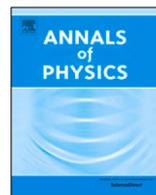




ELSEVIER

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Exact eigenvalue order statistics for the reduced density matrix of a bipartite system



B. Sharmila^{a,b,*}, V. Balakrishnan^a, S. Lakshmibala^a

^a Department of Physics, Indian Institute of Technology Madras, Chennai 600036, India

^b Department of Physics, University of Warwick, Coventry CV4 7AL, UK

ARTICLE INFO

Article history:

Received 24 January 2022

Accepted 21 August 2022

Available online 27 August 2022

Keywords:

Bipartite system

Random complex pure state

Gaussian ensemble

Reduced density matrix

Eigenvalue order statistic

ABSTRACT

We consider the reduced density matrix $\rho_A^{(m)}$ of a bipartite system AB of dimensionality mn ($n \geq m$ without loss of generality) in a Gaussian ensemble of random, complex pure states of the composite system. For a given dimensionality m of the subsystem A , the eigenvalues $\lambda_1^{(m)}, \dots, \lambda_m^{(m)}$ of $\rho_A^{(m)}$ are correlated random variables because their sum equals unity. The following quantities are known, among others: The joint probability density function (PDF) of the eigenvalues $\lambda_1^{(m)}, \dots, \lambda_m^{(m)}$ of $\rho_A^{(m)}$, the PDFs of the smallest eigenvalue $\lambda_{\min}^{(m)}$ and the largest eigenvalue $\lambda_{\max}^{(m)}$, and the family of average values $\langle \text{Tr}(\rho_A^{(m)q}) \rangle$ parametrized by q . Using these as inputs, we find the exact eigenvalue order statistics for any arbitrary value of m and n , i.e., explicit analytic expressions for the PDFs of each of the m eigenvalues arranged in ascending order from the smallest to the largest one. For the sake of clarity, we first present the eigenvalue order statistics for values of m running from 2 to 6, before going on to the general expressions. When $m = n$ (respectively, $m < n$) these PDFs are polynomials of order $m^2 - 2$ (respectively, $mn - 2$) with support in specific sub-intervals of the unit interval, demarcated by appropriate unit step functions. Our exact results are fully corroborated by numerically generated histograms of the ordered set of eigenvalues corresponding to ensembles of over 10^5 random complex pure states of the bipartite system.

© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license

(<http://creativecommons.org/licenses/by/4.0/>).

* Corresponding author at: Department of Physics, University of Warwick, Coventry CV4 7AL, UK.

E-mail address: Sharmila.Balamurugan@warwick.ac.uk (B. Sharmila).

1. Introduction

A basic motif in the study of entanglement involves a bipartite system AB comprising subsystems A and B with Hilbert space dimensions m and $n(\geq m)$, respectively. Given a specified Gaussian ensemble of random pure states of the composite system, the task is to deduce the statistical properties of the reduced density matrix $\rho_A^{(m)}$. A fairly extensive literature exists in this regard (see Refs. [1–32] and references therein). The properties investigated include the joint probability density function (PDF) of the set of eigenvalues $\{\lambda_k^{(m)}\}$ ($1 \leq k \leq m$) of $\rho_A^{(m)}$ [2,3], the leading large- m behaviour of the average values of the set of eigenvalues [12], the PDFs of the smallest and largest eigenvalues $\lambda_{\min}^{(m)}$ [4,12,14,20] and $\lambda_{\max}^{(m)}$ [22], their mean values and higher moments $\langle (\lambda_{\min}^{(m)})^q \rangle$ and $\langle (\lambda_{\max}^{(m)})^q \rangle$, the average $\langle \text{Tr}(\rho_A^{(m)})^q \rangle$ where q is any positive integer [14,22,32], and the associated entropies that quantify the extent of entanglement of A and B such as the average subsystem von Neumann entropy (SVNE) $-\langle \text{Tr}(\rho_A^{(m)} \ln \rho_A^{(m)}) \rangle$ [5–8] and the subsystem linear entropy $1 - \langle \text{Tr}(\rho_A^{(m)})^2 \rangle$ [2]. The main technical complication in these studies arises from the fact that the eigenvalues are correlated random variables.

The studies listed in the foregoing deal, by and large, with the set of eigenvalues of $\rho_A^{(m)}$ rather than individual ones, with the exception of the extreme values $\lambda_{\min}^{(m)}$ and $\lambda_{\max}^{(m)}$ as already stated. The natural extension of extreme value statistics is the *order statistics* of the eigenvalues, the task being to deduce the PDFs of the individual eigenvalues identified by their positions in an *ordered* sequence. The complications involved in the statistics of extreme values in the case of correlated random variables persist, of course, for order statistics as well.

In a different but related context, extensive investigations have been carried out on the eigenvalue order statistics of similar random matrix ensembles, however, without imposing the trace condition $\sum_{i=1}^m \lambda_i = 1$ (see, for instance, [16] and the references therein). Analytical expressions for the PDFs of the order statistics in such ensembles have also been obtained [17]. Further, in [25,26,28], the large deviations of the bulk and the extreme eigenvalues, and the order statistics of the eigenvalues, have been explained using a rate function in the limit of infinitely large random Gaussian matrices without imposing the trace condition. As an aside, we also note that a similar investigation has been carried out in the context of Cauchy random matrices [24]. The statistical properties of the eigenvalues corresponding to different random matrix ensembles such as the Bures ensemble [10] and the polynomial ensemble [29] have also been investigated. Order statistics in such random matrix ensembles have widespread applications, e.g., in data processing techniques such as the principal component analysis [23], in obtaining an optimal linear processing structure in multicarrier multiple-input multiple-output channels [9], in evaluating the statistical properties of the information flowing processes in multicarrier continuous-variable quantum key distribution [33], in testing the equality of two covariance matrices when the number of potentially dependent data vectors is large and proportional to the size of the vectors [34], and in multivariate time-series analysis [30]. Further, the order statistics of eigenvalues of covariance matrices [1] and the order statistics arising from a sample of random variables with random sample size [35] have applications in curve fitting, regression and financial modelling. Imposing the trace condition causes interesting and nontrivial correlations between the eigenvalues and therefore affects its statistical properties significantly.

The motivation for our investigation is two-fold. First, to obtain the PDFs of the full set of ordered eigenvalues of $\rho_A^{(m)}$ with the trace condition imposed, which is the task carried out in this paper. This is very different from earlier works [16,17,30], where such a condition is not imposed as the authors examine the PDFs of eigenvalues of measurable observables. Imposition of the trace condition allows for an elegant procedure, which we report in this paper, for finding the PDFs of not merely the extremal eigenvalues, but all the eigenvalues of the subsystem density matrix. We present the general solution of the PDFs of the full set of ordered eigenvalues, for arbitrary values of the subsystem dimensions m and $n \geq m$. However, it is instructive to first illustrate our procedure for values of m running from 2 to 6 in subsequent sections, before presenting the general solution.

The second aspect of our calculation is the following. The importance of PDFs of all eigenvalues has been emphasized in the context of observables. However, we note that in several physical

applications, particularly in hybrid quantum models of atom–field interaction, the extent of bipartite entanglement (always defined in terms of the subsystem density matrix) is mimicked to a great extent by the dynamics of a suitably chosen observable. For instance, when a three-level Λ atom interacts with two radiation fields which are initially in a coherent state, the dynamics of the mean photon number of either field mimics that of the extent of atomic entanglement with the fields, to such a degree that even the appearance of a bifurcation cascade is reflected in the dynamics of SVNE, which is the entanglement measure considered (see, for instance, [27,36]). Hence, the importance of the PDFs of all eigenvalues of observables gets translated to the PDFs of the full set of ordered eigenvalues of the subsystem density matrix. For arbitrarily large but finite number of atomic levels, as is relevant for real experiments, we have obtained PDFs of the full ordered set of eigenvalues of the subsystem density matrix.

In order to avoid any confusion, we shall denote the eigenvalues $\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_m^{(m)}$ arranged in ascending order by the sequence

$$\Lambda_1^{(m)}, \Lambda_2^{(m)}, \dots, \Lambda_m^{(m)}. \tag{1}$$

Thus $\Lambda_1^{(m)} \equiv \lambda_{\min}^{(m)}$ and $\Lambda_m^{(m)} \equiv \lambda_{\max}^{(m)}$. We seek the normalized PDF $p_k^{(m)}(x)$ (where $x \in [0, 1]$) corresponding to each k th-order statistic $\Lambda_k^{(m)}$, where $1 \leq k \leq m$. The investigations cited in the opening paragraph above rely on the following basic result [3]. Let $\lambda_1^{(m)}, \dots, \lambda_m^{(m)}$ be the eigenvalues of $\rho_A^{(m)}$ (listed in no particular order). Their joint PDF is then given by

$$P(\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_m^{(m)}) = C_{m,n}^{(\beta)} \delta \left(\sum_{i=1}^m \lambda_i^{(m)} - 1 \right) \prod_{i=1}^m (\lambda_i^{(m)})^\alpha \times \prod_{j < k} |\lambda_j^{(m)} - \lambda_k^{(m)}|^\beta. \tag{2}$$

Here the Dyson index $\beta = 2$ (respectively, 1) for a Gaussian ensemble of complex (respectively, real) pure states,

$$\alpha = (\beta/2)(n - m + 1) - 1, \tag{3}$$

and the normalization constant is

$$C_{m,n}^{(\beta)} = \frac{\Gamma(mn\beta/2)[\Gamma(1 + (\beta/2))]^m}{\prod_{j=0}^{m-1} \Gamma((n - j)\beta/2)\Gamma(1 + (m - j)\beta/2)}. \tag{4}$$

As is well known in random matrix theory, the eigenvalues $\{\lambda_k^{(m)}\}$ form a set of correlated random variables both because of the bunching effect arising from the requirement that their sum be equal to unity, as well as the level repulsion implied by the presence of the factor $|\lambda_j^{(m)} - \lambda_k^{(m)}|^\beta$ in the joint PDF of Eq. (2). In principle, the PDF $p_k^{(m)}(x)$ of the k th eigenvalue in the ordered eigenvalue sequence can be found by multiplying the joint PDF in Eq. (2) by the product of step functions

$$\prod_{j=1}^{k-2} \Theta(\lambda_{j+1} - \lambda_j) \Theta(x - \lambda_{k-1}) \Theta(\lambda_{k+1} - x) \prod_{l=k+1}^{m-1} \Theta(\lambda_{l+1} - \lambda_l), \tag{5}$$

integrating over $\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_m$, and normalizing the resulting function of x to unity. This approach, however, presents formidable technical problems, and is not feasible. We shall see that there is an alternative, simpler procedure to arrive at the result sought.

It is evident from Eqs. (2) and (4) that some simplification occurs when $\beta = 2$ (complex pure states), and we shall consider this case. While we shall finally present exact expressions for the PDFs of the ordered eigenvalues for arbitrary subsystem dimensions m and n , those expressions are somewhat complicated. The manner in which the structure of the solutions arises is best elucidated by explicit illustration for small values of m . Accordingly, we shall start with $m = 2$ and increase it step by step up to $m = 6$, to demonstrate how the PDFs build up. In the interests of clarity, we

shall further set $n = m$ for the most part in this demonstration, in order to take advantage of the simplification that ensues from the fact that $\alpha = 0$ in this case.

The plan of the paper is as follows. In Section 2, we write down the ranges of the random variables $\{\Lambda_k^{(m)}\}$, followed by some properties of Mellin transforms that will be used in the sequel. We also list three specific, already known results that will be used to deduce the PDFs of the ordered eigenvalues. Next, in Section 3 we state (for the sake of completeness) the existing results in the trivial case $m = n = 2$. We then discuss in Section 4 the case $m = n = 3$, which is again special because there is just one eigenvalue in between the smallest and largest eigenvalues. In Sections 5 and 6, we illustrate our procedure to obtain the analytical expressions for PDFs corresponding to $m = 4, 5$, and 6 when $m = n$. We also show that our procedure works for the case $m \neq n$ by considering the case $m = 4, n = 5$. Finally, in Section 7 we present the solution for the PDFs $\{p_k^{(m)}(x)\}$, $1 \leq k \leq m$, for arbitrary values of the subsystem dimensions m and $n \geq m$. We conclude with brief remarks on the behaviour of the PDFs at the end points of their domains.

2. Preliminaries

Integrating $P(\lambda_1^{(m)}, \dots, \lambda_m^{(m)})$ over all but any one of the set $\{\lambda_k^{(m)}\}$ is a technically complicated task. It yields a so-called ‘single-particle’ PDF for a single eigenvalue [18]. But this procedure automatically averages over the location of the eigenvalue in the ordered set of eigenvalues. Thus, it leads, for instance, to the expected result that the average value $\langle \lambda_k^{(m)} \rangle$ is just $1/m$. It is evident that this single-particle PDF is quite different from the individual PDFs corresponding to the ordered eigenvalues.

We start with some general properties of the ordered set of eigenvalues $\{\Lambda_k^{(m)}\}$ that will be needed to identify and isolate the corresponding set of PDFs of the individual members of the set. Since $\text{Tr } \rho_A^{(m)} = 1$, the ordered set of non-negative numbers $\{\Lambda_k^{(m)}\}$ satisfies the relation $\sum_{k=1}^m \Lambda_k^{(m)} = 1$. It follows at once that $\Lambda_1^{(m)}$ cannot exceed $1/m$, while $\Lambda_2^{(m)}$ cannot exceed $1/(m-1)$, and so on. That is,

$$0 \leq \Lambda_k^{(m)} \leq 1/(m+1-k), \quad k = 1, 2, \dots, m-1. \tag{6}$$

In particular, $\Lambda_{m-1}^{(m)} \leq 1/2$. Moreover, it is evident that the largest eigenvalue cannot be smaller than $1/m$, so that its range is given by

$$1/m \leq \Lambda_m^{(m)} \leq 1. \tag{7}$$

The ranges in Eqs. (6) and (7) therefore specify the support $I_k^{(m)}$ of the normalized PDF $p_k^{(m)}(x)$ of each $\Lambda_k^{(m)}$, for $1 \leq k \leq m$.

As we shall also be concerned with the higher moments of the eigenvalues $\Lambda_k^{(m)}$, it is helpful to recall very briefly some properties of Mellin transforms. The Mellin transform $\tilde{f}(q)$ of a function $f(x)$ ($x \in [0, 1]$) and its inverse are defined as

$$\tilde{f}(q) = \int_0^1 x^q f(x) dx, \quad f(x) = \frac{1}{2\pi i} \int_C x^{-q-1} \tilde{f}(q) dq, \tag{8}$$

where the Bromwich contour C runs from $c - i\infty$ to $c + i\infty$ to the right of all the singularities of $\tilde{f}(q)$. The following easily established results will be used in the sequel:

- (a) If $\tilde{f}(q)$ is a rational function of q with simple poles at the negative integers $q = -1, \dots, -\nu$, then $f(x)$ is a polynomial in x of order $\nu - 1$, multiplied by the unit step function $\Theta(1 - x)$.
- (b) Let r be a positive integer > 1 . If $\tilde{f}(q)$ is r^{-q} times a rational function of q with simple poles at the negative integers $q = -1, \dots, -\nu$, then $f(x)$ is a polynomial in x of order $\nu - 1$, multiplied by the unit step function $\Theta(1 - rx)$. In the present context, we note that the PDF $p_k^{(m)}(x)$ and the q th moment of $\Lambda_k^{(m)}$, namely,

$$\langle (\Lambda_k^{(m)})^q \rangle \equiv \int_{I_k^{(m)}} x^q p_k^{(m)}(x) dx, \tag{9}$$

comprise a Mellin transform pair.

We turn now to the three known results that we require to deduce the PDFs of the ordered eigenvalues, setting $\beta = 2$ and $m = n$.

(i) The PDF $p_1^{(m)}(x)$ of the smallest eigenvalue $\Lambda_1^{(m)}$ is given by [14]

$$p_1^{(m)}(x) = m(m^2 - 1)(1 - mx)^{m^2-2} \Theta(1 - mx), \tag{10}$$

where Θ denotes the unit step function. The corresponding moments of the smallest eigenvalue are then given by

$$\langle (\Lambda_1^{(m)})^q \rangle = \frac{\Gamma(q + 1)\Gamma(m^2)}{m^q \Gamma(m^2 + q)}. \tag{11}$$

(ii) The average $\langle \text{Tr}(\rho_A^{(m)})^q \rangle$ is given by [32]

$$\langle \text{Tr}(\rho_A^{(m)})^q \rangle = \frac{\Gamma(m^2)}{\Gamma(m^2 + q)} \sum_{i,p=0}^{m-1} \frac{\Gamma(p + q + 1)[\Gamma(q + 1)]^2}{[\Gamma(1 + i - p)\Gamma(1 + q + p - i)]^2 p!}. \tag{12}$$

We note that this average is the Mellin transform of the sum of PDFs, $\sum_{k=1}^m p_k^{(m)}(x)$. Inverting the transform will therefore yield that sum.

(iii) The third and most crucial ingredient is the PDF $p_m^{(m)}(x)$ of the largest eigenvalue $\Lambda_m^{(m)}$, for which an implicit formula has been derived in Ref. [22]. As pointed out therein, the determination of $p_m^{(m)}(x)$ involves several technical complications that are not present in the determination of $p_1^{(m)}(x)$. The procedure [22] leading to the result sought may be summarized in the following sequence of steps. First, one defines the set of m^2 functions

$$\Psi_{jl}(s) = \int_0^1 e^{-su} u^{j+l} du \quad (j, l = 0, 1, \dots, m - 1) \tag{13}$$

and evaluates the $(m \times m)$ determinant

$$\tilde{\mathcal{P}}(s) = \det [\Psi_{jl}(s)]_{j,l=0,1,\dots,m-1}. \tag{14}$$

The inverse Laplace transform $\mathcal{P}(t)$ of $\tilde{\mathcal{P}}(s)$ is then obtained, and t is set equal to $1/x$ in the result. Then, the quantity

$$Q_m(x) = (m^2 - 1)! \prod_{j=0}^{m-1} \frac{(j + 1)!(j!)^2}{(m - 1 - j)!(m - j)!(m + j)!} x^{m^2-1} \mathcal{P}(1/x) \tag{15}$$

is the cumulative distribution function of $\Lambda_m^{(m)}$, from which its PDF follows according to

$$p_m^{(m)}(x) = dQ_m(x)/dx. \tag{16}$$

The Mellin transform of $p_m^{(m)}(x)$ yields the moment $\langle (\Lambda_m^{(m)})^q \rangle$.

We note, for future reference, that the counterparts of Eq. (12) and Eqs. (13)–(15) to general values of m and $n (\geq m)$ are also available (Refs. [22,32], respectively). We proceed now to find the full set of PDFs $\{p_k^{(m)}(x)\}$ for different values of m . Symbolic manipulation in *Mathematica 11* has been used to carry out all the calculations in what follows.

3. Two qubits: $m = n = 2$

A bipartite system of two qubits is a trivial case as far as order statistics are concerned, since there are only two eigenvalues $\Lambda_1^{(2)}$ and $\Lambda_2^{(2)} = 1 - \Lambda_1^{(2)}$. Setting $m = 2$ in Eq. (10), the PDF of the smaller eigenvalue is given, in this case, by

$$p_1^{(2)}(x) = 6(1 - 2x)^2 \Theta(1 - 2x). \tag{17}$$

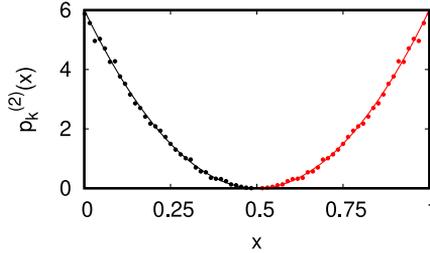


Fig. 1. PDFs $p_k^{(2)}(x)$ of the ordered eigenvalues in the case $m = n = 2$ with $k = 1$ (black) and $k = 2$ (red). In this figure and in all the figures that follow, the solid-line curves correspond to the exact analytical expressions derived. The dots represent numerical histograms obtained from an ensemble of over 10^5 random pure states. (Color figure online).

Working out the steps outlined in Eqs. (13) to (16) with m set equal to 2, we obtain

$$p_2^{(2)}(x) = 6(1 - 2x)^2 [\Theta(1 - x) - \Theta(1 - 2x)]. \tag{18}$$

Apart from the step functions, these are polynomials of order $m^2 - 2 = 2$, with support in $[0, 1/2]$ and $[1/2, 1]$ respectively. The expression for $p_2^{(2)}(x)$ could have been written down from that for $p_1^{(2)}(x)$ in this case: Since $\Lambda_1^{(2)} + \Lambda_2^{(2)} = 1$, it follows that $p_2^{(2)}(x) = p_1^{(2)}(1 - x)$.

In order to verify the expressions above and those to be obtained for $p_k^{(m)}(x)$ in subsequent sections, we generate the histograms of $\Lambda_k^{(m)}$ computed from an ensemble of random complex pure states of the composite system. We consider a randomly chosen pure state of the full system AB to be an mn -dimensional column vector, with the real and imaginary parts of each element of the vector drawn from a standard normal distribution. The state is then normalized. The moments (and hence the cumulants) of the numerically generated histograms, obtained with 1.001×10^5 random pure states, agree up to the third decimal place with those computed from the analytical expressions. We have also verified that this agreement improves on increasing the number of random pure states in the ensemble to 10^6 . These statements remain valid in all the cases to be considered in the sections that follow. Fig. 1 shows that there is excellent agreement between the numerically generated histograms of $\Lambda_1^{(2)}$ and $\Lambda_2^{(2)}$, and the analytical expressions of Eqs. (17) and (18) for the corresponding PDFs. We note that the histograms in this figure and in subsequent figures have been normalized in order to enable direct comparison with the calculated PDFs.

The average values of the two eigenvalues are $\langle \Lambda_1^{(2)} \rangle = 1/8$ and $\langle \Lambda_2^{(2)} \rangle = 7/8$, while the variance is $3/320$ in both cases. From the Mellin transforms of the PDFs in Eqs. (17) and (18) we get, for the q th moments of the eigenvalues,

$$\langle (\Lambda_1^{(2)})^q \rangle = 3!q!2^{-q}/(q + 3)! \tag{19}$$

and

$$\langle (\Lambda_2^{(2)})^q \rangle = 3!q!(q^2 + q + 2 - 2^{-q})/(q + 3)! \tag{20}$$

Their sum

$$\langle (\Lambda_1^{(2)})^q \rangle + \langle (\Lambda_2^{(2)})^q \rangle = 3!q!(q^2 + q + 2)/(q + 3)! \tag{21}$$

tallies with the corresponding expression for $\langle \text{Tr}(\rho_A^{(2)})^q \rangle$ obtained from Eq. (12). A similar agreement with the known result for $\langle \text{Tr}(\rho_A^{(m)})^q \rangle$ will serve as a further check on the correctness of all the PDFs to be derived in what follows.

4. Two qutrits: $m = n = 3$

A bipartite system of two qutrits is the first non-trivial case owing to the existence of an intermediate eigenvalue $\Lambda_2^{(3)}$ in between the smallest and largest eigenvalues $\Lambda_1^{(3)}$ and $\Lambda_3^{(3)}$. There

is, however, a simple strategy to find the PDF of $\Lambda_2^{(3)}$ in this instance. In brief, we first find $\langle (\Lambda_1^{(3)})^q \rangle$ and $\langle (\Lambda_3^{(3)})^q \rangle$, and use these results along with that for $\langle \text{Tr}(\rho_A^{(3)})^q \rangle$ to deduce $\langle (\Lambda_2^{(3)})^q \rangle$. The inverse Mellin transform of the latter then yields the PDF $p_2^{(3)}(x)$.

When $m = n = 3$, Eq. (10) gives for the PDF of the smallest eigenvalue the expression

$$p_1^{(3)}(x) = 24(1 - 3x)^7 \Theta(1 - 3x). \tag{22}$$

As before, working through the steps in Eqs. (13) to (16) with $m = 3$, we obtain

$$p_3^{(3)}(x) = 24(1 - 3x)^7 \Theta(1 - 3x) - 48(1 - 2x)^3(156x^4 - 165x^3 + 87x^2 - 15x + 1)\Theta(1 - 2x) + 24(1 - x)^3(309x^4 - 354x^3 + 132x^2 - 18x + 1)\Theta(1 - x). \tag{23}$$

The Mellin transforms of the two PDFs in Eqs. (22) and (23) yield, respectively,

$$\langle (\Lambda_1^{(3)})^q \rangle = \frac{8!q!3^{-q}}{(q + 8)!} \tag{24}$$

and

$$\langle (\Lambda_3^{(3)})^q \rangle = \frac{8!q!}{2^6(q + 8)!} \left\{ 2^4(q^4 + 2q^3 + 11q^2 + 10q + 12) - 2^{-q}(q^4 + 14q^3 + 83q^2 + 70q + 192) + 2^6 3^{-q} \right\}. \tag{25}$$

On the other hand, setting $m = 3$ in Eq. (12) gives

$$\langle \text{Tr}(\rho_A^{(3)})^q \rangle = \frac{8!q!}{4(q + 8)!} (q^4 + 2q^3 + 11q^2 + 10q + 12). \tag{26}$$

An important point that we note here for future reference is the following. After the ratio $q!/(q + 8)!$ is simplified, the expression on the right-hand side of Eq. (26) is a rational function of q . There are no transcendental functions like r^{-q} present in $\langle \text{Tr}(\rho_A^{(3)})^q \rangle$. Hence its inverse Mellin transform does not have any step functions of the form $\Theta(1 - rx)$ where $r > 1$.

It follows from Eqs. (24)–(26) that

$$\begin{aligned} \langle (\Lambda_2^{(3)})^q \rangle &= \langle \text{Tr}(\rho_A^{(3)})^q \rangle - \langle (\Lambda_1^{(3)})^q \rangle - \langle (\Lambda_3^{(3)})^q \rangle \\ &= \frac{8!q!}{2^6(q + 8)!} \left\{ 2^{-q}(q^4 + 14q^3 + 83q^2 + 70q + 192) - 2^7 3^{-q} \right\}. \end{aligned} \tag{27}$$

Inverting the Mellin transform, we obtain for the PDF of the middle eigenvalue the explicit expression

$$p_2^{(3)}(x) = 48(1 - 2x)^3(156x^4 - 165x^3 + 87x^2 - 15x + 1)\Theta(1 - 2x) - 48(1 - 3x)^7 \Theta(1 - 3x). \tag{28}$$

Fig. 2 again shows that the three PDFs $p_k^{(3)}(x)$, $k = 1, 2, 3$ are in excellent agreement with the numerically generated histograms.

As the PDFs are essentially polynomials with compact support in ranges whose end-points are rational numbers, all the moments of these PDFs (and hence their cumulants) are rational numbers. (This feature remains valid for all values of m and n .) Table 1 lists the values of the basic descriptors of the distributions concerned in terms of the corresponding cumulants κ_r : the mean κ_1 , the variance κ_2 , the skewness κ_3^2/κ_2^3 , and the excess of kurtosis κ_4/κ_2^2 . Here

$$\frac{\kappa_4}{\kappa_2^2} = \frac{C_4 - 3C_2^2}{C_2^2} \tag{29}$$

with C_k being the k -central moment. Also, we note that $\kappa_3 = C_3$ and $\kappa_2 = C_2$

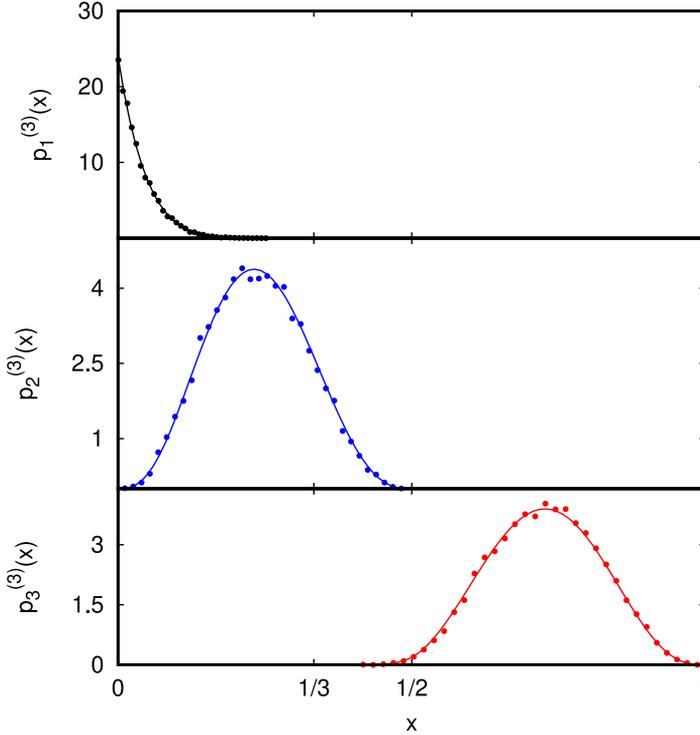


Fig. 2. PDFs of the ordered eigenvalues for $m = n = 3$. Solid curves: analytical expressions; dots: histograms from a Gaussian ensemble of random pure states.

Table 1

Values of the descriptors corresponding to the PDFs $p_k^{(3)}(x)$.

	$p_1^{(3)}(x)$	$p_2^{(3)}(x)$	$p_3^{(3)}(x)$
κ_1	$\frac{1}{27}$	$\frac{103}{432}$	$\frac{313}{432}$
κ_2	$\frac{4}{3645}$	$\frac{6499}{933120}$	$\frac{8179}{933120}$
κ_3^2/κ_2^3	$\frac{245}{121}$	$\frac{241916407220}{33214290609379}$	$\frac{80059327220}{66204269040019}$
κ_4/κ_2^2	$\frac{201}{88}$	$-\frac{248949138}{464607011}$	$-\frac{387186258}{735856451}$

5. The case $m = n = 4$

It is clear that the simple argument used in the case $m = 3$ is no longer applicable when there is more than a single intermediate eigenvalue, i.e., for any $m \geq 4$. There is, however, a way to deduce the PDF $p_k^{(m)}(x)$ for every one of these eigenvalues. The case $m = 4$ serves as the simplest illustration of this method. As before, we start with the PDF of the smallest eigenvalue, obtained by setting $m = n = 4$ in Eq. (10). We have

$$p_1^{(4)}(x) = 60(1 - 4x)^{14}\Theta(1 - 4x). \tag{30}$$

Next, we find the explicit expression for the PDF $p_4^{(4)}(x)$ of the largest eigenvalue from Eqs. (13)–(16) for $m = 4$. It is convenient to introduce the notation $A_j^{(4,4)}(x)$ ($j = 1, 2, 3, 4$) for the polynomial that

is the coefficient of $\Theta(1 - jx)$ in this expression. (The superscripts indicate the values of m and n .) We then find that

$$p_4^{(4)}(x) = -A_4^{(4,4)}(x)\Theta(1 - 4x) + A_3^{(4,4)}(x)\Theta(1 - 3x) - A_2^{(4,4)}(x)\Theta(1 - 2x) + A_1^{(4,4)}(x)\Theta(1 - x), \tag{31}$$

where

$$\begin{aligned} A_4^{(4,4)}(x) &= 60(1 - 4x)^{14}, \\ A_3^{(4,4)}(x) &= 60(1 - 3x)^8(3 - 96x + 1308x^2 - 6128x^3 + 29818x^4 - 70160x^5 + 67812x^6), \\ A_2^{(4,4)}(x) &= 30(1 - 2x)^6(6 - 264x + 5208x^2 - 45920x^3 + 229936x^4 - 859040x^5 + 2706592x^6 - 5570528x^7 + 5517256x^8), \\ A_1^{(4,4)}(x) &= 60(1 - x)^8(1 - 48x + 1044x^2 - 9904x^3 + 44934x^4 - 94128x^5 + 73116x^6). \end{aligned} \tag{32}$$

We observe that

$$-A_4^{(4,4)}(x) + A_3^{(4,4)}(x) - A_2^{(4,4)}(x) + A_1^{(4,4)}(x) = 0, \tag{33}$$

ensuring that $p_4^{(4)}(x)$ vanishes identically for $x < 1/4$. (As we know, its support is $[1/4, 1]$.) Next, setting $m = 4$ in Eq. (12), we get

$$\langle \text{Tr}(\rho_\lambda^{(4)})^q \rangle = \frac{15!q!}{36(q + 15)!} (144 + 156q + 184q^2 + 57q^3 + 31q^4 + 3q^5 + q^6). \tag{34}$$

Once again, we note that the expression on the right-hand side of Eq. (34) is a rational function of q (after the ratio $q!/(q + 15)!$ is simplified). Hence, by the property (a) of Mellin transforms noted in Section 2, the step functions $\Theta(1 - 4x)$, $\Theta(1 - 3x)$ and $\Theta(1 - 2x)$ cannot appear in its inverse Mellin transform $\sum_{k=1}^4 p_k^{(4)}(x)$. The coefficients of these step functions must therefore vanish identically when the individual PDFs are added up. Note also that $p_1^{(4)}(x) = A_4^{(4,4)}(x)\Theta(1 - 4x)$. These facts lead us naturally to the ansatz that $p_2^{(4)}(x)$ and $p_3^{(4)}(x)$ must have the forms

$$p_2^{(4)}(x) = -c_1 A_4^{(4,4)}(x)\Theta(1 - 4x) + c_2 A_3^{(4,4)}(x)\Theta(1 - 3x), \tag{35}$$

$$p_3^{(4)}(x) = c_1 A_4^{(4,4)}(x)\Theta(1 - 4x) - (c_2 + 1)A_3^{(4,4)}(x)\Theta(1 - 3x) + A_2^{(4,4)}(x)\Theta(1 - 2x), \tag{36}$$

where c_1 and c_2 are constants. They are determined from the normalization (to unity) of $p_2^{(4)}(x)$ and $p_3^{(4)}(x)$ in the ranges $[0, 1/3]$ and $[0, 1/2]$, respectively. Using the fact that $\int_0^{1/4} A_4^{(4,4)}(x)dx = 1$, $\int_0^{1/3} A_3^{(4,4)}(x)dx = 4$, $\int_0^{1/2} A_2^{(4,4)}(x)dx = 6$ and $\int_0^1 A_1^{(4,4)}(x)dx = 4$, we get $c_1 = 3$, $c_2 = 1$. Hence

$$p_2^{(4)}(x) = -3A_4^{(4,4)}(x)\Theta(1 - 4x) + A_3^{(4,4)}(x)\Theta(1 - 3x), \tag{37}$$

$$p_3^{(4)}(x) = 3A_4^{(4,4)}(x)\Theta(1 - 4x) - 2A_3^{(4,4)}(x)\Theta(1 - 3x) + A_2^{(4,4)}(x)\Theta(1 - 2x). \tag{38}$$

We observe from the foregoing (and from all the cases to be considered in the sequel) that the constants multiplying the coefficients $A_j^{(m,n)}(x)$ for a given j in the different PDFs $p_k^{(m)}(x)$ ($m - j + 1 \leq k \leq m$) are the binomial coefficients $\binom{j-1}{m-k}$ with alternating signs. This fact also guarantees that the step functions (other than $\Theta(1 - x)$) do not appear in $\sum_{k=1}^4 p_k^{(4)}(x)$, and more generally in $\sum_{k=1}^m p_k^{(m)}(x)$.

The four PDFs $p_k^{(4)}(x)$, $1 \leq k \leq 4$, are plotted in Fig. 3. Once again, there is excellent agreement with the numerically generated histograms of the ordered eigenvalues. The mean values of the four

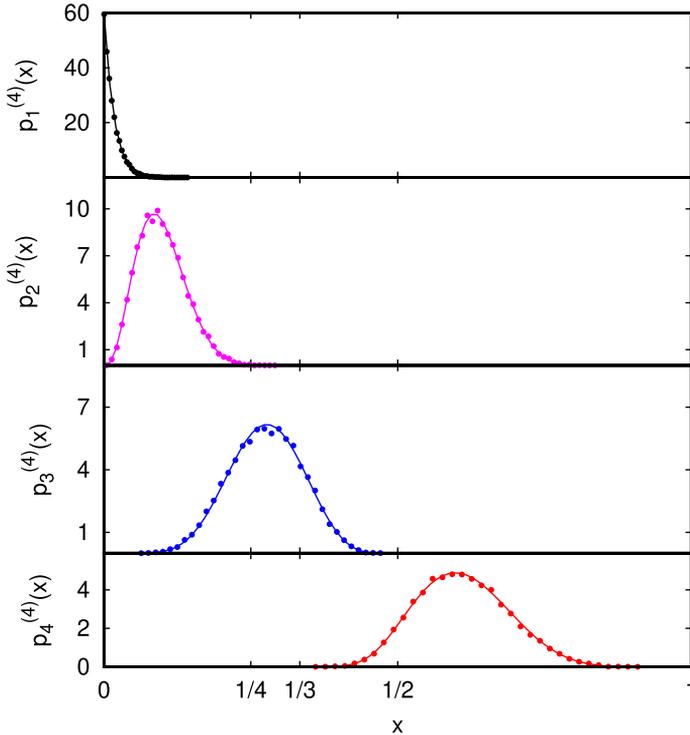


Fig. 3. PDFs of the ordered eigenvalues for $m = n = 4$. Solid curves: analytical expressions; dots: histograms from a Gaussian ensemble of random pure states.

eigenvalues are found to be

$$\langle \Lambda_1^{(4)} \rangle = \frac{1}{64}, \langle \Lambda_2^{(4)} \rangle = \frac{13727}{139968}, \langle \Lambda_3^{(4)} \rangle = \frac{617057}{2239488}, \langle \Lambda_4^{(4)} \rangle = \frac{1367807}{2239488}. \tag{39}$$

The higher cumulants can also be calculated, and they are all rational numbers. We have also verified that the sum of the q th moments of the eigenvalues tallies with the known expression for $\langle \text{Tr}(\rho_A^{(4)})^q \rangle$.

6. Other cases

As further checks of the method used, we have carried out similar calculations to determine the PDFs of the ordered eigenvalues in the cases $m = n = 5, 6$ and 7 , respectively. The algebraic expressions become considerably more lengthy as m increases. The expressions for the PDFs when $m = n = 5$ are given in [Appendix A](#), and these expressions agree very well with the numerically generated histograms, as shown in [Fig. 4](#). As already pointed out, we find that the constants multiplying the coefficient functions $A_j^{(5,5)}(x)$ are appropriate binomial coefficients with alternating signs.

The expressions obtained for the PDFs in the case $m = n = 6$ are also recorded in [Appendix A](#). Once again, we have also verified that there is very good agreement between the analytical expressions for the PDFs and the numerically generated histograms. Similarly, the expressions for $m = n = 7$ are also precisely along expected lines, and will not be given here. To illustrate the close match between our analytical results and numerical simulation for relatively large values of m , [Fig. 5](#) depicts the PDFs $p_k^{(7)}(x)$ in the case $m = n = 7$.

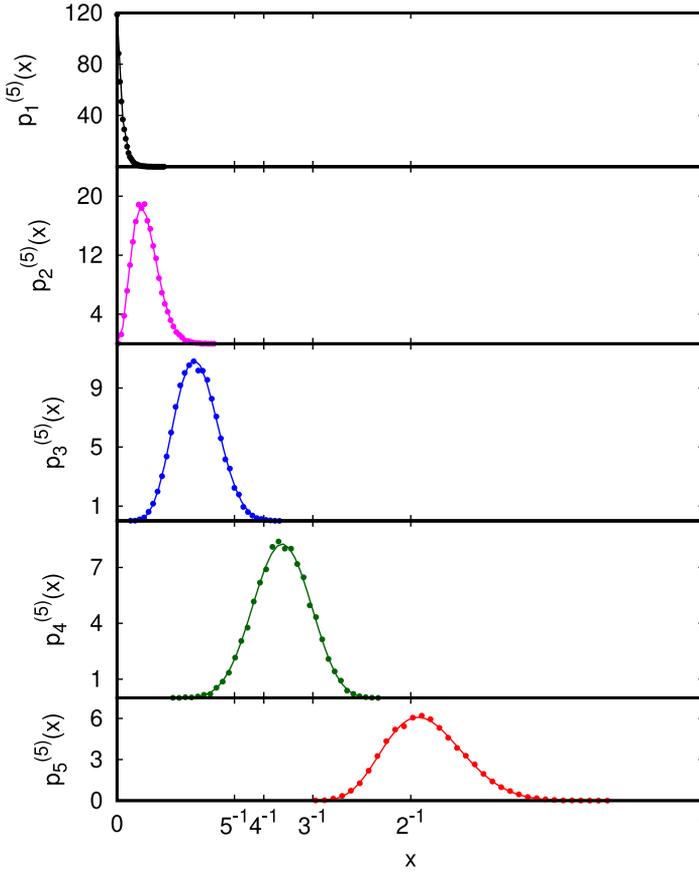


Fig. 4. PDFs of the ordered eigenvalues for $m = n = 5$.

Finally, in order to show that our method works even when $m \neq n$, we have found the analytical expressions for the PDFs when $m = 4$ and $n = 5$. We must now take into account the fact that the index $\alpha = 1$ in this case, and use the corresponding generalizations of Eqs. (12)–(16). The details are given in Appendix B. Once again, the plots of the calculated PDFs are in complete agreement with the numerical histograms, as shown in Fig. 6. Table 2 lists the averages $\langle \Lambda_k^{(m)} \rangle$ for the three cases considered in this section.

7. Solution for general m and n

We now proceed to the exact formal expression for the PDF $p_k^{(m,n)}(x)$ of the k th eigenvalue order statistic $\Lambda_k^{(m)}$, $1 \leq k \leq m$, for general values of the subsystem dimensions m and $n \geq m$. The procedure followed is the same as that for the case $m = n$. As already mentioned, the counterparts of Eqs. (12) [32] and Eqs. (13)–(15) [22] for the case $n \geq m$ are now required. The pattern in the structure of the PDFs found in the foregoing sections aids us considerably in deducing the structure for general m and n . We obtain, finally,

$$p_k^{(m,n)}(x) = \frac{1}{\mathcal{N}} \sum_{j=m-k+1}^m (-1)^{m-k+j+1} \binom{j-1}{m-k} A_j^{(m,n)}(x) \Theta(1-jx), \tag{40}$$

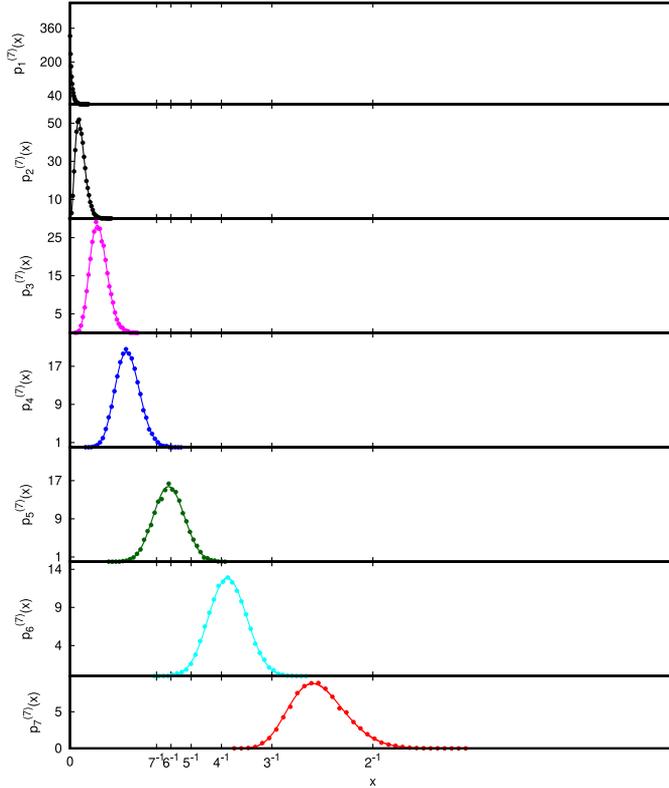


Fig. 5. PDFs of the ordered eigenvalues for $m = n = 7$.

Table 2

Mean values of the ordered eigenvalues for $m = 4, n = 5$; $m = n = 5$; and $m = n = 6$.

Mean	$m = 4, n = 5$	$m = n = 5$	$m = n = 6$
$\langle \Lambda_1^{(m)} \rangle$	$\frac{125}{4096}$	$\frac{1}{125}$	$\frac{1}{216}$
$\langle \Lambda_2^{(m)} \rangle$	$\frac{3188009}{26873856}$	$\frac{813587}{16384000}$	$\frac{301301927}{10546875000}$
$\langle \Lambda_3^{(m)} \rangle$	$\frac{7552985}{26873856}$	$\frac{1182578796887}{8707129344000}$	$\frac{873543307049548733}{11324620800000000000}$
$\langle \Lambda_4^{(m)} \rangle$	$\frac{15312737}{26873856}$	$\frac{2440637328617}{8707129344000}$	$\frac{75602489231060183229976073}{487487792008396800000000000}$
$\langle \Lambda_5^{(m)} \rangle$		$\frac{4581882694877}{8707129344000}$	$\frac{132969850997476498208010743}{487487792008396800000000000}$
$\langle \Lambda_6^{(m)} \rangle$			$\frac{225128892964655720357665283}{487487792008396800000000000}$

where $A_j^{(m,n)}(x)$ is a polynomial in x of order $mn - 1$, to be specified in Eqs. (41)–(43). The constant \mathcal{N} is determined by normalizing $p_k^{(m,n)}(x)$ to unity in the sub-interval $I_k^{(m)}$ of the unit interval in which it has a support.

Let $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{m-1})$ be a permutation of the sequence $\{0, 1, \dots, m - 1\}$, with $\text{sgn } \sigma = \pm 1$ depending on whether σ is an even or odd permutation of the natural order, and let S denote the

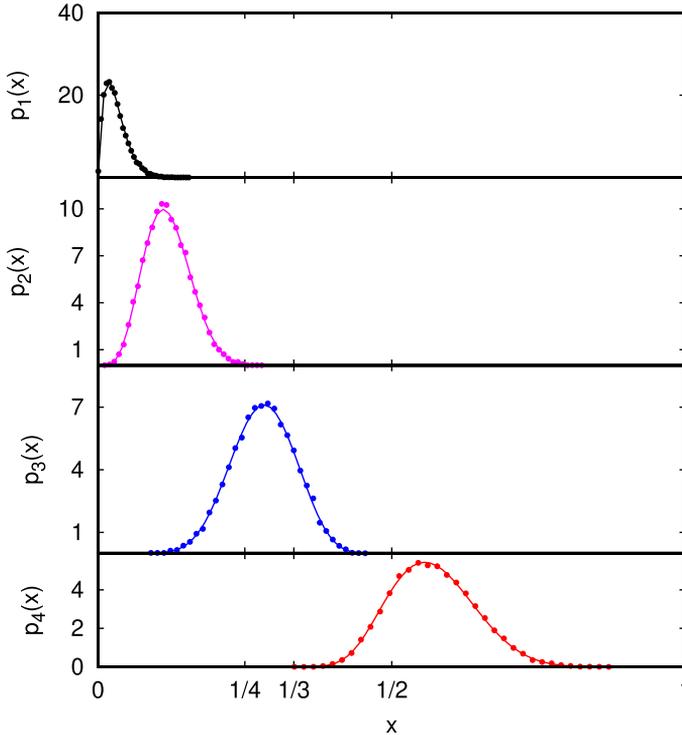


Fig. 6. PDFs of the ordered eigenvalues when $m = 4, n = 5$.

set of all permutations σ . Setting $a = m - j$ where $1 \leq j \leq m$, we have

$$\begin{aligned}
 A_j^{(m,n)}(x) &= -(mn - 1)! \prod_{u=0}^{m-1} \frac{(u + 1)!u!(n - m + u)!}{(n - 1 - u)!(m - u)!(n + u)!} \times \\
 &\quad (d/dx) \left[x^{mn-1} \sum_{\sigma \in S} (\text{sgn } \sigma) \left\{ \sum_{k_1=0}^{m-a} \sum_{k_2=k_1+1}^{m-a+1} \cdots \sum_{k_a=k_{a-1}+1}^{m-1} \times \right. \right. \\
 &\quad \left. \left. \prod_{i=0}^{m-1} \sum_{\ell_i=0}^{n-m+i+\sigma_i} \xi (x^{-1} - j)^\eta / \eta! \right\} \right], \tag{41}
 \end{aligned}$$

where

$$\xi = \binom{n-m+i+\sigma_i}{\ell_i} \ell_i! \left(1 - \sum_{b=1}^a \delta_{i,k_b} \right) + \sum_{c=1}^a \delta_{i,k_c} \delta_{\ell_i,0} (n - m + i + \sigma_i)! \tag{42}$$

and

$$\eta = m - 1 + \sum_{i=0}^{m-1} \left\{ \left(1 - \sum_{b=1}^a \delta_{i,k_b} \right) \ell_i + \sum_{c=1}^a \delta_{i,k_c} (n - m + i + \sigma_i) \right\}. \tag{43}$$

It is evident that the general solution for the PDF $p_k^{(m,n)}(x)$, while exact and explicit, is algebraically quite involved. This fact further corroborates the usefulness of displaying in detail the results for several small values of m , as has been done in the foregoing.

To summarize: We have obtained the probability density functions of the eigenvalue order statistics $\Lambda_k^{(m)}$ ($1 \leq k \leq m$) corresponding to the reduced density matrices for a Gaussian ensemble of random complex pure states of a bipartite system, where m is the smaller subsystem dimensionality. The PDF $p_k^{(m,n)}(x)$ of the ordered eigenvalue $\Lambda_k^{(m)}$ is a linear combination of unit step functions $\Theta(1 - mx), \dots, \Theta(1 - (m + 1 - k)x)$, each multiplied by a polynomial of order $m^2 - 2$ when $n = m$, and of order $mn - 2$ when $n > m$. The support of $p_k^{(m,n)}(x)$ is $[0, 1/(m - k + 1)]$ for $1 \leq k \leq m - 1$, and $[1/m, 1]$ for $k = m$. In all the cases considered, the analytic expressions obtained for the PDFs are in excellent agreement with the numerically generated histograms of the eigenvalues concerned. As further corroboration, we also find that, in every case, the Mellin transform of the sum of the q th moments of these PDFs matches the known expression [32] for $(\text{Tr}(\rho_A^{(m)})^q)$.

Based on the explicit analytic solutions in the cases $m = 3, 4, 5, 6$ and 7 , we deduce the following general properties. When $m = n$, the PDF $p_1^{(m)}(x)$ of the smallest eigenvalue decreases monotonically from the value $p_1^{(m)}(0) = m(m^2 - 1)$ to the value $p_1^{(m)}(1/m) = 0$ as x increases from 0 to $1/m$. When $m < n$, however, $p_1^{(m)}(0) = 0$. Reverting to $m = n$, every $p_k^{(m)}(x)$ (where $2 \leq k \leq m - 1$) vanishes like x^{k^2-1} as $x \rightarrow 0$. In the limit $x \rightarrow 1/(m - 1)$, $p_2^{(m)}(x)$ vanishes like $(1 - (m - 1)x)^{m^2-2m}$. The PDF $p_k^{(m)}(x)$ for both $k = 2 + j$ and $k = m - j$ ($j = 1, 2, \dots, \lfloor \frac{m}{2} \rfloor - 1$) vanishes like $(1 - (m + 1 - k)x)^r$ as $x \rightarrow 1/(m + 1 - k)$, where $r = m^2 - 2m - 2 \sum_{i=1}^j (m - 2i - 1)$. The PDF $p_m^{(m)}(x)$ of the largest eigenvalue $\Lambda_m^{(m)}$ vanishes like $(1 - mx)^{m^2-2}$ as $x \rightarrow 1/m$ from above, and like $(1 - x)^{m^2-2m}$ as $x \rightarrow 1$ from below.

CRedit authorship contribution statement

B. Sharmila: Software, Validation, Formal analysis, Data curation, Visualization, Writing – original draft. **V. Balakrishnan:** Conceptualization, Methodology, Formal analysis, Writing – review & editing, Supervision, Project administration. **S. Lakshimbala:** Conceptualization, Methodology, Formal analysis, Writing – review & editing, Supervision, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We would like to thank Arul Lakshminarayan for drawing our attention to Ref. [31]. SL and VB were supported, in part, by a grant from Mphasis to the Centre for Quantum Information, Communication, and Computing (CQulCC). BS acknowledges the UK STFC “Quantum Technologies for Fundamental Physics” program (Grant No. ST/T006404/1) for partial support.

Appendix A

We first present the analytic expressions of the PDFs of the ordered eigenvalues $\{\Lambda_k^{(5)}\}$ for $m = n = 5$. The PDFs $p_k^{(5)}(x)$ ($k = 1, 2, \dots, 5$) are written in terms of the functions $A_j^{(5,5)}(x)$ ($j = 1, 2, \dots, 5$) that are the coefficients of the respective step functions $\Theta(1 - jx)$. These coefficient functions are as listed below.

$$\begin{aligned}
 A_5^{(5,5)}(x) &= 120(1 - 5x)^{23}, \\
 A_4^{(5,5)}(x) &= 240(1 - 4x)^{15} (2 - 110x + 2690x^2 - 20600x^3 + 304595x^4 \\
 &\quad - 1558835x^5 + 4852905x^6 - 10365975x^7 + 11082660x^8), \\
 A_3^{(5,5)}(x) &= 720(1 - 3x)^{11} (1 - 82x + 3124x^2 - 55528x^3 + 656656x^4 \\
 &\quad - 6833200x^5 + 60965520x^6 - 390601200x^7 + 1733312295x^8 \\
 &\quad - 5065359970x^9 + 10140970180x^{10} - 13794793180x^{11}
 \end{aligned}$$

$$\begin{aligned}
 &+11635970460x^{12}), \\
 A_2^{(5,5)}(x) &= 240(1-2x)^{11}(2-186x+8118x^2-167464x^3+2021877x^4 \\
 &-18428355x^5+161532525x^6-1281331755x^7+7805513430x^8 \\
 &-33503168380x^9+94797708060x^{10}-158275026540x^{11} \\
 &+119326518320x^{12}), \\
 A_1^{(5,5)}(x) &= 120(1-x)^{15}(1-100x+4720x^2-104200x^3+1215160x^4 \\
 &-7812880x^5+27619440x^6-49896300x^7+35838555x^8). \tag{A.1}
 \end{aligned}$$

In terms of these polynomials, the PDFs $\{p_k^{(5)}(x)\}$ are found to be

$$p_1^{(5)}(x) = A_5^{(5,5)}(x)\theta(1-5x), \tag{A.2}$$

$$p_2^{(5)}(x) = -4A_5^{(5,5)}(x)\theta(1-5x) + A_4^{(5,5)}(x)\theta(1-4x), \tag{A.3}$$

$$\begin{aligned}
 p_3^{(5)}(x) &= 6A_5^{(5,5)}(x)\theta(1-5x) - 3A_4^{(5,5)}(x)\theta(1-4x) \\
 &+ A_3^{(5,5)}(x)\theta(1-3x), \tag{A.4}
 \end{aligned}$$

$$\begin{aligned}
 p_4^{(5)}(x) &= -4A_5^{(5,5)}(x)\theta(1-5x) + 3A_4^{(5,5)}(x)\theta(1-4x) \\
 &- 2A_3^{(5,5)}(x)\theta(1-3x) + A_2^{(5,5)}(x)\theta(1-2x), \tag{A.5}
 \end{aligned}$$

$$\begin{aligned}
 p_5^{(5)}(x) &= A_5^{(5,5)}(x)\theta(1-5x) - A_4^{(5,5)}(x)\theta(1-4x) \\
 &+ A_3^{(5,5)}(x)\theta(1-3x) - A_2^{(5,5)}(x)\theta(1-2x) \\
 &+ A_1^{(5,5)}(x)\theta(1-x). \tag{A.6}
 \end{aligned}$$

As before, the constants multiplying $A_j^{(m,n)}(x)$ for a given j in different PDFs $p_k^{(m)}(x)$ ($1 \leq k \leq m$) are the binomial coefficients $\binom{j-1}{m-k}$ with alternating signs.

We also report the analytic expressions of the PDFs of the ordered eigenvalues $\{A_k^{(6)}\}$ for $m = n = 6$. As before, the PDFs $p_k^{(6)}(x)$ ($k = 1, 2, \dots, 6$) are written in terms of the functions $A_j^{(6,6)}(x)$ ($j = 1, 2, \dots, 6$), which are listed below.

$$\begin{aligned}
 A_6^{(6,6)}(x) &= 210(1-6x)^{34}, \\
 A_5^{(6,6)}(x) &= 210(1-5x)^{24}(5-420x+16080x^2-160680x^3+6469230x^4 \\
 &-40658112x^5+261366628x^6-1595391672x^7+5683720173x^8 \\
 &-11348219292x^9+11273058660x^{10}), \\
 A_4^{(6,6)}(x) &= 420(1-4x)^{18}(5-660x+41220x^2-1199200x^3 \\
 &+26188080x^4-541359744x^5+9132924768x^6-109228380096x^7 \\
 &+969229595664x^8-6384003186176x^9+35245566675264x^{10} \\
 &-168039178157376x^{11}+674535601042864x^{12}-2058660341189376x^{13} \\
 &+4315240551175584x^{14}-5476040960131392x^{15}+3527358922055856x^{16}), \\
 A_3^{(6,6)}(x) &= 420(1-3x)^{16}(5-780x+58140x^2-2125680x^3 \\
 &+50119740x^4-1004003136x^5+19201278456x^6-311887564848x^7 \\
 &+3949780543830x^8-38228455420056x^9+283595869865088x^{10} \\
 &-1648254166845840x^{11}+7876735652844396x^{12}-32847798731822496x^{13} \\
 &+122296710227124168x^{14}-385032778740807120x^{15}+925473909342876741x^{16} \\
 &-1465716247992173916x^{17}+1154580059692232388x^{18}), \\
 A_2^{(6,6)}(x) &= 210(1-2x)^{18}(5-840x+67680x^2-2641760x^3+60875040x^4 \\
 &-1041040128x^5+17346863424x^6-289429058688x^7+4135247214912x^8 \\
 &-46316923954048x^9+395768314525056x^{10}-2538512485868160x^{11}
 \end{aligned}$$

$$\begin{aligned}
 &+ 11970268586045536x^{12} - 40164721924654464x^{13} + 90652008902870976x^{14} \\
 &- 123384033397219200x^{15} + 76712186285087664x^{16} \Big), \\
 A_1^{(6,6)}(x) = &210(1-x)^{24} (1 - 180x + 15600x^2 - 657000x^3 + 15307350x^4 \\
 &- 210235104x^5 + 1758025460x^6 - 8979492600x^7 + 27172972425x^8 \\
 &- 44490525420x^9 + 30241971348x^{10}) . \tag{A.7}
 \end{aligned}$$

In terms of these polynomials, the PDFs $\{p_k^{(6)}(x)\}$ are found to be

$$p_1^{(6)}(x) = A_6^{(6,6)}(x)\vartheta(1 - 6x), \tag{A.8}$$

$$p_2^{(6)}(x) = -5A_6^{(6,6)}(x)\vartheta(1 - 6x) + A_5^{(6,6)}(x)\vartheta(1 - 5x), \tag{A.9}$$

$$\begin{aligned}
 p_3^{(6)}(x) = &10A_6^{(6,6)}(x)\vartheta(1 - 6x) - 4A_5^{(6,6)}(x)\vartheta(1 - 5x) \\
 &+ A_4^{(6,6)}(x)\vartheta(1 - 4x), \tag{A.10}
 \end{aligned}$$

$$\begin{aligned}
 p_4^{(6)}(x) = &-10A_6^{(6,6)}(x)\vartheta(1 - 6x) + 6A_5^{(6,6)}(x)\vartheta(1 - 5x) \\
 &- 3A_4^{(6,6)}(x)\vartheta(1 - 4x) + A_3^{(6,6)}(x)\vartheta(1 - 3x), \tag{A.11}
 \end{aligned}$$

$$\begin{aligned}
 p_5^{(6)}(x) = &5A_6^{(6,6)}(x)\vartheta(1 - 6x) - 4A_5^{(6,6)}(x)\vartheta(1 - 5x) \\
 &+ 3A_4^{(6,6)}(x)\vartheta(1 - 4x) - 2A_3^{(6,6)}(x)\vartheta(1 - 3x) \\
 &+ A_2^{(6,6)}(x)\vartheta(1 - 2x). \tag{A.12}
 \end{aligned}$$

$$\begin{aligned}
 p_6^{(6)}(x) = &-A_6^{(6,6)}(x)\vartheta(1 - 6x) + A_5^{(6,6)}(x)\vartheta(1 - 5x) - A_4^{(6,6)}(x)\vartheta(1 - 4x) \\
 &+ A_3^{(6,6)}(x)\vartheta(1 - 3x) - A_2^{(6,6)}(x)\vartheta(1 - 2x) \\
 &+ A_1^{(6,6)}(x)\vartheta(1 - x). \tag{A.13}
 \end{aligned}$$

As already pointed out, we find that the constants multiplying the coefficient functions $A_j^{(6,6)}(x)$ are appropriate binomial coefficients with alternating signs.

Appendix B

We consider the case $m = 4$ and $n = 5$, in order to show that our method works even when $m \neq n$. We must take into account the fact that the index α , defined in Eq. (3), is now equal to 1. Using the corresponding generalizations of Eqs. (12)–(16), we find the explicit expression for the PDF $p_4^{(4,5)}(x)$ of the largest eigenvalue [22]. (In the general case $m < n$, this PDF is found to be a linear combination of polynomials of order $mn - 2$ multiplied by appropriate step functions.) We use the notation $A_j^{(4,5)}(x)$ ($j = 1, 2, 3, 4$) for the polynomial that is the coefficient of $\vartheta(1 - jx)$ in this expression. We then find that

$$\begin{aligned}
 p_4^{(4,5)}(x) = &-A_4^{(4,5)}(x)\vartheta(1 - 4x) + A_3^{(4,5)}(x)\vartheta(1 - 3x) \\
 &- A_2^{(4,5)}(x)\vartheta(1 - 2x) + A_1^{(4,5)}(x)\vartheta(1 - x), \tag{B.14}
 \end{aligned}$$

where each $A_j^{(4,5)}(x)$ is a polynomial of order $mn - 2 = 18$, given by

$$\begin{aligned}
 A_4^{(4,5)}(x) &= 3420x(1 - 4x)^{14}(1 + 5x - 20x^2 + 4x^3), \\
 A_3^{(4,5)}(x) &= 3420x(1 - 3x)^9(3 - 72x + 552x^2 + 360x^3 - 19846x^4 \\
 &\quad + 145224x^5 - 430948x^6 + 580728x^7 - 188941x^8), \\
 A_2^{(4,5)}(x) &= 3420x(1 - 2x)^8(3 - 105x + 1452x^2 - 8340x^3 + 8632x^4 + 174904x^5 \\
 &\quad - 1372976x^6 + 5366608x^7 - 11247836x^8 + 10332628x^9), \\
 A_1^{(4,5)}(x) &= 3420x(1 - x)^{11}(1 - 40x + 661x^2 - 5256x^3 + 21231x^4 \\
 &\quad - 41520x^5 + 31111x^6). \tag{B.15}
 \end{aligned}$$

As in the case $m = n$, the PDFs $p_k^{(4,5)}(x)$ is written in terms of $A_j^{(4,5)}(x)$ ($j, k = 1, 2, 3, 4$) where the constants multiplying these coefficient functions are appropriate binomial coefficients with alternating signs. We get

$$p_1^{(4,5)}(x) = A_4^{(4,5)}(x)\Theta(1 - 4x), \quad (\text{B.16})$$

$$p_2^{(4,5)}(x) = -3A_4^{(4,5)}(x)\Theta(1 - 4x) + A_3^{(4,5)}(x)\Theta(1 - 3x), \quad (\text{B.17})$$

$$p_3^{(4,5)}(x) = 3A_4^{(4,5)}(x)\Theta(1 - 4x) - 2A_3^{(4,5)}(x)\Theta(1 - 3x) + A_2^{(4,5)}(x)\Theta(1 - 2x) \quad (\text{B.18})$$

The manifest agreement between the plots of the calculated PDFs and the numerical histograms validates these expressions. We have also verified that the analytical expression of the sum of the q th moments of the eigenvalues matches the known expression [32] for $\langle \text{Tr}(\rho_A^{(4)})^q \rangle$ in this case.

References

- [1] M.A. Stephens, *Biometrika* 62 (1975) 23–28.
- [2] E. Lubkin, *J. Math. Phys.* 19 (1978) 1028–1031.
- [3] S. Lloyd, H. Pagels, *Ann. Phys., NY* 188 (1988) 186–213.
- [4] A. Edelman, *Math. Comp.* 58 (1992) 185–190.
- [5] D.N. Page, *Phys. Rev. Lett.* 71 (1993) 1291–1294.
- [6] S.K. Foong, S. Kanno, *Phys. Rev. Lett.* 72 (1994) 1148–1151.
- [7] J. Sánchez-Ruiz, *Phys. Rev. E* 52 (1995) 5653–5655.
- [8] S. Sen, *Phys. Rev. Lett.* 77 (1996) 1–3.
- [9] D.P. Palomar, J.M. Cioffi, M.A. Lagunas, *IEEE Trans. Signal Process.* 51 (2003) 2381–2401.
- [10] H.-J. Sommers, K. Zyczkowski, *J. Phys. A: Math. Gen.* 37 (2004) 8457–8466.
- [11] I. Bengtsson, K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, Cambridge University Press, Cambridge, 2006.
- [12] M. Žnidarič, *J. Phys. A* 40 (2006) F105–F111.
- [13] O. Giraud, *J. Phys. A* 40 (2007) F1053–F1062.
- [14] S.N. Majumdar, O. Bohigas, A. Lakshminarayan, *J. Stat. Phys.* 131 (2008) 33–49.
- [15] H. Kubotani, S. Adachi, M. Toda, *Phys. Rev. Lett.* 100 (2008) 240501.
- [16] L.G. Ordonez, D.P. Palomar, J.R. Fonollosa, *IEEE Trans. Signal Process.* 57 (2009) 672–689.
- [17] A. Zanella, M. Chiani, M.Z. Win, *IEEE Trans. Commun.* 57 (2009) 1050–1060.
- [18] S. Adachi, M. Toda, H. Kubotani, *Ann. Phys., NY* 324 (2009) 2278–2358.
- [19] D.-Z. Liu, D.-S. Zhou, *Int. Math. Res. Not.* 2011 (2010) 725–766.
- [20] Y. Chen, D.-Z. Liu, D.-S. Zhou, *J. Phys. A* 43 (2010) 315303.
- [21] P. Vivo, *J. Phys. A* 43 (2010) 405206.
- [22] P. Vivo, *J. Stat. Mech.: Theory E.* 2011 (2011) P01022.
- [23] S.N. Majumdar, P. Vivo, *Phys. Rev. Lett.* 108 (2012) 200601.
- [24] R. Marino, S.N. Majumdar, G. Schehr, P. Vivo, *J. Phys. A* 47 (2014) 055001.
- [25] R. Marino, S.N. Majumdar, G. Schehr, P. Vivo, *Phys. Rev. Lett.* 112 (2014) 254101.
- [26] I.P. Castillo, *Phys. Rev. E* 90 (2014) 040102.
- [27] P. Laha, B. Sudarsan, S. Lakshimbala, V. Balakrishnan, *Internat. J. Theoret. Phys.* 55 (2016) 4044–4059.
- [28] I.P. Castillo, *J. Stat. Mech.* 2016 (2016) 063207.
- [29] Y. Wang, *Electron. Commun. Probab.* 23 (2018) 1–14.
- [30] M. Vyas, T. Guhr, T.H. Seligman, *Sci. Rep.* 8 (2018) 14620.
- [31] P.J. Forrester, S. Kumar, *J. Phys. A* 52 (2019) 42LT02.
- [32] E. Bianchi, P. Donà, *Phys. Rev. D* 100 (2019) 105010.
- [33] L. Gyongyosi, *Int. J. Commun. Syst.* 33 (2020) e4314.
- [34] R. Mariétan, S. Morgenthaler, 2020, arXiv:2002.12741.
- [35] V.M. Koutras, M.V. Koutras, *Methodol. Comput. Appl. Probab.* 22 (2020) 1539–1558.
- [36] P. Laha, S. Lakshimbala, V. Balakrishnan, *Phys. Lett. A* 384 (2020) 126565.