Nonlinear Behaviors of First and Second Order Complex Digital Filters with Two's Complement Arithmetic

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***Abstract*—For first order complex digital filters with two's complement arithmetic, it is proved in this paper that overflow does not occur at the steady state if the eigenvalues of the system matrix are inside or on the unit circle. However, if the eigenvalues of the system matrix are outside the unit circle, chaotic behaviors would occur. For both cases, a limit cycle behavior does not occur. For second order complex digital filters with two's complement arithmetic, if all eigenvalues are on the unit circle, then there are two ellipses centered at the origin of the phase portraits when overflow does not occur. When limit cycle occurs, the number of ellipses exhibited on the phase portraits is no more than two times the periodicity of the symbolic sequences. If the symbolic sequences are aperiodic, some state variables may exhibit fractal behaviors, at the same time, irregular chaotic behaviors may occur in other phase variables.**

***Index Terms*—Complex digital filters, two's complement arithmetic, limit cycle behaviors, fractal behaviors, chaotic behaviors.**

I. Introduction

FOR practical reasons, digital filters are commonly implemented in hardware using two's complement arithmetic for the addition operation. Because of the adder overflow, the physical digital filters are actually nonlinear discrete time systems. Some strange phenomena, such as the occurrence of chaotic behaviors when all eigenvalues of the system matrix being *inside* the unit circle [1], as well as the convergence of state vectors to some fixed points or the occurrence of limit cycle behaviors when all eigenvalues of the system matrix being *outside* the unit circle [2]-[3], would occur in real digital filters with two's complement arithmetic. In this paper, we would investigate whether these strange phenomena would occur in complex digital filters with two's complement arithmetic. The differences between the real and complex digital filters with two's complement arithmetic are compared.

For complex digital filters with two's complement arithmetic, the order of the system matrices and the number of symbolic variables are doubled, as well as these matrices are no longer realized in the direct form. Hence, all the existing results for real digital filters with two's complement arithmetic [1]-[4] do not apply. In fact, behaviors of real and complex digital filters with two's complement arithmetic could be very different. For example, it will be shown in this paper that chaotic behavior never occurs if the eigenvalues of the system matrix are inside or on the unit circle when the system matrix is realized in the normal form, while it was reported in [1]-[4] that chaotic behaviors may occur for same eigenvalues when the system matrix is realized in the direct form.

Recently, some analysis has been done on second order complex digital filters with two's complement arithmetic [5]. In [5], a sufficient condition for the asymptotic stability of second order digital filters with two's complement arithmetic, as well as bounds on limit cycles, have been derived. Although complex signals were considered in [5], only real filter coefficients were considered. Also, only fixed point and limit cycle behaviors have been investigated. In this paper, we would generalize the analysis in [5] in such a way that the filter coefficients are complex, and chaotic behaviors will be explored.

The outline for this paper is as follows: the results on the first order and second order complex digital filters with two's complement arithmetic are, respectively, represented in Section II and Section III. Finally, a conclusion is presented in Section IV.

II. First Order Complex Digital Filters with Two's Complement Arithmetic

Consider the following first order difference equation:

$$y(k)+ay(k-1)=u(k), \text{ for } k\geq 0, \quad (1)$$

where a is a complex number, $u(k)$ and $y(k)$ are, respectively, the input and output of the difference equation. In this paper, we assume that $u(k)=0$ for $k\geq 0$ and the whole system is only influenced by the initial state. Let $y_{real}(k)$ and $y_{imag}(k)$ be, respectively, the real and imaginary parts of $y(k)$; a_{real} and

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a_{imag} be, respectively, the real and imaginary parts of a. Then (1) implies that:

$$y_{real}(k)+a_{real}y_{real}(k-1)-a_{imag}y_{imag}(k-1)=0, \text{ for } k\geq 0, \quad (2)$$

and

$$y_{imag}(k)+a_{imag}y_{real}(k-1)+a_{real}y_{imag}(k-1)=0, \text{ for } k\geq 0. \quad (3)$$

For practical implementation of (2) and (3), accumulators with two's complement arithmetic are employed for the addition operation. Define

$$\mathbf{x}(k)\equiv[x_1(k)\quad x_2(k)]^T\equiv[y_{real}(k-1)\quad y_{imag}(k-1)]^T.$$

Then the complex digital filter with two's complement arithmetic can be implemented by the following state space equation:

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+2\mathbf{s}(k), \text{ for } k\geq 0, \quad (4)$$

where $\mathbf{A}\equiv\begin{bmatrix}-a_{real} & a_{imag}\\ -a_{imag} & -a_{real}\end{bmatrix}$ is represented in the normal form, and $\mathbf{s}(k)\equiv\begin{bmatrix}s_1(k)\\ s_2(k)\end{bmatrix}$ is the vector consists of the corresponding symbolic sequences $s_i(k)\in Z$ for $i=1,2$ and Z denotes the set of integers. Define r and θ in such a way that $a_{real}=-r\cos\theta$ and $a_{imag}=-r\sin\theta$. Let the eigenvalues of $\mathbf{A}$ be, respectively, λ_1 and λ_2. Then it can be shown easily that $\lambda_1=re^{j\theta}$ and $\lambda_2=re^{-j\theta}$. Denoting Z^+ as the set of positive integers, we have the following lemmas:

Lemma 1

For $r\leq 1$ and $\forall\mathbf{x}(0)\in I^2\equiv[-1,1)\times[-1,1)$, $\exists k_1\in Z^+\bigcup\{0\}$ such that $\mathbf{s}(k)=\mathbf{0}$ for all $k\geq k_1$.

Proof:

Since $r\leq 1$,

$$\mathbf{s}(k)\in\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}-1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix}\right\}.$$

Consider the case when $\mathbf{s}(k_0)=\mathbf{0}$. Denoting $\mathbf{x}(k_0)\equiv\rho\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}$, then $\mathbf{x}(k_0+1)=r\rho\begin{bmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{bmatrix}$ and we have $\|\mathbf{x}(k_0+1)\|_2=r\rho\leq\rho=\|\mathbf{x}(k_0)\|_2$. Now consider the case when $\mathbf{s}(k_0)=\begin{bmatrix}-1\\0\end{bmatrix}$. We have $\mathbf{x}(k_0+1)=r\rho\begin{bmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{bmatrix}-\begin{bmatrix}2\\0\end{bmatrix}$ and $\|\mathbf{x}(k_0+1)\|_2=\sqrt{r^2\rho^2+4(1-r\rho\cos(\theta+\phi))}$. Since $r\rho\cos(\theta+\phi)>1$ because of the occurrence of overflow, we have $\|\mathbf{x}(k_0+1)\|_2<r\rho\leq\rho=\|\mathbf{x}(k_0)\|_2$. For other values of $\mathbf{s}(k_0)$, using the same argument, it can be shown that $\|\mathbf{x}(k_0+1)\|_2<r\rho\leq\rho=\|\mathbf{x}(k_0)\|_2$. Hence $\|\mathbf{x}(k_0+n)\|_2\leq r^n\rho$ for $n\geq 0$. For $r<1$, there exists $n_0\geq 0$ such that $r^{n_0}\rho<1$, so $\mathbf{s}(k)=\mathbf{0}$ for $k\geq k_0+n_0$. For $r=1$, since $\|\mathbf{x}(k_0+1)\|_2=\|\mathbf{x}(k_0)\|_2$ only when $\mathbf{s}(k_0)=\mathbf{0}$ and $0\leq\|\mathbf{x}(k_0+1)\|_2<\|\mathbf{x}(k_0)\|_2$ for $\mathbf{s}(k_0)\neq\mathbf{0}$, so there exists $n_0\geq 0$ such that $\mathbf{s}(k)=\mathbf{0}$ for $k\geq k_0+n_0$. This completes the proof. ■

Lemma 1 tells us that overflow does not occur at the steady state if $r\leq 1$, even though it may occur during the transient state. This result is different from that discussed in [1] in which chaotic behaviors may occur even though the eigenvalues of these two system matrices are the same. This is because the occurrence of nonlinear behaviors depends on the realization of the system. In [1], the system matrix is realized in the direct form, while in this paper it is realized in the normal form.

Now, let's consider the case when the system matrix is unstable.

Lemma 2

For $r>1$, there does not exist $k_0\in Z^+\bigcup\{0\}$, $M\in Z^+$ and $\mathbf{x}(k_0)\in I^2\setminus\{\mathbf{0}\}$ such that $\mathbf{s}(k)=\mathbf{s}(k+M)$ for $k\geq k_0$.

Proof:

Suppose $\exists k_0\in Z^+\bigcup\{0\}$, $M\in Z^+$ and $\mathbf{x}(k_0)\in I^2\setminus\{\mathbf{0}\}$ such that $\mathbf{s}(k)=\mathbf{s}(k+M)$ for $k\geq k_0$. Define $\mathbf{v}\equiv 2\sum_{j=0}^{M-1-j}\mathbf{A}^{M-1-j}\mathbf{s}(k_0+j)$ and $\mathbf{D}\equiv\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$. Let $\mathbf{T}$ be a 2×2 matrix such that $\mathbf{A}=\mathbf{T}\mathbf{D}\mathbf{T}^{-1}$. Then we have:

$$\begin{aligned}\mathbf{x}(k_0+kM)&=\mathbf{A}^{kM}\mathbf{x}(k_0)+2\sum_{j=0}^{kM-1-j}\mathbf{A}^{kM-1-j}\mathbf{s}(k_0+j)\\ &=\mathbf{T}\begin{bmatrix}\lambda_1^{kM} & 0\\ 0 & \lambda_2^{kM}\end{bmatrix}\mathbf{T}^{-1}\mathbf{x}(k_0)+\mathbf{T}\begin{bmatrix}\frac{1-\lambda_1^{kM}}{1-\lambda_1^{M}} & 0\\ 0 & \frac{1-\lambda_2^{kM}}{1-\lambda_2^{M}}\end{bmatrix}\mathbf{T}^{-1}\mathbf{v}\end{aligned}, \quad (5)$$

for $k\geq 1$. Since M is finite, $\mathbf{v}$ is a vector with finite magnitude. Since $\mathbf{T}$ and $\mathbf{x}(k_0)$ are independent of r, if $\mathbf{T}\begin{bmatrix}\lambda_1^{kM} & 0\\ 0 & \lambda_2^{kM}\end{bmatrix}\mathbf{T}^{-1}\mathbf{x}(k_0)\neq\mathbf{0}$, then the first term will grow faster than the second term because $r>1$. As a result, $\mathbf{x}(k)$ will eventually unbound. As

$$\mathbf{T}\begin{bmatrix}\lambda_1^{kM} & 0\\ 0 & \lambda_2^{kM}\end{bmatrix}\mathbf{T}^{-1}\mathbf{x}(k_0)=r^{kM}\rho\begin{bmatrix}\cos(kM\theta+\phi)\\ \sin(kM\theta+\phi)\end{bmatrix}\neq\mathbf{0}$$

for $\mathbf{x}(k_0)\neq\mathbf{0}$, so $\mathbf{x}(k)$ will eventually unbound. However, $\mathbf{x}(k)\in I^2$ for $k\geq 0$, there is a contradiction. This implies that for $r>1$, there does not exist $k_0\in Z^+\bigcup\{0\}$, $M\in Z^+$ and $\mathbf{x}(k_0)\in I^2\setminus\{\mathbf{0}\}$ such that $\mathbf{s}(k)=\mathbf{s}(k+M)$ for $k\geq k_0$, and it completes the proof. ■

Lemma 2 says that the symbolic sequences are aperiodic for $r>1$ no matter what values of θ and initial conditions are (except the case when the initial state is at the origin). Hence, chaotic behaviors occur and the trajectory will neither converge to some fixed points nor exhibit limit cycle behaviors. This result is also different from that discussed in [2]-[3], in which the trajectory may converge to some fixed points or limit cycle behaviors occur even though the eigenvalues of these two system matrices are the same. This is because these two system matrices are realized in different forms. The importance of Lemma 1 and 2 is that they provide information for engineers to avoid or utilize chaotic behaviors because chaotic behaviors can be guaranteed to be avoided if the system matrix is stable and occurred only if the system matrix is unstable, which is independent of the initial conditions (except zero initial condition).

III. Second Order Complex Digital Filters with Two's Complement Arithmetic

Now, consider the following second order difference equation:

$$y(k)+a\,y(k-1)+b\,y(k-2)=u(k),\ \text{for } k\geq 0, \tag{6}$$

where a and b are complex numbers and $u(k)=0$ for $k\geq 0$. Let b_{real} and b_{imag} be, respectively, the real and imaginary parts of b. Then we have:

$$\begin{aligned}&y_{real}(k)+a_{real}\,y_{real}(k-1)-a_{imag}\,y_{imag}(k-1)\\&+b_{real}\,y_{real}(k-2)-b_{imag}\,y_{imag}(k-2)=0\end{aligned}, \tag{7}$$

for $k\geq 0$, and

$$\begin{aligned}&y_{imag}(k)+a_{imag}\,y_{real}(k-1)+a_{real}\,y_{imag}(k-1)\\&+b_{imag}\,y_{real}(k-2)+b_{real}\,y_{imag}(k-2)=0\end{aligned}, \tag{8}$$

for $k\geq 0$. Define

$$\begin{aligned}\mathbf{x}(k)&\equiv[x_1(k)\quad x_2(k)\quad x_3(k)\quad x_4(k)]^T\\&\equiv[y_{real}(k-2)\quad y_{real}(k-1)\quad y_{imag}(k-2)\quad y_{imag}(k-1)]^T\end{aligned};$$

the second order complex digital filter with two's complement arithmetic can be represented by the following state space equation:

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+2\mathbf{B}\mathbf{s}(k),\ \text{for } k\geq 0, \tag{9}$$

where

$$\mathbf{A}\equiv\begin{bmatrix}0&1&0&0\\-b_{real}&-a_{real}&b_{imag}&a_{imag}\\0&0&0&1\\-b_{imag}&-a_{imag}&-b_{real}&-a_{real}\end{bmatrix},\quad \mathbf{B}\equiv\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}\ \text{and}$$

$\mathbf{s}(k)\equiv\begin{bmatrix}s_1(k)\\s_2(k)\end{bmatrix}$. It is worth noting that the system matrix $\mathbf{A}$ is neither represented in the direct form nor in the normal form. Let the eigenvalues of $\mathbf{A}$ be λ_i, for $i=1,2,3,4$. Then it can be shown easily that:

$$\lambda_1=\frac{a_{imag}\,j-a_{real}+\sqrt{a_{real}^{\ 2}-a_{imag}^{\ 2}-4b_{real}-2\,j\,a_{real}\,a_{imag}+4\,j\,b_{imag}}}{2}, \tag{10}$$

$$\lambda_2=\frac{a_{imag}\,j-a_{real}-\sqrt{a_{real}^{\ 2}-a_{imag}^{\ 2}-4b_{real}-2\,j\,a_{real}\,a_{imag}+4\,j\,b_{imag}}}{2}, \tag{11}$$

$$\lambda_3=\frac{-a_{imag}\,j-a_{real}+\sqrt{a_{real}^{\ 2}-a_{imag}^{\ 2}-4b_{real}+2\,j\,a_{real}\,a_{imag}-4\,j\,b_{imag}}}{2}, \tag{12}$$

and

$$\lambda_4=\frac{-a_{imag}\,j-a_{real}-\sqrt{a_{real}^{\ 2}-a_{imag}^{\ 2}-4b_{real}+2\,j\,a_{real}\,a_{imag}-4\,j\,b_{imag}}}{2}. \tag{13}$$

In this paper, we only consider the case when $|\lambda_i|=1$ for $i=1,2,3,4$. This implies that there exists $\theta_1\in[-\pi,\pi]$ and $\theta_2\in[-\pi,\pi]$ such that the set of eigenvalues $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ can be represented by the set $\{e^{j\theta_1},e^{-j\theta_1},e^{j\theta_2},e^{-j\theta_2}\}$. Assuming that $\mathbf{A}$ is diagonalizable, then there exists a real matrix $\mathbf{T}\equiv\begin{bmatrix}\mathbf{T}_1&\mathbf{T}_2\\\mathbf{T}_3&\mathbf{T}_4\end{bmatrix}$ and another real matrix $\mathbf{R}\equiv\begin{bmatrix}\mathbf{R}_1&\mathbf{0}\\\mathbf{0}&\mathbf{R}_2\end{bmatrix}$ such that $\mathbf{A}=\mathbf{T}\mathbf{R}\mathbf{T}^{-1}$, $\mathbf{R}_1\equiv\begin{bmatrix}\cos\theta_1&\sin\theta_1\\-\sin\theta_1&\cos\theta_1\end{bmatrix}$, $\mathbf{R}_2\equiv\begin{bmatrix}\cos\theta_2&\sin\theta_2\\-\sin\theta_2&\cos\theta_2\end{bmatrix}$ and

$$\mathbf{x}(n)=\mathbf{A}^n\,\mathbf{x}(0)+2\sum_{k=0}^{n-1}\mathbf{A}^{n-k-1}\mathbf{B}\mathbf{s}(k),\ \text{for } n\geq 1. \tag{14}$$

A. Linear case

Lemma 3

If $\mathbf{s}(k)=\mathbf{0}$ for $k\geq 0$, then by plotting the phase portraits $x_i(k)$ against $x_j(k)$ for $i\neq j$, there are two ellipses centered at the origin exhibited on the phase portraits.

Proof:

Since

$$\mathbf{T}^{-1}=\begin{bmatrix}-\mathbf{T}_3^{-1}\mathbf{T}_4\left(\mathbf{T}_2-\mathbf{T}_1\mathbf{T}_3^{-1}\mathbf{T}_4\right)^{-1}&-\mathbf{T}_1^{-1}\mathbf{T}_2\left(\mathbf{T}_4-\mathbf{T}_3\mathbf{T}_1^{-1}\mathbf{T}_2\right)^{-1}\\\left(\mathbf{T}_2-\mathbf{T}_1\mathbf{T}_3^{-1}\mathbf{T}_4\right)^{-1}&\left(\mathbf{T}_4-\mathbf{T}_3\mathbf{T}_1^{-1}\mathbf{T}_2\right)^{-1}\end{bmatrix} \tag{15}$$

and $\mathbf{s}(k)=\mathbf{0}$ for $k\geq 0$, then

$$\mathbf{x}(k)=\begin{bmatrix}\mathbf{T}_2\mathbf{R}_2^{\ k}\mathbf{p}_1-\mathbf{T}_1\mathbf{R}_1^{\ k}\mathbf{p}_2\\\mathbf{T}_4\mathbf{R}_2^{\ k}\mathbf{p}_1-\mathbf{T}_3\mathbf{R}_1^{\ k}\mathbf{p}_2\end{bmatrix}, \tag{16}$$

where

$$\begin{aligned}\mathbf{p}_1&\equiv\left(\mathbf{T}_2-\mathbf{T}_1\mathbf{T}_3^{-1}\mathbf{T}_4\right)^{-1}[x_1(0)\quad x_2(0)]^T\\&+\left(\mathbf{T}_4-\mathbf{T}_3\mathbf{T}_1^{-1}\mathbf{T}_2\right)^{-1}[x_3(0)\quad x_4(0)]^T\end{aligned} \tag{17}$$

and

$$\begin{aligned}\mathbf{p}_2&\equiv\mathbf{T}_3^{-1}\mathbf{T}_4\left(\mathbf{T}_2-\mathbf{T}_1\mathbf{T}_3^{-1}\mathbf{T}_4\right)^{-1}[x_1(0)\quad x_2(0)]^T\\&+\mathbf{T}_1^{-1}\mathbf{T}_2\left(\mathbf{T}_4-\mathbf{T}_3\mathbf{T}_1^{-1}\mathbf{T}_2\right)^{-1}[x_3(0)\quad x_4(0)]^T\end{aligned}. \tag{18}$$

Since $\mathbf{p}_1$ and $\mathbf{p}_2$ are real vectors, $\mathbf{R}_1$ and $\mathbf{R}_2$ are rotation matrices, and each of the signals $x_i(k)$ for $i=1,2,3,4$, is a superposition of two sinusoidal signals. Hence, by plotting the phase portrait $x_i(k)$ against $x_j(k)$ for $i\neq j$, there are two ellipses centered at the origin. And this completes the proof. ■

As the orientations of the ellipses depend on the matrices $\mathbf{T}_i$, the orientations of these two ellipses may be different.

B. Limit cycle case

Assume that there exists $M\in Z^+$ such that $\mathbf{s}(k)=\mathbf{s}(k+M)$ for $k\geq 0$. Let

$$\mathbf{x}_0^{\ *}\equiv 2\left(\mathbf{I}-\mathbf{A}^M\right)^{-1}\sum_{j=0}^{M-1}\mathbf{A}^{M-1-j}\mathbf{B}\mathbf{s}(j), \tag{19}$$

$$\mathbf{x}_{i+1}^{\ *}\equiv\mathbf{A}\mathbf{x}_i^{\ *}+2\mathbf{B}\mathbf{s}(i),\ \text{for } i=0,1,\cdots,M-2, \tag{20}$$

and

$$\hat{\mathbf{x}}_i(k)\equiv\mathbf{T}^{-1}\left(\mathbf{x}(kM+i)-\mathbf{x}_i^{\ *}\right),\ \text{for } i=0,1,\cdots,M-1,\ \text{and for } k\geq 0. \tag{21}$$

Then

$$\hat{\mathbf{x}}_i(k+1)=\mathbf{R}^M\,\hat{\mathbf{x}}_i(k),\ \text{for } i=0,1,\cdots,M-1,\ \text{and for } k\geq 0. \tag{22}$$

This implies that $\mathbf{x}(kM+i)=\mathbf{T}\mathbf{R}^{kM}\,\hat{\mathbf{x}}_i(0)+\mathbf{x}_i^{\ *}$, for $i=0,1,\cdots,M-1$, and for $k\geq 0$. The trajectories $\mathbf{x}(k)$ can be grouped into M sub-trajectories $\mathbf{x}(kM+i)$ for $i=0,1,\cdots,M-1$. Denoting

$$\mathbf{x}(kM+i)=[x_1(kM+i)\quad x_2(kM+i)\quad x_3(kM+i)\quad x_4(kM+i)]^T \tag{23}$$

for $i = 0,1,\cdots,M-1$ and for $k \geq 0$, and

$$\mathbf{x}_i^* = \begin{bmatrix} x_{i,1}^* & x_{i,2}^* & x_{i,3}^* & x_{i,4}^* \end{bmatrix}^T \text{ for } i = 0,1,\cdots,M-1, \qquad (24)$$

the plot of the phase portraits of the sub-trajectory $x_m(kM+i)$ against $x_n(kM+i)$ for $m \neq n$ consist of two ellipses centered at $\begin{bmatrix} x_{i,m}^* & x_{i,n}^* \end{bmatrix}^T$ exhibited on the phase portraits. And, the plots of the phase portraits of the trajectory $x_m(k)$ against $x_n(k)$ for $m \neq n$ consist of no more than $2M$ ellipses exhibited on the phase portraits.

Compared to the case of second order real digital filters with two's complement arithmetic [4], there are exactly M ellipses exhibited on the phase portrait. Since $\mathbf{x}_i^* \neq \mathbf{x}_j^*$ for $i \neq j$, these ellipses are distinct. However, for the case of complex digital filters with two's complement arithmetic, it is found that some of the ellipses 'overlapped' in the plot of phase portraits of $x_i(k)$ against $x_j(k)$ for $i \neq j$. However, these 'overlapped' ellipses will correspond to different ellipses in the *other* plots of the phase portraits of $x_m(k)$ against $x_n(k)$, where $m \neq n$ and $m, n \notin \{i, j\}$. For example, when $a_{real} = 0$, $a_{imag} = 0.5$, $b_{real} = -1$, $b_{imag} = 0$ and $\mathbf{x}(0) = [-0.616 \quad 0.616 \quad 0.616 \quad -0.616]^T$, it can be checked easily that $M = 20$. However, by plotting the phase portraits of $x_1(k)$ against $x_2(k)$, and that of $x_3(k)$ against $x_4(k)$, there are only 12 ellipses exhibited on the phase portraits.

C. Chaotic case

When $\mathbf{s}(k)$ is aperiodic, fractal behaviors may exhibit on some phase variables, at the same time, irregular chaotic behaviors may occur in other phase variables. For example, when $a_{real} = 0$, $a_{imag} = 0.5$, $b_{real} = -1$, $b_{imag} = 0$ and $\mathbf{x}(0) = [-0.6135 \quad 0.6135 \quad 0.6135 \quad -0.6135]^T$, it can be checked easily that if $x_1(k)$ is plotted against $x_4(k)$ or $x_2(k)$ is plotted against $x_3(k)$, then fractal patterns may exhibit on these phase portraits. However, random like chaotic patterns are exhibited on the phase portraits plotting other phase variables. This result is also different from that of second order real digital filters with two's complement arithmetic [4], in which only fractal patterns are exhibited on the phase portrait. This is because there are only two state variables in the system.

IV. Conclusion

For first order complex digital filters with two's complement arithmetic, if the eigenvalues of the system matrix are inside or on the unit circle, then overflow does not occur at the steady state. If the eigenvalues are outside the unit circle, then chaotic behavior would occur. For both cases, limit cycle behavior does not occur. For second order complex digital filters with two's complement arithmetic, if all eigenvalues are on the unit circle, then there are two ellipses exhibited on the phase portraits centered at the origin when overflow does not occur. When the symbolic sequences are periodic, the number of ellipses exhibited on the phase portraits is no more than two times the periodicity of the symbolic sequences. If the symbolic sequences are aperiodic, fractal behaviors may exhibit on some phase variables, at the same time, irregular chaotic behaviors may exhibit in other phase variables.

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