Bounding failure probability with the SIVIA algorithm

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The accuracy of Monte Carlo simulation methods depends on the computational effort invested in reducing the estimator variance. Typically reducing such variance requires invoking Monte Carlo with as many samples as one can afford. When the system is complex and the failure event is rare, it can be challenging to establish the correctness of the failure probability estimate. To combat this verification problem, we present an adaptation of the SIVIA algorithm (Set Inversion Via Interval Analysis) that computes rigorous bounds on the failure probability of rare events. With this method, the nonlinearity of the system and the magnitude of the failure event no longer constitute a limitation. This method can therefore be used for verification, when it is of interest to know the rigorous bounds of the very small target failure probability of complex systems, for example in benchmark problems. The method is rigorous i.e. inclusive and outside-in, so the more computational effort is invested the tighter the bounds. Because full separation is exercised between the engineering and the probability problem, the input uncertainty model can be changed without a re-evaluation of the physical function which opens avenues towards computing rigorous imprecise failure probability. For example, the reliability could be formulated without making dependency or distributional statements.

Keywords: Failure probability, Bounding, SIVIA algorithm, Reliability analysis, Rigorous probability

1. Introduction

With the emergence of ever more advanced Monte Carlo simulation methods for estimating the failure probability, it is also ever more challenging to prove the correctness of such methods in a rigorous and automatic way. In Au and Patelli (2016), it is pointed out that problems of practical significance currently poses three main challenges: (i) small failure probability, (ii) high dimension (i.e., a large number of input random variables) and (iii) high complexity (e.g., nonlinearity) in the input–output relationship. When it comes to estimating failure probability, there is no panacea to all challenges. Advanced Monte Carlo simulation methods developed in recent years, like subset simulation (Au and Patelli, 2016), importance sampling (Papaioannou et al., 2016), line sampling (Valdebenito et al., 2021), and regression based (Gasser and Bucher, 2018), and despite their solid mathematical bases, are notoriously not general purpose. For example, line sampling simulation works well on reliability problems with multiple random variables, but on quasi linear performance functions (as opposed to highly nonlinear); while subset simulation may be limited by the dimensionality and complexity of the input–output relationship. The correctness of these methods is almost always tested via Monte Carlo simulation. However, these tests are limited to cases for which Monte Carlo provides an accurate estimation, whilst the cost of running MC increases as the failure probability gets smaller.

The ability to separate the engineering from the probability problem may be considered desirable in reliability analyses. This is due to the fact that a fully defined joint probability distribution may be difficult to obtain in practice. Moreover, with this separation the same engineering problem could be adapted to different scenarios just by changing the joint probability distribution. Existing work to address the crude reliability problem without probability specifications has recently been presented in the literature (Crespo et al., 2009). The extension

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to allow for probability statements has also been proposed by the same authors in Crespo et al. (2013). Our work can be regarded concurrent to Crespo et al. (2013), while removing the assumption of polynomial input-output relationship, and shifting the goal from robust design optimization to verification.

This paper introduces a method based on an adaptation of the SIVIA algorithm to rigorously bound the failure probability. A recent review of the SIVIA algorithm and its implementation in high-level languages is provided in Herrero et al. (2012). The proposed method, that we name Set Inversion REliability (SIRE), fully separates the engineering problem from the probability problem as follows: first SIVIA is deployed to produce a subtiling that rigorously bounds the failure domain under study, and second the probability measure is computed on each element of such subtiling. The advantages of such procedure can be summarised as follows: (1) the information about the geometry of the failure domain can be extracted and stored independently of the probability statement; (2) once the subtiling is complete, the model is no longer needed and the failure probability can be computed at any time without invoking the model again; (3) the method is insensitive to the magnitude of the failure probability; (4) The method is insensitive to the nonlinearity of the performance function. (5) The obtained failure probability is deterministic (no noise), and its error is rigorously bounded. SIRE is a deterministic method to compute probabilities using rigorous partition of the space of interest.

There are of course drawbacks of using the proposed method. Unlike Monte Carlo, the method is sensitive to the dimensionality of the problem, see Table 1. In fact, subtiling is an operation whose complexity increases exponentially with the cardinality of the set being subtiled. Table 1 also shows that SIRE can be executed in parallel; while this is true, it is by no means as obvious as in the case of Monte Carlo. Because of the cardinality limitation we think that the method can be used as a verification tool on low-dimensional and highly-nonlinear models with very small failure probability, i.e. those reliability problems that plain Monte Carlo simulation cannot reach.

Table 1. Overall comparison between SIRE (Set-Inversion REliability) and Monte Carlo simulation for computing failure probability.

	SIRE	Monte Carlo
Rare events	Yes	No
Nonlinear system	Yes	Yes
High dimension	No	Yes
Parallel execution	Yes	Yes

The paper is organised as follows: §2 gives a short introduction to the proposed method, §3 presents some results of the method applied to some benchmark reliability problems. In §4 the results obtained in the previous section are discussed and in §5 coordinates to reproduce the paper are provided.

2. Set-inversion reliability analysis

In this section the main method is presented. It is shown how the SIVIA algorithm can be used to perform reliability analysis, while rigorously bounding the failure probability.

2.1. *The SIVIA algorithm*

The SIVIA algorithm (Set Inversion Via Interval Analysis) is a popular algorithm for constraint back propagation (Jaulin and Walter, 1993). The algorithm was applied to areas of engineering such as control and robotics (Jaulin et al., 2001).

In this section, we provide a brief introduction to the theory. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function, and let $y := f(x)$, be the image of x under f. Let $X \subset \mathbb{R}^d$ be a n-box and $\mathcal{Y}$ the image of X under f. Now, let us define $Y_{\text{target}} \subset \mathcal{Y}$ a given sub interval of $\mathcal{Y}$. The SIVIA algorithm uses a progressive bisection procedure to bound the preimage $\mathcal{A} \subset X$ such that $f(\mathcal{A}) = Y_{\text{target}}$. Because of f, the preimage $\mathcal{A}$ is not a box, but rather a *united set* (de Angelis, 2022). By means of progressive bisection, the SIVIA algorithm effectively creates a subtiling of the original domain X known to contain $\mathcal{A}$. The subtiling is constructed from a list of n-boxes that are a subset of X and share mutual boundaries. On each sub-box X_i, f is evaluated with the rules of interval arithmetic. The preimage $\mathcal{A}$ is given by the union of all sub-boxes whose image is fully contained in the target:

$$\mathcal{A} = \bigcup_{i}^{\infty} \{X_i : f(X_i) \subset Y_{\text{target}}\}, \quad (1)$$

The subtiling is a rigorous inner approximation of the preimage $\mathcal{A}$ in the sense that when a finite number of boxes n replaces ∞, the following holds:

$$\bigcup_{i}^{n} \{X_i : f(X_i) \subset Y_{\text{target}}\} \subset \mathcal{A}. \quad (2)$$

2.1.1. *Divide and conquer*

A subtiling is a collection of binned sub-boxes that satisfies the imposed constrain, for example the constrain expressed by the condition in (1). The SIVIA algorithm outputs three lists of sub-boxes: (1) a list of sub-boxes whose image is fully contained in the target: $\text{S} = \text{List}(\{S_i\})$, where $S_i = \{X_i : f(X_i) \subset Y_{\text{target}}\}$; (2) a list of sub-boxes whose image is not contained in the target: $\text{N} = \text{List}(\{N_i\})$, where $N_i = \{X_i : f(X_i) \subset \overline{Y}_{\text{target}}\}$, and $\overline{Y}_{\text{target}}$ is the complement of Y_{target}; (3) and a list of all the remaining sub-boxes whose image has non-empty intersection with the target: $\text{E} = \text{List}(\{E_i\})$, where $E_i = \{X_i : f(X_i) \cap Y_{\text{target}} \neq \varnothing, X_i \notin \text{S}\}$. To obtain a subtiling that is as close to $\mathcal{A}$ as possible, the size of each sub-box has to be made sufficiently small, so that the volume of the subtiling E complies with the desired accuracy. The SIVIA algorithm can be briefly summarised as follows:

(1) Fix Y_{target}
(2) Determine X such that $Y_{\text{target}} \subset f(X)$
(3) Bisect X into two sub-boxes X_1, X_2
(4) Evaluate $Y_i = f(X_i), \ i = 1, 2$
(5) Append X_i to S, if $f(X_i) \subset Y_{\text{target}}$
(6) Append X_i to N, if $f(X_i) \subset \overline{Y}_{\text{target}}$
(7) Append X_i to E, if $f(X_i) \cap Y_{\text{target}} \neq \varnothing$
(8) Terminate, if $X_i \in \text{E}$ is sufficiently small
(9) Otherwise, keep bisecting each X_i, go to (3) and set $X := X_i$.

The algorithm described here will output three lists of sub-boxes each belonging to either S, N, E. The overall size of the sub-boxes in E is an indicator of the accuracy of the subtiling, and it is merely determined by the number of evaluations of f. In what follows the algorithm is used to address a reliability problem, so it will need some adaptation.

2.2. *Reliability analysis with SIVIA*

The probability that a *continuous* random variable Y (element of a Borel topological space), is greater than a given threshold y_t, in formulas: $\mathbb{P}_Y(\text{Y} > y_t)$, is called a failure probability when associated with a rare undesirable event. Let $f : \mathbb{R}^d \to \mathbb{R}$, be a function such that $y = f(x)$. A failure probability is the probability that the output random variable Y is in the interval $[y_t, \infty)$, that is $p_{\text{F}} = \mathbb{P}_Y(\text{Y} \in [y_t, \infty))$. We will denote such undesirable event by $Y_{\text{target}} = [y_t, \infty)$. In reliability problems, the analytical closed-form distribution of Y is unknown, because f is often complex. For this reason, and because the distribution of the random variable vector X is known, it is often preferable to compute the failure probability as follows: $p_{\text{F}} = \mathbb{P}_X(\text{X} \in \Omega_{\text{F}})$, where Ω_{F} is the preimage of Y_{target} over f, which is also called *failure domain*. Running the SIVIA algorithm on the target: $Y_{\text{target}} = [y_t, \infty)$, will output a rigorous subtiling of the failure domain. Note that SIVIA need not have a bounded target in order to proceed. A lower subtiling of the failure domain Ω_F is obtained after progressive bisection of X, as follows:

$$\underline{\Omega_{\text{F}}} = \bigcup_{i}^{n} \{X_i : f(X_i) \subset Y_{\text{target}}\}, \quad (3)$$

where $X_i, \ i = 1, ..., n$ are sub-boxes as a result of the progressive bisecting. The subtiling (3) is called an inner subtiling because a subset of the actual failure domain, $\underline{\Omega_{\text{F}}} \subset \Omega_{\text{F}}$. The SIVIA algorithm also outputs the following subtiling

$$\Omega_{\text{E}} = \bigcup_{i}^{n} \{X_i : f(X_i) \cap Y_{\text{target}} \neq \varnothing, X_i \notin \underline{\Omega_{\text{F}}}\}, \quad (4)$$

which is an outer approximation of the boundary between the failure domain and its complement. The more computational investment is put in SIVIA, the smaller Ω_E is in size, thus the more accurate is this subtiling. Because the subtilings of (3) and (4) are the union of a finite number of boxes, their probability measure can be computed conveniently on each sub-box. Ideally, the volume of the subtiling Ω_{E} tends to zero, when the failure domain is exactly represented. However, this is only possible in theory, i.e. with an infinite number of sub-boxes and function evaluations. The lower bound failure probability $\underline{p_{\mathrm{F}}} = \mathbb{P}_X(\underline{\Omega_{\mathrm{F}}})$, is

$$\mathbb{P}_X(\underline{\Omega_{\mathrm{F}}}) = \sum_i^n \mathbb{P}_X\left(\{X_i : f(X_i) \subset Y_{\text{target}}\}\right), \quad (5)$$

whilst the upper failure probability is

$$\overline{p_{\mathrm{F}}} = \mathbb{P}_X(\underline{\Omega_{\mathrm{F}}}) + \mathbb{P}_X(\Omega_{\mathrm{E}}) \quad (6)$$

where,

$$\mathbb{P}_X(\Omega_{\mathrm{E}}) = \sum_i^n \mathbb{P}_X(\{X_i : f(X_i) \cap Y_{\text{target}} \neq \varnothing, X_i \notin \underline{\Omega_{\mathrm{F}}}\}).$$

The probability measure on each individual box is computed by means of the H-volume, knowing the joint probability distribution H_X. In the next section, how this measure, inclusive of considering correlation, is computed is further explored.

2.3. *Rigorous integration of uncertainty*

No constraint in shape, dimension, or dependence is placed on the input probability distribution H_X (and subsequently $\mathbb{P}_X$), other than that we know it precisely. The lower (5) and upper (6) bounds on the measure could be computed from the probability density function h_X, by integrating it on the subtiling perhaps using quadrature, e.g.

$$\underline{p_{\mathrm{F}}} = \sum_i^n \mathbb{P}_X(S_i) = \sum_i^n \int_{S_i} h_X dx, \quad (7)$$

where $S_i = \{X_i : f(X_i) \subset Y_{\text{target}}\}$. Computing (7) by quadrature however, will only produce an approximation of both probability bounds, whilst we seek a method which is also rigorous in the probability estimation. A simpler approach is to directly use the cumulative distribution function (cdf) H_X. The cdf is related to the probability measure by $H_X(x) = \mathbb{P}_X(X \leqslant x)$, and as such it is simple to compute the measure on a hyper-rectangle S by evaluating the cdf on the vertices and performing an iterative sum

$$V_{H_X}(S) = \sum_{s \in \text{vertices}(S)} \text{sign}_S(s) H_X(s), \quad (8)$$

where the sum is taken over all the vertices of S, and where

$$\text{sign}_S(s) = \begin{cases} 1 & \text{if } \#\text{le}(s) \text{ is even} \\ -1 & \text{if } \#\text{le}(s) \text{ is odd.} \end{cases} \quad (9)$$

The function V_{H_X} is called the H-volume by Schweizer and Sklar (2011), and generalises our usual notion of volume. It is defined for measures other than the Lebesgue measure (uniform distribution), for example the multivariate distribution H_X. It is also defined for sets larger than three dimensions, e.g., when using the Lebesgue measure in 1D it gives length, area in 2D, volume in 3D, and hyper-volume beyond that. In (9), $\#\text{le}(s)$ is the number of left endpoints of the vertex s. As an example, in two dimensions the (hyper-)rectangle $S = [\underline{s}_1, \overline{s}_1] \times [\underline{s}_2, \overline{s}_2]$ has four vertices $(\underline{s}_1, \underline{s}_2)$, $(\underline{s}_1, \overline{s}_2)$, $(\overline{s}_1, \underline{s}_2)$, and $(\overline{s}_1, \overline{s}_2)$, which each has 2 (even), 1 (odd), 1 (odd), and 0 (even) number of left endpoints $\#$le, respectively. The two dimensional calculation is thus

$$\begin{aligned} V_{H_X}([\underline{s}_1, \overline{s}_1] \times [\underline{s}_2, \overline{s}_2]) = \\ H_X(\overline{s}_1, \overline{s}_2) - H_X(\overline{s}_1, \underline{s}_2) \\ -H_X(\underline{s}_1, \overline{s}_2) + H_X(\underline{s}_1, \underline{s}_2). \end{aligned} \quad (10)$$

The probability measure of each of the sub-boxes S_i may be computed using (8), and since they are all disjoint and $\mathbb{P}_X$ is precise (and thus additive), the total failure probability is the sum of the measures of each sub-box

$$\underline{p_{\mathrm{F}}} = \sum_i^n V_{H_X}(S_i). \quad (11)$$

An identical calculation is performed for the upper bound (6). Note that this calculation can be performed directly on the copula C of H_X, since using Sklar's theorem the joint distribution H_X can be expressed in terms of C and marginals H_{X_j}

$$H_X(x) = C(H_{X_1}(x_1), \ldots, H_{X_d}(x_d)), \quad (12)$$

where C captures all dependence information of the random variable X. The integration can be performed directly on the copula by iso-probabilistically transforming each sub-box S_i

through the inverse cdfs of the marginal distributions, e.g.,

$$U_i = \bigtimes_{j=1}^{d} [H_{X_j}^{-1}(\underline{s}_{ji}), H_{X_j}^{-1}(\overline{s}_{ji})],$$

where j indexes the dimension and i the subboxes, and evaluating

$$\underline{p_{\mathrm{F}}} = \sum_{i}^{n} V_C(U_i). \quad (13)$$

2.4. *Performance function*

The function f is called a performance function g, when the failure domain is determined by its sign or by a fixed interval, such as $[0,1]$ or $[1,\infty]$. In this paper, the failure domain will be denoted by $\Omega_F = \{x : g(x) \leqslant 0\}$.

3. Application

In this section, set-inversion reliability analysis is applied to a few problems form the so called TNO reliability challenge, which is a set of reliability challenge problems released by Rozsas et al. (2018). The problems can be consulted publicly at https://rprepo.readthedocs.io/.
Among these problems we select a few for the purpose of demonstrating the proposed method.

3.1. *Nonlinear system with failure islands (RP57)*

This reliability problem displays multiple failure islands, as shown in Figure 1. A summary of this problem specifications is given in Table 2. The performance function is a multi-output nonlinear function provided as a combination of parallel and series functions. The performance function $g : \mathbb{R}^2 \to \mathbb{R}^3 \to \mathbb{R}$ has the following form:

$$\begin{aligned} g_1(x) &= -x_1^2 + x_2^3 + 3 \\ g_2(x) &= 2 - x_1 - 8x_2 \\ g_3(x) &= (x_1+3)^2 + (x_2+3)^2 - 4 \\ g(x) &= \min(\max(g_1, g_2), g_3) \end{aligned}$$

Figure 1 shows the isodensities of the joint standard normal with peak density at (0,0). The blue domain (or safe domain) is subtiled with boxes on which the model is guaranteed to evaluate in a positive interval $g > 0$; while the red-box region is where the model is certainly negative $g < 0$. The

Table 2. Reliability problem *RP57*

Target p_{F}	# R.v.	# P. functions	Distributions
2.84e-2	2	3	$N(0,1)$

Table 3. Set-inversion reliability results *RP57*

# Evaluations	p_{F} bounds (10^{-2})	Width (10^{-2})
456	[0.44, 11.61]	11.17
1996	[1.91, 3.80]	1.19
4046	[2.32, 3.32]	1.00
10000	[2.60, 3.06]	0.46

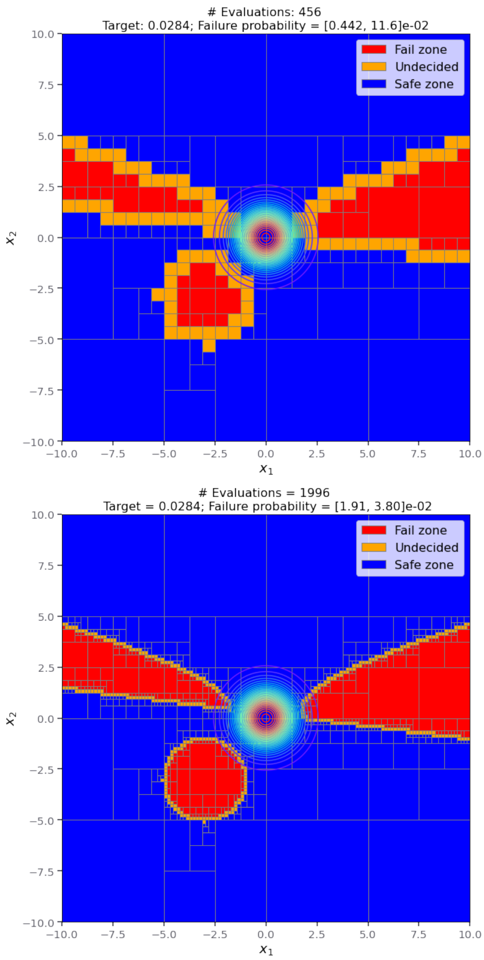

Fig. 1. Challenge problem *RP57* obtained with 456 and 1996 function evaluations. The upper figure corresponding to 456 evaluations shows a coarser subtiling (size of the orange boxes), resulting in a broader failure probability interval.

region where the algorithm is undecided (orange boxes), are an outer subtiling of the failure boundary. On the orange boxes, the interval evaluation results in a value of the performance that straddles zero: $0 \in g$. The results for increasing number of function evaluations are provided in Table 3.

3.2. *Nonlinear series system (RP35)*

This reliability problem is a series system with two nonlinear performance functions. From the subtiling shown in Figure 2, the geometry and shape of the failure domain can be appreciated. The problem displays two predominant failure modes. A summary of the problem specifications is given in Table 4, while results for increasing number of function evaluations are in Table 5.

Table 4. Reliability problem *RP35*

Target p_F	# R.v.	# P. functions	Distributions
3.54e-2	2	2	$N(0,1)$

The performance function of problem RP35 $g : \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$ has the following form:

$$\begin{aligned} g_1(x) &= 2 - x_2 + \exp(-0.1\, x_1^2) + (0.2\, x_1)^4 \\ g_2(x) &= 4.5 - x_1 x_2 \\ g(x) &= \min(g_1, g_2) \end{aligned}$$

Table 5. Set-inversion reliability results *RP35*

# Evaluations	p_F bounds (10^{-3})	Width (10^{-3})
374	[1.56, 17.13]	15.57
1584	[2.44, 4.71]	2.27
3198	[2.93, 4.07]	1.14
10000	[3.26, 3.73]	0.47

3.3. *Effect of correlation*

Correlation can dramatically change the results presented in the previous two examples.

The effect of correlation is explored on the previous two examples, introducing a Gaussian copula with a few correlation coefficients. As expected, The failure probability changes by orders of magnitude when the correlation coefficient is varied, as shown in Table 6.

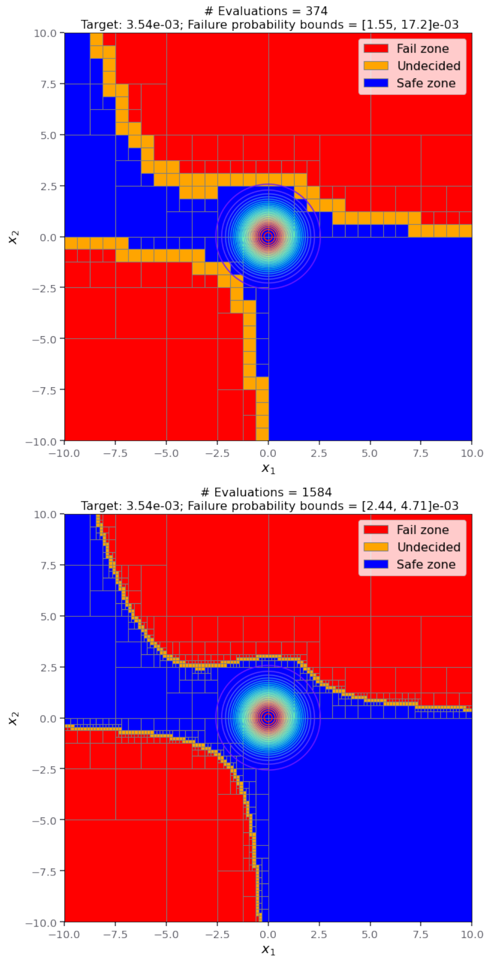

Fig. 2. Challenge problem *RP35* obtained with 374 and 1584 function evaluations. The upper figure corresponding to 374 evaluations shows a coarser subtiling (size of the orange boxes), resulting in a broader failure probability interval.

Figure 3 shows the isodensities with correlation $\rho = -0.99$ and $\rho = 0.99$ for problem *RP57*. From the isodensities shown in Figure 3, it appears clear why such a dramatic change in the failure probability is found. Note that because of the separation between the reliability and the engineering problem, the results in Table 6 could be obtained with a single run of SIVIA; in other words, exploring different correlation coefficients has been done “for free”, i.e. requiring repeated model evaluations for different values of ρ. This is possible thanks to the aforementioned separation, because once the subtiling is completed, the prob-

Table 6. Failure probability bounds for problems *RP57* and *RP35* with varying correlation coefficient. The results have been obtained with a total of 10^4 evaluations. Values are reported using outward rounding with 3 signif. digits.

ρ	p_{F} *RP57*	*RP35* p_{F}
-0.99	[0.0,0.0]	[0.00454, 0.00513]
-0.95	[4.29, 9.48]e-05	[0.00434, 0.00489]
-0.80	[0.00433, 0.00584]	[0.00363, 0.00410]
-0.50	[0.0127,0.0159]	[0.00252, 0.00287]
-0.30	[0.0175,0.0214]	[0.00218, 0.00251]
indep. (0)	[0.0260, 0.0306]	[0.00325, 0.00373]
0.30	[0.0356, 0.0407]	[0.00746, 0.00834]
0.50	[0.0417, 0.0469]	[0.0123, 0.0137]
0.80	[0.0476, 0.0522]	[0.0230, 0.0250]
0.95	[0.0516, 0.0550]	[0.0300, 0.0324]
0.99	[0.0541, 0.0575]	[0.0320, 0.0345]

ability measure can be computed with any given probability distribution.

4. Discussion

The results presented in this paper show some common features. The width of the error (or imprecision) in approximating the failure probability decreases with the number of function evaluations. So the more computational effort is invested the tighter the failure probability bound. When considering unbounded distributions, extra care needs to be placed to choose the size of the initial box. The initial box needs to be large enough to include all the boxes with a non-negligible probability measure. This can be done in each coordinate dimension by using the quantile function of each marginal distribution. If the initial box excludes regions with non-negligible probability measures, rigour may be compromised. For unbounded distributions, it is advised to start the algorithm with an initial box that stretches far beyond the 10^{-8} and $1-10^{-8}$ percentile; this is because the size of the initial box only marginally affects the subtiling process, and the efficiency of the algorithm. Even though the presented examples have mainly shown standard normal distributions, there is no restriction about the distribution family. The examples are taken from a repository of reliability challenge problems, and have been selected to showcase complexity rather than exotic distributions.

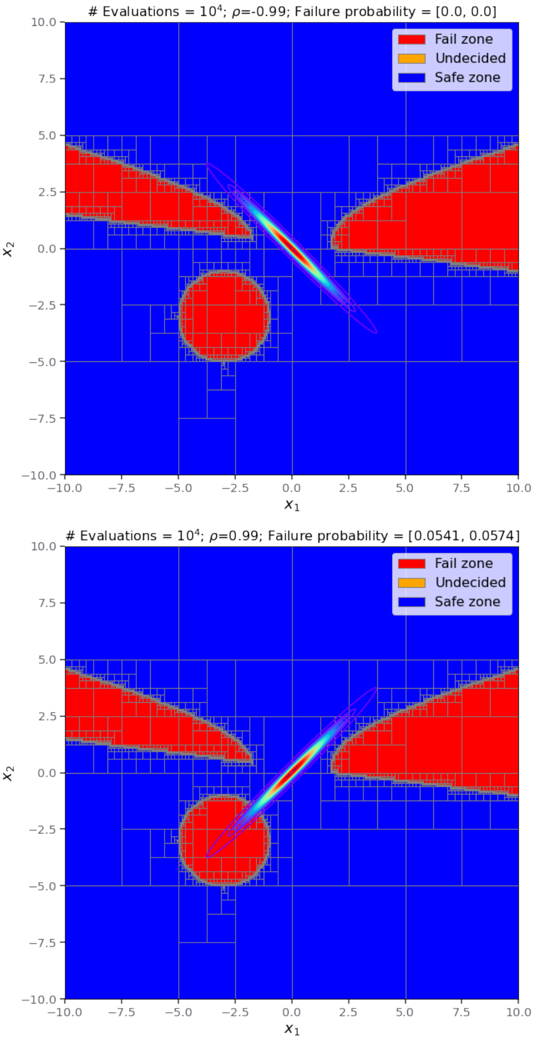

Fig. 3. Challenge problem *RP57* obtained with correlation coefficients $\rho = -0.99$ (upper figure), and $\rho = 0.99$ (lower figure).

We note that this method assumes that the input joint cdf H_X is exactly (rigorously) evaluable. Often such functions are not known analytically, as is the case with the multivariate normal cdf, and subsequently the Gaussian copula, and require a numerical method of some form. In this work, the multivariate cdfs is evaluated with the algorithm from Genz (1992), which although is considered an industry standard, is not rigorous and uses a mix of quadrature and Monte Carlo. The bounds obtained from the independent examples are rigorous, as they use the independent copula which is analytic. The correlated examples using the Gaussian copula rely on this non-rigorous algorithm, and as such may be subject to slight variation when re-evaluated. We emphasise that this paper presents a method to rigorously compute probabilities from a *known* H_X, and does not comment on how H_X is rigorously computed.

4.1. *Efficiency*

While SIRE is insensitive to the nonlinearity and the magnitude of the failure probability, its major barrier to efficiency is the cardinality of the input variable set. For convenience, we divide all reliability problems into four macro classes based on their input set cardinality: 1-5 small, 6-10 moderate, 11-25 large, $>$25 grand (or extra large). With this classification, the efficiency of the method can be better framed. The method as presented in this paper, can only be used on small to moderate reliability problems. Modification of the method could be used on large reliability problems if rigour is compromised. On small to moderate reliability problems, the efficiency of the method can be improved: (1) with a bisection criterion based on the magnitude of the probability measure in each subbox; (2) by improving the interval computations to minimise inflated uncertainty; (3) by burn-in subintervalisation combined with parallel processing. On moderate to large reliability problems, the SIVIA algorithm cannot terminate due to the limitations of subtiling high-dimensional spaces. Nonetheless, the algorithm can still be used with a termination criterion based on a maximum total number of evaluations. In this scenario, even though the subtiling is not satisfactory, the information from such partial subtiling can still be used to characterise the geometry of the boundary of the failure domain, a.k.a. limit-state boundary. This information can then be used to deploy other algorithms, like importance sampling or line sampling simulation that typically require cues about the geometry of such a boundary.

5. Reproducibility

The code and algorithms used in this document are available at

`https://github.com/marcodeangelis/SIRE`.

Interval computations were run using *intervals* `https://github.com/marcodeangelis/intervals` a code library for interval computing in Python. Both repositories were last accessed in June 2022.

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