A multi-splitting method to solve 2D parabolic reaction-diffusion singularly perturbed systems

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Abstract

In this paper we design and analyze a numerical method to solve a type of reaction-diffusion 2D parabolic singularly perturbed systems. The method combines the central finite difference scheme on an appropriate piecewise uniform mesh of Shishkin type to discretize in space, and the fractional implicit Euler method together with a splitting by directions and components of the reaction-diffusion operator to integrate in time. We prove that the method is uniformly convergent of first order in time and almost second order in space. The use of this time integration technique has the advantage that only tridiagonal linear systems must be solved to obtain the numerical solution at each time step; because of this, our method provides a remarkable reduction of computational cost, in comparison with other implicit methods which have been previously proposed for the same type of problems. Full details of the uniform convergence are given only for systems with two equations; nevertheless, our ideas can be easily extended to systems with an arbitrary number of equations as it is shown in the numerical experiences performed. The numerical results show in practice the qualities of our proposal.

Key words: coupled 2D parabolic systems, reaction-diffusion, fractional step methods, splitting by components, piecewise uniform Shishkin meshes, uniformly convergent methods
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1 Introduction

Reaction-diffusion systems appear in the modeling of multiple biological and physical problems like the models of predator-prey in [13], some models of

turbulent interaction between waves and currents described in [23], infiltration of liquids in porous media [2], nuclear reactor dynamics [19] as well as many chemical processes [10,12,25]. Frequently, the solutions of these problems show rapid variations in some narrow regions (layers) when the size of the diffusion parameters is very small. This behavior provokes that standard approximation techniques fail, specially in providing a good approach of the solution at these layers.

In this paper we deal with the robust and efficient numerical resolution of two dimensional parabolic singularly perturbed coupled reaction-diffusion systems given by

$$\begin{cases} \mathcal{L}_\varepsilon(t)\mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \mathcal{L}_{\mathbf{x},\varepsilon}(t)\mathbf{u}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t),\ (\mathbf{x},t) \in Q \equiv \Omega \times (0,T], \\ \mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t),\ (\mathbf{x},t) \in \partial\Omega \times [0,T],\ \mathbf{u}(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}),\ \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

where $\Omega = (0,1)^2$ and the spatial differential operator $\mathcal{L}_{\mathbf{x},\varepsilon}(t)$ is defined as

$$\mathcal{L}_{\mathbf{x},\varepsilon}(t)\mathbf{u} \equiv -\mathcal{D}\Delta\mathbf{u} + \mathcal{A}(\mathbf{x},t)\mathbf{u}, \tag{2}$$

with $\mathcal{D} = \mathrm{diag}(\varepsilon,\varepsilon)$ and $\mathcal{A}(\mathbf{x},t) = (a_{rp}(\mathbf{x},t)),\ r,p = 1,2.$

Here we only consider the case of systems for which the diffusion parameters are the same, or there are diffusion parameters of similar sizes in the equations of the system. Note that when the diffusion parameters are similar in size a simple scaling reduces the problem to (1). This case is not the most general, but it has been considered in many papers (see [12,13,19,22,25] for instance). The theoretical analysis for problems with diffusion parameters of very different sizes is much more complicated because of the presence of overlapping boundary layers in their exact solutions; these cases will be the subject of future works.

We assume that the diffusion parameter ε can be very small, $0 < \varepsilon \ll 1$, and the reaction matrix $\mathcal{A}$ is an M-matrix, i.e., it holds

$$\sum_{p=1}^{2} a_{rp} \geq \alpha > 0,\ a_{rr} > \beta > 0,\ r = 1,2,\ a_{rp} \leq 0,\ \text{ if } \ r \neq p, \forall (\mathbf{x},t) \in \overline{Q}. \tag{3}$$

Moreover, we assume that the source term $\mathbf{f}(\mathbf{x},t) = (f_1,f_2)^T$, the initial condition $\boldsymbol{\varphi}(\mathbf{x}) = (\varphi_1,\varphi_2)^T$, the boundary conditions $\mathbf{g}(\mathbf{x},t) = (g_1,g_2)^T$ and the reaction matrix $\mathcal{A}$ are composed by sufficiently smooth functions satisfying sufficient compatibility conditions in order to assure that the exact solution $\mathbf{u} \in C^{4,2}(\overline{Q})$ (see [22] for a detailed discussion).

There are many works (see for instance [4,11,16] and references therein) where suitable numerical methods are constructed for solving elliptic and parabolic

one dimensional coupled systems of reaction-diffusion type. The exact solution of these problems has, in general, boundary layers close to the boundary of the spatial domain. In all of those papers where a time dependance was considered, a time integration, via the backward Euler method, was performed, combined with a discretization of the spatial variable by means of the classical central finite difference scheme defined on piecewise uniform Shishkin meshes; using this combination, several uniformly convergent methods (see [18]) were constructed, i.e., methods which give reliable solutions for any value of the diffusion parameter ε. The main drawback of such methods is related to the coupling of the components of the exact solution, which provokes that banded linear systems must be solved at each time level of the discretization. To avoid this inconvenient and reduce substantially the computational cost, in [3] an additive scheme (see [24]) was used to discretize in time. A similar idea based on a splitting by components was introduced in [7]; this technique permits to decouple the components in such way that only tridiagonal systems must be solved at each time level; therefore, a reduction of the computational cost is obtained which, besides, becomes more remarkable as long as the number of components in the system increases.

As well, in other papers (see [14,15,21]) 2D elliptic singularly perturbed systems of type (1) were considered; from those studies, it is well known that parabolic boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ appear at the boundary $\partial\Omega$ of the spatial domain. The time dependent case for systems with two equations has been studied in [22]; in those papers, the asymptotic behavior of the exact solution was given and the uniform convergence of a classical implicit method defined on a Shishkin mesh was proven; such method has a large computational cost because large block tridiagonal systems must be solved at each time step.

The use of alternating direction implicit methods (see [17]) is a well known alternative to reduce the computational cost of solving multidimensional scalar parabolic problems, because only tridiagonal systems are involved in the integration in time. In [9], a method of this type was proposed for solving 2D scalar singularly perturbed diffusion-reaction problems, proving that it was uniformly convergent of first order in time and almost second order in space. Such method can be applied to the systems considered here but its computational complexity is not optimal in terms of the number of components because, in such case, linear systems with banded matrices should be solved to advance in time.

To avoid the drawbacks of the existing algorithms mentioned before, here we combine the ideas of [7] and [9] to design a new numerical algorithm which uses the central difference scheme on a suitable special mesh to discretize in space and the fractional implicit Euler method, together with a splitting by directions and components, to integrate in time. In this way only tridiagonal

linear systems must be solved at each time level getting a remarkable computational cost reduction with respect to any classical implicit method which has been used in this context.

The presence of non homogeneous boundary data is an additional difficulty which we consider here, because their discretization provokes an order reduction phenomenon in many time integration processes (see [1] and references therein). This order reduction enlarges the errors in time and complicates the analysis of the uniform convergence when this phenomenon is present. In the algorithm proposed here, we remove this drawback by making a simple modification of the evaluations of the boundary data.

The paper is organized as follows. In Section 2 we show the asymptotic behavior of the exact solution $\mathbf{u}$ of (1) with respect to ε and we give appropriate estimates for its derivatives. In Section 3 we construct the spatial discretization and we prove its uniform convergence when it is defined on an adequate Shishkin mesh. In Section 4 we give the time discretization by using the fractional Euler method and the splitting by components, proving that the fully discrete scheme is a uniformly convergent method of first order in time and almost second order in space; moreover, we show that suitable evaluations of the boundary data are essential to avoid the loss of consistency caused by the order reduction phenomenon. In Section 5 we show the numerical results obtained for different test problems, which corroborate in practice the theoretical results and the advantages of the proposed algorithm. We finish with some concluding remarks.

Henceforth, we denote by $\|\mathbf{f}\|_D = \max\{\|f_1\|_D, \|f_2\|_D\}$, where $\|\cdot\|_D$ is the maximum norm on the domain D, $|\mathbf{v}| = (|v_1|, |v_2|)^T$, $\mathbf{v} \geq \mathbf{w}$ (analogously $\mathbf{v} \leq \mathbf{w}$) means $v_r \geq w_r, r = 1, 2$, C denotes a generic positive constant which is independent of the diffusion parameter ε and the discretization parameters N and M, and $\mathbf{C} = (C, C)^T$.

2 Asymptotic behavior of the exact solution

In this section we provide estimates for the derivatives of the solution $\mathbf{u}$ of problem (1), which show its asymptotic behavior with respect to the diffusion parameter ε. The ideas to obtain those estimates follow the works [5,9,21,22], where the decomposition solution and the extended domains technique are used; then, the regular and the boundary layer functions (see below) are the restriction of the solution of parabolic problems with appropriate boundary and initial conditions, which are defined on suitable extended domains; moreover, the corner layers (see below) are the solution of a similar problem to (1) where the right-hand side is zero, the initial condition is also zero and the

boundary conditions depend on the value on $\partial\Omega$ of the regular and boundary layer functions. The regularity of each component is guaranteed using appropriate extended domains and the estimates of the derivatives of the exact solution of these problems are obtained using the barrier function technique.

To prove the existence of a unique solution of problem (1) and the maximum principle, we introduce the scalar uncoupled differential operators

$$\mathcal{L}_{r,\varepsilon}v(\mathbf{x},t) \equiv v_t(\mathbf{x},t) - \varepsilon(v_{xx}(\mathbf{x},t) + v_{yy}(\mathbf{x},t)) + a_{rr}v(\mathbf{x},t),\ r = 1,2,$$

which satisfy a maximum principle [20]. In [15] the next result was proved.
Lemma 1. *Let $w \in C(\bar{Q}) \cap C^2(Q)$ be such that $\mathcal{L}_{r,\varepsilon}w = \psi, r = 1,2$ on Q, $w = g$ on $\partial\Omega \times [0,T]$ and $w = \varphi$ on Ω. Then, it holds*

$$\|w\|_{\bar{Q}} \leq \|\psi/a_{rr}\|_{\bar{Q}} + \|g\|_{\partial\Omega\times[0,T]} + \|\varphi\|_{\overline{\Omega}}.$$

Moreover, following closely the proof in [15], the next result can be proved.
Lemma 2. *Let $\mathbf{u}$ be a solution of (1). For $k = 0,1,2,\ldots$, define the sequence of functions $\mathbf{u}^{[k]} = (u_1^{[k]}, u_2^{[k]})$ as follows: let $\mathbf{u}^{[0]}$ be any function in $(C(\overline{Q}))^2$, and for $k = 0,1,2,\ldots$, let $\mathbf{u}^{[k]}$ such that*

$$\begin{cases} \mathcal{L}_{r,\varepsilon}u_r^{[k]} = f_r - \sum_{p\neq r} a_{rp}u_r^{[k-1]},\ \text{on } Q,\ u_r^{[k]} = g_r \text{ on } \partial\Omega\times[0,T], \\ \qquad\qquad u_r^{[k]}(\mathbf{x},0) = \varphi_r(\mathbf{x}), \mathbf{x}\in\Omega, r = 1,2; \end{cases}$$

then, it holds $\lim_{k\to\infty} \mathbf{u}^{[k]} = \mathbf{u}$. Moreover, we have

$$\|\mathbf{u}\|_{\bar{Q}} \leq \frac{1}{\beta(1-\gamma)}\|\mathbf{f}\|_{\bar{Q}},$$

where $0 \leq \gamma \equiv \max_r \gamma_r < 1$ and $\gamma_r = \max_{\bar{Q}} \left\{ \frac{1}{a_{rr}} \sum_{p\neq r} |a_{rp}| \right\},\ r = 1,2.$

From Lemma 2 the existence of a unique solution of problem (1) and the following inverse positivity property are deduced.
Lemma 3. *Let $\mathbf{v} \in (C(\bar{Q}) \cap C^2(Q))^2$ be such that $\mathcal{L}_\varepsilon\mathbf{v} \geq \mathbf{0}$ on Q and $\mathbf{v} \geq \mathbf{0}$ on $\partial\Omega\times[0,T] \cup \Omega\times\{0\}$. Then, $\mathbf{v} \geq \mathbf{0}$ on $\bar{Q}$.*

To obtain suitable estimates for $\mathbf{u}$ and its derivatives, we consider the following subsets

$$\Gamma_1 = \{(x,0),\ 0 \leq x \leq 1\},\ \Gamma_2 = \{(0,y),\ 0 \leq y \leq 1\},$$
$$\Gamma_3 = \{(x,1),\ 0 \leq x \leq 1\},\ \Gamma_4 = \{(1,y),\ 0 \leq y \leq 1\},$$

which are the edges of $\partial\Omega$ and the four corners of the spatial domain are

denoted by c_l, $l = 1, 2, 3, 4$, being

$$c_1 = \Gamma_1 \cap \Gamma_2,\ c_2 = \Gamma_2 \cap \Gamma_3,\ c_3 = \Gamma_3 \cap \Gamma_4,\ c_4 = \Gamma_4 \cap \Gamma_1.$$

Now, following the ideas of [9], we decompose the solution of (1) as

$$\mathbf{u} = \mathbf{v} + \sum_{l=1}^{4} \mathbf{w}_l + \sum_{l=1}^{4} \mathbf{z}_l, \tag{4}$$

where $\mathbf{v}$ is the regular component of $\mathbf{u}$ which satisfies

$$\left| \frac{\partial^{k+k_0}}{\partial x^{k_1} \partial y^{k_2} \partial t^{k_0}} \mathbf{v}(\mathbf{x}, t) \right| \leq \mathbf{C}(1 + \varepsilon^{(2-k)/2}), \quad (\mathbf{x}, t) \in \bar{Q}, \quad 0 \leq k + 2k_0 \leq 4, \tag{5}$$

with $k = k_1 + k_2$, $\mathbf{w}_l, l = 1, 2, 3, 4$ are the boundary layer components of $\mathbf{u}$, associated to Γ_l, which satisfy

$$|\mathbf{w}_l(\mathbf{x}, t)| \leq \mathbf{C} e^{-\alpha_0 \varepsilon^{-1/2} r(\mathbf{x}, \Gamma_i)}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad l = 1, 2, 3, 4, \tag{6}$$

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} \mathbf{w}_l(\mathbf{x}, t) \right| \leq \mathbf{C}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad 1 \leq k_0 \leq 2, \quad l = 1, 2, 3, 4, \tag{7}$$

$$\left| \frac{\partial^{k}}{\partial x^{k_1} \partial y^{k_2}} \mathbf{w}_l(\mathbf{x}, t) \right| \leq \mathbf{C} \varepsilon^{-k_1/2}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad 1 \leq k \leq 4, \quad l = 2, 4, \tag{8}$$

$$\left| \frac{\partial^{k}}{\partial x^{k_1} \partial y^{k_2}} \mathbf{w}_l(\mathbf{x}, t) \right| \leq \mathbf{C} \varepsilon^{-k_2/2}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad 1 \leq k \leq 4, \quad l = 1, 3, \tag{9}$$

and $\mathbf{z}_l$, $l = 1, 2, 3, 4$ are the corner layer components which satisfy

$$|\mathbf{z}_l(\mathbf{x}, t)| \leq \mathbf{C} e^{-\alpha_0 \varepsilon^{-1/2} r(\mathbf{x}, c_l)}, \quad (\mathbf{x}, t) \in \bar{Q},\ l = 1, 2, 3, 4, \tag{10}$$

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} \mathbf{z}_l(\mathbf{x}, t) \right| \leq \mathbf{C}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad 1 \leq k_0 \leq 2,\ l = 1, 2, 3, 4, \tag{11}$$

$$\left| \frac{\partial^{k}}{\partial x^{k_1} \partial y^{k_2}} \mathbf{z}_l(\mathbf{x}, t) \right| \leq \mathbf{C} \varepsilon^{-k/2}, \quad (\mathbf{x}, t) \in \bar{Q}, \quad 1 \leq k \leq 4,\ l = 1, 2, 3, 4, \tag{12}$$

where α_0 is an arbitrary positive constant such that $0 < \alpha_0 < \alpha$, and $r(\mathbf{x}, \Gamma_l)$ and $r(\mathbf{x}, c_l)$ are the distances from the point $\mathbf{x}$ to the sets Γ_l and the corners $c_l, l = 1, 2, 3, 4$, respectively.

3 Spatial semidiscretization

To discretize in space problem (1), we use the classical central finite difference scheme defined on a rectangular mesh of Shishkin type. For simplicity, we take

the same number of grid points in both spatial directions. As the exact solution of the continuous problem has parabolic boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ on $\partial\Omega$, we propose a mesh which is the tensor product of one dimensional piecewise uniform meshes of Shishkin type, i.e., $\overline{\Omega}^N = \overline{I}_x^N \times \overline{I}_y^N$, with $\overline{I}_x^N = \{0 = x_0 < x_1 < \ldots, < x_N = 1\}$, $\overline{I}_y^N = \{0 = y_0 < y_1 < \ldots, < y_N = 1\}$, where, for the variable x (and the same for the variable y) the grid points of $\bar{I}_x^N$ are given by

$$x_i = \begin{cases} ih, & i = 0, \ldots, N/4, \\ x_{N/4} + (i - N/4)H, & i = N/4 + 1, \ldots, 3N/4, \\ x_{3N/4} + (i - 3N/4)h, & i = 3N/4 + 1, \ldots, N, \end{cases} \tag{13}$$

where $h = 4\sigma/N,\ H = 2(1-2\sigma)/N$, and the transition parameter σ is defined by

$$\sigma = \min\left\{1/4, 2\sqrt{\varepsilon}\ln N\right\}. \tag{14}$$

We denote by $h_{x,i} = x_i - x_{i-1}, h_{y,j} = y_j - y_{j-1}, i, j = 1, \ldots, N$, and by $\overline{h}_{x,i} = (h_{x,i} + h_{x,i+1})/2, \overline{h}_{y,j} = (h_{y,j} + h_{y,j+1})/2, i, j = 1, \ldots, N-1$.

Let us denote by Ω^N the subgrid of $\overline{\Omega}^N$ composed only by the interior points of it, i.e., by $\overline{\Omega}^N \bigcap \Omega$, $\partial\Omega^N \equiv \overline{\Omega}^N \backslash \Omega^N$, by $[\mathbf{v}]_{\Omega^N}$ (analogously $[v]_{\Omega^N}$ for scalar functions) the restriction operators, applied to vector functions defined in Ω, to the mesh Ω^N, and by $[\mathbf{v}]_{\partial\Omega^N}$ (analogously $[v]_{\partial\Omega^N}$ for scalar functions) the restriction operators, applied to vector functions defined in $\partial\Omega$, to the mesh $\partial\Omega^N$. For all $(x_i, y_j) \in \Omega^N$ we introduce the semidiscrete approach $\mathbf{U}^N(t) \equiv \mathbf{U}_{ij}^N(t),\ i, j = 1, \ldots, N-1$, with $\mathbf{U}_{ij}^N(t) \approx \mathbf{u}(x_i, y_j, t)$, as the solution of the following Initial Value Problem

$$\begin{cases} \dfrac{d\mathbf{U}^N}{dt}(t) + \mathcal{L}_\varepsilon^N(t)\overline{\mathbf{U}}^N(t) = [\mathbf{f}(\mathbf{x}, t)]_{\Omega^N},\ \text{in } \Omega^N \times [0, T], \\ \overline{\mathbf{U}}^N(t) = [\mathbf{g}(\mathbf{x}, t)]_{\partial\Omega^N}, \text{in } \partial\Omega^N \times [0, T], \\ \mathbf{U}^N(0) = [\boldsymbol{\varphi}(\mathbf{x})]_{\Omega^N}, \end{cases} \tag{15}$$

being $\overline{\mathbf{U}}^N(t)$ the natural extension of the semidiscrete functions $\mathbf{U}^N(t)$, defined in $\Omega^N \times [0, T]$, to $\overline{\Omega}^N \times [0, T]$ by adding the Dirichlet boundary values of $\mathbf{g}$ in $\partial\Omega^N \times [0, T]$ and being $\mathcal{L}_\varepsilon^N(t)$ the discretization, via central differences, of the diffusion-reaction operator $\mathcal{L}_{\mathbf{x},\varepsilon}(t)$, i.e.,

$$\begin{aligned} &(\mathcal{L}_\varepsilon^N(t)\overline{\mathbf{U}}^N)_{ij,1} = c_{ij,l}U_{i-1j,1}^N + c_{ij,r}U_{i+1j,1}^N + c_{ij,d}U_{ij-1,1}^N + \\ &c_{ij,u}U_{ij+1,1}^N + c_{ij,c}U_{ij,1}^N + a_{11}(t)U_{ij,1}^N + a_{12}(t)U_{ij,2}^N, \end{aligned} \tag{16}$$

and

$$
\begin{aligned}
&(\mathcal{L}_\varepsilon^N(t)\overline{\mathbf{U}}^N)_{ij,2} = c_{ij,l}U^N_{i-1j,2} + c_{ij,r}U^N_{i+1j,2} + c_{ij,d}U^N_{ij-1,2} + \\
&c_{ij,u}U^N_{ij+1,2} + c_{ij,c}U^N_{ij,2} + a_{21}(t)U^N_{ij,1} + a_{22}(t)U^N_{ij,2},
\end{aligned}
\tag{17}
$$

with

$$
\begin{aligned}
&c_{ij,l} = \frac{-\varepsilon}{h_{x,i}\overline{h}_{x,i}}, c_{ij,r} = \frac{-\varepsilon}{h_{x,i+1}\overline{h}_{x,i}}, c_{ij,d} = \frac{-\varepsilon}{h_{y,j}\overline{h}_{y,j}}, c_{ij,u} = \frac{-\varepsilon}{h_{y,j+1}\overline{h}_{y,j}}, \\
&c_{ij,c} = -(c_{ij,l} + c_{ij,r} + c_{ij,d} + c_{ij,u}),
\end{aligned}
\tag{18}
$$

for $i, j = 1, \ldots, N-1$.

The uniform well-posedness of (15) is a consequence of the following semidiscrete maximum principle.

Theorem 1. *Under the assumption* $[\mathbf{f}(\mathbf{x},t)]_{\Omega^N} \leq \mathbf{0}$*, it holds that* $\overline{\mathbf{U}}^N(t)$ *reaches its maximum componentwise value at the semidiscrete boundary* $\partial\Omega^N \times [0,T] \bigcup \Omega^N \times \{0\}$.

The proof of this result is similar to the proof of the semidiscrete maximum principle stated in [7]. From Theorem 1 the next result follows.

Theorem 2. *If* $[\mathbf{f}(\mathbf{x},t)]_{\Omega^N} \geq \mathbf{0}$*,* $[\mathbf{g}(\mathbf{x},t)]_{\partial\Omega^N} \geq \mathbf{0}$ *and* $[\boldsymbol{\varphi}(\mathbf{x})]_{\Omega^N} \geq \mathbf{0}$*, then* $\overline{\mathbf{U}}^N(t) \geq \mathbf{0}$*,*

Using now a well known barrier-function technique, the following result can be proved.

Theorem 3. *(Uniform stability for (15)). The unique solution of problem (15) satisfies the uniform estimate*

$$
\begin{aligned}
&\|\overline{\mathbf{U}}^N(t)\|_{\overline{\Omega}^N \times [0,T]} \leq \\
&\max\{\|[\boldsymbol{\varphi}(\mathbf{x})]_{\Omega^N}\|_{\Omega^N}, \|[\mathbf{g}(\mathbf{x},t)]_{\partial\Omega^N}\|_{\partial\Omega^N \times [0,T]}, \frac{1}{\alpha}\|[\mathbf{f}(\mathbf{x},t)]_{\Omega^N}\|_{\Omega^N \times [0,T]}\}.
\end{aligned}
\tag{19}
$$

In previous result the norms are the corresponding maximum discrete or semidiscrete norms; for instance, $\|\mathbf{U}^N(t)\|_{\overline{\Omega}^N \times [0,T]} \equiv \max\limits_{0 \leq i,j,N,\ 0 \leq t \leq T} \|\mathbf{U}^N_{ij}(t)\|$.

The last result in this section proves the uniform convergence of the spatial discretization.

Theorem 4. *Under the previous smoothness assumptions for* $\mathbf{u}$*, the error associated to the spatial discretization on the Shishkin mesh satisfies*

$$
\|\overline{\mathbf{U}}^N(t) - [\mathbf{u}(\mathbf{x},t)]_{\overline{\Omega}^N}\|_{\overline{\Omega}^N} \leq CN^{-2}\ln^2 N,\ t \in [0,T],
\tag{20}
$$

where C *is independent of* ε *and* N *and therefore the spatial discretization is uniformly convergent of almost second order.*

Proof. The proof follows the ideas of [7] where a 1D version of problem (1)

was considered. We only include the minimal details to understand how the estimate for the error can be deduced for the 2D problem considered here.

Firstly, we decompose the semidiscrete solution using an analogous decomposition to the one which we considered in previous section for the solution $\mathbf{u}$:

$$\overline{\mathbf{U}}^N(t) = \overline{\mathbf{V}}^N(t) + \sum_{l=1}^{4} \overline{\mathbf{W}}_l^N(t) + \sum_{l=1}^{4} \overline{\mathbf{Z}}_l^N(t); \tag{21}$$

these grid functions are the solution of the semidiscrete problems

$$\begin{cases} \dfrac{d\mathbf{V}^N}{dt}(t) + \mathcal{L}_\varepsilon^N(t)\overline{\mathbf{V}}^N(t) = [\mathcal{L}_\varepsilon(t)\mathbf{v}]_{\Omega^N}, \text{ in } \Omega^N \times [0,T], \\ \overline{\mathbf{V}}^N(t) = [\mathbf{v}(\mathbf{x},t)]_{\partial\Omega^N}, \text{in } \partial\Omega^N \times [0,T], \\ \mathbf{V}^N(0) = [\mathbf{v}(\mathbf{x},0)]_{\Omega^N}, \end{cases}$$

$$\begin{cases} \dfrac{d\mathbf{W}_l^N}{dt}(t) + \mathcal{L}_\varepsilon^N(t)\overline{\mathbf{W}}_l^N(t) = [\mathbf{0}]_{\Omega^N}, \text{ in } \Omega^N \times [0,T], \\ \overline{\mathbf{W}}_l^N(t) = [\mathbf{w}_l(\mathbf{x},t)]_{\partial\Omega^N}, \text{in } \partial\Omega^N \times [0,T], \\ \mathbf{W}_l^N(0) = [\mathbf{w}_l(\mathbf{x},0)]_{\Omega^N}, \end{cases}$$

and

$$\begin{cases} \dfrac{d\mathbf{Z}_l^N}{dt}(t) + \mathcal{L}_\varepsilon^N(t)\overline{\mathbf{Z}}_l^N(t) = [\mathbf{0}]_{\Omega^N}, \text{ in } \Omega^N \times [0,T], \\ \overline{\mathbf{Z}}_l^N(t) = [\mathbf{z}_l(\mathbf{x},t)]_{\partial\Omega^N}, \text{in } \partial\Omega^N \times [0,T], \\ \mathbf{Z}_l^N(0) = [\mathbf{z}_l(\mathbf{x},0)]_{\Omega^N}, \end{cases}$$

for $l = 1, 2, 3, 4$.

On Ω^N, for any $t \in (0,T]$, the vector of truncation errors is given by

$$\tau^N(t)(\mathbf{u}) \equiv [(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{x},\varepsilon}(t))\mathbf{u}(\mathbf{x},t)]_{\Omega^N} - (\frac{d}{dt}[\mathbf{u}(\mathbf{x},t)]_{\Omega^N} + \mathcal{L}_\varepsilon^N(t)[\mathbf{u}(\mathbf{x},t)]_{\overline{\Omega}^N}).$$

For the regular component, the truncation error satisfies

$$|\tau_{ij}^N(t)(\mathbf{v})| \leq |\mathcal{D}\left(\Delta\mathbf{v}(x_i,y_j,t) - \left(\delta_x^2 + \delta_y^2\right)\mathbf{v}(x_i,y_j,t)\right)|,$$

where δ_x^2 and δ_y^2 are the discretization on a nonuniform mesh of the second derivatives respect to x and y respectively. Using the estimates (5), it is not difficult to obtain

$$|\tau_{ij,k}^N(t)(\mathbf{v})| \leq \begin{cases} CN^{-1}\sqrt{\varepsilon}, \text{ if } x_i = \sigma, 1-\sigma, \text{ or } y_j = \sigma, 1-\sigma, \ k = 1,2, \\ CN^{-2}, \text{ otherwise.} \end{cases} \tag{22}$$

Using (22) and Theorem 3 in a classical way we cannot prove the almost second order of uniform convergence in space. To get this, following to [3,5] we define the barrier function

$$\Psi = \mathbf{C}\sigma^2\varepsilon^{-1}N^{-2}(\theta(x_i)+\theta(y_j)),$$

where $\theta(z)$ is the usual piecewise linear polynomial given by

$$\theta(z) = \begin{cases} \dfrac{z}{\sigma},\ 0 \le z \le \sigma, \\ 1,\ \sigma \le z \le 1-\sigma, \\ \dfrac{1-z}{\sigma},\ 1-\sigma \le z \le 1. \end{cases}$$

From the choice of transition points, it follows that $|\Psi| \le \mathbf{C}(N^{-1}\ln N)^2$ and, from Theorem 2, we obtain

$$|(\mathbf{V}^N - \mathbf{v})(x_i, y_j, t)| \le \Psi \le \mathbf{C}(N^{-1}\ln N)^2. \tag{23}$$

In second place, we analyze the error associated to the boundary layer functions $\mathbf{w}_l, l = 1,2,3,4$ (we only give details for $\mathbf{w}_1$). The analysis depends on the location of grid point y_j. First, we assume that $y_j \ge \sigma$; then, as in [3], using the barrier function

$$(B_{w_1;ij})_m = \begin{cases} \displaystyle\prod_{s=1}^{j}\left(1 + h_{y,s}\alpha_0/\sqrt{\varepsilon}\ \right)^{-1},\ j \neq 0, \\ 1,\ j = 0, \end{cases}$$

for $0 \le i,j \le N$ and $1 \le m \le 2$, the estimates (6) and that $(B_{w_1;ij})_m \le CN^{-2}$, for $y_j \ge \sigma$ it follows

$$|(\mathbf{W}_1^N - \mathbf{w}_1)(x_i, y_j, t)| \le \mathbf{C}\ N^{-2}, \quad \text{if } \sigma \le y_j \le 1. \tag{24}$$

For the grid points $(x_i, y_j) \in (0,1)\times(0,\sigma)$, using (7) and (9), the local error satisfies

$$|\tau_{ij}^N(t)(\mathbf{w}_1)| \le \mathbf{C}\ \varepsilon^{-1}h^2,$$

and as $h \le C\ \varepsilon^{1/2}N^{-1}\ln N$ it follows

$$|(\mathbf{W}_1^N - \mathbf{w}_1)(x_i, y_j, t)| \le \mathbf{C}(N^{-1}\ln N)^2, \quad \text{if } 0 \le y_j \le \sigma. \tag{25}$$

Finally, for the corner layer functions $\mathbf{z}_l, l = 1,2,3,4$ (we only show the details for $\mathbf{z}_1$), if $x_i \ge \sigma$, we proceed as for the layer function $\mathbf{w}_1$ to obtain

$$|(\mathbf{Z}_1^N - \mathbf{z}_1)(x_i, y_j, t)| \le \mathbf{C}e^{-\alpha_0\varepsilon^{-1/2}y_j} + \mathbf{B}_{w_1} \le \mathbf{C}N^{-2}, \ \text{ if } y_j \ge \sigma. \tag{26}$$

On the other hand, if $x_i \geq \sigma$, using that $\mathbf{z}_1$ decays exponentially from $x = 0$, it holds

$$|(\mathbf{Z}_1^N - \mathbf{z}_1)(x_i, y_j, t)| \leq \mathbf{C}e^{-\alpha_0 \varepsilon^{-1/2} x_i} + \mathbf{B}_{w_2} \leq \mathbf{C}N^{-2}, \quad \text{if } x_i \geq \sigma, \tag{27}$$

where

$$(B_{w_2;ij})_m = \begin{cases} \prod_{s=1}^{j} \left(1 + h_{x,s}\alpha_0/\sqrt{\varepsilon}\,\right)^{-1}, \; j \neq 0, \\ 1, \; j = 0, \end{cases}$$

for $0 \leq i, j \leq N$ and $1 \leq m \leq 2$.

At last, for the grid points $(x_i, y_j) \in (0, \sigma) \times (0, \sigma)$, as the mesh is fine in both spatial directions. a classical truncation error analysis permits to prove

$$|(\mathbf{Z}_1^N - \mathbf{z}_1)(x_i, y_j, t)| \leq \mathbf{C}(N^{-1} \ln N)^2, \quad \text{if } 0 \leq x_i, y_j \leq \sigma. \tag{28}$$

From (23)-(28) the required result follows. □

4 Time integration: the fully discrete scheme

In this section we propose an efficient fully discrete scheme to solve (1), by applying an appropriate time integrator to the semidiscrete problems (15). To simplify the notation, we introduce the difference operators

$$\begin{aligned} &\mathcal{L}_{1,x}^N(t)v^N \equiv -\varepsilon\partial_{xx}v^N + a_{11,x}(t)v^N, \;\; \mathcal{L}_{1,y}^N(t)v^N \equiv -\varepsilon\partial_{yy}v^N + a_{11,y}(t)v^N, \\ &\mathcal{L}_{2,x}^N(t)v^N \equiv -\varepsilon\partial_{xx}v^N + a_{22,x}(t)v^N, \;\; \mathcal{L}_{2,y}^N(t)v^N \equiv -\varepsilon\partial_{yy}v^N + a_{22,y}(t)v^N, \end{aligned} \tag{29}$$

being ∂_{xx} and ∂_{yy} the classical second order central differences, on the corresponding one dimensional Shishkin meshes, with $a_{rr,x}(x, y, t) + a_{rr,y}(x, y, t) = a_{rr}(x, y, t), r = 1, 2$ with the restriction that $a_{rr,z}(x, y, t) \geq 0, \; r = 1, 2, z = x, y$ holds. Analogously, we decompose the non diagonal coefficients of the reaction matrix in the form $a_{rp,x}(x, y, t) + a_{rp,y}(x, y, t) = a_{rp}(x, y, t), \; r, p = 1, 2, r \neq p$, and now the restriction to these splitting is that $a_{rp,z}(x, y, t) \leq 0, \; r, p = 1, 2, r \neq p, z = x, y$, and $\sum_{p=1}^{2} a_{rp,z}(x, y, t) \geq \alpha_z > 0, r = 1, 2, z = x, y$ hold, being $\alpha_x + \alpha_y = \alpha$ and α is the term which was introduced in (3). Notation $a_{rp,z}(t)v^N$ must be understood as follows: $(a_{rp,z}(t)v^N)_{ij} \equiv a_{rp,z}(x_i, y_j, t)v_{ij}^N$; as well, we decompose the right-hand side $\mathbf{f}(\mathbf{x}, t) \equiv (f_1, f_2)^T$, in the form $\mathbf{f}_x + \mathbf{f}_y \equiv (f_{1,x}, f_{2,x})^T + (f_{1,y}, f_{2,y})^T$.

Let $\tau \equiv T/M$ be the time step, $t_m = m\tau, m = 0, \ldots, M$ are the intermediate times where the semidiscrete solution $\overline{\mathbf{U}}(t_m)$ is going to be approached by

$\mathbf{U}^{N,m}$ and $\Omega_x^N \equiv I_x^N \times \overline{I}_y^N$, $\Omega_y^N \equiv \overline{I}_x^N \times I_y^N$; then, the fully discrete scheme is given by

$$
\begin{aligned}
&\text{(initialize)}\ \ \mathbf{U}^{N,0} = [\boldsymbol{\varphi}]_{\overline{\Omega}^N},\\
&\begin{cases}
\begin{cases}
\text{(calculus of } \mathbf{U}^{N,m+1}),\\
(I + \tau L_{\varepsilon,l}^N(t_{m+1}))\mathbf{U}^{N,m+l/4} = \mathbf{U}^{N,m+(l-1)/4} + \tau \mathbf{F}^{m+l/4}, \quad \text{in } \Omega_l^N,\\
\mathbf{U}^{N,m+l/4} = \mathbf{G}^{N,m+l/4}, \text{in } \partial\Omega_l^N,\\
l = 1,2,3,4,
\end{cases}\\
m = 0,\ldots,M-1,
\end{cases}
\end{aligned}
\tag{30}
$$

where

$$
L_{\varepsilon,1}^N(t_{m+1}) \equiv \begin{pmatrix} \mathcal{L}_{1,x}^N(t_{m+1}) & a_{12,x}(t_{m+1}) \\ 0 & I \end{pmatrix},\ L_{\varepsilon,2}^N(t_{m+1}) \equiv \begin{pmatrix} I & 0 \\ a_{21,x}(t_{m+1}) & \mathcal{L}_{2,x}^N(t_{m+1}) \end{pmatrix},
$$

$$
L_{\varepsilon,3}^N(t_{m+1}) \equiv \begin{pmatrix} I & 0 \\ a_{21,y}(t_{m+1}) & \mathcal{L}_{2,y}^N(t_{m+1}) \end{pmatrix},\ L_{\varepsilon,4}^N(t_{m+1}) \equiv \begin{pmatrix} \mathcal{L}_{1,y}^N(t_{m+1}) & a_{12,y}(t_{m+1}) \\ 0 & I \end{pmatrix},
$$

$$
\mathbf{F}^{m+1/4} \equiv \begin{pmatrix} F_{1,x}^{m+1} \\ 0 \end{pmatrix}, \mathbf{F}^{m+2/4} \equiv \begin{pmatrix} 0 \\ F_{2,x}^{m+1} \end{pmatrix},\ \mathbf{F}^{m+3/4} \equiv \begin{pmatrix} 0 \\ F_{2,y}^{m+1} \end{pmatrix},\ \mathbf{F}^{m+1} \equiv \begin{pmatrix} F_{1,y}^{m+1} \\ 0, \end{pmatrix},
$$

$\mathbf{G}^{N,m+1/2}$, $\mathbf{G}^{N,m+1}$ are given by

$$
\begin{aligned}
\mathbf{G}_0^{N,m+1/2} = \Big(&(I + \tau\mathcal{L}_{1,y}^N(t_{m+1}))[g_1(0,y,t_{m+1})]_{\bar{I}_y} - \tau[f_{1,y}(0,y,t_{m+1})]_{I_y} + \\
&[\tau a_{12,y}(0,y,t_{m+1}) g_2(0,y,t_{m+1})]_{I_y}, \\
&\quad (I + \tau\mathcal{L}_{2,y}^N(t_{m+1}))[g_2(0,y,t_{m+1})]_{\bar{I}_y} - \tau[f_{2,y}(0,y,t_{m+1})]_{I_y} + \\
&[\tau a_{21,y}(0,y,t_{m+1}) g_1(0,y,t_{m+1})]_{I_y} \Big)^T, \\
\mathbf{G}_N^{N,m+1/2} = \Big(&(I + \tau\mathcal{L}_{1,y}^N(t_{m+1}))[g_1(1,y,t_{m+1})]_{\bar{I}_y} - \tau[f_{1,y}(1,y,t_{m+1})]_{I_y} + \\
&[\tau a_{12,y}(1,y,t_{m+1}) g_2(1,y,t_{m+1})]_{I_y}, \\
&\quad (I + \tau\mathcal{L}_{2,y}^N(t_{m+1}))[g_2(1,y,t_{m+1})]_{\bar{I}_y} - \tau[f_{2,y}(1,y,t_{m+1})]_{I_y} + \\
&[\tau a_{21,y}(1,y,t_{m+1}) g_1(1,y,t_{m+1})]_{I_y} \Big)^T, \\
\mathbf{G}_0^{N,m+1} = [\mathbf{g}(x,0,t_{m+1})]_{I_x} &\text{ and } \mathbf{G}_N^{N,m+1} = [\mathbf{g}(x,1,t_{m+1})]_{I_x}.
\end{aligned}
\tag{31}
$$

$\mathbf{G}^{N,m+1/4} \equiv (\mathbf{G}_1^{N,m+1/2}, [U_2^{N,m}]_{\partial\Omega_1^N})^T$ and $\mathbf{G}^{N,m+3/4} \equiv ([U_1^{N,m+1/2}]_{\partial\Omega_3^N}, \mathbf{G}_2^{N,m+1})^T$, being $\partial\Omega_1^N = \partial\Omega_2^N \equiv \{0,1\} \times \bar{I}_y$ and $\partial\Omega_3^N = \partial\Omega_4^N \equiv \bar{I}_x \times \{0,1\}$.

Note that in the fractionary steps of (30), only tridiagonal linear systems are involved to obtain the numerical approaches $\mathbf{U}^{N,m},\ m=1,\ldots,M$. Therefore, the computational cost of advancing one step in time with our algorithm is similar to the cost of any one step explicit method. If we chose, for example, the implicit Euler method to integrate in time then a complicated block tridiagonal system should be solved to advance in time. Therefore, we obtain a remarkable cost reduction with our method when we compare it with classical choices.

With respect to the boundary data, it is important to remark that our proposal improves the accuracy given by the following apparently more natural choice

$$\begin{array}{l}\mathbf{G}_0^{N,m+1/2}=[\mathbf{g}(0,y,t_{m+1})]_{\bar{I}_y},\mathbf{G}_N^{N,m+1/2}=[\mathbf{g}(1,y,t_{m+1})]_{\bar{I}_y},\\ \mathbf{G}_0^{N,m+1}=[\mathbf{g}(x,0,t_{m+1})]_{I_x}\text{ and }\mathbf{G}_N^{N,m+1}=[\mathbf{g}(x,1,t_{m+1})]_{I_x}.\end{array}\tag{32}$$

Below, we give both theoretical and practical reasons to clarify that our choice for the boundary data is better than (32).

Let us study the qualities of our algorithm. In first place we prove an inverse positivity result which is the discrete analogue of Theorem 2 for the fully discrete scheme.

Theorem 5. *If all of the data* $(\mathbf{G},\mathbf{F}_1,\mathbf{F}_2,[\boldsymbol{\varphi}]_{\Omega^N})$*, which take part in (30), have non-negative components, then the solutions* $\mathbf{U}^{N,m},\ m=1,\ldots,M$ *of (30) have non-negative components.*

Proof. It is similar to the proof which was made in Theorem 5 of [7] for 1D singularly perturbed systems of reaction–diffusion type. □

Joint to the previous inverse positivity result, we are ready to state the uniform stability and the uniform consistency of our time integration process.

Corollary 1. *(Contractivity of the time integrator). If* $\mathbf{G}=0,\mathbf{F}_1=0$ *and* $\mathbf{F}_2=0$*, it holds*

$$\|\mathbf{U}^{N,m+1}\|_{\Omega_N}\leq\|\mathbf{U}^{N,m}\|_{\Omega_N},\ m=0,1,\ldots,M-1.\tag{33}$$

Proof. We use a four step inductive reasoning on the fractional steps of our scheme. Notice that now we are assuming that $\mathbf{U}^{N,m+1}$ is the solution of

$$\begin{cases}(I+\tau L_{\varepsilon,l}^N(t_{m+1}))\mathbf{U}^{N,m+l/4}=\mathbf{U}^{N,m+(l-1)/4},\ \text{ in }\Omega_l^N,\\ \mathbf{U}^{N,m+l/4}=\mathbf{0},\text{in }\partial\Omega_l^N,\\ l=1,2,3,4.\end{cases}\tag{34}$$

Let us consider the barrier function $\chi^{\mathbf{N}}$ defined as follows $\chi^{\mathbf{N}}{}_{ij,r}=C,\ i,j=0,\ldots,N,r=1,2$, being C a positive constant. It is easy to check that $(I+$

$\tau L^N_{\varepsilon,l}(t_{m+1}))\chi^{\mathbf{N}} \geq \chi^{\mathbf{N}}, l = 1,2,3,4$, i.e.,

$$\hat{\mathbf{F}}^{m+l/4} \equiv (I + \tau L^N_{\varepsilon,l}(t_{m+1}))\chi^{\mathbf{N}} - \chi^{\mathbf{N}} \geq \mathbf{0}, l = 1,2,3,4.$$

Now we take $C = \|\mathbf{U}^{N,m}\|_{\Omega_N}$ and we define $\mathbf{U}^{N,m+l/4}_- \equiv \chi^{\mathbf{N}} - \mathbf{U}^{N,m+l/4}, l = 1,2,3,4$. Such grid functions satisfy the following

$$\begin{array}{l} \mathbf{U}^{N,m}_- \geq \mathbf{0}, \\ \left\{ \begin{array}{l} (I + \tau L^N_{\varepsilon,l}(t_{m+1}))\mathbf{U}^{N,m+l/4}_- = \mathbf{U}^{N,m+(l-1)/4}_- + \tau \hat{\mathbf{F}}^{m+l/4}, \quad \text{in } \Omega^N_l, \\ \mathbf{U}^{N,m+l/4}_- = \chi^{\mathbf{N}}, \text{in } \partial\Omega^N_l, \\ l = 1,2,3,4. \end{array} \right. \end{array} \tag{35}$$

Now, resorting to the inverse positivity property of the operators $(I+\tau L^N_{\varepsilon,l}(t_{m+1}))$, we can deduce that $\mathbf{U}^{N,m+1}_- \geq \mathbf{0}$, i. e., $\chi^{\mathbf{N}} \geq \mathbf{U}^{N,m+1}$. We can repeat the reasoning with $U^{N,m+l/4}_+ \equiv \chi^{\mathbf{N}} + U^{N,m+l/4}$, $l = 1,2,3,4$, to deduce that also $\chi^{\mathbf{N}} \geq -\mathbf{U}^{N,m+1}$. From the last two inequalities, (33) trivially follows. □

To analyze the uniform consistency, we introduce the local error in time at time t_{m+1}, $m = 0,\ldots,M-1$, as usual

$$\mathbf{e}^{N,m+1} \equiv \overline{\mathbf{U}}^N(t_{m+1}) - \widehat{\mathbf{U}}^{N,m+1},$$

being $\widehat{\mathbf{U}}^{N,m+1}$ the result given by the step m of scheme (30) if we change $\mathbf{U}^{N,m}$ by $\overline{\mathbf{U}}^N(t_m)$.

Theorem 6. *(Uniform consistency of the time integrator). Under the assumption* $\mathbf{u} \in C^{4,2}(\overline{Q})$*, it holds*

$$\|\mathbf{e}^{N,m+1}\|_{\overline{\Omega}_N} \leq CM^{-2}, \quad \forall\ \tau\ \in (0,\tau_0] \quad \textit{and } \forall\ m = 0,1,\ldots,M-1. \tag{36}$$

Proof. It is analogue to the proof of Theorem 6 in [8]. □

Remark 1. *Note that if we choose the boundary data given in (32) instead of (31), a perturbation of size* $\mathcal{O}(\tau)$ *should appear, in general, in the boundary data of (30), causing a loss of accuracy in most of cases. Nevertheless, in the case of having homogeneous boundary conditions, a suitable partition of the source terms permits that (32) and (31) coincide and, in such case, the classical choice for the boundary conditions given in (32) will not cause a loss of precision.*

A classical combination of the previous consistency and stability results, permits us to prove that the time integration process is uniformly convergent of first order. The global error in time at t_m, $(\overline{\mathbf{U}}^N(t_m) - \mathbf{U}^{N,m})$ for $m = 1,\ldots,M$,

can be bounded as

$$\|\overline{\mathbf{U}}^N(t_m) - \mathbf{U}^{N,m}\|_{\overline{\Omega}^N} \leq \|\mathbf{e}^{N,m}\|_{\overline{\Omega}^N} + \|\widehat{\mathbf{U}}^{N,m} - \mathbf{U}^{N.m}\|_{\overline{\Omega}^N}.$$

Now, the stability result (33) ensures that

$$\|\widehat{\mathbf{U}}^{N,m} - \mathbf{U}^{N,m}\|_{\overline{\Omega}_N} \leq \|\overline{\mathbf{U}}^N(t_{m-1}) - \mathbf{U}^{N,m-1}\|_{\overline{\Omega}_N},$$

and therefore

$$\|\overline{\mathbf{U}}^N(t_m) - \mathbf{U}^{N,m}\|_{\overline{\Omega}^N} \leq \|\mathbf{e}^{N,m}\|_{\overline{\Omega}^N} + \|\overline{\mathbf{U}}^N(t_{m-1}) - \mathbf{U}^{N,m-1}\|_{\overline{\Omega}^N}.$$

Applying now m times this recurrent bound and (36) it is immediate to deduce that

$$\|\overline{\mathbf{U}}^N(t_m) - \mathbf{U}^{N,m}\|_{\overline{\Omega}^N} \leq \sum_{i=1}^{m} \|\mathbf{e}^{N,i}\|_{\overline{\Omega}^N} \leq CM^{-1}. \tag{37}$$

Therefore, the time integration process is uniformly convergent of first order.

Combining this result and the uniform convergence of the spatial discretization process, the main uniform convergence result of the paper follows.

Theorem 7. *(Uniform convergence). Assuming that* $\mathbf{u} \in C^{4,2}(\overline{Q})$ *, the global error associated to the numerical method defined by (30), (31) satisfies*

$$\max_{0\leq m\leq M} \|\mathbf{U}^{N,m} - [\mathbf{u}(\mathbf{x}, t_m)]_{\overline{\Omega}^N}\|_{\overline{\Omega}^N} \leq C\left((N^{-1}\ln N)^2 + M^{-1}\right), \tag{38}$$

being C *a positive constant independent of the diffusion parameter* $\boldsymbol{\varepsilon}$ *and the discretization parameters* N *and* M*.*

Proof. Using the triangle inequality, it is straightforward that it holds

$$\|\mathbf{U}^{N,m} - [\mathbf{u}(\mathbf{x}, t_m)]_{\overline{\Omega}^N}\|_{\overline{\Omega}^N} \leq \|\mathbf{U}^{N,m} - \overline{\mathbf{U}}^N(t_m)\|_{\overline{\Omega}^N} + \|\overline{\mathbf{U}}^N(t_m) - [\mathbf{u}(\mathbf{x}, t_m)]_{\overline{\Omega}^N}\|_{\overline{\Omega}^N}.$$

Now, applying the uniform bounds (20) and (37) the required result follows. □

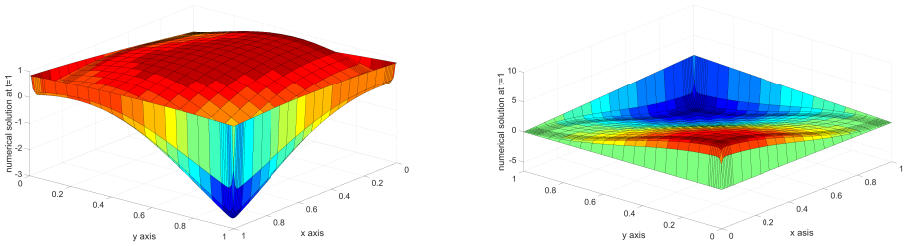

Fig. 1. Components u_1 (left) and u_2 (right) at $t = 1$ for $\varepsilon = 10^{-4}$ with $N = M = 32$

5 Numerical results

In this section we show the numerical results obtained for some test problems of type (1). In the first one the data are given by

$$\begin{aligned}
&\mathcal{A} = \begin{pmatrix} 2 + xyt & -(x^2 + y^2) \\ -\sin(x + y) & 1 + \sin(x + y)t^2 \end{pmatrix}, \\
&f_1 = (3\sin(\pi x) + \sin(\pi y))t^2, \;\; f_2 = 3(\cos(\pi x) + \cos(\pi y))e^t, \\
&g_1(x, y, t) = \sin(x + y)t^2, \;\; g_2(x, y, t) = xy(e^t - 1), \;\; (x, y) \in \partial\Omega, \\
&\varphi_r(x, y) = 0, \;\; r = 1, 2,
\end{aligned} \tag{39}$$

and $T = 1$. Figure 1 displays the numerical solution at $t = 1$ with $N = M = 32$ and the diffusion parameter $\varepsilon = 10^{-4}$. From it, we clearly see the boundary layers at the four sides of the spatial domain.

As the algorithm requires a suitable smooth partition of the coefficients of the reaction matrix, here, for simplicity, we have chosen

$$a_{rp,x}(x, y, t) = a_{rp,y}(x, y, t) = a_{rp}(x, y, t)/2, \;\; r, p = 1, 2. \tag{40}$$

As well, it is required a partition of the source term $\mathbf{f}(\mathbf{x}, t) \equiv (f_1, f_2)^T$, in the form $\mathbf{f}_x + \mathbf{f}_y \equiv (f_{1,x}, f_{2,x})^T + (f_{1,y}, f_{2,y})^T$. Here we take

$$f_{r,x}(x, y, t) = f_{r,y}(x, y, t) = f_r(x, y, t)/2, \;\; r = 1, 2,$$

The exact solution of problem (1) is unknown; then, we cannot calculate exactly the errors; to approximate them we use a variant of the double-mesh principle (see [5,18]). The maximum errors for each value of ε are approximated by

$$\mathbf{d}_{\boldsymbol{\varepsilon}}^{N,M} = \max_{0 \le m \le M} \max_{0 \le i,j \le N} ||\mathbf{U}_{ij}^{N,m} - \widehat{\mathbf{U}}_{2i\,2j}^{2N,2m}|,$$

where $\{\widehat{\mathbf{U}}_{ij}^{2N,m}\}$ is the numerical solution on a finer mesh $\{(\hat{x}_i, \hat{y}_j, \hat{t}_m)\}$, which has the mesh points of the coarse mesh and their midpoints. From the maxi-

Table 1
Maximum and uniform two-mesh differences and orders of convergence for the component u_1 in problem (39)

ε	N=16	N=32	N=64	N=128	N=256
	M=4	M=16	M=64	M=256	M=1024
10^{-1}	8.6216E-2	2.6793E-2	7.1193E-3	1.8085E-3	4.5397E-4
	1.6861	1.9120	1.9769	1.9941	
10^{-2}	1.4341E-1	4.3562E-2	1.1492E-2	2.9181E-3	7.3228E-4
	1.7190	1.9225	1.9775	1.9946	
10^{-3}	2.6012E-1	1.0823E-1	3.7068E-2	9.5571E-3	2.4156E-3
	1.2651	1.5459	1.9555	1.9842	
10^{-4}	2.7122E-1	1.1435E-1	4.1927E-2	1.3841E-2	4.3706E-3
	1.2460	1.4475	1.5989	1.6631	
10^{-5}	2.7436E-1	1.1611E-1	4.2566E-2	1.4057E-2	4.4409E-3
	1.2406	1.4477	1.5984	1.6624	
10^{-6}	2.7531E-1	1.1665E-1	4.2761E-2	1.4124E-2	4.4626E-3
	1.2389	1.4478	1.5982	1.6622	
10^{-7}	2.7561E-1	1.1681E-1	4.2823E-2	1.4145E-2	4.4694E-3
	1.2384	1.4478	1.5981	1.6621	
10^{-8}	2.7570E-1	1.1687E-1	4.2842E-2	1.4151E-2	4.4715E-3
	1.2382	1.4478	1.5981	1.6621	
$d_1^{N,M}$	2.7570E-1	1.1687E-1	4.2842E-2	1.4151E-2	4.4715E-3
$q_1^{N,M}$	1.2382	1.4478	1.5981	1.6621	

mum two-mesh differences $\mathbf{d}_{\boldsymbol{\varepsilon}}^{N,M}$, we obtain the ε-uniform two-mesh differences by

$$\mathbf{d}^{N,M} = \max_{\varepsilon} \mathbf{d}_{\boldsymbol{\varepsilon}}^{N,M}.$$

From $\mathbf{d}_{\boldsymbol{\varepsilon}}^{N,M}$, the numerical orders of convergence for each value of ε are calculated by

$$\mathbf{q}_{\varepsilon}^{N,M} = \log\left(\mathbf{d}_{\boldsymbol{\varepsilon}}^{N,M}/\mathbf{d}_{\boldsymbol{\varepsilon}}^{2N,2M}\right)/\log 2,$$

and from $\mathbf{q}^{N,M}$ the numerical uniform orders of uniform convergence are calculated by

$$\mathbf{p}^{N,M} = \log\left(\mathbf{d}^{N,M}/\mathbf{d}^{2N,2M}\right)/\log 2.$$

Tables 1 and 2 show the results when the improved evaluations of the boundary conditions are used. In this case, to observe the order of convergence in space, we multiply the discretization parameter M by 4 and the discretization parameter N by 2 in order that the errors associated to the spatial discretization dominate in the global errors. Using this procedure a uniformly convergent behavior of second order, up to a logarithmic factor, is observed. As well, the use of improved boundary data which we propose makes that the decomposition of the source term of the system has no influence in the uniformly convergent behavior of the algorithm, as is was shown in [6] for scalar reaction-diffusion problems. This quality permits simpler partitions of $\mathbf{f}$.

Table 2
Maximum and uniform two-mesh differences and orders of convergence for the component u_2 in problem (39)

ε	N=16	N=32	N=64	N=128	N=256
	M=4	M=16	M=64	M=256	M=1024
10^{-1}	1.4607E-1	6.1631E-2	2.0372E-2	5.8965E-3	1.5934E-3
	1.2449	1.5970	1.7887	1.8878	
10^{-2}	2.7052E-1	9.5415E-2	2.9267E-2	8.2135E-3	2.1842E-3
	1.5034	1.7050	1.8332	1.9109	
10^{-3}	5.2112E-1	2.8578E-1	1.1094E-1	3.2243E-2	8.5353E-3
	0.8667	1.3651	1.7828	1.9175	
10^{-4}	5.3051E-1	2.8876E-1	1.2089E-1	4.5015E-2	1.5306E-2
	0.8775	1.2562	1.4252	1.5563	
10^{-5}	5.3269E-1	2.8952E-1	1.2107E-1	4.5092E-2	1.5328E-2
	0.8797	1.2578	1.4249	1.5567	
10^{-6}	5.3330E-1	2.8974E-1	1.2112E-1	4.5115E-2	1.5334E-2
	0.8802	1.2583	1.4248	1.5568	
10^{-7}	5.3348E-1	2.8981E-1	1.2114E-1	4.5122E-2	1.5336E-2
	0.8804	1.2584	1.4248	1.5569	
10^{-8}	5.3354E-1	2.8983E-1	1.2115E-1	4.5124E-2	1.5337E-2
	0.8804	1.2584	1.4248	1.5569	
$d_2^{N,M}$	5.3354E-1	2.8983E-1	1.2115E-1	4.5124E-2	1.5337E-2
$q_2^{N,M}$	0.8804	1.2584	1.4248	1.5569	

To show that our technique can be easily extended to systems of larger dimensions, we have considered a second example which has three components. Now, the unknown solution is $\mathbf{u} \equiv (u_1(x,y,t), u_2(x,y,t), u_3(x,y,t))^T$, the diffusion term is $-\mathrm{diag}(\varepsilon,\varepsilon,\varepsilon)\Delta\mathbf{u}$ and the rest of the data are

$$\mathcal{A} = \begin{pmatrix} e^{x+y} & -(x+y) & -tx \\ -(x+y) & (t+1)(3+x+y) & -t\sin(y) \\ -tx & -t\sin(y) & e^t(2+\cos(x+y)) \end{pmatrix},$$
$$f_1 = 10e^{-t}xy(1-x)(1-y),\ f_2 = 5e^{-t}\sin(\pi xy),$$
$$f_3 = 10(1-t)\cos(x+y)x^2y^2(1-x)(1-y),$$
$$g_r(x,y,t) = 0,\ \varphi_r(x,y) = 0,\ r = 1,2,3. \tag{41}$$

The computed solutions at $t = 1$ are displayed in Figure 2. These surfaces show the presence of boundary layers in the three components of the solution.

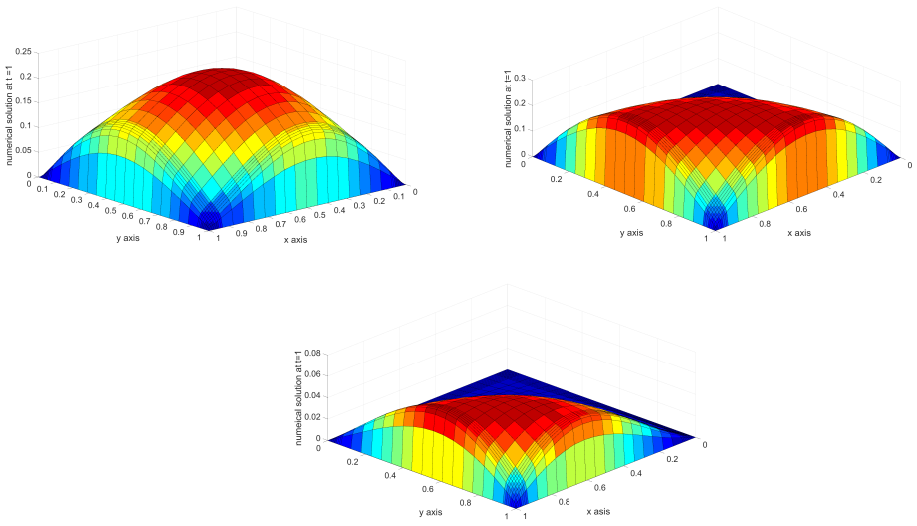

Fig. 2. Components u_1 (first row left), u_2 (first row right) and u_3 (second row left) at $t = 1$ for $\varepsilon = 10^{-4}$ with $N = M = 32$

In this example we use again

$$a_{rp,x}(x, y, t) = a_{rp,y}(x, y, t) = a_{rp}(x, y, t)/2, \ \ r, p = 1, 2, 3,$$

and

$$f_{r,x}(x, y, t) = f_{r,y}(x, y, t) = f_r(x, y, t)/2, \ \ r = 1, 2, 3.$$

As in this example the boundary conditions are homogenous, in the same way as in [9], to avoid the order reduction using classical (homogeneous) evaluations of the boundary data, it suffices to perform a suitable splitting for $(f_1, f_2, f_3)^T \equiv (f_{1,x}, f_{2,x}, f_{3,x})^T + (f_{1,y}, f_{2,y}, f_{3,y})^T$ in such way that $f_{r,x} = 0$ in $\{0, 1\} \times [0, 1] \times [0, T]$ and $f_{r,y} = 0$ in $[0, 1] \times \{0, 1\} \times [0, T]$ for $r = 1, 2, 3$. This restriction agrees completely with the analysis which we have developed here, because in such cases the classical and the improved evaluations of the boundary data coincide. As this system has 3 components, the corresponding multi-splitting algorithm has six fractional steps per time step (see [9,6] for capturing the main ideas of its construction).

Similarly to the previous example, we show the numerical results for each component in separate tables. The maximum two-mesh differences and the orders of convergence for $u_r, r = 1, 2, 3$ are given in Tables 3-5 respectively, where the discretization parameters N and M are multiplied by a factor of 2. The ε-uniform computed orders of convergence shows first order, because the errors associated to the time discretization dominate in the global errors, in agreement with Theorem 7.

Table 3
Maximum and uniform two-mesh differences and orders of convergence for the component u_1 in problem (41)

ε	N=16	N=32	N=64	N=128	N=256
	M=4	M=8	M=16	M=32	M=64
10^{-1}	2.1919E-2	1.4141E-2	7.9642E-3	4.2370E-3	2.1890E-3
	0.6323	0.8283	0.9105	0.9528	
10^{-2}	3.2557E-2	1.8072E-2	9.7599E-3	5.0356E-3	2.5603E-4
	0.8492	0.8888	0.9547	0.9758	
10^{-3}	3.3384E-2	1.8540E-2	9.9717E-3	5.1467E-3	2.6153E-3
	0.8485	0.8947	0.9542	0.9767	
10^{-4}	3.3180E-2	1.8575E-2	9.9948E-3	5.1593E-3	2.6217E-3
	0.8369	0.8941	0.9540	0.9767	
10^{-5}	3.3331E-2	1.8562E-2	9.9942E-3	5.1604E-3	2.6222E-3
	0.8445	0.8932	0.9536	0.9767	
10^{-6}	3.3375E-2	1.8555E-2	9.9940E-3	5.1596E-3	2.6222E-3
	0.8469	0.8927	0.9538	0.9765	
10^{-7}	3.3388E-2	1.8552E-2	9.9965E-3	5.1599E-3	2.6224E-3
	0.8478	0.8921	0.9541	0.9765	
10^{-8}	3.3392E-2	1.8551E-2	9.9971E-3	5.1598E-3	2.6224E-34
	0.8480	0.8919	0.9542	0.9765	
$d_1^{N,M}$	3.3392E-2	1.8575E-2	9.9971E-3	5.1604E-3	2.6224E-4
$q_1^{N,M}$	0.8461	0.8938	0.9540	0.9766	

Conclusions

In this work we have designed and analyzed a numerical algorithm to approximate the solution of 2D parabolic singularly perturbed systems of reaction–diffusion type, which have the same diffusion parameter at all equations. Although this is not the most general case, it is sufficiently interesting because it covers many applications (notice that the models with diffusion parameters of similar size can be included in our analysis by scaling the equations by convenient factors). The algorithm uses the central finite difference scheme to discretize in space and a splitting implicit method to integrate in time. We only show full details in the case of systems with two components; nevertheless, in the section devoted to numerical experiences, it is shown that the technique can be easily extended to systems with a larger number of equations. If a suitable Shishkin mesh is chosen, then the fully discrete scheme is uniformly convergent of first order in time and almost second order in space. The multi-splitting technique used to discretize in time provokes that only linear tridiagonal systems must be solved to advance in time, which is a remarkable advantage, in terms of computational cost, with respect to classical implicit methods. Some numerical experiences corroborate the good numerical behavior of our method predicted by the theory.

Table 4
Maximum and uniform two-mesh differences and orders of convergence for the component u_2 in problem (41)

ε	N=16	N=32	N=64	N=128	N=256
	M=4	M=8	M=16	M=32	M=64
10^{-1}	6.4165E-2	4.1108E-2	2.4925E-2	1.3664E-2	7.1928E-3
	0.6424	0.7218	0.8672	0.9257	
10^{-2}	7.3122E-2	4.4998E-2	2.5368E-2	1.3592E-2	7.0553E-3
	0.7004	0.8268	0.9003	0.9459	
10^{-3}	8.0875E-2	5.6540E-2	3.1474E-2	1.5561E-2	7.5501E-3
	0.5164	0.8451	1.0162	1.0434	
10^{-4}	8.0649E-2	5.6422E-2	3.2557E-2	1.6808E-2	8.1763E-
	0.5154	0.7933	0.9538	1.0397	
10^{-5}	8.0527E-2	5.6368E-2	3.2542E-2	1.6803E-2	8.1728E-3
	0.5146	0.7926	0.9536	1.0398	
10^{-6}	8.0484E-2	5.6350E-2	3.2537E-2	1.6801E-2	8.1717E-3
	0.5143	0.7923	0.9535	1.0398	
10^{-7}	8.0469E-2	5.6344E-2	3.2535E-2	1.6800E-2	8.1714E-3
	0.5142	0.7923	0.9535	1.0398	
10^{-8}	8.0465E-2	5.6342E-2	3.2535E-2	1.6800E-2	8.1712E-3
	0.5142	0.7922	0.9535	1.0398	
$d_2^{N,M}$	8.0875E-2	5.6540E-2	3.2557E-2	1.6808E-2	8.1763E-
$q_2^{N,M}$	0.5164	0.7963	0.9538	1.0397	

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Table 5
Maximum and uniform two-mesh differences and orders of convergence for the component u_3 in problem (41)

ε	N=16	N=32	N=64	N=128	N=256
	M=4	M=8	M=16	M=32	M=64
10^{-1}	4.2603E-3	1.7604E-3	8.2975E-4	4.0239E-4	1.9812E-4
	1.2751	1.0851	1.0441	1.0223	
10^{-2}	6.4783E-3	3.0674E-3	1.4897E-3	7.3356E-4	3.6392E-4
	1.0786	1.0420	1.0220	1.0113	
10^{-3}	7.2577E-3	3.3799E-3	1.5634E-3	7.7000E-4	3.8237E-4
	1.1025	1.1123	1.0218	1.0099	
10^{-4}	7.1638E-3	3.3293E-3	1.5703E-3	7.7538E-4	3.8510E-4
	1.1055	1.0842	1.0180	1.0097	
10^{-5}	7.1058E-3	3.3117E-3	1.5717E-3	7.7587E-4	3.8538E-4
	1.1014	1.0753	1.0184	1.0095	
10^{-6}	7.0849E-3	3.3056E-3	1.5711E-3	7.7599E-4	3.8540E-4
	1.0999	1.0731	1.0177	1.0097	
10^{-7}	7.0781E-3	3.3036E-3	1.5712E-3	7.7588E-4	3.8539E-
	1.0993	1.0722	1.0180	1.0095	
10^{-8}	7.0759E-3	3.3030E-3	1.5712E-3	7.7591E-4	3.8539E-4
	1.0991	1.0719	1.0179	1.0096	
$d_3^{N,M}$	7.2577E-3	3.3799E-3	1.5717E-3	7.7599E-4	3.8540E-4
$q_3^{N,M}$	1.1025	1.1047	1.0182	1.0097	

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