prime ideals have often been viewed as a generalized class of associated prime ideals in the noncommutative setting, which contain more information

than associated prime ideals. With this in mind, we shift the focus of our study from associated prime ideals to affiliated prime ideals. We also show that there is a very simple relationship between the prime divisors of an ideal in an Azumaya algebra R and the prime divisors of the corresponding ideal of the centre of R. We give an example to illustrate this result.

In chapter 4 we introduce localizations of rings and modules, giving basic properties. We give a method of 'going up and down' between ideals of the original ring and ideals of localizations of that ring, and give conditions under which this method is applicable. We then use this to derive a relationship between the prime divisors of the original ring and the prime divisors of localizations of that ring.

In chapter 5 we attempt to find a ring satisfying the two criteria set out above, using a localization as the overring. We give conditions under which this localization is an Azumaya algebra, enabling us to make use of the results in chapter 3. First we give necessary and sufficient conditions for asymptotic stability of prime divisors to hold in a prime Noetherian polynomial identity ring. We then give alternative sufficient conditions for asymptotic stability to hold in the same class of rings. We conclude with a discussion of some results in the general Noetherian case.

Introduction

Let I be an ideal in a commutative Noetherian ring R. The usage of the terminology *prime divisor of I* to mean an associated prime ideal of $(R/I)_R$ arises from the fact that the associated prime ideals of $(\mathbb{Z}/a\mathbb{Z})_{\mathbb{Z}}$ are precisely those ideals $p\mathbb{Z}$ such that p is a prime divisor of a. A study of the behaviour of the sets $Ass(R/I^n)_R$ as n gets larger was initiated by M. Nagata ([14]), who noticed the following fact: if R is a commutative Noetherian integral domain, P is a prime ideal of R and b is a non-zero element of P, then P is an associated prime ideal of R/bR implies that P is an associated prime ideal of R/cR, for *any* non-zero element c of P. It is an easy consequence of this that P is an associated prime ideal of R/bR if and only if P is an associated prime ideal of R/b^nR for any positive integer n.

Observations made by D. Rees in [17] led to various attempts to generalize M. Nagata's result to all ideals in a commutative Noetherian ring. The first such attempt was made by L.J. Ratliff, who conjectured the general result in [6]. This attempt failed, and the conjecture was subsequently shown to be false by M. Brodmann in [2]. However, L.J. Ratliff did prove a modified result for large powers of ideals which contain regular elements ([6, 2.11]), and also proved a corresponding statement concerning the integral closure of ideals ([6, 2.5]). The latter result was then improved in a later paper of L.J. Ratliff ([9, Theorems 2.4 and 2.8]).

The main stability result for prime divisors in a commutative Noetherian ring R was proved by M. Brodmann in [2]: that the sequences $Ass(R/I^n)_R$ and $Ass(I^{n-1}/I^n)_R$ eventually become stable as n gets large. S. McAdam and P. Eakin then provided characterizations of the stable values of these sequences in [11]. An excellent overview of the development of this theory is provided by L.J. Ratliff in [8].

Chapter 2 gives a detailed exposition of the proof of M. Brodmann's theorem (2.3.8) given in [10], which uses the associated graded ring, $gr_I(R)$. The proof has three main stages. Firstly, it is shown that it is sufficient to show that the sequence $Ass(I^{n-1}/I^n)_R$ is eventually stable. Secondly, it is shown that the union of all prime ideals in all terms of this sequence is a finite set. This is achieved by showing that there is a relationship between the associated prime ideals of $(I^{n-1}/I^n)_R$ and the associated prime ideals of $(gr_I(R))_{gr_I(R)}$, and by using the fact that $gr_I(R)$ is a Noetherian ring. Lastly, it is shown that the sequence $Ass(I^{n-1}/I^n)_R$ is eventually monotonically increasing. In section 2.4 we discuss the extent to which this argument can be generalized to a noncommutative setting, using some results of A.J. Gray ([5]).

Chapter 3 marks the beginning of our focus on noncommutative rings, and study of affiliated prime ideals instead of associated prime ideals. In Corollary 3.5.1 we give a generalization of M. Brodmann's theorem (Corollary 2.3.8) to a class of rings called Azumaya algebras. This is achieved by using the fact that there is a one-to-one correspondence between ideals of an Azumaya algebra and ideals of its centre. It is shown that this correspondence preserves the 'affiliated' property of prime ideals (3.3.6). An example is provided to illustrate this result in section 3.4.

A study of localizations of rings at suitable sets is made in chapter 4 and we determine in Theorem 4.7.5 and Theorem 4.7.6 conditions under which a localization preserves the 'affiliated' property of prime ideals. The conditions sufficient to preserve this property when 'going up' differ from those sufficient to preserve the property when 'going down': however we apply a result of T.H. Lenagan and R.B. Warfield ([7, Theorem 1.2], Theorem 4.7.10) to show that in the situation we are interested in the conditions are the same (Corollary 4.7.13).

The results in chapters 3 and 4 are tied together in chapter 5, where we make use of a theorem due separately to M. Artin ([1]) and C. Procesi ([16]), which determines a situation in which localizations are Azumaya algebras. This enables us to give sufficient conditions for asymptotic stability of prime divisors in a prime Noetherian polynomial identity ring to hold, in Proposition 5.2.3. We also give a necessary condition for the sets of prime divisors of two ideals in a prime Noetherian polynomial identity ring to coincide, in Proposition 5.2.2. In section 5.3 we give different sufficient conditions for asymptotic stability to hold, (Proposition 5.3.1). This thesis concludes by giving a characterization of the situation in which the sets of prime divisors of two ideals in a general Noetherian ring coincide (Proposition 5.4.1) in terms of the prime divisors of a localization. We then give an equivalent condition for asymptotic stability to hold in a general Noetherian ring (Theorem 5.4.2), also in terms of asymptotic stability in a localization.

Chapter 1

Preliminaries

Throughout this thesis we will assume that all rings are associative and have an identity element and that all modules are unital. If R is a ring and M is a right (left) R-module then we will denote this by M_R (${}_RM$). Unless stated otherwise all modules over rings will by convention be taken to be right modules.

1.1 Chain conditions

Let R be a ring. Let M be an R-module. If M satisfies the ascending chain condition (ACC) on submodules then M is called a *Noetherian* module. If M satisfies the descending chain condition (DCC) on submodules then M is called an *Artinian* module. If R_R (${}_RR$) is a Noetherian module then R is called a *right Noetherian* (*left Noetherian*) ring. Similarly, if R_R (${}_RR$) is an Artinian module then R is called a *right Artinian* (*left Artinian*) ring. If N is a submodule of an R-module M then M is Noetherian if and only if N and the factor module M/N are both Noetherian. A finite direct sum of Noetherian modules is again Noetherian. M is called a *finitely generated* module if there exist elements $m_1, \ldots, m_n$, for some positive integer n, such that $M = m_1R + \cdots + m_nR$. Any finitely generated right module over a right Noetherian ring is Noetherian.

Examples of noncommutative Noetherian rings can be generated from commutative Noetherian rings by using the following result.

Proposition 1.1.1. [4, Proposition 1.8] *Let R be a right Noetherian ring. Let n be* *a positive integer. Let S be a subring of the matrix ring $M_n(R)$. If S contains the subset:*

$$\left\{ \begin{bmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{bmatrix} \mid r \in R \right\}$$

then S is a right Noetherian ring. In particular $M_n(R)$ is a right Noetherian ring.

The two-sided ideals of $M_n(R)$ can be easily described in terms of the ideals of R as follows. A subset X of $M_n(R)$ is an ideal of $M_n(R)$ if and only if $X = M_n(I)$ for some ideal I of R.

1.2 Prime and semi-prime ideals

An ideal P of a ring R is called a *prime ideal* if $P \neq R$ and whenever I and J are ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. Equivalently, P is a prime ideal of R if $P \neq R$ and whenever I and J are right (left) ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. An ideal M of R is called *maximal ideal* if $M \neq R$ and whenever M' is an ideal of R such that $R \supsetneq M' \supseteq M$, then $M' = M$. Any maximal ideal of R is a prime ideal of R. R is called a *prime ring* if 0 is a prime ideal of R. An ideal P of R is a prime ideal if and only if R/P is a prime ring.

A prime ring R is right Artinian if and only if it is left Artinian; rings of this class are called *simple Artinian* rings. An element $x \in R$ is called *invertible* if there is an element $y \in R$ such that $xy = yx = 1_R$. If such an element $y \in R$ exists then it is unique and is denoted by x^{-1}. A ring R with at least two elements such that every non-zero element of R is invertible is called a *division ring*. Note that the centre of a division ring is a field. A ring R is a simple Artinian ring if and only if there is a division ring D and a positive integer n such that $R \cong M_n(D)$. It follows from this that the centre of a simple Artinian ring is a field.

A prime ideal P of R is called a *minimal prime ideal* if whenever Q is a prime ideal of R such that $Q \subseteq P$, then $Q = P$. It should be noted that it is a consequence of Zorn's lemma that any ring contains a minimal prime ideal. Thus given an ideal

I of R, the ring R/I has at least one minimal prime ideal. Any prime ideal P of R such that $I \subseteq P$ and such that P/I is a minimal prime ideal of R/I is called a *prime ideal minimal over* I.

An intersection (finite or infinite) of prime ideals of R is called a *semi-prime ideal.* R is called a *semi-prime* ring if 0 is a semi-prime ideal of R. An ideal I of R is a semi-prime ideal if and only if R/I is a semi-prime ring. The semi-prime ideal of R which is the intersection of *all* prime ideals of R is called the *prime radical* of R. It is denoted by $N(R)$.

1.3 Annihilator, associated and affiliated prime ideals

Throughout this section let R be a ring and let M be a right R-module.

Let X be a subset of M. Then $r_R(X)$, the *(right) annihilator of* X *in* R, is defined by:

$$r_R(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}.$$

If the ring R in which the annihilator is being taken is clear from the context then the subscript R is usually suppressed and the notation $r(X)$ is used. If $X = \{x\}$ is a singleton then we abbreviate the cumbersome notation $r_R(\{x\})$ to $r_R(x)$. Note that $r_R(X)$ is always a right ideal of R, and any right ideal of R of this form is called a *right annihilator.* If further X is a submodule of M then $r_R(X)$ is a (two-sided) ideal of R.

Let Y be a subset of R. Then $l_M(Y)$, the *(left) annihilator of* Y *in* M, is defined by:

$$l_M(Y) = \{m \in M \mid my = 0 \text{ for all } y \in Y\}.$$

Again, if $Y = \{y\}$ is a singleton we simply denote the annihilator by $l_M(y)$. Note that $l_M(Y)$ is always a subgroup of M considered as an additive group. If further Y is a left ideal of R then $l_M(Y)$ is a submodule of M.

We remark that if X is a subset of M and Y is a subset of R then we have the relations $X \subseteq l_M(r_R(X))$ and $Y \subseteq r_R(l_M(Y))$. Further we have the equalities $r_R(X) = r_R(l_M(r_R(X)))$ and $l_M(Y) = l_M(r_R(l_M(Y)))$.

M is called a *faithful* module if $r_R(M) = 0$. M is called a *fully faithful* module if it is faithful and every non-zero submodule of M is faithful. Note that M is always a faithful right module over the ring $R/r_R(M)$. If further M is a fully faithful right $R/r_R(M)$-module, then M is called a *prime module*. It is easy to see that $r_R(M)$ is a prime ideal of R if M is a non-zero prime module. For, if A and B are ideals of R such that $AB \subseteq r_R(M)$ then either $MA = 0$, in which case $A \subseteq r_R(M)$, or $MA \neq 0$, in which case $r_{R/r_R(M)}(MA) = 0$, which is equivalent to the equality $r_R(MA) = r_R(M)$. Then $B \subseteq r_R(MA)$ (since $AB \subseteq r_R(M)$), and thus $B \subseteq r_R(M)$. This explains the terminology.

Definition 1.3.1. A prime ideal P of R is called an *annihilator prime of M* if there is a non-zero submodule N of M such that $P = r_R(N)$. In this case $l_M(P)$ is a non-zero faithful right R/P-module.

A right R-module M is called *irreducible* if it has no submodules other than 0 and M. A special class of annihilator primes can be defined, arising from irreducible modules.

Definition 1.3.2. Let P be an ideal of R. P is called a *right primitive* ideal of R if there is an *irreducible* right R-module M such that $P = r_R(M)$. *Left primitive* ideals of R are defined analogously.

We note that any right (or left) primitive ideal of R is a prime ideal, and that any maximal ideal of R is both right and left primitive.

The following result gives us a situation in which the existence of annihilator primes is guaranteed.

Proposition 1.3.1. [4, Proposition 2.12] *If the set:*

$$\{r_R(N) \mid N \text{ is a non-zero submodule of } M\}$$

of ideals of R has a maximal element P, then the following two statements hold:

1. *P is a prime ideal of R;*

2. $l_M(P)$ is a fully faithful right R/P-module.

Thus if R is a right (or left) Noetherian ring and M is a non-zero right R-module then annihilator primes do exist. The annihilator primes which arise as maximal elements among the set of right annihilators of non-zero submodules of M are called *maximal annihilator primes* of M.

Definition 1.3.3. Let P be an annihilator prime ideal of M_R. Let N be a non-zero submodule of M such that $P = r_R(N)$. If N is a fully faithful right R/P-module then P is called an *associated prime* of M. In other words, $P = r_R(N)$ is an associated prime of M if whenever N' is a non-zero submodule of N we have $P = r_R(N')$.

The set of all associated primes of M will be denoted by $Ass(M)$, or by $Ass(M)_R$ if the ring R needs to be made explicit.

Definition 1.3.4. A submodule N of M is called an *affiliated submodule* of M if there is a maximal annihilator prime ideal P of M_R such that $N = l_M(P)$. We note that since P is an annihilator prime we have in this case that $P = r_R(l_M(P))$. A series of submodules of M:

$$0 = M_0 \subsetneqq M_1 \subsetneqq \dots \subsetneqq M_n = M$$

is called an *affiliated series for M* if for each $i \in \{1, \dots, n\}$, M_i/M_{i-1} is an affiliated submodule of M/M_{i-1}. The series of prime ideals of R:

$$P_1 = r_R(M_1/M_0), \dots, P_n = r_R(M_n/M_{n-1})$$

is called the *series of affiliated primes corresponding to the affiliated series.* The prime ideal P_n of R is occasionally called the *top affiliated prime of the affiliated series.* Finally a prime ideal P of R is called an *affiliated prime* of M if P appears among the series of affiliated primes corresponding to some affiliated series for M.

The set of all affiliated primes of M will be denoted by $Aff(M)$, or by $Aff(M)_R$ if the ring R needs to be made explicit.

There is a large class of modules which have an affiliated series, as the following result demonstrates.

Proposition 1.3.2. [4, Proposition 2.13] *Let R be a right Noetherian ring. Let M be a finitely generated right R-module. Then there exists an affiliated series for M.*

Lemma 1.3.3. [7, Lemma 1.1] *Let R be a Noetherian ring. Let M be a finitely generated right R-module. Let N be a submodule of M. Suppose that:*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

is an affiliated series for M with corresponding affiliated primes $P_1, \ldots, P_n$. Then the distinct terms in the chain:

$$0 = N \cap M_0 \subseteq N \cap M_1 \subseteq \ldots \subseteq N \cap M_n = N$$

form an affiliated series for N. The corresponding affiliated primes of this affiliated series are precisely the elements of the set:

$$\{P_i \mid i \in \{1, \ldots, n\} \text{ and } (N \cap M_{i-1}) \neq (N \cap M_i)\}.$$

In particular we note that if $N \nsubseteq M_{n-1}$ then P_n is an affiliated prime of N and $P_n = r_R((N \cap M)/(N \cap M_{n-1}))$.

Proof. Choose an index $i \in \{1, \ldots, n\}$ such that $(N \cap M_{i-1}) \neq (N \cap M_i)$. Now we have:

$$\begin{aligned} \frac{N \cap M_i}{N \cap M_{i-1}} &= \frac{N \cap M_i}{(N \cap M_i) \cap M_{i-1}} \\ &\cong \frac{(N \cap M_i) + M_{i-1}}{M_{i-1}} \\ &\subseteq \frac{M_i}{M_{i-1}}. \end{aligned}$$

Thus $P_i = r_R(M_i/M_{i-1}) \subseteq r_R((N \cap M_i)/(N \cap M_{i-1}))$. The above also gives that $(N \cap M_i)/(N \cap M_{i-1})$ is isomorphic to a (non-zero) submodule of M/M_{i-1}. So since P is a maximal annihilator prime ideal of M/M_{i-1} we have that $P_i = r_R((N \cap M_i)/(N \cap M_{i-1}))$. This gives that $l_{N/(N\cap M_{i-1})}(P_i) \supseteq (N \cap M_i)/(N \cap M_{i-1})$. To prove the opposite inclusion, take an element $n \in N$ such that $nP_i \subseteq N \cap M_{i-1}$. Then $nP_i \subseteq M_{i-1}$ and so $n + M_{i-1} \in l_{M/M_{i-1}}(P_i) = M_i/M_{i-1}$. This gives that $n \in N \cap M_i$ as required. Finally, since $N/(N \cap M_{i-1})$ is isomorphic to a (non-zero) submodule of M/M_{i-1} it follows from the fact that P_i is a maximal annihilator prime ideal of M/M_{i-1} that P_i is also a maximal annihilator prime ideal of $N/(N \cap M_{i-1})$. □

It is clear that any maximal annihilator prime of M is an associated prime, and that any associated prime of M is an annihilator prime. We provide a link between these latter two sets of primes and the set of affiliated primes of M as follows.

Lemma 1.3.4. *Let R be a right Noetherian ring. Let M be a finitely generated right R-module. Let P be an annihilator prime ideal of M_R. Then P is an affiliated prime ideal of M_R.*

Proof. Choose any affiliated series for M, with corresponding affiliated primes $P_1, \ldots, P_n$. Let P be an annihilator prime ideal of M_R, and choose a non zero submodule N of M such that $P = r_R(N)$. By 1.3.3 it is possible to choose an affiliated series for N:

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_m = N$$

with corresponding affiliated primes $Q_1, \ldots, Q_m$ such that $\{Q_1, \ldots, Q_m\}$ is a subset of $\{P_1, \ldots, P_n\}$.

Now it follows from the fact that $Q_m = r_R(N_m/N_{m-1})$ that $N_m Q_m \subseteq N_{m-1}$. Similarly, it follows from the fact that $Q_{m-1} = r_R(N_{m-1}/N_{m-2})$ that $N_{m-1}Q_{m-1} \subseteq$

N_{m-2}, so that $N_m Q_m Q_{m-1} \subseteq N_{m-1} Q_{m-1} \subseteq N_{m-2}$. This process continues all of the way down the chain of affiliated submodules, eventually yielding the following:

$$N_m Q_m Q_{m-1} \ldots Q_1 \subseteq N_0 = 0.$$

This shows that $Q_m Q_{m-1} \ldots Q_1 \subseteq r_R(N_m) = P$. Thus we must have $Q_i \subseteq P$, for some $i \in \{1, \ldots, m\}$, because P is a prime ideal of R. However it is clear that $P = r_R(N) \subseteq r_R(N_j) \subseteq r_R(N_j/N_{j-1}) = Q_j$, for all $j \in \{1, \ldots, m\}$. Thus $P = Q_i$. We have established that $P \in \{Q_1, \ldots, Q_m\} \subseteq \{P_1, \ldots, P_n\}$, which shows that P is an affiliated prime of M. □

In addition we note that it is clear from the definition that any affiliated prime of a right R-module M is a maximal annihilator prime of some factor module of M.

Examples exist of affiliated primes which are not annihilator primes, and of annihilator primes which are not associated primes. However the sets of annihilator, associated and affiliated primes of a finitely generated module over a ring R coincide in the case when R is a commutative Noetherian ring.

Proposition 1.3.5. *Let R be a commutative Noetherian ring and let M be a finitely generated right R-module. If P is an affiliated prime ideal of M_R then P is an associated prime ideal of M_R.*

1.4 The singular submodule

Let M be a right module over a ring R. A submodule E of M is called *essential (in M)* if whenever N is a non-zero submodule of M, the submodule $N \cap E$ of M is also non-zero. In this case M is additionally called an *essential extension* of E. The module M is called *uniform* if M is non-zero and every non-zero submodule of M is essential.

M is said to be a *finite dimensional* module if there are uniform submodules $E_1, \ldots, E_n$ of M, for some positive integer n, such that the direct sum of these submodules, $E_1 \oplus \cdots \oplus E_n$, is an essential submodule of M. Note that M is finite dimensional if and only if M does not contain an infinite direct sum of non-zero submodules. Any Noetherian module is finite dimensional.

Let R be a ring. Essential submodules of the module R_R are called *essential right ideals* of R. Note that if R is a prime ring then every non-zero two-sided ideal of R is essential as a right (or left) ideal.

Lemma 1.4.1. [4, Lemma 3.25] *Let R be a ring. Let M be a right R-module. Then the following subset of M:*

$$\begin{aligned} & \{m \in M \mid mI = 0 \text{ for some essential right ideal } I \text{ of } R\} \\ = \; & \{m \in M \mid r_R(m) \text{ is an essential right ideal of } R\} \end{aligned}$$

is a submodule of M.

The submodule given by the above result is called the *singular submodule* of M and is usually denoted by $Z(M)$. M is called a *singular* module if $Z(M) = M$. The module M is called *non-singular* if $Z(M) = 0$.

1.5 Regular elements and orders

Let R be a ring. An element $x \in R$ is called *right regular* if $r_R(x) = 0$, *left regular* if $l_R(x) = 0$, and *regular* if it is both right and left regular. Elements of R which are not regular are called *zero-divisors*. Let I be an ideal of R. An element $x \in R$ is called *right regular modulo I* if the element $x + I \in R/I$ is right regular, *left regular modulo I* if the element $x + I \in R/I$ is left regular, and *regular modulo I* if the element $x + I \in R/I$ is regular. The sets of all right regular modulo I, left regular modulo I and regular modulo I elements are denoted by $\mathcal{C}'(I)$, $'\mathcal{C}(I)$ and $\mathcal{C}(I)$ respectively. Thus the set of regular elements of R is sometimes denoted by $\mathcal{C}(0)$. If P is a prime ideal of R and S is the centre of R then we have the following useful properties:

1. $S \subseteq P \cup \mathcal{C}(P)$;

2. $S \backslash P = \mathcal{C}(P) \cap S$;

3. $S \backslash \mathcal{C}(P) = P \cap S$.

Definition 1.5.1. Let R be a ring. An overring $Q \supseteq R$ is called a *right quotient ring of* R if the following two conditions are satisfied:

1. every regular element of R is an invertible element of Q;

2. every element of Q has the form rc^{-1} for some elements $r \in R$ and $c \in \mathcal{C}(0)$.

In this case R is called a *right order in* Q. *Left quotient rings of* R are defined analogously.

There are important theorems of A.W. Goldie and L.W. Small which give criteria for R to be, respectively, a right order in a semi-simple Artinian ring and a right order in an arbitrary right Artinian ring. We first define a *right Goldie ring* to be a ring R such that R_R is finite dimensional and which satisfies the ACC on right annihilators.

Theorem 1.5.1. (Goldie) [4, Theorem 5.10] *Let R be any ring. Then R has a semi-simple Artinian right quotient ring if and only if R is a semi-prime right Goldie ring.*

Theorem 1.5.2. (Small) [4, Theorem 10.9] *Let R be a right Noetherian ring. Let $N(R)$ denote the prime radical of R. Then R has a right Artinian right quotient ring if and only if $\mathcal{C}(0) = \mathcal{C}(N(R))$.*

1.6 Tensor products

Let R be a ring. Let M, N and P be right R-modules. Let $f : M \times N \longrightarrow P$ be a map. Then f is called *R-bilinear* if the following two conditions hold:

1. for every element $x \in M$, the map $g_x : N \longrightarrow P$ defined by $g_x(y) = f(x, y)$, for all $y \in N$, is a right R-module homomorphism;

2. for every element $y \in N$, the map $h_y : M \longrightarrow P$ defined by $h_y(x) = f(x, y)$, for all $x \in M$, is a right R-module homomorphism.

Let S be a commutative ring. Given arbitrary S-modules M and N, there is a pair (T, g) such that T is a right S-module and $g : M \times N \longrightarrow T$ is a S-bilinear map satisfying the following property. If P is *any* right S-module and $f : M \times N \longrightarrow P$ is *any* S-bilinear map, then there is a unique S-module homomorphism $f' : T \longrightarrow P$ such that $f = f' \circ g$. Further, if (T, g) and (T', g') are two such pairs then there is a (unique) S-module isomorphism $j : T \longrightarrow T'$ such that $j \circ g = g'$. This leads to the following definition.

Definition 1.6.1. Let S be a commutative ring. Let M and N be S-modules. Then the S-module T given in the above discussion is called the *tensor product of M and N* and is denoted by $M \otimes_S N$, or simply by $M \otimes N$ if the ring S is clear from the context.

Given elements $x \in M$ and $y \in N$ the element $g(m, n) \in M \otimes N$ is denoted by $x \otimes y$ and is called an *elementary tensor*. The following relations on the elements of $M \otimes N$ follow from the fact that the map g is S-bilinear:

1. $(x + x') \otimes y = (x \otimes y) + (x' \otimes y)$, for all elements $x, x' \in M$ and $y \in N$;

2. $x \otimes (y + y') = (x \otimes y) + (x \otimes y')$, for all elements $x \in M$ and $y, y' \in N$;

3. $xs \otimes y = x \otimes ys = (x \otimes y)s$, for all elements $x \in M$, $y \in N$ and $s \in S$.

Remark 1.6.1. Let S be a commutative subring of rings R_1 and R_2. Then the S-module $R_1 \otimes_S R_2$ has a ring structure under the multiplication defined on elementary tensors by $(r_1 \otimes r_2)(r_1' \otimes r_2') = (r_1 r_1' \otimes r_2 r_2')$.

1.7 Links between prime ideals

Let R be a Noetherian ring. Let P and Q be prime ideals of R. Suppose that there is an ideal A of R satisfying the following three properties:

1. $QP \subseteq A \subsetneq Q \cap P$;
2. the right R/P-module $(Q \cap P)/A$ is non-zero and non-singular;
3. the left R/Q-module $(Q \cap P)/A$ is non-zero and non-singular.

Then there is said to be a *link from Q to P*. This fact is denoted by $Q \rightsquigarrow P$. The $(R/Q, R/P)$-bimodule $(Q \cap P)/A$ is called a *linking bimodule between Q and P* and the link from Q to P is said to be *via* $(Q \cap P)/A$.

The *graph of links* of R is defined to be the diagram with the prime ideals of R as the vertices and with the property that there is an arrow from one prime ideal Q to another prime ideal P if and only if there is a link from Q to P.

The connected components of the graph of links of R are called *cliques*. The (unique) clique containing a prime ideal P is denoted by $Cl(P)$.

Lemma 1.7.1. [4, Lemma 11.7] *Let R be a Noetherian ring. Let S be the centre of R. Let P be a prime ideal of R. Let c be an element of S. If $c \in P$, then $c \in Q$ for every prime ideal $Q \in Cl(P)$.*

1.8 Polynomial identity rings

Throughout this section we will use the notation $\mathbb{Z}\langle x\rangle$ to denote the *free $\mathbb{Z}$-algebra on countably many elements* $\{x_i \mid i \in \mathbb{Z},\, i \geq 0\}$.

Definition 1.8.1. Let R be a ring. An element $f(x_1, \ldots, x_n) \in \mathbb{Z}\langle x\rangle$ is called a *polynomial identity* of R if for any choice of n elements $r_1, \ldots, r_n$ of R we have $f(r_1, \ldots, r_n) = 0$. In this case R is said to *satisfy* f.

f is called *monic* if at least one of the words of highest degree in the support of f has coefficient 1.

Alternatively, a polynomial identity of R can be defined as an element of $\mathbb{Z}\langle x\rangle$ which is a member of $ker(\theta)$ for *every* ring homomorphism $\theta : \mathbb{Z}\langle x\rangle \longrightarrow R$.

The following polynomials play an important role in the theory of polynomial identities.

Definition 1.8.2. Let n be a positive integer. Define the *nth standard identity*, s_n, as follows:

$$s_n = \Sigma_{\sigma \in S_n} (\text{sign } \sigma) x_{\sigma(1)} \dots x_{\sigma(n)},$$

where S_n denotes the nth symmetric group. Note that s_n is a monic polynomial of degree n.

Definition 1.8.3. A ring R is called a *polynomial identity ring (PI ring)* if R satisfies some monic polynomial identity f.

Note that any commutative ring is a PI ring because it satisfies the identity $s_2 = x_1 x_2 - x_2 x_1$. Also any subring, or any homomorphic image, of a PI ring is again a PI ring.

Definition 1.8.4. A polynomial $f \in \mathbb{Z}\langle x_1, \dots, x_n\rangle$ is called *multilinear of degree n* if it is non-zero and has the following form:

$$f = \Sigma_{\sigma \in S_n} a_\sigma x_{\sigma(1)} \dots x_{\sigma(n)},$$

where, for each $\sigma \in S_n$, a_σ is an integer, and S_n denotes the nth symmetric group.

An important property exhibited by a multilinear polynomial f of degree n is that for any choice of n integers $c_1, \dots, c_n$ we have:

$$f(c_1 x_1, \dots, c_n x_n) = c_1 \dots c_n f(x_1, \dots, x_n).$$

Definition 1.8.5. Let $f \in \mathbb{Z}\langle x_1, \dots, x_n\rangle$ be a multilinear polynomial of degree n. Then f is said to be *alternating* if given any two distinct integers $i, j \in \{1, \dots, n\}$, we have that $f(x_1, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_n) = 0$.

It is clear that, for any positive integer n, the nth standard identity s_n is multilinear of degree n. It is also alternating.

Definition 1.8.6. Let R be a ring and let $g(x_1, \ldots, x_n) \in \mathbb{Z}\langle x \rangle$ be a polynomial of degree n. An element of R of the form $g(r_1, \ldots, r_n)$ for some elements $r_1, \ldots, r_n$ of R, is called an *evaluation of g*. The set of all evaluations of g is denoted by $g(R)$, and the additive subgroup of R generated by $g(R)$ is denoted by $g(R)^+$.

Definition 1.8.7. Let R be a ring. Let S be the centre of R. Let $g \in \mathbb{Z}\langle x \rangle$ be a polynomial. Express g in the form $g = a + f$, where a is an integer (the constant term of g) and $f \in \mathbb{Z}\langle x \rangle$ is a polynomial which has zero constant term. Then g is called a *central polynomial* for R if the following two conditions hold:

1. $g(R) \subseteq S$;

2. f is not a polynomial identity of R.

Chapter 2

The Commutative Case

In this chapter we outline the commutative ring theory result concerning asymptotic stability of prime divisors which provides the motivation for our study. M. Brodmann was the first to prove this result ([2]), but the proof we outline is the one given by S. McAdam and P. Eakin in [10].

2.1 Associated prime ideals

Lemma 2.1.1. *Let R be a ring. Let M be a right R-module. Let N be a submodule of M. Then we have the following:*

1. $Ass(N)_R \subseteq Ass(M)_R$;
2. $Ass(M)_R \subseteq Ass(N)_R \cup Ass(M/N)_R$.

Proof. Let P be an associated prime ideal of N_R. Choose a non-zero submodule N' of N such that $P = r_R(N')$ and such that N' is a fully faithful R/P-module. It follows that P is an associated prime of M_R, since N' is also a non-zero submodule of M.

Now let P be an associated prime ideal of M_R and choose a non-zero submodule M' of M such that $P = r_R(M')$ and such that M' is a fully faithful R/P-module.

Consider $M' \cap N$. Suppose that $M' \cap N$ is non-zero. Then since $M' \cap N$ is a submodule of M', we have $P = r_R(M' \cap N)$. Any non-zero submodule of a fully faithful module is again fully faithful. Thus $M' \cap N$ is a fully faithful R/P-module, and a non-zero submodule of N. This shows that P is an associated prime of N.

Now suppose that $M' \cap N = 0$. Then we have:

$$\frac{(M' \oplus N)}{N} \cong \frac{M'}{(M' \cap N)} \cong M'.$$

Thus $P = r_R((M' \oplus N)/N)$, which shows that P is an associated prime of $(M/N)_R$.

□

Proposition 2.1.2. *Let M be a right module over a ring R. Suppose that M is finite dimensional. Then $Ass(M)_R$ is a finite set.*

In particular $Ass(M)_R$ is finite if M is a Noetherian module.

Proof. Let $P_1, \ldots, P_k$ be distinct associated prime ideals of M_R, where k is some positive integer. Renumber these primes if necessary so that P_1 is minimal among the set $\{P_1, \ldots, P_k\}$. For each index $i \in \{1, \ldots, k\}$ choose a non-zero submodule B_i of M such that $P_i = r_R(B_i)$ and which is a fully faithful R/P_i -module. We claim that the sum $B_1 + \cdots + B_k$ is a direct sum. To establish this we use induction on k.

The result is trivial for $k = 1$ so suppose that $k \geq 2$ and assume that the claim is true for all positive integers j such that $j < k$. It suffices to show that the following holds:

$$B_1 \cap (B_2 \oplus \cdots \oplus B_k) = 0.$$

Suppose for a contradiction that $B_1 \cap (B_2 \oplus \cdots \oplus B_k)$ is non-zero. Since $B_1 \cap (B_2 \oplus \cdots \oplus B_k)$ is a non-zero submodule of B_1 we have:

$$P_1 = r(B_1 \cap (B_2 \oplus \cdots \oplus B_k)).$$

Now take an element $x \in P_2 \cap \ldots \cap P_k$ and an element:

$$b_1 = b_2 + \cdots + b_k \in B_1 \cap (B_2 \oplus \cdots \oplus B_k),$$

where $b_i \in B_i$ for each $i \in \{1, \ldots, n\}$. Now since we have:

$$x \in P_2 \cap \ldots \cap P_k = r(B_2) \cap \ldots \cap r(B_k),$$

we must have $b_1 x = b_2 x + \cdots + b_k x = 0$. This shows that:

$$x \in r(B_1 \cap (B_2 \oplus \cdots \oplus B_k)).$$

But we have that $r(B_1 \cap (B_2 \oplus \cdots \oplus B_k)) = P_1$. Thus $x \in P_1$, and hence $P_2 \cap \ldots \cap P_k \subseteq P_1$. Now we have:

$$P_2 \ldots P_k \subseteq P_2 \cap \ldots \cap P_k \subseteq P_1,$$

from which it follows that $P_j \subseteq P_1$ for some index $j \in \{2, \ldots, k\}$, since P_1 is a prime ideal of R. This is a contradiction to the minimality of P_1, which establishes the claim.

Finally we note that this implies that the existence of an arbitrarily large collection of distinct elements of $Ass(M)_R$ would yield an arbitrarily large direct sum of non-zero submodules of M, which would contradict the fact that M is finite dimensional. So the set $Ass(M)_R$ is finite. □

Proposition 2.1.3. *Let R be a right Noetherian ring. Let I be a right ideal of R. Suppose that the sequence:*

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes.

Then the sequence:

$$\left\{ Ass\left(\frac{R}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes.

Proof. Let n be a sufficiently large positive integer so that the sequence:

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\} \tag{2.1.1}$$

stabilizes.

Since I^n/I^{n+1} is a right R-submodule of R/I^{n+1} we have the inclusion:

$$Ass\left(\frac{R}{I^{n+1}}\right)_R \subseteq Ass\left(\frac{I^n}{I^{n+1}}\right)_R \cup Ass\left(\frac{R}{I^n}\right)_R,$$

by 2.1.1. Now we have:

$$Ass\left(\frac{I^n}{I^{n+1}}\right)_R = Ass\left(\frac{I^{n-1}}{I^n}\right)_R,$$

since the sequence (2.1.1) is stable. But we also have:

$$Ass\left(\frac{I^{n-1}}{I^n}\right)_R \subseteq Ass\left(\frac{R}{I^n}\right)_R,$$

by 2.1.1, so in fact we have:

$$Ass\left(\frac{R}{I^{n+1}}\right)_R \subseteq Ass\left(\frac{R}{I^n}\right)_R.$$

So, when n is sufficiently large, the sequence:

$$\left\{ Ass\left(\frac{R}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

is decreasing.

But we have that R is a right Noetherian ring, so that R/I^n is a Noetherian right R-module. Thus $Ass(R/I^n)_R$ is a finite set, by 2.1.2. Thus the sequence:

$$\left\{ Ass\left(\frac{R}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

must eventually stabilize. □

2.2 Graded rings

The proof of M. Brodmann's result ([2]) given in [10] makes crucial use of the *associated graded ring* of a ring R at an ideal I, which is defined in this section.

Definition 2.2.1. A ring R is called a *graded ring* if there is a collection of additive subgroups of R, indexed by the integers, $\{R_q \mid q \in \mathbb{Z}\}$, which satisfies the following two conditions:

1. $R = \bigoplus_{q\in\mathbb{Z}} R_q$;

2. $R_q R_{q'} \subseteq R_{q+q'}$, for all integers q and q'.

Definition 2.2.2. Let $R = \bigoplus_{q\in\mathbb{Z}} R_q$ be a graded ring. An element $r \in R$ is called *homogeneous* if there is an integer q such that $r \in R_q$.

Further, an element $r \in R$ is called *homogeneous of degree* q if $r \in R_q$ and $r \neq 0$. In this case we use the notation $deg(r) = q$.

Remark 2.2.1. Let $R = \bigoplus_{q\in\mathbb{Z}} R_q$ be a graded ring. Let $a, b \in R$ be non-zero homogeneous elements. Then we have the following properties:

1. $deg(ab) = deg(a) + deg(b)$;

2. whenever $a + b$ is homogeneous, we have $deg(a) = deg(b)$ and either $a + b = 0$ or $deg(a + b) = deg(a) = deg(b)$.

This second property can be extended by induction to give that whenever k is a positive integer and $a_1, \ldots, a_k$ are homogeneous elements of R such that $a_1 + \cdots + a_k$ is homogeneous, we have that either $a_1 + \cdots + a_k = 0$ or $deg(a_1) = \cdots = deg(a_k) = deg(a_1 + \cdots + a_k)$.

Definition 2.2.3. Let R be a ring. Let I be an ideal of R. Define the *associated graded ring of R at I*, denoted by $gr_I(R)$, by:

$$gr_I(R) = \bigoplus_{q\in\mathbb{Z},\, q\geq 0} \frac{I^q}{I^{q+1}}$$

$$= \left\{ (x_0 + I, x_1 + I^2, \ldots) \mid \begin{array}{l} \text{only finitely many components are non-zero} \\ \text{and } x_i \in I^i, \text{ for all positive integers } i \end{array} \right\},$$

where $I^0 = R$. The component-wise addition is defined as follows. If we have elements $x, y \in gr_I(R)$ with qth components $(x)_q = x_q + I^{q+1}$ and $(y)_q = y_q + I^{q+1}$ respectively, then the qth component of their sum is given by:

$$(x + y)_q = x_q + y_q + I^{q+1}.$$

The multiplication is defined as follows. If we have, as above, elements $x, y \in gr_I(R)$ with qth components $(x)_q = x_q + I^{q+1}$ and $(y)_q = y_q + I^{q+1}$ respectively, then the qth component of their product is given by:

$$(xy)_q = \sum_{i,j\in\mathbb{Z},\, i+j=q} x_i y_j + I^{q+1}.$$

It can be checked that $gr_I(R)$ is a ring with identity element $1_R + I$ and is graded with the grading given by the following rules:

$$\begin{cases} R_q = 0 & \text{if } q < 0 \\ R_q = I^q/I^{q+1} & \text{if } q \geq 0 \end{cases}.$$

Clearly $gr_I(R)$ is a commutative ring whenever R is a commutative ring.

2.3 Asymptotic stability in the commutative case

Now we outline the proof of M. Brodmann's result given in [10]. We first give a characterization of associated primes of a module over a commutative ring. Note that an ideal P of a commutative ring R is a prime ideal if and only if whenever a and b are elements of R such that $ab \in P$, we have either $a \in P$ or $b \in P$.

Lemma 2.3.1. *Let R be a commutative ring. Let M be a right R-module. Let P be a prime ideal of R. Then P is an associated prime ideal of M_R if and only if $P = r_R(m)$ for some non-zero element $m \in M$.*

Proof. Let P be an associated prime ideal of M_R. Choose a non-zero submodule M' of M such that $P = r(M')$ and such that M' is a fully faithful module over R/P. Choose any non-zero element $m' \in M'$. Since R is a commutative ring we have $r(m') = r(m'R)$. But $m'R$ is a non-zero submodule of M', so we have $P = r(m'R) = r(m')$.

Conversely, suppose that $P = r(m)$ for some non-zero element $m \in M$. As above $r(m) = r(mR)$ so that $P = r(mR)$. Suppose that N is a non-zero submodule of mR. Trivially we have that $P \subseteq r(N)$. Take an element $x \in r(N)$. Choose a non-zero element $n \in N$. Since $n \in mR$ we can write $n = mr$ for some element $r \in R$. Now $nx = 0$ so that $mrx = 0$, which means that $rx \in r(m) = P$. So since P is a prime ideal of R we must have either $r \in P$ or $x \in P$. But if $r \in P$ then $r \in r(m)$, from which we deduce that $mr = n = 0$. This is a contradiction, so we must have that $x \in P$. Hence we have $P \supseteq r(N)$ and thus $P = r(N)$. This shows that P is an associated prime of M_R. □

Lemma 2.3.2. [10, Lemma 1.1] *Let R be a commutative ring and let I be an ideal of R. Suppose that $gr_I(R)$ is a Noetherian ring. Let $c \in gr_I(R)$ be a homogeneous element. Let S be a subset of R/I with the property that the product of any two elements of S is again an element of S. Suppose that $r_{gr_I(R)}(c) \cap S = \emptyset$. Then there is a homogeneous element $d \in gr_I(R)$ such that $r_{gr_I(R)}(cd) \cap S = \emptyset$ and such that*

$r_{gr_I(R)}(cd)$ *is a prime ideal of* $gr_I(R)$.

Proof. The set:

$$\{r_{gr_I(R)}(cd) \mid d \in gr_I(R) \text{ is homogeneous and } r_{gr_I(R)}(cd) \cap S = \emptyset\}$$

is a non-empty set of ideals of $gr_I(R)$ by hypothesis, since we can take d to be the identity element in the ring $gr_I(R)$. Thus it has a maximal element, since $gr_I(R)$ is a Noetherian ring. Let $r_{gr_I(R)}(cd)$ be such a maximal element, where d is a homogeneous element of $gr_I(R)$ such that $r_{gr_I(R)}(cd) \cap S = \emptyset$. Take homogeneous elements $x, y \in gr_I(R)$ such that $xy \in r_{gr_I(R)}(cd)$ and suppose that $x \notin r_{gr_I(R)}(cd)$ and $y \notin r_{gr_I(R)}(cd)$. Then $x \in r_{gr_I(R)}(cdy) \backslash r_{gr_I(R)}(cd)$ and so we have $r_{gr_I(R)}(cd) \subsetneq r_{gr_I(R)}(cdy)$. Thus there is an element $s \in S \cap r_{gr_I(R)}(cdy)$, by the maximality of $r_{gr_I(R)}(cd)$.

Now $y \in r_{gr_I(R)}(cds) \backslash r_{gr_I(R)}(cd)$, so we have $r_{gr_I(R)}(cd) \subsetneq r_{gr_I(R)}(cds)$. Thus, again by the maximality of $r_{gr_I(R)}(cd)$, there is an element $s' \in S \cap r_{gr_I(R)}(cds)$. But then we have $ss' \in S \cap r_{gr_I(R)}(cd)$. This is a contradiction, since d was chosen so that $S \cap r_{gr_I(R)}(cd) = \emptyset$.

It can be checked that this is sufficient to show that $r_{gr_I(R)}(cd)$ is a prime ideal of R. □

Lemma 2.3.3. *Let R be a commutative ring. Let I be an ideal of R. Suppose that $gr_I(R)$ is a Noetherian ring. Then the set:*

$$\bigcup_{n \in \mathbb{Z},\, n \geq 1} Ass\left(\frac{I^{n-1}}{I^n}\right)_{R/I}$$

is finite.

Proof. [10, Proposition 1.3] Take an element $P \in \bigcup_{n \in \mathbb{Z},\, n \geq 1} Ass(I^{n-1}/I^n)_{R/I}$. Choose a positive integer n such that P is an associated prime ideal of $(I^{n-1}/I^n)_{R/I}$. Then by 2.3.1 there is a non-zero element $c \in (I^{n-1}/I^n)$ such that $P = r_{R/I}(c)$. Clearly we have $P = r_{gr_I(R)}(c) \cap R/I$.

Now by applying 2.3.2 (with $S = (R/I)\backslash P$), there is a homogeneous element $d \in gr_I(R)$ such that $r_{gr_I(R)}(cd)$ is a prime ideal of $gr_I(R)$ and $r_{gr_I(R)}(cd) \cap S = \emptyset$. Denote $r_{gr_I(R)}(cd)$ by P^*. Then we have $P^* \cap R/I = P$. Now by applying 2.3.1 again we see that P^* is an associated prime of $gr_I(R)_{gr_I(R)}$. In summary we have now shown that:

$$\bigcup_{n\in\mathbb{Z}, n\geq 1} Ass\left(\frac{I^{n-1}}{I^n}\right)_{R/I} \subseteq \left\{P^* \cap R/I \mid P^* \in Ass(gr_I(R))_{gr_I(R)}\right\}.$$

Now, by 2.1.2, the set $Ass(gr_I(R))_{gr_I(R)}$ is finite. Thus the set on the right hand side of the inclusion above is also finite. This shows that the set:

$$\bigcup_{n\in\mathbb{Z}, n\geq 1} Ass\left(\frac{I^{n-1}}{I^n}\right)_{R/I}$$

is finite, completing the proof. □

Lemma 2.3.4. [10, Lemma 1.1 (a)] *Let R be a commutative ring. Let I be an ideal of R. Suppose that $gr_I(R)$ is a Noetherian ring. Then there is a positive integer l such that whenever n is a positive integer such that $n \geq l$, we have:*

$$r_{gr_I(R)}\left(\frac{I}{I^2}\right) \cap \left(\frac{I^n}{I^{n+1}}\right) = 0.$$

Proof. We have that $r_{gr_I(R)}(I/I^2)$ is an ideal of the Noetherian ring $gr_I(R)$ so we can choose a finite set of elements $a_1, \ldots, a_s$ (for some positive integer s) which generate $r_{gr_I(R)}(I/I^2)$ as an ideal. Since every element of $gr_I(R)$ can be expressed as a finite sum of homogeneous elements, we may assume that the elements $a_1, \ldots, a_s$ are all homogeneous.

Set $l = 1 + max\{deg(a_i) \mid i \in \{1, \ldots, s\}\}$. Let n be an integer such that $n \geq l$ and take an element $x \in r_{gr_I(R)}(I/I^2) \cap (I^n/I^{n+1})$. Since $x \in r_{gr_I(R)}(I/I^2)$, we can write $x = a_{i_1}c_{i_1} + \cdots + a_{i_k}c_{i_k}$ for some positive integer k and some indices $i_1, \ldots, i_k \in \{1, \ldots, s\}$, where $c_{i_1}, \ldots, c_{i_k}$ are homogeneous elements of $gr_I(R)$. Since x is homogeneous with $deg(x) = n$, either $x = 0$, in which case the result follows, or we must have that $deg(a_{i_j}c_{i_j}) = n$, for every $j \in \{1, \ldots, k\}$. Then we can deduce the following:

$$\begin{aligned} n &= deg(a_{i_j}c_{i_j}) \\ &= deg(a_{i_j}) + deg(c_{i_j}) \\ &< l + deg(c_{i_j}) \\ &\leq n + deg(c_{i_j}). \end{aligned}$$

Thus $deg(c_{i_j}) > 0$, for every $j \in \{1, \ldots, k\}$. It follows that $c_{i_j} \in (I/I^2)gr_I(R)$ and hence that $a_{i_j}c_{i_j} = 0$, for every $j \in \{1, \ldots, k\}$. This shows that $x = 0$. Hence $r_{gr_I(R)}(I/I^2) \cap (I^n/I^{n+1}) = 0$ as required. □

Lemma 2.3.5. ([10]) *Let R be a commutative ring. Let I be an ideal of R. Suppose that $gr_I(R)$ is a Noetherian ring. Then there is a positive integer l such that whenever n is a positive integer such that $n \geq l$, we have:*

$$Ass\left(\frac{I^n}{I^{n+1}}\right)_{R/I} \subseteq Ass\left(\frac{I^{n+1}}{I^{n+2}}\right)_{R/I}.$$

Proof. Choose the positive integer l given by 2.3.4 and let n be an integer such that $n \geq l$. Let P be an associated prime ideal of $(I^n/I^{n+1})_{R/I}$. Choose a non-zero element $c \in (I^n/I^{n+1})$ such that $P = r_{R/I}(c)$. It follows from 2.3.4 that $P = r_{R/I}(c(I/I^2))$, and since we have $c(I/I^2) \subseteq (I^{n+1}/I^{n+2})$ we deduce that P is an associated prime ideal of $(I^{n+1}/I^{n+2})_{R/I}$. □

Corollary 2.3.6. ([10]) *Let R be a commutative ring. Let I be an ideal of R. Suppose that $gr_I(R)$ is a Noetherian ring. Then the sequence:*

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes.

Proof. It is clear from 2.3.3 and 2.3.5 that the sequence:

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_{R/I} \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes. It is straightforward to check that for a right R-module M such that $MI = 0$, we have that P/I is an associated prime ideal of $M_{R/I}$ if and only if P is an associated prime ideal of M_R and $I \subseteq P$. So there is a one-to-one correspondence between elements of $Ass(I^{n-1}/I^n)_{R/I}$ and elements of $Ass(I^{n-1}/I^n)_R$, for each positive integer n. Thus the sequence:

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes. □

Proposition 2.3.7. *Let R be a commutative Noetherian ring. Let I be an ideal of R. Then $gr_I(R)$ is a Noetherian ring.*

Proof. This is a consequence of the Hilbert basis theorem ([18, Theorem 4.48]). □

Thus we have the result giving asymptotic stability of prime divisors of powers of an ideal in a commutative Noetherian ring.

Corollary 2.3.8. (M. Brodmann) [2] *Let R be a commutative Noetherian ring. Let I be an ideal of R. Then the sequence:*

$$\left\{ Ass\left(\frac{I^{n-1}}{I^n}\right)_R \mid n \in \mathbb{Z}, n \geq 1 \right\}$$

stabilizes.

2.4 Prime ideals in the associated graded ring

We now briefly discuss the consequences of an attempt to generalize the above argument to a noncommutative setting. Let R be an arbitrary ring and let I be an ideal of R. It is clear that a method is needed for producing a prime ideal P^* of the associated graded ring of R at I, $gr_I(R)$. It is also clear that the method to do this used in the above argument (2.3.2) cannot be generalized in a straightforward way. There is however a method for producing such a prime ideal, in a certain setting, due to A.J. Gray ([5]). We first define a particular ring, named after D. Rees.

Definition 2.4.1. Let R be a ring. Let I be an ideal of R. Let t be a commuting indeterminate. Then the *Rees ring of R at I*, denoted by $R[t^{-1}, It]$, is defined to be the set of all polynomials of the form:

$$r_{-m}t^{-m} + \cdots + r_{-1}t^{-1} + r_0 + x_0 t + \cdots + x_n t^n,$$

where m and n are positive integers, $r_j \in R$ for each $j \in \{-m, \ldots, 0\}$, and $x_i \in I^i$ for each $i \in \{1, \ldots, n\}$.

It is straightforward to describe the Rees ring in terms of the associated graded ring, as follows.

Lemma 2.4.1. [5, Lemma 3.11.1] *Let R be a ring. Let I be an ideal of R. Then:*

$$gr_I(R) \cong \frac{R[t^{-1}, It]}{t^{-1}R[t^{-1}, It]}.$$

Proof. Define a map $\theta : R[t^{-1}, It] \longrightarrow gr_I(R)$ on an element $r_{-m}t^{-m} + \cdots + r_{-1}t^{-1} + r_0 + x_0 t + \cdots + x_n t^n$ of $R[t^{-1}, It]$ as follows:

$$\theta(r_{-m}t^{-m} + \cdots + r_{-1}t^{-1} + r_0 + x_0 t + \cdots + x_n t^n) = (r_0 + I, x_1 + I^2, \ldots, x_n + I^{n+1}).$$

It is straightforward to check that θ is a ring homomorphism which is surjective and is such that $ker(\theta) = t^{-1}R[t^{-1}, It]$. The result now follows from the first isomorphism theorem. □

Thus calculating the prime ideals of $gr_I(R)$ is equivalent to calculating the prime ideals of $R[t^{-1}, It]$ which contain the ideal $t^{-1}R[t^{-1}, It]$.

Definition 2.4.2. Let R be a ring. Let S be an overring of R. Suppose that R and S share the same identity element. Let I be an ideal of R. Then define the following two subsets of S:

1. $I^* = \{s \in S \mid sI \subseteq R\}$;

2. $I^+ = \{s \in S \mid Is \subseteq R\}$.

It is easy to verify the following basic properties of the sets I^* and I^+:

1. I^* and I^+ are additive subgroups of S;

2. $1 \in I^* \cap I^+$;

3. $I^*I \subseteq R$ and $II^+ \subseteq R$;

4. $I^* = I^*R$ and $I^* = RI^*$;

5. $I^+ = I^+R$ and $I^+ = RI^+$.

Definition 2.4.3. Let R be a ring. Let S be an overring of R. Suppose that R and S share the same identity element. Let I be an ideal of R. Then I is called an *invertible ideal* of R if $I^*I = II^+ = R$.

The properties listed above can be used to prove the following result.

Proposition 2.4.2. *Let R be a ring. Let S be an overring of R. Suppose that R and S share the same identity element. Let I be an invertible ideal of R. Then $I^* = I^+$, and this invariant is denoted by I^{-1}.*

Proof. We have $I^* = I^*R = I^*II^+ = RI^+ = I^+$. □

In attempting to generalize 2.3.8 to invertible ideals in a Noetherian ring, we have the following advantage.

Theorem 2.4.3. (A.J. Gray) [5, Corollary 4.2] *Let R be a right Noetherian ring. Let I be an invertible ideal of R. Then $gr_I(R)$ is a right Noetherian ring.*

Theorem 2.4.4. (A.J. Gray) [5, Lemma 4.4, (a)] *Let R be a ring. Let I be an invertible ideal of R. Let P be a prime ideal of R such that $PI = IP$. Then the following subset of $R[t^{-1}, It]$:*

$$P^* = \left(\bigoplus_{j \in \mathbb{Z},\, j<0} (PI^j \cap R)t^j \right) \oplus P \oplus \left(\bigoplus_{i \in \mathbb{Z},\, i>0} PI^i t^i \right)$$

is a prime ideal of $R[t^{-1}, It]$.

However to apply this result to a particular associated prime ideal P of $(R/I^n)_R$, for some positive integer n, we would need to establish that $PI = IP$. It is unclear whether this condition would hold in these circumstances. To conclude this discussion, we consider the following result.

Lemma 2.4.5. [3, Lemma 4.2] *Let R be a ring. Let I be an invertible ideal of R. Let P be a prime ideal of R. Suppose that $I \not\subseteq P$. Then $PI = IP$.*

Proof. First note that since $(P \cap I)I^{-1} \subseteq II^{-1} = R$, and since $P \cap I$ is an ideal of R, we must have that $(P \cap I)I^{-1}$ is a left ideal of R. Further we have that $(P \cap I)I^{-1}I = P \cap I \subseteq P$. Now P is a prime ideal of R, and $I \not\subseteq P$ by hypothesis, so we must have $(P \cap I)I^{-1} \subseteq P$. Multiplying both sides of this inclusion by I, we see that $P \cap I \subseteq PI$. Thus we have $P \cap I = PI$. A symmetrical argument shows that we also have $P \cap I = IP$. Hence $PI = IP$ as required. □

Note that if P is an associated prime ideal of $(R/I^n)_R$, for any positive integer n, then we have that $I^n \subseteq P$, and so $I \subseteq P$. In view of the above result, it is not true that $PI = IP$ under the general hypothesis that $I \subseteq P$, for we would arrive at the contradiction that $PI \neq IP$ implies that $PI = IP$.

Chapter 3

Azumaya Algebras

In this chapter we give a generalization of M. Brodmann's theorem (2.3.8) to Azumaya algebras, transferring the focus of our study from the associated primes of $(R/I)_R$ to the affiliated primes of $(R/I)_R$. In doing so we show that there is a very nice relationship between the affiliated primes of $(R/I)_R$ and the affiliated primes of $(S/I \cap S)_S$ (Corollary 3.3.6), where S is the centre of a Noetherian Azumaya algebra R and I is any ideal of R.

3.1 Introduction to Azumaya algebras

Let R be a ring. Let M be a right R-module. An *R-endomorphism of M* is a right R-module homomorphism $\theta : M \longrightarrow M$. The set of all R-endomorphisms of M forms a ring and is denoted by $End_R(M)$.

The *opposite ring* of R is denoted by R^{op} and is defined to be the set R together with the same addition as in R and the multiplication defined as follows. If juxtaposition denotes the multiplication in R then the product of two elements $a, b \in R^{op}$ is defined to be $a \times b = ba$.

Definition 3.1.1. Let R be a ring. Let S be the centre of R. Let $E = End_S(R)$, the ring of all endomorphisms of the S-module R. If a and b are elements of R then we can define a map $\phi_{a,b} \in E$ on an element $r \in R$ by:

$$\phi_{a,b}(r) = arb.$$

Using this definition we can define a ring homomorphism $\theta : R \otimes_S R^{op} \longrightarrow E$ on an elementary tensor $a \otimes b$ (where $a, b \in R$) by:

$$\theta(a \otimes b) = \phi_{a,b},$$

and then extending this definition to the whole of $R \otimes_S R^{op}$ by linearity.

Then R is called an *Azumaya algebra (over S)* if the following hold:

1. R is a finitely generated and projective S-module;

2. the map θ as defined above is a ring isomorphism.

Examples of Azumaya algebras include the class of central simple algebras (see chapter 5). In addition the ring $M_n(A)$, where n is any positive integer and A is any commutative ring, is an Azumaya algebra. As will be seen, Azumaya algebras have some very desirable properties in connection with their centres.

Proposition 3.1.1. [12, Proposition 13.7.9] *Let R be an Azumaya algebra. Let S be the centre of R. Then we have the following:*

1. *If I is an ideal of R then $I \cap S$ is an ideal of S;*

2. *If J is an ideal of S then JR is an ideal of R.*

Further we have:

1. *If I is an ideal of R then $(I \cap S)R = I$;*

2. *If J is an ideal of S then $JR \cap S = J$.*

In other words there is a one-to-one correspondence between ideals of R and ideals of S.

The following result establishes that this correspondence preserves products.

Lemma 3.1.2. *Let R be an Azumaya algebra with centre S. Then:*

1. *If J_1 and J_2 are ideals of S then $(J_1R)(J_2R) = (J_1J_2R)$;*

2. *If I_1 and I_2 are ideals of R then $(I_1 \cap S)(I_2 \cap S) = I_1I_2 \cap S$.*

Proof. Let J_1 and J_2 be ideals of S. That $(J_1R)(J_2R) = (J_1J_2R)$ follows easily from the fact that elements of S commute, since S is the centre of R. Now let I_1 and I_2 be ideals of R. We have the following:

$$\begin{aligned}(I_1 \cap S)(I_2 \cap S) &= ((I_1 \cap S)(I_2 \cap S))R \cap S && \text{(by 3.1.1)}\\ &= (I_1 \cap S)R(I_2 \cap S)R \cap S \\ &= I_1I_2 \cap S && \text{(by 3.1.1 again).}\end{aligned}$$

□

This result is easily extended by induction, a fact we state explicitly due to its relevance to the subject of this thesis.

Corollary 3.1.3. *Let R be an Azumaya algebra with centre S. Then:*

1. *If J is an ideal of S and n is a positive integer then $(JR)^n = J^nR$;*

2. *If I is an ideal of R and n is a positive integer then $(I \cap S)^n = I^n \cap S$.*

Using the above result we obtain a crucial property of ideals in Azumaya algebras.

Corollary 3.1.4. *Let R be an Azumaya algebra with centre S. Then:*

1. *If P is a prime ideal of R then $P \cap S$ is a prime ideal of S;*

2. *If Q is a prime ideal of S then QR is a prime ideal of R.*

The above results do not hold for one-sided ideals, as the following example illustrates.

Remark 3.1.1. Let R be the ring $M_2(\mathbb{Z})$. This ring has centre:

$$S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{Z} \right\},$$

and R is an Azumaya algebra over S, since it is a matrix ring over a commutative ring. The set:

$$I = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$$

is a right ideal of R. But clearly we have $I \cap S = 0$.

The next result establishes that chain conditions pass between an Azumaya algebra and its centre.

Corollary 3.1.5. [12, Corollary 13.7.10] *Let R be an Azumaya algebra with centre S. Then the following three conditions are equivalent:*

1. *R has the ascending chain condition on ideals;*

2. *R is a (two-sided) Noetherian ring;*

3. *S is a Noetherian ring.*

3.2 Affiliated submodules

Let I be an ideal in a ring R. In this section we give a result characterizing the affiliated submodules of $(R/I)_R$. This result arises from the fact that R/I is an (R, R)-bimodule. The result in this section is a special case of a result concerning bimodules given, for example, in [4, Chapter 7].

Lemma 3.2.1. *Let R be a ring. Let I be an ideal of R. Let $J/I = l_{R/I}(P)$ be an affiliated submodule of $(R/I)_R$, where P is a maximal annihilator prime ideal of*

$(R/I)_R$. Then J/I is a two-sided ideal of R/I, and P is a maximal element of the set:

$$\left\{ r_R\left(\frac{X}{I}\right) \mid \frac{X}{I} \text{ is an ideal of } \frac{R}{I} \text{ such that } I \subsetneq X \right\}.$$

Conversely, if P is a maximal element of the set:

$$\left\{ r_R\left(\frac{X}{I}\right) \mid \frac{X}{I} \text{ is an ideal of } \frac{R}{I} \text{ such that } I \subsetneq X \right\},$$

then $l_{R/I}(P)$ is an affiliated submodule of $(R/I)_R$.

Proof. By definition J/I is a submodule of $(R/I)_R$, and so is a right ideal of R/I. Thus it is enough to show that J/I is a left ideal of R/I. Take elements $j + I \in J/I$ and $r + I \in R/I$. Then we have $jP \subseteq I$, and so $rjP \subseteq I$. Thus we have:

$$(r+I)(j+I) = rj + I \in l_{R/I}(P) = J/I.$$

Thus J/I is a left ideal, and hence a two-sided ideal, of R/I.

To establish the second part of the necessary condition, we note that P is a maximal element of the set:

$$\left\{ r_R\left(\frac{X}{I}\right) \mid \frac{X}{I} \text{ is a right ideal of } \frac{R}{I} \text{ such that } I \subsetneq X \right\},$$

again by definition. Now since we have established that P belongs to the set:

$$\left\{ r_R\left(\frac{X}{I}\right) \mid \frac{X}{I} \text{ is an ideal of } \frac{R}{I} \text{ such that } I \subsetneq X \right\},$$

it must actually be a maximal element of this set as required.

To prove the converse, we need to show that P is a maximal annihilator prime ideal of $(R/I)_R$. Choose an ideal X of R such that $I \subsetneq X$ and $P = r_R(X/I)$. Suppose that $P \subseteq r_R(Y/I)$ for some right ideal Y of R such that $I \subsetneq Y$. Then RY is an ideal of R and $I \subsetneqq Y \subseteq RY$. So RY/I is a non-zero ideal of R/I. But it is easily seen that:

$$P \subseteq r_R\left(\frac{Y}{I}\right) \subseteq r_R\left(\frac{RY}{I}\right).$$

We must have equality in this chain, by the maximality of P among annihilators of non-zero ideals of R/I. This completes the proof. □

3.3 Affiliated primes and Azumaya algebras

Let R be an Azumaya algebra with centre S. Let I be an ideal of R. In this section we outline the relationship between the affiliated prime ideals of $(R/I)_R$ and the affiliated prime ideals of $(S/(I \cap S))_S$.

Let R be a Noetherian ring. Let I be an ideal of R. We note that by 1.3.2 there does exist an affiliated series:

$$I/I = J_0/I \subsetneqq J_1/I \subsetneqq \ldots \subsetneqq J_n/I = R/I$$

for $(R/I)_R$.

Proposition 3.3.1. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Suppose that:*

$$I/I = J_0/I \subsetneqq J_1/I \subsetneqq \ldots \subsetneqq J_n/I = R/I$$

is an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1, \ldots, P_n$. Then:

$$\frac{I \cap S}{I \cap S} = \frac{J_0 \cap S}{I \cap S} \subsetneqq \frac{J_1 \cap S}{I \cap S} \subsetneqq \ldots \subsetneqq \frac{J_n \cap S}{I \cap S} = \frac{S}{I \cap S}$$

is an affiliated series for $(S/(I \cap S))_S$ with corresponding affiliated primes $P_1 \cap S, \ldots, P_n \cap S$.

Proof. The given chain clearly consists of (right) S-submodules of $S/(I \cap S)$. To see that the chain is strictly ascending, we note that $J_0, \ldots, J_n$ are ideals of R by 3.2.1 and so if $J_{i-1} \cap S = J_i \cap S$ for any index $i \in \{1, \ldots, n\}$ then we would have $(J_{i-1} \cap S)R = (J_i \cap S)R$ from which it follows that $J_{i-1} = J_i$ by 3.1.1. This is a contradiction.

Now, for any $i \in \{1, \ldots, n\}$, we have that $P_i = r_R(J_i/J_{i-1})$ by definition, and we claim that $P_i \cap S = r_S((J_i \cap S)/(J_{i-1} \cap S))$. Take an element $p \in P_i \cap S$. Then $(J_i \cap S)p \subseteq J_i p \cap S \subseteq J_{i-1} \cap S$, so that $p \in r_S((J_i \cap S)/(J_{i-1} \cap S))$. This establishes that $P_i \cap S \subseteq r_S((J_i \cap S)/(J_{i-1} \cap S))$. For the reverse inclusion, let $s \in r_S((J_i \cap S)/(J_{i-1} \cap S))$. Then we have:

$$\begin{aligned} J_i s &= (J_i \cap S)Rs && \text{(by 3.1.1)} \\ &= (J_i \cap S)sR && \text{(since } s \in S\text{)} \\ &\subseteq (J_{i-1} \cap S)R \\ &= J_{i-1} && \text{(by 3.1.1).} \end{aligned}$$

Thus $s \in r_R(J_i/J_{i-1}) \cap S = P_i \cap S$. This proves the claim. It remains to show that $(J_i \cap S)/(J_{i-1} \cap S)$ is an affiliated right S-submodule of $S/(J_{i-1} \cap S)$.

Since $P_i \cap S = r_S((J_i \cap S)/(J_{i-1} \cap S))$, we have the following inclusion:

$$l_{S/(J_{i-1} \cap S)}(P_i \cap S) \supseteq (J_i \cap S)/(J_{i-1} \cap S).$$

The reverse inclusion is proved as follows. Take an element $s + (J_{i-1} \cap S) \in l_{S/(J_{i-1} \cap S)}(P_i \cap S)$. Then $s(P_i \cap S) \subseteq J_{i-1} \cap S$. Multiplying both sides of this inclusion by R, we have that $s(P_i \cap S)R \subseteq (J_{i-1} \cap S)R$. Then $sP_i \subseteq J_{i-1}$, by 3.1.1. This shows that $s + J_{i-1} \in l_{R/J_{i-1}}(P_i) = J_i/J_{i-1}$ and hence that $s \in J_i \cap S$.

Thus $l_{S/(J_{i-1} \cap S)}(P_i \cap S) = (J_i \cap S)/(J_{i-1} \cap S)$, and by 3.2.1 it is now sufficient to show that $P_i \cap S$ is a maximal element of the set:

$$\left\{ r_S\left(\frac{X}{J_{i-1} \cap S}\right) \mid X \text{ is an ideal of } S \text{ such that } J_{i-1} \cap S \subsetneq X \right\}.$$

We have established that $P_i \cap S = r_S((J_i \cap S)/(J_{i-1} \cap S))$, so that $P_i \cap S$ *is* a member of the above set. Suppose that $P_i \cap S \subseteq r_S(X/(J_{i-1} \cap S))$ for some ideal X of S such that $J_{i-1} \cap S \subsetneq X$. Then we have:

$$P_i = (P_i \cap S)R \subseteq r_S\left(\frac{X}{J_{i-1} \cap S}\right) R.$$

We note that given elements $s \in r_S(X/(J_{i-1} \cap S))$ and $r \in R$ we have $Xs \subseteq J_{i-1} \cap S$, and so we have:

$$XRsr = XsRr \subseteq (J_{i-1} \cap S)Rr = J_{i-1}r \subseteq J_{i-1}.$$

This shows that:

$$r_S\left(\frac{X}{J_{i-1} \cap S}\right) R \subseteq r_R\left(\frac{XR}{J_{i-1}}\right).$$

Now we have:

$$P_i \subseteq r_S\left(\frac{X}{J_{i-1} \cap S}\right) R \subseteq r_R\left(\frac{XR}{J_{i-1}}\right).$$

But we must have equality in this chain by 3.2.1. It follows, by taking intersections in S, that we have:

$$P_i \cap S = r_S\left(\frac{X}{J_{i-1} \cap S}\right).$$

This completes the proof. □

Corollary 3.3.2. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Then:*

$$Aff\left(\frac{R}{I}\right)_R \cap S \subseteq Aff\left(\frac{S}{I \cap S}\right)_S.$$

Proof. This follows directly from 3.3.1. □

Before proving the reverse inclusion, we need the following result.

Lemma 3.3.3. *Let R be an Azumaya algebra with centre S. Let I be an ideal of R. Let J be an ideal of R such that $J \supseteq I$, so that J/I is an ideal of R/I. Then:*

$$\left(\frac{J}{I} \cap \frac{S+I}{I}\right)\frac{R}{I} = \frac{J}{I}.$$

Proof. It is clear that:

$$\left(\frac{J}{I} \cap \frac{S+I}{I}\right)\frac{R}{I} \subseteq \frac{J}{I}.\frac{R}{I} = \frac{J}{I}.$$

To prove the reverse inclusion, note that it follows from the fact that J is an ideal of R that $(J \cap S)R = J$. So we have the following:

$$\frac{J}{I} = \frac{(J \cap S)R}{I} \subseteq \frac{(J \cap (S+I))R}{I} = \frac{J \cap (S+I)}{I}.\frac{R}{I} = \left(\frac{J}{I} \cap \frac{S+I}{I}\right)\frac{R}{I}.$$

□

We can now prove a result which amounts to a converse of 3.3.1.

Proposition 3.3.4. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Suppose that:*

$$\frac{I \cap S}{I \cap S} = \frac{J_0}{I \cap S} \subsetneq \frac{J_1}{I \cap S} \subsetneq \dots \subsetneq \frac{J_n}{I \cap S} = \frac{S}{I \cap S}$$

is an affiliated series for $(S/(I \cap S))_S$ with corresponding affiliated primes $Q_1, \dots, Q_n$. Then:

$$\frac{I}{I} = \frac{J_0 R}{I} \subsetneq \frac{J_1 R}{I} \subsetneq \dots \subsetneq \frac{J_n R}{I} = \frac{R}{I}$$

is an affiliated series for $(R/I)_R$ with corresponding affiliated primes $Q_1 R, \dots, Q_n R$.

Proof. Take an index $i \in \{1, \ldots, n\}$. We first establish that:

$$Q_i R = r_R\left(\frac{J_i R}{J_{i-1} R}\right). \tag{3.3.1}$$

Take elements $q \in Q_i$ and $a \in R$. Using the fact that $Q_i = r_S(J_i/J_{i-1})$, we have that:

$$\begin{aligned} J_i R q a &= J_i q R a \quad (\text{since } q \in Q_i \subseteq S) \\ &\subseteq J_{i-1} R a \\ &\subseteq J_{i-1} R. \end{aligned}$$

Thus $qa \in r_R(J_i R/J_{i-1} R)$, from which it follows that:

$$Q_i R \subseteq r_R\left(\frac{J_i R}{J_{i-1} R}\right).$$

For the reverse inclusion, note that for any element $s \in r_R(J_i R/J_{i-1} R) \cap S$, we have that $J_i R s \subseteq J_{i-1} R$. Thus $J_i s R \subseteq J_{i-1} R$, since $s \in S$. Intersecting both sides of this inclusion in S gives that $J_i s R \cap S \subseteq J_{i-1} R \cap S$. Now since $J_i s$ is an ideal of S, we can apply 3.1.1, giving that $J_i s \subseteq J_{i-1}$. Thus $s \in r_S(J_i/J_{i-1}) = Q_i$. This shows that:

$$r_R\left(\frac{J_i R}{J_{i-1} R}\right) \cap S \subseteq Q_i.$$

Now apply 3.1.1 again to conclude that:

$$r_R\left(\frac{J_i R}{J_{i-1} R}\right) \subseteq Q_i R.$$

This establishes the initial claim (3.3.1).

From the claim (3.3.1) we can immediately deduce that:

$$l_{\frac{R}{J_{i-1} R}}(Q_i R) \supseteq \frac{J_i R}{J_{i-1} R}. \tag{3.3.2}$$

Then we have the following:

$$\frac{J_i}{J_{i-1}} = l_{\frac{S}{J_{i-1}}}(Q_i).$$

Thus, using 3.1.1, we have:

$$\frac{J_i}{J_{i-1}R \cap S} = l_{\frac{S}{J_{i-1}R\cap S}}(Q_i),$$

and so, by the second isomorphism theorem:

$$\frac{J_i + J_{i-1}R}{J_{i-1}R} = l_{\frac{S+J_{i-1}R}{J_{i-1}R}}(Q_i).$$

Then:

$$\frac{(J_iR \cap S) + J_{i-1}R}{J_{i-1}R} = l_{\frac{R}{J_{i-1}R}}(Q_i) \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right).$$

Applying the Dedekind modular law to this gives that:

$$\frac{J_iR \cap (S + J_{i-1}R)}{J_{i-1}R} = l_{\frac{R}{J_{i-1}R}}(Q_i) \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right).$$

Thus:

$$\frac{J_iR}{J_{i-1}R} \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right) = l_{\frac{R}{J_{i-1}R}}(Q_i) \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right).$$

So since $Q_i \subseteq Q_iR$, we have:

$$\frac{J_iR}{J_{i-1}R} \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right) \supseteq l_{\frac{R}{J_{i-1}R}}(Q_iR) \cap \left(\frac{S + J_{i-1}R}{J_{i-1}R}\right). \tag{3.3.3}$$

We claim that $l_{R/J_{i-1}R}(Q_iR)$ is an ideal of $R/J_{i-1}R$. Q_iR is an ideal of R, which shows that $l_{R/J_{i-1}R}(Q_iR)$ is a submodule of $(R/J_{i-1}R)_R$. In other words $l_{R/J_{i-1}R}(Q_iR)$ is a right ideal of $R/J_{i-1}R$. Given an arbitrary element $r \in R$ and an element $a \in R$ such that $aQ_iR \subseteq J_{i-1}R$, clearly we have that $raQ_iR \subseteq J_{i-1}R$ (since $J_{i-1}R$ is an ideal of R). Thus $ra + J_{i-1}R \in l_{\frac{R}{J_{i-1}R}}(Q_iR)$. This establishes that $l_{R/J_{i-1}R}(Q_iR)$ is an ideal of $R/J_{i-1}R$. Now by applying Lemma 3.3.3 to the inclusion 3.3.3 we can conclude that:

$$l_{\frac{R}{J_{i-1}R}}(Q_iR) \subseteq \frac{J_iR}{J_{i-1}R},$$

which together with the inclusion 3.3.2 gives that:

$$l_{\frac{R}{J_{i-1}R}}(Q_iR) = \frac{J_iR}{J_{i-1}R}.$$

So, by using 3.2.1, it is now sufficient to show that Q_iR is a maximal element of the set:

$$\left\{ r_R\left(\frac{X}{J_{i-1}R}\right) \mid X \text{ is an ideal of } R \text{ such that } J_{i-1}R \subsetneq X \right\}.$$

Q_iR does belong to this set, since $Q_iR = r_R(J_iR/J_{i-1}R)$. Now suppose that $Q_iR \subseteq r_R(X/J_{i-1}R)$ for some ideal X of R such that $J_{i-1}R \subsetneq X$. Then $Q_iR \cap S \subseteq r_R(X/J_{i-1}R) \cap S$, and so $Q_i \subseteq r_S(X/J_{i-1}R)$, by 3.1.1. Now if we take an element $s \in S$ such that $Xs \subseteq J_{i-1}R$, then we have:

$$\begin{aligned} (X \cap S)s &\subseteq Xs \cap S \\ &\subseteq J_{i-1}R \cap S \\ &= J_{i-1}. \end{aligned}$$

This gives the chain of inclusions:

$$Q_i \subseteq r_S\left(\frac{X}{J_{i-1}R}\right) \subseteq r_S\left(\frac{X \cap S}{J_{i-1}}\right).$$

Finally, note that Q_i is maximal among annihilators of non-zero ideals of S/J_{i-1}, by 3.2.1, so equality must hold across the above chain. This gives that:

$$Q_i = r_S\left(\frac{X}{J_{i-1}R}\right).$$

Thus, by 3.1.1, we have:

$$Q_iR \cap S = r_R\left(\frac{X}{J_{i-1}R}\right) \cap S.$$

Applying 3.1.1 again, this time to both sides of the inclusion, gives that:

$$Q_iR = r_R\left(\frac{X}{J_{i-1}R}\right).$$

Thus Q_iR is a maximal annihilator prime ideal of $(R/J_{i-1}R)_R$ as required. This completes the proof. □

Corollary 3.3.5. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Then:*

$$Aff\left(\frac{R}{I}\right)_R \cap S \supseteq Aff\left(\frac{S}{I \cap S}\right)_S.$$

Proof. This follows directly from 3.3.4. □

Corollary 3.3.6. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Then:*

$$Aff\left(\frac{R}{I}\right)_R \cap S = Aff\left(\frac{S}{I \cap S}\right)_S,$$

and:

$$Aff\left(\frac{R}{I}\right)_R = Aff\left(\frac{S}{I \cap S}\right)_S R.$$

Proof. The first statement follows from 3.3.2 and 3.3.5. The second statement follows directly from 3.1.1. □

3.4 An example

In this section we provide an example to illustrate 3.3.6.

Let A be a commutative Noetherian ring. Then $M_2(A)$ is a Noetherian Azumaya algebra. Suppose that $M_2(A)$ is a Noetherian ring. Let I be an ideal of $M_2(A)$. Then $I = M_2(J)$ for some ideal J of A. We aim to use 3.3.6 to determine the elements of the set:

$$Aff\left(\frac{M_2(A)}{M_2(J)}\right)_{M_2(A)}.$$

We now introduce some notation. If S_1, S_2, S_3 and S_4 are subsets of A, then the notation:

$$\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$$

will normally denote the set:

$$\left\{ \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \mid s_i \in S_i \text{ for each } i = 1,2,3,4 \right\}.$$

However, in this example the notation:

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix},$$

where S is a subset of A, will denote the set:

$$\left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \mid s \in S \right\}.$$

Let X denote the centre of $M_2(A)$. Then:

$$X = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

and clearly $X \cong A$ via the ring isomorphism $\theta : X \longrightarrow A$ defined by:

$$\theta\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = a,$$

for all $a \in A$. Now since X is a commutative Noetherian ring, we have:

$$Aff\left(\frac{X}{M_2(J) \cap X}\right)_X = Ass\left(\frac{X}{M_2(J) \cap X}\right)_X,$$

by 1.3.5. Further we have $\theta(M_2(J) \cap X) = J$, and so:

$$\theta\left(Ass\left(\frac{X}{M_2(J) \cap X}\right)_X\right) = Ass\left(\frac{A}{J}\right)_A.$$

Thus:

$$Ass\left(\frac{X}{M_2(J)\cap X}\right)_X = \left\{\begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \mid P \in Ass\left(\frac{A}{J}\right)_A\right\}.$$

We claim that:

$$Aff\left(\frac{M_2(A)}{M_2(J)}\right)_{M_2(A)} = \left\{M_2(P) \mid P \in Ass\left(\frac{A}{J}\right)_A\right\}.$$

Take an element:

$$Y \in Aff\left(\frac{M_2(A)}{M_2(J)}\right)_{M_2(A)}.$$

Then there is an ideal P of A such that $Y = M_2(P)$. Then, by 3.3.6, we have that:

$$M_2(P) = \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} M_2(A),$$

for some associated prime ideal P' of $(A/J)_A$. So it is enough to show that $P = P'$. This is true as follows. Take an element $p \in P$. Then we have:

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \in M_2(P) = \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} M_2(A).$$

Then it follows that there is a positive integer n, elements $p_1, \ldots, p_n \in P'$ and elements $a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, a_{13}, \ldots, a_{n3}, a_{14}, \ldots, a_{n4} \in A$ such that:

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_1a_{11} + \cdots + p_na_{n1} & p_1a_{12} + \cdots + p_na_{n2} \\ p_1a_{13} + \cdots + p_na_{n3} & p_1a_{14} + \cdots + p_na_{n4} \end{bmatrix}.$$

Thus $p \in P'$. For the opposite inclusion, let $p' \in P'$. Then clearly we have:

$$\begin{bmatrix} p' & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} \subseteq \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} M_2(A) = M_2(P).$$

Thus $p' \in P$.

Conversely, let P be an associated prime ideal of $(A/J)_A$. Then clearly we have that:

$$M_2(P) \cap X = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}.$$

Hence we have:

$$M_2(P) \cap X \in Ass\left(\frac{X}{M_2(J) \cap X}\right)_X = Aff\left(\frac{X}{M_2(J) \cap X}\right)_X.$$

Hence:

$$M_2(P) \in Aff\left(\frac{X}{M_2(J) \cap X}\right)_X M_2(A).$$

Thus by 3.3.6 we have:

$$M_2(P) \in Aff\left(\frac{M_2(A)}{M_2(J)}\right)_{M_2(A)}.$$

Remark 3.4.1. In the above example we have used 2 x 2 matrices, but the argument is equally valid for matrices of higher dimensions. Thus we have the following. Let n be a positive integer. Let A be a commutative ring such that $M_n(A)$ is a Noetherian ring and let $M_n(J)$ be an ideal of $M_n(A)$, where J is an ideal of A. Then:

$$Aff\left(\frac{M_n(A)}{M_n(J)}\right)_{M_n(A)} = \left\{M_n(P) \mid P \in Ass\left(\frac{A}{J}\right)_A\right\}.$$

Remark 3.4.2. Let $A = \mathbb{Z}$ in the above example. Then $\mathbb{Z}$ is a Noetherian ring, so $M_n(\mathbb{Z})$ is also a Noetherian ring. Now $\mathbb{Z}$ is a principal ideal domain, so that ideals of $\mathbb{Z}$ are of the form $a\mathbb{Z}$, for some integer $a \in \mathbb{Z}$. Thus ideals of $M_n(\mathbb{Z})$ are of the form $M_n(a\mathbb{Z})$, for some integer $a \in \mathbb{Z}$.

Now let $M_n(a\mathbb{Z})$ be an ideal of $M_n(\mathbb{Z})$. It can be checked that the prime ideals of the factor ring $\mathbb{Z}/a\mathbb{Z}$ are precisely the elements of the set $\{p\mathbb{Z}/a\mathbb{Z} \mid p \text{ is a prime divisor of } a\}$. Now every prime ideal of $\mathbb{Z}/a\mathbb{Z}$ is an associated prime ideal of $(\mathbb{Z}/a\mathbb{Z})_{(\mathbb{Z}/a\mathbb{Z})}$ as follows. Let $p\mathbb{Z}/a\mathbb{Z}$ be a prime ideal of $\mathbb{Z}/a\mathbb{Z}$. Then since p is a prime divisor of a there is an integer $x \in \mathbb{Z}$ such that $a = xp$. Then it is easy to check that $p\mathbb{Z}/a\mathbb{Z} = r_{\mathbb{Z}/a\mathbb{Z}}(x + a\mathbb{Z})$, so that $p\mathbb{Z}/a\mathbb{Z}$ is an associated prime ideal of $(\mathbb{Z}/a\mathbb{Z})_{(\mathbb{Z}/a\mathbb{Z})}$ by 2.3.1. Thus we have that:

$$Ass\left(\frac{\mathbb{Z}}{a\mathbb{Z}}\right)_{\mathbb{Z}} = \{p\mathbb{Z} \mid p \text{ is a prime divisor of } a\}.$$

Hence we can conclude that:

$$Aff\left(\frac{M_n(\mathbb{Z})}{M_n(a\mathbb{Z})}\right)_{M_n(\mathbb{Z})} = \{M_n(p\mathbb{Z}) \mid p \text{ is a prime divisor of } a\}.$$

3.5 Asymptotic stability for Azumaya algebras

Having now established that affiliated prime ideals go up and down between an Azumaya algebra and its centre in the manner we desire, we are in a position to prove an asymptotic stability result for Azumaya algebras. This result generalizes 2.3.8 to right Noetherian Azumaya algebras.

Corollary 3.5.1. *Let R be an Azumaya algebra with centre S. Suppose that R is a right Noetherian ring. Let I be an ideal of R. Then there is a positive integer p such that whenever n is a positive integer such that $n \geq p$, we have:*

$$Aff\left(\frac{R}{I^n}\right)_R = Aff\left(\frac{R}{I^{n+1}}\right)_R.$$

Proof. Since R is a right Noetherian ring, S is also a Noetherian ring by 3.1.5. Thus by 2.3.6 we can choose a positive integer p such that whenever n is a positive integer such that $n \geq p$, we have:

$$Aff\left(\frac{S}{(I\cap S)^n}\right)_S = Aff\left(\frac{S}{(I\cap S)^{n+1}}\right)_S.$$

Take a prime ideal:

$$P \in Aff\left(\frac{R}{I^n}\right)_R.$$

Then, by 3.3.6, we have:

$$P\cap S \in Aff\left(\frac{S}{I^n\cap S}\right)_S.$$

Thus, by applying 3.1.3, we have:

$$P\cap S \in Aff\left(\frac{S}{(I\cap S)^n}\right)_S.$$

By hypothesis:

$$P \cap S \in Aff\left(\frac{S}{(I \cap S)^{n+1}}\right)_S.$$

By applying 3.1.3 again, we have:

$$P \cap S \in Aff\left(\frac{S}{I^{n+1} \cap S}\right)_S.$$

By applying 3.3.6 again, we have:

$$P \in Aff\left(\frac{R}{I^{n+1}}\right)_R.$$

This shows that:

$$Aff\left(\frac{R}{I^n}\right)_R \subseteq Aff\left(\frac{R}{I^{n+1}}\right)_R.$$

The reverse inclusion is proved analogously. □

Chapter 4

Affiliated primes and localization

Let R be a Noetherian PI ring. Our general strategy is to find a overring R' which satisfies the following two criteria:

1. affiliated primes 'go up and down' between R' and R;

2. affiliated primes 'go up and down' between R' and the centre of R'.

It would then be possible to formulate a theorem concerning asymptotic stability of affiliated primes in R, using the fact that asymptotic stability of associated primes holds in the centre of R'.

4.1 Localizations

Recall in chapter 1 that we defined the right quotient ring of a ring. If a ring R does have a right quotient ring then the set of all regular elements of R form what is called a right denominator set, and the right quotient ring can be viewed as a *localization* of R at the set of regular elements. In this section we generalize this concept and define a localization at *any* right denominator set in a ring R.

Definition 4.1.1. A subset X of a ring R is called a *multiplicative set* if the following conditions hold:

1. whenever we have elements $x_1, x_2 \in X$, we have $x_1x_2 \in X$;

2. $1_R \in X$.

Definition 4.1.2. Let X be a multiplicative subset of R. Then X is called:

1. a *right Ore set* if given arbitrary elements $x \in X$ and $r \in R$, there exist elements $x' \in X$ and $r' \in R$ such that $xr' = rx'$;

2. a *right reversible set* if whenever $r \in R$ and $x \in X$ are elements such that $xr = 0$, then there is an element $x' \in X$ such that $rx' = 0$;

3. a *right denominator set* if it is both a right Ore set and a right reversible set.

Left Ore sets, *left reversible sets* and *left denominator sets* are defined analogously.

Definition 4.1.3. Let R be a ring. Let X be a multiplicative subset of R. Then a *right ring of fractions* (or *right Ore quotient ring*, or *right Ore localization*) *for R with respect to X* is a pair (S, ϕ), where S is a ring and $\phi : R \longrightarrow S$ is a ring homomorphism which satisfies the following conditions:

1. $\phi(x)$ is a unit in S whenever $x \in X$;

2. every element of S has the form $\phi(r)\phi(x)^{-1}$ for some elements $r \in R$ and $x \in X$;

3. $ker(\phi) = \{r \in R \mid rx = 0 \text{ for some } x \in X\}$.

Remark 4.1.1.

1. In this case S is usually referred to as the right ring of fractions for R with respect to X, and an element $\phi(r)\phi(x)^{-1}$ of S (where $r \in R$ and $x \in X$) is abbreviated to simply rx^{-1}.

2. A *left ring of fractions for R with respect to X* is defined symmetrically.

3. If X consists of regular elements of R then the map $\phi : R \longrightarrow S$ is a monomorphism. In fact, $\phi : R \longrightarrow S$ is a monomorphism if and only if X consists of left regular elements.

4. The map ϕ is occasionally called the *natural localization map.*

The next two results establish for which multiplicative sets it is possible to form a right ring of fractions, and that when we can form such a ring it must be unique.

Theorem 4.1.1. [4, Theorem 9.7] *Let R be a ring. Let X be a multiplicative subset of R. Then there exists a right ring of fractions for R with respect to X if and only if X is a right denominator set.*

Theorem 4.1.2. [4, Corollary 9.5] *Let R be a ring. Let X be a right denominator set in R. Suppose that $\phi_1 : R \longrightarrow S_1$ and $\phi_2 : R \longrightarrow S_2$ are both right rings of fractions for R with respect to X. Then there is a unique ring isomorphism $\eta : S_1 \longrightarrow S_2$ such that $\eta \circ \phi_1 = \phi_2$.*

Thus given a right denominator set X in a ring R, the right ring of fractions for R with respect to X, which exists and is unique up to isomorphism by 4.1.1 and 4.1.2 can be denoted by RX^{-1} without ambiguity.

If X is a *denominator* set in a ring R, that is to say if X is both a right and a left denominator set in R, then we can form both a right ring of fractions, RX^{-1}, and a left ring of fractions, $X^{-1}R$, for R with respect to X. Furthermore, by 4.1.2 and the left-handed version of 4.1.2, both RX^{-1} and $X^{-1}R$ are unique (up to isomorphism). In fact, we have the following.

Proposition 4.1.3. [4, Proposition 9.8] *Let X be a (two-sided) denominator set in a ring R. Then any right ring of fractions for R with respect to X is also a left ring of fractions for R with respect to X. Symmetrically, any left ring of fractions for R with respect to X is also a right ring of fractions for R with respect to X.*

Thus if X is a (two-sided) denominator set in a ring R, then RX^{-1} and $X^{-1}R$ are isomorphic and thus it is possible to speak of a *ring of fractions for R with respect to X* which is unique up to isomorphism.

Suppose that R is a ring which has a right quotient ring Q. Let X be the set of all regular elements of R. We note that in this case X is a right denominator set and that the right ring of fractions for R with respect to X is (isomorphic to) Q. Thus a right ring of fractions can be regarded as a generalization of a right quotient ring.

4.2 The X-torsion submodule

Let R be a ring and let M be a right R-module. If R is a semi-prime right Goldie ring then it is standard to use 'torsion' terminology for the singular submodule of M. That is: the singular submodule of M is called the *torsion submodule* of M; if M is a singular right R-module then it is called a *torsion* right R-module; if M is a non-singular right R-module then it is called a *torsion-free* right R-module. The reason for using this terminology is that in the case when R is a semi-prime right Goldie ring the singular submodule of a right R-module is analogous to the torsion submodule of a module over a commutative integral domain, as the following result shows.

Proposition 4.2.1. [4, Proposition 6.9] *Let R be a semi-prime right Goldie ring. Let M be a right R-module. Let $Z(M)$ denote the singular submodule of M. Then we have:*

$$Z(M) = \{m \in M \mid mx = 0 \textit{ for some regular element } x \in R\}.$$

Thus, generally speaking, it only makes sense to use 'torsion' terminology when R is a semi-prime right Goldie ring. However, using the following result, we are able to make a more general definition.

Lemma 4.2.2. [4, Lemma 9.3] *Let R be a ring. Let X be a subset of R such that X is a right Ore set. Let M be a right R-module. Then the subset of M:*

$$N = \{m \in M \mid mx = 0 \textit{ for some } x \in X\}$$

is a submodule of M.

Definition 4.2.1. Let R be a ring. Let X be a subset of R such that X is a right Ore set. Let M be a right R-module. Then the submodule described above:

$$N = \{m \in M \mid mx = 0 \text{ for some } x \in X\}$$

of M is called the *X-torsion submodule of M* and is denoted by $t_X(M)$.

Definition 4.2.2. Let R be a ring. Let X be a subset of R such that X is a right Ore set. Let M be a right R-module. Then:

1. M is said to be *X-torsion* if $t_X(M) = M$;

2. M is said to be *X-torsion-free* if $t_X(M) = 0$.

Remark 4.2.1. Let R be a ring. Let X be a right denominator set in R, so that the right ring of fractions for R with respect to X, RX^{-1}, exists. Let $rx^{-1} \in RX^{-1}$ (where $r \in R$ and $x \in X$). Then $rx^{-1} = 0$ if and only if $r \in t_X(R_R)$.

If R is a semi-prime right Goldie ring and X is the set of regular elements of R then the X-torsion submodule of a right R-module is just the torsion submodule. Thus the above is a generalization of the definition of torsion and torsion-free modules. There are corresponding generalizations of some of the important results concerning torsion and torsion-free modules. We give those that are relevant to our study.

Proposition 4.2.3. *Let R be a ring. Let X be a right Ore set in R. Let M be an X-torsion right R-module. Then every submodule of M is X-torsion, and every factor module of M is X-torsion.*

Proof. Take a submodule N of M. Take an element $n \in N$. Then $n \in M$ also, so there is an element $x \in X$ such that $nx = 0$. This shows that N is X-torsion. Now take an element $m + N \in M/N$, where $m \in M$. Since M is X-torsion, there is an element $x \in X$ such that $mx = 0$. Then $(m+N)x = mx + N = N$, showing that M/N is X-torsion. □

Proposition 4.2.4. *Let R be a ring. Let X be a right Ore set in R. Let M be an X-torsion-free right R-module. Then every submodule of M is X-torsion-free, and every essential extension of M is X-torsion-free.*

Proof. Take a submodule N of M. Let $n \in N$ be an element such that $nx = 0$ for some element $x \in X$. Then $n \in M$, and thus $n = 0$ since M is X-torsion-free. This shows that N is X-torsion-free. Now let M' be a right R-module such that M is an essential submodule of M'. Clearly we have that $t_X(M') \cap M = t_X(M)$. So since M is X-torsion-free we have that $t_X(M) = 0$ and hence that $t_X(M') \cap M = 0$. Now $t_X(M')$ is a submodule of M' so the fact that M is essential in M' shows that $t_X(M') = 0$. Thus M' is X-torsion-free. □

For the results of which 4.2.3 and 4.2.4 are generalizations, see [4, Proposition 6.10, (a) and (c)].

4.3 Localizations of modules

Throughout this section let X be a right denominator set in a ring R, so that the right ring of fractions for R with respect to X, RX^{-1}, exists. We generalize the concepts in the above section further, and define the localization of a right R-module M at X.

Remark 4.3.1. Let $\theta : R \longrightarrow S$ be any ring homomorphism. Note that any right S-module M can be made to form a right R-module under the module action $mr = m\theta(r)$. Thus even if the natural localization map $\phi : R \longrightarrow RX^{-1}$ is not a monomorphism it is still true that any right module over RX^{-1} is also a right module over R.

This allows a greater level of generality to be assumed in the following theory.

Definition 4.3.1. Let M be a right R-module. Then a *module of fractions for M with respect to X* is defined to be a pair (N, f), where N is a right RX^{-1}-module and $f : M \longrightarrow N$ is a right R-module homomorphism which satisfies the following:

1. every element of N has the form $f(m)x^{-1}$ for some elements $m \in M$ and $x \in X$;

2. $ker(f) = t_X(M) = \{m \in M \mid mx = 0 \text{ for some } x \in X\}$.

As with localizations of rings, we abuse the notation and simply call N a module of fractions for M with respect to X, and denote an element $f(a)x^{-1}$ (where $a \in M$ and $x \in X$) of the module of fractions simply by ax^{-1}. The map f is occasionally called the *natural localization map*.

There are existence and uniqueness theorems for modules of fractions which are similar to those which exist for localizations of rings.

Theorem 4.3.1. [4, Theorem 9.13] *Let M be a right R-module. Then there exists a module of fractions for M with respect to X.*

Theorem 4.3.2. [4, Corollary 9.11] *Let M be a right R-module. Suppose that $f_1 : M \longrightarrow N_1$ and $f_2 : M \longrightarrow N_2$ are both modules of fractions for M with respect to X. Then there is a unique right RX^{-1}-module isomorphism $h : N_1 \longrightarrow N_2$ such that $h \circ f_1 = f_2$.*

Given a right R-module M, the module of fractions for M with respect to X, which exists and is unique up to isomorphism by 4.3.1 and 4.3.2, is denoted by MX^{-1}. As with localizations of rings, there is no ambiguity in this notation because of the uniqueness of the module of fractions.

Remark 4.3.2. Considering R as a right module over itself, it is clear that the right ring of fractions RX^{-1} is also a module of fractions for R_R with respect to X. Thus the localization of R as a ring at X and the localization of R as a right module over itself at X are isomorphic.

4.4 Extensions and contractions

In order to be able to make use of localizations, or rings or modules of fractions, we need to be able to relate the ideal structure of a ring with the ideal structure of a localization of that ring. To do this, mechanisms for going up and down between the ideals of a ring and the ideals of a localization of that ring are needed. This is the purpose of this section.

Definition 4.4.1. Let X be a right denominator set in a ring R. Let M be a right R-module.

1. For any submodule N of M_R, define the *extension* of N, denoted by N^e, as follows:

$$N^e = \{nx^{-1} \mid n \in N, x \in X\}.$$

2. For any submodule A of $(MX^{-1})_{RX^{-1}}$, define the *contraction* of A, denoted by A^c, as follows:

$$A^c = \{m \in M \mid m1^{-1} \in A\}.$$

Proposition 4.4.1. *Let X be a right denominator set in a ring R. Let M be a right R-module. Let N be a submodule of M_R and A a submodule of $(MX^{-1})_{RX^{-1}}$. Then N^e is a submodule of $(MX^{-1})_{RX^{-1}}$ and A^c is a submodule of M_R.*

Proof. [4, Theorem 9.17, (a) and (b)]. □

Lemma 4.4.2. *Let X be a right denominator set in a ring R. Let M be a right R-module. Let N be a submodule of M_R and let A be a submodule of $(MX^{-1})_{RX^{-1}}$. Then we have the following:*

1. $A^{ce} = A$;

2. $N^{ec} \supseteq N$;

3. $N^{ec} = N$ *if and only if M/N is a torsion-free right R-module.*

Proof. [4, Theorem 9.17, (a) and (b)]. □

An important consequence of these results is the following.

Corollary 4.4.3. *Let X be a right denominator set in a ring R. Let M be a right R-module. Suppose that M is a Noetherian right R-module. Then MX^{-1} is a Noetherian right RX^{-1}-module.*

Proof. [4, Corollary 9.18, (a)]. □

Remark 4.4.1. Let X be a (two-sided) denominator set in a (two-sided) Noetherian ring R. Then we have by 4.4.3 that RX^{-1} is a right Noetherian ring. We also have that the left ring of fractions for R with respect to X, $X^{-1}R$, is a left Noetherian ring, by the left-handed version of 4.4.3. But by 4.1.2 and 4.1.3, we have that $X^{-1}R \cong RX^{-1}$. Thus RX^{-1} is a (two-sided) Noetherian ring.

Lemma 4.4.4. *Let X be a right denominator set in a ring R. Suppose that X consists of central elements of R. Then the set:*

$$\{x^{-1} \mid x \in X\} \subseteq RX^{-1}$$

consists of central elements of RX^{-1}.

Proof. Let $x \in X$. Since X consists of central elements of R, we have $rx = xr$ for any element $r \in R$. So $x^{-1}r = rx^{-1}$ for any element $r \in R$. Now for any $ry^{-1} \in RX^{-1}$ (where $r \in R$ and $y \in X$), we have $ry^{-1}x^{-1} = r(xy)^{-1} = r(yx)^{-1} = rx^{-1}y^{-1} = x^{-1}ry^{-1}$. Thus the element $x^{-1} \in RX^{-1}$ is central as required. □

We next need to investigate the circumstances in which the extension and contraction operations preserve products.

Lemma 4.4.5. *Let X be a right denominator set in a ring R. Let I_1 and I_2 be right ideals of R. Then:*

$$(I_1 I_2)^e \subseteq {I_1}^e {I_2}^e.$$

Let J_1 and J_2 be right ideals of RX^{-1}. Then:

$${J_1}^c {J_2}^c \subseteq (J_1 J_2)^c.$$

Proof. Take an element $cx^{-1} \in (I_1 I_2)^e$ where $c \in I_1 I_2$ and $x \in X$. Since $c \in I_1 I_2$ we can write $c = a_1 b_1 + \cdots + a_n b_n$, where $a_i \in I_1$ and $b_i \in I_2$ for each $i \in \{1, \ldots, n\}$. Then $cx^{-1} = a_1 b_1 x^{-1} + \cdots + a_n b_n x^{-1} = a_1 1^{-1} b_1 x^{-1} + \cdots + a_n 1^{-1} b_n x^{-1} \in {I_1}^e {I_2}^e$. Thus $(I_1 I_2)^e \subseteq {I_1}^e {I_2}^e$ as required.

To prove the second statement, note that by 4.4.1, ${J_1}^c$ and ${J_2}^c$ are right ideals of R. Thus we can apply the first statement, which gives $({J_1}^c {J_2}^c)^e \subseteq {J_1}^{ce} {J_2}^{ce} = J_1 J_2$. Then, contracting both sides of this inclusion, we have $({J_1}^c {J_2}^c)^{ec} \subseteq (J_1 J_2)^c$ and so ${J_1}^c {J_2}^c \subseteq (J_1 J_2)^c$, by 4.4.2. □

Lemma 4.4.6. *Let X be a right denominator set in a ring R. Suppose that X consists of central elements of R. Let I_1 and I_2 be right ideals of R. Then $(I_1 I_2)^e = {I_1}^e {I_2}^e$.*

Proof. The inclusion in one direction is shown by 4.4.5. For the inclusion in the opposite direction, take elements $ax^{-1} \in {I_1}^e$, where $a \in I_1$ and $x \in X$, and $by^{-1} \in {I_2}^e$, where $b \in I_2$ and $y \in X$. Then:

$$\begin{aligned} ax^{-1}by^{-1} &= abx^{-1}y^{-1} \quad \text{(by 4.4.4)} \\ &= ab(yx)^{-1} \\ &\in (I_1I_2)^e. \end{aligned}$$

It follows from this that ${I_1}^e{I_2}^e \subseteq (I_1I_2)^e$, as required. □

Corollary 4.4.7. *Let X be a right denominator set in a ring R. Suppose that X consists of central elements of R. Let n be any positive integer. Let $I_1, \ldots, I_n$ be right ideals of R. Then $(I_1 \ldots I_n)^e = {I_1}^e \ldots {I_n}^e$.*

Proof. This is a straightforward extension of 4.4.6 by induction. □

We now turn our consideration to two-sided ideals.

Proposition 4.4.8. *Let X be a right denominator set in a ring R. Let J be an ideal of RX^{-1}. Then J^c is an ideal of R.*

Proof. [4, Proposition 9.19 (a)]. □

However if I is an ideal of R then I^e is not necessarily an ideal of RX^{-1}. The reader is referred to [4, Exercise 9N] for an example to demonstrate this. However, we have the following.

Proposition 4.4.9. *Let X be a right denominator set in a ring R such that RX^{-1} is a right Noetherian ring. Let I be an ideal of R. Then I^e is an ideal of RX^{-1}.*

Proof. [4, Theorem 9.20 (a)]. □

There are alternative circumstances in which we can 'go up' from ideals in a ring to ideals in a localization of that ring, provided by the following.

Proposition 4.4.10. *Let X be a right denominator set in a ring R. Suppose that X consists of central elements of R. Let I be an ideal of R. Then I^e is an ideal of RX^{-1}.*

Proof. By 4.4.1 we have that I^e is a right ideal of RX^{-1}. Take elements $ax^{-1} \in I^e$, where $a \in I$ and $x \in X$, and $ry^{-1} \in RX^{-1}$, where $r \in R$ and $y \in X$. Then, using 4.4.4, we have $ry^{-1}ax^{-1} = ray^{-1}x^{-1} = ra(xy)^{-1} \in I^e$. □

We note that a number of the preceeding and following results require X to be a subset of R which is 'a right denominator set which consists of central elements of R'. In fact it is clear from the definition that any multiplicative set which consists of central elements is automatically a right denominator set. Hence the sets we require are precisely the central multiplicative sets.

Proposition 4.4.11. *Let X be a central multiplicative set in a ring R. Let P be a prime ideal of R. Then the following three conditions are equivalent:*

1. *$P = Q^c$ for some prime ideal Q of RX^{-1};*

2. *the elements of X are regular modulo P: in other words we have $X \subseteq \mathcal{C}(P)$;*

3. *R/P is an X-torsion-free right R-module.*

Proof. The equivalence of the first and the third statements is given by [4, Theorem 9.20 (d)]. We proceed to show that the second and the third statements are equivalent.

Suppose that $X \subseteq \mathcal{C}(P)$ and take an element $r+P \in t_X((R/P)_R)$. Then there is an element $x \in X$ such that $rx \in P$. But $x \in \mathcal{C}(P)$, so we must have $r \in P$. Conversely, suppose that R/P is an X-torsion-free right R-module and take an element $x \in X$. Let $r \in R$ be an element such that $rx \in P$. Then $(r + P)x = P$ which means that $r + P \in t_X((R/P)_R) = 0$. Thus $r \in P$. Since the elements of X are central, $xr \in P$ implies that $rx \in P$, which implies that $r \in P$ as above. So $x \in \mathcal{C}(P)$ as required. □

4.5 Linking maximal annihilator primes

It is now necessary to approach the problem of finding a relationship between the affiliated primes of two modules. In order to do this we must first consider the relationship between the maximal annihilator primes of the two modules. An affiliated prime of a module is a prime ideal that occurs as an annihilator of a factor in an affiliated series for that module. Each of these factors is an affiliated submodule of the original module, which arise as annihilators of maximal annihilator primes. This is the relationship which we must explore to make progress.

We first note the following.

Lemma 4.5.1. *Let R be a ring. Let M be an X-torsion-free right R-module. Then $(R/r_R(M))$ is also an X-torsion-free right R-module.*

Proof. We must show that $t_X(R/r_R(M)) = 0$. Let $r + r_R(M) \in t_X(R/r_R(M))$. Then there exists an element $x \in X$ such that $rx \in r_R(M)$. So $Mrx = 0$ and thus $Mr \subseteq t_X(M)$. Thus $Mr = 0$, since M is X-torsion-free. Hence $r \in r_R(M)$, and so $(R/r_R(M))_R$ is X-torsion-free. □

We now establish how annihilators behave under the extension and contraction operations defined in the previous section.

Lemma 4.5.2. *Let X be a central multiplicative set in a ring R. Let M be a right R-module. Let A be a right RX^{-1}-module. Then we have the following inclusions:*

1. $r_R(M)^e \subseteq r_{RX^{-1}}(M^e)$;

2. $r_R(A^c) \subseteq r_{RX^{-1}}(A)^c$.

Proof. Take an element $rx^{-1} \in r_R(M)^e$, where $r \in r_R(M)$ and $x \in X$. Now take an arbitrary element $ay^{-1} \in M^e$, where $a \in M$ and $y \in X$. Then we have $ay^{-1}rx^{-1} = ary^{-1}x^{-1} = 0$, using 4.4.4. So we have $rx^{-1} \in r_{RX^{-1}}(M^e)$ as required.

To prove the second part, we note that A^c is a right R-module by 4.4.1. Thus we can apply the first statement to A^c, to give:

$$\begin{aligned} r_R(A^c)^e &\subseteq r_{RX^{-1}}(A^{ce}) \\ &= r_{RX^{-1}}(A) \qquad \text{(by 4.4.2).} \end{aligned}$$

Then, contracting both sides of this inclusion, we have:

$$r_R(A^c)^{ec} \subseteq r_{RX^{-1}}(A)^c,$$

and hence:

$$r_R(A^c) \subseteq r_{RX^{-1}}(A)^c,$$

by 4.4.2. This is as required. □

We provide the circumstances under which the opposite inclusions hold.

Lemma 4.5.3. *Let X be a right denominator set in a ring R. Let M be an X-torsion-free right R-module. Let A be a right RX^{-1}-module such that A^c is an X-torsion-free right R-module. Then the following inclusions hold:*

1. $r_R(M)^e \supseteq r_{RX^{-1}}(M^e)$;

2. $r_R(A^c) \supseteq r_{RX^{-1}}(A)^c$.

Proof. Take an element $rx^{-1} \in r_{RX^{-1}}(M^e)$, where $r \in R$ and $x \in X$. Now for an element $m \in M$ we have $m1^{-1} \in M^e$ and so $m1^{-1}rx^{-1} = 0$. Then $mrx^{-1} = 0$ and hence $mry = 0$ for some $y \in X$. Thus $mr \in t_X(M) = 0$. This shows that $r \in r_R(M)$ and hence that $rx^{-1} \in r_R(M)^e$.

For the second statement, we note that applying the first statement with $M = A^c$ gives that:

$$\begin{aligned} r_R(A^c)^e &\supseteq r_{RX^{-1}}(A^{ce}) \\ &= r_{RX^{-1}}(A) \qquad \text{(by 4.4.2).} \end{aligned}$$

Thus we have:

$$r_R(A^c)^{ec} \supseteq r_{RX^{-1}}(A)^c.$$

By 4.5.1, $(R/r_R(A^c))$ is an X-torsion-free right R-module. So we have $r_R(A^c)^{ec} = r_R(A^c)$, by 4.4.2. Thus $r_R(A^c) \supseteq r_{RX^{-1}}(A)^c$ as required. □

Corollary 4.5.4. *Let X be a central multiplicative set in a ring R. Let M be an X-torsion-free right R-module. Then:*

$$r_R(M)^e = r_{RX^{-1}}(M^e).$$

Proof. The two inclusions are given by 4.5.2 and 4.5.3. □

Corollary 4.5.5. *Let X be a central multiplicative set in a ring R. Let A be a right RX^{-1}-module such that A^c is an X-torsion-free right R-module. Then:*

$$r_R(A^c) = r_{RX^{-1}}(A)^c.$$

Proof. The two inclusions are given by 4.5.2 and 4.5.3. □

The above two results establish conditions under which we may 'go up' and 'go down' between annihilator ideals of a module and annihilator ideals of a localization of that module. We can use these results to outline conditions under which there is a desirable relationship between the *maximal* annihilator (prime) ideals of the respective modules. We first need one further observation.

Lemma 4.5.6. *Let X be a right denominator set in a ring R.*

1. *Suppose that M is a right R-module that is not X-torsion. If M is non-zero then M^e is non-zero.*

2. *Suppose that A is a right RX^{-1}-module. If A is non-zero then A^c is non-zero.*

Proof. We have that $t_X(M) \subsetneq M$ so take an element $m \in M \backslash t_X(M)$. Then $m1^{-1}$ is a non-zero element of MX^{-1} for otherwise we would have $m \in t_X(M)$, a contradiction.

For the second statement, simply note that if $A^c = 0$, then $A^{ce} = 0$ and so $A = 0$. □

Proposition 4.5.7. *Let X be a central multiplicative set in a ring R. Let M be an X-torsion-free right R-module. Suppose that P is a maximal annihilator prime ideal of M_R. Then P^e is a maximal annihilator prime ideal of $(MX^{-1})_{RX^{-1}}$.*

Proof. P is a maximal element of the set:

$$\{r_R(N) \mid N \text{ is a non-zero submodule of } M\}. \tag{4.5.1}$$

Thus we can choose a non-zero submodule N of M such that $P = r_R(N)$. Then $P^e = r_R(N)^e = r_{RX^{-1}}(N^e)$. This follows from 4.5.4, since N is X-torsion-free, by 4.2.4. Now N^e is non-zero by 4.5.6, so that P^e is a member of the set:

$$\{r_{RX^{-1}}(A) \mid A \text{ is a non-zero submodule of } MX^{-1}\}.$$

It remains to show that P^e is a *maximal* element of this set. Suppose that B is a non-zero submodule of MX^{-1} such that $P^e \subseteq r_{RX^{-1}}(B)$. Then:

$$P \subseteq P^{ec} \subseteq r_{RX^{-1}}(B)^c = r_R(B^c), \tag{4.5.2}$$

by 4.4.2 and 4.5.5 (the latter applies because B^c is a submodule of M, and so is X-torsion-free, by 4.2.4). Now if $B^c = 0$ then $B = B^{ce} = 0$, a contradiction. So by the maximality of P in 4.5.1 we have equality in the chain of inclusions (4.5.2). Thus:

$$P^{ec} = r_{RX^{-1}}(B)^c.$$

From this we have, by extending both sides:

$$P^{cece} = r_{RX^{-1}}(B)^{ce},$$

and so:

$$P^{e} = r_{RX^{-1}}(B),$$

by 4.4.2. This is as required. □

Proposition 4.5.8. *Let X be a central multiplicative set in a ring R. Let M be an X-torsion-free right R-module. Suppose that Q is a maximal annihilator prime ideal of $(MX^{-1})_{RX^{-1}}$. Then Q^c is a maximal annihilator prime ideal of M_R.*

Proof. Q is a maximal element of the set:

$$\{r_{RX^{-1}}(A) \mid A \text{ is a non-zero submodule of } MX^{-1}\}. \tag{4.5.3}$$

Thus we can choose a non-zero submodule A of MX^{-1} such that $Q = r_{RX^{-1}}(A)$. Then $Q^c = r_{RX^{-1}}(A)^c = r_R(A^c)$. This follows from 4.5.5, since A^c is an X-torsion-free right R-module, by 4.2.4. Now A^c is non-zero by 4.5.6, so that Q^c is a member of the set:

$$\{r_R(N) \mid N \text{ is a non-zero submodule oi } M\}.$$

It remains to show that Q^c is a maximal element of this set. Suppose that N is a non-zero submodule of M such that $Q^c \subseteq r_R(N)$. Then:

$$Q^{ce} \subseteq r_R(N)^e.$$

Thus:

$$\begin{aligned} Q &\subseteq r_R(N)^e && \text{(by 4.4.2)} \\ &= r_{RX^{-1}}(N^e) && \text{(by 4.5.4).} \end{aligned}$$

Now since N is non-zero and X-torsion-free, by 4.5.6 we have that N^e is non-zero. Thus by the maximality of Q in the set (4.5.3) we have $Q = r_{RX^{-1}}(N^e)$. Thus $Q^c = r_{RX^{-1}}(N^e)^c$ and so $Q^c = r_R(N)^{ec}$ by 4.5.4. Now by 4.5.1, $(R/r_R(N))$ is an X-torsion-free right R-module and so 4.4.2 gives that $Q^c = r_R(N)$. Thus Q^c is a maximal annihilator prime ideal of M_R as required. □

4.6 Left annihilator submodules

Let I be an ideal of a ring R and let M be a right R-module. We would like to study what happens when we extend the submodule $l_M(I)$ of M into the localization of M at X. This is addressed in this section.

Lemma 4.6.1. *Let X be a central multiplicative set in a ring R. Let M be a right R-module. Then:*

$$l_M(I)^e \subseteq l_{MX^{-1}}(I^e).$$

If further the submodule MI of M is X-torsion-free then:

$$l_M(I)^e = l_{MX^{-1}}(I^e).$$

Proof Take elements $ax^{-1} \in l_M(I)^e$, where $a \in l_M(I)$ and $x \in X$, and $wy^{-1} \in I^e$, where $w \in I$ and $y \in X$. Then:

$$\begin{aligned} ax^{-1}wy^{-1} &= awx^{-1}y^{-1} && \text{(by 4.4.4)} \\ &= 0 && \text{(since } a \in l_M(I) \text{ and } w \in I). \end{aligned}$$

Thus $ax^{-1} \in l_{MX^{-1}}(I^e)$ as required.

Now suppose that the submodule MI of M is X-torsion-free and take an element $ax^{-1} \in l_{MX^{-1}}(I^e)$, where $a \in M$ and $x \in X$. We claim that $a \in l_M(I)$. Now for any element $w \in I$ we have $w1^{-1} \in I^e$, so since $ax^{-1} \in l_{MX^{-1}}(I^e)$ we have $ax^{-1}w1^{-1} = 0$. Thus $awx^{-1}1^{-1} = 0$, by 4.4.4. This gives that:

$$aw \in t_X(MI) = 0.$$

So $a \in l_M(I)$, as claimed, and $ax^{-1} \in l_M(I)^e$. □

4.7 Going up and down

Our aim is to gain information about suitable localizations of the right R-module R/I, where I is an ideal of a ring R. We have two possible approaches: given a right denominator set X in R we can either localize the ring R and then factor out the extended ideal I^e, or we can localize the *module* $(R/I)_R$. It transpires that it does not make any difference which of these approaches we take.

Lemma 4.7.1. *Let I be an ideal of a ring R. Then the map $\theta : (RX^{-1}/I^e) \longrightarrow (R/I)X^{-1}$ defined by:*

$$\theta(rx^{-1} + I^e) = (r + I)x^{-1},$$

for each $r \in R$ and $x \in X$, is an isomorphism of right RX^{-1}-modules. Further if J is an ideal of R then we have:

$$\theta\left(\frac{J^e + I^e}{I^e}\right) = \left(\frac{J + I}{I}\right)^e.$$

Proof. Let $r_1x_1^{-1} + I^e$ and $r_2x_2^{-1} + I^e$ be elements of RX^{-1}/I^e, where r_1 and r_2 are elements of R and x_1 and x_2 are elements of X. Suppose that $r_1x_1^{-1} + I^e = r_2x_2^{-1} + I^e$. Then $r_1x_1^{-1} - r_2x_2^{-1} \in I^e$. Since X is a right denominator set we may choose elements $x_3 \in X$ and $r_3 \in R$ such that $x_1x_3 = x_2r_3$. Set $y = x_1x_3 = x_2r_3 \in X$. Then $x_1^{-1} = x_3y^{-1}$ and $x_2^{-1} = r_3y^{-1}$. Hence we have $r_1x_3y^{-1} - r_2r_3y^{-1} \in I^e$, and so $(r_1x_3 - r_2r_3)y^{-1} \in I^e$. Thus there are elements $a \in I$ and $x_4 \in X$ such that

$(r_1x_3 - r_2r_3)y^{-1} = ax_4^{-1}$, which implies that $r_1x_3 - r_2r_3 = ax_4^{-1}y$. Using the fact that X is a right denominator set again we can choose elements $x_5 \in X$ and $r_4 \in R$ such that $yx_5 = x_4r_4$, from which it follows that $x_4^{-1}y = r_4x_5^{-1}$. Hence we now have that $r_1x_3 - r_2r_3 = ar_4x_5^{-1}$. Then there exists an element $x_6 \in X$ such that $(r_1x_3 - r_2r_3)x_5x_6 = ar_4x_6 \in I$. Now $r_1x_3x_5x_6 - r_2r_3x_5x_6 \in I$, which implies that $r_1x_3x_5x_6 + I = r_2r_3x_5x_6 + I$. Thus $(r_1 + I)x_3x_5x_6 = (r_2 + I)r_3x_5x_6$. Multiplying both sides of this equation by $(x_5x_6)^{-1}y^{-1}$ gives that $(r_1 + I)x_1^{-1} = (r_2 + I)x_2^{-1}$. This shows that θ is a well-defined map. It is straightforward to check that θ is an isomorphism, and that the image under θ of the right RX^{-1}-submodule $(J^e + I^e)/I^e$ of RX^{-1}/I^e is $((J + I)/I)^e$. □

Proposition 4.7.2. [4, Proposition 7.5] *Let M be an (R, S)-bimodule where S is a right Noetherian ring and R is any ring. Suppose that M is Noetherian as a left R-module. Then there is an affiliated series:*

$$0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_n = M$$

for M_S with corresponding affiliated primes $P_1, \ldots, P_n$ such that for each $i \in \{1, \ldots, n\}$, (M_i/M_{i-1}) is a torsion-free right (S/P_i)-module.

Remark 4.7.1. Under the above hypotheses, if:

$$0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_n = M$$

is *any* affiliated series for M_S, then (M_i/M_{i-1}) is a torsion-free right (S/P_i)-module for each $i \in \{1, \ldots, n\}$, where $P_1, \ldots, P_n$ are the corresponding affiliated primes.

We now apply this result to the special case that we are interested in.

Corollary 4.7.3. *Let R be a Noetherian ring. Let:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

be an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1 = r_R(J_1/I), \ldots, P_n = r_R(R/J_{n-1})$. Then for each $i \in \{1, \ldots, n\}$, (J_i/J_{i-1}) is a torsion-free right (R/P_i)-module.

Proposition 4.7.4. *Let I be an ideal of a ring R. Let:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

be an affiliated series for $(R/I)_R$. Suppose that X is a right Ore set in R such that for each $i \in \{1, \ldots, n\}$, (J_i/J_{i-1}) is an X-torsion-free right R-module. Then for each $i \in \{1, \ldots, n\}$, (R/J_{i-1}) is an X-torsion-free right R-module.

Proof. Let $i \in \{1, \ldots, n\}$ and take an element $r + J_i \in t_X(R/J_i)$. Choose an element $x \in X$ such that $rx \in J_i$. Then we have:

$$rx \in J_i \subseteq J_{i+1} \subseteq \ldots \subseteq J_{n-1} \subseteq J_n = R.$$

Thus $r + J_{n-1}$ is an element of R/J_{n-1} such that $rx \in J_{n-1}$. Hence $r + J_{n-1} \in t_X(R/J_{n-1})$. But $t_X(R/J_{n-1}) = 0$. Hence $r \in J_{n-1}$. Now $r + J_{n-2}$ is an element of J_{n-1}/J_{n-2} such that $rx \in J_{n-2}$. Hence $r + J_{n-2} \in t_X(J_{n-1}/J_{n-2})$. But $t_X(J_{n-1}/J_{n-2}) = 0$. Hence $r \in J_{n-2}$. We can continue this process until eventually we get that $r \in J_i$. This proves that R/J_i is an X-torsion-free right R-module. □

We are now in a position to prove a 'going up' theorem for affiliated primes.

Theorem 4.7.5. *Let R be a Noetherian ring. Let I be an ideal of R. Let:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

be an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1, \ldots, P_n$. Let X be a central multiplicative set in R such that $X \subseteq \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n)$. Then:

$$I^e/I^e = {J_0}^e/I^e \subsetneqq {J_1}^e/I^e \subsetneqq \ldots \subsetneqq {J_n}^e/I^e = RX^{-1}/I^e$$

is an affiliated series for $(RX^{-1}/I^e)_{RX^{-1}}$ with corresponding affiliated primes ${P_1}^e, \ldots, {P_n}^e$.

Proof. Let $i \in \{1, \ldots, n\}$. Applying 4.7.3 we see that J_i/J_{i-1} is a torsion-free right R/P_i-module. This is equivalent to saying that J_i/J_{i-1} is a $\mathcal{C}(P_i)$-torsion-free right R-module. So since $X \subseteq \mathcal{C}(P_i)$, the module $(J_i/J_{i-1})_R$ is X-torsion-free. This is true for arbitrary $i \in \{1, \ldots, n\}$, so by 4.7.4 we have that R/J_{i-1} is an X-torsion-free right R-module for every $i \in \{1, \ldots, n\}$.

Now observe that if ${J_{i-1}}^e = J_i^e$ for any $i \in \{1, \ldots, n\}$ then ${J_{i-1}}^{ec} = J_i^{ec}$ and thus $J_{i-1} = J_i$, since both R/J_{i-1} and R/J_i are X-torsion-free right R-modules. This contradicts our hypothesis, so we must have that ${J_{i-1}}^e \subsetneqq J_i^e$ for every i. This shows that the chain is strictly ascending as claimed. Further we have:

$$P_i = r_R(J_i/J_{i-1}),$$

from which it follows that:

$$P_i^e = r_R(J_i/J_{i-1})^e.$$

Now by 4.5.4 we have:

$$P_i^e = r_{RX^{-1}}((J_i/J_{i-1})^e),$$

and applying 4.7.1 gives:

$$P_i^e = r_{RX^{-1}}(J_i^e/{J_{i-1}}^e).$$

It remains to show that, for each $i \in \{1, \ldots, n\}$, the factor:

$$\frac{J_i^e/I^e}{{J_{i-1}}^e/I^e}$$

is an affiliated submodule of the module:

$$\frac{RX^{-1}/I^e}{{J_{i-1}}^e/I^e}.$$

This is equivalent to showing that $J_i^e/{J_{i-1}}^e$ is an affiliated submodule of $RX^{-1}/{J_{i-1}}^e$.

By hypothesis we have:

$$l_{\frac{R}{J_{i-1}}}(P_i) = \frac{J_i}{J_{i-1}}.$$

Extending both sides into the localization of the right R-module R/J_{i-1} at X gives:

$$l_{\frac{R}{J_{i-1}}}(P_i)^e = \left(\frac{J_i}{J_{i-1}}\right)^e.$$

Now since R/J_{i-1} is an X-torsion-free right R-module we can apply 4.6.1 to give:

$$l_{\left(\frac{R}{J_{i-1}}\right)X^{-1}}(P_i^e) = \left(\frac{J_i}{J_{i-1}}\right)^e.$$

Now we can apply 4.7.1 to conclude that:

$$l_{\frac{RX^{-1}}{{J_{i-1}}^e}}(P_i^e) = \frac{J_i^e}{{J_{i-1}}^e}.$$

Finally by hypothesis we have that P_i is a maximal annihilator prime ideal of $(R/J_{i-1})_R$. So by 4.5.7, P_i^e is a maximal annihilator prime ideal of $((R/J_{i-1})X^{-1})_{RX^{-1}}$. Thus P_i^e is also a maximal annihilator prime ideal of the isomorphic module $(RX^{-1}/{J_{i-1}}^e)_{RX^{-1}}$. This shows that $J_i^e/{J_{i-1}}^e$ is an affiliated submodule of $RX^{-1}/{J_{i-1}}^e$ and completes the proof. □

The hypotheses we need to prove the dual 'going down' theorem are slightly different, although we will see later (see 4.7.7 and 4.7.12) that in the situation we are interested in they turn out to be the same.

Theorem 4.7.6. *Let R be a Noetherian ring. Let I be an ideal of R. Let X be a central multiplicative set in R such that $X \subseteq \mathcal{C}(I)$. Suppose that:*

$$I^e/I^e = J_0/I^e \subsetneq J_1/I^e \subsetneq \ldots \subsetneq J_n/I^e = RX^{-1}/I^e$$

is an affiliated series for $(RX^{-1}/I^e)_{RX^{-1}}$ *with corresponding affiliated primes* $Q_1, \ldots, Q_n$. *Then:*

$$I/I = J_0{}^c/I \subsetneq J_1{}^c/I \subsetneq \ldots \subsetneq J_n{}^c/I = R/I$$

is an affiliated series for $(R/I)_R$ *with corresponding affiliated primes* $Q_1{}^c, \ldots, Q_n{}^c$.

Proof. First note that since $X \subseteq \mathcal{C}(I)$, the module $(R/I)_R$ is X-torsion-free. Thus $I^{ec} = I$ and hence $J_0{}^c = I$. It is now clear that the chain is as claimed, for if $J_{i-1}{}^c = J_i{}^c$ for any $i \in \{1, \ldots, n\}$ then $J_{i-1}{}^{ce} = J_i{}^{ce}$ and hence $J_{i-1} = J_i$, which is a contradiction. Now we have $Q_i = r_{RX^{-1}}(J_i/J_{i-1})$ and we proceed to prove that $Q_i{}^c = r_R(J_I{}^c/J_{i-1}{}^c)$ directly. Take an element $r \in Q_i{}^c$. Then $r1^{-1} \in Q_i$. Take an element $s \in J_i{}^c$. Then $s1^{-1} \in J_i$. Now $Q_i = r_{RX^{-1}}(J_i/J_{i-1})$, so that $s1^{-1}r1^{-1} \in J_{i-1}$, which shows that $sr \in J_{i-1}{}^c$. Thus $r \in r_R(J_i{}^c/J_{i-1}{}^c)$. For the reverse inclusion, take an element $r \in r_R(J_i{}^c/J_{i-1}{}^c)$. Now let $sx^{-1} \in J_i = J_i{}^{ce}$, where $s \in J_i{}^c$ and $x \in X$. Since $s \in J_i{}^c$ we have $sr \in J_{i-1}{}^c$. Then:

$$\begin{aligned} sx^{-1}r1^{-1} &= srx^{-1} \quad \text{(by 4.4.4)} \\ &\in J_{i-1}{}^{ce} \quad (\text{since } sr \in J_{i-1}{}^c) \\ &= J_{i-1}. \end{aligned}$$

This shows that $r1^{-1} \in Q_i$ and hence that $r \in Q_i{}^c$.

We now need to show that for each $i \in \{1, \ldots, n\}$, the factor:

$$\frac{J_i{}^c/I}{J_{i-1}{}^c/I}$$

is an affiliated submodule of the module:

$$\frac{R/I}{J_{i-1}{}^c/I}.$$

We show, equivalently, that ${J_i}^c/{J_{i-1}}^c$ is an affiliated submodule of $R/{J_{i-1}}^c$. We already have that ${Q_i}^c = r_R({J_i}^c/{J_{i-1}}^c)$ which, by taking left annihilators in the right R-module $R/{J_{i-1}}^c$, gives the following inclusion:

$$l_{\frac{R}{{J_{i-1}}^c}}({Q_i}^c) \supseteq \frac{{J_i}^c}{{J_{i-1}}^c}.$$

We claim that in fact equality holds. By hypothesis we have the following:

$$l_{\frac{RX^{-1}}{J_{i-1}}}(Q_i) = \frac{J_i}{J_{i-1}}.$$

Thus:

$$l_{\frac{RX^{-1}}{{J_{i-1}}^{ce}}}(Q_i) = \frac{{J_i}^{ce}}{{J_{i-1}}^{ce}}.$$

Applying 4.7.1 gives:

$$l_{\left(\frac{R}{{J_{i-1}}^c}\right)X^{-1}}(Q_i) = \left(\frac{{J_i}^c}{{J_{i-1}}^c}\right)^e,$$

so that:

$$l_{\left(\frac{R}{{J_{i-1}}^c}\right)X^{-1}}({Q_i}^{ce}) = \left(\frac{{J_i}^c}{{J_{i-1}}^c}\right)^e.$$

Now it follows from the equality ${J_{i-1}}^{cec} = {J_{i-1}}^c$ that $R/{J_{i-1}}^c$ is an X-torsion-free right R-module and so we can apply 4.6.1 to give the following:

$$l_{\frac{R}{{J_{i-1}}^c}}({Q_i}^c)^e = \left(\frac{{J_i}^c}{{J_{i-1}}^c}\right)^e.$$

Contracting back both sides gives:

$$l_{\frac{R}{{J_{i-1}}^c}}({Q_i}^c)^{ec} = \left(\frac{{J_i}^c}{{J_{i-1}}^c}\right)^{ec}.$$

Now $R/{J_i}^c$ is an X-torsion-free right R-module and since we have:

$$\frac{R}{{J_i}^c} \cong \frac{R/{J_{i-1}}^c}{{J_i}^c/{J_{i-1}}^c},$$

it follows that:

$$l_{\frac{R}{J_{i-1}{}^c}}(Q_i{}^c)^{ec} = \frac{J_i{}^c}{J_{i-1}{}^c}.$$

Thus we have:

$$l_{\frac{R}{J_{i-1}{}^c}}(Q_i{}^c) \subseteq \frac{J_i{}^c}{J_{i-1}{}^c},$$

as required. Finally by hypothesis Q_i is a maximal annihilator prime ideal of the module $(RX^{-1}/J_{i-1})_{RX^{-1}}$, and so is a maximal annihilator prime ideal of the isomorphic module $((R/J_{i-1}{}^c)X^{-1})_{RX^{-1}}$. Again, using the fact that $R/J_{i-1}{}^c$ is an X-torsion-free right R-module we can apply 4.5.8 to conclude that $Q_i{}^c$ is a maximal annihilator prime ideal of $(R/J_{i-1}{}^c)_R$. This shows that $J_i{}^c/J_{i-1}{}^c$ is an affiliated submodule of $R/J_{i-1}{}^c$. The proof is now complete. □

Proposition 4.7.7. *Let R be a Noetherian ring. Let S be the centre of R. Let I be an ideal of R. Let:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

be an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1, \ldots, P_n$. Then:

$$\mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n) \subseteq {}'\mathcal{C}(I),$$

and:

$$\mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n) \cap S \subseteq \mathcal{C}(I) \cap S.$$

Proof. Take an element $x \in \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n)$. We claim that for each $i \in \{1, \ldots, n\}$, the following implication holds:

$$(ax \in J_{i-1} \text{ and } a \in J_i) \Rightarrow a \in J_{i-1}.$$

The fact that $a \in J_i$ ensures that $a + J_{i-1} \in J_i/J_{i-1}$, while the fact that $x \in \mathcal{C}(P_i)$ gives that $x + P_i \in R/P_i$ is a regular element. So $a + J_{i-1}$ is a torsion element of $(J_i/J_{i-1})_{R/P_i}$, because $ax \in J_{i-1}$. But this module is torsion-free, by 4.7.3. Thus $a \in J_{i-1}$, proving the claim. Now take an element $a \in R$ such that $ax \in I$. Then since we have:

$$ax \in I = J_0 \subsetneq J_1 \subsetneq \dots \subsetneq J_n = R,$$

we can apply our claim (with $i = n$) to give that $a \in J_{n-1}$. Now we can apply the claim (with $i = n-1$) to give that $a \in J_{n-2}$. This process continues until eventually we get that $a \in I$. This proves that $x \in {}'\mathcal{C}(I)$. The second statement is now obvious. □

In aiming to prove the reverse inclusion, we will need the following results.

Corollary 4.7.8. *Let R be a Noetherian ring. Let M be a finitely generated right R-module. Let:*

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$$

be an affiliated series for M with corresponding affiliated primes $P_1, \dots, P_n$. Then for any $i \in \{1, \dots, n\}$, the submodule series:

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_i$$

is an affiliated series for M_i with corresponding affiliated primes $P_1, \dots, P_i$.

Proof. The result follows from applying 1.3.3 with $N = M_i$. □

Lemma 4.7.9. [4, Lemma 7.3] *Let R_1 be any ring and let R_2 be a prime right Noetherian ring. Let M be a (R_1, R_2)-bimodule such that ${}_{R_1}M$ is a Noetherian module. Let N be a sub-bimodule of M. Let L be a right R_2-submodule of M such that $L \supseteq N$. Suppose that L/N is a torsion right R_2-module. Then there is a non-zero ideal I of R_2 such that $LI \subseteq N$.*

This lemma is crucial in proving the following result.

Theorem 4.7.10. [7, Theorem 1.2] *Let R be a Noetherian ring. Let M be a finitely generated right R-module. Let:*

$$0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$$

be an affiliated series for M with corresponding affiliated primes $P_1, \ldots, P_n$. Then one of the following conditions holds:

1. *P_n is an annihilator prime ideal of M_R;*

2. *$P_n \rightsquigarrow P_i$ for some integer i such that $1 \leq i < n$.*

Proof. We first show that it is sufficient to prove the result for a submodule D of M such that $D \not\subseteq M_{n-1}$. In this case, by 1.3.3, we have that the distinct terms in the submodule series:

$$0 = D \cap M_0 \subseteq D \cap M_1 \subseteq \ldots \subseteq D \cap M_{n-1} \subseteq D \cap M_n = D$$

form an affiliated series for D. Further, the corresponding affiliated prime ideals are a subset of $\{P_1, \ldots, P_n\}$. Since $D \not\subseteq M_{n-1}$, we have $D \cap M_{n-1} \neq D$ and so the top affiliated prime corresponding to this affiliated series is P_n. Since the result holds for D, either P_n is an annihilator prime ideal of D_R or $P_n \rightsquigarrow P_i$ for some integer i such that $1 \leq i < n$. But clearly P_n is an annihilator prime ideal of D_R implies that P_n is an annihilator prime ideal of M_R. Thus the result holds for M.

Now consider the set:

$$\{r_R(X) \mid X \text{ is a submodule of } M \text{ such that } X \not\subseteq M_{n-1}\}.$$

Clearly this set is non-empty, since $r_R(M)$ is a member. Thus since M is a Noetherian right R-module we may choose a maximal element $r_R(D)$ of this set. By the above argument it is sufficient to prove the result for D. So we may assume without loss of generality that $D = M$ and that $r_R(M)$ is maximal in the above set. We claim that if M' is a submodule of M such that $M' \not\subseteq M_{n-1}$, then for every $i \in \{0, \ldots, n-1\}$ we have that:

$$r_R\left(\frac{M}{M_i}\right) = r_R\left(\frac{M' + M_i}{M_i}\right). \tag{4.7.1}$$

To establish this, fix $i \in \{0, \ldots, n-1\}$. That:

$$r_R\left(\frac{M}{M_i}\right) \subseteq r_R\left(\frac{M' + M_i}{M_i}\right)$$

follows directly from the fact that $M' \subseteq M$. For the opposite inclusion, let $x \in r_R((M' + M_i)/M_i)$. Then $M'x \subseteq M_i$. From this $M'xP_i \subseteq M_iP_i \subseteq M_{i-1}$, and so $M'xP_iP_{i-1} \subseteq M_{i-1}P_{i-1} \subseteq M_{i-2}$ and so on until $M'xP_iP_{i-1}\ldots P_1 = 0$. Hence we have $xP_iP_{i-1}\ldots P_1 \subseteq r_R(M')$. By the maximality of $r_R(M)$ we have $r_R(M) = r_R(M')$. So in fact $xP_iP_{i-1}\ldots P_1 \subseteq r_R(M)$ and $MxP_iP_{i-1}\ldots P_1 = 0$. It follows that $MxP_iP_{i-1}\ldots P_2 \subseteq l_M(P_1) = M_1$. Then $MxP_iP_{i-1}\ldots P_3 \subseteq l_{M/M_1}(P_2) = M_2$ and so on until eventually $MxP_i \subseteq l_{M/M_{i-2}}(P_{i-1}) = M_{i-1}$ and $Mx \subseteq l_{M/M_{i-1}}(P_i) = M_i$. Thus $x \in r_R(M/M_i)$ as required.

Since $MP_n \subseteq M_{n-1}$, there exists an integer $0 \leq r \leq n-1$ such that there is a submodule M' of M with $M' \not\subseteq M_{n-1}$ and $M'P_n \subseteq M_r$. Let r be the least such integer. If $r = 0$ then $M'P_n = 0$ for some submodule M' of M with $M' \not\subseteq M_{n-1}$. Thus $P_n \subseteq r_R(M')$. Now consider the following chain of inclusions:

$$P_n \subseteq r_R(M') \subseteq r_R\left(\frac{M' + M_{n-1}}{M_{n-1}}\right).$$

Now $(M' + M_{n-1})/M_{n-1}$ is a non-zero submodule of M/M_{n-1}, so equality in the above chain of inclusions follows from the fact that P_n is a maximal annihilator prime of $(M/M_{n-1})_R$. Thus $P_n = r_R(M')$ and P_n is an annihilator prime ideal of M as required.

Now suppose that $r > 0$. To establish the result it is enough to show that $P_n \rightsquigarrow P_r$. Let $A = r_R(M/M_{r-1})$, and set $P = P_n$ and $Q = P_r$. Claim that $P \rightsquigarrow Q$ via A. We first show that $PQ \subseteq A \subsetneq P \cap Q$. Claim that $P = r_R(M/M_r)$. Now there is a submodule M' of M with $M' \nsubseteq M_{n-1}$ and $M'P \subseteq M_r$. Thus $P \subseteq r_R((M' + M_r)/M_r)$ and so $P \subseteq r_R(M/M_r)$ by equation (4.7.1). The reverse inclusion follows because $M_r \subseteq M_{n-1}$ and $P = r_R(M/M_{n-1})$. Then we have the following:

$$MPQ \subseteq M_rQ \subseteq M_{r-1}.$$

Thus $PQ \subseteq r_R(M/M_{r-1}) = A$. Further we have that $A = r_R(M/M_{r-1}) \subseteq r_R(M/M_r) = P$, since $M_{r-1} \subseteq M_r$, and that $A = r_R(M/M_{r-1}) \subseteq r_R(M_r/M_{r-1}) = Q$. Thus $A \subseteq P \cap Q$. Now suppose that $A = P \cap Q$. If $Q \subseteq P$ then $Q = Q \cap P = A$, and so $r_R(M/M_{r-1}) = Q$. Taking the left annihilator of both sides of this equation in the module $(M/M_{r-1})_R$ gives that $M/M_{r-1} = l_{(M/M_{r-1})}(Q)$. But we also have that $M_r/M_{r-1} = l_{(M/M_{r-1})}(Q)$. This is a contradiction, since $M_r \subsetneq M$. Thus we must have $Q \nsubseteq P$. From this it follows that $MQ \nsubseteq M_{n-1}$. Hence by equation (4.7.1) we have that:

$$r_R\left(\frac{M}{M_{r-1}}\right) = r_R\left(\frac{MQ + M_{r-1}}{M_{r-1}}\right),$$

and thus:

$$r_R\left(\frac{M}{M_{r-1}}\right) = r_R\left(\left(\frac{M}{M_{r-1}}\right)Q\right).$$

Then $(M/M_{r-1})QP \subseteq (M/M_{r-1})(Q \cap P) = (M/M_{r-1})A = 0$, which gives that $P \subseteq r_R((M/M_{r-1})Q) = r_R(M/M_{r-1})$. But then $MP \subseteq M_{r-1}$, contradicting the minimality of the integer r. Thus our initial supposition must be false, so that $A \subsetneqq P \cap Q$.

Suppose that $(P \cap Q)/A$ is not a torsion-free left R/P-module. Let ${}_{R/P}(B/A)$ be the torsion submodule of ${}_{R/P}((P \cap Q)/A)$. Then $B \supsetneqq A$. Now by applying the left-handed version of 4.7.9 with $M = (P \cap Q)/PQ$, $R_1 = R/P$, $R_2 = R/Q$, $N = A/PQ$ and $L = B/PQ$ we can conclude that there is an ideal X of R such that $X \supsetneqq P$ and such that $XB \subseteq A$. Thus $(M/M_{r-1})XB = 0$. Now MX is a submodule of M and if $MX \subseteq M_{n-1}$ then $X \subseteq P$, a contradiction. Thus by equation (4.7.1) we have that:

$$r_R\left(\frac{M}{M_{r-1}}\right) = r_R\left(\left(\frac{M}{M_{r-1}}\right)X\right).$$

Hence we have that $B \subseteq r_R((M/M_{r-1})X) = r_R(M/M_{r-1}) = A$. But B was constructed so that $A \subsetneqq B$. This is a contradiction, so $(P \cap Q)/A$ is a torsion-free left R/P-module.

Finally, suppose that $(P \cap Q)/A$ is not a torsion-free right R/Q-module, and let $(B/A)_{R/Q}$ be the non-zero torsion submodule of $((P \cap Q)/A)_{R/Q}$. Again by applying 4.7.9 we have that there is an ideal Y of R such that $Y \supsetneqq Q$ and such that $BY \subseteq A$. Thus $(M/M_{r-1})BY = 0$. This gives that $(M/M_{r-1})BQ = 0$ and hence $r_R(M_r/M_{r-1}) = Q \subseteq r_R((M/M_{r-1})B)$. Now $(M/M_{r-1})B$ must be non-zero, for otherwise $B \subseteq A$, and so since Q is a maximal annihilator prime of $(M/M_{r-1})_R$ we have that $r_R((M/M_{r-1})B) = r_R(M/M_{r-1})$. This gives that:

$$Y \subseteq r_R((M/M_{r-1})B) = r_R(M/M_{r-1}) = Q,$$

a contradiction. So $(P \cap Q)/A$ is a torsion-free right R/Q-module. This completes the proof. □

Corollary 4.7.11. *Let R be a Noetherian ring. Let M be a finitely generated right R-module. Let P be an affiliated prime ideal of M_R. Then there are affiliated prime ideals $P_1, \ldots, P_k$ of M_R such that:*

$$P \rightsquigarrow P_1 \rightsquigarrow \ldots \rightsquigarrow P_{k-1} \rightsquigarrow P_k,$$

where P_k is an annihilator prime ideal of M_R.

Proof. Suppose that:

$$0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_n = M$$

is an affiliated series for M with corresponding affiliated primes $P_1, \ldots, P = P_i, \ldots, P_n$. By 4.7.8 we have that:

$$0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_i$$

is an affiliated series for M_i with corresponding affiliated primes $P_1, \ldots P_i$. By applying 4.7.10 we have the following two possible cases:

1. $P_i = r_R(M')$ for some non-zero submodule M' of M, proving the result;

2. $P_i \rightsquigarrow P_j$ for some positive integer j such that $j < i$.

If the second case applies, we then apply 4.7.8 and 4.7.10 as above with P_i replaced by P_j. Continuing this process, we must eventually get:

$$P_i \rightsquigarrow \ldots \rightsquigarrow P_k,$$

where k is a positive integer such that $P_k = r_R(M')$ for some non-zero submodule M' of M so that P_k is an annihilator prime ideal of M_R. □

Proposition 4.7.12. *Let R be a Noetherian ring. Let S be the centre of R. Let I be an ideal of R. Suppose that:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

is an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1, \ldots, P_n$. Then we have:

$$\mathcal{C}(I) \cap S \subseteq \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n).$$

Proof. Take an element $c \in \mathcal{C}(I) \cap S$. Suppose that $c \in P_i$ for some $i \in \{1, \ldots, n\}$. By 4.7.11 there is a subset $\{P_{i_1}, \ldots, P_{i_k}\}$ of the set of affiliated primes $\{P_1, \ldots, P_n\}$ such that:

$$P_i = P_{i_1} \rightsquigarrow \ldots \rightsquigarrow P_{i_k} = P_j,$$

and such that P_j is an annihilator prime ideal of $(R/I)_R$. Thus there is a right ideal A of R such that $I \subsetneq A$ and $P_j = r_R(A/I)$. Since $c \in P_i$ we have by 1.7.1 that $c \in P_j$. Hence $Ac \subseteq I$. From the fact that $c \in \mathcal{C}(I)$ it follows that $A \subseteq I$, which is a contradiction. Thus c is not a member of P_i, for any $i \in \{1, \ldots, n\}$. Hence $c \in \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n) \cap S$, as required. □

Corollary 4.7.13. *Let R be a Noetherian ring. Let S be the centre of R. Let I be an ideal of R. Suppose that:*

$$I/I = J_0/I \subsetneq J_1/I \subsetneq \ldots \subsetneq J_n/I = R/I$$

is an affiliated series for $(R/I)_R$ with corresponding affiliated primes $P_1, \ldots, P_n$. Then:

$$\mathcal{C}(I) \cap S = \mathcal{C}(P_1) \cap \ldots \cap \mathcal{C}(P_n) \cap S.$$

Proof. The two inclusions are shown by 4.7.7 and 4.7.12. □

In summary, then, we have proved the following.

Theorem 4.7.14. *Let R be a Noetherian ring. Let I be an ideal of R. Let X be a central multiplicative set in R such that $X \subseteq \mathcal{C}(I)$. Then:*

$$Aff\left(\frac{R}{I}\right)_R = Aff\left(\frac{RX^{-1}}{I^e}\right)^c_{RX^{-1}},$$

and:

$$Aff\left(\frac{R}{I}\right)^e_R = Aff\left(\frac{RX^{-1}}{I^e}\right)_{RX^{-1}}.$$

Proof. It is clear from 4.7.13 that the hypotheses of 4.7.5 and 4.7.6 both hold. □

Chapter 5

Conditions for stability

Chapter 4 outlines under which conditions a localization behaves sufficiently well to enable us to go up and down between the two sets of affiliated primes in which we are interested. We would like to find a localization which behaves well enough as a ring to allow going up and down between the prime divisors of its ideals and the prime divisors of the ideals of its centre. This is the aim of this chapter. We first need to introduce some terminology.

Let R be a ring. An ideal I of R is called a *Brodmann ideal* of R if there is a positive integer p such that whenever n is a positive integer such that $n \geq p$, we have:

$$Aff\left(\frac{R}{I^n}\right)_R = Aff\left(\frac{R}{I^{n+1}}\right)_R .$$

In this case the smallest positive integer p it is possible to take in the above definition is called the *Brodmann stabilizing point* of I and is denoted by $Brod(R, I)$. Note that in the commutative ring theory literature this integer is sometimes called the *index of stability* of I (cf. [13]). R is called a *Brodmann ring* if every ideal of R is a Brodmann ideal. Thus, by 2.3.8, a commutative Noetherian ring is a Brodmann ring.

5.1 The PI degree

Let R be a prime polynomial identity ring. Then we can define a degree function on R, called the *PI degree* of R and denoted by $PIdeg(R)$. We begin by defining this degree for a class of PI rings called *central simple algebras* and then extend this definition to include the larger class of all prime PI rings.

Definition 5.1.1. Let R be a simple Artinian ring with centre S. If R is finite dimensional (as a vector space) over (the field) S then R is called a *central simple algebra (over S).*

Lemma 5.1.1. [12, Lemma 13.3.4] *Let R be a simple Artinian ring. Let D be a division ring and t be a positive integer such that $R = M_t(D)$. Choose a maximal subfield H of D. Let S be the centre of D (so that $S \subseteq H$). Let V be an irreducible right R-module. It follows from this that $End(V_R) \cong D$. Then the following two statements hold:*

1. *$T = R \otimes_S H$ is a simple ring and V forms an irreducible right T-module with $End(V_T) \cong H$;*
2. *if V is finite dimensional (as a vector space) over H with $dim(V_H) = m$ then $T \cong M_m(H)$, $dim(D_H) = dim(H_S) = m/t$ and $dim(R_S) = m^2$.*

Corollary 5.1.2. [12, Corollary 13.3.5] *Let R be a central simple algebra. Let D be a division ring and t be a positive integer such that $R = M_t(D)$. Let S be the centre of D and let H be a maximal subfield of D. Then the following hold:*

1. *$D \otimes_S H \cong M_m(H)$, where $m^2 = dim(D_S)$;*
2. *$R \otimes_S H \cong M_n(H)$, where $n^2 = (mt)^2 = dim(R_S)$;*
3. *R satisfies s_{2n} and does not satisfy s_{2n-1}.*

Definition 5.1.2. Let R be a central simple algebra with centre S. Then the integer n given by 5.1.2 is called the *PI degree* of R. Property 2 in 5.1.2 shows that n^2 is the vector space dimension of R over S.

To extend the definition of PI degree to the class of all prime PI rings we will require a well-known result, which is due to E.C. Posner ([15, 13.11.6]).

Theorem 5.1.3. (E.C. Posner) *Let R be a prime PI ring. Let S be the centre of R. Let X denote the set $S\backslash\{0\}$. Let $Q = RX^{-1}$ and let $Z = SX^{-1}$. Then Q is a central simple algebra with centre Z, R is a right and left order in Q and $Q = RZ$.*

Definition 5.1.3. Let R be a prime PI ring and let Q be the quotient ring of R given by E.C. Posner's Theorem (5.1.3). Then we can now define the *PI degree* of R, $PIdeg(R)$, by $PIdeg(R) = PIdeg(Q)$.

Corollary 5.1.4. [12, Corollary 13.6.7] *Let n be a positive integer. Then there is a multilinear polynomial g_n which satisfies the following three conditions for every prime PI ring R with $PIdeg(R) = n$:*

1. *for every non-zero ideal I of R we have $g_n(I) \neq 0$;*
2. *if X is the set $g_n(R)\backslash\{0\}$ then $Q = RX^{-1}$;*
3. *g_n is a central polynomial for R, but if m is an integer such that $m > n$ then g_m is a polynomial identity for R.*

Lemma 5.1.5. [12, Lemma 11.7.2, (i)] *Let R be a prime PI ring. Suppose that $PIdeg(R) = n$. Let P be a prime ideal of R. Then $PIdeg(R/P) \leq PIdeg(R)$. Further we have $PIdeg(R/P) = PIdeg(R)$ if and only if $g_n(R) \not\subseteq P$.*

Definition 5.1.4. Let R be a prime PI ring. Let P be a prime ideal of R. If $PIdeg(R/P) = PIdeg(R)$ then P is called a *regular* prime ideal.

Regular prime ideals play an important part in the definitive result for Azumaya algebras, a characterization due to M. Artin ([1]) and C. Procesi ([16]).

Theorem 5.1.6. (M. Artin-C. Procesi) *Let R be a prime ring. Then the following three conditions are equivalent:*

1. *R is an Azumaya algebra;*

2. *R is a prime PI ring, $PIdeg(R) = n$ and every prime ideal of R is regular;*

3. *$g_n(R)R = R$.*

Thus if we could find a way of transferring to a ring in which every prime ideal is regular, then it would follow that this ring is a Brodmann ring by the above result. The next result gives a class of localizations in which this situation occurs.

Proposition 5.1.7. [12, Proposition 13.7.4] *Let R be a prime PI ring. Suppose that $PIdeg(R) = n$. Suppose that X is a central multiplicative set in R such that $0 \notin X$. Suppose that $X \cap g_n(R) \neq \emptyset$. Then every prime ideal of RX^{-1} is regular.*

5.2 Necessary and sufficient conditions for stability

Given an arbitrary prime Noetherian PI ring we are able to find a class of localizations in which asymptotic stability of prime divisors holds, by using 5.1.7, 5.1.6 and 3.5.1. We also know a class of multiplicative sets at which we can localize and preserve some desirable 'going up and down' properties (4.7.14). We now aim to use these observations to determine necessary and sufficient conditions for a version of asymptotic stability to hold in a prime Noetherian PI ring. We first need a slight variation on a previous result.

Proposition 5.2.1. *Let R be a Noetherian ring. Let S be the centre of R. Let I be an ideal of R. Then we have:*

$$\mathcal{C}(I) \cap S = \bigcap_{P \in Ass(R/I)_R} \mathcal{C}(P) \cap S.$$

Proof. This follows directly from 4.7.13. □

This result can be used to derive a straightforward necessary condition for the sets of prime divisors of two ideals to coincide.

Proposition 5.2.2. *Let R be a Noetherian ring. Let S be the centre of R. Let I and J be ideals of R. Suppose that:*

$$Aff\left(\frac{R}{I}\right)_R = Aff\left(\frac{R}{J}\right)_R.$$

Then:

$$\mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S.$$

Proof. We have the following:

$$\begin{aligned} \mathcal{C}(I) \cap S &= \bigcap_{P \in Aff(R/I)_R} \mathcal{C}(P) \cap S \quad \text{(by 5.2.1)} \\ &= \bigcap_{P \in Aff(R/J)_R} \mathcal{C}(P) \cap S \quad \text{(by hypothesis)} \\ &= \mathcal{C}(J) \cap S \quad \text{(by 5.2.1).} \end{aligned}$$

□

We now give sufficient conditions for an ideal I of a prime Noetherian PI ring R to be a Brodmann ideal.

Theorem 5.2.3. *Let R be a prime Noetherian PI ring. Suppose that $PIdeg(R) = n$. Let S be the centre of R. Let I be an ideal of R. Suppose that there is a positive integer m such that the following two conditions hold:*

1. *$\mathcal{C}(I^m) \cap S = \mathcal{C}(I^{m+i}) \cap S$, for every positive integer i;*

2. *$g_n(R) \cap \mathcal{C}(I^m) \neq \emptyset$.*

Then there is a positive integer p such that whenever k is a positive integer such that $k \geq p$ we have:

$$Aff\left(\frac{R}{I^k}\right)_R = Aff\left(\frac{R}{I^{k+1}}\right)_R.$$

Proof. Set $X = \mathcal{C}(I^m) \cap S$. Note that X is a central multiplicative set in R. Since RX^{-1} is a Noetherian ring (by 4.4.1), it follows from 5.1.7, 5.1.6 and the hypotheses that RX^{-1} is an Azumaya algebra. Thus by 3.5.1 we have that I^e is a Brodmann ideal of RX^{-1}. Set $p = max\{m, Brod(RX^{-1}, I^e)\}$. Let k be an integer such that $k \geq m$. Let P be an affiliated prime ideal of $(R/I^k)_R$. Then we have:

$$P^e \in Aff\left(\frac{RX^{-1}}{(I^k)^e}\right)_{RX^{-1}},$$

by 4.7.14. This follows from the fact that since $k \geq p \geq m$ we have $X = \mathcal{C}(I^k) \cap S$, and so $X \subseteq \mathcal{C}(I^k)$. By 4.4.7 we have that $(I^k)^e = (I^e)^k$ and thus we have:

$$P^e \in Aff\left(\frac{RX^{-1}}{(I^e)^k}\right)_{RX^{-1}}.$$

Now since $k \geq p \geq Brod(RX^{-1}, I^e)$ we have:

$$P^e \in Aff\left(\frac{RX^{-1}}{(I^e)^{k+1}}\right)_{RX^{-1}},$$

and using 4.4.7 again we deduce that:

$$P^e \in Aff\left(\frac{RX^{-1}}{(I^{k+1})^e}\right)_{RX^{-1}}.$$

Applying 4.7.14 again, we have that:

$$P^{ec} \in Aff\left(\frac{R}{I^{k+1}}\right)_R.$$

We finally note that since $X \subseteq \mathcal{C}(P)$, by 5.2.1, we have $P = P^{ec}$, by 4.4.11 and 4.4.2, and so we have:

$$P \in Aff\left(\frac{R}{I^{k+1}}\right)_R,$$

which gives that:

$$Aff\left(\frac{R}{I^k}\right)_R \subseteq Aff\left(\frac{R}{I^{k+1}}\right)_R.$$

The same proof works to establish the reverse inclusion. □

5.3 Regular prime ideals

We now state a result with the same conclusion as 5.2.3, but under hypotheses which involve the regularity of appropriate prime divisors.

Definition 5.3.1. Let R be a PI ring. Suppose that $PIdeg(R) = n$. Let $\mathcal{S}$ be a non-empty collection of prime ideals of R which has the property that for each element $P \in \mathcal{S}$, we have $g_n(R) \cap \mathcal{C}(P) \neq \emptyset$. Then $\mathcal{S}$ is said to satisfy the $g_n(R)$*-intersection condition* if the following condition holds:

$$g_n(R) \cap \left(\bigcap_{P\in\mathcal{S}} \mathcal{C}(P)\right) \neq \emptyset.$$

Using the above condition we can obtain another set of sufficient conditions involving regularity of prime ideals for asymptotic stability of prime divisors to hold in a prime Noetherian PI ring.

Proposition 5.3.1. *Let R be a prime Noetherian PI ring. Suppose that $PIdeg(R) = n$. Let S be the centre of R. Let I be an ideal of R. Suppose that there is a positive integer m such that the following two conditions hold:*

1. *whenever l is a positive integer such that $l \geq m$, we have $\mathcal{C}(I^l) \cap S = \mathcal{C}(I^{l+1}) \cap S$;*

2. *there is an integer m' such that $m' \geq m$ and such that every affiliated prime ideal of $(R/I^{m'})_R$ is regular and $Aff(R/I^{m'})_R$ satisfies the $g_n(R)$-intersection condition.*

Then there is a positive integer p such that whenever k is a positive integer such that $k \geq p$ we have:

$$Aff\left(\frac{R}{I^k}\right)_R = Aff\left(\frac{R}{I^{k+1}}\right)_R.$$

Proof. Let $X = \mathcal{C}(I^m) \cap S$. Then X is a central multiplicative set. We claim that RX^{-1} is an Azumaya algebra. A prime ideal P of R is regular if and only if $g_n(R) \not\subseteq P$ (5.1.5), which in turn is true if and only if $g_n(R) \cap \mathcal{C}(P) \neq \emptyset$. So since every prime ideal in $Aff(R/I^{m'})_R$ is regular, we have that $g_n(R) \cap \mathcal{C}(P) \neq \emptyset$ for each prime ideal $P \in Aff(R/I^{m'})_R$. Now since the set $Aff(R/I^{m'})_R$ satisfies the $g_n(R)$-intersection condition, we have the following:

$$g_n(R) \cap \left(\bigcap_{P \in Aff(R/I^{m'})_R} \mathcal{C}(P) \right) \neq \emptyset.$$

Now we proceed as follows:

$$\begin{aligned}
\emptyset &\neq g_n(R) \cap \left(\textstyle\bigcap_{P \in Aff(R/I^{m'})_R} \mathcal{C}(P)\right) \\
&= g_n(R) \cap S \cap \left(\textstyle\bigcap_{P \in Aff(R/I^{m'})_R} \mathcal{C}(P)\right) \\
&\quad \text{(since } g_n \text{ is a central polynomial for } R\text{)} \\
&= g_n(R) \cap \mathcal{C}(I^{m'}) \cap S \\
&\quad \text{(by 5.2.1)} \\
&= g_n(R) \cap \mathcal{C}(I^m) \cap S \\
&\quad \text{(since } m' \geq m\text{)} \\
&= g_n(R) \cap \mathcal{C}(I^m).
\end{aligned}$$

The result now follows from 5.2.3. □

5.4 General equivalent conditions

To conclude our discussion, we move away from the prime PI case and consider asymptotic stability of prime divisors in a general Noetherian ring.

Proposition 5.4.1. *Let R be a Noetherian ring. Let S be the centre of R. Suppose that I and J are ideals of R. Then the following two conditions are equivalent:*

1. $Aff(R/I)_R = Aff(R/J)_R$;

2. $\mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$ *and* $Aff(RX^{-1}/I^e)_{RX^{-1}} = Aff(RX^{-1}/J^e)_{RX^{-1}}$, *where* X *is the set* $\mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$.

Proof. Suppose that $Aff(R/I)_R = Aff(R/J)_R$. Then $\mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$ by 5.2.2. To establish the second condition, set $X = \mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$. Then we have the following equivalences:

$$\begin{aligned} Q \in Aff\left(\tfrac{RX^{-1}}{I^e}\right)_{RX^{-1}} &\iff Q^c \in Aff\left(\tfrac{R}{I}\right)_R && \text{(by 4.7.14)} \\ &\iff Q^c \in Aff\left(\tfrac{R}{J}\right)_R && \text{(by hypothesis)} \\ &\iff Q^{ce} \in Aff\left(\tfrac{RX^{-1}}{J^e}\right)_R && \text{(by 4.7.14)} \\ &\iff Q \in Aff\left(\tfrac{RX^{-1}}{J^e}\right)_R. \end{aligned}$$

Conversely, suppose that we have $\mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$ and:

$$Aff\left(\frac{RX^{-1}}{I^e}\right)_{RX^{-1}} = Aff\left(\frac{RX^{-1}}{J^e}\right)_{RX^{-1}},$$

where $X = \mathcal{C}(I) \cap S = \mathcal{C}(J) \cap S$. Then the following equivalences hold:

$$\begin{aligned} P \in Aff\left(\tfrac{R}{I}\right)_R &\iff P^e \in Aff\left(\tfrac{RX^{-1}}{I^e}\right)_{RX^{-1}} && \text{(by 4.7.14)} \\ &\iff P^e \in Aff\left(\tfrac{RX^{-1}}{J^e}\right)_{RX^{-1}} && \text{(by hypothesis)} \\ &\iff P^{ec} \in Aff\left(\tfrac{R}{J}\right)_R && \text{(by 4.7.14).} \end{aligned}$$

But since $X \subseteq \mathcal{C}(P)$, we have $P^{ec} = P$, and so in fact we have the following equivalence:

$$P \in Aff\left(\frac{R}{I}\right)_R \iff P \in Aff\left(\frac{R}{J}\right)_R .$$

This gives the result. ☐

Using this result we can obtain an equivalent condition for asymptotic stability to hold.

Theorem 5.4.2. *Let R be a Noetherian ring. Let S be the centre of R. Let I be an ideal of R. Then I is a Brodmann ideal of R if and only if there is a positive integer m such that the following hold:*

1. *whenever n is a positive integer such that $n \geq m$, we have $\mathcal{C}(I^n) \cap S = \mathcal{C}(I^{n+1}) \cap S$;*

2. *if X is the set $\mathcal{C}(I^m) \cap S$ then I^e is a Brodmann ideal of RX^{-1}.*

Proof. Suppose that I is a Brodmann ideal of R and choose a positive integer m such that whenever n is a positive integer such that $n \geq m$, we have:

$$Aff\left(\frac{R}{I^n}\right)_R = Aff\left(\frac{R}{I^{n+1}}\right)_R .$$

Take a positive integer n such that $n \geq m$. Now by 5.4.1 we have that $\mathcal{C}(I^n) \cap S = \mathcal{C}(I^{n+1}) \cap S$, and, letting X be the set $\mathcal{C}(I^n) \cap S = \mathcal{C}(I^{n+1}) \cap S$, we also have that:

$$Aff\left(\frac{RX^{-1}}{(I^n)^e}\right)_{RX^{-1}} = Aff\left(\frac{RX^{-1}}{(I^{n+1})^e}\right)_{RX^{-1}} .$$

By 4.4.7 we have that for any ideal I of R and any positive integer n we have $(I^n)^e = (I^e)^n$. Thus for any integer n such that $n \geq m$, we have:

$$Aff\left(\frac{RX^{-1}}{(I^e)^n}\right)_{RX^{-1}} = Aff\left(\frac{RX^{-1}}{(I^e)^{n+1}}\right)_{RX^{-1}} .$$

Hence I^e is a Brodmann ideal of RX^{-1}.

Conversely, suppose that there is a positive integer m such that whenever n is a positive integer such that $n \geq m$, we have $\mathcal{C}(I^n) \cap S = \mathcal{C}(I^{n+1}) \cap S$, and such that I^e is a Brodmann ideal of RX^{-1}, where X is the set $\mathcal{C}(I^m) \cap S$. Choose a positive integer p such that whenever n is a positive integer such that $n \geq p$, we have:

$$Aff\left(\frac{RX^{-1}}{(I^e)^n}\right)_{RX^{-1}} = Aff\left(\frac{RX^{-1}}{(I^e)^{n+1}}\right)_{RX^{-1}}.$$

By 4.4.7 again, we have that:

$$Aff\left(\frac{RX^{-1}}{(I^n)^e}\right)_{RX^{-1}} = Aff\left(\frac{RX^{-1}}{(I^{n+1})^e}\right)_{RX^{-1}},$$

for every integer n such that $n \geq p$. Now set $l = max\{m, p\}$. Then, by 5.4.1, we have:

$$Aff\left(\frac{R}{I^n}\right)_R = Aff\left(\frac{R}{I^{n+1}}\right)_R,$$

whenever n is an integer such that $n \geq l$. So I is a Brodmann ideal of R, as required. □

Finally, we note an immediate consequence of the above result.

Corollary 5.4.3. *If the equivalent statements given in Theorem 5.4.2 hold, we have the following relation:*

$$Brod(RX^{-1}, I^e) \leq Brod(R, I).$$

Little is known about the Brodmann stabilizing point of an ideal I in a Brodmann ring R. The subject has been touched upon by S. Morey, and the reader is referred to [13] for more information.

Bibliography

[1] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J.Alg. **11** (1969), 532–63.

[2] M. Brodmann, *Asymptotic stability of* $Ass(M/I^nM)$, Proc. Amer. Math. Soc. **74** (1979), 16–18.

[3] A.W. Chatters, A.W. Goldie, C.R. Hajarnavis, and T.H. Lenagan, *Reduced rank in Noetherian rings*, J. Alg. **61** (1979), 582–589.

[4] K.R. Goodearl and R.B. Warfield, *An introduction to noncommutative Noetherian rings*, vol. 16, London Mathematical Society Student Texts, no. 16, Cambridge University Press, Cambridge, 1989.

[5] A.J. Gray, *Centres and invertible ideals of Noetherian rings*, Ph.D. thesis, Mathematics Institute, University of Warwick, 1984.

[6] L.J. Ratliff Jr., *On prime divisors of* I^n*, n large*, Michigan Math. J. **23** (1976), 337–352.

[7] T.H. Lenagan and R.B. Warfield, *Affiliated series and extensions of modules*, J. Alg. **142** (1991), 164–187.

[8] Jr. L.J. Ratliff, *A brief survey and history of asymptotic prime divisors*, Rocky Mountain Math. J. **13** (1983), no. 3.

[9] ———, *On asymptotic prime divisors*, Pacific J. Math. **111** (1984), no. 2.

[10] S. McAdam, *Asymptotic prime divisors*, vol. 1023, Lecture Notes in Mathematics, no. 1023, Springer-Verlag, 1983.

[11] S. McAdam and P. Eakin, *The asymptotic Ass*, J. Alg. **61** (1979), 71–81.

[12] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian rings*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1987.

[13] S. Morey, *Stability of associated primes and equality of ordinary and symbolic powers of ideals*, Comm. Alg. **27** (1999), no. 7.

[14] M. Nagata, *On the derived normal rings of Noetherian integral domains*, Mem. Coll. Sc. Univ. Kyoto **29** (1955), 293–303.

[15] E.C. Posner, *Prime rings satisfying a polynomial identity*, Proc. Amer. Math. Soc. **11** (1960), 180–183.

[16] C. Procesi, *On a theorem of M. Artin*, J. Alg. **22** (1972), 309–315.

[17] D. Rees, *Valuations associated with ideals*, Proc. London Math. Soc. **6** (1956), 161–174.

[18] L.H. Rowen, *Polynomial identities in ring theory*, Academic Press, 1980.