Constructive Approximation
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CONSTRUCTIVE APPROXIMATION

Wall Polynomials on the Real Line: A Classical Approach to OPRL Khrushchev's Formula

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Abstract

The standard proof of Khrushchev's formula for orthogonal polynomials on the unit circle given in Khrushchev (J Approx Theory 108:161–248, 2001, J Approx Theory 116:268–342, 2002) combines ideas from continued fractions and complex analysis, depending heavily on the theory of Wall polynomials. Using operator theoretic tools instead, Khrushchev's formula has been recently extended to the setting of orthogonal polynomials on the real line in the determinate case (Grünbaum and Velázquez in Adv Math 326:352–464, 2018). This paper develops a theory of Wall polynomials on the real line, which serves as a means to prove Khrushchev's formula for any sequence of orthogonal polynomials on the real line. This real line version of Khrushchev's formula is used to rederive the characterization given in Simon (J Approx Theory 126:198–217, 2004) for the weak convergence of $p_n^2 \mathrm{d}\mu$, where p_n are the orthonormal polynomials with respect to a measure μ supported on a bounded subset of the real line (Theorem 8.1). The generality and simplicity of such a Khrushchev's formula also permits the analysis of the unbounded case. Among other results, we use this tool to prove that no measure μ supported on an unbounded subset of the real line yields a weakly convergent sequence $p_n^2 \mathrm{d}\mu$ (Corollary 8.10), but there exist instances such that $p_n^2 \mathrm{d}\mu$ becomes vaguely convergent (Example 8.5 and Theorem 8.6). Some other asymptoptic results related to the convergence of $p_n^2 \mathrm{d}\mu$ in the unbounded case are obtained via Khrushchev's formula (Theorems 8.3, 8.7, 8.8, Proposition 8.4, Corollary 8.9). In the bounded case, we include a simple diagrammatic proof of Khrushchev's formula on the real line which sheds light on its graph theoretical meaning, linked to Pólya's recurrence theory for classical random walks.

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1 Introduction

The beginning of this century has witnessed a revolution in the theory of orthogonal polynomials on the unit circle (OPUC) by the hand of Sergei Khrushchev [9, 10]. In Appendix C—"Twelve Great Papers"—of his OPUC monograph [16], Barry Simon highlights these contributions among the most significant ones in OPUC history. Khrushchev's approach exploits the OPUC connections with continued fractions and complex analysis provided by certain analytical functions known as Schur functions. S. Khrushchev realized the interest in translating OPUC questions to the language of Schur functions, what led to many innovative techniques and novel findings for OPUC, a body of results currently known as Khrushchev's theory. The cornerstone of this theory is Khrushchev's formula, a factorization of the Schur function for $|\varphi_n|^2 d\mu$, where φ_n are the orthonormal polynomials with respect to a measure μ on the unit circle.

Surprisingly, apart from isolated results—see for instance [15]—no analog of the whole Khrushchev's theory has been established for orthogonal polynomials on the real line (OPRL). Behind this, there is the absence of a clear real line counterpart of the Schur function for a measure on the unit circle, in terms of which an OPRL Khrushchev's formula should look like as simple as its OPUC version. A proposal for such a real line extension of the standard link between Schur functions and measures on the unit circle has been established in [5]. It yields a new connection between measures on the real line and the so called Nevanlinna functions. The strongest argument supporting the relevance of this connection is the simplicity of an OPRL Khrushchev's formula originated by it, also obtained in [5]. It takes the form of a decomposition of the Nevanlinna function for $p_n^2 d\mu$ as a sum of two other ones, with p_n the orthonormal polynomials with respect to a measure μ on the real line.

The referred OPRL Khrushchev's formula was uncovered in [5] in an operator theoretic way, resorting to the self-adjointness of the operator defined by the Jacobi matrix encoding the OPRL recurrence relation. Therefore, its validity is limited to OPRL related to determinate moment problems. In contrast, a continued fraction expansion of Schur functions based on the so called Schur algorithm was the key to obtain OPUC Khrushchev's formula in the seminal papers [9, 10]. In this strategy the Wall polynomials—numerators and denominators of the approximants for the continued fraction expansion of Schur functions—play an essential role.

This work presents a continued fraction approach to OPRL Khrushchev's formula in complete parallelism with [9, 10]. This approach is based on the real line extension proposed in [5] for the notion of Schur function and the corresponding Schur algorithm. This leads to the development of the real line version for the Wall polynomials. Apart from the novelty of this result, in contrast with [5], this approach allows us to prove Khrushchev's formula for arbitrary OPRL, no matter if the corresponding moment problem is determinate or not. This constitutes an essential advantage regarding the eventual use of Khrushchev's formula to establish a general Khrushchev's theory for OPRL which could cover even the indeterminate case.

Among the results which such OPRL Khrushchev's theory should include is the characterization of the measures μ on real line which make $p_n^2 \mathrm{d}\mu$ vaguely convergent (as it is shown in Sect. 4, this is the natural convergence notion when dealing with measures on the real line with not necessarily bounded support). This was answered by B. Simon [15,Theorem 2] in the case of measures with bounded support, a situation which makes vague convergence equivalent to weak convergence. The simplicity of our OPRL Khrushchev's formula not only permits us to rederive this result, but the general validity of such a formula also allows us to take the first steps towards the analysis of the vague convergence of $p_n^2 \mathrm{d}\mu$ for measures μ with unbounded support. The translation of this problem—via Khrushchev's formula—as a convergence question about Nevanlinna functions will be the key to prove in this work that the unbounded case includes new instances of vaguely convergent sequences $p_n^2 \mathrm{d}\mu$, not considered in [15,Theorem 2]. This points to an OPRL Khrushchev's theory which is expected to become particularly interesting in the unbounded case—the most relevant difference with respect to OPUC– where new qualitative results should arise, requiring the use of novel techniques. Although the full development of such a general OPRL Khrushchev's theory will be the goal of subsequent publications, this work presents new results on the convergence of $p_n^2 \mathrm{d}\mu$ in the unbounded case, which illustrate the effectiveness of our real line version of Khrushchev's formula, and are by themselves central contributions of the paper: see Theorems 8.3, 8.6, 8.7, 8.8, Proposition 8.4 and Corollaries 8.9, 8.10.

The content of the paper is structured as follows: Sect. 2 summarizes some results on Schur functions, Khrushchev's formula and Wall polynomials on the unit circle, to compare with the analogues on the real line that will appear in the rest of the paper. Section 3 details the relation between measures on the real line and Nevanlinna functions used throughout this work, including the real line version of the Schur algorithm. This establishes a triple connection between the sequences of real parameters coming from such an algorithm, Nevanlinna functions and measures on the real line. Different convergence notions in the corresponding spaces and their relations are analyzed in Sect. 4, including the quirks of the unbounded case. The Wall polynomials on the real line are introduced in Sect. 5, which also discusses their properties and relations with OPRL. Section 6 identifies the Wall polynomials as the numerators and denominators of the approximants for a continued fraction expansion of Nevanlinna functions, whose convergence is also addressed. The results of the previous sections give rise to OPRL Khrushchev's formula in Sect. 7, which is exploited in Sect. 8 for the analysis of the convergence of $p_n^2 \mathrm{d}\mu$, including examples and general results for the case of measures with unbounded support. Finally, in the bounded case, Sect. 9 gives a diagrammatic approach to OPRL Khrushchev's formula based on a graph theoretic interpretation of Nevanlinna functions.

2 OPUC Khrushchev's Formula and Wall Polynomials

Before introducing the Wall polynomials on the real line and proving OPRL Khrushchev's formula, we will summarize the OPUC precedents of these results. This will permit the reader to appreciate the extent to which the present work is a natural OPRL extension of that one developed in [9, 10] for OPUC. For a detailed exposition the reader may consult the original Khrushchev's papers [9, 10], for a very quick introduction one may resort to Section 1.3 of Simon's OPUC monograph [16].

Schur functions are the analytic mappings f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $|f(z)| \leq 1$ for $z \in \mathbb{D}$. A Schur function f is associated to each measure μ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ such that $\mu(\mathbb{T}) = 1$—also called a probability measure on $\mathbb{T}$—via the equality

$$\frac{1 + zf(z)}{1 - zf(z)} = \int_{\mathbb{T}} \frac{t + z}{t - z}\, \mathrm{d}\mu(t), \tag{1}$$

which may be rewritten as

$$\frac{1}{1 - z\overline{f(\overline{z})}} = \int_{\mathbb{T}} \frac{\mathrm{d}\mu(t)}{1 - zt}. \tag{2}$$

Indeed, (2) establishes a one-to-one correspondence between the set of Schur functions and the set of probability measures on $\mathbb{T}$. It is worth stressing that (1) actually defines an analytic function on $\mathbb{C} \setminus \mathbb{T}$ such that

$$\overline{f(\overline{z})} = f(z^{-1})^{-1}, \qquad \frac{1 - |f(z)|}{1 - |z|} \geq 0. \qquad z \in \mathbb{C} \setminus \mathbb{T}. \tag{3}$$

This constitutes a natural way of extending Schur functions to $\mathbb{C} \setminus \mathbb{T}$, which thus may be equivalently defined as the analytic functions on $\mathbb{C} \setminus \mathbb{T}$ satisfying (3).

The simplest examples of the above relation between Schur functions and measures on $\mathbb{T}$ are given by

$$f(z) = \begin{cases} c, & |z| < 1, \\ 1/\overline{c} & |z| > 1, \end{cases} \quad c \in \mathbb{C}, \quad |c| \leq 1, \quad \Rightarrow \quad \mathrm{d}\mu(x) = \begin{cases} \delta(x - c), & c \in \mathbb{T}, \\ \frac{1}{2\pi} \frac{1-|c|^2}{|t-\overline{c}|^2}\, dt, & c \in \mathbb{C} \setminus \mathbb{T}, \end{cases} \tag{4}$$

where $\delta(x - c)$ is the Dirac delta at c.

Given a Schur function f, the Schur algorithm

$$f_{n+1}(z) = \frac{1}{z} \frac{f_n(z) - f_n(0)}{1 - \overline{f_n(0)}\, f_n(z)}, \qquad f_0 = f,$$

generates a sequence of Schur functions f_n called the Schur iterates of f. The sequence is finite if $|f_n(0)| = 1$ for some n, which stops the algorithm at the n-th step and implies that f_n is constant due to Schwarz's lemma. The Schur function f is determined by the—finite or infinite—sequence $\alpha_n = f_n(0)$, known as the sequence of Schur parameters of f. This yields a one-to-one correspondence between the set of Schur functions and the set of sequences of Schur parameters,

$$\mathbb{D}^\infty \bigcup \left(\bigcup_{n=0}^{\infty} \mathbb{D}^n \times \mathbb{T} \right).$$

Equipped with the pointwise convergence, the above set of sequences becomes homeomorphic to the set of Schur functions with the uniform convergence on compacts subsets of $\mathbb{C} \setminus \mathbb{T}$, and also to the set of probability measures on $\mathbb{T}$ with the weak convergence.

A result that helps to exploit the above homeomorphims is a factorization of the Schur function for the measure $|\varphi_n|^2 d\mu$, where φ_n are the orthonormal polynomials with respect to the measure μ on $\mathbb{T}$. Such a Schur function factorizes as

$$f_n \frac{\varphi_n}{\varphi_n^*}, \tag{5}$$

where $\varphi_n^*(z) = z^n \overline{\varphi_n(1/\overline{z})}$ and f_n are the iterates of the Schur function f of μ. While the iterates f_n are characterized by the Schur parameters $(\alpha_n, \alpha_{n+1}, \dots)$, the rational functions φ_n/φ_n^*, known as the inverse Schur iterates of f, are the Schur functions with Schur parameters $(-\overline{\alpha_{n-1}}, -\overline{\alpha_{n-2}}, \dots, -\overline{\alpha_0}, 1)$. The factorization (5) is known as Khrushchev's formula.

Inverse iterates provide the universal form of the Schur functions for finitely supported measures, which may be identified as the Schur functions with a finite number of Schur parameters and are given by finite Blaschke products

$$\zeta \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}, \qquad z_k \in \mathbb{D}, \qquad \zeta \in \mathbb{T}.$$

A finite number of steps of the Schur algorithm may be condensed into a direct relation between a Schur function f and its iterates,

$$f = \frac{A_n + z X_n f_{n+1}}{B_n + z Y_n f_{n+1}},$$

where $X_n = B_n^*$, $Y_n = A_n^*$ and A_n, B_n are the so called Wall polynomials, given by

$$\begin{aligned}
X_n &= zX_{n-1} + \overline{\alpha_n} A_{n-1}, & X_0 &= \overline{\alpha_0},\\
A_n &= \alpha_n z X_{n-1} + A_{n-1}, & A_0 &= \alpha_0,\\
Y_n &= zY_{n-1} + \overline{\alpha_n} B_{n-1}, & Y_0 &= 1,\\
B_n &= \alpha_n z Y_{n-1} + B_{n-1}, & B_0 &= 1.
\end{aligned}$$

Rewriting the Schur algorithm as

$$(f_n - \alpha_n)\left(\overline{\alpha_n} z + \frac{1}{f_{n+1}}\right) = \rho_n^2 z, \qquad \rho_n = \sqrt{1-|\alpha_n|^2},$$

leads to the following continued fraction expansion of the Schur function f,

$$f = \alpha_0 + \frac{\rho_0^2 z|}{|\overline{\alpha_0} z} + \frac{1|}{|\alpha_1} + \frac{\rho_1^2 z|}{|\overline{\alpha_1} z} + \cdots + \frac{1|}{|\alpha_n} + \frac{\rho_n^2 z|}{|\overline{\alpha_n} z} + \frac{1|}{|f_{n+1}}.$$

As a consequence, A_n/B_n and X_n/Y_n are respectively the even and odd approximants of the Wall continued fraction

$$\alpha_0 + \frac{\rho_0^2 z|}{|\overline{\alpha_0} z} + \frac{1|}{|\alpha_1} + \frac{\rho_1^2 z|}{|\overline{\alpha_1} z} + \cdots + \frac{1|}{|\alpha_n} + \frac{\rho_n^2 z|}{|\overline{\alpha_n} z} + \cdots .$$

Besides,

$$f^{[n]} = \begin{cases} A_n/B_n, & |z| < 1,\\ X_n/Y_n, & |z| > 1, \end{cases}$$

defines a sequence of Schur functions with Schur parameters $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ which converges to f uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{T}$.

3 Schur Algorithm on $\mathbb{R}$

The Wall polynomials associated to a measure on $\mathbb{T}$ arise from the continued fraction expansion induced for the corresponding Schur function by the Schur algorithm. Therefore, the identification of the Wall polynomials for a measure on $\mathbb{R}$ calls for a right definition of the analogue of a Schur function for such a measure, a proper version of the Schur algorithm for this kind of function and the analysis of the continued fraction expansion generated by such an algorithm. These are the main objectives of the present section, as well as Sects. 5 and 6.

The natural real line analogue of the Schur functions are the so called Nevanlinna functions, i.e. the analytic mappings f on $\mathbb{C} \setminus \mathbb{R}$ which satisfy the real line version of (3), namely,

$$\overline{f(\overline{z})} = f(z), \qquad \frac{\operatorname{Im} f(z)}{\operatorname{Im} z} \geq 0, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{6}$$

We will denote by $\mathfrak{N}$ the set of Nevanlinna functions. To prove that a given function is a Nevanlinna one is often aided by simple invariance properties such as

$$f \in \mathfrak{N} \setminus \{0\}, \; g \in \mathfrak{N}, \; \lambda > 0 \; \Rightarrow \; f + g, \; \lambda f, \; -1/f \in \mathfrak{N} \setminus \{0\}. \tag{7}$$

The analyticity of $-1/f$ follows from the fact that the imaginary part of a Nevanlinna function cannot vanish on $\mathbb{C} \setminus \mathbb{R}$ unless it is a real constant, thus $f \in \mathfrak{N} \setminus \{0\}$ means that $f \in \mathfrak{N}$ does not vanish at any non-real point.

Concerning integral representations, it has been argued [16,App. B.2]—for a more detailed presentation, see [14]—that the Schur function of a measure on $\mathbb{T}$ may be viewed as a unit circle counterpart of the m-function of a measure μ on $\mathbb{R}$, a kind of Nevanlinna functions defined by

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{8}$$

However, this comparison has a serious drawback: the above relation does not exhaust the set $\mathfrak{N}$ of Nevanlinna functions, in contrast with the role of (1) concerning Schur functions.

A new proposal [5] takes a real line version of (2) as a starting point for another connection between Nevanlinna functions and measures. For details of the subsequent discussion, see [5].

Definition 3.1 The Nevanlinna function of a measure μ on $\mathbb{R}$ such that $0 < \mu(\mathbb{R}) \leq 1$, is the function f defined by

$$\frac{1}{1 - zf(z)} = \int_{\mathbb{R}} \frac{d\mu(x)}{1 - zx}, \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{9}$$

The above definition is consistent with the fact that $1 - zf(z) = -z(f(z) - z^{-1})$ does not vanish on $\mathbb{C} \setminus \mathbb{R}$ for any $f \in \mathfrak{N}$ because $|\operatorname{Im}(f(z) - z^{-1})| \geq |\operatorname{Im} z^{-1}|$. Nevertheless, that (9) defines a Nevanlinna function f is a non-trivial fact which follows from general properties of Nevanlinna functions [8]. Also, in contrast with (8) and more general integral representations of Nevanlinna functions (see e.g. [4] and references therein), the relation (9) establishes a one-to-one correspondence between the whole set $\mathfrak{N}$ of Nevanlinna functions and $\mathfrak{M} \setminus \{0\}$, where $\mathfrak{M}$ is the set of subprobability measures on $\mathbb{R}$, constituted by those measures μ such that $\mu(\mathbb{R}) \leq 1$ [5,Thm. 4.1]. Although it is not the aim of this paper, we should mention that other linear fractional transformations might be combined with (9) to improve the behaviour of the measure in the integral representation.

The freedom $0 < \mu(\mathbb{R}) \leq 1$ in the normalization of the measure for (9)—which makes a difference with respect to the case of the unit circle—is linked to the presence of the Nevanlinna function $-z^{-1}$. Actually, applying dominated convergence to (9), we find that

$$\lim_{\substack{z \to 0 \\ z \in i\mathbb{R}}} zf(z) = 1 - \frac{1}{\mu(\mathbb{R})}, \tag{10}$$

which implies that, for $z \in i\mathbb{R}$ around the origin,

$$f(z) = -cz^{-1} + o(z^{-1}), \qquad \begin{cases} c = 0, & \text{if } \mu(\mathbb{R}) = 1, \\ c > 0, & \text{if } \mu(\mathbb{R}) < 1. \end{cases}$$

This shows that no Nevanlinna function has on the imaginary axis an asymptotic behaviour around the origin with a divergence greater than $-z^{-1}$, and such a maximum divergence is absent only when the related subprobability measure μ is a probability one, i.e. $\mu(\mathbb{R}) = 1$. The case $c = 0$ does not exclude other lower order singularities at the origin, as it is the case for instance of $f(z) = \sqrt{z}$, $f(z) = \ln z$ or $f(z) = \pm i$, $z \in \mathbb{C}_\pm$. Bearing in mind that any real translation $f(z) \to f(z-x)$, $x \in \mathbb{R}$, preserves the set $\mathfrak{N}$ of Nevanlinna functions, we also conclude that $-(z-x)^{-1}$ is the maximum singularity that may arise in a Nevanlinna function when approaching to a real point x orthogonally to the real line.

A direct application of (9) yields, for instance, the Nevanlinna functions for the simplest non-null subprobability measures on $\mathbb{R}$,

$$d\mu(x) = \lambda\delta(x-c), \quad c \in \mathbb{R}, \quad \lambda \in (0,1] \quad \Rightarrow \quad f(z) = \lambda^{-1}c - (\lambda^{-1}-1)z^{-1}. \quad (11)$$

As expected, the term proportional to $-z^{-1}$ appears only for $\mu(\mathbb{R}) = \lambda < 1$. On the other hand, (10) implies that $\mu(\mathbb{R}) = 1$ for the measure μ of the simplest Nevanlinna functions, i.e. those which are constant in each half-plane,

$$f(z) = \begin{cases} c, & \operatorname{Im} z > 0, \\ \overline{c} & \operatorname{Im} z < 0, \end{cases} \quad c \in \mathbb{C}, \quad \operatorname{Im} c \geq 0 \quad \Rightarrow \quad d\mu(x) = \begin{cases} \delta(x-c), & c \in \mathbb{R}, \\ \frac{1}{\pi}\frac{\operatorname{Im} c}{|x-c|^2}\,dx, & c \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

These examples should be compared with (4). More instances of the correspondence between measures and Nevanlinna functions established by (9) can be found in [5].

For convenience, we will distinguish the following types of Nevanlinna functions according to the qualitative properties of the related measure.

Definition 3.2 A Nevanlinna function f with subprobability measure μ will be called:
Normalized if μ is a probability measure.
In this case, it will be called:
Regular if the moments $\mu_n = \int_{\mathbb{R}} x^n \, d\mu(x)$ are finite for $n \in \mathbb{N}$.
Trivial if μ is finitely supported.
Degenerate if μ is supported on a single point.

Obviously, the above classes of Nevanlinna functions are progressively smaller. As (11) shows, the degenerate situation corresponds to the constant Nevanlinna functions, which must be real due to the first condition in (6). This parallels the case of Schur functions. Schwarz's lemma implies that the only Nevanlinna functions whose imaginary part vanishes on a non-real point are the degenerate ones. Trivial Nevanlinna functions play the same role as Blaschke products do regarding Schur functions. Concerning regular Nevanlinna functions, as we will show, they have a close relation with the Schur algorithm on the real line (see Proposition 3.6). According to (10), normalized Nevanlinna functions may be characterized by their behaviour around the origin,

$$f \in \mathfrak{N}: \quad f \text{ is normalized} \quad \Leftrightarrow \quad \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} zf(z) = 0. \tag{12}$$

The following result provides a simple transformation that "normalizes" any Nevanlinna function by simply deleting any $-z^{-1}$ behaviour at the origin. This allows us to restrict our attention to normalized Nevanlinna functions, i.e. to probability measures on $\mathbb{R}$.

Proposition 3.3 *If f is a Nevanlinna function and $f(z) = -cz^{-1} + o(z^{-1})$ for $z \to 0$, $z \in i\mathbb{R}$, then $\hat{f}(z) = f(z) + cz^{-1}$ is a normalized Nevanlinna function. We refer to the transformation $f \mapsto \hat{f}$ as the* **normalization** *of the Nevanlinna function f.*

Proof Bearing in mind the characterization (12) of normalized Nevanlina functions, we only need to show that $\hat{f}$ is indeed a Nevanlinna function. To prove this, consider the measure μ of f. From (10) we know that $c = 1/\mu(\mathbb{R}) - 1$. On the other hand, (9) implies that the probability measure $\nu = \mu/\mu(\mathbb{R})$ corresponds to the Nevanlinna function $g(z) = \mu(\mathbb{R})f(z) + (1-\mu(\mathbb{R}))z^{-1} = \mu(\mathbb{R})\hat{f}(z)$. Therefore, $\hat{f}$ is a Nevanlinna function too. □

The relation between the Nevanlinna function f of a measure and the corresponding m-function follows from (9),

$$f(z) = z^{-1} + m(z^{-1})^{-1}. \tag{13}$$

It reveals a curious fact about the set $\mathfrak{N}$ of Nevanlinna functions: the mapping $m \mapsto f$ transforms a proper subset of $\mathfrak{N}$, the m-functions generated by $\mathfrak{M} \setminus \{0\}$, onto the whole set $\mathfrak{N}$. As we will see, the integral representation (9) of Nevanlinna functions has other advantages: it will be key to establish a very simple Khrushchev's formula for OPRL, which will come with the development of the corresponding Wall polynomials originated by a suitable version of the Schur algorithm for Nevanlinna functions.

To introduce such an algorithm, let us consider a normalized Nevanlinna function f whose restriction to the imaginary axis is derivable, i.e. such that there exist

$$f(0) := \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} f(z), \qquad f'(0) := \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} \frac{f(z)-f(0)}{z}. \tag{14}$$

Note that $f(0) \in \mathbb{R}$ because $\overline{f(\overline{z})} = f(z)$, while $\operatorname{Im} f(z)/\operatorname{Im} z \geq 0$ guarantees that $f'(0) \geq 0$ since

$$f'(0) = \lim_{\substack{y\to 0\\ y\in\mathbb{R}}} \frac{f(iy)-f(-iy)}{2iy} = \lim_{\substack{y\to 0\\ y\in\mathbb{R}}} \frac{\operatorname{Im} f(iy)}{y}.$$

The equality $f'(0) = 0$ characterizes the degeneracy of f [5,Appendix B]. Therefore, assuming $f'(0) > 0$ implies that $f(z) - f(0) \in \mathfrak{N} \setminus \{0\}$, so that

$$g(z) = -\frac{f'(0)}{f(z) - f(0)}$$

defines a Nevanlinna function with asymptotic behaviour $g(z) = -z^{-1} + o(z^{-1})$ for $z \in i\mathbb{R}$ around the origin. According to Proposition 3.3, we may define a new Nevanlinna function f_1 by normalizing g,

$$f_1(z) = \hat{g}(z) = -\frac{f'(0)}{f(z) - f(0)} + z^{-1} = \frac{1}{z} \frac{f(z) - f(0) - f'(0)z}{f(z) - f(0)}.$$

This transformation may be iterated to give an algorithm generating a sequence f_n of normalized Nevanlinna functions

$$f_{n+1}(z) = \frac{1}{z} \frac{f_n(z) - f_n(0) - f_n'(0)z}{f_n(z) - f_n(0)}, \qquad f_0 = f, \tag{15}$$

as long as each $f_n(z)$, $z \in i\mathbb{R}$, has a non-null derivative $f_n'(0)$ in the sense of (14). If $f_n'(0) = 0$ then f_n is degenerate, i.e. $f_n(z) = f_n(0)$, and the algorithm terminates at the n-th step.

Definition 3.4 Following [5], we refer to (15) as the **'Schur' algorithm on the real line**. If μ is the measure related to f, the Nevanlinna functions f_n will be called the **'Schur' iterates** of f or μ, while

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \dots) = (b_0, a_0, b_1, a_1, \dots), \qquad b_n = f_n(0), \qquad a_n = f_n'(0)^{1/2},$$

will be named the sequence of **'Schur' parameters** of f or μ. If $a_n = 0$, then $f_n(z) = b_n$ and the algorithm terminates at the n-th step, leading to a finite sequence of Schur parameters, $(b_0, a_0, b_1, a_1, \dots, b_{n-1}, a_{n-1}, b_n, a_n) = (b_0, a_0, b_1, a_1, \dots, b_{n-1}, a_{n-1}, b_n, 0)$.

In terms of the Schur parameters, the forward and backward Schur algorithm read as

$$f_{n+1}(z) = \frac{1}{z} \frac{f_n(z) - b_n - a_n^2 z}{f_n(z) - b_n}, \qquad f_n(z) = b_n + \frac{a_n^2 z}{1 - z f_{n+1}(z)}. \tag{16}$$

Another useful way of expressing these relations is

$$(f_n(z) - b_n)(1 - z f_{n+1}(z)) = a_n^2 z. \tag{17}$$

The Schur algorithm on the real line does not apply to every Nevanlinna function. For instance, $-z^{-1}$, $\ln z$, z^r with $r \in (0, 1)$ or the non-degenerate Nevanlinna functions which are constant in each half-plane, are examples for which even the first step of the algorithm is not possible. Regarding the class of Nevanlinna functions for which the Schur algorithm makes sense, it is convenient to introduce the following notation.

Definition 3.5 In what follows, by a slight abuse of notation, given a Nevanlinna function f we will denote by $f^{(n)}(0)$ the n-th derivative at $z = 0$ of $f(z)$ restricted to $z \in i\mathbb{R}$.

Then, we have the following result.

Proposition 3.6 *Given a Nevanlinna function f, the following statements are equivalent:*

(i) *f is regular.*
(ii) *$f^{(n)}(0)$ exists for every $n \in \mathbb{N}$.*
(iii) *The Schur algorithm on the real line applies to f.*

Proof The finiteness of the moments $\mu_0, \mu_1, \dots, \mu_n$ for the measure μ related to f is equivalent to the existence of the following asymptotic expansion of order n around the origin,

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1 - xz} = \mu_0 + \mu_1 z + \cdots + \mu_n z^n + o(z^n), \qquad z \in i\mathbb{R}. \tag{18}$$

When $\mu_0 = 1$, this is the same as stating that the Nevanlinna function f, given by (9), has an asymptotic expansion of order $n - 1$ around the origin,

$$f(z) = s_0 + s_1 z + \cdots + s_{n-1} z^{n-1} + o(z^{n-1}), \qquad z \in i\mathbb{R} \setminus \{0\}, \tag{19}$$

proving the equivalence (i) $\Leftrightarrow$ (ii).

The statement *(iii)* means that, for each iterate f_n, there exist $f_n(0)$ and $f_n'(0)$ in the sense of (14). In other words, $f_{n-1}(z)$, $z \in i\mathbb{R}$, has an asymptotic expansion around the origin of order 1. Hence, rewriting (16) for the relation between f_{n-1} and f_{n-2} shows that $f_{n-2}(z)$, $z \in i\mathbb{R}$, has an asymptotic expansion around the origin of order 3. By induction we conclude that $f(z)$, $z \in i\mathbb{R}$, has an asymptotic expansion around the origin of order $2n - 1$, i.e. there exist the first $2n - 1$ derivatives of f at the origin in the sense of Definition 3.5. If the Schur algorithm does not stop, then all the derivatives $f^{(n)}(0)$ exist. Otherwise, there is an iterate f_n which is a real constant and therefore f_n, and hence $f(z)$, $z \in i\mathbb{R}$, are derivable infinitely many times at the origin. This proves (iii) $\Rightarrow$ (ii).

Conversely, *(ii)* means that $f(z)$, $z \in i\mathbb{R}$, has an asymptotic expansion (19) around the origin of any order. Then, using (16) one gets by induction that $f_n(z)$, $z \in i\mathbb{R}$, has also an asymptotic expansion around the origin of any order for every $n \in \mathbb{N}$. This means that every iterate f_n has derivatives of all orders at the origin in the sense of Definition 3.5. $\square$

Although the above Schur algorithm only applies to a regular Nevanlinna function f, the first N steps of the algorithm make sense in a broader situation: it suffices that the moments of the related measure μ satisfy $\mu_0 = 1$ and $\mu_1, \dots, \mu_{2N} < \infty$, which in view of the proof of Proposition 3.6, is equivalent to the existence of the first $2N - 1$ derivatives of $f(z)$, $z \in i\mathbb{R}$, at $z = 0$. This condition guarantees that f_1 exists and has an asymptotic expansion around the origin of order $2N - 3$, as follows from (15).

By induction we find that f_N exists, i.e. the first N steps of the Schur algorithm make sense, originating $2N$ Schur parameters b_n, a_n, $0 \leq n \leq N-1$.

The proof of Proposition 3.6 shows that, if the first N steps of the Schur algorithm go well with a Nevanlinna function f, the $2N$ Schur parameters generated by these steps determine the coefficients of a Taylor expansion like (19) for f of order $2N-1$. The following result gives more details on this dependence.

Proposition 3.7 *If a Nevanlinna function f admits N steps of the Schur algorithm with Schur parameters $(\gamma_n)_{n=0}^{2N}$, then $f(z) = \sum_{n=0}^{2N-1} s_n z^n + o(z^{2N-1})$ for $z \in i\mathbb{R}$, with $s_0 = b_0$ and $s_n = s_n(\gamma_1, \dots, \gamma_n)$ an homogeneous polynomial of degree $n+1$ in the Schur parameters for $n \geq 1$. More precisely, if $\Upsilon_n = a_0 a_1 \cdots a_n$,*

$$\begin{aligned} s_{2n+1} &= \Upsilon_n^2 + a_0^2\, r_{2n+1}(b_1, a_1, \dots, b_n), \\ s_{2n+2} &= \Upsilon_n^2 b_{n+1} + a_0^2\, r_{2n+2}(b_1, a_1, \dots, b_n, a_n), \end{aligned} \qquad n \geq 0, \tag{20}$$

where $r_1 = r_2 = 0$ and $r_n = r_n(\gamma_2, \dots, \gamma_{n-1})$ is an homogeneus polynomial of degree $n-1$ in the Schur parameters for $n \geq 3$. The Taylor coefficients of f are related to the moments μ_n of the corresponding measure μ by

$$\mu_n = s_0 \mu_{n-1} + s_1 \mu_{n-2} + \cdots + s_{n-2} \mu_1 + s_{n-1}. \tag{21}$$

Proof We know that $s_0 = b_0$ and $s_1 = a_0^2$. The rest of the Taylor coefficients follow from (16), which for $n = 0$ is

$$(f - b_0)(1 - z f_1) = a_0^2 z.$$

Denoting by $s_n^{(1)}$ the Taylor coefficients of f_1, the above identity leads to

$$s_{n+1} = s_n s_0^{(1)} + s_{n-1} s_1^{(1)} + \cdots + s_1 s_{n-1}^{(1)}, \qquad n \geq 1. \tag{22}$$

Using this recurrence relation, all the properties of the relations between the Taylor coefficients and the Schur parameters follow by induction, using as induction hypothesis that they hold for any previous Taylor coefficient of any Nevanlinna function, in particular for f and f_1.

The relation between the Taylor coefficients of f and the moments μ_n of the corresponding measure μ arises by introducing the asymptotic expansions (18) and (19) into (9). □

The polynomials r_n become quite involved soon, something that can be observed from the list below, which summarizes the first non-null ones:

$$r_3 = b_1^2, \qquad r_4 = b_1^3 + 2b_1 a_1^2, \qquad r_5 = b_1^4 + 3b_1^2 a_1^2 + a_1^4 + 2b_1 a_1^2 b_2 + a_1^2 b_2^2, \qquad \dots. \tag{23}$$

When the Nevanlinna function f is trivial, $f_N = b_N$ is a real constant for some index N. Then, the expressions for r_n follow from the general ones (23) assuming that $a_N = 0$, which amounts to the cancellation of all the terms including a_N.

Proposition 3.6 states that a Nevanlinna function f admits the Schur algorithm on the real line when the corresponding measure μ is a probability one with finite moments. These are precisely the measures having an associated sequence of orthonormal polynomials p_n with $p_0 = 1$. Such polynomials are given by a three term recurrence relation

$$\mathcal{J} p(x) = x p(x), \qquad p = (p_0, p_1, \dots)^{\mathsf{T}}, \tag{24}$$

encoded by a Jacobi matrix $\mathcal{J}$.

On the other hand, rewriting the Schur algorithm on the real line for the m-functions m_n of the iterates f_n leads to the relation

$$m_n(z) = \frac{1}{b_n - z - a_n^2 m_{n+1}(z)}, \qquad m_0 = m. \tag{25}$$

This identifies a_n and b_n as the parameters of the three term recurrence relation for the orthonormal polynomials [13,Sect. 5], so that the corresponding Jacobi matrix is given by

$$\mathcal{J} = \begin{pmatrix} b_0 & a_0 & & & \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & a_2 & \\ & & a_2 & b_3 & a_3 \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad a_n > 0, \qquad b_n \in \mathbb{R}. \tag{26}$$

The Schur algorithm on the real line stops at the n-th step if the n-th iterate f_n is degenerate, which means that f_n must be the real constant $b_n = f_n(0)$ and $a_n = f_n'(0)^{1/2} = 0$. This leads to a finite sequence of Schur parameters, $(b_0, a_0, b_1, a_1, \dots, b_{n-1}, a_{n-1}, b_n, 0)$, which identifies f as a trivial Nevanlinna function whose finitely supported measure μ is associated to the finite Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} b_0 & a_0 & & & & \\ a_0 & b_1 & a_1 & & & \\ & a_1 & b_2 & a_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & b_{n-1} & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix}.$$

Otherwise, the algorithm generates infinitely many Schur parameters giving an infinite Jacobi matrix, hence μ is supported on infinitely many points and f is non-trivial. Every regular Nevanlinna function determines a—finite or infinite—sequence of Schur parameters. However, the Schur parameters determine a unique Nevanlinna function or infinitely many ones depending whether the corresponding Jacobi matrix is related to a determinate or indeterminate moment problem.

The following example of trivial Nevanlinna functions is an especially important one.

Proposition 3.8 *Let P_n be a sequence of monic orthogonal polynomials on the real line with three term recurrence relation given by*

$$xP_{n-1}(x) = P_n(x) + b_{n-1}P_{n-1}(x) + a_{n-2}^2 P_{n-2}(x),$$
$$P_0 = 1, \; P_{-1} = 0, \qquad a_n > 0, \; b_n \in \mathbb{R}.$$

Defining the reversed polynomials by

$$P_n^*(z) = z^n P_n(z^{-1}),$$

the rational function

$$R_n(z) = b_n + a_{n-1}^2 z \frac{P_{n-1}^*(z)}{P_n^*(z)} = b_n + a_{n-1}^2 \frac{P_{n-1}(z^{-1})}{P_n(z^{-1})} \tag{27}$$

is a trivial Nevanlinna function with Schur parameters $(b_n, a_{n-1}, b_{n-1}, \ldots, a_0, b_0, 0)$ and iterates $R_{n-1}, R_{n-2}, \ldots, R_0$.

Proof Rewritten in terms of the reversed polynomials, the three term recurrence relation becomes

$$P_{n-1}^* = P_n^* + b_{n-1}zP_{n-1}^* + a_{n-2}^2 z^2 P_{n-2}^*. \tag{28}$$

Dividing by P_{n-1}^*, this gives

$$1 = \frac{a_{n-1}^2 z}{R_n - b_n} + zR_{n-1}.$$

Equivalently,

$$R_{n-1} = \frac{1}{z}\frac{R_n - b_n - a_{n-1}^2 z}{R_n - b_n}.$$

This proves the result, bearing in mind that $R_n(0) = b_n$, $R_n'(0) = a_{n-1}^2$ and $R_0 = b_0$. $\square$

Definition 3.9 In analogy with the terminology in [10], if a probability measure μ is an orthogonality measure for P_n—i.e., $(b_0, a_0, b_1, a_1, \ldots)$ are the Schur parameters of μ—and f is the related Nevanlinna function, R_n will be called the **inverse 'Schur' iterates** of f or μ. Bearing in mind the compact form (17) of the Schur iterations, the inverse iterates may be viewed as the Nevanlinna functions generated by the recurrence relation

$$(R_{n+1}(z) - b_{n+1})(1 - zR_n(z)) = a_n^2 z, \qquad R_0 = b_0. \tag{29}$$

The monic orthogonal polynomials P_n are related to the orthonormal polynomials p_n satisfying (24) by

$$P_n = \Upsilon_{n-1} p_n, \qquad \Upsilon_n = a_0 a_1 \cdots a_n. \tag{30}$$

Therefore, the inverse Schur iterates (27) may be also rewritten as

$$R_n(z) = b_n + a_{n-1} z \frac{p_{n-1}^*(z)}{p_n^*(z)} = b_n + a_{n-1} \frac{p_{n-1}(z^{-1})}{p_n(z^{-1})}, \tag{31}$$

where $p_n^*(z) = z^n p_n(z^{-1})$.

Inverse Schur iterates constitute more than mere examples of trivial Nevanlinna functions, but characterize such kind of functions, which are the Nevanlinna version of the Schur functions given by finite Blaschke products. The next result provides several characterizarions of trivial Nevanlinna functions which, analogously to the case of the unit circle, will play a prominent role in Khrushchev's formula on the real line.

Proposition 3.10 *The following statements are equivalent:*

(i) f *is a trivial Nevanlinna function.*
(ii) f *has the form* (27) *for some sequence of monic orthogonal polynomials on the real line.*
(iii) f *is a real rational function analytic at the origin such that* $f'(0) > 0$ *and* $f(z) - f(0)$ *has simple zeros and poles in the extended complex plane, all of them lying in the extended real line and interlacing.*
(iv) $f(z) = b + \sum_{k=1}^{n} \frac{N_k}{z^{-1} - x_k}$ *with* $b, x_k \in \mathbb{R}$ *and* $N_k > 0$.

Proof (i) $\Leftrightarrow$ (ii) is a direct consequence of the fact that the rational functions R_n given in (27) exhaust all the trivial Nevanlinna functions with $2n + 1$ Schur parameters by running $b_k \in \mathbb{R}$ and $a_k > 0$.

(ii) $\Leftrightarrow$ (iii) follows from the properties of the zeros of orthogonal polynomials on $\mathbb{R}$. As a consequence of Geronimus-Wendroff's theorem, a rational function proportional to the quotient P_{n-1}/P_n of consecutive orthogonal polynomials on $\mathbb{R}$ is characterized by having n simple poles and $n - 1$ simple zeros on $\mathbb{R}$ which interlace. In other words, such a rational function is characterized by having simple interlacing poles and zeros on the extended real line $\overline{\mathbb{R}}$, being the infinity one of the simple zeros (which means that substituting z by $1/z$ gives a simple zero at the origin because $\deg P_{n-1} = \deg P_n - 1$). Equivalently, a rational function proportional to $zP_{n-1}^*(z)/P_n^*(z) = P_{n-1}(z^{-1})/P_n(z^{-1})$ is characterized by having simple interlacing poles and zeros on $\overline{\mathbb{R}}$, the origin being one of the zeros.

On the other hand, the parameters $b_n = R_n(0) \in \mathbb{R}$ and $a_{n-1} = R_n'(0) > 0$ in (27) are independent of P_{n-1}/P_n, which only depends on $b_0, a_0, \ldots, b_{n-2}, a_{n-2}, b_{n-1}$. A rational function f satisfies *(iii)* iff $f = b + ag$ where $a > 0$, $b \in \mathbb{R}$ and g is a real rational function vanishing at the origin, whose zeros and poles lie on $\overline{\mathbb{R}}$, are simple and interlace. Therefore, the previous comments prove that f has the form (27) iff it satisfies *(iii)*.

(i) $\Leftrightarrow$ (iv) may be proved resorting to the fact that a Nevanlinna function f and its iterate f_1 are simultaneously trivial. Hence, f is trivial iff the measure $\mu^{(1)}$ of f_1 is finitely supported, i.e.

$$d\mu^{(1)}(x) = \sum_{k=1}^{n} M_k\, \delta(x - x_k), \qquad x_k \in \mathbb{R}, \qquad M_k > 0, \qquad \sum_{k=1}^{n} M_k = 1.$$

From (16) and (9) we find that $\mu^{(1)}$ has the above form iff

$$\begin{aligned} f(z) &= f(0) + \frac{f'(0)z}{1 - zf_1(z)} = f(0) + f'(0)z \int_{\mathbb{R}} \frac{d\mu^{(1)}(x)}{1 - zx} \\ &= b + \sum_{k=1}^{n} \frac{N_k z}{1 - zx_k}, \qquad \begin{array}{l} b = f(0) \in \mathbb{R}, \\ N_k = f'(0)M_k > 0, \end{array} \end{aligned}$$

which proves the equivalence. □

4 Convergence Properties

In this section we will discuss some convergence properties relating Nevanlinna functions, measures on $\mathbb{R}$ and Schur parameters. Apart from their own interest, they are useful for the analysis of the convergence of the continued fraction generated by the Schur algorithm on the real line, as well as for the development and applications of Khrushchev's formula on the real line.

We are dealing with the set $\mathfrak{N}$ of Nevanlinna functions, the set $\mathfrak{M}$ of subprobability measures on $\mathbb{R}$ and the set of—finite or infinite– sequences of Schur parameters, given by

$$\mathfrak{S} = (\mathbb{R} \times \mathbb{R}^+)^\infty \bigcup \left(\bigcup_{n=0}^{\infty} (\mathbb{R} \times \mathbb{R}^+)^n \times (\mathbb{R} \times \{0\}) \right), \qquad \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}.$$

Several convergence notions will be considered in these sets. Given $f \in \mathfrak{N}$, $\mu \in \mathfrak{M}$, $\boldsymbol{\gamma} \in \mathfrak{S}$ and sequences $f^{[k]} \in \mathfrak{N}$, $\mu^{[k]} \in \mathfrak{M}$, $\boldsymbol{\gamma}^{[k]} \in \mathfrak{S}$, we will use the following notation:

$f^{[k]} \to f \quad \equiv \quad$ **pointwise convergence** on $\mathbb{C} \setminus \mathbb{R}$,

$f^{[k]} \rightrightarrows f \quad \equiv \quad$ **local uniform convergence** on $\mathbb{C} \setminus \mathbb{R}$, i.e. uniform convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$,

$\mu^{[k]} \xrightarrow{v} \mu \quad \equiv \quad$ **vague convergence**, i.e. $\lim_{k\to\infty} \int_{\mathbb{R}} h\, d\mu^{[k]} = \int_{\mathbb{R}} h\, d\mu$ for every continuous function h on $\mathbb{R}$ such that $\lim_{|x|\to\infty} h(x) = 0$,

$\mu^{[k]} \xrightarrow{w} \mu \quad \equiv \quad$ **weak convergence**, i.e. $\lim_{k\to\infty} \int_{\mathbb{R}} h\, d\mu^{[k]} = \int_{\mathbb{R}} h\, d\mu$ for every bounded continuous function h on $\mathbb{R}$,

$\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma} \quad \equiv \quad$ **pointwise convergence** on $\mathfrak{S}$, i.e. $\lim_{k\to\infty} \gamma_n^{[k]} = \gamma_n$ for each n such that γ_n exists.

The pointwise convergence in $\mathfrak{S}$ makes sense even if the number of Schur parameters of $\boldsymbol{\gamma}^{[k]}$ depends on k and is different from the number of Schur parameters of $\boldsymbol{\gamma}$. In that case the convergence requires implicitly that, for each n such that γ_n exits, there is an index $k_0(n)$ such that $\gamma_n^{[k]}$ exists for $k \geq k_0(n)$. For instance, if $\boldsymbol{\gamma}^{[k]}$ are finite but of increasing length, they may converge to an infinite sequence of Schur parameters. Also, the convergence of $\boldsymbol{\gamma}^{[k]} = (b_0^{[k]}, a_0^{[k]}, b_1^{[k]}, a_1^{[k]}, \dots)$ to a finite sequence $(b_0, a_0, b_1, a_1, \dots, b_n, 0)$ means that

$$\lim_{k\to\infty} b_j^{[k]} = b_j, \quad 0 \leq j \leq n; \quad \lim_{k\to\infty} a_j^{[k]} = a_j, \quad 0 \leq j \leq n-1; \quad \lim_{k\to\infty} a_n^{[k]} = 0.$$

The presence of the last condition is necessary to guarantee the uniqueness of the limit, otherwise the convergence of $\boldsymbol{\gamma}^{[k]}$ to a finite or infinite sequence $\boldsymbol{\gamma}$ would imply also the convergence of $\boldsymbol{\gamma}^{[k]}$ to any finite part of $\boldsymbol{\gamma}$.

Concerning the vague and weak convergence in $\mathfrak{M}$, they are related by

$$\mu^{[k]} \xrightarrow{w} \mu \quad \Leftrightarrow \quad \mu^{[k]} \xrightarrow{v} \mu, \quad \lim_{k\to\infty} \mu^{[k]}(\mathbb{R}) = \mu(\mathbb{R}). \tag{32}$$

The right implication is obvious. The left one may be reduced to the case of probability measures by considering $\mu^{[k]}/\mu^{[k]}(\mathbb{R})$ and $\mu/\mu(\mathbb{R})$, in which case a proof can be found in [2,Theorem 4.4.2]. The vague convergence only has the following consequence,

$$\mu^{[k]} \xrightarrow{v} \mu \quad \Rightarrow \quad \liminf_{k\to\infty} \mu^{[k]}(\mathbb{R}) \geq \mu(\mathbb{R}), \tag{33}$$

as follows by taking $n \to \infty$ in $\mu([-n,n]) = \lim_{k\to\infty} \mu^{[k]}([-n,n]) \leq \liminf_{k\to\infty} \mu^{[k]}(\mathbb{R})$. Obviously, a situation where vague convergence becomes weak convergence is when the supports of the measures $\mu^{[k]}$ lie on the same bounded subset of $\mathbb{R}$.

As a consequence of Montel's normality criteria [11] and Helly's selection theorem [6], local uniform convergence and vague convergence yield similar compactness properties to $\mathfrak{N}$ and $\mathfrak{M}$:

- Let $f^{[k]} \in \mathfrak{N}$ be a **locally uniformly bounded** sequence, i.e. uniformly bounded on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. Then, $f^{[k]}$ has a locally uniformly convergent subsequence [12,Section 2.2]. Besides, if all the locally uniformly convergent subsequences of $f^{[k]}$ have the same limit function f, then $f^{[k]} \rightrightarrows f$ [12,Theorem 2.4.2].
- Every sequence $\mu^{[k]} \in \mathfrak{M}$ has a vaguely convergent subsequence [2,Theorem 4.3.3]. Besides, if all the vaguely convergent subsequences of $\mu^{[k]}$ have the same limit measure μ, then $\mu^{[k]} \xrightarrow{v} \mu$ [2,Theorem 4.3.4].

These results shed light on the relation between pointwise and local uniform convergence,

$$f^{[k]} \rightrightarrows f \quad \Leftrightarrow \quad f^{[k]} \to f, \quad f^{[k]} \text{ locally uniformly bounded.} \tag{34}$$

The right implication is trivial. The left one follows by noticing that any locally uniformly convergent subsequence of $f^{[k]}$ must have the limit f if $f^{[k]} \to f$.

Although the boundedness condition in (34) cannot be avoided for general analytic functions, pointwise and local uniform convergence become equivalent in $\mathfrak{N}$. Despite being a known result, the following proposition includes a new proof based on the representation (13) for Nevanlinna functions. The aim of the proposition is a relation with the convergence notions in $\mathfrak{M}$ which will be of interest for the discussion of Khrushchev's formula on the real line.

Proposition 4.1 *If $f^{[k]}$, f are Nevanlinna functions with associated measures $\mu^{[k]}$, μ, respectively, then*

$$f^{[k]} \rightrightarrows f \quad \Leftrightarrow \quad f^{[k]} \to f \quad \Leftrightarrow \quad \mu^{[k]} \xrightarrow{v} \mu.$$

Furthermore, when f is normalized, $\mu^{[k]} \xrightarrow{v} \mu$ becomes $\mu^{[k]} \xrightarrow{w} \mu$ in the above equivalence.

Proof Let $m^{[k]}$, m be the m-functions of $\mu^{[k]}$, μ, respectively.

Suppose that $\mu^{[k]} \xrightarrow{v} \mu$. Then $m^{[k]} \to m$ because, for $z \in \mathbb{C} \setminus \mathbb{R}$, $1/(x-z)$ is a continuous function of $x \in \mathbb{R}$ which vanishes for $|x| \to \infty$. The relation (13) shows that $f^{[k]} \to f$ because, as the m-function of a non-null measure μ, m is a non-null Nevanlinna function which thus vanishes at no point of $\mathbb{C} \setminus \mathbb{R}$. Besides, since $\mu^{[k]}$ is a subprobability measure, $|m^{[k]}(z)| \leq 1/|\operatorname{Im} z|$ so that $m^{[k]}$ is locally uniformly bounded. This implies that $m^{[k]} \rightrightarrows m$. Also, $-1/m$ is bounded on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ because it is a Nevanlinna function, hence the relation $|m^{[k]}| \geq |m| - |m^{[k]} - m|$ shows that the sequence $-1/m^{[k]} \in \mathfrak{N}$ is locally uniformly bounded. In consequence, (13) yields $f^{[k]} \rightrightarrows f$.

Assume now that $f^{[k]} \to f$. Then, (13) shows that $m^{[k]} \to m$ because $f - z^{-1}$ is a non-null Nevanlinna function. If a subsequence $\mu^{[k_j]}$ converges vaguely to a measure $\tilde{\mu}$ with m-function $\tilde{m}$, the previous arguments show that $m^{[k_j]} \to \tilde{m}$. Therefore, $\tilde{m} = m$ and $\tilde{\mu} = \mu$. That is, the limit of every vaguely convergent subsequence of $\mu^{[k]}$ must be μ, which proves that $\mu^{[k]} \xrightarrow{v} \mu$.

Finally, if $f^{[k]} \to f$ with f normalized, then $\mu^{[k]} \xrightarrow{v} \mu$ with μ a probability measure. Since $\liminf_{k\to\infty} \mu^{[k]}(\mathbb{R}) \geq \mu(\mathbb{R}) = 1$, we get $\lim_{k\to\infty} \mu^{[k]}(\mathbb{R}) = 1 = \mu(\mathbb{R})$ because $\mu^{[k]}(\mathbb{R}) \leq 1$. Therefore, the vague convergence is indeed weak convergence. □

Remark 4.2 It is worth remarking that, in the above result, the assumption that μ is a non-null measure—as a measure of a Nevanlinna function—is crucial. For instance, $\mu^{[k]} \xrightarrow{v} 0$ for $\mu^{[k]}$ the Dirac delta at $x = k$, while $f^{[k]} = k$ is non-convergent in this case. Actually, this example illustrates a general rule: if $f^{[k]}$ and $m^{[k]}$ are respectively the Nevanlinna functions and m-functions of the measures $\mu^{[k]}$, similar arguments to those given in the proof of Proposition 4.1 show that

$$\mu^{[k]} \xrightarrow{v} 0 \quad \Leftrightarrow \quad m^{[k]} \rightrightarrows 0 \quad \Leftrightarrow \quad m^{[k]} \to 0 \quad \Leftrightarrow \quad f^{[k]} \to \infty.$$

□

As a consequence of Proposition 4.1 and (7),

$$f^{[k]}, f \in \mathfrak{N}: \quad f^{[k]} \to f \neq 0 \quad \Rightarrow \quad 1/f^{[k]} \rightrightarrows 1/f. \tag{35}$$

Another consequence of the previous proposition is that $f^{[k]} \to f$ in $\mathfrak{N}$ implies that $f^{[k]}$ is locally uniformly bounded. The following proposition yields another locally uniformly boundedness condition in $\mathfrak{N}$. It will be key for the subsequent discussion of the relation between the convergence in $\mathfrak{N}$ and $\mathfrak{S}$.

Proposition 4.3 *Let $\mathfrak{F}$ be a family of normalized Nevanlinna functions. If $f(0)$ and $f'(0)$ exist—in the sense of Definition 3.5—and are bounded for $f \in \mathfrak{F}$, then $\mathfrak{F}$ is locally uniformly bounded.*

Proof We need to prove that every $f \in \mathfrak{F}$ is bounded on any compact $K \subset \mathbb{C} \setminus \mathbb{R}$ by a bound depending on K but not on f. If $f'(0) = 0$, then $f(z) = f(0)$ because it is degenerate [5,Appendix B]. Otherwise, $f'(0) > 0$ and the first step of the Schur algorithm on the real line applies to f. If f_1 is the first iterate of f, (16) yields

$$f(z) = f(0) - \frac{f'(0)}{f_1(z) - z^{-1}}.$$

Hence, bearing in mind that $|f_1(z) - z^{-1}| \geq |\operatorname{Im}(f_1(z) - z^{-1})| \geq |\operatorname{Im} z^{-1}|$ because f_1 is a Nevanlinna function, we find that

$$\sup_{z \in K} |f(z)| \leq |f(0)| + \frac{|f'(0)|}{\inf\limits_{z \in K} \left|\operatorname{Im} z^{-1}\right|}.$$

The desired bound follows from that of $f(0)$ and $f'(0)$ for $f \in \mathfrak{F}$. Obviously, the degenerate case $f(z) = f(0)$ is also covered by such a bound. □

Regarding the relation between the convergence notions in $\mathfrak{N}$ and $\mathfrak{S}$, similarly to the case of Schur functions, one could expect that the convergence $f^{[k]} \rightrightarrows f$ for regular Nevanlinna functions implies the convergence $\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}$ for the related Schur parameters. This is not the case, as the following simple example clearly shows.

Example 4.4 Consider the sequence of Nevanlinna functions given by

$$f^{[k]}(z) = \frac{1}{1 - kz},$$

which satisfies $f^{[k]} \rightrightarrows f = 0$. The corresponding Schur parameters are

$$\boldsymbol{\gamma}^{[k]} = (1, \sqrt{k}, k, 0), \qquad \boldsymbol{\gamma} = (0, 0),$$

thus, not only $\boldsymbol{\gamma}^{[k]} \nrightarrow \boldsymbol{\gamma}$, but $\boldsymbol{\gamma}^{[k]}$ is non-convergent.

In the previous example the lack of convergence of $\boldsymbol{\gamma}^{[k]}$ is due to the divergence of some of the coefficients. This suggests to try a boundedness condition on $\boldsymbol{\gamma}^{[k]}$ necessary for its convergence, as a minimal requirement to get such a convergence. This idea is materialized in the next theorem.

Theorem 4.5 *Let $f^{[k]}$, f be regular Nevanlinna functions with Schur parameters $\boldsymbol{\gamma}^{[k]}$, $\boldsymbol{\gamma}$, respectively. If, for each fixed $n \geq 0$ such that γ_n exists, the sequence $\gamma_n^{[k]}$ is bounded, then*

$$f^{[k]} \to f \quad \Rightarrow \quad \boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}, \quad f_n^{[k]} \rightrightarrows f_n \text{ if } f_n \text{ exists}. \tag{36}$$

Conversely, if the moment problem for $\boldsymbol{\gamma}$ is determinate, then

$$\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma} \quad \Rightarrow \quad f^{[k]} \rightrightarrows f. \tag{37}$$

Proof Assume that $f^{[k]} \to f$ and $\gamma_n^{[k]}$ is bounded in k for each n such that γ_n exists. Although we know that actually $f^{[k]} \rightrightarrows f$, we will rederive this using a kind of argument that will be necessary later on in situations where no pointwise convergence is available. Since f has at least the first two Schur parameters, b_0 and a_0—this last one vanishing in the degenerate case–, the boundedness hypothesis on the Schur parameters guarantees that Proposition 4.3 applies to the sequence $f^{[k]}$, hence $f^{[k]}$ is locally uniformly bounded and $f^{[k]} \rightrightarrows f$. Denoting by $f_n^{[k]}$ the iterates of $f^{[k]}$, (16) leads to

$$f^{[k]} = b_0^{[k]} + \frac{a_0^{[k]2} z}{1 - z f_1^{[k]}} = b_0^{[k]} + a_0^{[k]2} z + a_0^{[k]2} z^2 \frac{f_1^{[k]}}{1 - z f_1^{[k]}}. \tag{38}$$

This expression remains valid when $f^{[k]}$ is degenerate because, although this means that it has no iterates, in that case $a_0^{[k]} = 0$ and we can take for instance $f_1^{[k]} = 0$. Let $\tilde{b}_0$ be a limit point of the bounded sequence $b_0^{[k]}$. Since $a_0^{[k]}$ is also bounded, we may suppose without loss that there exists a subsequence k_j such that $b_0^{[k_j]}$ converges to $\tilde{b}_0$ and $a_0^{[k_j]}$ converges to some $\tilde{a}_0$. Due to the boundedness of the Schur parameters, Proposition 4.3 also applies to the sequence $f_1^{[k]}$. Hence, $f_1^{[k]}$ is locally uniformly bounded, thus we also may assume without loss that $f_1^{[k_j]} \rightrightarrows g$ for some $g \in \mathfrak{N}$. Since $f_1^{[k_j]} - z^{-1} \rightrightarrows g - z^{-1} \in \mathfrak{N} \setminus \{0\}$, in view of (35), we get $1/(1 - z f_1^{[k_j]}) \rightrightarrows 1/(1 - zg)$. Therefore, taking subsequential limits in (38) gives

$$\tilde{b}_0 + \tilde{a}_0^2 z + \tilde{a}_0^2 z^2 \frac{g}{1 - zg} = b_0 + a_0^2 z + a_0^2 z^2 \frac{f_1}{1 - z f_1},$$

so that $\tilde{b}_0 + \tilde{a}_0^2 z + o(z) = b_0 + a_0^2 z + o(z)$. This implies that $\tilde{b}_0 = b_0$ and $\tilde{a}_0 = a_0$, proving that b_0 is the only limit point of $b_0^{[k]}$, i.e. $b_0^{[k]}$ converges to b_0. A similar reasoning starting with a limit point of $a_0^{[k]}$ shows that this sequence converges to a_0. Also, if $a_0 > 0$—so that f_1 is the true iterate of f—the same arguments, but starting with a subsequential limit of $f_1^{[k]}$ with respect to the local uniform convergence, yield $f_1^{[k]} \rightrightarrows f_1$.

At this stage, if $a_0 > 0$, there is a second iterate f_2 of f, which is the first iterate of f_1. Then, the sequence $f^{[k]}$ may be replaced in the above arguments by $f_1^{[k]}$ to prove that $b_1^{[k]}$, $a_1^{[k]}$ and $f_2^{[k]}$ converge to b_1, a_1 and f_2, respectively. The general result (36) follows by induction.

Suppose now that $\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}$. Then, the boundedness hypothesis on $\boldsymbol{\gamma}^{[k]}$ is guaranteed, thus Proposition 4.3 applies to $f^{[k]}$, which is therefore locally bounded. Consider a locally uniformly convergent subsequence $f^{[k_j]} \rightrightarrows \tilde{f}$. In view of (36), $\boldsymbol{\gamma}^{[k]} \to \tilde{\boldsymbol{\gamma}}$ where $\tilde{\boldsymbol{\gamma}}$ is the sequence of Schur parameters of $\tilde{f}$. Therefore, $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}$ and the determinacy of the moment problem related to $\boldsymbol{\gamma}$ ensures that $\tilde{f} = f$. Since $\tilde{f}$ was any subsequential limit of the locally bounded sequence $f^{[k]}$, (37) is proved. $\square$

The above theorem is the central result of this section. It will trivialize the convergence of the continued fraction expansion of a regular Nevanlinna function discussed in Sect. 6. Even more important, as in the case of the unit circle, in Sect. 7 it will be key to extend the real line version of Khruschev's formula beyond the case of absolutely continuous measures. The following restatements of the convergence of the Schur parameters are useful for this purpose.

Proposition 4.6 *Let $f^{[k]}$, f be regular Nevanlinna functions with measures $\mu^{[k]}$, μ and Schur parameters $\boldsymbol{\gamma}^{[k]}$, $\boldsymbol{\gamma}$, respectively. If $s_n^{[k]}$, s_n are the Taylor coefficients for the asymptotic expansion around the origin of $f^{[k]}(z)$, $f(z)$, $z \in i\mathbb{R}$, and $\mu_n^{[k]}$, μ_n are the moments of $\mu^{[k]}$, μ, respectively, the following conditions are equivalent:*

(i) $\lim_{k\to\infty} \gamma_n^{[k]} = \gamma_n$ *for* $n \le N$.
(ii) $\lim_{k\to\infty} s_n^{[k]} = s_n$ *for* $n \le N$.
(iii) $\lim_{k\to\infty} \mu_n^{[k]} = \mu_n$ *for* $n \le N+1$.

Proof The equivalence (i) $\Leftrightarrow$ (ii) follows by induction from Proposition 3.7, since it gives $\gamma_0 = s_0$, $\gamma_1^2 = s_1$, while (20) yields

$$\gamma_{2n+1}^2 = \frac{s_{2n+1} - \gamma_1^2 r_{2n+1}(\gamma_2, \dots, \gamma_{2n})}{\gamma_1^2 \cdots \gamma_{2n-1}^2}, \quad \gamma_{2n+2} = \frac{s_{2n+2} - \gamma_1^2 r_{2n+2}(\gamma_2, \dots, \gamma_{2n+1})}{\gamma_1^2 \cdots \gamma_{2n+1}^2}.$$

Relation (21) from the same proposition also leads by induction to (ii) $\Leftrightarrow$ (iii). $\square$

5 Wall Polynomials on $\mathbb{R}$

The previous real line version of the Schur algorithm will be used as a starting point for the introduction of the Wall polynomials on $\mathbb{R}$. For this purpose, given a regular Nevanlinna function f, we express the relation (16) between consecutive iterates in the matrix form

$$\begin{pmatrix} f_n(z) \\ 1 \end{pmatrix} = \frac{1}{1 - z f_{n+1}(z)} \begin{pmatrix} -b_n z & b_n + a_n^2 z \\ -z & 1 \end{pmatrix} \begin{pmatrix} f_{n+1}(z) \\ 1 \end{pmatrix},$$

which we denote in short by

$$f_n(z) \doteq \begin{pmatrix} -b_n z & b_n + a_n^2 z \\ -z & 1 \end{pmatrix} f_{n+1}(z).$$

Iterating this relation yields

$$f(z) \doteq \begin{pmatrix} -b_0 z & b_0 + a_0^2 z \\ -z & 1 \end{pmatrix} \begin{pmatrix} -b_1 z & b_1 + a_1^2 z \\ -z & 1 \end{pmatrix} \cdots \begin{pmatrix} -b_n z & b_n + a_n^2 z \\ -z & 1 \end{pmatrix} f_{n+1}(z).$$

Equivalently,

$$f \doteq \begin{pmatrix} C_n & A_n \\ D_n & B_n \end{pmatrix} f_{n+1}, \tag{39}$$

with A_n, B_n, C_n and D_n real polynomials given by

$$\begin{pmatrix} C_n & A_n \\ D_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & A_{n-1} \\ D_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} -b_n z & b_n + a_n^2 z \\ -z & 1 \end{pmatrix}, \qquad \begin{pmatrix} C_{-1} & A_{-1} \\ D_{-1} & B_{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The relation (39) means that

$$f = \frac{A_n + C_n f_{n+1}}{B_n + D_n f_{n+1}}. \tag{40}$$

Since z divides C_n and D_n, we can define new polynomials X_n and Y_n by

$$C_n = zX_n, \qquad D_n = zY_n.$$

We conclude that

$$f = \frac{A_n + zX_n f_{n+1}}{B_n + zY_n f_{n+1}}, \tag{41}$$

where the polynomials A_n, B_n, X_n and Y_n satisfy

$$\begin{pmatrix} X_n & A_n \\ Y_n & B_n \end{pmatrix} = \begin{pmatrix} X_{n-1} & A_{n-1} \\ Y_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} -b_n z & c_n z \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} X_{-1} & A_{-1} \\ Y_{-1} & B_{-1} \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{42}$$

$$c_n = b_n + a_n^2 z,$$

Writing (42) explicitly yields

$$X_n = -b_n z X_{n-1} - A_{n-1}, \qquad X_0 = -b_0, \tag{43}$$

$$A_n = c_n z X_{n-1} + A_{n-1}, \qquad A_0 = c_0, \tag{44}$$

$$Y_n = -b_n z Y_{n-1} - B_{n-1}, \qquad Y_0 = -1, \tag{45}$$

$$B_n = c_n z Y_{n-1} + B_{n-1}, \qquad B_0 = 1. \tag{46}$$

We call A_n, B_n, X_n and Y_n the **Wall polynomials for the Nevanlinna function** f, in short, the Wall polynomials on $\mathbb{R}$. The existence of all the Wall polynomials only holds for regular Nevanlinna functions. Nevertheless, as it is the case for the first $n+1$ steps of the Schur algorithm on the real line, the Wall polynomials A_n, B_n, X_n and Y_n exist as long as μ has finite moments $\mu_1, \dots, \mu_{2n+2}$—equivalently, there exist the first $2n+1$ derivatives at the origin of $f(z)$, $z \in i\mathbb{R}$—which is the only requirement for the validity of (41).

We should remark that, as follows from (42), the Wall polynomials only depend on the Schur parameters of f, hence, in the indeterminate case, they are common to the Nevanlinna functions of all the probability measures related to the same Jacobi matrix. More precisely, (43)–(46) show that X_n and Y_n are determined by the Schur parameters $(b_0, a_0, \dots, b_{n-1}, a_{n-1}, b_n)$, while A_n and B_n also depend on the additional parameter a_n. This dependence is clarified in the next proposition, which establishes some of the properties of the Wall polynomials on the real line, which should be compared with those of their unit circle counterparts.

Theorem 5.1 *Let A_n, B_n, X_n, Y_n be the Wall polynomials of a Nevanlinna function f with Schur parameters $(b_0, a_0, b_1, a_1, \dots)$ and iterates f_n. If Υ_n is given by (30), then:*

(i) $X_n B_n - Y_n A_n = \Upsilon_n^2 z^{2n+1}$.

(ii) $\deg X_n, \deg Y_n \le n$ *and* $\deg A_n, \deg B_n \le n+1$ *for* $n \ge 0$.

(iii) The zeros of A_n, B_n, X_n and Y_n are real for $n \ge 1$. These zeros lie on $\mathbb{R} \setminus \{0\}$ except for X_n and A_n when $b_0 = 0$, in which case both have a simple zero at the origin.

(iv) The rational functions A_n/B_n, X_n/Y_n, zY_n/B_n and zX_n/A_n are irreducible for $n \ge 1$, except for the last one when $b_0 = 0$, in which case the irreducible representation is $z\hat{X}_n/\hat{A}_n$ with $A_n = z\hat{A}_n$ and $X_n = z\hat{X}_n$.

(v) A_n/B_n, X_n/Y_n, $-zY_n/B_n$ and $-zX_n/A_n$ are regular Nevanlinna functions.

(vi) A_n/B_n has Schur parameters $(b_0, a_0, \dots, b_n, a_n, 0, 0)$, X_n/Y_n has Schur parameters $(b_0, a_0, \dots, b_{n-1}, a_{n-1}, b_n, 0)$.

(vii) $A_n + zX_n f_{n+1} = f \prod_{k=1}^{n+1}(1 - zf_k)$, $B_n + zY_n f_{n+1} = \prod_{k=1}^{n+1}(1 - zf_k)$.

(viii)
$$B_n f - A_n = \frac{\Upsilon_n^2 z^{2n+2} f_{n+1}}{\prod_{k=1}^{n+1}(1 - zf_k)} = z^{n+1} f_{n+1} \prod_{k=0}^{n}(f_k - b_k),$$
$$X_n - Y_n f = \frac{\Upsilon_n^2 z^{2n+1}}{\prod_{k=1}^{n+1}(1 - zf_k)} = z^n \prod_{k=0}^{n}(f_k - b_k).$$

Proof Denoting

$$\Delta_n = \det \begin{pmatrix} X_n & A_n \\ Y_n & B_n \end{pmatrix},$$

(42) yields $\Delta_n = a_n^2 z^2 \Delta_{n-1}$ with $\Delta_{-1} = z^{-1}$. This implies that $\Delta_n = \Upsilon_n^2 z^{2n+1}$, proving *(i)*.

Statement *(ii)* follows from (43)–(46) by an inductive argument.

To prove *(iv)* note that, in view of *(i)*, the origin is the only possible common zero of the numerator and the denominator for each of the rational functions. Evaluating (43)–(46) at $z = 0$ we find that $B_n(0) = 1$, $Y_n(0) = -1$, $A_n(0) = b_0$ and $X_n(0) = -b_0$

for $n \geq 0$. This shows that no Wall polynomial vanishes at $z = 0$, unless $b_0 = 0$, in which case A_n and X_n are the only ones with a zero at the origin. If $b_0 = 0$, the polynomials $\hat{A}_n = z^{-1}A_n$ and $\hat{X}_n = z^{-1}X_n$ satisfy the same recurrence relations (43), (44) as X_n and A_n, but with initial conditions, $\hat{X}_0 = 0$ and $\hat{A}_0 = a_0^2$. Evaluating such recurrence relations at $z = 0$ we get $\hat{X}_n(0) = -a_0^2$ and $\hat{A}_n(0) = a_0^2$ for $n \geq 1$, which are non-null. This proves *(iv)* and the second part of *(iii)*. To complete the proof of *(iii)* it only remains to see that the n-th Wall polynomials do not vanish on $\mathbb{C} \setminus \mathbb{R}$ for $n \geq 1$. This is a consequence of the previous results and *(v)*—which we are about to prove below—because a Nevanlinna function is analytic out of the real line and cannot vanish there unless it is null.

To prove that A_n/B_n and X_n/Y_n are regular Nevanlinna functions, let us exploit the fact that (41) is valid for any Nevanlinna function for which the first $n+1$ steps of the Schur algorithm on the real line make sense. This is the case of the regular Nevanlinna function g with Schur parameters $(b_0, a_0, \dots, b_n, a_n, 0, 0)$. Since $g_{n+1} = 0$, the application of (41) to g identifies it with A_n/B_n. A similar reasoning proves that the regular Nevanlinna function with Schur parameters $(b_0, a_0, \dots, b_{n-1}, a_{n-1}, b_n, 0)$ is $(A_{n-1} + b_n z X_{n-1})/(B_{n-1} + b_n z Y_{n-1})$, which coincides with X_n/Y_n, as follows from (43) and (45). Hence, *(vi)* is true.

In the case of the rational function $\phi_n = -zY_n/B_n$ we will use a different strategy. Starting from the non-null Nevanlinna function $\phi_0 = z$, we will show by induction that ϕ_n is obtained by a finite number of the operations (7) which preserve the set $\mathfrak{N} \setminus \{0\}$ of non-null Nevanlinna functions. Indeed, it suffices to see that the relation between ϕ_n and ϕ_{n-1} is a composition of such Nevanlinna preserving operations. This is accomplished by expressing such a relation in the following way, using (45) and (46),

$$\phi_n = z\frac{b_n z Y_{n-1} + B_{n-1}}{b_n z Y_{n-1} + a_n^2 z^2 Y_{n-1} + B_{n-1}} = \frac{1}{\dfrac{1}{z} + \dfrac{a_n^2 z Y_{n-1}}{b_n z Y_{n-1} + B_{n-1}}} = \frac{-1}{-\dfrac{1}{z} - \dfrac{a_n^2}{b_n - \dfrac{1}{\phi_{n-1}}}}.$$

Since $B_n(0) \neq 0$, the rational Nevanlinna function ϕ_n is analytic at the origin and therefore regular.

A similar reasoning proves that $-zX_n/A_n \in \mathfrak{N}$, starting from the non-null Nevanlinna function

$$\begin{aligned} -zX_0/A_0 &= \frac{b_0 z}{b_0 + a_0^2 z} = \frac{-1}{-\dfrac{1}{z} - \dfrac{a_0^2}{b_0}}, && b_0 \neq 0, \\ -zX_1/A_1 &= z, && b_0 = 0. \end{aligned}$$

Besides, the rational Nevanlinna function $-zX_n/A_n$ is regular because in its irreducible form its denominator does not vanish at the origin. This completes the proof of *(v)*.

The identities in *(vii)* follow from (42), which implies that

$$\begin{pmatrix} A_n + zX_n f_{n+1} \\ B_n + zY_n f_{n+1} \end{pmatrix} = \begin{pmatrix} X_n & A_n \\ Y_n & B_n \end{pmatrix} \begin{pmatrix} zf_{n+1} \\ 1 \end{pmatrix} = \begin{pmatrix} X_{n-1} & A_{n-1} \\ Y_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} a_n^2 z^2 + b_n z(1 - zf_{n+1}) \\ 1 - zf_{n+1} \end{pmatrix}$$
$$= (1 - zf_{n+1}) \begin{pmatrix} X_{n-1} & A_{n-1} \\ Y_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} zf_n \\ 1 \end{pmatrix} = (1 - zf_{n+1}) \begin{pmatrix} A_{n-1} + zX_{n-1} f_n \\ B_{n-1} + zY_{n-1} f_n \end{pmatrix},$$

where we have used (17) in the second line. The above relation, together with the values of A_{-1}, B_{-1}, X_{-1} and Y_{-1}, give *(vii)*.

Finally, introducing (41) into $B_n f - A_n$, $X_n - Y_n f$, and resorting to *(i)*, *(vii)*, we arrive at the left equalitites in *(viii)*. The right ones follow by using (17). ◻

The Wall polynomials may be expressed in terms of more familiar ones. Consider for instance the pair X_n and A_n. Adding (43) and (44) yields

$$X_n + A_n = a_n^2 z^2 X_{n-1}.$$

Combining this with (43) to eliminate A_n leads to

$$X_n = X_{n+1} + b_{n+1} z X_n + a_n^2 z^2 X_{n-1}.$$

Finally, introducing the new polynomials $Q_n(x) = z^{-n} X_n(z)$, $x = z^{-1}$, the above relation becomes

$$x Q_n = Q_{n+1} + b_{n+1} Q_n + a_n^2 Q_{n-1}. \tag{47}$$

Due to the similarity between the recurrence relations for the pairs X_n, A_n and Y_n, B_n, defining $Q_n(x) = z^{-n} Y_n(z)$, $x = z^{-1}$, also leads to (47).

The recurrence relation (47) has two distinguished independent solutions which span the the rest of them: the monic orthogonal polynomials $Q_n = P_{n+1}$ whose Jacobi matrix $\mathcal{J}$ is given by the Schur parameters of the Wall polynomials, and the associated polynomials $Q_n = P_n^{(1)}$, related to the stripped Jacobi matrix $\mathcal{J}^{(1)}$ obtained by deleting the first row and column in $\mathcal{J}$. These two solutions are generated respectively by the initial conditions

$$Q_0 = P_1 = x - b_0, \quad Q_{-1} = P_0 = 1,$$
$$Q_0 = P_0^{(1)} = 1, \quad Q_{-1} = P_{-1}^{(1)} = 0.$$

The general solution of (47) is $Q_n = \alpha P_{n+1} + \beta P_n^{(1)}$, with α and β arbitrary functions of x. Bearing in mind the initial conditions for X_n and Y_n, pointed out in (42), (43) and (45), and using the notation $P^*(z) = z^n P(z^{-1})$ for a polynomial P of degree n, we find that

$$X_n = \frac{P_{n+1}^* - P_n^{(1)*}}{z}, \qquad Y_n = -P_n^{(1)*}. \tag{48}$$

Then, (43) and (45) allow us to express the the remaining Wall polynomials in terms of the monic orthogonal polynomials and the associated ones,

$$A_n = -\frac{P_{n+2}^* - P_{n+1}^{(1)*}}{z} - b_{n+1}(P_{n+1}^* - P_n^{(1)*}), \qquad B_n = P_{n+1}^{(1)*} + b_{n+1} z P_n^{(1)*}. \tag{49}$$

6 Wall Continued Fraction on $\mathbb{R}$

The Schur algorithm on the real line provides a continued fraction expansion for every regular Nevanlinna function f. If a_n and b_n are the Schur parameters of f, rewriting the backward n-th step of the algorithm in (16) as

$$f_n = b_n + \frac{a_n^2}{z^{-1} - f_{n+1}}, \tag{50}$$

and iterating it starting from f, leads to

$$\begin{aligned} f &= b_0 + \frac{a_0^2}{z^{-1} - f_1} = b_0 + \cfrac{a_0^2}{z^{-1} - b_1 - \cfrac{a_1^2}{z^{-1} - f_2}} = b_0 + \cfrac{a_0^2}{z^{-1} - b_1 - \cfrac{a_1^2}{z^{-1} - b_2 - \cfrac{a_2^2}{z^{-1} - f_3}}} \\ &= \cdots = b_0 + \frac{a_0^2|}{|z^{-1} - b_1} - \frac{a_1^2|}{|z^{-1} - b_2} - \frac{a_2^2|}{|z^{-1} - b_3} - \cdots - \frac{a_{n-1}^2|}{|z^{-1} - b_n} - \frac{a_n^2|}{|z^{-1} - f_{n+1}}, \end{aligned} \tag{51}$$

which is a continued fraction version of (41). Therefore, the Schur algorithm for a regular Nevanlinna function is closely related to a continued fraction built up out of its Schur parameters, namely,

$$K(z) = b_0 + \frac{a_0^2|}{|z^{-1} - b_1} - \frac{a_1^2|}{|z^{-1} - b_2} - \frac{a_2^2|}{|z^{-1} - b_3} - \cdots - \frac{a_{n-1}^2|}{|z^{-1} - b_n} - \cdots. \tag{52}$$

We will refer to K as the Wall continued fraction for the regular Nevanlinna function f. It has a simple connection with the standard continued fraction for the associated m-function. Rewriting (13) as

$$m(z) = -\frac{1}{z - f(z^{-1})}, \tag{53}$$

and substituting f by K in the above expression yields

$$-\frac{1}{z - K(z^{-1})} = -\frac{1|}{|z - b_0} - \frac{a_0^2|}{|z - b_1} - \frac{a_1^2|}{|z - b_2} - \frac{a_2^2|}{|z - b_3} - \cdots - \frac{a_{n-1}^2|}{|z - b_n} - \cdots, \tag{54}$$

which is the known Jacobi continued fraction related to m. One should not underestimate the importance of the new continued fraction (52) due to the simplicity of its relation with the Jacobi one. This small change will be key to uncover an extremely simple OPRL version of Khrushchev's formula.

The approximants of the continued fraction (52) are closely related to the Wall polynomials. The results *(vi)* and *(vii)* of Theorem 5.1, combined with (51), imply that

$$K_n^-(z) = b_0 + \frac{a_0^2|}{|z^{-1}-b_1} - \frac{a_1^2|}{|z^{-1}-b_2} - \frac{a_2^2|}{|z^{-1}-b_3} - \cdots - \frac{a_{n-1}^2|}{|z^{-1}-b_n} = \frac{X_n(z)}{Y_n(z)},$$
$$K_n^+(z) = b_0 + \frac{a_0^2|}{|z^{-1}-b_1} - \frac{a_1^2|}{|z^{-1}-b_2} - \frac{a_2^2|}{|z^{-1}-b_3} - \cdots - \frac{a_{n-1}^2|}{|z^{-1}-b_n} - \frac{a_n^2|}{|z^{-1}} = \frac{A_n(z)}{B_n(z)}.$$

That is, A_n/B_n and X_n/Y_n are the approximants of K obtained by taking $f_{n+1} = 0, \infty$ in (41)–equivalently, in (51)—respectively.

The convergence of the above approximants in the determinate case is a direct consequence of Theorem 4.5.

Corollary 6.1 *Let A_n, B_n, X_n, Y_n be Wall polynomials of a regular Nevanlinna function f whose measure solves a determinate moment problem. Then,*

$$\frac{A_n}{B_n} \rightrightarrows f, \qquad \frac{X_n}{Y_n} \rightrightarrows f.$$

Proof If f has Schur parameters $\boldsymbol{\gamma} = (b_0, a_0, b_1, a_1, \dots)$, Theorem 5.1 shows that A_n/B_n and X_n/Y_n are trivial Nevanlinna functions with Schur parameters $\boldsymbol{\gamma}^{[n]} = (b_0, a_0, \dots, b_n, a_n, 0, 0)$ and $\tilde{\boldsymbol{\gamma}}^{[n]} = (b_0, a_0, \dots, b_{n-1}, a_{n-1}, b_n, 0)$, respectively. Both, $\boldsymbol{\gamma}^{[n]}$ and $\tilde{\boldsymbol{\gamma}}^{[n]}$, converge trivially to $\boldsymbol{\gamma}$, hence the result follows from Theorem 4.5. □

From (48) we find that

$$\frac{X_n(z)}{Y_n(z)} = z^{-1} - \frac{P_{n+1}(z^{-1})}{P_n^{(1)}(z^{-1})}.$$

Therefore, the convergence $X_n/Y_n \rightrightarrows f$ is equivalent to

$$-P_n^{(1)}/P_{n+1} \rightrightarrows m, \tag{55}$$

where m is the m-function (53) of the underlying measure on $\mathbb{R}$. In other words, the convergence of the approximants X_n/Y_n of the Wall continued fraction (52) is equivalent to that of the standard approximants $-P_n^{(1)}/P_{n+1}$ of the Jacobi continued fraction (54). On the other hand, (49) yields

$$\frac{A_n(z)}{B_n(z)} = z^{-1} - \frac{P_{n+2}(z^{-1}) + b_{n+1}P_{n+1}(z^{-1})}{P_{n+1}^{(1)}(z^{-1}) + b_{n+1}P_n^{(1)}(z^{-1})},$$

so that the convergence $A_n/B_n \rightrightarrows f$ means that

$$-\frac{P_n^{(1)} + b_n P_{n-1}^{(1)}}{P_{n+1} + b_n P_n} \rightrightarrows m.$$

This result does not follow naively from (55) because the asymptotic behaviour of x_n/y_n does not necessarily coincide with that of $(x_n + b_n x_{n-1})/(y_n + b_n y_{n-1})$ for $x_n, y_n \in \mathbb{C}, b_n \in \mathbb{R}$, as it is shown for instance by the example $x_n = n + a(n \mod 2)$, $y_n = n, b_n = -1$, with $a \in \mathbb{C} \setminus \{0\}$.

The error of the approximation of a regular Nevanlinna function by the approximants of the corresponding Wall continued fraction may be compactly expressed by using Theorem 5.1.*(viii)* and relations (48), (49),

$$f(z) - \frac{X_n(z)}{Y_n(z)} = \frac{\prod_{k=0}^{n}(f_k(z) - b_k)}{P_n^{(1)}(z^{-1})},$$
$$f(z) - \frac{A_n(z)}{B_n(z)} = \frac{f_{n+1}(z)\prod_{k=0}^{n}(f_k(z) - b_k)}{P_{n+1}^{(1)}(z^{-1}) + b_{n+1}P_n^{(1)}(z^{-1})}.$$

The use of these expressions to obtain a bound for the rate of convergence remains as a challenge.

7 OPRL Khrushchev's Formula

Analogously to the case of Schur functions, the normal boundary values $\lim_{y\downarrow 0} f(x + iy)$ of a Nevanlinna function f exist for almost every $x \in \mathbb{R}$ (see for instance [4] and references therein). Actually, except for $f = 0$, these normal boundary values must be non-null a.e. in $\mathbb{R}$, otherwise the Nevanlinna function $-1/f$ would not have normal boundary values a.e. in $\mathbb{R}$. Upper and lower normal boundary values are related by $\lim_{y\uparrow 0} f(x + iy) = \lim_{y\downarrow 0} \overline{f(x + iy)}$, as follows from (6).

In the subsequent discussions, a key role will be played by the relation between the boundary values of the m-function m of a measure μ on $\mathbb{R}$ and its Radon-Nikodym derivative μ' with respect to the Lebesgue measure. This relation states that

$$\mu'(x) = \frac{1}{\pi}\lim_{y\downarrow 0} \operatorname{Im} m(x + iy), \qquad \text{a.e. in } \mathbb{R}, \tag{56}$$

a relation that holds for every $x \in \mathbb{R}$ if μ is absolutely continuous and μ' is continuous on $\mathbb{R}$. In view of (53), if f is the Nevanlinna function of μ, (56) also reads as

$$\mu'(x) = \frac{1}{\pi}\lim_{y\downarrow 0} \operatorname{Im}\left(\frac{1}{f(z^{-1}) - z}\right) = -\frac{1}{\pi}\lim_{y\downarrow 0} \frac{\operatorname{Im} f(z^{-1})}{|f(z^{-1}) - z|^2}, \quad z = x+iy, \quad \text{a.e. in } \mathbb{R}. \tag{57}$$

Concerning the denominators in the above expressions, note that $f(z^{-1})-z$ must have non-null normal boundary values a.e. in $\mathbb{R}$ since $f(z)-z^{-1}$ is a non-null Nevanlinna function.

The real line version of Khrushchev's formula on the unit circle would answer the following question: given a regular Nevanlinna function f whose measure μ has a sequence p_n of orthonormal polynomials, what is the Nevanlinna function of the measure $p_n^2 d\mu$? This question makes sense because the modified measure is a probability one and has finite moments whenever the original one does. In analogy with the case of the unit circle, we expect a close relationship between the Nevanlinna function of $p_n^2 d\mu$ and the n-th iterate f_n of f. The first step to Khrushchev's formula will be to find a relation between f_n and $p_n^2\mu'$ in the spirit of (57). Such a relation is given by the following theorem, which may be considered as a preliminary version of Khrushchev's formula.

Theorem 7.1 *Let p_n be the orthonormal polynomials related to a probability measure μ on $\mathbb{R}$ with finite moments. If the regular Nevanlinna function f associated to μ has iterates f_n and Schur parameters $(b_0, a_0, b_1, a_1, \dots)$, then,*

$$p_n(x)^2\,\mu'(x) = -\frac{1}{\pi}\lim_{y\downarrow 0}\frac{\operatorname{Im} f_n(z^{-1})}{|f_n(z^{-1})+g_n(z^{-1})-z|^2}, \qquad z=x+iy, \qquad \text{a.e. in } \mathbb{R}. \tag{58}$$

where

$$g_n(z) = a_{n-1}z\,\frac{p^*_{n-1}(z)}{p^*_n(z)} = a_{n-1}\frac{p_{n-1}(z^{-1})}{p_n(z^{-1})}. \tag{59}$$

Proof For convenience we will prove the relation for the index $n+1$ instead of n. This proof consists simply in rewriting (57) in terms of f_{n+1} by using its relation with f given in (41). In this process, the Wall polynomials appearing in (41) should be also expressed in terms of the orthogonal polynomials via (48) and (49).

First, from (41) we get

$$f - z^{-1} = \frac{A_n + zX_n f_{n+1}}{B_n + zY_n f_{n+1}} - z^{-1} = \frac{z^{-1}(zA_n - B_n) + (zX_n - Y_n)f_{n+1}}{B_n + zY_n f_{n+1}}.$$

Besides, (48) and (49), combined with (28), yield

$$zX_n - Y_n = P^*_{n+1}, \qquad zA_n - B_n = -P^*_{n+2} - b_{n+1}zP^*_{n+1} = -P^*_{n+1} + a_n^2 z^2 P^*_n.$$

Bearing in mind (30), the above expressions lead to

$$\begin{aligned} f - z^{-1} &= \frac{P^*_{n+1}}{B_n + zY_n f_{n+1}}\left(f_{n+1} + a_n^2 z\frac{P^*_n}{P^*_{n+1}} - z^{-1}\right) \\ &= \frac{\Upsilon_n\, p^*_{n+1}}{B_n + zY_n f_{n+1}}\left(f_{n+1} + g_{n+1} - z^{-1}\right). \end{aligned} \tag{60}$$

On the other hand, bearing in mind that the Wall polynomials have real coefficients and that the Nevanlinna functions f_{n+1} have normal boundary values a.e. in $\mathbb{R}$, (41) and Theorem 5.1.*(i)* give for $z = x + iy$,

$$\lim_{y\downarrow 0} \operatorname{Im} f = \lim_{y\downarrow 0} \frac{\operatorname{Im}\,(z(B_nX_n - A_nY_n)f_{n+1})}{|B_n + zY_nf_{n+1}|^2} = \lim_{y\downarrow 0} \frac{\Upsilon_n^2 z^{2n+2}\operatorname{Im} f_{n+1}}{|B_n + zY_nf_{n+1}|^2}, \qquad \text{a.e. in } \mathbb{R}. \tag{61}$$

Here we have used that, due to Theorem 5.1.*(vii)*, $B_n + zY_nf_{n+1} = (-z)^{n+1}\prod_{k=1}^{n+1}(f_k - z^{-1})$ is a product of non-null Nevanlinna functions up to the factor $(-z)^{n+1}$, hence it has non-zero normal boundary values a.e. in $\mathbb{R}$. Therefore, $N/|B_n + zY_nf_{n+1}|^2$ has null normal boundary values a.e. in $\mathbb{R}$ whenever the numerator N does.

Combining (57), (60) and (61) ends in

$$\mu'(x) = -\frac{1}{\pi}\lim_{y\downarrow 0} \frac{1}{p_{n+1}(z)^2} \frac{\operatorname{Im} f_{n+1}(z^{-1})}{|f_{n+1}(z^{-1}) + g_{n+1}(z^{-1}) - z|^2}, \quad z = x + iy, \quad \text{a.e. in } \mathbb{R},$$

where we have substituted z by z^{-1}, which causes no problem a.e. in $\mathbb{R}$ and does not change the equalities because, according to (6), $f(x - iy) = \overline{f(x + iy)}$ and similarly for f_{n+1}. The last equality proves the theorem. □

As we will see, the above result becomes Khrushchev's formula for OPRL related to absolutely continuous measures on $\mathbb{R}$. Extending Khrushchev's formula to any measure on $\mathbb{R}$ relies on a limiting process based on the following result.

Theorem 7.2 *For every regular Nevanlinna function f there exists a sequence $f^{[k]}$ of regular Nevanlinna functions with absolutely continuous measures $\mu^{[k]}$ such that $f^{[k]} \rightrightarrows f$. In addition, this sequence may be chosen such that*

$$\mu^{[k]} \xrightarrow{w} \mu, \qquad \boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}, \qquad \lim_{k\to\infty} s_n^{[k]} = s_n, \qquad \lim_{k\to\infty} \mu_n^{[k]} = \mu_n, \qquad n \ge 0, \tag{62}$$

with μ the measure of f; $\mu_n^{[k]}$, μ_n the moments of $\mu^{[k]}$, μ respectively; $s_n^{[k]}$, s_n the Taylor coefficients for the expansion around the origin of $f^{[k]}$, f respectively and $\boldsymbol{\gamma}^{[k]}$, $\boldsymbol{\gamma}$ the corresponding Schur parameters.

Proof Let $\tilde{m}$ be the m-function of a measure $\tilde{\mu}$ on $\mathbb{R}$ with finite moments. Then, $f^{[k]} \rightrightarrows f$ for the sequence of Nevanlinna functions

$$f^{[k]}(z) = f(z) - \frac{1}{k}\tilde{m}(z^{-1}) = f(z) + \frac{z}{k}\int_{\mathbb{R}} \frac{d\tilde{\mu}(x)}{1 - xz}.$$

Besides, the asymptotic expansion (18) for $\tilde{\mu}$ shows that, according to Proposition 3.6, $f^{[k]}$ is regular with Taylor coefficients of the expansion around the origin given by

$$s_0^{[k]} = s_0, \qquad s_n^{[k]} = s_n + \frac{1}{k}\tilde{\mu}_{n-1}, \quad n \ge 1.$$

Obviously, $\lim_{k\to\infty} s_n^{[k]} = s_n$, thus Proposition 4.6 ensures that $\lim_{k\to\infty} \mu_n^{[k]} = \mu_n$ and $\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}$. Besides, $\mu^{[k]} \xrightarrow{w} \mu$ as a consequence of Proposition 4.1.

It only remains to see that $\tilde{\mu}$ may be chosen to guarantee that $\mu^{[k]}$ is absolutely continuous for every k. Let $\tilde{\mu}$ be given by a continuous weight which is strictly positive on $\mathbb{R}$, i.e.

$$d\tilde{\mu}(x) = w(x)\,dx, \qquad w \text{ continuous on } \mathbb{R}, \qquad w(x) > 0 \ \ \forall x \in \mathbb{R}.$$

This weight follows from its m-function $\tilde{m}$ by the inversion formula

$$w(x) = \frac{1}{\pi} \lim_{y\downarrow 0} \operatorname{Im} \tilde{m}(x+iy).$$

Therefore, for every $x \in \mathbb{R}$, there exists $\epsilon(x) > 0$ such that

$$\operatorname{Im} \tilde{m}(x+iy) > \frac{\pi}{2} w(x), \qquad 0 < y < \epsilon(x).$$

On the other hand, the singular part of $\mu^{[k]}$ is concentrated on the set of points $x \in \mathbb{R}$ satisfying

$$\lim_{y\downarrow 0} \operatorname{Im} m^{[k]}(x+iy) = \infty,$$

which, due to the relation

$$m^{[k]}(z) = -\frac{1}{z - f^{[k]}(z^{-1})},$$

is contained in the set of points $x \in \mathbb{R}$ such that

$$\lim_{y\downarrow 0} f^{[k]}(z^{-1}) = x, \qquad z = x+iy.$$

Bearing in mind that $-f(z^{-1}) \in \mathfrak{N}$, we find that

$$\left|\operatorname{Im} f^{[k]}(z^{-1})\right| \geq \frac{1}{k} \operatorname{Im} \tilde{m}(z) > \frac{\pi}{2k} w(x), \quad z = x+iy, \quad 0 < y < \epsilon(x), \quad x \in \mathbb{R}.$$

This proves that $\mu^{[k]}$ has no singular part whenever $\tilde{\mu}$ is given by a continuous weight which is strictly positive on the whole real line. □

The above two results lead to Khruschchev's formula for arbitrary OPRL.

Theorem 7.3 *(Khrushchev's formula for OPRL) Let p_n be the orthonormal polynomials related to a probability measure μ on $\mathbb{R}$ with finite moments. If the Nevanlinna* *function f of μ has Schur parameters $(b_0, a_0, b_1, a_1, \dots)$, then the Nevanlinna function of the probability measure $p_n^2 \mathrm{d}\mu$ is*

$$h^{[n]} = f_n + g_n, \tag{63}$$

where f_n stands for the n-th iterate of f and g_n is given by (59). In other words,

$$\frac{1}{1 - z(f_n(z) + g_n(z))} = \int_{\mathbb{R}} \frac{p_n(x)^2 \, \mathrm{d}\mu(x)}{1 - zx},$$

where f_n and g_n are Nevanlinna functions with Schur parameters $(b_n, a_n, b_{n+1}, a_{n+1}, \dots)$ and $(0, a_{n-1}, b_{n-1}, \dots, a_0, b_0, 0)$ respectively.

Proof Let us see first that (58) becomes Khrushchev's formula when μ is absolutely continuous, i.e. when $\mathrm{d}\mu(x) = \mu'(x)\,dx$. Taking into account that g_n is a quotient of real polynomials and, as a non-null Nevanlinna function, $f_n + g_n - z^{-1}$ has non-zero normal boundary values a.e. in $\mathbb{R}$, (58) may be rewritten as

$$p_n(x)^2 \, \mu'(x) = -\frac{1}{\pi} \lim_{y \downarrow 0} \frac{\operatorname{Im}(f_n(z^{-1}) + g_n(z^{-1}))}{|f_n(z^{-1}) + g_n(z^{-1}) - z|^2}, \qquad z = x + iy, \qquad \text{a.e. in } \mathbb{R}.$$

If ν is the measure related to the Nevanlinna function $f_n + g_n$, in view of (57), the above relation implies that $\nu' = p_n^2 \mu'$. Since f_n and g_n are normalized Nevanlinna functions, the same applies to $f_n + g_n$ due to (12). Therefore, ν is a probability measure, which implies that

$$1 = \int_{\mathbb{R}} d\nu(x) \geq \int_{\mathbb{R}} \nu'(x)\,dx = \int_{\mathbb{R}} p_n(x)^2 \, \mu'(x)\,dx = \int_{\mathbb{R}} p_n(x)^2 \, \mathrm{d}\mu(x) = 1.$$

Hence, $d\nu = p_n^2 \mathrm{d}\mu$, which identifies $f_n + g_n$ as the Nevanlinna function of $p_n^2 \mathrm{d}\mu$.

Suppose now that μ is an arbitrary measure on $\mathbb{R}$ with finite moments. Theorem 7.2 guarantees the existence of a sequence $\mu^{[k]}$ of absolutely continuous measures with finite moments such that (62) holds and $f^{[k]} \rightrightarrows f$ for the related Nevanlinna functions. Using the supersript $[k]$ for the objects related to the measure $\mu^{[k]}$, we already have proved that

$$\frac{1}{1 - z(f_n^{[k]}(z) + g_n^{[k]}(z))} = \int_{\mathbb{R}} \frac{p_n^{[k]}(x)^2 \, \mathrm{d}\mu^{[k]}(x)}{1 - zx}. \tag{64}$$

The convergence $\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}$ of the Schur parameters ensures that $\gamma_n^{[k]}$ is bounded for each fixed n such that γ_n exists. Hence, $f_n^{[k]} \rightrightarrows f_n$ whenever f_n exists, as a consequence of Theorem 4.5. On the other hand, bearing in mind (24), $\boldsymbol{\gamma}^{[k]} \to \boldsymbol{\gamma}$ also implies that $p_n^{[k]} \rightrightarrows p_n$, thus $g_n^{[k]} \rightrightarrows g_n$ because p_n has no zeros on $\mathbb{C} \setminus \mathbb{R}$. Since $1 - z(f_n + g_n)$ does not vanish on $\mathbb{C} \setminus \mathbb{R}$ because $f_n + g_n - z^{-1} \in \mathfrak{N} \setminus \{0\}$, we find that

$$\frac{1}{1 - z(f_n^{[k]} + g_n^{[k]})} \rightrightarrows \frac{1}{1 - z(f_n + g_n)}.$$

Besides, the equality

$$\frac{x^n z^n}{1-xz} = \frac{1}{1-xz} - (1 + xz + x^2 z^2 + \cdots + x^{n-1} z^{n-1})$$

yields

$$\int_{\mathbb{R}} \frac{x^n \, \mathrm{d}\mu(x)}{1-xz} = z^{-n} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{1-xz} - (z^{-n} + \mu_1 z^{1-n} + \mu_2 z^{2-n} + \cdots + \mu_{n-1} z^{-1}).$$

A similar relation for $\mu^{[k]}$, together with the weak convergence $\mu^{[k]} \xrightarrow{w} \mu$ and the convergence $\lim_{k\to\infty} \mu_n^{[k]} = \mu_n$ of the moments, leads to

$$\int_{\mathbb{R}} \frac{x^n \, \mathrm{d}\mu^{[k]}(x)}{1-zx} \rightrightarrows \int_{\mathbb{R}} \frac{x^n \, \mathrm{d}\mu(x)}{1-zx}, \qquad n = 0, 1, 2, \ldots$$

This result, combined with the convergence $p_n^{[k]} \rightrightarrows p_n$, shows that

$$\int_{\mathbb{R}} \frac{p_n^{[k]}(x)^2 \, \mathrm{d}\mu^{[k]}(x)}{1-zx} \rightrightarrows \int_{\mathbb{R}} \frac{p_n(x)^2 \, \mathrm{d}\mu(x)}{1-zx}.$$

As a consequence of the previous results, taking limits in (64) we conclude that

$$\frac{1}{1 - z(f_n(z) + g_n(z))} = \int_{\mathbb{R}} \frac{p_n(x)^2 \, \mathrm{d}\mu(x)}{1-zx},$$

which is Khrushchev's formula for the measure μ.

Finally, the identification of the Schur parameters of g_n follows from Proposition 3.8. ◻

Maybe a more natural way of expressing Khrushchev's formula for OPRL is in terms of the Nevanlinna functions with Schur parameters $(b_n, a_{n-1}, b_{n-1}, \ldots, a_0, b_0, 0)$, i.e. the inverse Schur iterates R_n of μ given in (31). Since $R_n = b_n + g_n$, Khrushchev's formula also reads as

$$h^{[n]} = f_n + R_n - b_n.$$

Khrushchev's formula may be translated to the m-function m of a measure μ by using its relation (13) with the Nevanlinna function f of μ. This results in the identification of the m-function of $p_n^2 \mathrm{d}\mu$ as

$$\frac{1}{\frac{1}{m_n} + a_{n-1} \frac{p_{n-1}}{p_n}} = \frac{p_n m_n}{p_n + a_{n-1} p_{n-1} m_n},$$

where m_n is the m-function related to the iterate f_n. The intricacy of the above relation, compared with (63), is a clear indication that Definition 3.1 for the Nevanlinna function of a measure establishes a much more natural relation between Nevanlinna functions and measures on $\mathbb{R}$ than the standard connection given by m-functions.

8 Applications

As a first application of Khrushchev's formula for OPRL, we will give a new proof of a result obtained by Barry Simon in [15]. It extends to OPRL a result for OPUC due to Sergei Khrushchev [10]. Khrushchev's formula for OPRL, as stated in the previous section, is a ready-made tool to tackle this kind of questions.

More precisely, if p_n are the orthonormal polynomials related to a probability measure μ on $\mathbb{R}$, one can ask about the vague convergence of the sequence of probability measures given by $p_n^2 \mathrm{d}\mu$. Proposition 4.1 and Remark 4.2 state that this question can be translated into the analysis of the convergence or divergence of the corresponding sequence of Nevanlinna functions, which by Khrushchev's formula has the form $h^{[n]} = f_n + g_n$, with f_n, g_n Nevanlinna functions with Schur parameters $(b_n, a_n, b_{n+1}, a_{n+1}, \dots)$ and $(0, a_{n-1}, b_{n-1}, \dots, a_0, b_0, 0)$ respectively. The Nevanlinna function g_n is always determined by its (finite) sequence of Schur parameters, while for f_n this is the case for instance when a_n and b_n are bounded because this means that μ has bounded support, so that the moment problem for $p_n^2 \mathrm{d}\mu$ is determinate. In the bounded case we know that vague convergence becomes weak convergence, while Theorem 4.5 states that the convergence of $h^{[n]}$ holds simultaneously to that of the corresponding Schur parameters which, in view of Proposition 3.7, yields the convergence of the Taylor coefficients of $h^{[n]}$. These remarks lead to the following result (see [15]).

Theorem 8.1 *Let p_n be the orthonormal polynomials with respect to a probability measure μ supported on a bounded subset of $\mathbb{R}$. Then, the sequence $p_n^2 \mathrm{d}\mu$ converges weakly if and only if the sequences b_n, a_{2n} and a_{2n+1} are convergent, where a_n and b_n are the coefficients of the recurrence relation* (24) *for p_n.*

Proof In the bounded case, weak convergence is equivalent to vague convergence, and the limit of $p_n^2 \mathrm{d}\mu$ is again a probability measure. Thus, according to the comments at the beginning of this section, we only need to characterize the convergence of the sequence $h^{[n]} = f_n + g_n$ of Nevanlinna functions which, due to the boundedness of the support of $p_n^2 \mathrm{d}\mu$, implies the convergence of the Taylor coefficients $s_k^{[n]}$ of $h^{[n]}$. Using Proposition 3.7 for f_n and g_n we get

$$s_0^{[n]} = b_n, \qquad s_1^{[n]} = a_n^2 + a_{n-1}^2, \qquad s_3^{[n]} = a_n^2(a_{n+1}^2 + b_{n+1}^2) + a_{n-1}^2(a_{n-2}^2 + b_{n-1}^2). \tag{65}$$

Therefore, the convergence of $h^{[n]}$ implies that of $b_n, a_n^2 + a_{n-1}^2$ and $a_{n+1}^2 a_n^2 + a_{n-1}^2 a_{n-2}^2$, which guarantees the convergence of a_{2n} and a_{2n+1} (see [15,pp 208-209]).

Suppose now that $b_n \to b$, $a_{2n} \to a$ and $a_{2n+1} \to a'$. Let us rewrite $h^{[n]} = f_n + R_n - b_n$, where R_n are the inverse iterates (31) with Schur parameters $(b_n, a_{n-1}, b_{n-1}, \dots, a_0, b_0, 0)$. From Theorem 4.5 we find that

$$f_{2n} \rightrightarrows f_{b,a,a'}, \qquad f_{2n+1} \rightrightarrows f_{b,a',a}, \qquad R_{2n} \rightrightarrows f_{b,a',a}, \qquad R_{2n+1} \rightrightarrows f_{b,a,a'},$$

where $f_{b,a,a'}$ stands for the Nevanlinna function with Schur parameters $(b, a, b, a', b, a, b, a', \dots)$. Here we understand that $f_{b,0,a'}$ and $f_{b,a,0}$ $(a \neq 0)$ are actually defined by the Schur parameters $(b, 0)$ and $(b, a, b, 0)$ respectively. Therefore, we conclude that

$$h^{[n]} = f_n + R_n - b_n \rightrightarrows f_{b,a,a'} + f_{b,a',a} - b.$$

□

Khrushchev's formula for OPRL not only gives a proof of Simon's result in [15], but paves the way to other kinds of results which could constitute the OPRL extension of Khrushchev's theory for OPUC [9, 10] (see also [16,Chapter 9]). Such an extension may present some subtleties in the unbounded case, absent for OPUC. An example of this is the generalization of the previous result to measures with unbounded support. Khrushchev's formula allows one to translate the vague convergence of $p_n^2 \mathrm{d}\mu$ into the convergence/divergence of their Nevanlinna functions $h^{[n]} = f_n + R_n - b_n$ even in the unbounded case, but in this situation vague convergence might not imply weak convergence—indeed, we will see that this is always the case. Also, such a convergence might not be accompanied by that of the Schur parameters or the Taylor coefficients of $h^{[n]}$, as it is shown by Example 4.4. Therefore, although (65) proves that the boundedness of the support of μ is guaranteed by the convergence—even by the boundedness—of the first two Taylor coefficients of $h^{[n]}$—equivalently, by the convergence of the moment $\mu_2^{[n]} = b_n^2 + a_n^2 + a_{n-1}^2$ of $\mathrm{d}\mu^{[n]} = p_n^2 \mathrm{d}\mu$, as it was already noticed in [15]–, the existence of vaguely convergent sequences $p_n^2 \mathrm{d}\mu$ when μ has an unbounded support is not excluded by Theorem 8.1. Actually, we will see that such sequences do exist.

Our first example will show the difficulties in finding vaguely convergent sequences $p_n^2 \mathrm{d}\mu$ for measures μ with unbounded support. According to Proposition 4.1 and Remark 4.2, this is equivalent to finding convergent or divergent sequences $h^{[n]} = f_n + R_n - b_n$ when a_n or b_n are unbounded. To understand the constraints imposed by such a situation, let us first derive a general relation between consecutive functions, $h^{[n]}$ and $h^{[n+1]}$, which will provide asymptotic necessary conditions for the convergence of $h^{[n]}$ in the unbounded case.

Lemma 8.2 *For any probability measure μ on $\mathbb{R}$ with orthonormal polynomials p_n, the Nevanlinna function $h^{[n]}$ of $p_n^2 \mathrm{d}\mu$ satisfies*

$$\frac{h^{[n+1]} - z^{-1}}{h^{[n]} - z^{-1}} = \frac{f_{n+1} - z^{-1}}{R_n - z^{-1}} = \frac{(f_{n+1} - z^{-1})(f_{n+1} - h^{[n+1]})}{a_n^2}$$
$$= \frac{a_n^2}{(R_n - z^{-1})(R_n - h^{[n]})},$$

where a_n, b_n are the coefficients of the recurrence relation (24) for p_n and f_n, R_n are the iterates and inverse iterates of μ.

Proof Rewritting the Schur algorithm (17) for the iterates and (29) for the inverse iterates as

$$
\begin{aligned}
a_n^2 z &= (f_n - b_n)(1 - z f_{n+1}) = (h^{[n]} - R_n)(1 - z f_{n+1}),\\
a_n^2 z &= (R_{n+1} - b_{n+1})(1 - z R_n) = (h^{[n+1]} - f_{n+1})(1 - z R_n),
\end{aligned}
\tag{66}
$$

leads to

$$
[(h^{[n+1]} - z^{-1}) - (f_{n+1} - z^{-1})](R_n - z^{-1}) = [(h^{[n]} - z^{-1}) - (R_n - z^{-1})](f_{n+1} - z^{-1}).
$$

This yields the first equality of the lemma which, combined with (66), gives the remaining ones. □

The above relations permits us to shows that, in the unbounded case, the convergence of $h^{[n]}$—even certain subsequential convergence—has strong consequences. As a first result, we find that the convergence of $h^{[n]}$ cannot hold if only the sequence b_n is unbounded.

Theorem 8.3 *Let a_n, b_n be the coefficients of the recurrence relation (24) for the orthonormal polynomials p_n with respect to a probability measure μ with unbounded support on $\mathbb{R}$. Then, the vague convergence of $p_n^2 \mathrm{d}\mu$ to a non-null measure implies that the sequence a_n is unbounded.*

Proof Suppose $h^{[n]}$ convergent, which is equivalent to the vague convergence of $p_n^2 \mathrm{d}\mu$ to a non-null limit. Then, Lemma 8.2 shows that a_n bounded guarantees that f_n and R_n are locally uniformly bounded. Thus,

$$
b_n = f_n + \frac{a_n^2}{f_{n+1} - z^{-1}}
$$

is also bounded because

$$
|b_n| \le |f_n| + \frac{a_n^2}{|f_{n+1} - z^{-1}|} \le |f_n| + \frac{a_n^2}{|\operatorname{Im} z^{-1}|}.
$$

We conclude that, when μ has unbounded support, the convergence of $h^{[n]}$ needs a_n unbounded.

The previous theorem states that the vague convergence of $p_n^2 \mathrm{d}\mu$ to a non-null measure when μ has unbounded support requires the presence of a divergent subsequence of a_n. On the contrary, the next result shows that such a divergence gives information about the asymptotic behaviour of related subsequences of iterates and inverse iterates, even assuming only some subsequential vague convergence for $p_n^2 \mathrm{d}\mu$.

Proposition 8.4 *Let a_n, b_n be the coefficients of the recurrence relation (24) for the orthonormal polynomials p_n with respect to a probability measure μ on $\mathbb{R}$. If a subsequence a_{n_j} diverges and the contiguous subsequences $p_{n_j}^2 \mathrm{d}\mu$ and $p_{n_j+1}^2 \mathrm{d}\mu$ are* *vaguely convergent to non-null measures, then the iterates f_n and inverse iterates R_n of μ satisfy*

$$\kappa f_{n_j+1}^2 \sim \frac{1}{\kappa} R_{n_j}^2 \sim a_{n_j}^2, \qquad R_{n_j} \sim \kappa f_{n_j+1}, \qquad \kappa > 0.$$

In particular, $\kappa = 1$ when $p_{n_j}^2 \mathrm{d}\mu$ and $p_{n_j+1}^2 \mathrm{d}\mu$ have the same non-null vague limit.

Proof The convergence $p_{n_j}^2 \mathrm{d}\mu \xrightarrow{v} \nu \neq 0$ and $p_{n_j+1}^2 \mathrm{d}\mu \xrightarrow{v} \rho \neq 0$ translates into the convergence $h^{[n_j]} \to h_1$ and $h^{[n_j+1]} \to h_2$ for some Nevanlinna functions h_1, h_2. Then, from Lemma 8.2 we find that

$$\left(\frac{R_{n_j}}{a_{n_j}} - \frac{z^{-1}}{a_{n_j}}\right)\left(\frac{R_{n_j}}{a_{n_j}} - \frac{h^{[n_j]}}{a_{n_j}}\right) \to \Delta = \frac{h_1 - z^{-1}}{h_2 - z^{-1}},$$

with Δ a quotient of non-null Nevanlinna functions. This implies that $R_{n_j}^2/a_{n_j}^2 \to \Delta$ due to the divergence of a_{n_j}. Analogously, Lemma 8.2 yields $f_{n_j+1}^2/a_{n_j}^2 \to 1/\Delta$. The same Lemma gives

$$\frac{f_{n_j+1} - z^{-1}}{R_{n_j} - z^{-1}} \to \frac{1}{\Delta},$$

which implies that $f_{n_j+1}/R_{n_j} \to 1/\Delta$ because we have proved that f_{n_j+1} and R_{n_j} diverge.

The above results imply that any limit point of the sequence of Nevanlinna functions R_{n_j}/a_{n_j} must be a square root $\sqrt{\Delta}$. Then, $f_{n_j+1}/R_{n_j} \to 1/\Delta$ implies that the inverse $1/\sqrt{\Delta}$ of the same square root must be a limit point of the sequence of Nevanlinna functions f_{n_j+1}/a_{n_j}. Therefore, $\sqrt{\Delta}$ and $1/\sqrt{\Delta}$ must be Nevanlinna functions, which only holds in the non-null degenerate case, i.e. when $\sqrt{\Delta}$ is a non-null real constant. As a consequence, Δ must be a positive constant κ. This finishes the proof of the main statement of the proposition.

When $p_{n_j}^2 \mathrm{d}\mu$ and $p_{n_j+1}^2 \mathrm{d}\mu$ have the same vague limit, $h_1 = h_2$ so that $\kappa = 1$. □

These results will be used later on to obtain general convergence properties of $p_n^2 \mathrm{d}\mu$ when μ has unbounded support. Nevertheless, our first goal is to use them as a guide to surmise an example of a vaguely convergent sequence $p_n^2 \mathrm{d}\mu$ in the unbounded case.

To simplify things, let us assume the divergence of the whole sequence a_n. According to Proposition 8.4, assuming the vague convergence of $p_n^2 \mathrm{d}\mu$ to a non-null measure yields

$$f_{n+1} \sim R_n, \qquad f_{n+1}^2 \sim a_n^2 \sim R_n^2.$$

A situation compatible with these requirements is, for instance,

$$f_{n+1} \sim a_n \sim R_n,$$

which also gives

$$b_n \sim a_{n-1} + a_n,$$

because

$$\begin{aligned}\left|\frac{b_n + h^{[n]}}{a_{n-1}+a_n} - 1\right| &= \left|\frac{f_n + R_n}{a_{n-1}+a_n} - 1\right| \leq \left|\frac{f_n - a_{n-1}}{a_{n-1}+a_n}\right| + \left|\frac{R_n - a_n}{a_{n-1}+a_n}\right| \\ &\leq \left|\frac{f_n}{a_{n-1}} - 1\right| + \left|\frac{R_n}{a_n} - 1\right|.\end{aligned}$$

Since $g_{n+1} = R_{n+1} - b_{n+1} = -a_n^2/(R_n - z^{-1})$, these conditions also lead to

$$g_n \sim -a_{n-1}, \qquad -\frac{p_n}{p_{n+1}} \sim 1.$$

There is a well known example which fits with the above constraints: the Laguerre polynomials. It should be an ideal candidate to provide an example of the kind of convergence we are looking for in the unbounded case.

Example 8.5 **Laguerre Polynomials**
The Laguerre polynomials correspond to the measure $e^{-x}dx$ on $[0,\infty)$, whose associated Schur parameters are

$$a_n = n+1, \qquad b_n = a_n + a_{n-1} = 2n+1.$$

The ratio asymptotics of the orthonormal Laguerre polynomials L_n with positive leading coefficients is given by [3]

$$-\frac{L_{n+1}(z)}{L_n(z)} = 1 + \frac{\sqrt{-z}}{\sqrt{n}} - \left(\frac{1}{4} + \frac{z}{2}\right)\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$

uniformly on compacts of $\mathbb{C} \setminus [0,\infty)$, the branch of the square root being $(-\infty, 0]$. Therefore,

$$g_n(z) = n\frac{L_{n-1}(z^{-1})}{L_n(z^{-1})} = -n + \sqrt{-z^{-1}}\sqrt{n} + \left(\frac{1}{2z} - \frac{1}{4}\right) + \mathcal{O}(n^{-1/2}).$$

On the other hand, the m-function of the measure associated to the iterate f_n is [7]

$$m_n(z) = \frac{\Psi(n+1, 1, -z)}{\Psi(n, 1, -z)},$$

where $\Psi(a,b,z)$ is the confluent hypergeometric function of the second kind. As for the large n asymptotics, we have that, uniformly on compacts of the Riemann surface of the logarithm [17, 18],

$$\Psi(n,1,z^2)=e^{z/2}\frac{2}{(n-1)!}\left(K_0(u_nz)\left[1+\frac{\alpha(z)}{u_n^2}+\mathcal{O}(u_n^{-4})\right]-K_1(u_nz)\left[\frac{\beta(z)}{u_n}+\mathcal{O}(u_n^{-3})\right]\right),$$
$$u_n=2\sqrt{n-\frac{1}{2}},\qquad \alpha(z)=\frac{z^6}{72}-\frac{z^2}{6},\qquad \beta(z)=\frac{z^3}{6},$$

where K_m are the modified Bessel functions of the second kind. Resorting to the asymptotics of these functions for large argument,

$$K_m(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{4m^2-1}{8z}+\frac{16m^4-40m^2+9}{128z^2}+\mathcal{O}(z^{-3})\right),$$
$$|\arg z|<\frac{3\pi}{2},\quad z\to\infty,$$

a tedious but straightforward calculation gives

$$f_n(z)=z^{-1}+\frac{1}{m_n(z^{-1})}=z^{-1}+\frac{\Psi(n,1,-z^{-1})}{\Psi(n+1,1,-z^{-1})}=n+\sqrt{-z^{-1}}\sqrt{n}+\frac{1}{4}+\mathcal{O}(n^{-1/2}).$$

Hence,

$$h^{[n]}(z)=f_n(z)+g_n(z)=2\sqrt{-z^{-1}}\sqrt{n}+\frac{1}{2z}+\mathcal{O}(n^{-1/2})$$

becomes finally divergent.

One may wonder if the coefficients $\sqrt{-z^{-1}}$ of $\sqrt{n}$ could have opposite sign for f_n and g_n, so that they cancel each other and $h^{[n]}$ converges. If this were the case then the sequence $h^{[n]}$ of Nevanlinna functions would converge to $1/(2z)$, which is not possible because it is not a Nevanlinna function. Another way to understand this is by noticing that both,

$$\frac{g_n(z)+n}{\sqrt{n}}=\sqrt{-z^{-1}}+\mathcal{O}(n^{-1/2}),\qquad \frac{f_n(z)-n}{\sqrt{n}}=\sqrt{-z^{-1}}+\mathcal{O}(n^{-1/2}),$$

are convergent sequences of Nevanlinna functions whose limit $\sqrt{-z^{-1}}$ must be a Nevanlinna function too, thus the corresponding square roots cannot have opposite sign.

We conclude that the dominant divergences f_n and g_n, with a behaviour $\propto n$, cancel each other in $h^{[n]}=f_n+g_n$. However, this is not the case for the subdominant divergences with a behaviour $\propto\sqrt{n}$. This causes the divergence of $h^{[n]}$, which means that the sequence of measures $L_n^2(x)e^{-x}dx$ on $[0,\infty)$ converges vaguely to the null measure, as follows from Remark 4.2.

The Laguerre example was selected to fit with a vague convergence $p_n^2\mathrm{d}\mu\xrightarrow{v}\nu\neq0$. In spite of this, it surprised us as an example with $p_n^2\mathrm{d}\mu\xrightarrow{v}0$ due to the presence of subdivergences in f_n and g_n which made $h^{[n]}$ finally divergent. The message from this example is that one should avoid subdivergences in f_n and g_n to guarantee the vague convergence of $p_n^2 \mathrm{d}\mu$ to a non-null measure in the unbounded case. The next theorem provides a recipe to build examples falling in this situation. This theorem not only shows the use of OPRL Khrushchev's formula for the study of the vague asymptotics of $p_n^2 \mathrm{d}\mu$ in the non-convergent case, but also provides new examples of measures μ with unbounded support such that $p_n^2 \mathrm{d}\mu$ converges vaguely. Both, the Laguerre example and the theorem below, prove that the measures on the real line covered by Theorem 8.1 are not the unique instances which make $p_n^2 \mathrm{d}\mu$ vaguely convergent.

Theorem 8.6 *Let p_n be the orthonormal polynomials with respect to a probability measure μ on the real line. Suppose that the coefficients a_n and b_n of the recurrence relation (24) for p_n satisfy*

$$\begin{aligned} &a_{2n} \to \infty, \quad a_{2n+1} \to 0, \quad b_{2n} - c a_{2n} \to d, \\ &b_{2n+1} - c' a_{2n} \to d', \quad c, c' \in \mathbb{R} \setminus \{0\}, \quad d, d' \in \mathbb{R}. \end{aligned}$$

Then, denoting by $\delta(x - x_0)$ the Dirac delta at $x_0 \in \mathbb{R}$, we have the following cases:

(i) If $cc' \neq 1$, the sequence $p_n^2 \mathrm{d}\mu$ converges vaguely to the null measure.
(ii) If $cc' = 1$, the even and odd subsequences of $p_n^2 \mathrm{d}\mu$ are vaguely convergent. Actually,

$$p_{2n}^2(x)\,\mathrm{d}\mu(x) \xrightarrow{v} \frac{1}{1+c^2}\,\delta(x - \tfrac{d+c^2 d'}{1+c^2}), \qquad p_{2n+1}^2(x)\,\mathrm{d}\mu(x) \xrightarrow{v} \frac{c^2}{1+c^2}\,\delta(x - \tfrac{d+c^2 d'}{1+c^2}).$$

In particular, the sequence $p_n^2 \mathrm{d}\mu$ is vaguely convergent when $c' = c = \pm 1$, in which case

$$p_n^2(x)\,\mathrm{d}\mu(x) \xrightarrow{v} \frac{1}{2}\,\delta(x - \tfrac{d+d'}{2}).$$

Proof From (66) we find that

$$|R_{2n} - b_{2n}| = \frac{a_{2n-1}^2}{|R_{2n-1} - z^{-1}|} \leq \frac{a_{2n-1}^2}{|\operatorname{Im} z^{-1}|}, \qquad R_{2n+1} = b_{2n+1} - \frac{a_{2n}^2}{R_{2n} - z^{-1}},$$

$$|f_{2n+1} - b_{2n+1}| = \frac{a_{2n+1}^2}{|f_{2n+2} - z^{-1}|} \leq \frac{a_{2n+1}^2}{|\operatorname{Im} z^{-1}|}, \qquad f_{2n} = b_{2n} - \frac{a_{2n}^2}{f_{2n+1} - z^{-1}}.$$

Therefore,

$$R_{2n} = b_{2n} + o(1), \qquad f_{2n+1} = b_{2n+1} + o(1),$$

so that

$$\begin{aligned} h^{[2n]} &= f_{2n} + R_{2n} - b_{2n} = f_{2n} + o(1), \\ h^{[2n+1]} &= f_{2n+1} + R_{2n+1} - b_{2n+1} = R_{2n+1} + o(1). \end{aligned}$$

Also, inserting the asymptotic conditions for b_n, one obtain

$$R_{2n} = ca_{2n} + d + o(1), \qquad f_{2n+1} = c'a_{2n} + d' + o(1),$$

and

$$\begin{aligned} h^{[2n]} &= b_{2n} - \frac{a_{2n}^2}{f_{2n+1} - z^{-1}} + o(1) = ca_{2n} + d - \frac{a_{2n}/c'}{1 + (d' - z^{-1})/c'a_{2n} + o(a_{2n}^{-1})} + o(1) \\ &= ca_{2n} + d - \frac{a_{2n}}{c'}\left(1 - \frac{d' - z^{-1}}{c'a_{2n}} + o(a_{2n}^{-1})\right) + o(1) \\ &= \left(c - \frac{1}{c'}\right)a_{2n} + d + \frac{d' - z^{-1}}{(c')^2} + o(1), \\ h^{[2n+1]} &= b_{2n+1} - \frac{a_{2n}^2}{R_{2n} - z^{-1}} + o(1) = c'a_{2n} + d' - \frac{a_{2n}/c}{1 + (d - z^{-1})/ca_{2n} + o(a_{2n}^{-1})} + o(1) \\ &= c'a_{2n} + d' - \frac{a_{2n}}{c}\left(1 - \frac{d - z^{-1}}{ca_{2n}} + o(a_{2n}^{-1})\right) + o(1) \\ &= \left(c' - \frac{1}{c}\right)a_{2n} + d' + \frac{d - z^{-1}}{c^2} + o(1). \end{aligned}$$

The above results imply that $h^{[n]}$ diverges whenever $cc' \neq 1$, which proves *(i)*. On the other hand, if $cc' = 1$, then

$$h^{[2n]} \to d + c^2(d' - z^{-1}), \qquad h^{[2n+1]} \to d' + \frac{d - z^{-1}}{c^2}.$$

Bearing in mind Proposition 4.1, this yields *(ii)* because, according to (9), the measure ν of the Nevanlinna function $\beta - \alpha z^{-1}$, $\beta \in \mathbb{R}$, $\alpha > 0$, is given by the relation

$$\int_{\mathbb{R}} \frac{d\nu(x)}{1 - zx} = \frac{1}{1 + \alpha - \beta z} = \frac{1}{1 + \alpha}\frac{1}{1 - \frac{\beta}{1+\alpha}z},$$

hence

$$d\nu(x) = \frac{1}{1 + \alpha}\,\delta(x - \tfrac{\beta}{1+\alpha}).$$

□

It is worth remarking that, in contrast to the bounded case covered by Teorem 8.1, the convergent measures $p_n^2 \mathrm{d}\mu$ provided by the above theorem have a vague limit which is not a probability measure but a strictly subprobability one. According to (32), this implies that such vague limits are not weak limits. A natural question arises: is this a general feature of the asymptotic behaviour of $p_n^2 \mathrm{d}\mu$ in the unbounded case? The following results not only answer this in the affirmative, but give even more information about this situation.

Theorem 8.7 *Let a_n, b_n be the coefficients of the recurrence relation (24) for the orthonormal polynomials p_n with respect to a probability measure μ on $\mathbb{R}$. If a subsequence a_{n_j} diverges and the contiguous subsequences $p_{n_j}^2 \mathrm{d}\mu$ and $p_{n_j+1}^2 \mathrm{d}\mu$ converge vaguely to the measures ν and ρ respectively, then $\nu(\mathbb{R}) + \rho(\mathbb{R}) \leq 1$. Therefore, $\nu(\mathbb{R}), \rho(\mathbb{R}) < 1$ when ν and ρ are non-null, while $\nu(\mathbb{R}) \leq 1/2$ if $\nu = \rho$.*

Proof Assume $p_{n_j}^2 \mathrm{d}\mu \overset{v}{\to} \nu$ and $p_{n_j+1}^2 \mathrm{d}\mu \overset{v}{\to} \rho$. From (33) we know that $\mu(\mathbb{R}), \rho(\mathbb{R}) \leq 1$. Therefore, the result holds if $\nu = 0$ or $\rho = 0$.

Assume therefore $\nu, \rho \neq 0$. If R_n are the inverse iterates of μ, Proposition 8.4 guarantees that $R_{n_j}^2/a_{n_j}^2 \sim \kappa$ for some $\kappa > 0$. Hence, the sequence R_{n_j}/a_{n_j} of Nevanlinna functions is locally uniformly bounded, and its possible limit points—which must be Nevanlinna functions again—are $\pm\sqrt{\kappa}$. By restricting to a new subsequence if necessary, we may suppose that R_{n_j}/a_{n_j} has a limit $c \in \{\sqrt{\kappa}, -\sqrt{\kappa}\}$. Then, Proposition 8.4 shows that f_{n_j}/a_{n_j} converges to $1/c$. Hence,

$$f_{n_j+1} = c^{-1} a_{n_j} + \zeta_j, \qquad R_{n_j} = c a_{n_j} + \xi_j, \qquad \zeta_j, \xi_j = o(a_{n_j}),$$

ζ_j and ξ_j being Nevanlinna functions. Using (66) we can write

$$\begin{aligned} h^{[n_j]} &= R_{n_j} + f_{n_j} - b_{n_j} = R_{n_j} - \frac{a_{n_j}^2}{f_{n_j+1} - z^{-1}} = c a_{n_j} + \xi_j - \frac{a_{n_j}^2}{c^{-1} a_{n_j} + \zeta_j - z^{-1}} \\ &= c a_{n_j} + \xi_j - \frac{c a_{n_j}}{1 + c \frac{\zeta_j - z^{-1}}{a_{n_j}}}. \end{aligned}$$

Applying the equality $1/(1+x) = 1 - x/(1+x)$ in the following way

$$\frac{c a_{n_j}}{1 + c \frac{\zeta_j - z^{-1}}{a_{n_j}}} = c a_{n_j} - c^2 \frac{\zeta_j - z^{-1}}{1 + c \frac{\zeta_j - z^{-1}}{a_{n_j}}},$$

and bearing in mind that

$$\frac{\zeta_j - z^{-1}}{1 + c \frac{\zeta_j - z^{-1}}{a_{n_j}}} - \left(\frac{\zeta_j}{1 + c \frac{\zeta_j}{a_{n_j}}} - z^{-1} \right) = \frac{c z^{-1}}{1 + c \frac{\zeta_j - z^{-1}}{a_{n_j}}} \left(\frac{\frac{\zeta_j}{a_{n_j}}}{1 + c \frac{\zeta_j}{a_{n_j}}} + \frac{\zeta_j - z^{-1}}{a_{n_j}} \right) \to 0,$$

we conclude that

$$h^{[n_j]} = \xi_j + \frac{c^2 \zeta_j}{1 + c \frac{\zeta_j}{a_{n_j}}} - c^2 z^{-1} + o(1) = \xi_j + \frac{c^2}{\frac{1}{\zeta_j} + \frac{c}{a_{n_j}}} - c^2 z^{-1} + o(1).$$

The vague convergence of $p_{n_j}^2\,\mathrm{d}\mu$ means that $h^{[n_j]}$ converges. In light of the above equality, this reads as the convergence of the sequence of Nevanlinna functions

$$\xi_j + \frac{c^2}{\frac{1}{\zeta_j} + \frac{c}{a_{n_j}}}.$$

Let us denote by η the limit of this sequence, which must be a Nevanlinna function too. According to (10), for $z \in i\mathbb{R}$ around the origin, $\eta = -\epsilon z^{-1} + o(z^{-1})$ with $\epsilon \geq 0$, so that

$$h^{[n_j]} \to h = \eta - c^2 z^{-1} = -(c^2+\epsilon)z^{-1} + o(z^{-1}).$$

Since h is the Nevanlinna function for the vague limit ν of $p_{n_j}^2\,\mathrm{d}\mu$, applying (10) to h we find that

$$1 - \frac{1}{\nu(\mathbb{R})} = \lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} zh(z) = -(c^2+\epsilon) \leq -c^2,$$

which gives the inequality

$$\nu(\mathbb{R}) \leq \frac{1}{1+c^2}.$$

Proceeding analogously with $h^{[n_j+1]}$ we find that

$$\begin{aligned}
h^{[n_j+1]} &= f_{n_j+1} + R_{n_j+1} - b_{n_j+1} = f_{n_j+1} - \frac{a_{n_j}^2}{R_{n_j} - z^{-1}} \\
&= c^{-1}a_{n_j} + \zeta_j - \frac{a_{n_j}^2}{ca_{n_j} + \xi_j - z^{-1}} \\
&= c^{-1}a_{n_j} + \zeta_j - \frac{c^{-1}a_{n_j}}{1 + c^{-1}\frac{\zeta_j - z^{-1}}{a_{n_j}}} = \zeta_j + c^{-2}\frac{\xi_j - z^{-1}}{1 + c^{-1}\frac{\zeta_j - z^{-1}}{a_{n_j}}} \\
&= \zeta_j + \frac{c^{-2}\,\xi_j}{1 + c^{-1}\frac{\xi_j}{a_{n_j}}} - c^{-2}z^{-1} + o(1).
\end{aligned}$$

Similar arguments to those used in the case of $h^{[n_j]}$ yield now

$$1 - \frac{1}{\rho(\mathbb{R})} \leq -c^{-2},$$

leading to the inequality

$$\rho(\mathbb{R}) \leq \frac{1}{1+c^{-2}} = \frac{c^2}{1+c^2}.$$

Combining both results gives

$$\nu(\mathbb{R}) + \rho(\mathbb{R}) \leq \frac{1}{1+c^2} + \frac{c^2}{1+c^2} = 1,$$

which proves the theorem. □

The combination of Theorem 8.3 with the above one leads to an important result.

Theorem 8.8 *Let p_n be the orthonormal polynomials with respect to a probability measure μ supported on an unbounded subset of $\mathbb{R}$. If the sequence $p_n^2 \mathrm{d}\mu$ converges to a vague limit ν, then $\nu(\mathbb{R}) \leq 1/2$.*

Proof Assuming that $p_n^2 \mathrm{d}\mu$ has a vague limit ν, the statement is obviously true when $\nu = 0$. Suppose $\nu \neq 0$. If a_n, b_n are the coefficients of the recurrence relation (24) for the orthonormal polynomials p_n, Theorem 8.3 ensures the presence of a divergent subsequence a_{n_j}. Then, Theorem 8.7 implies that $\nu(\mathbb{R}) \leq 1/2$. □

In light of the relation (32) between weak and vague convergence, these last two theorems have a couple of immediate consequences.

Corollary 8.9 *Let a_n, b_n be the coefficients of the recurrence relation (24) for the orthonormal polynomials p_n with respect to a probability measure μ on $\mathbb{R}$. If a subsequence a_{n_j} diverges, then the contiguous subsequences $p_{n_j}^2 \mathrm{d}\mu$ and $p_{n_j+1}^2 \mathrm{d}\mu$ cannot be simultaneosuly weakly convergent.*

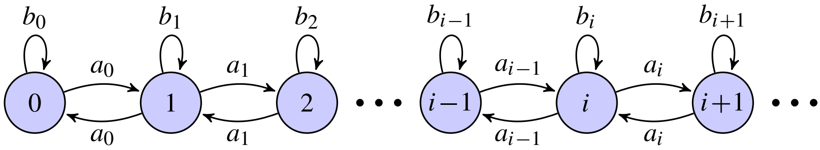
Corollary 8.10 *If p_n are the orthonormal polynomials with respect to a probability measure μ supported on an unbounded subset of $\mathbb{R}$, the sequence $p_n^2 \mathrm{d}\mu$ cannot be weakly convergent.*

The above corollary means that the only probability measures μ on the real line giving weakly convergent sequences $p_n^2 \mathrm{d}\mu$ are those obtained by B. Simon in [15], and summarized in Theorem 8.1. Nevertheless, we have proved that this is not the end of the story because the unbounded case brings us instances of vaguely convergent sequences $p_n^2 \mathrm{d}\mu$ which are not weakly convergent. The complete classification of the measures μ supported on unbounded subsets of the real line such that $p_n^2 \mathrm{d}\mu$ is vaguely convergent remains as a challenge for a future work.

9 Graph Theory Approach

In this section we present a different approach to Khrushchev's formula inspired by the classical theory of recurrence in random walks founded by George Pólya. This new approach uncovers a graph theoretic interpretation of the Nevanlinna function of a measure on the real line, and provides a simplified proof of Khrushchev's formula in the bounded case. This simpler diagrammatic approach is therefore more limited in scope than the one previously presented, based on the development of the Wall polynomials on the real line. Nevertheless, this new approach, being less dependent on the particularities of OPRL theory, opens a way to exporting Khrushchev's formula to other contexts (see [1, 5]).

To build this new approach we need to look at a Jacobi matrix $\mathcal{J} = (\mathcal{J}_{ij})_{i,j=0}^{\infty}$, such as (26), as the weight matrix of the following directed graph with set of nodes $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$,



(67)

The edges of this graph are the ordered pairs $(i, j) \in \mathbb{Z}_+^2$ satisfying $|i - j| \leq 1$, the corresponding weight being $\omega(i, j) = \mathcal{J}_{ij}$, i.e.

$$\omega(i, i) = b_i, \qquad \omega(i, i+1) = \omega(i+1, i) = a_i.$$

This corresponds essentially to the random walk model used in [15,p 207] to prove Theorem 8.1. The fact that the original proof of such a result already used random walk techniques similar to those developed in this section, highlights even more the importance of the graph theory approach to OPRL problems.

A **path of length** ℓ in this graph is an ordered set $(i_0, i_1, \dots, i_\ell)$ of contiguous nodes $i_k \in \mathbb{Z}_+$, i.e. such that $|i_k - i_{k+1}| \leq 1$. To each path we associate the weight

$$\omega(i_0, i_1, \dots, i_\ell) = \omega(i_0, i_1)\, \omega(i_1, i_2) \cdots \omega(i_{\ell-1}, i_\ell) = \mathcal{J}_{i_0 i_1} \mathcal{J}_{i_1 i_2} \cdots \mathcal{J}_{i_{\ell-1} i_\ell}.$$

A path may be split as a product of paths, an operation defined by

$$(i_0, \dots, i_k, \dots, i_\ell) = (i_0, \dots, i_k)(i_k, \dots, i_\ell),$$

so that the length is additive and the weight is multiplicative for such a product.

Given two nodes $m, n \in \mathbb{Z}_+$, the quantity

$$\Omega_\ell(m, n) = (\mathcal{J}^\ell)_{mn} = \sum_{i_1, \dots, i_{\ell-1} \in \mathbb{Z}_+} \mathcal{J}_{m i_1} \mathcal{J}_{i_1 i_2} \cdots \mathcal{J}_{i_{\ell-1} n}$$

may be viewed as a sum of weights over the paths connecting the nodes m and n,

$$\Omega_\ell(m, n) = \sum_{i_1, \dots, i_{\ell-1} \in \mathbb{Z}_+} \omega(m, i_1, \dots, i_{\ell-1}, n).$$

We will give the name n**-loops** to the paths with the same initial and final node n, while $\Omega_\ell(n) := \Omega_\ell(n, n)$ will be called the weight of the n-loops of length ℓ. If p_n are the orthonormal polynomials corresponding to $\mathcal{J}$ –given by (24)—and μ is an orthogonality measure for them, (24) yields the integral representation

$$\Omega_\ell(n) = (\mathcal{J}^\ell)_{nn} = \int_{\mathbb{R}} x^\ell p_n(x)^2 \mathrm{d}\mu(x),$$

thus the weight of the n-loops of length ℓ becomes the ℓ-th moment of the measure $p_n^2 \mathrm{d}\mu$. When μ has a bounded support—which will be assumed in what follows—$\mathcal{J}$ defines a bounded self-adjoint operator on the space of square-summable sequences, thus one finds that $|\Omega_\ell(n)| \leq \|\mathcal{J}\|^\ell$. Then, denoting by $\mathcal{I}$ the semi-infinite identity matrix, the above result leads to

$$\begin{aligned} \lambda_n(z) &= \int_{\mathbb{R}} \frac{p_n(x)^2 \mathrm{d}\mu(x)}{1 - xz} = \left((\mathcal{I} - z\mathcal{J})^{-1}\right)_{nn} \\ &= \sum_{\ell \geq 0} \Omega_\ell(n)\, z^\ell, \quad \Omega_0(n) := 1, \quad |z| < \|\mathcal{J}\|, \end{aligned} \tag{68}$$

so that λ_n is an analytic function around the origin whose Taylor coefficients are the weights of the n-loops. We will refer to λ_n as the **generating function of the n-loops**.

A n-loop $(n, i_1, \dots, i_{\ell-1}, n)$ will be called **simple** if the intermediate nodes are all different from n, i.e. $i_k \neq n$ for $0 < k < \ell$. Every n-loop is a product of simple ones obtained by splitting the original loop at the indices k such that $i_k = n$. The weight of the simple n-loops of length ℓ is defined by

$$\widehat{\Omega}_\ell(n) = \sum_{i_1, \dots, i_{\ell-1} \in \mathbb{Z}_+ \setminus \{n\}} \omega(n, i_1, \dots, i_{\ell-1}, n),$$

the corresponding generating function being

$$\sigma_n(z) = \sum_{\ell \geq 1} \widehat{\Omega}_\ell(n)\, z^\ell, \tag{69}$$

which will be called the **generating function of the simple n-loops**. That this power series defines an analytic function around the origin follows from the relation

$$\widehat{\Omega}_\ell(n) = \sum_{i_k \in \mathbb{Z}_+ \setminus \{n\}} \mathcal{J}_{n i_1} \mathcal{J}_{i_1 i_2} \cdots \mathcal{J}_{i_{\ell-1} n} = (\mathcal{J}(\mathcal{Q}_n \mathcal{J})^{\ell-1})_{nn},$$

where $\mathcal{Q}_n$ is the semi-infinite matrix given by $\mathcal{Q}_n = (\delta_{i,j} - \delta_{i,n}\delta_{j,n})_{i,j=0}^\infty$. This leads to the operator representation

$$\sigma_n(z) = z\left(\mathcal{J}(\mathcal{I} - z\mathcal{Q}_n\mathcal{J})^{-1}\right)_{nn},$$

where $\mathcal{J}$, $\mathcal{Q}_n$ and the semi-infinite identity matrix $\mathcal{I}$ are identified with the bounded operators that they define in the corresponding Hilbert space of square-summable sequences. The fact that $\mathcal{Q}_n\mathcal{J}$ is bounded guarantees the analyticity of σ_n around the origin. Actually, since $\|\mathcal{Q}_n\| = 1$, we find that (69) converges for $|z| < \|\mathcal{J}\|$, similarly to (68).

The expressions of λ_n and σ_n as power series with coefficients given by sums over n-loops make sense for any weighted graph, as long as the weight matrix is bounded. This ensures the existence of λ_n and σ_n as analytic functions around the origin, following similar arguments to those given previously for a bounded Jacobi matrix. In such a general setting, these generating functions are connected by a very simple relation which is the graph translation of the so called renewal equation for Markov chains.

Proposition 9.1 *For any graph with a bounded weight matrix and any node n of the graph, the generating functions λ_n and σ_n of the n-loops and the simple n-loops are related by*

$$\lambda_n(z) = \frac{1}{1-\sigma_n(z)}.$$

Proof Let us split every n-loop as $(n, i_1, \dots, i_{\ell-1}, n) = (n, i_1, \dots, i_{k-1}, n)(n, i_{k+1}, \dots, i_{\ell-1}, n)$, according to the smallest index k such that $i_k = n$. While the second factor $(n, i_{k+1}, \dots, i_{\ell-1}, n)$ is again a n-loop, the first factor $(n, i_1, \dots, i_{k-1}, n)$ is a simple n-loop. Hence, one may write

$$\begin{aligned}\Omega_\ell(n) &= \sum_{k=1}^{\ell} \sum_{\substack{i_1,\dots,i_{k-1} \neq n \\ i_{k+1},\dots,i_{\ell-1}}} \omega(n, i_1, \dots, i_{k-1}, n)\, \omega(n, i_{k+1}, \dots, i_{\ell-1}, n) \\ &= \sum_{k=1}^{\ell} \widehat{\Omega}_k(n)\, \Omega_{\ell-k}(n).\end{aligned}$$

Inserting this relation into λ_n gives

$$\lambda_n(z) = 1 + \sum_{\ell \geq 1} \sum_{k=1}^{\ell} \widehat{\Omega}_k(n)\, \Omega_{\ell-k}(n)\, z^\ell = 1 + \sigma_n(z)\lambda_n(z),$$

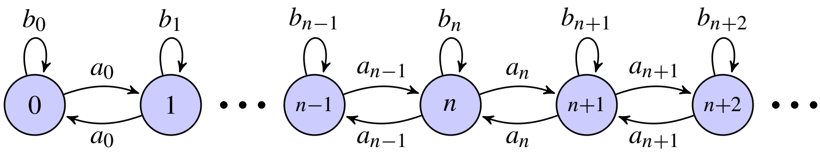
which proves the result. □

Combining the above result with (68) and Definition 3.1, we find a loop interpretation for the Nevanlinna function of a measure on the real line.

Corollary 9.2 *Let p_n be the orthonormal polynomials with respect to a measure μ supported on a bounded subset of the real line. If $h^{[n]}$ is the Nevanlinna function of the measure $p_n^2 \mathrm{d}\mu$, then $\sigma_n(z) = zh^{[n]}(z)$ is the generating function of the simple n-loops for the graph 67 associated to the corresponding Jacobi matrix. In particular, the generating function of the simple 0-loops is $\sigma_0(z) = zf(z)$, where f is the Nevanlinna function of the measure μ.*

The above interpretation of Nevanlinna functions is key for the following diagrammatic proof of Khrushchev's formula.

Proof (Diagrammatic proof of Theorem 7.3, bounded case)
Consider the node n of the following graph associated to the Jacobi matrix (26),

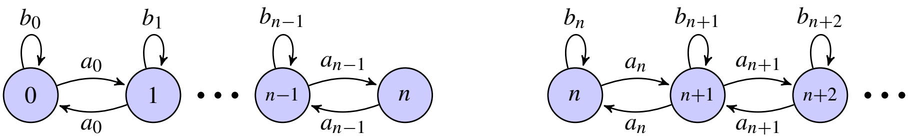


(70)

Any simple n-loop $(n, i_1, \dots, i_{\ell-1}, n)$ falls into one of three disjoint classes: since $i_1, \dots, i_{\ell-1} \neq n$ and $|i_k - i_{k+1}| \leq 1$ for all k, either $\ell = 1$ and there is no intermediate node, or $\ell > 1$ and all the intermediate nodes satisfy $i_1, \dots, i_{\ell-1} < n$ or $i_1, \dots, i_{\ell-1} > n$. In other words, any simple n-loop is a self-loop or, before returning to n, it moves only to the left of n, or only to the right of n. According to this picture, the generating function of the simple n-loops decomposes as

$$\sigma_n(z) = \sum_{\ell>1} \widehat{\Omega}_\ell^L(n)\, z^\ell + b_n z + \sum_{\ell>1} \widehat{\Omega}_\ell^R(n)\, z^\ell$$
$$\widehat{\Omega}_\ell^L(n) = \sum_{i_1,\dots,i_{\ell-1}<n} \omega(n, i_1, \dots, i_{\ell-1}, n), \quad \widehat{\Omega}_\ell^R(n) = \sum_{i_1,\dots,i_{\ell-1}>n} \omega(n, i_1, \dots, i_{\ell-1}, n).$$

The different parts of this decomposition may be identified resorting to the splitting of the previous graph into the following ones,



(71)

The generating functions of simple n-loops for the left and right graph are $\sigma_n^L(z) = \sum_{\ell>1} \widehat{\Omega}_\ell^L(n)\, z^\ell$ and $\sigma_n^R(z) = b_n z + \sum_{\ell>1} \widehat{\Omega}_\ell^R(n)\, z^\ell$, respectively. Hence, $\sigma_n = \sigma_n^L + \sigma_n^R$.

On the other hand, from Corollary 9.2 we know that $\sigma_0(z) = zf(z)$, with f the Nevanlinna function of a measure related to the Jacobi matrix associated to the graph 70, which has Schur parameters $(b_0, a_0, b_1, a_1, \dots)$. As for the right graph of 71, analogously, $\sigma_n^R(z) = zf^R(z)$, where f^R is a Nevanlinna function with Schur parameters $(b_n, a_n, b_{n+1}, a_{n+1}, \dots)$. Regarding the left graph of 71, looking at it from the right to the left shows that $\sigma_n^L(z) = zf^L(z)$, with f^L a Nevanlinna function having Schur parameters $(0, a_{n-1}, b_{n-1}, \dots, a_0, b_0, 0)$. Since in the bounded case the Schur parameters characterize the Nevanlinna functions, we conclude that f^R is the iterate f_n of f, while Proposition 3.8 shows that f^L is the Nevanlinna function $g_n(z) = a_{n-1} p_{n-1}(z^{-1}) / p_n(z^{-1})$. If $h^{[n]}$ is the Nevanlinna function of

$p_n^2 \mathrm{d}\mu$, we already know from Corollary 9.2 that $\sigma_n(z) = zh^{[n]}(z)$. Then, the equality $\sigma_n = \sigma_n^L + \sigma_n^R$ proves the identity $h^{[n]} = g_n + f_n$. $\square$

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References

1. Cedzich, C., Grünbaum, F.A., Velázquez, L., Werner, A.H., Werner, R.F.: A quantum dynamical approach to matrix Khrushchev's formulas. Commun. Pure Appl. Math. **69**, 909–957 (2016)
2. Chung, K.L.: A Course in Probability Theory, 3rd edn. Academic Press, San Diego (2000)
3. Dueñas, H., Huertas, E.J., Marcellán, F.: Asymptotic properties of Laguerre–Sobolev-type orthogonal polynomials. Numer. Algorithms **60**(1), 51–73 (2012)
4. Gesztesy, F., Tsekanovskii, E.: On matrix-valued Herglotz functions. Math. Nachr. **218**, 61–138 (2000)
5. Grünbaum, F.A., Velázquez, L.: A generalization of Schur functions: applications to Nevanlinna functions, orthogonal polynomials, random walks and unitary open quantum walks. Adv. Math. **326**, 352–464 (2018)
6. Helly, E.: Über lineare Funktionaloperationen. Wien. Ber. **121**, 265–297 (1912)
7. Ismail, M.E.H., Letessier, J., Valent, G.: Linear birth and death models and associated Laguerre and Meixner polynomials. J. Approx. Theory **56**, 337–348 (1988)
8. Kac, I.S., Krein, M. G., R-functions–analytic functions mapping the upper halfplane into itself. In: American Mathematical Society Translations, Series 2, vol. 103: Nine papers in analysis. AMS, Providence (1974)
9. Khrushchev, S.: Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in L^2(. J. Approx. Theory **108**, 161–248 (2001)
10. Khrushchev, S.: Classification theorems for general orthogonal polynomials on the unit circle. J. Approx. Theory **116**, 268–342 (2002)
11. Montel, P.: Leçons sur les familles normales de fonctions analytiques et leurs applications, Gauthier-Villars, Paris, 1927. Reprinted by Chelsea Publ. Co., New York (1974)
12. Schiff, J.L.: Normal families. Universitext. Springer, New York (1993)
13. Simon, B.: The classical moment problem as a self-adjoint finite difference operator. Adv. Math. **137**, 82–203 (1998)
14. Simon, B.: Analogs of the m-function in the theory of orthogonal polynomials on the unit circle. J. Comput. Appl. Math. **171**, 411–424 (2004)
15. Simon, B.: Ratio asymptotics and weak asymptotic measures for orthogonal polynomials on the real line. J. Approx. Theory **126**, 198–217 (2004)
16. Simon, B.: Orthogonal polynomials on the unit circle, part 1 and 2. Am. Math. Soc. Colloq. Publ. Ser. **54**, 1–2 (2005)
17. Temme, N.M.: Remarks on Slater's asymptotic expansions of Kummer functions for large values of the a-parameter. Adv. Dyn. Syst. Appl. **8**, 365–377 (2013)
18. Volkmer, H.: The asymptotic expansion of Kummer functions for large values of the a-parameter, and remarks on a paper by Olver. SIGMA **12**, 22 (2016)