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Boundary classes for graph problems involving non-local properties

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ABSTRACT

We continue the study of boundary classes for NP-hard problems and focus on seven NP-hard graph problems involving non-local properties: Hamiltonian Cycle, Hamiltonian Cycle Through Specified Edge, Hamiltonian Path, Feedback Vertex Set, Connected Vertex Cover, Connected Dominating Set and Graph $\mathrm{VC}_{\mathrm{CON}}$ Dimension. Our main result is the determination of the first boundary class for Feedback Vertex Set. We also determine boundary classes for Hamiltonian Cycle Through Specified Edge and Hamiltonian Path and give some insights on the structure of some boundary classes for the remaining problems.

1. Introduction

Many NP-hard graph problems remain NP-hard even for restricted classes of graphs, while they become polynomial-time solvable when further restrictions are applied. For example, the well-known graph problem Hamiltonian Cycle is NP-hard in general and it remains NP-hard for subcubic graphs (see, e.g., [26]). On the other hand, it is clearly trivial for graphs with maximum degree 2. It is therefore natural to ask when a certain "hard" graph problem becomes "easy": Is there any "boundary" separating "easy" and "hard" instances? Alekseev [3] considered this question in the case the instances are hereditary classes of graphs. Given a graph problem Π, a hereditary class of graphs X is Π-*hard* if Π is NP-hard for X, and Π-*easy* if Π is solvable in polynomial time for graphs in X. He introduced the notion of Π-boundary class, playing the role of the "boundary" separating Π-hard and Π-easy instances, and showed that a finitely defined (hereditary) class is Π-hard if and only if it contains a Π-boundary class. Moreover, he determined a boundary class for Independent Set, the first result in the systematic study of boundary classes for NP-hard graph problems (see, e.g., [6,7,38,50,51]). Note that here and throughout the paper we tacitly assume that $\mathrm{P} \neq \mathrm{NP}$, or else the notion of "boundary" becomes vacuous.

In this paper, we continue the study of boundary classes for NP-hard problems and focus on seven NP-hard graph problems involving non-local properties: Hamiltonian Cycle, Hamiltonian Cycle Through Specified Edge, Hamiltonian Path, Feedback Vertex Set, Connected Vertex Cover, Connected Dominating Set and Graph $\mathrm{VC}_{\mathrm{CON}}$ Dimension. Our main result is the determination of the first boundary class for Feedback Vertex Set.

In a first attempt to answer the meta-question posed above, one might be tempted to consider maximal Π-easy classes and minimal Π-hard classes. In fact, the first approach immediately turns out to be meaningless: there are no maximal Π-easy classes. Indeed, every Π-easy class X can be extended to another Π-easy class simply by adding to X a graph

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$G \notin X$ together with all its induced subgraphs. Even the approach through minimal Π-hard classes is not completely satisfactory, as for some problems they might not exist at all: for any $\ell \geq 3$, Hamiltonian Cycle is NP-hard for subcubic $(C_3, \ldots, C_\ell)$-free graphs (see, e.g., [38]) and this gives an infinite decreasing sequence of hard classes. Many other examples of this kind are known for problems like Independent Set and Dominating Set [3,6]. In other situations, minimal Π-hard classes indeed exist, as the following example due to Malyshev and Pardalos [52] shows. Consider the Traveling Salesman Problem: given a graph G, a weight function $w\colon E(G) \to \mathbb{R}$ and a number s, does there exist a Hamiltonian cycle C of G such that $\sum_{e \in E(C)} w(e) \leq s$? A simple reduction from Hamiltonian Cycle shows that the problem is NP-hard for the class of complete graphs. On the other hand, every proper hereditary subclass of the class of complete graphs is finite and so the problem can be clearly solved in polynomial time for such a subclass. This means that the class of complete graphs is a minimal (hereditary) hard class for the Traveling Salesman Problem.

The previous discussion suggests that the limit of a decreasing sequence of Π-hard classes should play a role in the search of a "boundary" between easy and hard classes. Alekseev [3] formalised this intuition by introducing the notions of limit class and boundary class for Independent Set. In fact, these concepts are completely general[1]:

Definition 1 *(Alekseev et al. [7]).* Let Π be an NP-hard graph problem and X a Π-hard class of graphs. A class of graphs Y is a *limit class for* Π *with respect to* X ((Π, X)-limit, in short) if there exists a sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of Π-hard subclasses of X such that $\bigcap_{n \geq 1} Y_n = Y$. The class Y is a *limit class for* Π (Π-limit) if there exists a Π-hard class X such that Y is (Π, X)-limit.

An inclusion-wise minimal (Π, X)-limit class is a *boundary class for* Π *with respect to* X ((Π, X)-boundary, in short). The class Y is a *boundary class for* Π (Π-boundary) if there exists a Π-hard class X such that Y is (Π, X)-boundary.

Note that Alekseev [3] originally defined a limit class and a boundary class for the Independent Set problem Π as a (Π, X)-limit class and a (Π, X)-boundary class, respectively, where X is the set of all graphs.

We remark that a Π-hard class is by definition hereditary and this is a fundamental requirement. Note also that in the definition of a (Π, X)-limit class, the Π-hard subclasses of X in a sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ need not be distinct. In particular, every Π-hard subclass of X is (Π, X)-limit. On the other hand, a Π-limit class need not be Π-hard. Indeed, consider again Hamiltonian Cycle. Denoting by Y_ℓ the class of $(C_3, \ldots, C_\ell)$-free graphs, we have that Hamiltonian Cycle is NP-hard for Y_ℓ, for any $\ell \geq 3$. Moreover, $Y_\ell \supseteq Y_{\ell+1}$ and $\bigcap_{n \geq 1} Y_n$ is the class of forests, for which the problem is clearly trivial. The following is another important remark (see also [7,38]):

Remark 2. A Π-limit subclass of a Π-hard class X is not necessarily (Π, X)-limit. Indeed, let Π be Hamiltonian Cycle. We have seen that this problem is NP-hard for graphs with arbitrarily large girth and that the class of forests is a Π-limit class. Moreover, Müller [54] showed that the class $X = Free(C_3, C_5, C_6, \ldots)$ of *chordal bipartite* graphs is Π-hard. The class of forests is clearly contained in X but it is not (Π, X)-limit. Indeed, suppose there exists a sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of Π-hard subclasses of X such that $\bigcap_{n \geq 1} Y_n$ coincides with the class of forests. Clearly, there exists a class Y_i not containing C_4. But then Y_i is the class of forests, which is Π-easy, a contradiction.

The existence of a boundary class with respect to every Π-hard class is guaranteed by the following theorem:

Theorem 3 (Alekseev et al. [7]). *A class X is Π-hard if and only if it contains a (Π, X)-boundary class.*

Note that the "if" direction in Theorem 3 is trivial: if $Y \subseteq X$ is a (Π, X)-boundary class, there exists a sequence $Y_1 \supseteq Y_2 \supseteq \ldots$ of Π-hard subclasses of X and so Π is NP-hard for X as well.

Theorem 3 shows that a boundary class with respect to a Π-hard class represents indeed a meaningful notion of "boundary" between Π-hard and Π-easy subclasses. Moreover, Π-boundary classes can be used to characterise the finitely defined graph classes which are Π-hard:

Theorem 4 (Alekseev et al. [7]). *A finitely defined class is Π-hard if and only if it contains a Π-boundary class.*

Alekseev [3] studied Independent Set and revealed the first boundary class for this problem: the class $\mathcal{T}$ of forests whose components have at most three leaves. This shows that Theorem 4 is not true in general: the class of forests contains $\mathcal{T}$ but it is easy for the problem. So far, $\mathcal{T}$ is the only known boundary class for Independent Set and in fact he conjectured no other boundary class exists. It is easy to see that this conjecture is equivalent to the following statement: for each $G \in \mathcal{T}$, Independent Set is not NP-hard for $Free(G)$. The conjecture seems to be very challenging since it is already a major open problem to determine whether Independent Set is NP-hard for P_k-free graphs with $k > 6$. On the other hand, Lokshantov et al. [43] showed that Independent Set is polynomial-time solvable for P_5-free graphs and recently Lokshantov et al. [44]

[1] We also invite the reader to notice that the notions of limit class and boundary class make sense for every partially ordered set (see, e.g., [37]).

provided a quasipolynomial-time algorithm for P_6-free graphs. Further evidence in favour of the conjecture is obtained when forbidding certain subdivisions of the claw (see, e.g., [5,47,49]).

Other problems have been studied in the context of boundary classes. For example, Alekseev et al. [6] revealed three boundary classes for Dominating Set, one of them being $\mathcal{T}$, and Malyshev [51] found a fourth boundary class. Alekseev et al. [7] further emphasised the special role played by $\mathcal{T}$ and showed that it is boundary for Independent Dominating Set, Induced Matching and Edge Dominating Set. On the other hand, $\mathcal{T}$ is not boundary for Hamiltonian Cycle and Korpelainen et al. [38] revealed two boundary classes for this problem. So far, the complete description of boundary classes has been obtained only for a single problem, the so-called List Edge-Ranking: Malyshev [50] showed it admits exactly ten boundary classes. Note that some problems may admit infinitely many boundary classes and it is known that there is a continuum set of boundary classes for Vertex k-Colouring [38,48].

In this paper, we consider seven NP-hard graph problems involving non-local properties. In Section 3, we show that the boundary class for Hamiltonian Cycle with respect to graphs with arbitrarily large girth determined in [38] is also boundary with respect to planar bipartite graphs. This is obtained by strengthening a complexity result in [38] which also allows us to determine a non-trivial limit class with respect to split graphs. In Sections 4 and 5, we provide the first boundary class for Hamiltonian Cycle Through Specified Edge and Hamiltonian Path. This class in fact coincides with the one obtained in Section 3 and our proofs follow the lines of [38]. We also show that the non-trivial limit class for Hamiltonian Cycle with respect to split graphs is limit for Hamiltonian Cycle Through Specified Edge and Hamiltonian Path as well. In Section 6, we consider Feedback Vertex Set and prove our main result: the determination of the first boundary class for this problem (with respect to planar bipartite graphs with maximum degree 4). As the main tool for our proof, we show that Feedback Vertex Set can be solved in polynomial time for graphs with maximum degree at most 4 and bounded number of 4-vertices. Moreover, we provide a non-trivial limit class with respect to line graphs. In Sections 7 to 9, we determine non-trivial limit classes for Connected Vertex Cover, Connected Dominating Set and Graph $\mathrm{VC_{CON}}$ Dimension with respect to planar bipartite graphs and line graphs. Finally, in Section 10, we provide a complexity dichotomy for the closely related Connected Dominating Set and Graph $\mathrm{VC_{CON}}$ Dimension when restricted to classes of graphs defined by a single forbidden induced subgraph.

We conclude this section by recalling a series of results which will be used throughout the paper. We have already seen that, in general, a (Π, X)-limit class is not Π-hard. On the other hand, this is the case if the limit class is defined by finitely many forbidden induced subgraphs with respect to X:

Lemma 5 *(Alekseev et al. [7]). If Y is a (Π, X)-limit class which is defined by finitely many forbidden induced subgraphs with respect to X, then it is Π-hard.*

The following stronger version of Theorem 3 holds. It clearly implies Theorem 4.

Theorem 6 *(Alekseev et al. [7]). A subclass $Y \subseteq X$ defined by finitely many forbidden induced subgraphs with respect to X is Π-hard if and only if Y contains a (Π, X)-boundary class.*

Proving the minimality of a certain limit class is in general not an easy task. Nevertheless, the following sufficient condition turns out to be useful and it will be employed in all our proofs:

Lemma 7 *(Alekseev and Malyshev [4]). A (Π, X)-limit class $Y = Free(M)$ is (Π, X)-boundary if for every $G \in Y$ there exists a finite set of graphs $A \subseteq M$ such that $Free(A \cup \{G\})$ is a Π-easy class.*

Proof. Suppose Y is not (Π, X)-boundary. This means there exists a (Π, X)-limit class $Z \subsetneq Y$ and let G be a graph in $Y \setminus Z$. By assumption, there exists a finite set $A \subseteq M$ such that $Z' = Free(A \cup \{G\})$ is Π-easy. Moreover, since Z is (Π, X)-limit, there exists a sequence $Z_1 \supseteq Z_2 \supseteq \dots$ of Π-hard subclasses of X such that $\bigcap_{n \geq 1} Z_n = Z$. But then, for each n, we have that $Z'_n = Z_n \cup Z'$ is Π-hard. Moreover, for each k, we have that $Z'_k \supseteq Z'_{k+1}$ and $\bigcap_{n \geq 1} Z'_n = Z'$. In other words, Z' is a Π-limit class as well. Since the set of forbidden induced subgraphs for $Z' = Free(A \cup \{G\})$ is finite, there exists a class Z'_n which is $A \cup \{G\}$-free. Therefore, Z'_n is Π-easy, a contradiction. □

2. Preliminaries

In this paper we consider only finite graphs. Given a graph G, we usually denote its vertex set by $V(G)$ and its edge set by $E(G)$ and we let $n(G) = |V(G)|$ and $m(G) = |E(G)|$. A *loop* is an edge whose endpoints are equal and *multiple edges* are edges having the same pair of endpoints. For the most part of this paper, we consider only *simple* graphs, namely graphs with no loops and no multiple edges. However, in some parts of Sections 6.1 and 6.2, we allow loops and multiple edges and this will be explicitly mentioned whenever it is the case. In Section 3, we also consider digraphs. A *digraph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the *tail* of the edge and the second is the *head*.

Neighbourhoods and degrees. For a vertex $v \in V(G)$, the *neighbourhood* $N_G(v)$ is the set of vertices adjacent to v in G. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v in G, with the exception that each loop counts as two edges. A *k-vertex* is a vertex of degree k. We refer to a 3-vertex as a *cubic* vertex and to a 0-vertex as an *isolated* vertex. We denote by $d_k(G)$ the set of k-vertices of G. The *maximum degree* $\Delta(G)$ of G is the number $\max\{d_G(v) : v \in V(G)\}$ and G is *subcubic* if $\Delta(G) \leq 3$. Similarly, the *minimum degree* $\delta(G)$ of G is the quantity $\min\{d_G(v) : v \in V(G)\}$. If all the vertices of G have the same degree k, then G is *k-regular* and a *cubic* graph is a 3-regular graph. A *k-factor* of a graph is a spanning k-regular subgraph.

Given a vertex v of a digraph, the *outdegree* $d^+(v)$ is the number of edges with tail v and the *indegree* $d^-(v)$ is the number of edges with head v.

Paths and cycles. A *path* is a non-empty graph $P = (V, E)$ with $V = \{x_0, x_1, \dots, x_k\}$ and $E = \left\{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\right\}$, and where the x_i's are all distinct. The vertices x_0 and x_k are *linked* by P and they are called the *ends* of P. The vertices $x_1, \dots, x_{k-1}$ are the *inner* vertices of P. The *length* of a path is its number of edges and the path on n vertices is denoted by P_n. We refer to a path P by a natural sequence of its vertices: $P = x_0x_1 \cdots x_k$. Such a path P is a path *between* x_0 *and* x_k, or a *x_0, x_k-path*. If $P = x_0 \cdots x_k$ is a path and $k \geq 2$, the graph with vertex set $V(P)$ and edge set $E(P) \cup \{x_kx_0\}$ is a *cycle*. The cycle on n vertices is denoted by C_n. A *Hamiltonian path* of a graph G is a path of G which is spanning. A *Hamiltonian cycle* of G is a spanning cycle of G and a graph is *Hamiltonian* if it contains a Hamiltonian cycle.

Denoting by x_ix_{i+1} the ordered edge with tail x_i and head x_{i+1}, the notions from the previous paragraph have obvious analogues for digraphs.

The *distance* $d_G(u, v)$ from a vertex u to a vertex v in a graph G is the length of a shortest path between u and v. If u and v are not linked by any path in G, we set $d_G(u, v) = \infty$. The *radius* of G is the quantity $\min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$. The *girth* of a graph containing a cycle is the length of a shortest cycle and a graph with no cycle has infinite girth.

Graph operations. Let $G = (V, E)$ be a graph and $V' \subseteq V$ and $E' \subseteq E$. The operation of *deleting the set of vertices* V' from G results in the graph $G - V' = G[V \setminus V']$. The operation of *deleting the set of edges* E' from G results in the graph $G - E' = (V, E \setminus E')$. The *complement* of a simple graph G is the graph $\overline{G}$ with vertex set $V(G)$ and such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *union* of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are disjoint, their union is denoted by $G + H$ and the union of k (disjoint) copies of G is denoted by kG. A *k-subdivision* of G is the graph obtained from G by adding k new vertices for each edge of G, i.e. each edge is replaced by a path of length $k + 1$.

Graph classes and special graphs. If a graph does not contain induced subgraphs isomorphic to graphs in a set Z, it is *Z-free* and the set of all Z-free graphs is denoted by *Free*(Z). A class of graphs is *hereditary* if it is closed under deletions of vertices. It is well-known and easy to see that a class of graphs X is hereditary if and only if it can be defined by a set of forbidden induced subgraphs, i.e. $X = \mathit{Free}(Z)$ for some set of graphs Z. The minimal set Z with this property is unique and it is denoted by *Forb*(X). If the set of minimal forbidden induced subgraphs for a hereditary class X is finite, then X is *finitely defined*. If $X \subseteq Y$ and $\mathit{Forb}(X) \setminus \mathit{Forb}(Y)$ is a finite set, then X is *defined by finitely many forbidden induced subgraphs with respect to* Y.

A *complete graph* is a graph whose vertices are pairwise adjacent and the complete graph on n vertices is denoted by K_n. A *triangle* is the graph K_3. A graph G is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. An r-partite graph in which every two vertices from different partition classes are adjacent is called *complete* and 2-partite graphs are usually called *bipartite*. A *split graph* is a graph whose vertices can be partitioned into a clique and an independent set (see below for the definitions of clique and independent set). A *cograph* (or *complement reducible* graph) is defined recursively as follows: K_1 is a cograph, the disjoint union of cographs is a cograph, the complement of a cograph is a cograph. In fact, the class of cographs coincides with that of P_4-free graphs.

Trees. A *tree* is a connected graph not containing any cycle as a subgraph and the vertices of degree 1 are its *leaves*. A spanning tree is a *maximum leaf spanning tree* (MLST) if there is no spanning tree with a larger number of leaves. Given a graph G, we denote by $\ell(G)$ the number of leaves in a MLST of G. The *Betti number* $\mu(G)$ (also known as the *cyclomatic number* or the *circuit rank*) is the minimum number of edges that must be deleted from G in order to make it acyclic. It is easy to see that if G has c components, then $\mu(G) = |E(G)| - |V(G)| + c$.

Graph parameters. A set of vertices or edges of a graph is *minimum* with respect to the property $\mathcal{P}$ if it has minimum size among all subsets having property $\mathcal{P}$. The term *maximum* is defined analogously. In this paper we often consider the following parameters of a graph G.

An *independent set* of a graph is a set of pairwise non-adjacent vertices. A *clique* of a graph is a set of pairwise adjacent vertices.

A *colouring* of G is a partition of $V(G)$ into independent sets and the minimum number of partition classes is the *chromatic number* $\chi(G)$. The graph G is *k-colourable* if $\chi(G) \leq k$.

A *matching* of a graph is a set of pairwise non-incident edges and the *matching number* $\alpha'(G)$ is the size of a maximum matching of G.

A *vertex cover* of a graph is a subset of vertices containing at least one endpoint of every edge. The size of a minimum vertex cover of G is denoted by $\beta(G)$. Clearly, $S \subseteq V(G)$ is a vertex cover of G if and only if $V(G) \setminus S$ is an independent set of G. A *connected vertex cover* of G is a vertex cover S of G such that $G[S]$ is connected and we denote by $\beta_c(G)$ the size of a minimum connected vertex cover of G.

An *edge cover* of a graph G is a subset $S \subseteq E(G)$ of edges such that every vertex of G is incident to an edge in S. We denote by $\beta'(G)$ the size of a minimum edge cover of G. A result known as one of the Gallai's identities asserts that $\alpha'(G) + \beta'(G) = |V(G)|$, for any graph G without isolated vertices.

A *dominating set* of G is a subset $D \subseteq V(G)$ such that each vertex in $V(G) \setminus D$ is adjacent to a vertex in D. The size of a minimum dominating set of G is denoted by $\gamma(G)$. A *connected dominating set* of G is a dominating set D of G such that $G[D]$ is connected and we denote by $\gamma_c(G)$ the size of a minimum connected dominating set of G.

A *feedback vertex set* of G is a subset $T \subseteq V(G)$ such that $G - T$ is acyclic and we denote by $\tau_c(G)$ the size of a minimum feedback vertex set of G.

A *vertex triangle-transversal* of G is a subset $T \subseteq V(G)$ such that $G - T$ is triangle-free and we denote by $\tau_\Delta(G)$ the size of a minimum vertex triangle-transversal of G.

A *non-separating independent set* of G is an independent set $I \subseteq V(G)$ such that there is no $X \subseteq I$ for which $G - X$ has more components than G. Therefore, for a connected graph G, if $I \subseteq V(G)$ is a non-separating independent set of G, then $V(G) \setminus I$ is a connected vertex cover of G. We denote by $z(G)$ the size of a maximum non-separating independent set of G.

Tree-width and clique-width. Graphs of bounded tree-width are particularly interesting from an algorithmic point of view: many NP-complete problems can be solved in linear time for them. The notion of tree-width was introduced by Robertson and Seymour [64] in their seminal work on graph minors:

A *tree decomposition* of a graph $G = (V, E)$ is a pair (X, T), where $T = (I, F)$ is a tree and $X = \{X_i : i \in I\}$ is a family of subsets of V such that:

- $\bigcup_{i\in I} X_i = V$;
- for all edges $vw \in E$, there is an $i \in I$ such that $\{v, w\} \subseteq X_i$;
- for all vertices $v \in V$, the set $\{i \in I : v \in X_i\}$ forms a connected subtree of T.

The *width* of the tree decomposition (X, T) is $\max_{i\in I} |X_i| - 1$ and the *tree-width* of a graph G is the minimum width among all tree decompositions of G. It is easy to see that forests have tree-width at most 1 and the tree-width measures, loosely speaking, how far a given graph is from a tree. In fact, the graphs having tree-width at most k are exactly the so-called partial k-trees (see, e.g., [11] for a proof and other characterisations).

The rough idea is that, for certain problems, once a tree decomposition of the input graph with small width is found, it can be used in a dynamic programming algorithm to solve the original problem (see, e.g., [23] for some examples). For a fixed k, it is in fact possible to test in linear time whether a graph has tree-width at most k and, if so, to find a tree-decomposition with width at most k [10]. A celebrated algorithmic meta-theorem of Courcelle [14] provides a way to quickly establish that a certain problem is decidable in linear time on graphs of bounded tree-width: all (graph) problems expressible in monadic second-order logic with edge-set quantification are decidable in linear time on graphs of bounded tree-width, assuming a tree decomposition is given (see also [8]). Let us briefly recall that monadic second-order logic is an extension of first-order logic by quantification over sets. The language of *monadic second-order logic of graphs* (MSO_1 in short) contains the expressions built from the following elements:

- Variables $x, y, \ldots$ for vertices and $X, Y, \ldots$ for sets of vertices;
- Predicates $x \in X$ and $adj(x, y)$;
- Equality for variables, standard Boolean connectives and the quantifiers $\forall$ and $\exists$.

By considering edges and sets of edges as other sorts of variables and the incidence predicate $inc(v, e)$, we obtain *monadic second-order logic of graphs with edge-set quantification* (MSO_2 in short). A notion related to tree-width is that of clique-width, introduced by Courcelle et al. [16]. The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following operations:

- Creation of a new vertex v with label i;
- Disjoint union of two labelled graphs G and H;
- Joining by an edge each vertex with label i to each vertex with label j;
- Renaming label i to j.

Every graph can be defined by an algebraic expression using these four operations and such an expression is a *k-expression* if it uses k different labels. As shown by Courcelle and Olariu [15], every graph of bounded tree-width has bounded clique-width but there are graphs of bounded clique-width having unbounded tree-width (for example, complete graphs). Therefore, clique-width can be viewed as a more general concept than tree-width. An important class of graphs having

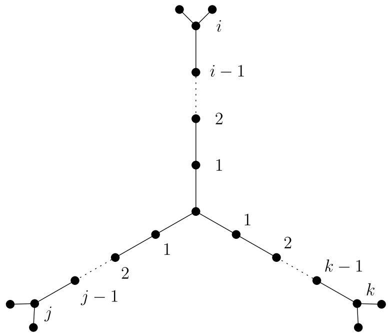


Fig. 1. A tribranch $Y_{i,j,k}$.

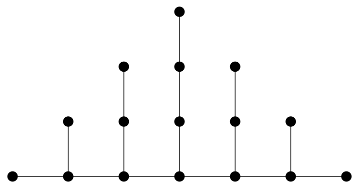


Fig. 2. A caterpillar with hairs of arbitrary length.

bounded clique-width is that of cographs: it directly follows from the definition that cographs have clique-width at most 2. We refer to [34] for other examples of graph classes of bounded clique-width.

Similarly to tree-width, having bounded clique-width has interesting algorithmic implications. If a graph property is expressible in the more restricted MSO_1, then Courcelle et al. [17] showed that it is decidable in linear time even for graphs of bounded clique-width, assuming a k-expression of the graph is explicitly given. On the other hand, for fixed k, Oum and Seymour [59] provided a polynomial-time algorithm that given a graph G either decides G has clique-width at least $k+1$ or outputs a $2^{3k+2}-1$-expression. Therefore, a graph property expressible in MSO_1 is decidable in polynomial time for graphs of bounded clique-width.

3. Hamiltonian Cycle

Hamiltonian Cycle is a well-known NP-complete graph problem which remains NP-hard even for subcubic graphs (see, e.g., [26]). Korpelainen et al. [38] determined the first boundary class for this problem (with respect to the class of subcubic graphs). In order to state their result, we have first to introduce some notation. For positive integers i, j and k, let $Y_{i,j,k}$ be the graph depicted in Fig. 1, called a *tribranch*. Moreover, let $\mathcal{Y}_p = \{Y_{i,j,k} : i, j, k \leq p\}$ and $\mathcal{C}_p = \{C_k : k \leq p\}$. Finally, denote by $\overline{\mathcal{Q}_p}$ the class of subcubic $\mathcal{Y}_p \cup \mathcal{C}_p$-free graphs such that each cubic vertex has a non-cubic neighbour.

Korpelainen et al. [38] showed that the class $\mathcal{Q} = \bigcap_{p \geq 1} \overline{\mathcal{Q}_p}$ is a boundary class for Hamiltonian Cycle:

Theorem 8 *(Korpelainen et al. [38]). For any $p \geq 1$,* Hamiltonian Cycle *is* NP*-complete for graphs in $\overline{\mathcal{Q}_p}$. Moreover, $\mathcal{Q}$ is a boundary class for* Hamiltonian Cycle.

The class $\mathcal{Q}$ is clearly contained in the class of forests and in fact it coincides with the class of graphs whose components are caterpillars with hairs of arbitrary length, where a *caterpillar with hairs of arbitrary length* is a subcubic tree in which all cubic vertices belong to a single path (see Fig. 2):

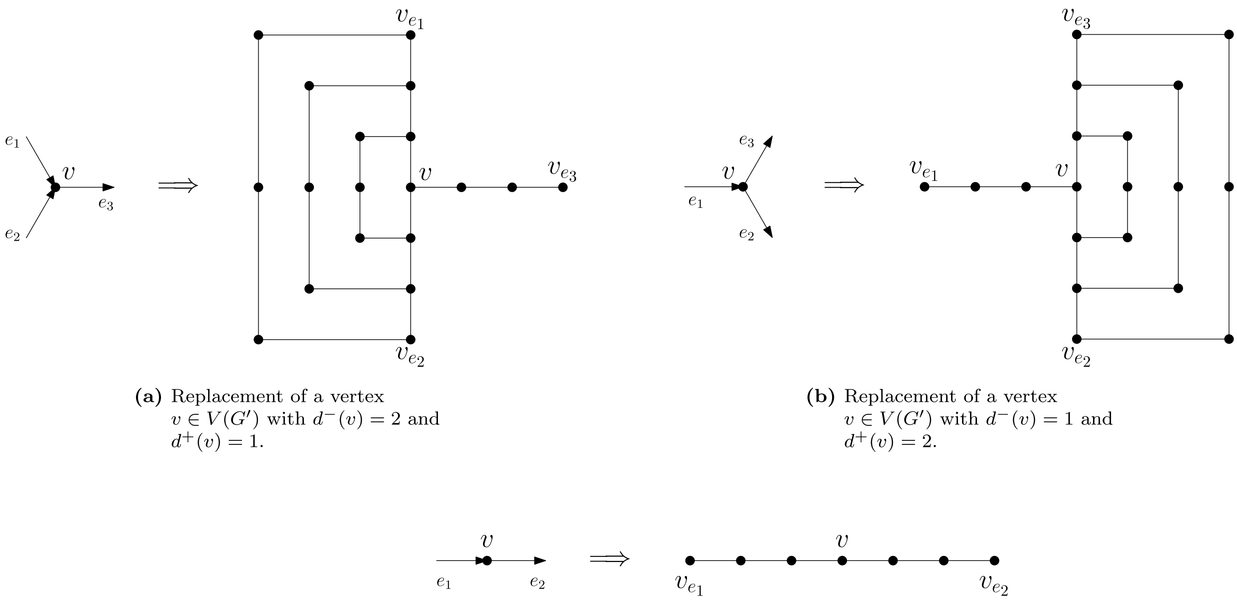
Lemma 9 *(Korpelainen et al. [38]). A graph G belongs to $\mathcal{Q}$ if and only if each component of G is a caterpillar with hairs of arbitrary length.*

Hamiltonian Cycle remains NP-hard for subcubic planar bipartite graphs (see, e.g., [2]) and in this section we show that the class $\mathcal{Q}$ is boundary for this problem even with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth.

We denote by $\mathcal{Q}_p$ the class of subcubic planar bipartite $\mathcal{Y}_p \cup \mathcal{C}_p$-free graphs such that each cubic vertex has a non-cubic neighbour. The following result strengthens Theorem 8 and implies that $\mathcal{Q}$ is limit for Hamiltonian Cycle with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth. Its proof is inspired by [9].

Lemma 10. *For any $p \geq 1$,* Hamiltonian Cycle *is* NP*-complete for graphs in $\mathcal{Q}_p$.*

Fig. 3. Construction of the digraph G' in the proof of Lemma 10.



(a) Replacement of a vertex $v \in V(G')$ with $d^-(v) = 2$ and $d^+(v) = 1$.

(b) Replacement of a vertex $v \in V(G')$ with $d^-(v) = 1$ and $d^+(v) = 2$.

(c) Replacement of a vertex $v \in V(G')$ with $d^-(v) = d^+(v) = 1$.

Fig. 4. Construction of the graph G'' in the proof of Lemma 10. Each vertex of G' is replaced by the gadgets depicted in (a), (b) and (c). Each vertex v_{e_i} in a gadget corresponds to the directed edge e_i incident to v in G' and vertices corresponding to the same directed edge are linked in G''.

Proof. We reduce from Hamiltonian Cycle restricted to planar digraphs such that each vertex has either indegree 1 and outdegree 2 or indegree 2 and outdegree 1, shown to be NP-complete by Plesńik [63]. Given such a digraph G, we construct in two steps an undirected graph G'' as follows.

First, we replace each vertex v of G with two vertices v_{in} and v_{out} as depicted in Fig. 3. Clearly, the resulting digraph G' is planar and $\{v_{\text{in}} : v \in V(G)\} \cup \{v_{\text{out}} : v \in V(G)\}$ is a bipartition of $V(G')$. Moreover, G' is Hamiltonian if and only if G is.

Now we replace each vertex of G' with the gadgets depicted in Fig. 4. The gadgets in Figs. 4(a) and 4(b) are obtained from a $2p$-subdivision of a claw by adding $2p+1$ paths of length $2p+2$ (in Fig. 4, we have $p=1$), while the gadget in Fig. 4(c) is a path of length $2(2p+1)$. Note that each vertex v_{e_i} in a gadget corresponds to the directed edge e_i incident to v in G'. The undirected graph G'' is then obtained by adding edges between the vertices v_{e_i} corresponding to the same directed edge in G'. Clearly, G'' is subcubic, planar and $\mathcal{C}_p$-free. Moreover, each cubic vertex has a non-cubic neighbour. Now note that G'' is obtained from the bipartite graph G' by a $2(2p+1)$-subdivision and then by adding disjoint paths of length $2p+2$ between vertices belonging to the same bipartition class. Therefore, G'' is bipartite as well. Finally, it is easy to see that G'' is $\mathcal{Y}_p$-free and so $G'' \in \mathcal{Q}_p$.

We now claim that G is Hamiltonian if and only if G'' is. Note that each vertex v of G is incident to an edge belonging to every Hamiltonian cycle (if any): the edge with head v, if $d^-(v)=1$, or the edge with tail v, if $d^+(v)=1$.

Suppose first G is Hamiltonian. Then G' has clearly a Hamiltonian cycle C. Moreover, each (directed) edge $e = u \rightarrow v$ of G' corresponds to the edge $u_e v_e$ of G'' and we select the edges of G'' corresponding to those in $E(C)$. Finally, we select all the edges of G'' inside gadgets corresponding to vertices v with $d^-(v) = d^+(v) = 1$ and those edges inside gadgets corresponding to vertices v with $d^-(v)+d^+(v)=3$ as depicted in Fig. 5. The selected edges clearly constitute a Hamiltonian cycle of G''.

Suppose now G'' is Hamiltonian and let C be a Hamiltonian cycle. It is easy to see that, for each gadget corresponding to a vertex $v \in V(G')$ with $d^-(v)+d^+(v)=3$, the cycle C can travers its vertices in exactly two ways, as depicted in Fig. 5. This means that, for each vertex $v \in V(G')$ with $d^-(v)+d^+(v)=3$, exactly two edges of the form $v_e u_e$ belong to $E(C)$. Therefore, we select the two corresponding directed edges of G'. In this way, we can easily obtain a Hamiltonian cycle of G' and then a Hamiltonian cycle of G. □

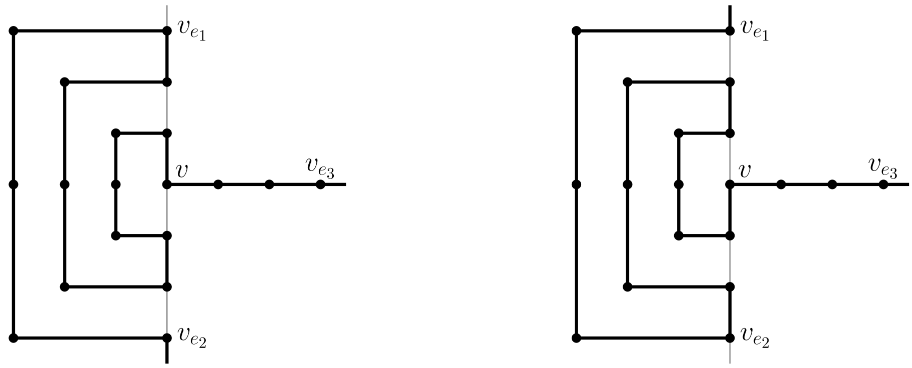


Fig. 5. The two possible ways a Hamiltonian cycle traverses a gadget in G'' corresponding to a vertex $v \in V(G')$ with $d^-(v) + d^+(v) = 3$.

The fact that $\mathcal{Q}$ is a minimal limit class for Hamiltonian Cycle with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth follows by Theorem 8:

Theorem 11. *$\mathcal{Q}$ is a boundary class for* Hamiltonian Cycle *with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth.*

In [7,38] it was observed that Hamiltonian Cycle admits a boundary class with respect to the class of split graphs. With the aid of Lemma 10, we can now determine a non-trivial limit class for the problem with respect to the class of split graphs. The result is based on the following operation: Given a bipartite graph G, a *clique implant* of G is the operation replacing one bipartition class of G with a clique. Given a subclass $\mathcal{X}$ of bipartite graphs, we denote by $S(\mathcal{X})$ the class of graphs obtained from every possible clique implant of graphs in $\mathcal{X}$. Clearly, if $\mathcal{X}$ is hereditary, $S(\mathcal{X})$ is hereditary as well. Moreover, it is easy to see that the following holds:

Lemma 12 *(Folklore). Let G be a bipartite graph having bipartition classes of equal size and G' be a graph obtained from a clique implant of G. We have that G is Hamiltonian if and only if G' is.*

Since a bipartite graph has a Hamiltonian cycle only if the two bipartition classes have equal size, Lemmas 12 and 10 immediately implies the following:

Lemma 13. *For any $p \geq 1$,* Hamiltonian Cycle *is* NP*-complete for graphs in $S(\mathcal{Q}_p)$.*

Remark 14. Note that the boundary class $\mathcal{C}$ whose existence is guaranteed by Lemma 13 is distinct from $\mathcal{Q}$. Indeed, $\mathcal{C}$ is a subclass of split graphs which must contain K_3 or else, by Theorem 6, Hamiltonian Cycle would be NP-hard for triangle-free split graphs.

4. Hamiltonian Cycle Through Specified Edge

In this section, we provide the first boundary class for Hamiltonian Cycle Through Specified Edge:

Hamiltonian Cycle Through Specified Edge	
Instance:	A graph $G = (V, E)$ and $e \in E$.
Question:	Does G contain a Hamiltonian cycle through e?

Hamiltonian Cycle Through Specified Edge is NP-complete for subcubic graphs (see, e.g., [57]) and we indeed begin this section by determining a boundary class with respect to the class of subcubic graphs. In fact, adapting the reasoning of Korpelainen et al. [38], we show that $\mathcal{Q}$ is a boundary class also for Hamiltonian Cycle Through Specified Edge (see Section 3 for the definition of $\mathcal{Q}$). We first show it is a limit class:

Lemma 15. *For any $p \geq 1$,* Hamiltonian Cycle Through Specified Edge *is* NP*-complete for graphs in $\mathcal{Q}_p$.*

Proof. We reduce from Hamiltonian Cycle for graphs in $\mathcal{Q}_p$, which is NP-complete by Lemma 10. Given an instance $G \in \mathcal{Q}_p$ of Hamiltonian Cycle, we construct an instance of Hamiltonian Cycle Through Specified Edge as follows. Clearly, we may assume G contains a cubic vertex v and no 1-vertex. Since each cubic vertex of G has a non-cubic neighbour, v has a neighbour v' with $d(v') = 2$. Therefore, we simply let $G' = G$ and $e = vv'$ be an instance of Hamiltonian Cycle Through Specified Edge. Clearly, G has a Hamiltonian cycle if and only if G' has a Hamiltonian cycle through e. □

Remark 16. The proof of Lemma 15 shows that Hamiltonian Cycle Through Specified Edge remains NP-hard for graphs in $\mathcal{Q}_p$ even when the edge $e = uv$ in the instance is such that $d(u) = 3$ and $d(v) = 2$. This fact will be used in the proof that Hamiltonian Path is NP-hard for graphs in $\mathcal{Q}_p$ (Lemma 19).

We now show that $\mathcal{Q}$ is a minimal limit class. The idea is to use Lemma 7. More precisely, we show that for every $G \in \mathcal{Q}$ there exists a constant p such that $Free(M \cup \{G\})$ is an easy class for Hamiltonian Cycle Through Specified Edge, where M is the set with $\overline{\mathcal{Q}_p} = Forb(M)$. The applicability of Lemma 7 follows from the fact that $\overline{\mathcal{Q}_p}$ is finitely defined. Indeed, the set of forbidden induced subgraphs for $\overline{\mathcal{Q}_p}$ contains finitely many cycles and tribranches. Similarly, the conditions that every graph in $\overline{\mathcal{Q}_p}$ is subcubic and that every cubic vertex has a non-cubic neighbour can be expressed by finitely many forbidden induced subgraphs.

Theorem 17. *$\mathcal{Q}$ is a boundary class for* Hamiltonian Cycle Through Specified Edge*.*

Proof. As remarked above, it is enough to show that for every $G \in \mathcal{Q}$ there exists a constant p such that Hamiltonian Cycle Through Specified Edge is solvable in polynomial time for G-free graphs in $\overline{\mathcal{Q}_p}$.

We say that an edge of a graph G is *good* if it belongs to every Hamiltonian cycle of G (if any), whereas it is *bad* if it does not belong to any Hamiltonian cycle of G. Korpelainen et al. [38] showed that, for each $G \in \mathcal{Q}$, there exists a constant p' such that the following holds: given a G-free graph $G' \in \overline{\mathcal{Q}_{p'}}$, for any cubic vertex $v \in V(G')$, there is a polynomial-time algorithm that labels at least two edges incident to v as good, or it returns as output that the graph has no Hamiltonian cycle.

We claim it is enough to take the constant p'. In other words, we show that for every $G \in \mathcal{Q}$, Hamiltonian Cycle Through Specified Edge is solvable in polynomial time for G-free graphs in $\overline{\mathcal{Q}_{p'}}$. Therefore, consider an input of this problem consisting of a G-free graph G' in $\overline{\mathcal{Q}_{p'}}$ and an edge $e \in E(G')$. Clearly, we may assume G' has no vertices of degree 1, or else G' has no Hamiltonian cycle. But then every vertex of G' has degree 2 or 3 and we can clearly label all the edges incident to vertices of degree 2 as good. Now, for each cubic vertex, we simply run the algorithm mentioned above, thus obtaining a labelling of the edges of G'. If e is labelled bad, then G' has no Hamiltonian cycle through e. Therefore, suppose e is labelled good. If there exists $v \in V(G')$ incident to three good edges, then G' has no Hamiltonian cycle. Otherwise, each vertex of G' is incident to exactly two good edges and these edges induce a collection of disjoint cycles. If the collection contains exactly one cycle, then there exists a Hamiltonian cycle in G' through e. On the other hand, if the collection contains more than one cycle, then G' contains no Hamiltonian cycle at all. □

The existence of a boundary class for Hamiltonian Cycle Through Specified Edge with respect to the class of split graphs is guaranteed by the following result similar to Lemma 13:

Lemma 18. *For any $p \geq 1$,* Hamiltonian Cycle Through Specified Edge *is* NP*-complete for graphs in $S(\mathcal{Q}_p)$.*

Proof. Let G be a bipartite graph having bipartition classes of equal size and G' be a graph obtained from a clique implant of G. It is easy to see that G has a Hamiltonian cycle through $e \in E(G)$ if and only if G' has a Hamiltonian cycle through the corresponding edge $e \in E(G')$. The fact that, for any $p \geq 1$, Hamiltonian Cycle Through Specified Edge is NP-complete for graphs in $\mathcal{Q}_p$ (Lemma 15) completes the proof of the lemma. □

5. Hamiltonian Path

In this section, we provide the first boundary class for Hamiltonian Path. This problem is NP-complete for subcubic planar bipartite graphs [57] and in fact we show that, similarly to Hamiltonian Cycle and Hamiltonian Cycle Through Specified Edge, the class $\mathcal{Q}$ defined in Section 3 is boundary for Hamiltonian Path as well.

We denote by $\mathcal{R}_p$ the subclass of $\mathcal{Q}_p$ consisting of the $(C_4, \ldots, C_{2p+2})$-free graphs G with $\delta(G) \geq 2$. The following lemma implies that $\mathcal{Q}$ is a limit class for Hamiltonian Path:

Lemma 19. *For any $p \geq 1$,* Hamiltonian Path *is* NP*-complete for graphs in $\mathcal{R}_p$.*

Note that the reason for the apparently useless restriction to $\mathcal{R}_p$ will appear in Lemma 34.

Proof. We reduce from Hamiltonian Cycle Through Specified Edge for graphs in $\mathcal{Q}_{2p+3}$, which is NP-complete by Lemma 15. Let $G = (V, E)$ and $uv \in E$ be an instance of this problem, where $G \in \mathcal{Q}_{2p+3}$. Clearly, we may assume $\delta(G) \geq 2$

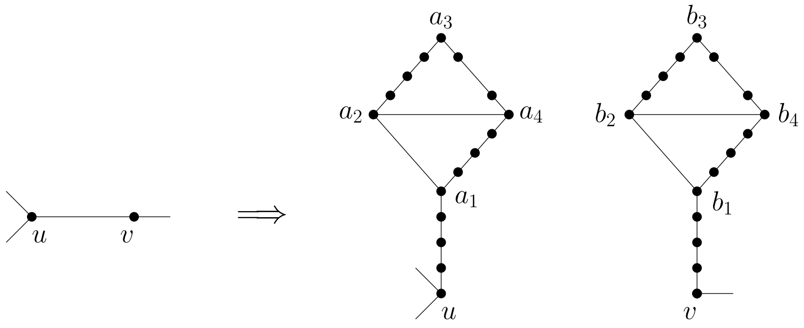


Fig. 6. Construction of the graph G', for $p = 1$, in the proof of Lemma 19.

and, by Remark 16, we may further assume that $d(u) = 3$ and $d(v) = 2$. We construct a graph $G' = (V', E')$ as follows (see Fig. 6). Set first $V' = V \cup \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$, where each a_i and b_i is a new vertex, and

$$E' = (E \setminus \{uv\}) \cup \{a_1u, b_1v, a_1a_2, a_1a_4, a_2a_3, a_2a_4, a_3a_4, b_1b_2, b_1b_4, b_2b_3, b_2b_4, b_3b_4\}.$$

Finally, subdivide each edge in $\{a_1u, b_1v, a_1a_4, a_2a_3, b_1b_4, b_2b_3\}$ with $2p+1$ new vertices and each edge in $\{a_3a_4, b_3b_4\}$ with $2p$ new vertices (thus belonging to V'). Note that the cubic vertices in $V' \setminus V$ are exactly the vertices in $\{a_1, a_2, a_4, b_1, b_2, b_4\}$.

Clearly, G' is a subcubic planar bipartite $\mathcal{C}_{2p+2}$-free graph with $\delta(G') \geq 2$. Moreover, it is easy to see that each cubic vertex of G' has a non-cubic neighbour.

Suppose now G' contains an induced tribranch $Y_{i,j,k}$ with $i, j, k \leq p$. Since G does not contain a tribranch in $\mathcal{Y}_p$ (even as a subgraph), we have that $Y_{i,j,k}$ is not contained in $G - uv \subseteq G'$ and so it must contain at least one vertex in $V' \setminus V$. On the other hand, it is easy to see that no vertex in $V' \setminus V$ can be a cubic vertex of $Y_{i,j,k}$ and so it must be that the unique vertex x in $N_{G'}(u) \setminus V$ is a 1-vertex of $Y_{i,j,k}$ and $V(Y_{i,j,k}) \cap (V' \setminus V) = \{x\}$. Moreover, if v does not belong to $Y_{i,j,k}$, then G contains a subgraph in $\mathcal{Y}_p$ (just replace x with v), a contradiction. Therefore, v belongs to $Y_{i,j,k}$. But then the distance between u and v in G' is at most $2p+1$ and so, since $uv \in E$, we have that G contains a cycle of length at most $2p+2$, a contradiction. This implies that $G' \in \mathcal{R}_p$.

Finally, we claim that G has a Hamiltonian cycle through uv if and only if G' has a Hamiltonian path. Suppose first G' has a Hamiltonian path P. It is easy to see that a_1 is a 2-vertex of P and so a vertex in the gadget attached to u is a 1-vertex of P. Similarly, a vertex in the gadget attached to v is a 1-vertex of P. Therefore, there exists a Hamiltonian path in G between u and v and so a Hamiltonian cycle through uv. Conversely, it is easy to see that if G has a Hamiltonian cycle through uv, then G' has a Hamiltonian path between a_2 and b_2. □

We now show the minimality of $\mathcal{Q}$. Our proof is based again on Lemma 7 and it is inspired by the proof in [38] that $\mathcal{Q}$ is boundary for Hamiltonian Cycle. The following special graphs in $\mathcal{Q}$ play a role in our arguments: for $d \geq 2$, the graph T_d is the caterpillar consisting of a path of length $2d$ and $2d-1$ consecutive hairs of lengths $1, 2, \dots, d-1, d, d-1, \dots, 2, 1$ (the caterpillar in Fig. 2 is in fact T_3). Clearly, every graph in $\mathcal{Q}$ is an induced subgraph of some T_d:

Observation 20 *(Korpelainen et al. [38]). Every graph in $\mathcal{Q}$ is an induced subgraph of T_d, for some $d \geq 2$.*

The idea is that for every $G \in \mathcal{Q}$, we can carefully choose a constant p such that a G-free graph in $\overline{\mathcal{Q}_p}$ is "locally" a graph in $\mathcal{Q}$. This allows to implement a labelling procedure as in the proof of Theorem 17. To determine the local structure, we make use of the following elementary result whose proof can be found in [18]:

Lemma 21. *If G is a graph of radius at most r and maximum degree at most $k \geq 3$, then $|V(G)| < \frac{k}{k-2}(k-1)^r$.*

Theorem 22. *$\mathcal{Q}$ is a boundary class for* Hamiltonian Path.

Proof. As remarked above, we show that for every $G \in \mathcal{Q}$ there exists a constant p such that Hamiltonian Path is solvable in polynomial time for G-free graphs in $\overline{\mathcal{Q}_p}$. By Lemma 7, this would conclude the proof.

Consider a graph $G \in \mathcal{Q}$. By Observation 20, we have that G is an induced subgraph of T_d, for some $d \geq 2$, and we define $p = 3 \cdot 2^d$. We claim we can decide in polynomial time whether a T_d-free graph in $\overline{\mathcal{Q}_p}$ contains a Hamiltonian Path. This would clearly imply the assertion in the paragraph above, thus concluding the proof. Therefore, let G' be a T_d-free graph in $\overline{\mathcal{Q}_p}$. For each pair of vertices u and v of G' and edges uu' and vv', we show that it is possible to check in polynomial time whether there exists a Hamiltonian u, v-path containing uu' and vv'. The conclusion would then follow by repeating this procedure $O(|V(G')|^2)$ times.

Clearly, we may assume G' contains a cubic vertex, or else the problem is trivial. Moreover, it is enough to prove our claim for graphs with no 1-vertex. Indeed, suppose G' contains a 1-vertex w and let P be a shortest path linking w to a cubic vertex w'. Clearly, any Hamiltonian path of G' contains P as a subpath. Therefore, it is enough to check whether there exists a Hamiltonian path in $G' - (V(P) \setminus \{w'\})$ having w' as one end. By the proof of Theorem 17, we may also assume that u and v are non-adjacent, or else a Hamiltonian u, v-path is equivalent to a Hamiltonian cycle through uv and the constant p we take is the same as the one in [38] and in Theorem 17 (note that we make this assumption just in order to shorten the proof).

We say that an edge of G' is *good* if it belongs to every Hamiltonian u, v-path of G' containing uu' and vv' (if any), whereas it is *bad* if it does not belong to any Hamiltonian u, v-path of G' through uu' and vv' (clearly, uu' and vv' are good). We provide a polynomial-time algorithm that either labels at least two edges incident to w as good, for each vertex $w \in V(G') \setminus \{u, v\}$, or returns as output that the graph has no Hamiltonian u, v-path through uu' and vv'. More precisely, we address the vertices sequentially and, if during the labelling process some edges are relabelled (i.e. a good edge becomes bad or vice versa), we have that G' does not contain any Hamiltonian u, v-path through uu' and vv'. Suppose now we have obtained such a labelling. If there exists $w \in V(G') \setminus \{u, v\}$ incident to three good edges, then G' has no Hamiltonian u, v-path through uu' and vv'. Otherwise, each vertex in $V(G') \setminus \{u, v\}$ is incident to exactly two good edges and these edges induce a path and possibly some cycles. If there are no cycles, we have found a desired Hamiltonian path. Otherwise, no such path exists.

Let us finally proceed with the description of the labelling algorithm. Clearly, uu' and vv' are good and the same holds for the edges incident to a 2-vertex $w \in V(G') \setminus \{u, v\}$. Moreover, the other edges incident to u and v are bad. Therefore, it remains to consider the cubic vertices in $V(G') \setminus \{u, v\}$. Let w be such a vertex and let H_w be the subgraph of G' induced by the set of vertices at distance at most d from w. Since H_w belongs to $\overline{\mathcal{Q}_p}$, it is a subcubic $\mathcal{Y}_p \cup \mathcal{C}_p$-free graph such that each cubic vertex has a non-cubic neighbour. On the other hand, since H_w is subcubic, we have that $|V(H_w)| < 3 \cdot 2^d = p$ (Lemma 21) and so H_w is $\mathcal{Y}_k \cup \mathcal{C}_k$-free for any $k > p$. Therefore, H_w belongs to $\mathcal{Q}$ and, being connected, it must be a caterpillar with hairs of arbitrary length. Moreover, each leaf of H_w is at distance exactly d from w, or else it would be a 1-vertex of G'. Consider now a path P of H_w connecting two leaves and containing all cubic vertices of H_w. Since G' is T_d-free, we have that P contains 2-vertices, and let w_i be a 2-vertex of P having shortest distance from w. Moreover, let $w_0w_1 \cdots w_i$ be the subpath between w_i and $w_0 = w$. Each vertex w_j with $j \neq i$ has a neighbour w'_j not on the path and which is a 2-vertex different from w_i. We denote by W the set $\{w_0, \ldots, w_{i-1}, w_i, w'_0, \ldots, w'_{i-1}\}$ and we distinguish several cases according to the size of the intersection $W \cap \{u, v\}$. In each case, we are going to argue that there exists a subpath $w_0w_1 \cdots w_j$ whose edges are labelled alternately good and bad. This suffices to label two edges incident to $w = w_0$ as good. Indeed, if w_0w_1 is bad, the other two edges incident to w_0 are good. On the other hand, suppose w_0w_1 is good and let $w''_0 \in N(w_0) \setminus \{w_1, w'_0\}$. If w'_0 is either u or v (say without loss of generality $w'_0 = u$) and $u' \neq w_0$, we have that $w_0w''_0$ is good. Otherwise, the remaining good edge must be $w_0w'_0$.

Suppose first $W \cap \{u, v\} = \varnothing$. This means that the edges incident to a 2-vertex in W are both labelled good. Since w_iw_{i-1} and $w_{i-1}w'_{i-1}$ are both good, $w_{i-1}w_{i-2}$ is bad. Moreover, since each cubic vertex in W has at least two good incident edges, $w_{i-2}w_{i-3}$ is good. Therefore, the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad (while traversing the path from w_i to w_0).

Suppose now W contains exactly one vertex from $\{u, v\}$, say without loss of generality $u \in W$. If u is a cubic vertex then, by assumption, we have that $u = w_j$ with $j > 0$ and $u' = w'_j$ (otherwise $v = w'_j$). Moreover, for each 2-vertex in W, its incident edges are labelled good. But then w_jw_{j-1} is bad and, similarly to the paragraph above, the edges on the subpath $w_0w_1 \cdots w_j$ are labelled alternately bad and good. Suppose now u is a 2-vertex in W. If $u = w_i$ and u' is the neighbour of u different from w_{i-1}, the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately bad and good. If $u = w_i$ and $u' = w_{i-1}$, the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad. If $u = w'_j$ and $u' = w_j$, for some $0 \leq j \leq i-1$, we have again that the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad. Finally, if $u = w'_j$ and u' is the neighbour of u different from w_j, for some $0 \leq j \leq i-1$, the edges on the subpath $w_0w_1 \cdots w_j$ are labelled alternately good and bad.

Finally, suppose that $\{u, v\} \subseteq W$. Consider the smallest index $j < i$ such that either w_j or w'_j is a vertex in $\{u, v\}$. Note that, since $uv \notin E(G')$, it cannot be that $\{u, v\} \subseteq \{w_j, w'_j\}$ and we assume, without loss of generality, that $u \in \{w_j, w'_j\}$. If $u = w_j$, then $u' = w'_j$ and, by minimality, each 2-vertex w'_k with $k < j$ is incident to two good edges. Therefore, w_jw_{j-1} is bad and the edges on the subpath $w_0w_1 \cdots w_j$ are labelled alternately bad and good. If $u = w'_j$ and u' is the neighbour of w'_j different from w_j, then $w_j \neq v$ and the edges w_jw_{j-1} and w_jw_{j+1} are both good. Moreover, by minimality, each 2-vertex w'_k with $k < j$ is incident to two good edges and so the edges on the subpath $w_0w_1 \cdots w_j$ are labelled alternately good and bad. It remains to consider the case $u = w'_j$ and $u' = w_j$. If $v = w_k$ and $v' = w'_k$ or $v = w'_k$ and $v' = w_k$, for some $j < k < i$, it is easy to see that the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad. If $v = w'_k$ and $v' = w_k$, for some $j < k < i$, the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad. If $v = w'_k$ and v' is the neighbour of w'_k different from w_k, for $j < k < i$, the edges on the subpath $w_0w_1 \cdots w_k$ are labelled alternately good and bad. Finally, in the case $v = w_i$, the edges on the subpath $w_0w_1 \cdots w_i$ are labelled alternately good and bad, if $v' = w_{i-1}$, or bad and good otherwise.

This concludes the proof. □

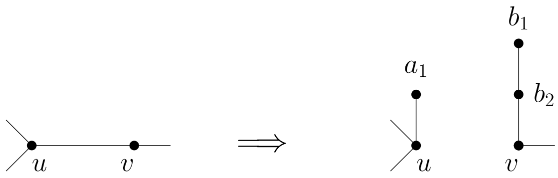


Fig. 7. Construction of the graph G' in the proof of Lemma 23.

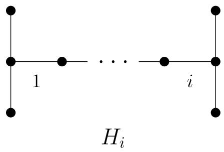


Fig. 8. The graph H_i.

As with Hamiltonian Cycle and Hamiltonian Cycle Through Specified Edge, we now show that $\bigcap_{p\geq 1} S(\mathcal{Q}_p)$ is a limit class for Hamiltonian Path:

Lemma 23. *For any $p \geq 1$,* Hamiltonian Path *is* NP*-complete for graphs in $S(\mathcal{Q}_p)$.*

Proof. We reduce from Hamiltonian Cycle Through Specified Edge for graphs in $\mathcal{Q}_{2p+3}$, which is NP-complete by Lemma 15. Let $G = (V, E)$ and $uv \in E$ be an instance of this problem, where $G \in \mathcal{Q}_{2p+3}$. By Remark 16, we may assume that $d(u) = 3$ and $d(v) = 2$. Moreover, we may assume that G has a bipartition $V = V_1 \cup V_2$ with $|V_1| = |V_2|$, or else G is not Hamiltonian. Without loss of generality, we have $v \in V_1$ and $u \in V_2$. We begin by constructing a graph $G' = (V', E')$ as follows (see Fig. 7): we set $V' = V \cup \{a_1, b_1, b_2\}$, where each a_i and b_i is a new vertex, and $E' = (E \setminus \{uv\}) \cup \{a_1u, b_1b_2, b_2v\}$. Therefore, $V_1 \cup \{a_1, b_1\}$ and $V_2 \cup \{b_2\}$ are bipartition classes of G'. We finally construct the graph G'' by adding to G' all possible edges with both endpoints in $V_2 \cup \{b_2\}$, thus turning the independent set $V_2 \cup \{b_2\}$ into a clique.

We claim that $G' \in \mathcal{Q}_p$. Clearly, G' is a subcubic planar bipartite $\mathcal{C}_p$-free graph. Moreover, each cubic vertex of G' has a non-cubic neighbour. Suppose now G' contains an induced tribranch $Y_{i,j,k}$ with $i, j, k \leq p$. Since G does not contain a tribranch in $\mathcal{Y}_p$ (even as a subgraph), we have that $Y_{i,j,k}$ is not contained in $G - uv$ and so it must contain at least one vertex in $\{a_1, b_1, b_2\}$. But no b_i can be a 1-vertex of $Y_{i,j,k}$ and so we have that a_1 is a 1-vertex of $Y_{i,j,k}$ and $V(Y_{i,j,k}) \cap \{a_1, b_1, b_2\} = \{a_1\}$. Moreover, if v does not belong to $Y_{i,j,k}$, then G contains a subgraph in $\mathcal{Y}_p$ (just replace a_1 with v), a contradiction. Therefore, v belongs to $Y_{i,j,k}$. But then the distance between u and v in G' is at most $2p + 1$ and so, since $uv \in E$, we have that G contains a cycle of length at most $2p + 2$, a contradiction. This implies that $G' \in \mathcal{Q}_p$ and so $G'' \in S(\mathcal{Q}_p)$.

Finally, we claim that G has a Hamiltonian cycle through uv if and only if G'' has a Hamiltonian path. Clearly, if G has a Hamiltonian cycle through uv, then G' has a Hamiltonian path between a_1 and b_1 and similarly for G''.

Conversely, suppose G'' has a Hamiltonian path P. Since a_1 and b_1 are 1-vertices of G'', P is a Hamiltonian a_1, b_1-path and $\{a_1u, b_1b_2\} \subseteq E(P)$. This means that all the vertices in V_1 (which is an independent set of G'') have degree 2 in P. Therefore, since P has $2|V_1| + 2$ edges, it does not contain any edge of G'' with both endpoints in $V_2 \cup \{b_2\}$ and so there exists a Hamiltonian a_1, b_1-path in G' and in turn a Hamiltonian cycle through uv in G. □

6. Feedback Vertex Set

In this section, we provide the first boundary class for Feedback Vertex Set. Ueno et al. [69] showed that Feedback Vertex Set (and Connected Vertex Cover) can be solved in polynomial time for subcubic graphs by a reduction to a matroid matching problem. On the other hand, Feedback Vertex Set is NP-hard for planar graphs with maximum degree at most 4, as first shown by Speckenmeyer [67], and so it admits a boundary class with respect to the class of planar graphs with maximum degree at most 4. We begin by showing that the class of forests whose components have at most four leaves and at most one vertex of degree three is in fact a limit class.

For $k \geq 1$, we denote by $\mathcal{S}_k$ the class of planar bipartite $(C_4, \ldots, C_{2k}, H_1, \ldots, H_k)$-free graphs with maximum degree at most 4 (see Fig. 8).

Lemma 24. *For any $k \geq 1$,* Feedback Vertex Set *is* NP*-complete for graphs in $\mathcal{S}_k$.*

Proof. We reduce from Feedback Vertex Set for planar graphs with maximum degree at most 4, which is known to be NP-complete (see, e.g., [57,67] and Section 6.3). Given a planar graph G with maximum degree at most 4, we denote by G' a $2k+1$-subdivision of G. It is easy to see that $G' \in \mathcal{S}_k$ and $\tau_c(G) = \tau_c(G')$. □

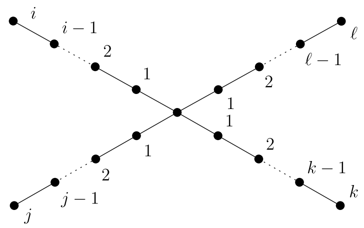


Fig. 9. The graph $S_{i,j,k,\ell}$.

We denote by $\mathcal{S}$ the class of forests whose components have at most four leaves and at most one vertex of degree three. Each graph in $\mathcal{S}$ has components of the form $S_{i,j,k,\ell}$, for some non-negative integers i, j, k, ℓ (see Fig. 9). It is not difficult to see that $\bigcap_{k\geq 1} \mathcal{S}_k = \mathcal{S}$ and so $\mathcal{S}$ is a limit class for Feedback Vertex Set with respect to the class of planar bipartite graphs with maximum degree at most 4.

In order to show the minimality of $\mathcal{S}$, we first prove that Feedback Vertex Set can be solved in polynomial time for graphs with maximum degree at most 4 and bounded number of 4-vertices (see Section 6.2). This result follows from the fact that even the weighted version of Feedback Vertex Set can be solved in polynomial time for cubic graphs (see Section 6.1) which, in turn, is an easy corollary of a deep result on the weighted linear matroid matching problem recently obtained by Iwata [30], Iwata and Kobayashi [32] and Pap [61].

6.1. Weighted feedback vertex set for cubic graphs

Let us begin by recalling the notion of polymatroid. We refer the reader to [46] for a complete introduction to 2-polymatroids. Note that in this section we allow graphs to contain loops and multiple edges.

A 2-*polymatroid* is a pair $P = (S, f)$, where S is a finite set and f is a function $f : 2^S \to \mathbb{Z}$ satisfying the following properties:

(P1) $f(\emptyset) = 0$;
(P2) $f(X) \leq f(Y)$, for any $X \subseteq Y \subseteq S$;
(P3) $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$, for any $X, Y \subseteq S$;
(P4) $f(\{x\}) \leq 2$, for any $x \in S$.

If in addition $f(\{x\}) \leq 1$, for any $x \in S$, then P is a *matroid*. A subset $X \subseteq S$ is a *matching* of P if $f(X) = 2|X|$ and it is a *spanning set* of P if $f(X) = f(S)$. The size of a maximum matching of P is denoted by $\nu(P)$ while the size of a minimum spanning set of P is denoted by $\rho(P)$.

It is easy to see that, given a (simple) graph $G = (V, E)$, the pair $P = (E, f)$ with $f(X) = |\bigcup_{e\in X} e|$ is a 2-polymatroid (note that $e \in E$ is a 2-element set). Moreover, a matching of P is a matching of G and a spanning set of P is an edge cover of G. A well-known Gallai's identity asserts that $\alpha'(G) + \beta'(G) = |V(G)|$, for any graph G without isolated vertices. In fact, Lovász provided the following generalisation (see [46]):

Theorem 25. *For any* 2*-polymatroid* $P = (S, f)$*, we have* $\nu(P) + \rho(P) = f(S)$.

Feedback vertex sets of (sub)cubic graphs have been extensively studied (see, e.g., [27,42,68]). In particular, Speckenmeyer [68] showed that $\tau_c(G) + z(G) = \mu(G)$, for any connected cubic simple graph G, and Ueno et al. [69] showed that the previous relation actually holds for every cubic graph (thus allowing loops and multiple edges):

Theorem 26 *(Ueno et al. [69]). If G is a cubic graph, then* $\tau_c(G) + z(G) = \mu(G)$.

Their proof is based on the fact that τ_c and z can be interpreted as the size of a minimum spanning set and the size of a maximum matching, respectively, of a suitably defined 2-polymatroid. Given a graph G, it is easy to see that the function $f : 2^{V(G)} \to \mathbb{Z}$ defined by $f(X) = \mu(G) - \mu(G - X)$ satisfies (P1) to (P3). Moreover, for any graph G and $v \in V(G)$, we have that $\mu(G) - \mu(G - v) \leq d_G(v) - 1$ and the inequality is strict if and only if v is a cut-vertex or the endpoint of a loop. Therefore, if G is a cubic graph, $P(G) = (V(G), f)$ is indeed a 2-polymatroid and the following crucial result holds:

Theorem 27 *(Ueno et al. [69]). Let G be a cubic graph. A subset $T \subseteq V(G)$ is a feedback vertex set of G if and only if it is a spanning set of the* 2*-polymatroid $P(G)$. Moreover, $I \subseteq V(G)$ is a non-separating independent set of G if and only if it is a matching of $P(G)$.*

At this point, Theorem 26 is an immediate consequence of the "generalised Gallai's identity" stated in Theorem 25.

Lovász [45] provided a polynomial-time algorithm that finds a maximum matching of a special class of 2-polymatroids, the so-called linearly represented 2-polymatroids (see also [24,46,58]). A 2-polymatroid (S, f) is *linearly representable* (over a field $\mathbb{F}$) if there exists a matrix $A = (A_e)_{e\in S} \in \mathbb{F}^{d\times 2|S|}$ obtained by concatenating $|S|$ matrices $A_e \in \mathbb{F}^{d\times 2}$ and such that $f(X) = \text{rank } A(X)$, for any $X \subseteq S$, where d is a positive integer and $A(X) = (A_e)_{e\in X}$ denotes the submatrix of A obtained by selecting the corresponding columns. In view of Theorem 25, Lovász's result implies we can find in polynomial time a minimum spanning set of a linearly represented 2-polymatroid. Moreover, Ueno et al. [69] showed that the 2-polymatroid $P(G)$ is linearly representable and that a linear representation can be obtained in polynomial time. Therefore, the mentioned results and Theorem 27 imply we can find in polynomial time a maximum non-separating independent set and a minimum feedback vertex set of any cubic graph. Note that the same conclusion holds for a minimum connected vertex cover, since the complement of a non-separating independent set of a connected graph is a connected vertex cover and vice versa. Moreover, applying standard cleaning procedures, it is easy to see that these results hold for subcubic graphs [69] (see also Theorem 30).

In a recent breakthrough, Iwata [30,31], Iwata and Kobayashi [32] and Pap [61] showed that even the following weighted version of the matroid matching problem can be solved in polynomial time:

Weighted Linear Matroid Matching	
Instance:	A linearly represented 2-polymatroid on S and a function $w\colon S \to \mathbb{R}$.
Task:	Find a matching $X \subseteq S$ with maximum $w(X) = \sum_{x\in X} w(x)$.

We now highlight the interesting consequences of this result for the weighted version of Feedback Vertex Set. Indeed, consider a 2-polymatroid (S, f) and its 2-*dual* (S, f^*), where $f^*\colon 2^S \to \mathbb{Z}$ is defined by $f^*(X) = 2|X| + f(S\setminus X) - f(S)$ (see, e.g., [60]). Clearly, X is a matching of (S, f^*) if and only if $S\setminus X$ is a spanning set of (S, f) and so the problem of finding a minimum-weight spanning set for (S, f) is equivalent to the weighted matroid matching problem for the 2-dual (S, f^*). Moreover, given a linear representation of (S, f), it is not difficult to find a linear representation of (S, f^*) in polynomial time (see, e.g., [60]). Therefore, by Theorem 27, the following holds:

Theorem 28. *The problem of finding a minimum-weight feedback vertex of a cubic graph can be solved in polynomial time.*

6.2. Towards the minimality of $\mathcal{S}$

In this section, we show that $\mathcal{S}$ is a boundary class for Feedback Vertex Set. As mentioned above, we first prove that Feedback Vertex Set can be solved in polynomial time for graphs with maximum degree at most 4 and bounded number of 4-vertices. The idea is to reduce this problem to the cubic case and then rely on Theorem 28. The following operation is crucial for our purposes:

Let $x \in V(G)$ be a 4-vertex of a graph G (which may contain loops and multiple edges). A *vertex stretching* with respect to x is the operation replacing x with two new vertices x_1 and x_2 as depicted in Fig. 10. Note that there is some freedom in determining the neighbours of x_1 and x_2. A vertex stretching with respect to x has the following fundamental properties:

- It does not alter the cycles of G, except possibly increasing the length of a cycle through x by 1;
- It does not modify the degrees of the vertices adjacent to x and it gives $d(x_1) = d(x_2) = 3$.

The following is an immediate but useful observation:

Lemma 29. *Let $x \in V(G)$ be a 4-vertex of G and let G' be the graph obtained from G by a vertex stretching with respect to x. We have that $\tau_c(G - x) = \tau_c(G' - \{x_1, x_2\})$. Moreover, G has a feedback vertex set avoiding x if and only if G' has a feedback vertex set avoiding $\{x_1, x_2\}$ and the size of a minimum feedback vertex set of G avoiding x is the same as that of a minimum feedback vertex set of G' avoiding $\{x_1, x_2\}$.*

We can finally prove the following:

Theorem 30. Feedback Vertex Set *can be solved in polynomial time for graphs with maximum degree at most 4 and bounded number of 4-vertices.*

Proof. Let G be a graph with $\Delta(G) \le 4$ and $|d_4(G)| \le c$, for some constant c. The following observations are immediate: if v is a 1-vertex of G, then $\tau_c(G) = \tau_c(G - v)$; if v is a 2-vertex of G which is the endpoint of a loop, then $\tau_c(G) = \tau_c(G - v) + 1$; if v is a 2-vertex of G adjacent to u and u' (note that we might have $u = u'$) then, denoting by H the graph obtained from G by deleting v and adding the edge uu', we have $\tau_c(G) = \tau_c(H)$. Therefore, we can reduce Feedback Vertex Set for G to the same problem for the graph G' obtained by the following cleaning procedure: if v is a 1-vertex or a 2-vertex which is

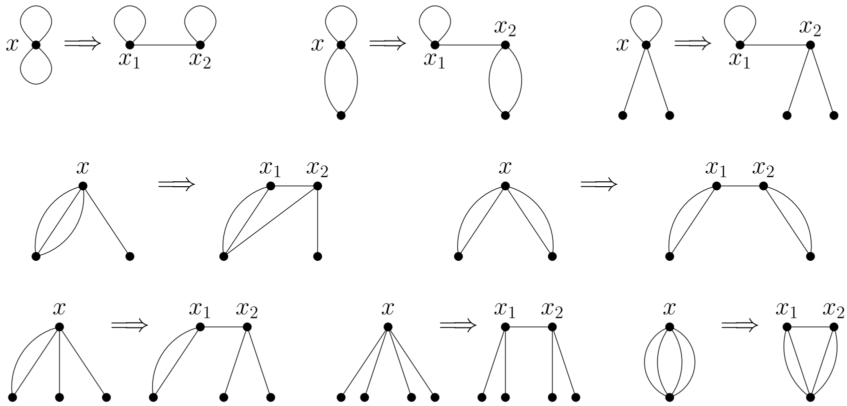


Fig. 10. Vertex stretching.

the endpoint of a loop, we delete v; if v is a 2-vertex adjacent to u and u' (possibly $u = u'$), we delete v from G and add the edge uu'. It is easy to see that the graph G' obtained by applying these operations as long as possible either contains only vertices of degree 3 or 4, or it is empty.

We now proceed by a brute force argument: for each subset $S \subseteq d_4(G')$, we find a minimum feedback vertex set T of G' subject to $T \cap d_4(G') = S$ (if any). This is done as follows. We fix $S \subseteq d_4(G')$ and we apply a vertex stretching to each 4-vertex of G' in order to obtain a cubic graph G''. Then we define a weight function $w\colon V(G'') \to \mathbb{R}$ as follows: $w(x) = 0$, if x is the result of a vertex stretching with respect to a vertex in S; $w(x) = |V(G'')| + 1$, if x is the result of a vertex stretching with respect to a vertex in $d_4(G') \setminus S$; $w(x) = 1$ otherwise.

By Theorem 28, we can find in polynomial time a minimum-weight feedback vertex set T_S of (G'', w). Note that, without loss of generality, we may assume that every zero-weight vertex belongs to T_S. If $w(T_S) > |V(G'')|$ then, by Lemma 29, there exists no feedback vertex set T of G' such that $T \cap d_4(G') \subseteq S$. Otherwise, we remove from T_S the vertices which are the result of a vertex stretching with respect to a 4-vertex v (which must belong to S) and we add v to T_S, in order to obtain a set T'_S. Clearly, T'_S is a minimum feedback vertex set of G' subject to $T'_S \cap d_4(G') = S$. We thus build a vector indexed by the at most 2^c subsets S of $d_4(G')$ and we return the minimum value of $|T'_S|$. Correctness and polynomiality are evident. □

We now have all the machinery to show that $\mathcal{S}$ is a boundary class for Feedback Vertex Set. Our proof relies once again on Lemma 7. Note that, in the following, all considered graphs are simple, i.e. loops and multiple edges are not allowed anymore.

Theorem 31. *$\mathcal{S}$ is a boundary class for* Feedback Vertex Set.

Proof. Let $\overline{\mathcal{S}_p}$ denote the class of $(C_3, \dots, C_p, H_1, \dots, H_p)$-free graphs with maximum degree 4. In view of Lemma 7, it is enough to show that, for each $H \in \mathcal{S}$, there exists a constant p such that Feedback Vertex Set is solvable in polynomial time for H-free graphs in $\overline{\mathcal{S}_p}$. Therefore, consider $H \in \mathcal{S}$ and let c be the number of its components. Since each component of H is of the form $S_{i,j,k,\ell}$, for some non-negative integers i, j, k and ℓ, there exists a non-negative integer d such that H is an induced subgraph of $cS_{d,d,d,d}$. We define $p = 2 \cdot 3^d$ and we claim that Feedback Vertex Set can be solved in polynomial time for $cS_{d,d,d,d}$-free graphs in $\overline{\mathcal{S}_p}$. Lemma 7 would then imply that $\mathcal{S}$ is a boundary class.

Let G be a $cS_{d,d,d,d}$-free graph in $\overline{\mathcal{S}_p}$ and consider a 4-vertex $v \in V(G)$ (if G is subcubic, we know the problem is solvable in polynomial time). Let G_v be the subgraph of G induced by the set of vertices at distance at most d from v. Since G_v belongs to $\overline{\mathcal{S}_p}$, it is a $(C_3, \dots, C_p, H_1, \dots, H_p)$-free graph with maximum degree 4. Moreover, since each vertex of G_v has degree at most 4, we have that $|V(G_v)| < 2 \cdot 3^d = p$ (Lemma 21) and so G_v is (C_k, H_k)-free for any $k > p$. Therefore, G_v belongs to $\mathcal{S}$ and, being connected, it must be of the form S_{i_1,i_2,i_3,i_4}, for some non-negative integers $i_1, i_2, i_3, i_4 \leq d$. Suppose now some i_j is strictly less than d and let v_{i_j} be the leaf of S_{i_1,i_2,i_3,i_4} at distance i_j from v. Clearly, no cycle of G contains a vertex belonging to the unique v, v_{i_j}-path P in S_{i_1,i_2,i_3,i_4} and different from v. Therefore, we may delete $V(P) \setminus \{v\}$ from G. Repeating this operation for each 4-vertex $v \in V(G)$, we obtain a graph G' with maximum degree at most 4. This graph is such that $\tau_c(G) = \tau_c(G')$ and, for each 4-vertex $v \in V(G')$, the induced subgraph G_v is isomorphic to $S_{d,d,d,d}$.

We now claim that G' has a bounded number of 4-vertices. In view of Theorem 30, this would conclude the proof. Let $F \subseteq d_4(G')$ be a subset of maximum size such that the corresponding induced copies of $S_{d,d,d,d}$ (for each $v \in F$, the induced subgraph G_v is isomorphic to $S_{d,d,d,d}$) are pairwise vertex-disjoint and denote by F' this corresponding set. Note that no two subgraphs in F' are connected by an edge, or else a copy of H_{2d+1} would arise in $G' \in \overline{\mathcal{S}_{2\cdot 3^d}}$. Therefore, since G' is $cS_{d,d,d,d}$-free, we have $|F| < c$. Moreover, we claim that $|d_4(G')| \leq 17|F|$. By definition, for each $v \in d_4(G') \setminus F$, we have that G_v intersects a graph in F'. It is therefore enough to show that each branch of $S_{d,d,d,d}$ can intersect at most four other copies of $S_{d,d,d,d}$, where a branch is the unique path between a leaf and the 4-vertex. To this end, let S be a copy of

$S_{d,d,d,d}$ with 4-vertex v and let P be a v, v_i-branch. The following are easy observations. If v belongs to another copy S' of $S_{d,d,d,d}$, then $V(P) \subseteq V(S')$ and v_i is the 4-vertex of S', and so P intersects at most four other copies of $S_{d,d,d,d}$. If v does not belong to another copy of $S_{d,d,d,d}$ but an inner vertex of P does, then P intersects at most one other copy of $S_{d,d,d,d}$. Finally, if v_i is the only vertex of P which belongs to another copy of $S_{d,d,d,d}$, then P intersects at most three other copies of $S_{d,d,d,d}$. $\square$

In [7] it was conjectured that, for any problem Π solvable in polynomial time for graphs of bounded clique-width, $\mathcal{T}$ and $L(\mathcal{T}) = \{L(G) : G \in \mathcal{T}\}$ are the only Π-boundary classes with respect to any Π-hard class of bounded vertex degree. Theorem 31 shows that this is not the case for FEEDBACK VERTEX SET (which is solvable in polynomial time for graphs of bounded clique-width by [12]). Note that the same conclusion can be obtained as follows. Let X be a boundary class for FEEDBACK VERTEX SET with respect to the class of graphs with maximum degree at most 4 and let X' be the class of subcubic graphs. If $X \subseteq X'$, then FEEDBACK VERTEX SET is NP-hard for graphs in X' by Theorem 6 (as X' is finitely defined), contradicting the fact that FEEDBACK VERTEX SET is polynomial for subcubic graphs [69]. Therefore, X contains a graph with vertices of degree 4 and so it is distinct from $\mathcal{T}$ and $L(\mathcal{T})$.

6.3. Line graphs

In [57], we showed that FEEDBACK VERTEX SET remains NP-hard for line graphs of planar cubic bipartite graphs. This was obtained by first showing that if G is the line graph of a cubic triangle-free graph H, then $\tau_c(G) \leq \frac{|V(G)|}{3} + 1$ if and only if H contains a Hamiltonian path. The conclusion follows then from the fact that HAMILTONIAN PATH is NP-complete for planar cubic bipartite graphs. In this section, we adapt the previous reasoning and show that, for any $p \geq 1$, FEEDBACK VERTEX SET is NP-complete for line graphs of graphs in $\mathcal{R}_p$. Recall that $\mathcal{Q}_p$ is the class of subcubic planar bipartite $\mathcal{Y}_p \cup \mathcal{C}_p$-free graphs such that each cubic vertex has a non-cubic neighbour and $\mathcal{R}_p$ is the subclass of $\mathcal{Q}_p$ consisting of the $(C_4, \ldots, C_{2p+2})$-free graphs G with $\delta(G) \geq 2$ (see Section 4).

We begin by considering (vertex) triangle-transversals of line graphs of subcubic triangle-free graphs. Given a subset of vertices S of a graph G, an *S-cover* of G is a subset of $E(G)$ covering each vertex in S. Clearly, $V(G)$-covers are the usual edge covers of G. Recall that $d_k(G) = \{v \in V(G) : d_G(v) = k\}$.

Lemma 32. *For a subcubic triangle-free graph H, there is a bijection between the $d_3(H)$-covers of H and the triangle-transversals of $L(H)$.*

Proof. Using the bijection between edges of H and vertices of $L(H)$, the assertion follows from the fact that the triangles of $L(H)$ are in bijection with the cubic vertices of H. $\square$

The following lower bound on the size of a feedback vertex set will be crucial for our NP-hardness proof:

Lemma 33. *Let H be a subcubic triangle-free graph with $\delta(H) \geq 2$ and let $G = L(H)$ be its line graph. We have $\tau_c(G) \geq \frac{|d_3(H)|}{2} + 1$, with equality if and only if H contains a Hamiltonian path.*

Proof. By Lemma 32, there exists a bijection between the triangle-transversals of G and the $d_3(H)$-covers of H. Therefore, we have $\tau_c(G) \geq \tau_\triangle(G) \geq \frac{|d_3(H)|}{2}$ and the last inequality is an equality if and only if the induced subgraph $H[d_3(H)]$ contains a 1-factor.

Suppose, to the contrary, that $\tau_c(G) = \frac{|d_3(H)|}{2}$. This means there exists a minimum triangle-transversal T of G (of size $\frac{|d_3(H)|}{2}$) which is also a feedback vertex set of G. Moreover, T corresponds to a 1-factor H' of $H[d_3(H)]$ and so, since $\delta(H) \geq 2$, we have that $H - E(H')$ is 2-regular. But then there exists a cycle in $G - T$, a contradiction. This implies that $\tau_c(G) \geq \frac{|d_3(H)|}{2} + 1$.

Suppose now equality holds, i.e. $\tau_c(G) = \frac{|d_3(H)|}{2} + 1$, and let T be a minimum feedback vertex set of G. Moreover, let $T_\triangle$ be a triangle-transversal of G having minimum size among those contained in T. Clearly, we have $\frac{|d_3(H)|}{2} \leq |T_\triangle| \leq \frac{|d_3(H)|}{2} + 1$. If $|T_\triangle| = \frac{|d_3(H)|}{2}$, then $T_\triangle$ corresponds to a 1-factor H' of $H[d_3(H)]$ and so $H - E(H')$ is a 2-factor F with $p \geq 1$ components. Since the components of F give rise to p vertex-disjoint cycles in $G - T_\triangle$, then $\tau_c(G) = \frac{|d_3(H)|}{2} + 1$ implies that $p = 1$ and so $F \subseteq H$ is a Hamiltonian cycle.

Suppose now that $T_\triangle$ has size $\frac{|d_3(H)|}{2} + 1$ (in particular, $T = T_\triangle$), and let T' be the corresponding $d_3(H)$-cover of H. We have that T' contains at most two edges with an endpoint not in $d_3(H)$. If T' contains only edges of $H[d_3(H)]$ then, by the minimality of $T_\triangle$, we have that T' consists of a maximum matching M of size $\frac{|d_3(H)|}{2} - 1$ together with two edges, each one covering exactly one vertex uncovered by M. If T' contains exactly one edge e with an endpoint not in $d_3(H)$, then $T' \setminus \{e\}$ consists of a maximum matching M of $H[d_3(H)]$ of size $\frac{|d_3(H)|}{2} - 1$ and an edge of $H[d_3(H)]$ covering the vertex uncovered by $M \cup \{e\}$. Finally, if T' contains exactly two edges e_1 and e_2 with an endpoint not in $d_3(H)$, then e_1 and e_2 cover distinct cubic vertices. Moreover, $T' \setminus \{e_1, e_2\}$ consists of a maximum matching of $H[d_3(H)]$ of size $\frac{|d_3(H)|}{2} - 1$. It is easy to see that, in all the three cases above, the graph $H - T'$ either contains a single isolated vertex and all the remaining vertices have degree 2, or it contains exactly two 1-degree vertices with all the remaining ones having degree 2. On the other hand, $H - T'$ is a forest, or else there would be a cycle in $G - T$. This implies that all the vertices of $H - T'$ have degree 2, except two of them having degree 1, and that $H - T'$ is a path. Therefore, H contains a Hamiltonian path.

Conversely, suppose that H contains a Hamiltonian path P. The number of edges in $E(H) \setminus E(P)$ is $|d_2(H)| + \frac{3}{2}|d_3(H)| - (|d_2(H)| + |d_3(H)| - 1) = \frac{|d_3(H)|}{2} + 1$ and these edges constitute a $d_3(H)$-cover of H. If T is the corresponding triangle-transversal of G of size $\frac{|d_3(H)|}{2} + 1$, we have that $G - T \subseteq L(P)$ and so T is in fact a feedback vertex set. □

By Lemma 19, Hamiltonian Path is NP-complete for graphs in $\mathcal{R}_p$, for any $p \geq 1$, and so we can finally prove the following:

Lemma 34. *For any $p \geq 1$, Feedback Vertex Set is NP-complete for line graphs of graphs in $\mathcal{R}_p$.*

Proof. We reduce from Hamiltonian Path for graphs in $\mathcal{R}_p$. Let $G = (V, E)$ be an instance of this problem. In particular, G is a subcubic triangle-free graph with $\delta(G) \geq 2$. Consider now its line graph $G' = L(G)$. By Lemma 33, we have that $\tau_c(G') \leq \frac{|d_3(G)|}{2} + 1$ if and only if G contains a Hamiltonian path. The conclusion immediately follows. □

Note that the class $\mathcal{Q}'_p$ of $(C_4, \ldots, C_{2p+2})$-free graphs in $\mathcal{Q}_p$ is *monotone*, i.e. it is closed under vertex and edge deletions. Moreover, by Lemma 34, Feedback Vertex Set is NP-complete for graphs in $L(\mathcal{Q}'_p) = \{L(G) : G \in \mathcal{Q}'_p\}$, for any $p \geq 1$. Therefore, $L(\mathcal{Q}'_p)$ is a hard class for Feedback Vertex Set and it is easy to see that $\bigcap_{p \geq 1} L(\mathcal{Q}'_p) = L(\mathcal{Q})$. In other words, $L(\mathcal{Q})$ is a limit class:

Corollary 35. *$L(\mathcal{Q})$ is a limit class for* Feedback Vertex Set.

We suspect that $L(\mathcal{Q})$ is indeed a minimal limit class and we leave this verification as an open problem.

7. Connected Vertex Cover

In this section, we consider the connected variant of the vertex cover problem which asks for a minimum-size vertex cover inducing a connected graph:

> Connected Vertex Cover
>
> *Instance:* A graph $G = (V, E)$ and a positive integer k.
> *Question:* Does $\beta_c(G) \leq k$ hold?

Connected Vertex Cover was introduced by Garey and Johnson [25], who showed it is NP-complete for planar graphs with maximum degree 4. Fernau and Manlove [22] strengthened this result by showing that it remains NP-hard even for planar bipartite graphs with maximum degree 4 (see also [21]). On the other hand, we have seen in Section 6 that it is solvable in polynomial time for subcubic graphs. In the following, we make some observations towards determining the first boundary classes for this problem.

Alekseev [3] showed that the class of forests whose components have at most three leaves is boundary for Vertex Cover[2] and conjectured no other boundary class exists. For Connected Vertex Cover, we show there are at least two boundary classes. One of them is a subclass of line graphs of bipartite graphs:

Lemma 36. Connected Vertex Cover *is NP-complete for line graphs of planar cubic bipartite graphs.*

Note that, on the contrary, Vertex Cover restricted to line graphs can be solved in polynomial time by a reduction to a matching problem [53,65].

Our proof of Lemma 36 is based on the following:

Lemma 37. *If G is the line graph of a cubic triangle-free graph H, then $\beta_c(G) \geq \frac{2}{3}|V(G)|$. Equality holds if and only if H contains a Hamiltonian cycle.*

[2] He actually stated this result for Independent Set but the two problems are polynomially equivalent.

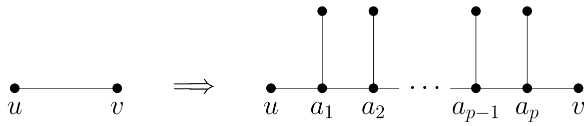


Fig. 11. The operation A_p.

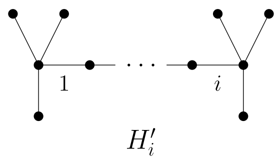


Fig. 12. The graph H_i'.

Proof. Clearly, there is a bijection between the vertices of H and the triangles of G. Moreover, any vertex cover of G contains at least two vertices for every triangle. Since there are $|V(H)| = \frac{2}{3}|V(G)|$ triangles in G and any two of them share at most one vertex, we have that $\beta_c(G) \geq \beta(G) \geq \frac{2}{3}|V(G)|$.

Suppose now that $\beta_c(G) = \frac{2}{3}|V(G)|$. This means there exists a connected vertex cover S of G containing exactly two vertices for each triangle of G. Consider now the set of edges $S' \subseteq E(H)$ corresponding to S. Since $G[S]$ is connected, we have that $H[S']$ is connected as well. Therefore, $S' \subseteq E(H)$ is a set of edges such that each vertex of H is incident to exactly two of them and $H[S']$ is connected and so S' induces a Hamiltonian cycle in H.

Conversely, suppose H contains a Hamiltonian cycle C and let S be the set of vertices of G corresponding to $E(C)$. Since every edge of H is incident to an edge in $E(C)$, we have that S is a vertex cover of G. Moreover, $G[S]$ is connected and so $\beta_c(G) \leq |V(H)| = \frac{2}{3}|V(G)|$. □

Proof of Lemma 36. We reduce from Hamiltonian Cycle for planar cubic bipartite graphs, which is known to be NP-complete [2]. Let G be an instance of this problem and consider its line graph $G' = L(G)$. By Lemma 37, we have that $\beta_c(G') \leq \frac{2}{3}|V(G')|$ if and only if G contains a Hamiltonian cycle. □

The results in [21,22] mentioned above show that the class of planar bipartite graphs with maximum degree 4 is limit. On the other hand, we now show it is not minimal. The following operation proves to be helpful: Given a graph $G = (V, E)$, an edge $uv \in E$ and an integer $p \geq 1$, the graph $A_p(G)$ is obtained from G by replacing uv with the gadget depicted in Fig. 11.

The fundamental property of A_p, which is left as an easy exercise, is that $\beta_c(A_p(G)) = \beta_c(G) + p$. We can now provide the other limit class (see Fig. 12):

Lemma 38. *For any $k \geq 1$,* Connected Vertex Cover *is* NP-*complete for planar bipartite $(C_4, \ldots, C_{2k}, H_1', \ldots, H_k')$-free graphs with maximum degree at most* 4.

Proof. We reduce from Connected Vertex Cover for planar graphs with maximum degree 4, which is NP-complete by Lemma 36. Given an instance G of this problem, we construct a graph G' by applying the operation A_p, with $p = 2k + 1$, to each edge of G. The statement follows from the fact that $\beta_c(A_p(G)) = \beta_c(G) + p$. □

Note that the two boundary classes whose existence is guaranteed by Lemmas 36 and 38 are distinct. Indeed, by Lemma 36, there exists a boundary class $\mathcal{C}_1$ which is a subclass of line graphs and, by Lemma 38, there exists a boundary class $\mathcal{C}_2$ which is a subclass of forests. On the other hand, $\mathcal{C}_1$ must contain K_3 or else, by Theorem 6, Connected Vertex Cover would be NP-hard for triangle-free line graphs and so for graphs with maximum degree 2.

The determination of these two boundary classes is left as an open problem.

8. Connected Dominating Set

In this section, we consider Connected Dominating Set and we begin by showing that the class of subcubic forests is a limit class for this problem. Alekseev et al. [6] showed that the class $\mathcal{T}$ of forests whose components have at most three leaves is boundary for the related Dominating Set. Clearly, $\mathcal{T}$ is contained in the class of subcubic forests but we provide some evidence for the fact that it may not be boundary for Connected Dominating Set.

Douglas [19] showed that the following variant of Connected Dominating Set is NP-hard even for subcubic planar graphs:

($\frac{|V|}{2} - 1$)-Connected Dominating Set

Instance: A graph $G = (V, E)$.
Question: Does $\gamma_c(G) \leq \frac{|V|}{2} - 1$ hold?

In particular, Connected Dominating Set is NP-hard for subcubic planar graphs and in the following we show that the same holds for the class of subcubic planar bipartite graphs with arbitrarily large girth. This would clearly imply that the class of subcubic forests is a limit class for Connected Dominating Set.

Recall that, for any graph G, we have $\gamma_c(G) = |V(G)| - \ell(G)$. Moreover, it is easy to see that for any subcubic graph G, we have $\ell(G) \leq \frac{|V(G)|}{2} + 1$, with equality if and only if G contains a $\{1,3\}$-*spanning tree*, i.e. a spanning tree with no vertices of degree 2. Therefore, ($\frac{|V|}{2} - 1$)-Connected Dominating Set restricted to subcubic graphs is polynomially equivalent to the following problem:

$\{1,3\}$-Spanning Tree

Instance: A graph $G = (V, E)$.
Question: Does there exist a spanning tree T of G such that $d_T(v)$ is either 1 or 3, for any $v \in V$?

In fact, Douglas [19] first showed that $\{1,3\}$-Spanning Tree is NP-hard for subcubic planar graphs and then used the equivalence above to deduce that the same holds for ($\frac{|V|}{2} - 1$)-Connected Dominating Set. In the following, we take the same path and show that $\{1,3\}$-Spanning Tree is NP-hard even for the class of subcubic planar bipartite graphs with arbitrarily large girth. We first need the following lemma:

Lemma 39. *For any $\ell \geq 2$,* Hamiltonian Path *is* NP*-complete for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs with exactly two vertices of degree* 1.

Proof. We reduce from Hamiltonian Cycle Through Specified Edge for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs, which is NP-complete by Lemma 15. Let $G = (V, E)$ and $uv \in E$ be an instance of this problem. Clearly, we may assume G has no vertices of degree 1. Our reduction constructs a graph $G' = (V', E')$ as follows. Let $V' = V \cup \{a, b\}$, where a, b are new vertices, and $E' = (E \setminus \{uv\}) \cup \{au, bv\}$. Clearly, G' is a subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graph and a and b are the only 1-vertices of G'.

It is easy to see that G has a Hamiltonian cycle through uv if and only if G' has a Hamiltonian path (between a and b). $\square$

Lemma 40. *For any $\ell \geq 2$, $\{1,3\}$-*Spanning Tree *is* NP*-complete for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs.*

Proof. We reduce from Hamiltonian Path for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs with exactly two vertices of degree 1, which is NP-complete by Lemma 39. Given such a graph $G = (V, E)$, our reduction constructs a graph $G' = (V', E')$ as follows. We replace each 3-vertex u of G with the gadget depicted in Fig. 13 by taking $k = 2\ell + 1$. We proceed similarly for any 2-vertex u of G (in this case, u_3 will be a 2-vertex in G'). Note that there is some freedom on the way the edges incident to u are attached to the gadget. It is easy to see that G' is a planar bipartite $(C_4, \ldots, C_{2\ell})$-free graph with maximum degree 3.

Let v_1 and v_2 be the vertices of degree 1 in G (and so in G'). We claim that G has a Hamiltonian path (between v_1 and v_2) if and only if there exists a spanning tree T of G' such that $d_T(v)$ is either 1 or 3, for any $v \in V'$.

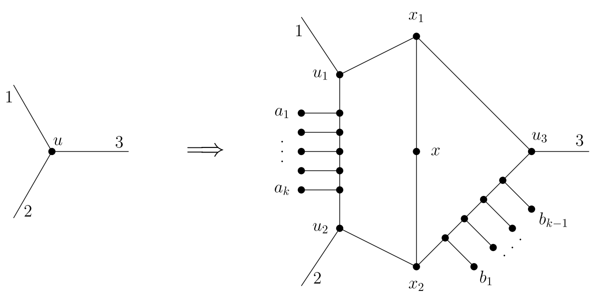


Fig. 13. Construction of the graph G' in the proof of Lemma 40: The vertex u is replaced by a gadget containing $2(2\ell + 1) + 2 \cdot 2\ell + 6$ vertices.

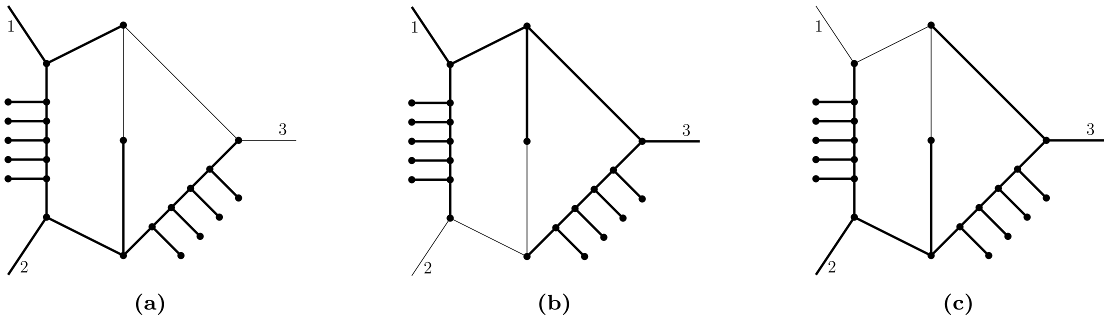


Fig. 14. Construction of the spanning tree T. If P passes through the edges 1 and 2, we select the bold edges in (a). Similarly for the other two cases illustrated in (b) and (c).

Suppose first G contains a Hamiltonian path P between v_1 and v_2. Each $u \in V \setminus \{v_1, v_2\}$ is incident to exactly two edges in $E(P)$. We select the corresponding edges in G' and, for each gadget, we select the bold edges as described in Fig. 14 (if the gadget replaces a 2-vertex of G, the construction is similar). In this way, we obtain a spanning tree T of G' such that $d_T(v)$ is either 1 or 3, for any $v \in V'$.

Conversely, suppose there exists a spanning tree T of G' such that $d_T(v)$ is either 1 or 3, for any $v \in V'$, and consider the gadget replacing a 3-vertex $u \in V$ (see Fig. 13). Clearly, the edges incident to the 1-vertices a_i's and b_i's are all in $E(T)$. Moreover, by the degree constraint and the connectedness of T, all the edges incident to the 3-neighbours of the a_i's and b_i's are in $E(T)$. We now claim that $|E(T) \cap \{1, 2, 3\}| = 2$. Since T is a spanning tree, $|E(T) \cap \{1, 2, 3\}| \geq 1$. Suppose first $\{1, 2, 3\} \subseteq E(T)$. By the degree constraint, we have that $\{u_1x_1, u_2x_2, u_3x_1\} \subseteq E(T)$ and so T contains a cycle, a contradiction. Suppose now $|E(T) \cap \{1, 2, 3\}| = 1$ and, without loss of generality, $E(T) \cap \{1, 2, 3\} = \{1\}$ (the other two cases are treated similarly). By the degree constraint, we have $u_1x_1 \in E(T)$, $u_2x_2 \notin E(T)$ and $u_3x_1 \notin E(T)$, a contradiction to the fact that at least one of xx_1 and xx_2 is in $E(T)$. Therefore, we have $|E(T) \cap \{1, 2, 3\}| = 2$. If $xx_1 \in E(T)$, then it must be $E(T) \cap \{1, 2, 3\} = \{1, 3\}$ (see Fig. 14(b)). Otherwise, i.e. if $xx_2 \in E(T)$, we either have $E(T) \cap \{1, 2, 3\} = \{1, 2\}$ or $E(T) \cap \{1, 2, 3\} = \{2, 3\}$ (see Figs. 14(a) and 14(c)). A similar reasoning applies to the gadget replacing a 2-vertex. But then, by contracting each gadget in G' to a single vertex, we obtain a connected spanning subgraph P of G such that, for each vertex of degree at least 2, exactly two of its incident edges are in $E(P)$. This implies that P is a Hamiltonian v_1, v_2-path in G. □

We have seen that a subcubic graph G contains a $\{1, 3\}$-spanning tree if and only if $\gamma_c(G) \leq \frac{|V(G)|}{2} - 1$. Therefore, Lemma 40 has the following two immediate consequences:

Corollary 41. *For any $\ell \geq 2$, $(\frac{|V|}{2} - 1)$-*Connected Dominating Set *is* NP-*complete for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs.*

Corollary 42. *The class of subcubic forests is a limit class for $(\frac{|V|}{2} - 1)$-*Connected Dominating Set *and* Connected Dominating Set.

We now show that the class $\mathcal{T}$ of forests whose components have at most three leaves (which is boundary for Dominating Set) is not boundary for $(\frac{|V|}{2} - 1)$-Connected Dominating Set. Clearly, $\mathcal{T}$ does not contain the graph H_1 depicted in Fig. 8.

Lemma 43. *If X is a class of subcubic forests which is boundary for $(\frac{|V|}{2} - 1)$-*Connected Dominating Set, *then $H_1 \in X$.*

Proof. We first claim that $(\frac{|V|}{2} - 1)$-Connected Dominating Set is solvable in polynomial time for graphs in the class Y of subcubic (C_3, C_4, H_1)-free graphs. Since this problem is polynomially equivalent to $\{1, 3\}$-Spanning Tree when restricted to subcubic graphs, it is enough to show the claim above for the latter problem. Therefore, consider a graph $G \in Y$. If G has maximum degree 2, then $\{1, 3\}$-Spanning Tree is trivial for G. Moreover, if G contains a cubic vertex v, then each neighbour of v has degree at most 2. This implies that v has degree 3 in a $\{1, 3\}$-spanning tree of G and so such a tree exists if and only if $G = K_{1,3}$. Therefore, $\{1, 3\}$-Spanning Tree is trivial for the class Y.

Suppose now $H_1 \notin X$. We have that $X \subseteq Y$ and Y is finitely defined. Therefore, by Theorem 6, $(\frac{|V|}{2} - 1)$-Connected Dominating Set is NP-hard for Y, a contradiction to the previous paragraph. □

Remark 44. Similarly to Lemma 43, it is not difficult to show that if X is a class of subcubic forests which is boundary for $(\frac{|V|}{2} - 1)$-Connected Dominating Set, then X contains a tribranch. Therefore, the class $\mathcal{Q}$ is not boundary for this problem either.

Unfortunately, we do not know whether Lemma 43 holds for Connected Dominating Set and the major open problem is to find a boundary class with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth. Another interesting problem (related to Lemma 43) is to determine the computational complexity of Connected Dominating Set for (C_3, C_4, H_1)-free subcubic graphs. We have seen that $(\frac{|V|}{2} - 1)$-Connected Dominating Set is trivial for this class and we know that Dominating Set is NP-hard for it [6]. Moreover, in Section 10, we show that Dominating Set and Connected Dominating Set belong to the same complexity class when restricted to $Free(H)$, for any graph H. If Connected Dominating Set is polynomial for (C_3, C_4, H_1)-free subcubic graphs, we would obtain a first example[3] of a non-trivial hereditary class for which Dominating Set and Connected Dominating Set belong to different complexity classes.

We now show that Connected Dominating Set admits a boundary class with respect to the class of line graphs. In order to show that the problem is NP-hard even for line graphs, it is useful to introduce the following notion. A *connected edge dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq E$ such that $G[D]$ is connected and every edge in $E \setminus D$ is incident to at least one edge in D. We denote by $\gamma'_c(G)$ the size of a minimum connected edge dominating set of G and consider the following natural problem:

Connected Edge Dominating Set	
Instance:	A graph G and a positive integer s.
Question:	Does $\gamma'_c(G) \leq s$ hold?

Lemma 45. *For any $k \geq 1$, Connected Edge Dominating Set is NP-complete for planar bipartite $(C_4, \dots, C_{2k})$-free graphs with maximum degree at most 4.*

Proof. Recall that a connected vertex cover of a graph G is a vertex cover S such that $G[S]$ is connected and we denote by $\beta_c(G)$ the size of a minimum connected vertex cover of G. We first claim that $\gamma'_c(G) = \beta_c(G) - 1$, for any graph G.[4]

Indeed, consider a minimum connected edge dominating set D of G. It is easy to see that $G[D]$ is a tree. But then the vertex set of $G[D]$ is a connected vertex cover of G of size $\gamma'_c(G) + 1$.

Conversely, let D be a minimum connected vertex cover of G and let T be a spanning tree of $G[D]$. We have that $E(T)$ is a connected edge dominating set of G of size $\beta_c(G) - 1$.

The claim above implies that Connected Edge Dominating Set and Connected Vertex Cover are polynomially equivalent and since the latter is NP-complete for planar bipartite $(C_4, \dots, C_{2k})$-free graphs with maximum degree at most 4 (Lemma 38), the conclusion follows. □

Since there is an obvious bijection between the connected edge dominating sets of a graph G and the connected dominating sets of $L(G)$, we have that Connected Edge Dominating Set polynomially reduces to Connected Dominating Set for line graphs. Therefore, Lemma 45 implies the following:

Lemma 46. *For any $k \geq 1$, Connected Dominating Set is NP-complete for line graphs of planar bipartite $(C_4, \dots, C_{2k})$-free graphs with maximum degree at most 4.*

9. Graph VC_{con} Dimension

In this section, we consider Graph VC_{CON} Dimension, a problem closely related to Connected Dominating Set. Graph VC_{CON} Dimension is the problem of deciding the VC-dimension of a certain set system arising from a graph. In order to give the precise definition, let us first recall the notion of VC-dimension.

Given a set system $\mathcal{H}$ on X, a subset $Y \subseteq X$ is *shattered by* $\mathcal{H}$ if $\{E \cap Y : E \in \mathcal{H}\} = 2^Y$ and the *VC-dimension* of $\mathcal{H}$ is defined as the maximum size of a set shattered by $\mathcal{H}$, or as ∞ if arbitrarily large subsets can be shattered. The notion of VC-dimension was introduced by Vapnik and Chervonenkis [70] and it represents a prominent measure of the "complexity" of a set system. Given a graph, we can consider set systems induced by a certain family of subgraphs. In this way we obtain several different notions of VC-dimension, each one related to a special family of subgraphs. Kranakis et al. [40] initiated a systematic study of these notions and adapted the definition of VC-dimension to the graph theoretic setting as follows:

Definition 47. Let $G = (V, E)$ be a graph and let $\mathcal{P}$ be a family of subgraphs of G. A subset $A \subseteq V$ is *$\mathcal{P}$-shattered* if every subset of A can be obtained as the intersection of $V(H)$ with A, for some $H \in \mathcal{P}$. The *VC-dimension* of G with respect to $\mathcal{P}$ is the size of a maximum $\mathcal{P}$-shattered subset and it is denoted by $\text{VC}_{\mathcal{P}}(G)$.

[3] To the best of our knowledge.
[4] This claim is in fact folklore.

According to Definition 47, we denote by VC_{con} the VC-dimension with respect to the family of connected subgraphs. Kranakis et al. [40] showed that $\mathsf{VC}_{\text{con}}(G)$ differs by at most 1 from the number of leaves $\ell(G)$ in a maximum leaf spanning tree of G:

Theorem 48 *(Kranakis et al. [40]). $\ell(G) \leq \mathsf{VC}_{con}(G) \leq \ell(G) + 1$, for any connected graph G.*

Papadimitriou and Yannakakis [62] considered the problem of deciding the VC-dimension: Given a set system $\mathcal{H}$ on X (by its incidence matrix) and an integer s, does $\mathcal{H}$ have VC-dimension at least s? Since the VC-dimension is at most $\log_2 |\mathcal{H}|$, it can clearly be computed by brute force in $O(|X|^{\log_2 |\mathcal{H}|})$ time and so the problem looks unlikely to be NP-complete. In fact, they introduced the complexity class LOGNP and showed that the problem in question is complete for it. Kranakis et al. [40] investigated the computational complexity of computing $\mathsf{VC}_{\mathcal{P}}(G)$, for a given graph G and a certain family of its subgraphs $\mathcal{P}$. They formulated the decision problem as follows:

Graph $\mathsf{VC}_{\mathcal{P}}$ Dimension	
Instance:	A graph G and a positive integer s.
Question:	Does $\mathsf{VC}_{\mathcal{P}}(G) \geq s$ hold?

Moreover, they showed that Graph VC_{con} Dimension is NP-complete. In [56], we strengthened this result and provided several others, as summarised in the following table:

Family $\mathcal{P}$	Graph G	Comp. Compl.	Reference
Star		P	Kranakis et al. [40]
Neighbourhood		LOGNP-complete	Kranakis et al. [40]
Path		Σ_3^{p}-complete	Schaefer [66]
Cycle		Σ_3^{p}-complete	Schaefer [66]
k-Connected	split	NP-complete	Munaro [56]
k-Connected	bounded clique-width	P	Munaro [56]
k-Connected	split, Dilworth number ≤ 2	P	Munaro [56]
Connected	planar, bipartite, $\Delta(G) = 3$	NP-complete	Munaro [56]
2-Connected	planar, bipartite, $\Delta(G) = 4$	NP-complete	Munaro [56]

Theorem 48 hints at the fact that Graph VC_{con} Dimension and Connected Dominating Set are two closely related problems[5] and we now show that the class of subcubic forests is limit for Graph VC_{con} Dimension as well.

Lemma 49. *For any $\ell \geq 2$,* Graph VC_{con} Dimension *is* NP-*complete for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs.*

Proof. It is easy to see that the complement of a connected dominating set can be shattered by connected subgraphs. Moreover, if G is a subcubic graph with sufficiently large order and VC-dimension, we showed in [56] that the converse holds as well: more precisely, for every subcubic graph G with $|V(G)| \geq 46$, $\gamma_c(G) \leq \frac{|V(G)|}{2} - 1$ if and only if $\mathsf{VC}_{\text{con}}(G) \geq \frac{|V(G)|}{2} + 1$ [56, Theorem 25]. But then the proof immediately follows from the fact that, for any $\ell \geq 2$, $(\frac{|V|}{2} - 1)$-Connected Dominating Set is NP-complete for subcubic planar bipartite $(C_4, \ldots, C_{2\ell})$-free graphs (Corollary 41). □

Corollary 50. *The class of subcubic forests is a limit class for* Graph VC_{con} Dimension.

Definition 47 has an analogue formulation for edge sets:

Definition 51 *(Kranakis et al. [40]).* Let $G = (V, E)$ be a graph and let $\mathcal{P}$ be a family of sets of edges of G. A subset $A \subseteq E$ is *$\mathcal{P}$-edge-shattered* if every subset of A can be obtained as the intersection of a $C \in \mathcal{P}$ with A. The *edge VC-dimension* of G with respect to $\mathcal{P}$ is defined as the size of a maximum $\mathcal{P}$-edge-shattered subset and it is denoted by $\mathsf{EVC}_{\mathcal{P}}(G)$.

We denote by $\mathsf{EVC}_{\text{con}}$ the edge VC-dimension with respect to the family of connected edge sets. Since a graph G is connected if and only if $L(G)$ is, it is easy to see that $\mathsf{EVC}_{\text{con}}(G) = \mathsf{VC}_{\text{con}}(L(G))$ [40]. Moreover, given a graph G and a positive integer s, it is NP-complete to decide whether $\mathsf{EVC}_{\text{con}}(G) \geq s$ holds [40]. It immediately follows that Graph VC_{con} Dimension is NP-complete for line graphs.

To further emphasise the connection between Graph VC_{con} Dimension, Connected Dominating Set and Dominating Set, we show in the next section that the complexities of these problems all agree in monogenic classes.

[5] Recall that $\ell(G) = |V(G)| - \gamma_c(G)$.

10. Complexity dichotomies in monogenic classes for Graph $\mathsf{VC}_{\mathsf{con}}$ Dimension and Connected Dominating Set

A class of graphs $\mathcal{G}$ is *monogenic* if it is defined by a single forbidden induced subgraph, i.e. $\mathcal{G} = Free(H)$, for some graph H. We say that a (decision) graph problem admits a dichotomy in monogenic classes if, for each monogenic class, the problem is either NP-complete or decidable in polynomial time. The first result in this direction was obtained by Korobitsin [36], who showed that Dominating Set is decidable in polynomial time if H is an induced subgraph of $P_4 + tK_1$, for $t \geq 0$, and NP-complete otherwise. Král' et al. [39] showed that Colouring is decidable in polynomial time if H is an induced subgraph of P_4 or of $P_3 + K_1$ and NP-complete otherwise. Kamiński [33] showed that Simple Max-Cut is decidable in polynomial time if H is an induced subgraph of P_4 and NP-complete otherwise. Other dichotomies were obtained by Golovach et al. [28] for Precolouring Extension and ℓ-List Colouring and by AbouEisha et al. [1] for Upper Dominating Set. Other problems, like k-Colouring and Independent Set, appear more stubborn and no complete dichotomy is available: we refer the reader to [29] for a survey on the status of k-Colouring and we just mention that a major question related to Independent Set is whether it is decidable in polynomial time for P_k-free graphs with $k \geq 6$ (see, e.g., [44]).

In the following, we enlarge the list above by providing dichotomies for the closely related Graph $\mathsf{VC}_{\mathsf{CON}}$ Dimension and Connected Dominating Set. Let us begin with Graph $\mathsf{VC}_{\mathsf{CON}}$ Dimension:

Theorem 52. Graph $\mathsf{VC}_{\mathsf{CON}}$ Dimension *restricted to H-free graphs is decidable in polynomial time if H is an induced subgraph of $P_4 + tK_1$ and* NP-*complete otherwise.*

Proof. Suppose first H contains an induced cycle C_k. If k is odd, then the problem is NP-complete since it is NP-complete when restricted to bipartite graphs [56]. If k is even, then the problem is again NP-complete since it is NP-complete when restricted to split graphs [56].

Suppose now H is a forest with a vertex of degree at least 3. Then H contains an induced claw and the problem is NP-complete since it is NP-complete when restricted to line graphs (see Section 9).

Finally, suppose H is the disjoint union of paths. If H contains at least two paths on at least 2 vertices, then H contains $2K_2$ and the problem is NP-complete since it is NP-complete when restricted to split graphs. The same conclusion holds if H contains a path on at least 5 vertices. It remains to consider the case of H being of the form $P_k + tK_1$, for some $k \leq 4$ and $t \geq 0$. Therefore, let G be such a $P_k + tK_1$-free graph. If G is in addition P_k-free, then it has bounded clique-width (see Section 2) and the problem is decidable in polynomial time [56]. On the other hand, if G contains an induced copy G' of P_k, then there are at most $t-1$ pairwise non-adjacent vertices of G none of which is adjacent to a vertex of G'. Denoting this set by S, we have that $V(G') \cup S$ is a dominating set of size at most $t+3$. Moreover, denoting by $\gamma(G)$ the domination number of G, Duchet and Meyniel [20] showed that $\gamma_c(G) \leq 3\gamma(G) - 2$ and so we have $\gamma_c(G) \leq 3t + 7$. If G has n vertices, Theorem 48 implies that $n - \gamma_c(G) \leq \mathsf{VC}_{\mathsf{con}}(G) \leq n - \gamma_c(G) + 1$. But then we simply check all the subsets of $V(G)$ of size $n - \gamma_c(G)$ and $n - \gamma_c(G) + 1$: their number is $\binom{n}{\gamma_c(G)} + \binom{n}{\gamma_c(G)-1} = O(n^{3t+7})$ and so, by the proof of [56, Theorem 17], we can compute $\mathsf{VC}_{\mathsf{con}}(G)$ in polynomial time. $\square$

Let us now consider Connected Dominating Set. We have seen in Section 2 that if a graph property is expressible in MSO_1, then it is decidable in polynomial time for graphs of bounded clique-width. We now show that being a connected dominating set is an example of such a property:

Lemma 53. *Being a connected dominating set is expressible in* MSO_1.

Proof. Let $G = (V, E)$ be a graph. The following MSO_2 sentence says that the subgraph induced by $X \subseteq V$ is connected:

$$\mathbf{conn}(X) = \forall_{Y \subseteq V} [(\exists_{u \in X}\, u \in Y \wedge \exists_{v \in X}\, v \notin Y) \rightarrow (\exists_{e \in E} \exists_{u \in X} \exists_{v \in X}\, inc(u, e) \wedge inc(v, e) \wedge u \in Y \wedge v \notin Y)].$$

Moreover, the quantification over single edges can be expressed by a MSO_1 sentence as follows:

$$\exists_{a \in V} \exists_{b \in V} \exists_{u \in X} \exists_{v \in X}\, adj(a, b) \wedge (u = a \vee u = b) \wedge (v = a \vee v = b) \wedge u \in Y \wedge v \notin Y.$$

Finally, the following MSO_1 sentence says that $D \subseteq V$ is a connected dominating set:

$$\mathbf{cds}(D) = \mathbf{conn}(D) \wedge \forall_{v \in V \setminus D} \exists_{u \in D}\, adj(u, v). \qquad \square$$

Corollary 54. Connected Dominating Set *is decidable in polynomial time for graphs of bounded clique-width. In particular, it is decidable in polynomial time for P_4-free graphs.*

We can finally prove our second dichotomy. The proof is similar to that of Theorem 52.

Theorem 55. Connected Dominating Set *restricted to H-free graphs is decidable in polynomial time if H is an induced subgraph of $P_4 + tK_1$ and* NP-*complete otherwise.*

Proof. Suppose first H contains an induced cycle C_k. If k is odd, then the problem is NP-complete since it is NP-complete when restricted to bipartite graphs (Corollary 41). If k is even, then the problem is again NP-complete since it is NP-complete when restricted to split graphs [41].

Suppose now H is a forest with a vertex of degree at least 3. Then H contains an induced claw and the problem is NP-complete since, by Lemma 46, it is NP-complete when restricted to line graphs.

Finally, suppose H is the disjoint union of paths. If H contains at least two paths on at least 2 vertices, then H contains $2K_2$ and the problem is NP-complete since it is NP-complete when restricted to split graphs. The same conclusion holds if H contains a path on at least 5 vertices. Therefore, it remains to consider the case of H being of the form $P_k + tK_1$, for some $k \leq 4$ and $t \geq 0$. Let G be such a $P_k + tK_1$-free graph. If G is in addition P_k-free, then the problem is decidable in polynomial time by Corollary 54. On the other hand, if G contains an induced copy of P_k, we have seen in the proof of Theorem 52 that $\gamma_c(G) \leq 3t + 7$. Therefore, it suffices to check all the subsets of $V(G)$ of size at most $3t + 7$, and this can be clearly done in polynomial time. □

11. Concluding remarks and open problems

In this paper, we considered seven NP-hard graph problems involving non-local properties and provided some new boundary classes for them. Unfortunately, we could not completely determine the set of boundary classes for any of these problems.

Korpelainen et al. [38] showed that the class $\mathcal{Q}$ is boundary for Hamiltonian Cycle (with respect to the class of subcubic graphs) and in Section 3 we showed that $\mathcal{Q}$ is boundary also with respect to the class of subcubic planar bipartite graphs. Moreover, we showed there exists a boundary class with respect to split graphs which is a subclass of $\bigcap_{p \geq 1} S(\mathcal{Q}_p)$ (Lemma 13) and which is distinct from $\mathcal{Q}$ (Remark 14). A boundary class with respect to line graphs was determined by Korpelainen et al. [38]. Their result is an application of the following operation, usually known as Y-extension (see, e.g., [57]). Given a graph G, the graph $R(G)$ is obtained by replacing each cubic vertex $u \in V(G)$ with a triangle T_u, where each $x \in V(T_u)$ corresponds to an edge incident to u, and by connecting the vertices of the triangles which correspond to the same edge. They showed that the hereditary closure $R(\mathcal{Q})$ of $\{R(G) : G \in \mathcal{Q}\}$ is a boundary class. The existence of one more boundary class for Hamiltonian Cycle follows from the fact that the problem is NP-hard for chordal bipartite graphs [54], i.e. $(C_3, C_5, C_6, \ldots)$-free graphs. Reasonings similar to Remarks 2 and 14 show that these boundary classes are distinct and so Hamiltonian Cycle admits at least four boundary classes. Determining all of them seems to be a challenging problem.

A similar situation occurs for Hamiltonian Path. In Section 5, we showed that $\mathcal{Q}$ is a boundary class with respect to the class of subcubic planar bipartite graphs with arbitrarily large girth. Moreover, we showed there exists a boundary class (with respect to split graphs) which is a subclass of $\bigcap_{p \geq 1} S(\mathcal{Q}_p)$. By a result in [57], we have that $R(\mathcal{Q})$ is a limit class with respect to line graphs and the existence of a boundary class with respect to chordal bipartite graphs follows by results in [54]. Once again, it is easy to see that these four boundary classes are distinct and we leave as an open problem their complete determination. In particular, it would be interesting to check whether Hamiltonian Path and Hamiltonian Cycle admit the same boundary classes.

In Section 6, we considered Feedback Vertex Set and provided the first boundary class for the problem (with respect to planar bipartite graphs with maximum degree 4 and arbitrarily large girth): the class $\mathcal{S}$ of forests whose components have at most four leaves and no two vertices of degree three. The existence of a boundary class with respect to line graphs was shown in Section 6.3, where we proved that $L(\mathcal{Q})$ is a limit class. In fact, we suspect that this class is minimal. Note that the problem is solvable in polynomial time for chordal graphs [13] (the $C_{3,1}$ problem for chordal graphs defined therein is equivalent to Feedback Vertex Set) and for chordal bipartite graphs [35].

In Section 7, we considered Connected Vertex Cover and showed it admits at least two boundary classes. The determination of these two classes is left as an open problem: one of them is a subclass of line graphs of subcubic planar bipartite graphs and the other is a subclass of planar bipartite graphs with maximum degree 4 and arbitrarily large girth. Note that Connected Vertex Cover is solvable in polynomial time for chordal graphs [21].

In Section 8, we considered Connected Dominating Set and showed that the class of subcubic forests is a limit class for this problem. Moreover, we showed that Connected Dominating Set admits a boundary class with respect to line graphs and gave some insights into its structure. Note that the problem is NP-hard for chordal bipartite graphs [55] and for split graphs [41] and so reasonings similar to Remarks 2 and 14 show that it admits at least four boundary classes.

In Section 9, we observed that Graph $\mathrm{VC}_{\mathrm{CON}}$ Dimension, a problem closely related to Connected Dominating Set, admits at least three boundary classes.

Finally, in Section 10, we provided a dichotomy for Graph $\mathrm{VC}_{\mathrm{CON}}$ Dimension and Connected Dominating Set in monogenic classes showing that the complexities of these two problems all agree with those of Dominating Set when restricted to monogenic classes.

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