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Paul Wijnbergen *Utrecht, 7 October 2022*

Contents

1 | Introduction

The modeling of dynamical processes plays an increasingly important role in science and engineering. Some processes change continuously over time and exhibit only continuous dynamics and are called continuous-time systems. Think for example of the water level in a leaking vessel, where the water level continuously decreases as the water is leaking out of the vessel. Other processes evolve only discretely in time, such as the average amount of cars parked in a parking garage each day. Such processes are called discrete-time systems. In the case a dynamical process exhibits both continuous- and discrete-time dynamics, it is called a hybrid system. In particular, such a system involves a class of linear or nonlinear systems and results from the interaction of continuous-time subsystems with discrete events. More precisely, the internal variable of each system is regulated by a set of differential equations and each of the separate subsystems is labeled as a discrete mode. As a consequence of the transitions between discrete states, the continuous variable may contain jumps.

Switched systems constitute a particular subclass of hybrid systems. Specifically, a switched system is a dynamical system that consists of a finite number of subsystems, referred to as modes and a logical rule that orchestrates the switching between these subsystems. The main property of switched systems is that these systems switch among a finite number of subsystems and the discrete events interacting with the subsystems are governed by a piecewise continuous function called the switching signal. However, one can classify switched systems based on the dynamics of their subsystems, for example continuous-time, discrete-time, linear or nonlinear and so on. In the case that a dynamical system is formed by a collection of linear continuous state space models and the switching among them is according to a discrete signal, the system is called a linear switched system. Due to the large number of applications of switched systems, they have been studied extensively during the past decades. Examples of applications can be found in aeronautical and mechanical systems, automotive industry, modeling of electronic circuits with physical switches and power converters, as discussed in *e.g.,* [25, 133].

The class of switched systems with a time depending switching signal constitutes a subclass of the general class of time-varying systems. In the case that each mode consists of a linear system, the overall switched system remains linear; the switched system can be regarded as a particular linear time-varying system. However, in the case of a state-dependent switching signal linearity is lost and a particular nonlinear system is obtained. In this thesis switched linear systems with a time depending switching signal are studied. Therefore, the standard qualitative and quantitative properties for standard systems cannot be applied, but it is necessary to develop specific tools for them. For a detailed discussion on switched systems, cf. [40,75,113].

In the case of switched linear systems, the mathematical model for each mode is typically given by an ordinary differential equation (ODE). However, in many applications the dynamics are restricted by some algebraic equations and hence both algebraic and differential equations are needed to model the dynamical system. Equations of this kind are called differential algebraic equations (DAEs), descriptor systems, singular systems or sometimes implicit systems and have been studied extensively, see *e.g.,* [32,33,64,132] for early works and [14, 16, 17, 24, 66, 101, 107] and the references therein. A DAE model is naturally obtained when modeling linear electrical circuits, mechanical systems or linear systems with additional linear algebraic constraints. For examples of DAE models used for the modeling of network structures such as electrical circuits and gas networks see [1,24,94,102,105,122] and [42], respectively, where the algebraic constraints are induced by the network topology. However, there are also applications in the treatment of semidiscretized partial differential equations such as the Navier-Stokes equation and fluid dynamics in general [2, 41, 82, 131], chemical engineering [26, 97], holonomically constrained mechanical systems [29, 90, 106], and in a vast variety of economical problems [85,86].

In the case a system undergoes abrupt structural changes due to physical switches in the system or component failure, one generally needs multiple DAE models to describe each mode of the system. In the case each subsystem of a linear system is governed by a DAE, the switched system is called a switched DAE. Whereas for each mode it could be possible to obtain an ODE model by solving for the algebriac constraints, there generally does not exists an ODE model for each mode with a common state variable. Therefore switched DAEs have been studied directly cf. [79,80,111,118,120,121,125].

1.1 Qualitative properties of switched systems

Given a mathematical model for a dynamical system, properties of the system can be derived from the mathematical model. Many dynamical systems can be influenced by applying an external input such as a force or a voltage. Alternatively, we can perform measurements on certain components of a system to obtain information on the state without influencing the system. One can for example measure the current running through a resistor or measure the velocity of a vehicle driving. These measurements and external inputs give rise to the question to what extent the behavior of a dynamical system can be shaped by means of applying a suitable input. With the introduction of the state space by Kalman [58] the research on control theory for linear ODE systems accelerated rapidly. Fundamental properties such as controllability and observability of ODE systems and their duality have been studied extensively, cf. [45,46,59,60,83,84,99,135,136] and similar concepts were studied for DAE systems [15–17, 20, 22, 73]. For many control purposes it is not necessary to be able to have full control over the system as long as the system can be stabilized around some operating point. In the case a system can be stabilized around the origin it is called stabilizable. As this property is of interest for many applications, it has been the topic of research for a long time and in particular the concept of stabilizability in terms of Lyapunov functions has been studied extensively, see *e.g.*, [5, 96]. Similar to the geometric control theory for linear systems as provided in [3, 139] and later in [127], a coordinate independent approach to DAEs has been given in [7, 74].

Control theoretical properties of switched linear ODEs such as controllability, stabilizability and observability have been studied during the past few decades as well, see *e.g.*, [18, 81, 98, 115, 116] and the references therein. Non-switched ODEs can be regarded as particular switched systems, namely those with a constant switching signal. Hence one would expect that the results for the non-switched case can easily be generalized to the switched case. This is true up to a certain extent, but there are some subtleties that have to be taken into account when considering switched systems. The results in the field of switched systems namely depend in general on the switching signal and hence one of the first questions to ask when considering control of switched systems is whether to regard the switching signal as a control input. In the case that the switching signal is not controllable, the question remains whether it is known a priori. The relevance of how to regard the switching signal is evidenced by the results on stabilizability, where it is shown that even though every subsystem of the switched ODE is stabilizable, the overall system does not need to be stabilizable for certain switching signals [13,77,78]. Moreover, there exist systems of which the state will grow unbounded for some switching signals, but could be stabilized given some other switching signals. Similarly, if none of the subsystems is observable, the overall system might be observable. [115]. Consequently, a system that is stabilizable or observable if the switching signal is a control input, is not necessarily stabilizable or observable for all switching signals. Another interesting aspect which has been studied is the case where the switching signal is assumed to be unknown. One generally assumes in such cases that each mode is active for a minimum amount of time, the so called dwell time. Various results for switched systems with a dwell time have been presented, cf. [30, 47, 53, 55].

Whereas the literature on fundamental properties of switched ODEs and non-switched linear systems is rich, the literature on properties of switched DAEs has not matured yet. Stabilizability of switched DAEs has been studied in [79, 80, 91, 92] and also some results on controllability have been presented [68]. This gap in the literature can be explained by the fact that a firm solution framework for switched DAE has recently been developed [123]. In contrast to solutions of switched ODEs, solutions of switched

DAEs typically contain Dirac impulses and jumps,which prevent classical solutions from existence. These discontinuities in the state are however not only a mathematical artifact. Dirac impulses can for example be observed in applications in the form of sparks induced at a switch when an inductor is disconnected or as hydraulic shocks in a water distribution network. In order to be able to incorporate these phenomena in a mathematical model, the distributional solutional framework has to be adopted. The jumps and Dirac impulses in the state can be beneficial for characterizing various observability properties. They can *e.g.*, be used in determining the state of a switched DAE [118–121], but also to detect which mode is active [67, 70, 71]. However, Dirac impulses are usually undesired in applications as they are prone to cause damage to components of the system or cause a hazardous situation for the environment of the system. Particularly in micro chips and water networks with fragile pipelines it is of utter importance to avoid Dirac impulses. However, to the best of the authors knowledge only properties for switched DAEs are studied where Dirac impulses in the state are allowed. Hence it is unclear on how to stabilize or control a switched DAE while avoiding Dirac impulses. For non-switched DAEs the property to avoid Dirac impulses is referred to as controllability at infinity or impulse-controllability and it has been studied in *e.g.*, [20]. Hence so far it has been an open research question on how to control a switched DAE while avoiding Dirac impulses.

1.2 Optimal control of switched systems

Alongside qualitative properties such as controllability and observability, quantitative properties of dynamical systems have been studied as well. By introducing a certain cost functional, the performance of the system can be measured and one can make quantitative statements about control inputs. For many applications it is of interest to design a controller which performs optimal such that a minimum cost can be guaranteed. From an economical perspective this might not need motivation, but even in a physical setup it might be of interest to control a vehicle such that as little fuel is consumed while driving.

In the context of linear systems the linear quadratic regulator (LQR) problem on both the finite and infinite horizon has been studied extensively, see [56, 61, 62, 134, 138] for results on ODEs and [6, 8, 21, 37, 38, 48, 65, 87–89] for DAEs. In the LQR problem a cost function that is quadratic in the state and input is considered and in the linear case it admits a surprisingly elegant solution. The optimal input turns out to be a feedback which can be computed by means of solving a Riccati equation or certain matrix inequalities.

Most recent studies regarding the optimal control problem for DAEs focus on finding solutions based on the Lure inequality or an extension of the Kalman-Yakubovich-Popov lemma [103, 104, 130]. More recently the concept of model predictive control has been studied. In model predictive control the future behavior of a system is predicted over a finite time horizon. Based on these predictions and the current measured or estimated state of the system, the optimal control inputs with respect to a defined cost functional are computed and applied. After a certain time interval, the measurement or estimation and computation processes are repeated with a shifted horizon. This subject has been investigated in [34, 49, 50, 93] and the references therein. Besides optimal control various other optimization problems such as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem have been studied for ODEs, see [27, 28, 57, 108] and DAEs, cf. [23, 52, 95, 114]. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem deal with finding an optimal feedback that minimizes the influence of process disturbances or measurement noise on an output of the system.

Optimal control of switched systems has also been studied. Given the results on the LQR problem for non-switched ODEs it is rather straightforward to solve the LQR problem for a switched ODE with a fixed and known switching signal using a dynamic programming approach [4, 11, 76]. Hence the goal in optimal control of switched systems is often to find an optimal switching signal. The cost may change for example if the switching times are changed, but also when the sequence in which the modes appear varies. The problem of finding an optimal switching signal has been studied in *e.g.*, [39, 140, 141, 143]. Also model predictive control for switched systems has been studied cf. [44, 129, 142]. However, so far optimal control of switched differential algebraic equations has not been studied yet. Hence there are no techniques available to measure the performance of a controller for switched DAEs.

Motivated by the gaps in the literature on fundamental qualitative and quantitative properties of switched DAEs, this thesis aims to contribute to the development of control theoretical tools for switched DAEs. First, impulse-free solutions of switched DAEs will be investigated. Then the problem of stabilizing a switched DAE while avoiding Dirac impulses in the state is considered. Finally, the linear quadratic regulator problem for switched DAEs will be considered.

1.3 Thesis outline and contributions

So far it can be concluded from the overview in the previous section that impulse-free solutions and optimal control of switched DAEs have not been studied yet in the existing literature. Hence this thesis aims to contribute with respect to these topics. More precisely, this thesis is concerned with control theoretical properties of switched DAEs. We confine our attention to switched DAEs for which the switching signal is fixed and known and adopt the piecewise-smooth distributional solution framework. To that extent, a brief introduction to mathematical distributions is given in Chapter 2, together with a brief overview of some crucial control theoretical concepts for non-switched DAEs.

The existence of impulse-free solutions of switched DAEs is studied in Chapter 3. First impulse-controllability of switched DAEs with a switching signal that induces finitely many switches is investigated. Necessary and sufficient conditions for such switched DAEs to be impulse-controllable are given in terms of an algorithm that involves subspaces and runs backward in time. Then sufficient conditions for switched DAEs with infinitely many switches to be impulse-controllable are given in terms of an algorithm that runs forward in time. Chapter 3 is concluded with a section on impulse-controllability of system classes of switched DAEs. A switched DAE can in principle be thought of as a system generated by some subsystems defined by triplets of system matrices and a switching signal. Hence systems generated by some matrix triplets and a class of switching signals define a system class. The concept of impulse-controllability of system classes is introduced and impulse-controllable system classes generated by some matrix triplets and the class arbitrary switching signals are characterized. For the classes of systems with the same mode sequence it is shown that either all or almost all systems or none or almost none of the systems in the class are impulse-controllable. Furthermore, it is shown that although every system in the system class is impulse-controllable, inputs that achieve impulse-free solutions are generally not independent of the switching signal.

Stabilizability and controllability of switched DAEs is studied in Chapter 4. First it is shown by means of an example that if a switched DAE is impulse-controllable and stabilizable it is not necessarily impulse-free stabilizable. However, it is shown that the concepts of null-controllability, reachability and controllability in the behavioral sense are equivalent for switched DAEs, but the same does not hold for the impulse-free versions of these concepts. In order to deal with switched DAEs with a large number of switches, a definition of stabilizability on a bounded interval is given, so called interval-stabilizability. Under certain assumptions global stabilizability can be concluded from interval-stabilizability. A similar counterpart for impulse-free interval-stabilizabilty is also given. Necessary and sufficient conditions for impulse-free interval-stabilizability are given in terms of an algorithm involving subspaces that runs forward in time. Based on this approach a novel characterization of impulse-free controllability is given. The chapter is concluded with the extension of the results to the case where Dirac impulses in the state and input are allowed.

In Chapter 5 the linear quadratic regulator (LQR) problem for switched DAEs is considered. It is shown that if there exists an input that solves the problem, the optimal input is linear in the state and the optimal cost is a quadratic function of the initial value. Given this result, it is shown how the finite horizon LQR problem can be motivated by the infinite horizon problem. In the case of a single switched DAE, the LQR problem on the infinite horizon can be reduced to a finite horizon LQR problem for non-switched DAEs with terminal constraints. As a consequence of the cost resulting from the second interval, a general positive semi-definite cost matrix that is not necessarily structurally related to the first mode has to be considered. Furthermore, the state at the end of the interval needs to be contained in a subspace. Necessary and sufficient conditions for solvability of this LQR problem with endpoint constraints are given and it is shown how to compute the optimal feedback matrix. Intuitively these results can be interpreted as follows. The optimal control problem for non-switched DAEs with endpoint constraints is solvable if and only if it is solved by the input that solves the unconstrained LQR problem for non-switched DAEs. Finally, necessary and sufficient conditions on solvability of the LQR problem for switched DAEs with an arbitrary number of switches are stated.

1.4 Publications

The following publications form the content of this thesis.

Conference publications

C3 P. Wijnbergen, S. Trenn, "Optimal control of DAEs with unconstrained terminal cost", *in Proceedings of the Conference on Decision and Control*, Austin, USA, 2021.

C2 P. Wijnbergen, M. Jeeninga, S. Trenn, "On stabilizability of switched differential algebraic equations", *in Proceedings of the IFAC World Congress 2020*, Berlin, Germany, 2020.

C1 P. Wijnbergen, S. Trenn, "Impulse controllability of switched differential algebraic equations", *in Proceedings of the European Control Conference 2019*, St. Petersburg, Russia, 2019.

Journal publications

J2 P. Wijnbergen, S. Trenn, "Impulse controllability of system classes of differential algebraic equations", submitted to *Mathematics of Control, Systems and Signals, 2022*

J1 P. Wijnbergen, S. Trenn. "Impulse-free interval-stabilization of switched differential algebraic equations." *Systems & Control Letters* 149 (2021): 104870.

Journal publications in preparation

J1* P. Wijnbergen, S. Trenn, "Optimal control of switched differential algebraic equations"

Peer reviewed extended abstracts

A1 P. Wijnbergen, S. Trenn, "Optimal control of switched differential algebraic equations", *in Proceedings of the MTNS 2022*, Bayreuth, Germany, 2022.

The following publications were published during the PhD project, but their content is unrelated to this thesis.

Other journal publications

O2 P. Wijnbergen, M. Jeeninga, B. Besselink, "Nonlinear spacing policies for vehicle platoons: a geometric approach to decentralized control", *Systems & Control Letters*, 145 (2020): 104796.

O1 P. Wijnbergen, B. Besselink, "Existence of decentralized controllers for vehicle platoons: On the role of spacing policies and available measurements", *Systems & Control Letters*, 153 (2021): 104954.

1.5 Notation

We close the introduction with the nomenclature used throughout the thesis:

Basic sets

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers
$\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$	set of real numbers, set of positive real numbers, set of negative real numbers
$\mathbb{C}, \mathbb{C}_+, \mathbb{C}_-$	set of comlex numbers, set of complex number with positive real part, set of complect numbers with negative real part
e_i	The standard base vector of $\mathbb{R}^n$, the j^{th} entry $e_{ij} = 1$ if $i = j$ and $e_{ij} = 0$ if $i \neq j$.
$\lvert\alpha\rvert$	Absolute value of $\alpha \in \mathbb{R}$

Matrices and subspaces

$A^\top$	the transpose of the matrix $A \in \mathbb{R}^{m\times n}$
I_n	the $n \times n$ identity matrix
$\operatorname{rank} A$	the rank of the matrix $A \in \mathbb{R}^{m\times n}$
$\operatorname{im} A$	the range of the linear map $A : \mathcal{X} \to \mathcal{Y}$, *i.e.*, $\operatorname{im} A = \{Ax \mid x \in \mathcal{X}\}$
$\ker A$	the kernel of the linear map A, *i.e.*, $\ker A = \{x \in \mathcal{X} \mid Ax = 0\}$
$A^{-1}\mathcal{V}$	the inverse image of a subspace $\mathcal{V} \subset \mathcal{Y}$ given the linear map A, *i.e.*, $A^{-1}\mathcal{V} = \{x \in \mathcal{X} \mid Ax \in \mathcal{V}\}$
$\mathcal{V}^\perp$	the orthogonal complement of $\mathcal{V} \subseteq \mathcal{X}$ in the inner product space $\mathcal{X}$, *i.e.*, $\mathcal{V}^\perp = \{x \in \mathcal{X} \mid x^\top y = 0,\ \forall v \in \mathcal{V}\}$
$\mathcal{V} + \mathcal{W}$	the sum of subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathcal{X}$, *i.e.*, $\mathcal{V} + \mathcal{W} = \{x \in \mathcal{X} \mid x = v + w,\ v \in \mathcal{V},\ w \in \mathcal{W}\}$
$\langle A \mid \mathcal{V}\rangle$	the smallest A invariant subspace generated by $\mathcal{V}$, *i.e.*, $\langle A \mid \mathcal{V}\rangle = \mathcal{V} + A\mathcal{V} + \cdots + A^{n-1}\mathcal{V}$
$\langle \mathcal{V} \mid A\rangle$	the largest A invariant subspace contained in $\mathcal{V}$, *i.e.*, $\langle \mathcal{V} \mid A\rangle = \{x \in \mathcal{V} \mid Ax \in \mathcal{V}\}$

Dynamical systems

(E, A, B)	shorthand notation for a differential algebraic system of the form $E\dot{x} = Ax + Bu$.
$(E_\sigma, A_\sigma, B_\sigma)$	shorthand notation for a switched differential algebraic system of the for $E_\sigma\dot{x} = A_\sigma x + B_\sigma u$.
$\Sigma_{\mathtt{n}}$	the system class of switched differential algebraic systems generated by the matrix triplets (E, A, B), *i.e.*, $\Sigma_\sigma = \{(E_\sigma, A_\sigma, B_\sigma) \mid \sigma \in \mathcal{S}_{\mathtt{n}}\}$

Functions and function spaces

$\mathcal{L}_1^{\mathrm{loc}}$	the space of locally integrable functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{L}_2$	the space of square integrable functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{C}^\infty$	the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$
$\mathcal{C}_0^\infty$	the space of test functions, *i.e.*, the space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ with support on a compact set $K \subseteq \mathbb{R}$
$\mathcal{C}_{\mathrm{pw}}^\infty$	the space of piecewise-smooth functions $f : \mathbb{R} \to \mathbb{R}$
$\mathbb{D}$	the space of distributions $D : \mathcal{C}_0^\infty \to \mathbb{R}$
$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$	the space of piecewise-smooth distributions

Norms

$\\| \cdot \\|$	The Euclidean norm
$\\| \cdot \\|_{\mathcal{L}^p}$	$:= \left(\int_0^\infty \lvert \cdot \rvert^p \, \mathrm{d}t\right)^{\frac{1}{p}}$

2 | Mathematical Preliminaries

The purpose of this chapter is to lay a theoretical foundation for the subsequent chapters. Before studying switched differential algebraic equations, the precise type of systems considered throughout the thesis will be described. Then the solutional framework adopted will be discussed. To do so, some theory on mathematical distributions is recalled and the space of piecewise-smooth distributions is introduced. Furthermore, some properties of regular matrix pencils in relation to differential algebraic equations are presented. Finally some control theoretical results for non-switched DAEs are considered.

2.1 Switched DAEs

Throughout this thesis we will study switched differential algebraic equations of the form

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{2.1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^n$ is the control input and $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal indicating which mode is active at which particular time instance. Unless stated differently, we will assume throughout this thesis that the switching signal is fixed and known a priori. Hence it is not regarded as a control input. Furthermore, to avoid chattering behavior, we assume the switching signal is right-continuous and induces locally finitely many switches. Specifically, we assume that the switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$ where $\mathcal{S}_{\mathtt{n}}$ is defined as follows.

Definition 2.1. The class of switching signals $\mathcal{S}_{\mathtt{n}}$ is defined as the set of all $\sigma : \mathbb{R} \to \{0, 1, ..., \mathtt{n}\}$ of the form

$$\sigma(t) = p \quad t \in [t_p, t_{p+1}), \tag{2.2}$$

where $p \in \{0, 1, ..., \mathtt{n}\}$ and $t_1 < t_2 < ... < t_{\mathtt{n}}$ are the $\mathtt{n} \in \mathbb{N}$ switching times in $(0, \infty)$ with $t_0 := 0$ and $t_{\mathtt{n}+1} := \infty$ for notational convenience. Furthermore, for a given sequence of switching times, let $\tau_i := t_{i+1} - t_i$, $i = 0, 1, \ldots, \mathtt{n} - 1$ and $\boldsymbol{\tau} := (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$,

As a consequence of $\sigma \in \mathcal{S}_{\mathtt{n}}$, we assume that on the interval $[t_0, \infty)$ at most $\mathtt{n} + 1$ modes are induced and each mode is described by a non-switched DAE with matrices $E_p, A_p \in \mathbb{R}^{n\times n}$ and $B_p \in \mathbb{R}^{n\times m}$ for $p \in \{0, 1, ..., \mathtt{n}\}$.

Solutions to (2.1) typically exhibit Dirac impulses which exclude classical solutions from existence. Hence we need to consider a distributional solution framework. To do so, we will first give a brief review of distributions and we will show that the general space of distributions is not suited as a solutional framework. Most of this section is based on [123] and [125] to which is referred to for a more extensive treatment of distributions. On occasion we include a proof for illustration or whenever the exact formulation is not found in the literature.

2.2 Classical distribution theory

In this section classical distributions as formalized by Schwartz [112], *i.e.,* linear functionals on the space of test functions, are considered and important properties are highlighted.

Definition 2.2. The space of test function is

$$\mathcal{C}_0^\infty := \{\varphi \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}) \mid \text{supp } \varphi \text{ compact } \},$$

where

$$\text{supp } \varphi := \text{cl } \{t \in \mathbb{R} \mid \varphi(t) \neq 0\}$$

is the support of φ and $\text{cl } M$ denotes the closure of a set $M \subseteq \mathbb{R}$.

Although the space of test function is equipped with a suitable locally convex topology, the definition of continuity in terms of this topology is rarely used. Instead, one often works with the following characterization of continuity.

Lemma 2.3 ([54], Thrm 12.7 and 14.2). *A linear map $D : \mathcal{C}_0^\infty \to \mathbb{R}$ is continuous if and only if $\lim_{n\to\infty} D(\varphi_n) = 0$ for all sequences $(\varphi_n) \in (\mathcal{C}_0^\infty)^{\mathbb{N}}$ with the following properties*

i) $\exists$ *compact* $K \subseteq \mathbb{R}\ \forall n \in \mathbb{N} : \text{supp } \varphi_n \subseteq K$, *and*

ii) $\forall m \in \mathbb{N} : \lim_{n\to\infty} \|\varphi_n^{(m)}\|_\infty = 0$

where $\|\cdot\|_\infty$ denotes the supremum norm of a bounded function.

Definition 2.4. The space of distributions is

$$\mathbb{D} := \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous}\}.$$

Definition 2.5. The space of locally integrable functions, *i.e.,* the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which the integral $\int_K |f|$ is finite for all compact sets $K \subseteq \mathbb{R}$ is given by

$$\mathcal{L}_1^{\text{loc}} := \mathcal{L}_1^{\text{loc}}(\mathbb{R} \to \mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is locally integrable}\}.$$

Distributions are sometimes referred to as generalized functions because of the following result.

Theorem 2.6 ([123], Proposition 2.1.5). *Each $f \in \mathcal{L}_1^{\text{loc}}$ induces a distribution $f_{\mathbb{D}} \in \mathbb{D}$ given by*

$$f_{\mathbb{D}} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}} \varphi(t) f(t)\, \mathrm{d}t$$

and for any $f, g \in \mathcal{L}_1^{\text{loc}}$, there is the one-to-one correspondence in the following sense

$$f_{\mathbb{D}} = g_{\mathbb{D}} \qquad \Longleftrightarrow \qquad f = g \text{ almost everywhere.}$$

Definition 2.7. The space of regular distribution is given by

$$\mathbb{D}^{\text{reg}} := \{ f_{\mathbb{D}} \mid f \in \mathcal{L}_1^{\text{loc}} \}.$$

A very important and useful property of distributions is that all distributions have a distributional derivative within $\mathbb{D}$.

Lemma 2.8 ([123] Definition 2.1.6 and Lemma 2.1.7). *The distributional derivative of $D \in \mathbb{D}$ is given by*

$$D'(\varphi) := -D(\varphi'), \quad D \in \mathbb{D}, \varphi \in \mathcal{C}_0^\infty$$

and is again a distribution. Furthermore, for differentiable $f : \mathbb{R} \to \mathbb{R}$ it holds that $(f')_{\mathbb{D}} = (f_{\mathbb{D}})'$.

Perhaps the most famous distribution which is not induced by a function is the Dirac impulse.

Definition 2.9. Let $\mathbb{1}_{[t,\infty)} : \mathbb{R} \to \mathbb{R}$ be the Heaviside step function at $t \in \mathbb{R}$, *i.e.*, $\mathbb{1}_{[t,\infty)}(\tau) = 1$ for $\tau \geqslant t$ and zero otherwise. Then the distributional derivative of the Heaviside step function is called Dirac impulse at t, denoted by δ_t, *i.e.*,

$$\delta_t := (\mathbb{1}_{[t,\infty)})'.$$

In the case $t = 0$ the index is omitted, *i.e.*, $\delta = \delta_0$.

The Dirac impulse can equivalently be defined in an alternative way. Namely as the distribution satisfying

$$\delta_t(\varphi) = \varphi(t), \quad \varphi \in \mathcal{C}_0^\infty, \forall t \in \mathbb{R},$$

which concludes the revision of classical distributions. The product of *smooth* functions and distributions is well-defined. This multiplication allows us to interpret linear differential equations as distributional differential equations. However, to do so, we will first need the following result.

Lemma 2.10 ([123] Proposition 2.1.10)**.** *Let $D \in \mathbb{D}$ be a distribution and let $\alpha \in \mathcal{C}^\infty$ be a smooth function. Then the product αD is given by*

$$\alpha D(\varphi) := D(\alpha\varphi), \quad D \in \mathbb{D},\ \alpha \in \mathcal{C}^\infty,\ \varphi \in \mathcal{C}_0^\infty$$

and is again a distribution. In particular

$$\begin{aligned} \alpha f_{\mathbb{D}} &= (\alpha f)_{\mathbb{D}}, \\ \alpha\delta^{(d)} &= \sum_{i=0}^{d} \binom{d}{i} (-1)^{d-i}\alpha^{(d-i)}(t)\delta_t^{(i)}, \quad \textit{for } d \in \mathbb{R} \end{aligned}$$

and

$$(\alpha D)' = \alpha' D + \alpha D'.$$

By interpreting constant matrices and scalars as smooth constant functions, we can interpret a linear differential equation as a distributional differential equation. This observation leads to the following definition.

Definition 2.11. A distribution $(x, u) \in (\mathbb{D})^{n+m}$ is said to solve

$$E\dot{x} = Ax + Bu \tag{2.3}$$

for some smooth constant functions E, A and B if for all $\varphi \in \mathcal{C}_0^\infty$

$$E\dot{x}(\varphi) = Ax(\varphi) + Bu(\varphi).$$

In the case that α is not smooth, then the product $\alpha\varphi$ is not smooth in general and the evaluation $D(\alpha\varphi)$ is not well-defined. Hence it is not clear whether the space of distributions can be used as a solution space for switched DAEs. In the case that the entries of the matrices $E_\sigma, A_\sigma, B_\sigma$ are regarded as piecewise-constant functions, it is tempting to think the distribution space can be used as solution space by restricting distributions to a time interval and regard the concatenation of each restricted distribution as a solution. However, the restriction of a distribution to an interval is not well-defined as the following example shows.

Example 2.12 (*cf.* [123], Lemma 2.2.3)**.** Consider the following distribution, which is well-defined:

$$D = \sum_{i \in \mathbb{N}} d_i \delta_{d_i}, \quad d_i = \frac{(-1)^i}{i+1}, \quad i \in \mathbb{N}.$$

The restriction to the interval $(0, \infty)$ should have the form

$$D_{(0,\infty)} = \sum_{k \in \mathbb{N}} \frac{1}{2k+1}\delta_{\frac{1}{2k+1}}.$$

However, for any test function $\varphi \in \mathcal{C}_0^\infty$, the series does not converge and hence the restricted distribution is not well-defined. ◇

From Example 2.12 we can conclude that the restriction of a distribution to an interval is generally not well-defined and consequently the whole space of distributions is too large to be a solution space for switched differential algebraic equation of the form (2.1). This means that we need to find a suitable subspace of $\mathbb{D}$. To that extent we introduce the following space.

Definition 2.13. The space of piecewise-smooth functions is

$$\mathcal{C}^\infty_{\mathrm{pw}} := \left\{ \alpha = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \;\middle|\; \begin{array}{l} \{t_i \in \mathbb{R} \mid i \in \mathbb{Z}\} \text{ locally finite,} \\ t_i < t_{i+1},\ (\alpha_i)_{i\in\mathbb{Z}} \in (\mathcal{C}^\infty)^{\mathbb{Z}} \end{array} \right\}.$$

Given the space of piecewise-smooth function, we can define the space of piecewise-smooth distributions.

Definition 2.14. The space of piecewise-smooth distributions is

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}},\ T \subset \mathbb{R} \text{ locally finite,} \\ \forall t \in T : D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', ...,\} \end{array} \right\}.$$

Clearly the first requirement is satisfied for this subspace of distributions. The second and third requirement also hold as follows from the next results.

Lemma 2.15 ([123] Proposition 2.3.4)**.** *For all $D \in \mathbb{D}_{pw\mathcal{C}^\infty}$ the derivative $D' \in \mathbb{D}_{pw\mathcal{C}^\infty}$.*

Lemma 2.16 ([123] Theorem 2.4.1)**.** *There exists a unique multiplication $\star : \mathbb{D}_{pw\mathcal{C}^\infty} \times \mathbb{D}_{pw\mathcal{C}^\infty} \to \mathbb{D}_{pw\mathcal{C}^\infty}$ which is distributive, compatible with scalar multiplication and satisfies*

i) $f_{\mathbb{D}} \star g_{\mathbb{D}} = (fg)_{\mathbb{D}}$ *for all* $f, g \in \mathcal{C}^\infty_{\mathrm{pw}}$,

ii) $(F \star G)' = F' \star G + F \star G'$ *for all* $F, G \in \mathbb{D}_{pw\mathcal{C}^\infty}$,

iii) $F \star (G \star H) = (F \star G) \star H$ *for all* $F, G, H \in \mathbb{D}_{pw\mathcal{C}^\infty}$.

Hence $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is a suitable solution space for switched differential algebraic equations of the form (2.1). However, in order to use the piecewise-smooth distributions in a meaningful way in the context of control theory, we need to define how to evaluate a distribution at a specific time instance. Before doing so, we state the following result, from which an intuitive definition will follow.

Lemma 2.17. *The distributions $X \in \{f_{\mathbb{D}} \mid f \in \mathcal{C}^\infty\}^n$ and $U \in \{f_{\mathbb{D}} \mid f \in \mathcal{C}^\infty\}^m$, i.e., X and U are induced by $x \in (\mathcal{C}^\infty)^n$ and $u \in (\mathcal{C}^\infty)^m$, solve* (2.3) *in the distributional sense if, and only if*

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{2.4}$$

for all $t \geqslant 0$.

Proof. After denoting $g(t) =: E\dot{x}(t) - (Ax(t) + Bu(t))$, we can write

$$\begin{aligned} EX'(\varphi) - (AX(\varphi) + BU(\varphi)) &= \int_0^\infty E\dot{x}(t)\varphi(t) - (Ax(t)\varphi(t) + Bu(t)\varphi(t))\ \mathrm{d}t \\ &= \int_0^\infty \big(E\dot{x}(t) - (Ax(t) + Bu(t))\big)\varphi(t)\ \mathrm{d}t \\ &= \int_0^\infty g(t)\varphi(t)\ \mathrm{d}t. \end{aligned} \tag{2.5}$$

If $x \in (\mathcal{C}^\infty)^n$ and $u \in (\mathcal{C}^\infty)^m$ solve (2.4) for all $t \geqslant 0$, it follows that $g(t) = 0$ for all $t \geqslant 0$. Consequently, (2.5) is zero for all $\varphi \in \mathcal{C}_0^\infty$, which implies that (X, U) solves (2.3).

Let (X, U) solve (2.4) in the distributional sense. Then (2.5) equals zero for all $\varphi \in \mathcal{C}^\infty$. Furthermore, observe that as $x, u \in \mathcal{C}^\infty$ it follows that $g \in \mathcal{C}^\infty$. Suppose that $g(t) \neq 0$ for some $t \geqslant 0$. Let t_p be such that $g(t_p) \neq 0$. Without loss of generality we can assume that $g(t_p) > 0$. By continuity there exists a $\varepsilon > 0$ such that $g(t) > \frac{1}{2}g(t_p)$ for all $t \in (t-\varepsilon, t+\varepsilon)$. Consequently, there exists a $\varphi \in \mathcal{C}^\infty$ such that (2.5) is unequal to zero, which contradicts the assumption that (X, U) solves (2.3). Hence $g(t) = 0$ for all $t \geqslant 0$, which implies that (x, u) solves (2.4). □

The result of Lemma 2.17 shows that smooth distributions do not behave much differently than classical solutions. However, a distribution $X \in \mathbb{D}^n$ generated by $x \in (\mathcal{C}^\infty)^n$, can equivalently be regarded as generated by $\bar{x} \in (\mathcal{L}_1^{\mathrm{loc}})^1$ if $x = \bar{x}$ almost everywhere. Therefore, $\bar{x}$ could also be thought as a solution in the distributional sense, although $\dot{\bar{x}}$ might not be well-defined everywhere. Consequently, more functions can be regarded as solutions by considering their induced distributions as solutions. On the other hand, information on the initial value is lost. To overcome this problem, there is a need to evaluate distributions with respect to the function that induces them.

Definition 2.18. Let $t \in \mathbb{R}$ and $D = f_{\mathbb{D}} + \sum_{\tau \in T} D_\tau$, then the left/right evaluation of D at t is given by

$$D(t^-) := f(t^-) = \lim_{\varepsilon \searrow 0} f(t-\varepsilon), \qquad D(t^+) := f(t^+) = f(t)$$

and the impulsive part of D at t is

$$D[t] = \begin{cases} D_t & t \in T, \\ 0 & t \notin T. \end{cases}$$

2.3 Existence and uniqueness of solutions

With the introduction of piecewise-smooth distributions for which a multiplication with piecewise constant coefficient matrices is well-defined, it is now possible to interpret

(2.1) as an equation within the space of piecewise-smooth distributions. The next step is to study existence and uniqueness of solutions within this distributional solution framework.

2.3.1 Regular matrix pairs

Regularity of the matrix pairs (E, A) will play a crucial roll in this investigation, which is defined as follows.

Definition 2.19. The matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called regular if $\det(sE - A)$ is not the zero polynomial.

In the case that the matrix pair (E, A) is regular, there exists a coordinate transformation that allows for a convenient expression with respect to a DAE. This transformation transforms the system in to the so called quasi-Weierstrass form [125].

Proposition 2.20 ([9] Theorem 2.6)**.** *The matrix pair (E, A) is regular if and only if there exist invertible matrices $T, S \in \mathbb{R}^{n\times n}$ such that (E, A) is transformed into the quasi-Weierstrass form (QWF):*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{2.6}$$

where $J \in \mathbb{R}^{n_1\times n_1}$, $0 \leqslant n_1 \leqslant n$ is some matrix and $N \in \mathbb{R}^{n_2\times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.

The matrices S and T that transform (E, A) into the quasi-Weierstrass form can be calculated by using the so-called *Wong sequences* [9, 137]:

$$\begin{aligned} &\mathcal{V}_0 := \mathbb{R}^n, && \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), && i = 0, 1, ... \\ &\mathcal{W}_0 := \{0\}, && \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), && i = 0, 1, ... \end{aligned} \tag{2.7}$$

The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i. \tag{2.8}$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (2.6) holds.

With a simple inductive argument it can be shown that the Wong sequences are nested and terminate, *i.e.*, there exist $i^*, j^* \leqslant n$ such that

$$\begin{aligned} \mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \cdots \supseteq \mathcal{V}^{i^*} = \mathcal{V}^{i^*+1} = \cdots \\ \mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \cdots \subseteq \mathcal{W}^{j^*} = \mathcal{W}^{j^*+1} = \cdots \end{aligned}$$

and in particular

$$\mathcal{V}^* = \mathcal{V}^{i^*} = A^{-1}(E\mathcal{V}^{i^*}) \quad \text{and} \quad \mathcal{W}^* = \mathcal{W}^{j^*} = E^{-1}(A\mathcal{W}^{j^*}).$$

Finally, it follows that

$$\ker A \subseteq \mathcal{V}^* \quad \text{and} \quad \ker E \subseteq \mathcal{W}^*.$$

Proposition 2.21 ([124] Proposition 17)**.** *Let (E, A) be a regular matrix pair. Then for any invertible matrices T, S the matrix pair (SET, SAT) is regular.*

2.3.2 Non-switched differential algebraic equations

A regular matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, an input matrix $B \in \mathbb{R}^{m\times n}$ and an input u can be associated with the differential algebraic equation

$$E\dot{x} = Ax + Bu. \tag{2.9}$$

The index of (2.9) is defined as follows.

Definition 2.22. Let (S, T) transform (E, A) into the quasi-Weierstrass form. The index of (2.9) is defined as the smallest integer $\nu \in \mathbb{N}$ for which $N^{\nu-1} = 0$.

To derive an explicit solution formula for (2.9) we define the following projectors and selectors based on the Wong sequences (2.7).

Definition 2.23. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (2.6). The *consistency projector* of (E, A) is given by

$$\Pi := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Furthermore, the *differential selector* and *impulse selector* are respectively given by

$$\Pi^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \quad \Pi^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Next, we define

$$A^{\text{diff}} := \Pi^{\text{diff}} A, \quad B^{\text{diff}} := \Pi^{\text{diff}} B, \quad E^{\text{imp}} := \Pi^{\text{imp}} E, \quad B^{\text{imp}} := \Pi^{\text{imp}} B.$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A).

Lemma 2.24. *A classical solution x solves* (2.9) *if and only if x satisfies $x = x^{\text{diff}} + x^{\text{imp}}$ where the differential component $x^{\text{diff}} := \Pi x$ solves*

$$\dot{x}^{\text{diff}} = A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u$$

and an algebraic component $x^{\text{imp}} := (I - \Pi)x$ satisfies

$$x^{\text{imp}} = -\sum_{i-1}^{\nu} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}.$$

Proof. Using the projectors and selectors resulting from the Wong sequences we can write $x = \Pi x + (I - \Pi)x$. The component $x^{\text{diff}} := \Pi x$ solves

$$\begin{aligned} \dot{x}^{\text{diff}}(t) &= \Pi \dot{x} \\ &= T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1} \dot{x} \\ &= T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S S^{-1} \left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} \dot{x} \\ &= \Pi^{\text{diff}} E \dot{x} \\ &= A^{\text{diff}} x(t) + B^{\text{diff}} u(t). \end{aligned} \tag{2.10}$$

To find a similar expression for x^{imp} it is observed that

$$\begin{aligned} E^{\text{imp}} = \Pi^{\text{imp}} E = T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S S^{-1} \left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} &= T \left[\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} \\ &= T \left[\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} = E^{\text{imp}}(I - \Pi) \end{aligned}$$

and

$$(I - \Pi) = T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} = T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S S^{-1} \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} = \Pi^{\text{imp}} A.$$

Consequently, we can write for x^{imp} the following

$$\begin{aligned} E^{\text{imp}} \dot{x}^{\text{imp}} &= E^{\text{imp}} (I - \Pi) \dot{x} \\ &= E^{\text{imp}} \dot{x} \\ &= \Pi^{\text{imp}} E \dot{x} \\ &= \Pi^{\text{imp}} A x + B^{\text{imp}} u \\ &= (I - \Pi) x + B^{\text{imp}} u \\ &= x^{\text{imp}} + B^{\text{imp}} u. \end{aligned} \tag{2.11}$$

The component x^{imp} can be expressed as an explicit function of the input and its derivatives as the operator $\left(E^{\text{imp}} \frac{\mathrm{d}}{\mathrm{d}t} - I\right)$ is invertible. Its inverse is given by $\left(E^{\text{imp}} \frac{\mathrm{d}}{\mathrm{d}t} - I\right)^{-1} = -\sum_{i=0}^{\nu-1} (E^{\text{imp}})^i \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, where $\nu \in \mathbb{N}$ is the nilpotency index of N, *i.e.*, the smallest integer value for which $(E^{\text{imp}})^{\nu} = 0$. Consequently

$$x^{\text{imp}}(t) = -\sum_{i=0}^{\nu-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t),$$

which concludes the proof. □

Corollary 2.25. *A classical solution to* (2.9) *is given by*

$$x(t) = e^{A^{\mathrm{diff}}t}\Pi c + \int_0^t e^{A^{\mathrm{diff}}(t-\tau)} B^{\mathrm{diff}} u(\tau)\, d\tau - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t), \tag{2.12}$$

for some value $c \in \mathbb{R}^n$.

Altogether this leads to the following result.

Corollary 2.26. *For every smooth input* u *there exists a smooth solution* x *to* (2.9) *which is uniquely determined by the value* $x(t_0^-)$ *for any fixed* $t_0 \in \mathbb{R}$ *if and only if the matrix pair* (E, A) *is regular.*

It follows as a direct consequence of the solution formula (2.12) that if x is smooth and solves (2.9), it satisfies

$$x(t_0) = \Pi c - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t_0).$$

In particular, we can conclude that the initial value problem (2.9) with $x(t_0) = x_0$ has a smooth solution if and only if

$$x_0 + \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t_0) \in \operatorname{im} \Pi = \mathcal{V}^*,$$

which characterizes consistency of the initial value.

Definition 2.27. Consider the DAE (2.9), then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \{x_0 \in \mathbb{R}^n \mid \exists \text{ smooth solution } x \text{ of } E\dot{x} = Ax, \text{ with } x(0) = x_0\}$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{x_0 \in \mathbb{R}^n \mid \exists \text{ smooth solutions } (x,u) \text{ of } E\dot{x} = Ax + Bu \text{ and } x(0) = x_0\right\}.$$

Proposition 2.28 ([10] Theorem 4.4)**.** *Consider the DAE* (2.9)*, then the consistency space* $\mathcal{V}_{(E,A)} = \mathcal{V}^*$ *and the augmented consistency space* $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}}\rangle$*, where* $\langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}}\rangle$ *is the largest* E^{imp} *invariant subspace generated by* $\operatorname{im} B^{\mathrm{imp}}$.

2.3.3 Inconsistent initial values

In the presence of switches, a consistent initial value cannot be assumed and a classical solution fails to exist. Implicitly it is thus assumed that the DAE (2.9) was not active before the initial time in the case of an inconsistent initial value. This gives rise to the following initial trajectory problem where $x^0 : (-\infty, t_0) \to \mathbb{R}^n$ is some initial trajectory:

$$\begin{aligned} x_{(-\infty,t_0)} &= x^0_{(-\infty,t_0)}, \\ (E\dot{x})_{[t_0,\infty)} &= (Ax + Bu)_{[t_0,\infty)}. \end{aligned} \tag{2.13}$$

If $x^0(t_0)$ is not consistent, a classical solution does not exist, however, it will be shown in the following that there exists a distributional solution. Therefore (2.13) is considered as an equation of piecewise-smooth distributions, in particular, the input u and the initial trajectory are also pieceswise smooth distributions.

Theorem 2.29 ([125] Theorem 5.1). *Let (E, A) be a regular matrix pair. Then for any initial trajectory $x^0 \in (\mathbb{D}_{pw\mathcal{C}^\infty})^n$ and any input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ the ITP* (2.13) *has a unique solution $x \in (\mathbb{D}_{pw\mathcal{C}^\infty})^n$. In particular the jump from $x^0(t_0^-)$ to $x(t_0^+)$ and the impulsive part $x[t_0]$ is uniquely determined.In the case that the input u is impulse-free, it follows that*

$$\begin{aligned} x(t_0^+) &= \Pi x^0(t_0^-) - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t_0^+), \\ x[t_0] &= -\sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^{i+1} \left(x^0(t_0^-)\delta^{(i)} + \sum_{j=0}^{i} B^{\mathrm{imp}} u^{(i-j)}(t_0^+)\delta^{(j)} \right) \\ &= -\sum_{i=0}^{\nu-1} \left(E^{\mathrm{imp}}\right)^{i+1} \left(x^0(t_0^-) - x(t_0^+) \right) \delta^{(i)} \end{aligned} \tag{2.14}$$

and for $t \in (t_0, \infty)$

$$x(t^-) = e^{A^{\mathrm{diff}} t} \Pi x^0(t_0^-) + \int_{t_0}^{t} e^{A^{\mathrm{diff}}(t-\tau)} B^{\mathrm{diff}} u(\tau)\, \mathrm{d}\tau - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t^-), \tag{2.15}$$

where Π is the consistency projector and $E^{\mathrm{imp}} = \Pi^{\mathrm{imp}} E$ with the impulse selector Π^{imp} as in Definition 2.23

Remark 2.30. As follows from the result in Theorem 2.29, the solution to the ITP (2.13) is uniquely determined by the input and the initial value $x^0(t_0^-)$. Consequently, the initial value problem

$$E\dot{x} = Ax + Bu, \qquad x(t_0^-) = x_0,$$

can be equivalently be considered, with the implicit assumption that the DAE is active only on $[t_0, \infty)$ and uniqueness of x is only considered on that interval.

As follows from the formula (2.14), the solution of an ITP (2.13) generally contains jumps and Dirac impulses, even in the case $u = 0$. For many applications, it is of interest to avoid Dirac impulses. For some initial trajectories, Dirac impulses can be avoided by means of a suitable input, but in general this is not possible for all initial trajectories.

Lemma 2.31. *There exists a solution of* (2.9) *that is impulse-free at* t_0*, if and only if the initial trajectory satisfies*

$$(I-\Pi)x^0(t_0^-) \in \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle + \ker E.$$

Proof. Assume that x is impulse-free at t_0 and solves (2.9). Then it follows from (2.14) that

$$(I-\Pi)\left(x^0(t_0^-) - x(t_0^+)\right) \in \ker (E^{\text{imp}})^i,$$

for $i \in \mathbb{N}$. Observe that

$$\begin{aligned}\ker E^{\text{imp}} = \ker T \left[\begin{smallmatrix} 0 & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} = \ker T \left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right] T^{-1} + \ker T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] T^{-1} \\ = \ker TSE + \operatorname{im} \Pi = \ker E + \operatorname{im} \Pi.\end{aligned}$$

Consequently, since $\operatorname{im} \Pi \cap \operatorname{im}(I-\Pi) = 0$, we can conclude that

$$\begin{aligned}(I-\Pi)x^0(t_0^-) &\in \ker E + (I-\Pi)x(t_0^+) \\ &\in \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle + \ker E.\end{aligned}$$

Conversely, assume that $(I-\Pi)x^0(t_0^-) \in \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle + \ker E$. Then $(I-\Pi)x^0(t_0^-) = v + w$ for some $w \in \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle$ and $w \in \ker E$. Observe that $v \in \ker E \subseteq \ker E^{\text{imp}}$. Hence for any u satisfying

$$\sum_{i=0}^{\nu-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t_0^+) = w,$$

it follows that $(I-\Pi)x(t_0^+) = -w$ and hence $x[t_0] = 0$. □

The result of Lemma 2.31 gives rise to the so called impulse-controllable space.

Definition 2.32. Consider the DAE (2.9), then the *impulse-controllable space* is defined as

$$\mathcal{C}^{\text{imp}} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \text{ of (2.13)} \\ \text{s.t. } x(0^-) = x_0 \text{ and } (x,u)[0] = 0. \end{array} \right\}.$$

Proposition 2.33 ([100] Proposition 1)**.** *Consider the DAE* (2.9)*, then the impulse-controllable space satisfies*

$$\mathcal{C}^{\text{imp}} = \mathcal{V}_{(E,A,B)} + \ker E.$$

Viewing (2.1) as a repeated ITP (where the switching times are the initial times), one obtains the following result regarding the existence and uniqueness of solutions of (2.1).

Theorem 2.34 ([125] Corollary 5.2)**.** *Consider the switched DAE* (2.1) *with a switching signal* $\sigma \in \mathcal{S}_{\text{n}}$ *and regular matrix pairs* $(E_p, A_p) \in \mathbb{R}^{n\times n}$*. Then there exists a globally defined solution* $x \in (\mathbb{D}_{pw\mathcal{C}^\infty})^n$ *to*

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \qquad x(t_0^-) = x_0,$$

which is uniquely determined by the value $x(t_0^-)$ *and the input* $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$*.*

2.4 Control theoretical concepts for DAEs

As each mode of (2.1) can be interpreted as a non-switched DAE which is active for some time, we review some important controllability properties of non-switched DAEs. To that extent consider the DAE

$$E\dot{x} = Ax + Bu \tag{2.16}$$

and assume that the matrix pair (E, A) is regular. Concentrating on the relation between x and u, the amount to which the solution x can be influenced by means of a suitable choice of input u gives rise to the reachable space defined as follows.

Definition 2.35. The reachable space of the regular DAE (2.16) is defined as

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \; \exists \text{ smooth solution } (x, u) \text{ of (2.16)} \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}.$$

It is easily seen that the reachable space for (2.16) coincides with the controllable space, i.e. the space of initial values which can be controlled to zero in a smooth manner.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \; \exists \text{ smooth solution } (x, u) \text{ of (2.16)} \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Given the reachable space, we can decompose a solution into a part resulting from the initial value and a part resulting from the input.

Lemma 2.36. *Consider the DAE* (2.16)*. Any solution (x, u) with $x(t_0^-) = x_0$ satisfies for each $t \in (t_0, \infty)$*

$$x(t^-) = e^{A^{\text{diff}} t} \Pi x_0 + \eta,$$

for some $\eta \in \mathcal{R}$.

Proof. Considering the solution formula (2.15) in relation to the reachable space, we can conclude that for all t

$$\int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau) \, \mathrm{d}\tau - \sum_{i=0}^{\nu-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t^-) := \eta \in \mathcal{R}.$$

Hence the result follows. □

The reachable space of (2.16) can be characterized as follows.

Lemma 2.37 ([10] Corollary 4.5)**.** *Consider the regular DAE* (2.16)*. Then the reachable space is given by*

$$\mathcal{R} = \langle A^{\text{diff}} \mid \operatorname{im} B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid \operatorname{im} B^{\text{imp}} \rangle.$$

Given the result of Lemma 2.37, we can express the augmented consistency space and the impulse-controllable space in terms of the reachable space.

Corollary 2.38. *The augmented consistency space of* (2.9) *satisfies*

$$\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} \oplus \langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}} \rangle = \mathcal{V}_{(E,A)} + \mathcal{R}.$$

Proof. Since $\operatorname{im} A^{\mathrm{diff}} \subseteq \mathcal{V}_{(E,A)}$ and $\operatorname{im} B^{\mathrm{diff}} \in \mathcal{V}_{(E,A)}$ it follows that $\langle A^{\mathrm{diff}} \mid \operatorname{im} B^{\mathrm{diff}} \rangle \subseteq \mathcal{V}_{(E,A)}$. Consequently

$$\begin{aligned}\mathcal{V}_{(E,A,B)} &= \mathcal{V}_{(E,A)} \oplus \langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}} \rangle \\ &= \mathcal{V}_{(E,A)} \oplus \langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}} \rangle + \langle A^{\mathrm{diff}} \mid \operatorname{im} B^{\mathrm{diff}} \rangle = \mathcal{V}_{(E,A)} + \mathcal{R}.\end{aligned}$$

where the first equality follows from Proposition 2.28. □

Corollary 2.39. *The impulse-controllable space of* (2.9) *satisfies*

$$\begin{aligned}\mathcal{C}^{\mathrm{imp}} &= \mathcal{V}_{(E,A,B)} + \ker E \\ &= \mathcal{V}_{(E,A)} + \langle E^{\mathrm{imp}} \mid \operatorname{im} B^{\mathrm{imp}} \rangle + \ker E \\ &= \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E.\end{aligned}$$

As the solutions of (2.16) are generally confined to a subspace, we can distinct the following definitions of controllability.

Definition 2.40. The DAE (2.16) is called

$i)$ completely controllable if $\mathcal{R} = \mathbb{R}^n$,

$ii)$ behaviorally controllable if $\mathcal{R} = \mathcal{V}_{(E,A,B)}$.

The following definition deals with the avoidance of Dirac impulses in the state trajectory.

Lemma 2.41 ([22] Proposition 3)**.** *The regular DAE* (2.9) *is impulse controllable if and only if*

$i)$ $\mathcal{C}^{\mathrm{imp}} = \mathbb{R}^n$,

$ii)$ $\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n$,

$iii)$ *there exists a matrix* K *such that* $\operatorname{im} E + (A + BK) \ker E = \mathbb{R}^n$.

Besides the coordinate transformation that puts a regular matrix pair (E, A) in the quasi-Weiserstrass form, there exists for every matrix triplet (E, A, B) a coordinate transformation that puts the matrices in the Quasi-Weierstrass Form-Kalman decomposition for differential algebraic equations.

Proposition 2.42 ([10] Proposition 4.2)**.** *For a regular matrix pair (E, A) there exist invertible matrices $T, S \in \mathbb{R}^{n\times n}$ such that (E, A) is transformed into*

$$(SET, SAT, SB) = \left(\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{22} \end{bmatrix}, \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & J_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \begin{bmatrix} B_{11} \\ 0 \\ B_{12} \\ 0 \end{bmatrix} \right), \tag{2.17}$$

where $\left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right], \left[\begin{smallmatrix} J_{11} & 0 \\ 0 & I \end{smallmatrix}\right], \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right]\right)$ is completely controllable and N_{11} and N_{22} are nilpotent.

Corollary 2.43. *Consider the matrices (E, A, B) in the form* (2.17)*. Then*

$$\mathcal{C}^{\text{imp}} = \operatorname{im} \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & V \end{smallmatrix}\right],$$

where V is any matrix with $\operatorname{im} V = \ker N_{12} \cap N_{22}$.

Proof. Since $\left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right], \left[\begin{smallmatrix} J_{11} & 0 \\ 0 & I \end{smallmatrix}\right], \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right]\right)$ is completely controllable, it follows that $\mathcal{R} = \operatorname{im} \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$. Furthermore, $\mathcal{V}_{(E,A)} = \operatorname{im} \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right]$. Then by Corollary 2.39 we obtain

$$\mathcal{C}^{\text{imp}} = \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E = \operatorname{im} \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] + \ker E = \operatorname{im} \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & V \end{smallmatrix}\right],$$

which proves the result. □

By transforming the matrices (E, A, B) into the form (2.17) we can prove the following results regarding solutions of a DAE satisfying $x(t_0^-) = x_0 \in \mathcal{C}^{\text{imp}}$.

Lemma 2.44. *Consider the matrix pair (E, A, B) and let T, S matrices that transform (E, A, B) into the form* (2.17)*. Then the matrix pair (EW, A) where*

$$W = T \left[\begin{smallmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right] T^{-1}, \tag{2.18}$$

is regular.

Proof. By Proposition 2.21 the matrix pair (E, A) is regular, then (SET, SAT) is regular. Consequently $\det(\lambda SET - SAT) \neq 0$ and hence we have

$$\begin{aligned} \det(\lambda SET - SAT) &= \det\left(\lambda I - \left[\begin{smallmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{smallmatrix}\right]\right) \det\left(\lambda \left[\begin{smallmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{smallmatrix}\right] - I\right) \\ &= \det\left(\lambda I - \left[\begin{smallmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{smallmatrix}\right]\right) \det(\lambda N_{11} - I) \det(\lambda N_{22} - I) \\ &\neq 0 \end{aligned}$$

and hence $\det\left(\lambda I - \left[\begin{smallmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{smallmatrix}\right]\right) \det(\lambda N_{11} - I) \neq 0$. As T, S are invertible, it follows that (EW, A) is regluar if and only if $(SEWT, SAT) = SETT^{-1}WT, SAT)$ is regular and thus we compute

$$\begin{aligned}
\det(\lambda SETT^{-1}WT - SAT) &= \det\left(\lambda \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & J_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\right) \\
&= \det\left(\lambda I - \left[\begin{smallmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{smallmatrix}\right]\right) \det\left(\lambda \left[\begin{smallmatrix} N_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] - I\right) \\
&= -\det\left(\lambda I - \left[\begin{smallmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{smallmatrix}\right]\right) \det(\lambda N_{11} - I) \\
&\neq 0.
\end{aligned}$$

This proves regularity of (EW, A). □

Lemma 2.45. *Consider* (2.16) *and assume it is regular. The pair* (x, u) *with* $x(t_0^-) = x_0 \in \mathcal{C}^{\mathrm{imp}}$ *solves* (2.16) *if and only if it solves*

$$EW\dot{x} = Ax + Bu, \tag{2.19}$$

where W *is given by* (2.18)*. Furthermore,* (2.19) *is impulse controllable.*

Proof. Without loss of generality we can assume that the matrices (E, A, B) are already in the form (2.17). Then we can decompose $x = \left[\, x_1^\top \; x_2^\top \; x_3^\top \; x_4^\top \,\right]^\top$. By Theorem 2.29 it follows that

$$x_4 = -\sum_{i=0}^{\nu-1} N_{22}^{i+1} x_4(t_0^-)\delta^{(i)}.$$

Since $x_0 \in \mathcal{C}^{\mathrm{imp}}$ it follows from Corollary 2.43 that $x_4(t_0^-) \in \operatorname{im} V = \ker N_{12} \cap \ker N_{22}$ and hence $x_4 = 0$ on $[t_0, \infty)$. Consequently $\dot{x}_4 = 0$ on (t_0, ∞) and hence $E\dot{x} = EW\dot{x}$ on (t_0, ∞). However, $x_4(t_0^-) \in \ker N_{12} \cap N_{22}$ it follows that at t_0

$$E\dot{x}[t_0] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1[t_0] \\ \dot{x}_2[t_0] \\ \dot{x}_3[t_0] \\ \dot{x}_4[t_0] \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1[t_0] \\ \dot{x}_2[t_0] \\ \dot{x}_3[t_0] \\ \dot{x}_4[t_0] \end{bmatrix} = EW\dot{x}[t_0],$$

from which we can conclude that $E\dot{x} = EW\dot{x}$ on $[t_0, \infty)$.

($\Leftarrow$) Conversely, assume that x solves $EW\dot{x} = Ax + Bu$. Then x solves

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{22} \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & J_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} x + \begin{bmatrix} B_{11} \\ 0 \\ B_{12} \\ 0 \end{bmatrix} u$$

and we can conclude that $x_4 = 0$ on $[t_0, \infty)$ and thus $\dot{x}_4 = 0$ on (t_0, ∞). Therefore x solves

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{22} \end{bmatrix} \dot{x} = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & J_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} x + \begin{bmatrix} B_{11} \\ 0 \\ B_{12} \\ 0 \end{bmatrix} u,$$

on (t_0, ∞). Furthermore, at t_0 we can conclude that $x_4(t_0^-) \in \ker N_{12} \cap \ker N_{22}$ as $x_0 \in \mathcal{C}^{\mathrm{imp}}$. Consequently, we obtain

$$EW\dot{x}[t_0] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1[t_0] \\ \dot{x}_2[t_0] \\ \dot{x}_3[t_0] \\ \dot{x}_4[t_0] \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & 0 & N_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1[t_0] \\ \dot{x}_2[t_0] \\ \dot{x}_3[t_0] \\ \dot{x}_4[t_0] \end{bmatrix} = E\dot{x}[t_0],$$

from which we can conclude that $EW\dot{x} = E\dot{x}$ on $[t_0, \infty)$.

It remains to show that (2.19) is impulse-controllable. Let n_1, n_2, n_3 and n_4 correspond to the sizes of the blocks in (2.17). Then $n_1+n_2+n_3+n_4 = n$. Since $\left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right], \left[\begin{smallmatrix} J_{11} & 0 \\ 0 & I \end{smallmatrix}\right], \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right] \right)$ is completely controllable, we can conclude that

$$\mathrm{im} \left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right] + \left[\begin{smallmatrix} J_{11} & 0 \\ 0 & I \end{smallmatrix}\right] \ker \left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right] + \mathrm{im} \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right] = \mathbb{R}^{n_1+n_3}.$$

Hence we can conclude that

$$\begin{aligned} \dim(\mathrm{im}\, EW + A \ker EW + \mathrm{im}\, B) &= \dim \left(\mathrm{im} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & J_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \ker \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & N_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right. \\ &\qquad \left. + \,\mathrm{im} \begin{bmatrix} B_{11} \\ 0 \\ B_{12} \\ 0 \end{bmatrix} \right) \\ &= \dim \left(\mathrm{im} \left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right] + \left[\begin{smallmatrix} J_{11} & 0 \\ 0 & I \end{smallmatrix}\right] \ker \left[\begin{smallmatrix} I & 0 \\ 0 & N_{11} \end{smallmatrix}\right] + \mathrm{im} \left[\begin{smallmatrix} B_{11} \\ B_{12} \end{smallmatrix}\right] \right) \\ &\qquad + \dim \left(\mathrm{im} \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{n_4} \end{bmatrix} \right) \\ &= n_1 + n_2 + n_3 + n_4 \\ &= n \end{aligned}$$

and hence we can conclude

$$\mathrm{im}\, EW + A \ker EW + \mathrm{im}\, B = \mathbb{R}^n.$$

Therefore (2.19) is impulse-controllable. □

We conclude this chapter with a geometric control concept for DAEs, namely controlled invariant subspaces.

Definition 2.46. Consider the DAE (2.9). A subspace $\mathcal{V}$ is called (E, A, B) invariant, or controlled invariant if for all $x_0 \in \mathcal{V}$ there exists an input u such that the solution (x, u) with $x(t_0^-) = x_0$ satisfies $x(t^-) \in \mathcal{V}$ for all $t \geqslant t_0$.

Lemma 2.47 ([74] Theorem 10)**.** *Let $\mathcal{W} \subseteq \mathbb{R}^n$ be a subspace. There exists a largest (E, A, B) invariant subspace contained in $\mathcal{W}$, denoted by $\langle \mathcal{W} \mid E, A, B \rangle$, in the sense that if $\mathcal{V}$ is some (E, A, B) invariant and contained in $\mathcal{W}$, then $\mathcal{V} \subseteq \langle \mathcal{W} \mid E, A, B \rangle$. Furthermore, $\langle \mathcal{W} \mid E, A, B \rangle$ equals the limit of the sequence*

$$\mathcal{V}_0 = \mathbb{R}^n, \tag{2.20}$$

$$\mathcal{V}_{i+1} = \mathcal{W} \cap A^{-1}(E\mathcal{V}_i + \mathrm{im}\, B). \tag{2.21}$$

Lemma 2.48. *Let $\mathcal{V}$ be an (E, A, B) controlled invariant subspace. Then*

$$A^{\text{diff}}\mathcal{V} \subseteq \mathcal{V} + \mathcal{R}.$$

Proof. Since $\mathcal{V}$ is controlled invariant, there exists for every $x_0 \in \mathcal{V}$ an input u such that the solution (x, u) with $x(t_0^-) = x_0 \in \mathcal{V}$ satisfies $x(t^-) \in \mathcal{V}$ for all $t > t_0$. Consequently, it follows from Lemma 2.36 that for each t there exists a $\eta \in \mathcal{R}$ such that

$$x(t^-) = e^{A^{\text{diff}}t}\Pi x_0 + \eta \in \mathcal{V} + \mathcal{R},$$

for all $t > t_0$. Hence $e^{A^{\text{diff}}t}\Pi x_0 \in \mathcal{V} + \mathcal{R}$ for all $t \geqslant t_0$ and thus it follows from Lemma A.1 that

$$\Pi x_0 \in \langle \mathcal{V} + \mathcal{R} \mid A^{\text{diff}} \rangle \subseteq \mathcal{V} + \mathcal{R}.$$

Since $A^{\text{diff}} = A^{\text{diff}}\Pi$, we can write

$$A^{\text{diff}}x_0 = A^{\text{diff}}\Pi x_0 \subseteq A^{\text{diff}}\langle \mathcal{V} + \mathcal{R} \mid A^{\text{diff}} \rangle \subseteq \langle \mathcal{V} + \mathcal{R} \mid A^{\text{diff}} \rangle \subseteq \mathcal{V} + \mathcal{R}.$$

Since $x_0 \in \mathcal{V}$ was arbitrary, the result follows. $\square$

Corollary 2.49. *Let $\mathcal{V}$ be an (E, A, B) invariant subspace. Then $\Pi\mathcal{V} \subseteq \langle \mathcal{V} + \mathcal{R} \mid A^{\text{diff}} \rangle$.*

3 | Switched DAEs and impulses

In this chapter we investigate the existence of impulse-free solutions of switched differential algebraic equations, *i.e.*, existence of solutions without the occurrence of Dirac impulses. In many engineering applications it is of utter importance that Dirac impulses in the state are avoided as they can damage components in the system or cause hazardous situations. First we will investigate impulse-controllability for switched differential algebraic equations with a known switching signal. Then we will investigate the dependence of impulse-controllability on the switching times, by considering impulse-controllability of system classes.

3.1 Impulse-controllability of DAEs

In this section we will introduce the concept of impulse-controllability of switched differential algebraic equations. Again consider the switched differential algebraic equation

$$E_\sigma x = A_\sigma x + B_\sigma u, \tag{3.1}$$

with a switching signal $\sigma \in \mathcal{S}_\mathrm{n}$ that induces finitely many mode changes. Hence we assume that the switching times $t_1 < t_2 < ... < t_\mathtt{n}$ are known and that mode $\mathtt{n}$ is active on $[t_\mathtt{n}, \infty)$. We will consider (3.1) on the interval $[t_0, \infty)$ for some $t_0 \in (-\infty, t_1)$, but assume that mode 0 was already active on $(-\infty, t_0)$. Consequently, any initial trajectory will satisfy $x(t_0^-) = x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. In order to define impulse-controllability we will consider the behavior of (3.1) defined as follows.

Definition 3.1. Consider the system (3.1) for some switching signal $\sigma \in \mathcal{S}_\mathrm{n}$. The behavior $\mathfrak{B}_\sigma$ is the set of all distribution pairs (x, u) that solve (4.3), *i.e.*,

$$\mathfrak{B}_\sigma := \{(x, u) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n+m} \mid E_\sigma \dot{x} = A_\sigma x + B_\sigma u\}.$$

The system (3.1) will roughly speaking be called impulse-controllable if it has the following property: for all initial trajectories there exists an impulse-free input $u \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ such that the resulting state trajectory is impulse-free for all time. Since we assume that the initial trajectory is a solution of (3.1) we will speak of *behavioral impulse-controllability*.

Definition 3.2. The switched DAE (4.3) with a switching signal $\sigma \in \mathcal{S}_\mathrm{n}$ is *behaviorally* *impulse-controllable* if the corresponding solution behavior $\mathfrak{B}_\sigma$ is impulse-controllable, *i.e.*

$$\begin{aligned}
&\forall (x,u) \in \mathfrak{B}_\sigma\ \exists (x^*,u^*) \in \mathfrak{B}_\sigma :\\
&(x^*,u^*)_{(-\infty,t_0)} = (x,u)_{(-\infty,t_0)},\\
&\text{and } (x^*,u^*)[t] = 0,\ \forall t > t_0.
\end{aligned}$$

Remark 3.3. The solution (x^*,u^*) in Definition 3.2 is allowed to contain Dirac impulses as long as they occur at $t < t_0$. These Dirac impulses can be induced by *e.g.*, discontinuities or even Dirac impulses in the input u. However, if there exists a solution (x^*,u^*) which is impulse-free on $[t_0,\infty)$ there also exists a solution $(\bar{x},\bar{u})$ which is impulse-free on the entire interval $(-\infty,\infty)$ and satisfies $(\bar{x},\bar{u})_{[t_0,\infty)} = (x^*,u^*)_{[t_0,\infty)}$.

As already mentioned, any initial trajectory satisfies $x(t_0^-) \in \mathcal{V}_{(E_0,A_0,B_0)}$. In fact, any initial trajectory satisfies $x(t^-) \in \mathcal{V}_{(E_0,A_0,B_0)}$ for $t \in (-\infty,t_0)$. This allows for defining impulse-controllability of (3.1) equivalently in terms of the initial values as follows.

Definition 3.4. The switched DAE (3.1) with a switching signal $\sigma \in \mathcal{S}_\mathrm{n}$ is called *impulse-controllable* on $[t_0,\infty)$, if for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an impulse-free solution $(x,u) \in \mathbb{D}^{n+m}_{\mathrm{pw}\mathcal{C}^\infty}$ of (3.1) with $x(t_0^-) = x_0$.

As an alternative for Definition 3.2 and 3.4, impulse-controllability could also be defined in terms of arbitrary initial trajectories or initial values $x_0 \in \mathbb{R}^n$. This would result in the immediate necessary condition that the first mode of a switched differential algebraic equation is impulse-controllable. However, the only interesting initial values with respect to the avoidance of Dirac impulses are those contained in $\mathcal{V}_{(E_0,A_0,B_0)}$. Indeed, for initial values $x_0 \in \ker E$, the trajectory will jump to the augmented consistency space and for initial values $x_0 \notin \mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0$ a Dirac impulse occurs inevitably.

As a consequence of defining impulse-controllability in the behavioral sense or in terms of $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$, all switched differential algebraic equations with a constant switching signal, *i.e.*, non-switched differential algebraic equations, are impulse-controllable according to Definition 3.4, due to the definition of the augmented consistency space in terms of smooth (in particular, impulse-free) solutions. This seems counter intuitive, because the active mode on that interval is not necessarily impulse-controllable; however, recall that impulse-controllability for a single mode governed by matrices E, A, B is formulated in terms of an initial trajectory problem (ITP), which can be interpreted as a switched system with one switch at $t_1 = 0$. In fact, letting $t_0 = -\varepsilon$, $t_f = \varepsilon$, $(E_0,A_0,B_0) = (I,0,0)$ and $(E_1,A_1,B_1) = (E,A,B)$, the DAE $E\dot{x} = Ax + Bu$ is impulse-controllable if, and only if, the corresponding ITP (reinterpreted as a switched DAE) is impulse-controllable on $(-\varepsilon,\varepsilon)$.

3.1.1 A backward approach

In this section we will investigate impulse-controllability of a switched DAE (3.1) with a switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$, which is assumed to be known a priori. Hence we assume that (3.1) has $\mathtt{n} \in \mathbb{N}$ modes and both the switching times and the switching sequence are known. Clearly, impulse-controllability of *each* mode is a sufficient condition for impulse-controllability of the overall switched DAE, however, the following example shows that this is in fact not necessary.

Example 3.5. Consider the switched DAE

$$\Sigma_\sigma : \begin{cases} \left[\begin{smallmatrix} 1&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right] \dot{x}(t) = x(t) + \left[\begin{smallmatrix} 0\\0\\1 \end{smallmatrix}\right] u(t), & 0 \leqslant t < t_1, \\ \left[\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&1 \end{smallmatrix}\right] \dot{x}(t) = x(t) + \left[\begin{smallmatrix} 0\\0\\1 \end{smallmatrix}\right] u(t), & t_1 \leqslant t. \end{cases}$$

The first mode in the example is impulse-controllable, whereas the second mode is not. Observe that as

$$\mathcal{C}_1^{\mathrm{imp}} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1\\0\\0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0\\0\\1 \end{smallmatrix}\right] \right\},$$

an impulse-free solution x has to satisfy $x_2(t_1^-) = 0$. However, as

$$\mathcal{R}_0 = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1\\0\\0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0\\1\\0 \end{smallmatrix}\right] \right\},$$

the state component x_2 can be controlled to zero in arbitrary time. Hence for any initial value $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an input such that the resulting trajectory is impulse-free and thus the system is impulse-controllable. $\diamond$

The reason that the system in Example 3.13 is impulse-controllable although not every mode is impulse-controllable, is that every state that possibly results in a Dirac impulse at the switch, can be controlled to zero before the switch occurs. Alternatively formulated, the impulse-controllable space of the final mode can be reached from every initial value in the augmented consistency space before the switch. Motivated by this observation, we will investigate the largest subspace from which the impulse-controllable space of the final mode can be reached without the occurrence of Dirac impulses. To that extent, we define the following sequence of subspaces which can be associated with the switched differential algebraic equation (3.1) for which the switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$:

$$\begin{aligned} \mathcal{K}_{\mathtt{n}}^{\tau} &= \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}, \\ \mathcal{K}_{i-1}^{\tau} &= \operatorname{im}\Pi_{i-1} \cap \left(e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \mathcal{K}_i^{\tau} + \mathcal{R}_{i-1} \right) + \langle E_{i-1}^{\mathrm{imp}} \mid \operatorname{im} B_{i-1}^{\mathrm{imp}} \rangle + \ker E_{i-1} \\ & \quad i = \mathtt{n}, \mathtt{n}-1, \dots, 1. \end{aligned} \tag{3.2}$$

Remark 3.6. Recall that $\operatorname{im}\Pi_i = \mathcal{V}_{(E_i,A_i)}$ and that $\mathcal{C}_i^{\mathrm{imp}} = \mathcal{V}_{(E_i,A_i)} + \langle E_i^{\mathrm{imp}} \mid \operatorname{im} B_i^{\mathrm{imp}}\rangle + \ker E_i$. Therefore we have that $\mathcal{K}_i^{\boldsymbol{\tau}} \subseteq \mathcal{C}_i^{\mathrm{imp}}$.

Each space $\mathcal{K}_{i-1}^{\boldsymbol{\tau}}$ is the largest space containing the initial values from which $\mathcal{K}_i^{\boldsymbol{\tau}}$ can be reached given the dynamics of (3.1) restricted to the interval $[t_{i-1}, t_i)$. In particular, we can prove the following result.

Lemma 3.7. *Consider the DAE* (3.1) *restricted to the interval* $[t_{i-1}, t_i)$. *Then there exists a local solution* (x, u) *that is impulse-free on* $[t_{i-1}, t_i)$ *and solves*

$$(E_\sigma \dot{x})_{[t_{i-1},t_i)} = (A_\sigma x + B_\sigma u)_{[t_{i-1},t_i)}, \qquad x(t_{i-1}^-) = x_0$$

and satisfies $x(t_i^-) \in \mathcal{K}_i^{\boldsymbol{\tau}}$ *if and only if* $x_0 \in \mathcal{K}_{i-1}^{\boldsymbol{\tau}}$.

Proof. $(\Rightarrow)$ Assume there exists an impulse-free solution x that satisfies $x(t_i^-) \in \mathcal{K}_i^{\boldsymbol{\tau}}$. The dynamics of (3.1) restricted to the interval $[t_{i-1}, t_i)$ are governed by

$$E_{i-1}\dot{x} = A_{i-1}x + B_{i-1}u.$$

Consequently, for some $\eta_{i-1} \in \mathcal{R}_{i-1}$ and some $k_i \in \mathcal{K}_i^{\boldsymbol{\tau}}$ we can write

$$x(t_i^-) = e^{A_{i-1}^{\mathrm{diff}}\tau_{i-1}}\Pi_{i-1}x_0 + \eta_{i-1} = k_i \in \mathcal{K}_i^{\boldsymbol{\tau}}, \tag{3.3}$$

which after rearranging terms yields

$$\Pi_{i-1}x_0 \in e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}}\mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1}. \tag{3.4}$$

As x is impulse-free, in particular at t_0, the result of Lemma 2.31 and (3.4) lead to

$$\begin{aligned} x_0 &= \Pi_{i-1}x_0 + (I - \Pi_{i-1})x_0 \\ &\in \operatorname{im}\Pi_{i-1} \cap (e^{-A_{i-1}^{\mathrm{diff}} i\tau_{i-1}}\mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1}) + \langle E_{i-1}^{\mathrm{imp}} \mid \operatorname{im} B_{i-1}^{\mathrm{imp}}\rangle + \ker E_{i-1} \\ &= \mathcal{K}_{i-1}^{\boldsymbol{\tau}}. \end{aligned}$$

Conversely, let $x_0 \in \mathcal{K}_{i-1}^{\boldsymbol{\tau}}$. We will construct an input u that is impulse-free on $[t_{i-1}, t_i)$ that results in a trajectory x which is impulse-free on $[t_{i-1}, t_i)$ too and satisfies $x(t_{i-1}^-) = x_0$ and $x(t_i^-) \in \mathcal{K}_i^{\boldsymbol{\tau}}$. To that extent, we regard a solution x to (3.3) as the sum of two solutions, *i.e,* $x = x_1 + x_2$ where $x_1(t_{i-1}^-) = \Pi_{i-1}x_0$ and $x_2(t_{i-1}^-) = (I - \Pi_{i-1})x_0$. Observe that $x_0 \in \mathcal{K}_{i-1}^{\boldsymbol{\tau}}$ implies

$$x_1(t_{i-1}^-) = \Pi_{i-1}x_0 \in \Pi_{i-1}\mathcal{K}_{i-1}^{\boldsymbol{\tau}} = \operatorname{im}\Pi_{i-1} \cap \left(e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}}\mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1}\right)$$

and

$$x_2(t_{i-1}^-) = (I - \Pi_{i-1})x_0 \in (I - \Pi_{i-1})\mathcal{K}_{i-1}^{\boldsymbol{\tau}} = \langle E_{i-1}^{\mathrm{imp}} \mid \operatorname{im} B_{i-1}^{\mathrm{imp}}\rangle + \ker E_{i-1}.$$

In particular, it follows that

$$e^{A_{i-1}^{\text{diff}}\tau_{i-1}}\Pi_{i-1}x_0 = k_i + \eta_{i-1}$$

and

$$x_2(t_{i-1}^-) = \bar{\eta}_{i-1} + e_{i-1},$$

for some $\eta_{i-1} \in \mathcal{R}_{i-1}$, $k_i \in \mathcal{K}_i^{\tau}$, $\bar{\eta}_{i-1} \in \langle E_{i-1}^{\text{imp}} \mid \operatorname{im} B_{i-1}^{\text{imp}} \rangle$ and $e_{i-1} \in \ker E_{i-1}$.

As $x_1(t_{i-1}^-) \in \operatorname{im} \Pi_{i-1}$ the solution is impulse-free for any smooth input. Let u_1 be a smooth input that steers the origin to the vector $-\eta_{i-1}$. The resulting trajectory x_1 is impulse-free on $[t_{i-1}, t_i)$then satisfies $x_1(t_i^-) = k_i$. Next, let u_2 be smooth solution such that

$$x_2(t_{i-1}^+) = \sum_{j=0}^{\nu_i - 1} \left(E_{i-1}^{\text{imp}}\right)^j B_{i-1}^{\text{imp}} u_2^{(j)}(t_{i-1}^+) = \bar{\eta}_{i-1}$$

and $u_2^{(j)}(t_i^-) = 0$ for $j = 0, 1, ..., \nu_i - 1$. Then it follows that x_2 is impulse-free on $[t_{i-1}, t_i)$ and satisfies $x_2(t_i^-) = 0$. Consequently the input $u := u_1 + u_2$ results in an impulse-free trajectory satisfying

$$x(t_i^-) = x_1(t_i^-) + x_2(t_i^-) = k_i \in \mathcal{K}_i^{\tau},$$

which concludes the proof. □

The result of Lemma 3.7 can be used inductively to show that each $\mathcal{K}_{i-1}^{\tau}$ space is in fact the largest set containing initial values from for which there exist impulse-free solutions to (3.1) restricted to $[t_{i-1}, \infty)$. Although this is conceptually rather intuitive, we provide a proof for the sake of completeness.

Corollary 3.8. *Consider the switched differential algebraic equation* (3.1) *restricted to the interval* $[t_{\mathtt{n}-i}, \infty)$ *for some* $i \in \{0, 1, ..., \mathtt{n}\}$. *There exists a solution* (x, u) *that is impulse-free on* $[t_{\mathtt{n}-i}, \infty)$ *and solves*

$$(E_\sigma \dot{x})_{[t_{\mathtt{n}-i}, \infty)} = (A_\sigma x + B_\sigma u)_{[t_{\mathtt{n}-i}, \infty)}, \qquad x(t_{\mathtt{n}-i}^-) = x_0, \tag{3.5}$$

if and only if $x_0 \in \mathcal{K}_{\mathtt{n}-i}^{\tau}$.

Proof. ($\Rightarrow$) For $i = 0$ it follows that $x_0 \in \mathcal{C}_{\mathtt{n}}^{\text{imp}}$ and hence there exists an impulse-free trajectory on $[t_{\mathtt{n}}, \infty)$ as no more switches occur. Hence we can assume that the statement holds for i. Together with Corollary 3.8 this means that we need to show that there exists an impulse-free solution x satisfying $x(t_{i-1}^-) = x_0$ and $x(t_i^-) \in \mathcal{K}_i^{\tau}$. By Lemma 3.7 this is the cases if and only if $x(t_{i-1}^-) \in \mathcal{K}_{i-1}^{\tau}$. □

Given these preliminary results we can present the following theorem, which characterizes impulse-controllability of (3.1).

Theorem 3.9. *Consider the switched system* (3.1) *with switching signal* $\sigma \in \mathcal{S}_{\mathrm{n}}$*. The system is impulse-controllable if and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^{\tau}. \tag{3.6}$$

Proof. ($\Leftarrow$) Assume that $\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^{\tau}$. This means that for all initial values $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an input u such that the resulting trajectory x is impulse-free. Hence the system is impulse-controllable.

($\Rightarrow$) Assume that the system is impulse-controllable. Then for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an input u such that the resulting trajectory is impulse-free. By Corollary 3.8 it follows that $x_0 \in \mathcal{K}_0^{\tau}$. As this hold for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ it follows that

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^{\tau},$$

which proves the desired result. $\square$

To illustrate the result from Theorem 3.9 we show the following example where we verify impulse-controllability.

Example 3.10. Consider the switched DAE with a switching signal σ with $\sigma(t_0^+) = 0$, $\sigma(t_1^+) = 1$ and $\sigma(t_2^+) = 2$ and the switching times are given by $t_0 = 0$, $t_1 = \ln(4)$ and $t_2 = t_1 + \frac{1}{2}\pi$. Furthermore, let the modes be given by

$$\begin{aligned}
(E_0, A_0, B_0) &= \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), \\
(E_1, A_1, B_1) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right), \\
(E_2, A_2, B_2) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right).
\end{aligned}$$

Since the second mode rotates the state, it is easy to calculate that:

$$\mathcal{C}_2^{\mathrm{imp}} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{K}_1^{\tau} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Then calculating the involved subspaces yields

$$\begin{aligned}
&\Pi_0 = \operatorname{im} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad e^{-A_0^{\mathrm{diff}} \ln(4)} \mathcal{K}_1^{\tau} = \operatorname{im} \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ -3 & -4 \end{bmatrix}, \\
&\mathcal{R}_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}, \quad \ker E_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}$$

From this it can be calculated that $\mathcal{K}_0^{\tau} = \mathbb{R}^n$ and hence it follows that the system is impulse-controllable. $\diamond$

In the case that the switching signal only induces a single switch, the conditions for impulse-controllability simplify considerably due to the A_0^{diff} invariance of the reachable space $\mathcal{R}_0$ and the augmented consistency space $\mathcal{V}_{(E_0,A_0,B_0)}$.

Lemma 3.11. *Consider the single switched DAE* (3.1) *with switching signal $\sigma \in \mathcal{S}_1$. Then the system is impulse-controllable if and only if*

$$\operatorname{im}\Pi_0 \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

Proof. ($\Rightarrow$) Since for all initial values there exists an input u such that the resulting trajectory is impulse-free, it follows that for all x_0 there exists an $\eta_0 \in \mathcal{R}_0$ such that

$$x(t_1^-) = e^{A_0^{\text{diff}}\tau_0}\Pi_0 x_0 + \eta_0 \in \mathcal{C}_1^{\text{imp}},$$

which implies

$$e^{A_0^{\text{diff}}\tau_0}\Pi_0 x_0 \in \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

Since this holds for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ and $e^{A_0^{\text{diff}}}$ is invertible and leaves $\operatorname{im}\Pi_0$ invariant we obtain

$$\operatorname{im}\Pi_0 \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

($\Leftarrow$) As $\operatorname{im}\Pi_0 \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0$ and $\operatorname{im}\Pi_0 + \mathcal{R}_0 = \mathcal{V}_{(E_0,A_0,B_0)}$ it follows that

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0.$$

The augmented consistency space $\mathcal{V}_{(E_0,A_0,B_0)}$ is A_0^{diff} invariant and thus

$$\mathcal{V}_{(E_0,A_0,B_0)} = e^{-A_0^{\text{diff}}\tau_0}\mathcal{V}_{(E_0,A_0,B_0)} \subseteq e^{-A_0^{\text{diff}}\tau_0}(\mathcal{C}_1^{\text{imp}} + \mathcal{R}_0)$$

and consequently

$$\operatorname{im}\Pi_0 \cap \mathcal{V}_{(E_0,A_0,B_0)} \subseteq \operatorname{im}\Pi_0 \cap e^{-A_0^{\text{diff}}\tau_0}(\mathcal{C}_1^{\text{imp}} + \mathcal{R}_0).$$

Recall that $\mathcal{V}_{(E_0,A_0,B_0)} = \operatorname{im}\Pi_0 + \langle E_0^{\text{imp}} \mid \operatorname{im} B_0^{\text{imp}}\rangle$ and observe that $\operatorname{im}\Pi_0 \cap \mathcal{V}_{(E_0,A_0,B_0)} = \operatorname{im}\Pi_0$ from which it follows that

$$\begin{aligned}
\mathcal{V}_{(E_0,A_0,B_0)} &= \operatorname{im}\Pi_0 \cap \mathcal{V}_{(E_0,A_0,B_0)} + \langle E_0^{\text{imp}} \mid \operatorname{im} B_0^{\text{imp}}\rangle \\
&\subseteq \operatorname{im}\Pi_0 \cap e^{-A_0^{\text{diff}}\tau_0}(\mathcal{C}_1^{\text{imp}} + \mathcal{R}_0) + \langle E_0^{\text{imp}} \mid \operatorname{im} B_0^{\text{imp}}\rangle + \ker E_0 \\
&= \mathcal{K}_0^{\tau}.
\end{aligned}$$

and thus by Theorem 3.9 the system is impulse-controllable. $\square$

3.1.2 A forward approach

The sequences (3.2) that led to the characterization of impulse-controllability of (3.1) run backward in time. Consequently, if a switching signal is to be designed and it is to be determined which modes should be induced next such that impulse-controllability is guaranteed, the computations of the subspaces in (3.2) would have to be done for each possible future mode. If the number of modes induced by the switching signal and the order of the system are small, these computations can be done relatively easily. However, if many modes are induced and the order of the system is large, doing these computations can become computationally expensive. Furthermore, the subspaces in (3.2) only contain the states from which $\mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$ can be reached impulse-freely and hence they are not necessarily very informative regarding states other than contained in $\mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$, that can be reached in an impulse-free way. Therefore, we aim to find conditions, which can be verified forward in time. To that extent, consider the following sequence of subspaces.

$$\begin{aligned} \mathcal{W}_0^{\tau} &= \mathcal{V}_{(E_0,A_0)} + \mathcal{R}_0, \\ \mathcal{W}_i^{\tau} &= e^{A_i^{\mathrm{diff}}\tau_i}\Pi_i\left(\mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\mathrm{imp}}\right) + \mathcal{R}_i, \end{aligned} \qquad i = 0,1,...,\mathtt{n}. \tag{3.7}$$

Note that these sequences are defined forward in time, in contrast to the sequence (3.2) which are defined backward in time.

Lemma 3.12. *Consider the switched system* (3.1) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$. Then for any $i \in \{0,1,2,\dots\}$ and all $w \in \mathcal{W}_i^{\tau}$ there exists a solution (x,u) satisfying $x(t_{i+1}^-) = w$ which is impulse-free on $[t_0, t_{i+1})$.*

Proof. The proof is by induction. By definition of the augmented consistency space (and taking into account the time-invariance of the definition), there exists a smooth solution (x,u) that is impulse-free on $[t_0,t_1)$ and satisfies $x(t_1^-) = w \in \mathcal{W}_0^{\tau} = \mathcal{V}_{(E_0,A_0)} + \mathcal{R}_0 = \mathcal{V}_{(E_0,A_0,B_0)}$. This proves the case for $i=0$.

Assuming now that the statement holds for $i-1$, we now consider the case for i. Let $w \in \mathcal{W}_i^{\tau}$, then by definition here exist elements $\bar{w} \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\mathrm{imp}}$ and $\eta_i \in \mathcal{R}_i$ such that

$$w = e^{A_i^{\mathrm{diff}}\tau_i}\Pi_i\bar{w} + \eta_i.$$

As $\bar{w} \in \mathcal{W}_{i-1}^{\tau}$, it follows from the induction assumption that there exists a solution $(\bar{x},\bar{u})$ of (3.1) with $\bar{x}(t_i^-) = \bar{w}$ which is impulse-free on (t_0,t_i). Since $\bar{x}(t_i^-) \in \mathcal{C}_i^{\mathrm{imp}}$, this solution can be assumed to be impulse-free also on $[t_i,t_{i+1})$. We will now alter this solution on $[t_i,t_{i+1})$ such that at t_{i+1}^- the desired value w is reached and no additional impulses occur. From the solution formula it follows that $\bar{\eta}_i := \bar{x}(t_{i+1}^-) - e^{A_i^{\mathrm{diff}}\tau_i}\Pi_i\bar{w} \in \mathcal{R}_i$ and hence $\eta_i - \bar{\eta}_i \in \mathcal{R}_i$. By definition of the reachable space of mode i there exists a (smooth) solution $(\widetilde{x}, \widetilde{u})$ of $E_i\dot{x} = A_i x + B_i u$ such that $\widetilde{x}(t_i^-) = 0$ and $\widetilde{x}(t_{i+1}^-) = \eta_i - \bar{\eta}_i$. In fact, it can be assumed that $(\widetilde{x}, \widetilde{u})$ is identically zero on (t_0, t_i), hence $(\widetilde{x}, \widetilde{u})$ is then also a solution of the switched DAE (3.1). By linearity, $(x, u) = (\bar{x} + \widetilde{x}, \bar{u} + \widetilde{u})$ is a solution of (3.1) that is impulse-free on (t_0, t_{i+1}) with $x(t_{i+1}^-) = \bar{x}(t_{i+1}^-) + \widetilde{x}(t_{i+1}^-) = (e^{A_i^{\text{diff}}\tau_i}\Pi_i\bar{w} + \bar{\eta}_i) + (\eta_i - \bar{\eta}_i) = w$, which concludes the proof. $\square$

The result of Lemma 3.12 yields that for all $w \in \mathcal{W}_i^\tau$ there exists an initial value $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ and an impulse-free input u such that the resulting trajectory x will satisfy $x(t_{i+1}^-) = w$. Hence the spaces $\mathcal{W}_i^\tau$ can be used in the case that a switching signal is to be designed and a particular state is necessarily reached at time t_{i+1}. However, although Lemma 3.12 shows that the $\mathcal{W}_i^\tau$ spaces contain all the states that can be reached from some initial condition with a trajectory that is impulse-free on $[t_0, t_{i+1})$, not all those trajectories can be continued impulse-freely on $[t_i, \infty)$.

Example 3.13. Consider the following switched DAE restricted to the interval $[t_0, t_3)$, where (A, B) is controllable.

$$\Sigma_\sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \quad 0 \leqslant t < t_1, \\ \dot{x}(t) &= 0 \qquad\qquad\qquad t_1 \leqslant t < t_2, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) &= x(t) \qquad\qquad\quad t_2 \leqslant t < t_3. \end{cases}$$

In order to have impulse-free solutions, the state of the system needs to be in $\operatorname{span}\{e_1, e_2\}$ at $t = t_2$. However, $\mathcal{W}_0^\tau = \mathcal{W}_1^\tau = \mathbb{R}^n$ and therefore we can reach e_3 impulse-freely on $(0, t_1]$, but this would lead to an impulse at $t = t_2$. $\diamond$

The example shows that the sequence (3.7) will not lead to a characterization of impulse-controllability. However, these spaces can be used to find a sufficient condition. The next result is concerned with the subspace of $\mathcal{W}_i^\tau$ containing the points which can be reached impulse-freely and which can be continued impulse-freely on the interval $[t_{i+1}, t_{i+2})$.

Lemma 3.14. *Consider the switched system* (3.1) *with the switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *If* (x, u) *is impulse-free on* $[t_0, \infty)$ *then*

$$x(t_{i+1}^-) \in \mathcal{W}_i^\tau \cap \mathcal{C}_{i+1}^{\text{imp}},$$

for all $i \in \{0, ..., \mathtt{n} - 1\}$.

Proof. We will prove the statement by induction. For $i = 0$ it follows directly that $x(t_1^-) = x_1 \in \mathcal{V}_{(E_0,A_0,B_0)} = \mathcal{W}_0^\tau$. As the solution x is impulse-free, it follows that at the switch $x(t_1^-) \in \mathcal{C}_1^{\text{imp}}$. Consequently, $x(t_1^-) \in \mathcal{W}_0^\tau \cap \mathcal{C}_1^{\text{imp}}$.

This shows the claim for $i = 0$ and we conclude the proof inductively by assuming that the statement holds for i and proving that it holds for $i+1$. Since u is impulse-free on $[t_i, t_{i+1})$ we have for some $\eta_i \in \mathcal{R}_i$

$$x(t_{i+1}^-) = e^{A_i^{\text{diff}}\tau_i}\Pi_i x(t_i^-) + \eta_i.$$

By assumption $x(t_i^-) \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\text{imp}}$ and thus $x(t_{i+1}^-) \in \mathcal{W}_i^{\tau}$. Furthermore the solution x does not exhibit impulses, therefore we have $x(t_{i+1}^-) \in \mathcal{C}_{i+1}^{\text{imp}}$. Combining these observations leads to $x(t_{i+1}^-) \in \mathcal{W}_i^{\tau} \cap \mathcal{C}_{i+1}^{\text{imp}}$ which completes the proof. □

Corollary 3.15. *Consider the switched DAE* (3.1) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$. Then $\mathcal{W}_{i-1}^{\tau} \cap \mathcal{K}_i^{\tau}$ is the smallest set containing states that can be reached in an impulse-free way on (t_0, t_i) and that can be extended in an impulse-free way on $[t_i, t_{\mathtt{n}+1})$.*

Proof. By Lemma 3.12 we have for all $x_i \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{K}_i^{\tau} \subseteq \mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\text{imp}}$ that there exists an impulse-free solution satisfying $x(t_i^-) = x_i$ and $x(t_0^-) = x_0$ for some $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. And by Corollary 3.8 there exists an input such that $\mathcal{C}_{\mathtt{n}}^{\text{imp}}$ is reached impulse-freely from x_i.

Let (x, u) be an impulse-free solution. Then by Lemma 3.14 we have that at t_i $x_i \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\text{imp}}$ and therefore $x_i \in \mathcal{W}_{i-1}^{\tau}$. Since we can reach $\mathcal{C}_{\mathtt{n}}^{\text{imp}}$ impulse-freely from x_i it must hold that $x_i \in \mathcal{K}_i^{\tau}$. Therefore $x_i \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{K}_i^{\tau}$, which proves the result. □

Given these intermediate result, we can now state the following sufficient condition for impulse-controllability of switched DAEs in termes of the $\mathcal{W}_i^{\tau}$ spaces.

Theorem 3.16. *Consider the switched system* (3.1) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$. If for all $i \in \{1, ..., \mathtt{n}\}$*

$$\mathcal{W}_{i-1}^{\tau} \subseteq \mathcal{C}_i^{\text{imp}} + \mathcal{R}_{i-1}, \tag{3.8}$$

then the system is impulse-controllable.

Proof. We prove the statement inductively. For $i = 1$ there exists for any $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ by definition a solution $(\widehat{x}, \widehat{u})$ with $\widehat{x}(t_0^-) = x_0$ which is smooth (and in particular impulse-free) on $[t_0, t_1)$. Furthermore, $\widehat{x}(t_1^-) \in \mathcal{W}_0^{\tau} \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0$, i.e. there exists $\xi \in \mathcal{C}_1^{\text{imp}}$ and $\eta \in \mathcal{R}_0$ such that $\widehat{x}(t_1^-) = \xi + \eta$. Since η is reachable in mode 0 there exists a solution $(\widetilde{x}, \widetilde{u})$ of (3.1) satisfying $\widetilde{x}(t_0^+) = 0$ and $\widetilde{x}(t_1^-) = -\eta$. Now $(x, u) := (\widehat{x} + \widetilde{x}, \widehat{u} + \widetilde{u})$ solves (3.1), is impulse-free on $[t_0, t_1)$ and satisfies $x(t_0^+) = x_0$ and $x(t_1^-) = \xi + \eta - \eta \in \mathcal{C}_1^{\text{imp}}$.

Now assume that any initial condition can be steered to $\mathcal{C}_i^{\text{imp}}$ impulse-freely on $[t_0, t_i)$. This solution can now be extended to an impulse-free solution $(\widehat{x}, \widehat{u})$ onto $[t_0, t_{i+1})$. Similar as in Lemma 4.11 we can conclude that $\widehat{x}(t_i^-) \in \mathcal{W}_{i-1}^{\tau} \cap \mathcal{C}_i^{\text{imp}}$, and hence $\widehat{x}(t_{i+1}^-) \in \mathcal{W}_i^{\tau} \subseteq \mathcal{C}_{i+1}^{\text{imp}} + \mathcal{R}_i$. Hence $\widehat{x}(t_{i+1}^-) = \xi + \eta$ for $\xi \in \mathcal{C}_{i+1}^{\text{imp}}$ and $\eta \in \mathcal{R}_i$. Similar as above we find a solution $(\widetilde{x}, \widetilde{u})$ which is smooth on $[t_0, t_{i+1})$, identically zero on (t_0, t_i)

and satisfies $\widetilde{x}(t_{i+1}^-) = -\eta$. Then $(x,u) = (\widehat{x}+\widetilde{x}, \widehat{u}+\widetilde{u})$ is a solution which is impulse-free on $[t_0, t_f)$, has the same initial value as $\widehat{x}$ satisfies $x(t_{i+1}^-) \in \mathcal{C}_{i+1}^{\mathrm{imp}}$.

Finally, from the fact that for any initial value there is a solution (x,u) with $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$ it can be concluded that this solution can be extended to $[t_0, \infty)$ in an impulse-free way, i.e. the switched system is impulse-controllable. □

Remark 3.17. Besides giving a sufficient condition for impulse-controllability of a system for a given switching signal, the result of Theorem 3.16 can also be used to design a specific switching signal for which the switched system becomes impulse-controllable.

To illustrate the result of Theorem 3.16 we will give an example where the subspaces involved become apparent.

Example 3.18. Consider the following switched DAE with a switching signal σ satisfying $\sigma(t_0^+) = 0$ and $\sigma(t_1^+) = 1$. Let the modes be given by:

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right),$$
$$(E_1, A_1, B_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right).$$

It follows from the computation of the consistency projector and the reachable space that

$$\mathcal{V}_{(E_0,A_0)} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},\ \mathcal{R}_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\},$$

such that we obtain $\mathcal{W}_0^{\tau} = \mathbb{R}^n$. The impulse-controllable space is given by

$$\mathcal{C}_1^{\mathrm{imp}} = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This means that $\mathcal{C}_1^{\mathrm{imp}} + \mathcal{R}_0 = \mathbb{R}^n$ and hence the condition that $\mathcal{W}_0^{\tau} \subseteq \mathcal{C}_1^{\mathrm{imp}} + \mathcal{R}_0$ of Theorem 3.16 is satisfied and we can conclude that the system is impulse-controllable. Indeed we see that for all elements of the consistency space there exists a reachable point such that the sum of the two are in the impulse-controllable space. ◇

3.2 Impulse-controllability of system classes

Given the results on impulse-controllability of switched DAEs with a switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$, we will now investigate to what extent impulse-controllability is dependent on the switching signal. To do so we will regard a switched DAE (3.1) as generated by a set of $\mathtt{n} \in \mathbb{N}$ matrix triplets $\{E_p, A_p, B_p\}_{p=0}^{\mathtt{n}}$ and a switching signal, for example $\sigma \in \mathcal{S}_{\mathtt{n}}$. Note that for each $\sigma \in \mathcal{S}_{\mathtt{n}}$ the mode sequence is fixed. To fully investigate impulse-controllability in relation to the switching signal, the sequence in which the modes are induced should also be considered. Therefore we also define the class of arbitrary switching signals.

Definition 3.19 (Arbitrary switching signals)**.** The class of (arbitrary) switching signals $\overline{\mathcal{S}}_{\mathtt{n}}$ is defined as the set of all $\sigma : \mathbb{R} \to \{0, 1, ..., \mathtt{n}\}$ of the form

$$\sigma(t) = q_p \quad t \in [t_p, t_{p+1}), \tag{3.9}$$

where $\mathbf{q} := (q_0, q_1, \dots, q_{\mathtt{n}}) \in \{0, 1, \dots, \mathtt{n}\}^{\mathtt{n}+1}$ is the *mode sequence* of σ and $t_1 < t_2 < ... < t_{\mathtt{n}}$ are the $\mathtt{n} \in \mathbb{N}$ switching times in $(0, \infty)$ with $t_0 := 0$ and $t_{\mathtt{n}+1} := \infty$ for notational convenience. Furthermore, for a given sequence of switching times, let $\tau_i := t_{i+1} - t_i$, $i = 0, 1, \dots, \mathtt{n} - 1$ and

$$\boldsymbol{\tau} := (\tau_0, \tau_1, \dots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}, \tag{3.10}$$

the sequence of (finite) mode durations.

Note that in the above definition, a switching signal for which $q_p = q_{p+1}$ for some p is not excluded, effectively leading to a switching signal with less then $\mathtt{n}$ switches. Consequently, for such a switching signal the mode duration $\boldsymbol{\tau}$ is not uniquely defined, as the switching time t_{p+1} can be altered without changing the actual switching signal. Nevertheless, this does not lead to any technical problems in the following and we will use $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$ and the pair $(\mathbf{q}, \boldsymbol{\tau}) \in \mathbb{N}^{\mathtt{n}+1} \times \mathbb{R}^{\mathtt{n}}_{>0}$ interchangeably.

By regarding the switched DAE (3.1) to be generated by $\{E_p, A_p, B_p\}_{p=0}^{\mathtt{n}}$ and a switching signal $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$, we can define a system class of switched DAEs as follows.

Definition 3.20 (System classes)**.** For a family of matrix triplets $\{(E_p, A_p, B_p)\}_{p=0}^{\mathtt{n}}$ with regular pairs (E_p, A_p), the *system class* $\overline{\Sigma}_{\mathtt{n}}$ of associated switched (regular) DAEs (3.1) under arbitrary switching is given by

$$\overline{\Sigma}_{\mathtt{n}} := \left\{(E_\sigma, A_\sigma, B_\sigma) \;\middle|\; \sigma \in \overline{\mathcal{S}}_{\mathtt{n}}\right\},$$

where $(E_\sigma, A_\sigma, B_\sigma)$ is understood as a triple of (piecewise-constant) time-varying matrices for each specific switching signal $\sigma : (t_0, \infty) \to \{0, 1, \dots, \mathtt{n}\}$.

The corresponding system class $\Sigma_{\mathtt{n}}$ of switched DAEs with fixed mode sequence $\mathbf{q} = (0, 1, \dots, \mathtt{n})$ is given by

$$\Sigma_{\mathtt{n}} := \{(E_\sigma, A_\sigma, B_\sigma) \mid \sigma \in \mathcal{S}_{\mathtt{n}}\}.$$

3.2.1 Strong impulse-controllability of $\overline{\Sigma}_{\mathtt{n}}$

For a particular switched DAE $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$, impulse-controllability has been studied in the previous sections. However, impulse-controllability of $(E_\sigma, A_\sigma, B_\sigma)$ might be dependent on the switching signal in the sense that some $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ are impulse-controllable, whereas some are not. In the case every system in the system class

$\overline{\Sigma}_{\mathtt{n}}$ is impulse-controllable, the property can be said to be independent of the switching signal. To that extent we define strong impulse-controllability of system classes as follows.

Definition 3.21. The whole system class $\overline{\Sigma}_{\mathtt{n}}$ associated to the family $\{(E_p, A_p, B_p)\}_{p=0}^{\mathtt{n}}$ and $\overline{\mathcal{S}}_{\mathtt{n}}$ is called strongly impulse-controllable, if $(E_\sigma, A_\sigma, B_\sigma)$ is impulse-controllable for all $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$.

Some system classes are trivially strongly impulse-controllable (*e.g.,* when each individual mode is impulse-controllable or the switched DAEs is in fact non-switching because $(E_p, A_p, B_p) = (E_q, A_q, B_q)$ for all p, q). However, the following example shows that there exists non-trivial example of strongly impulse-controllable system classes.

Example 3.22. Consider a switched DAE $(E_\sigma, A_\sigma, B_\sigma)$ generated by the mode triplets

$$\begin{aligned}(E_0, A_0, B_0) &= \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right),\\ (E_1, A_1, B_1) &= \left(\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right).\end{aligned} \tag{3.11}$$

It is easily seen that the corresponding augemented consistency and impulse-controllable spaces satisfy $\mathcal{V}_0 = \mathcal{C}_0^{\text{imp}} = \mathbb{R}^2$ and $\mathcal{V}_1 = \mathcal{C}_1^{\text{imp}} = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

The corresponding system class Σ_1 is strongly impulse-controllable, which can be seen by considering all possible cases for the switching signals: switching signals with $\mathbf{q} = (0, 0)$ or $\mathbf{q} = (1, 1)$ are trivially impulse-controllable as a non-switched DAE (with consistent initial values); for mode sequence $\mathbf{q} = (0, 1)$ it is possible to choose a smooth input on (t_0, t_1) such that $x_2(t_1^-) = 0$ and hence no impulse occurs at the switching time t_1; for the mode sequence $(1, 0)$ the input $u(t) = 0$ will result in an impulse-free solution for all initial values in $\mathcal{V}_1 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ ◇

In the case of switched DAEs with a single switch, recall the characterization of impulse-controllability for single switched DAEs in Lemma 3.11.The single-switch result can directly be used to arrive at a characterization of strong impulse-controllability as follows.

Theorem 3.23. *Consider the system class $\overline{\Sigma}_{\mathtt{n}}$ associated to $\overline{\mathcal{S}}_{\mathtt{n}}$ and $\{E_p, A_p, B_p\}_{p=0}^{\mathtt{n}}$ with corresponding (individual) consistency projectors Π_p, impulse-controllable spaces $\mathcal{C}_p^{\text{imp}}$ and reachability spaces $\mathcal{R}_p$. Then $\overline{\Sigma}_{\mathtt{n}}$ is strongly impulse-controllable if, and only if,*

$$\operatorname{im}\Pi_i \subseteq \mathcal{C}_j^{\text{imp}} + \mathcal{R}_i \tag{3.12}$$

for all $i, j \in \{0, 1, ..., \mathtt{n}\}$.

Proof. Necessity of (3.12) is clear by considering switching signals with mode sequences of the form $\mathbf{q} = (i, j, q_2, \dots, q_{\mathtt{n}})$ together with Lemma 3.11 and the obvious fact that an impulse-free solution needs to be impulse-free on the initial interval $[t_0, t_2)$ as well.

Sufficiency of (3.12) is also clear by considering each switched system $(E_\sigma, A_\sigma, B_\sigma)$ as a concatenation of single switch switched DAEs and the ability to choose the input independently around the switching times to ensure impulse-freeness at each individual switch (as a consequence of Lemma 3.11). $\square$

Remark 3.24. The characterization of strong impulse-controllability of $\overline{\Sigma}_{\mathtt{n}}$ via (3.12) is much simpler than the characterization of impulse-controllability of an individual switched system, which is based on a rather complicated recursive subspace sequence (discussed in detail in the next subsection, see (3.14)) and depends on the specific mode durations $\boldsymbol{\tau}$. The underlying reason is that strong impulse-controllability is by definition independent from the mode durations and, furthermore, can be reduced to the single switch case (as utilized in the proof of Theorem 3.23).

3.2.2 Impulse-controllability of Σ_n

As can be seen from Theorem 3.23, verifying whether a system class $\overline{\Sigma}_{\mathtt{n}}$ is strongly impulse-controllable can be done by verifying impulse-controllability of all possible single switch switched DAEs. However, if a mode sequence is fixed, these conditions are only sufficient and not necessary in general. In fact, defining strong impulse-controllability for Σ_n analogously as in Definition 3.21 (see also the forthcoming Defintion 3.27), we have the following consequence from Lemma 3.11.

Corollary 3.25. *The system class Σ_n of switched systems with fixed mode sequence $\mathbf{q} = (0, 1, 2, \dots, \mathtt{n})$ is strongly impulse-controllable if*

$$\operatorname{im} \Pi_k \subseteq \mathcal{C}_{k+1}^{\mathrm{imp}} + \mathcal{R}_k \quad \forall k \in \{0, 1, \dots, \mathtt{n} - 1\}. \tag{3.13}$$

The following example shows that (3.13) is indeed only sufficient and not necessary in general.

Example 3.26. Consider the system class $\Sigma_{\mathtt{n}}$ with $\mathtt{n} = 2$ and modes $(E_0, A_0, B_0) = (I, 0, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ $(E_1, A_1, B_1) = (I, 0, 0)$ $(E_2, A_2, B_2) = (\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], I, 0)$. It is easily seen that $\Sigma_{\mathtt{n}}$ is strongly impulse-controllable; in fact, for any switching time t_1 and any initial value it is possible to choose the input u on $[0, t_1)$ such $x_1(t_1^-) = 0$, in the second mode the state then remains constant and hence $x_1(t_2^-) = x_1(t_1^-) = 0$ which then implies that at the last switch x_1 does not jump and hence no Dirac impulse is induced. However, condition (3.13) is not satisfied for the mode pair $(1, 2)$; indeed $\operatorname{im} \Pi_1 = \mathbb{R}^2$ is not contained in $\mathcal{C}_2^{\mathrm{imp}} + \mathcal{R}_1 = \operatorname{im} \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] + \{0\}$.

The above example shows that characterization of impulse-controllability of $\Sigma_{\mathtt{n}}$ cannot simply be reduced to the single switch case anymore. In particular, it will turn out that it is possible that a switched system with fixed mode sequence has some isolated mode duration for which impulse-controllability is lost, but for all remaining mode duration it is impulse-controllable. Furthermore, for arbitrary switching signals it is not possible that *none* of the systems in $\overline{\Sigma}_{\mathtt{n}}$ are impulse-uncontrollable, however, for a fixed mode sequence it is indeed possible, that *all* of the systems in $\Sigma_{\mathtt{n}}$ are *not* impulse-controllable. Finally, it is also possible that for some specific mode durations a system in $\Sigma_{\mathtt{n}}$ is impulse-controllable, while for all remaining mode durations the systems are not impulse-controllable. This motivates us to introduce the following different notions of impulse-controllability for the system class $\Sigma_{\mathtt{n}}$.

Definition 3.27 (Strong and essential impulse-(un-)controllability for $\Sigma_{\mathtt{n}}$)**.** Consider the class $\Sigma_{\mathtt{n}}$ of switched systems (3.1) with fixed mode sequence $\mathbf{q} = (0, 1, 2, \ldots, \mathtt{n})$ and arbitrary mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$.

- $\Sigma_{\mathtt{n}}$ is called *strongly impulse-controllable* if all $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_{\mathtt{n}}$ are impulse-controllable.
- $\Sigma_{\mathtt{n}}$ is called *essentially impulse-controllable* if the set of all mode durations $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ of $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_{\mathtt{n}}$ which are not impulse-controllable has measure zero in $\mathbb{R}^{\mathtt{n}}_{>0}$.
- $\Sigma_{\mathtt{n}}$ is called *strongly impulse-uncontrollable* if all $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_{\mathtt{n}}$ are not impulse-controllable.
- $\Sigma_{\mathtt{n}}$ is called *essentially impulse-uncontrollable* if the set of all mode durations $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ of $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_{\mathtt{n}}$ which are impulse-controllable has measure zero in $\mathbb{R}^{\mathtt{n}}_{>0}$.

First note that clearly every strongly impulse-(un-)controllable system class is also essentially impulse-(un-)controllable.

Example 3.26 already provides a nontrivial example for a strongly impulse-controllable $\Sigma_{\mathtt{n}}$, and every $\Sigma_{\mathtt{n}}$ with two modes which do not satisfy the single-switch impulse-controllability condition (3.11) is an example for a strongly impulse-uncontrollable $\Sigma_{\mathtt{n}}$. In order to justify the introduction of the notion of essential impulse-(un-)controllability we will provide in the following examples which are essentially impulse-(un-)controllable but not strongly impulse-(un-)controllable.

Example 3.28 (Essentially, but not strongly, impulse-controllable class)**.** Consider the switched system class Σ_2 with modes

$$
\begin{aligned}
(E_0, A_0, B_0) &= (I, 0, \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]),\\
(E_1, A_1, B_1) &= (I, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], 0),\\
(E_2, A_2, B_2) &= (\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], I, 0).
\end{aligned}
$$

For any mode duration $\boldsymbol{\tau} = (\tau_0, \tau_1)$ we see that the solution of the corresponding switched DAE with initial value $x(0^-) = \left(\begin{smallmatrix} x_{01} \\ x_{02} \end{smallmatrix}\right)$ is given by

$$
\begin{aligned}
x(t) &= \begin{pmatrix} x_{01} + \int_0^t u \\ x_{02} \end{pmatrix}, \quad t \in (0, t_1), \\
x[t_1] &= 0, \\
x(t) &= \left[\begin{smallmatrix} \cos(t-t_1) & \sin(t-t_1) \\ -\sin(t-t_1) & \cos(t-t_1) \end{smallmatrix}\right] x(t_1^-), \quad t \in (t_1, t_2), \\
x[t_2] &= -\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(t_2^-)\delta_{t_2}, \\
x(t) &= 0, \quad t > t_2.
\end{aligned}
$$

For the specific mode duration $\tau_2 = 2\pi$ we see that $x(t_2^-) = x(t_1^-)$, hence the second component of $x(t_2^-)$ is x_{02}, independently of the choice of the input u. However, for $x_{02} \neq 0$ this leads to an unavoidable Dirac impulse at $t = t_2$, *i.e.*, $\Sigma_{\mathtt{n}}$ is not strongly impulse-controllable. On the other hand, for all $\tau_2 \neq k\pi$, it is easily seen that there exists an input u on $(0, t_1)$ resulting in a suitable first entry of $x(t_1^-)$ such that the rotation in mode 1 leads to $x_2(t_2^-)$ having a zero second component and hence resulting in an impulse-free switch at $t = t_2$. This shows that $\Sigma_{\mathtt{n}}$ is indeed essentially impulse-controllable. ◇

Example 3.29 (Essentially, but not strongly, impulse-uncontrollable class)**.** Consider the switched system class Σ_2 with modes

$$
\begin{aligned}
(E_0, A_0, B_0) &= \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], 0\right), \\
(E_1, A_1, B_1) &= \left(I, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], 0\right), \\
(E_2, A_2, B_2) &= \left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], I, 0\right).
\end{aligned}
$$

Note that for this example the input is not effecting the dynamics at all, so impulse-controllability reduces to impulse-freeness. Clearly, the solution in the initial mode is given by $x(t) = \left[\begin{smallmatrix} x_{01} \\ 0 \end{smallmatrix}\right]$ and afterwards the solutions are given as in Example 3.28 (because modes 1 and 2 are identical to the ones there). Consequently, for $\tau_1 = 2\pi$ we have $x(t_2^-) = x(t_1^-) = \left[\begin{smallmatrix} x_{01} \\ 0 \end{smallmatrix}\right]$, which results in an impulse-free solution of the switched DAE, *i.e.*, Σ_2 is not strongly impulse-uncontrollable. Nevertheless, for any $\tau_1 \neq k\pi$ we see that the second component of $x(t_2^-)$ is non-zero (if $x_{01} \neq 0$) and hence a Dirac impulse occurs at $t = t_2$. This means that Σ_2 is essentially impulse-uncontrollable. ◇

In order to make statements regarding essential impulse-controllability, we will make use of analytical matrix valued maps, so called analytic matrices. Loosely speaking a matrix valued map $M : \mathbb{R}^p \to \mathbb{R}^{n \times m}$ is called an analytic matrix if each entry of the map is an analytic function, *i.e.*, a function which if it is zero for some value, it is either identically zero, or it is nonzero almost everywhere. Such a matrix $M(\tau)$ has generically full rank if it has full rank for almost all values $\tau \in \mathbb{R}^p$. For precise definitions of matrix valued analytical maps which are generically full rank we refer to Appendix B. For the spaces $\mathcal{K}_i^{\boldsymbol{\tau}}$ we are able to construct an analytical matrix which has generically full rank as the following result shows.

Lemma 3.30. *Consider the sequence* (3.2). *Then for all* $i \in \{0, 1, ..., \mathtt{n}\}$ *there exists an analytic matrix* $N_i : \mathbb{R}^{\mathtt{n}-i} \to \mathbb{R}^{n \times k_i}$ *with generically full rank such that* $\operatorname{im} N_i(\boldsymbol{\tau}) = \mathcal{K}_i^{\boldsymbol{\tau}}$ *for a.a.* $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}-i}$.

Proof. For $i = \mathtt{n}$ we use the convention that a constant full rank matrix is interpreted as an analytic matrix depending on an empty tuple $\boldsymbol{\tau} = () \in \mathbb{R}^0$, then the claim is correct by simply choosing the columns of $N_{\mathtt{n}}(\boldsymbol{\tau})$ as a (constant) basis of $\mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$. We now proceed inductively and assume the claim is correct for some $i \in \{1, 2, \ldots, \mathtt{n}\}$. Let $\mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}} := e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \operatorname{im} N_i(\boldsymbol{\tau}) + \mathcal{R}_{i-1}$ and $\mathcal{R}_{i-1}^{\mathrm{imp}} := \langle E_{i-1}^{\mathrm{imp}} \mid \operatorname{im} B_{i-1}^{\mathrm{imp}} \rangle + \ker E_{i-1}$, then

$$\mathcal{K}_{i-1}^{(\tau_{i-1},\boldsymbol{\tau})} = \left(\operatorname{im} \Pi_{i-1} \cap \mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}}\right) + \mathcal{R}_{i-1}^{\mathrm{imp}},$$

for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}-i}$ and all $\tau_{i-1} \in \mathbb{R}$. Utilizing Lemmas B.5 and B.8 we find analytic and generically full rank matrices $\widetilde{N}_{i-1} : \mathbb{R}^{\mathtt{n}-(i-1)} \to \mathbb{R}^{n \times \widetilde{k}_i}$, $\overline{N}_{i-1} : \mathbb{R}^{\mathtt{n}-i+1} \to \mathbb{R}^{n \times \overline{k}_i}$, $N_{i-1} : \mathbb{R}^{\mathtt{n}-(i-1)} \to \mathbb{R}^{n \times k_i}$ such that a.a. $(\tau_{i-1}, \boldsymbol{\tau}) \in \mathbb{R}^{\mathtt{n}-(i-1)}$

$$\begin{aligned}
\operatorname{im} \widetilde{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}}, \\
\operatorname{im} \overline{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \operatorname{im} \Pi_{i-1} \cap \operatorname{im} \widetilde{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}), \\
\operatorname{im} N_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \operatorname{im} \overline{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) + \mathcal{R}_{i-1}^{\mathrm{imp}},
\end{aligned}$$

i.e., $\mathcal{K}_{i-1}^{(\tau_{i-1},\boldsymbol{\tau})} = \operatorname{im} N_{i-1}(\tau_{i-1}, \boldsymbol{\tau})$ as desired. □

We are now ready to formulate our first main result concerning impulse-controllability of the class of switched DAEs with fixed mode sequence.

Theorem 3.31. *Consider a class* $\Sigma_{\mathtt{n}}$ *of switched systems* (3.1) *with fixed mode sequence* $\mathbf{q} = (0, 1, 2, \ldots, \mathtt{n})$. *Then* $\Sigma_{\mathtt{n}}$ *is either essentially impulse-controllable or essentially impulse-uncontrollable.*

The proof utilizes properties of analytic matrices which are provided in Appendix B.

Proof. Case 1: All systems in $\Sigma_{\mathtt{n}}$ are impulse-controllable.
By definition $\Sigma_{\mathtt{n}}$ is then strongly impulse-controllable and in particular essentially impulse-controllable.

Case 2: There exists at least one impulse-uncontrollable system in $\Sigma_{\mathtt{n}}$.
In view of Lemma 3.30 we can choose an analytic matrix $N_0 : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{n \times k_0}$ with generically full rank such that $\operatorname{im} N_0(\boldsymbol{\tau}) = \mathcal{K}_0^{\boldsymbol{\tau}}$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}$.

Case 2a: For all impulse-uncontrollable mode durations $\overline{\boldsymbol{\tau}} \in \mathbb{R}_{>0}^{\mathtt{n}}$ we have that $\operatorname{im} N_0(\overline{\boldsymbol{\tau}}) \neq \mathcal{K}_0^{\overline{\boldsymbol{\tau}}}$ or $N_0(\overline{\boldsymbol{\tau}})$ does not have full rank.

In this case the set of impulse-uncontrollable mode durations is contained in a set of measure zero, hence $\Sigma_{\mathtt{n}}$ is essentially impulse-controllable.

Case 2b: There exists an impulse-uncontrollable mode duration $\overline{\boldsymbol{\tau}} \in \mathbb{R}^{\mathtt{n}}_{>0}$ such that $\operatorname{im} N_0(\overline{\boldsymbol{\tau}}) = \mathcal{K}_0^{\overline{\boldsymbol{\tau}}}$ and $N_0(\overline{\boldsymbol{\tau}})$ has full rank.
Since impulse-controllability for a specific switching signal is equivalent to (3.6) we have

$$\mathcal{V}_{(E_0,A_0,B_0)} \not\subseteq \mathcal{K}_0^{\overline{\boldsymbol{\tau}}} = \operatorname{im} N_0(\overline{\boldsymbol{\tau}}).$$

Hence there exists a vector $v \in \mathcal{V}_{(E_0,A_0,B_0)}$ such that $M(\boldsymbol{\tau}) := [N(\boldsymbol{\tau}), v]$ has full rank for $\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}$. In particular, M is an analytic matrix for which $\boldsymbol{\tau} \mapsto \det M(\boldsymbol{\tau})^\top M(\boldsymbol{\tau})$ is not identically zero, *i.e.*, M is generically full rank. Consequently, $v \notin \operatorname{im} N(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ and hence

$$\mathcal{V}_{(E_0,A_0,B_0)} \not\subseteq \operatorname{im} N_0(\boldsymbol{\tau}) = \mathcal{K}_0^{\boldsymbol{\tau}}, \quad \text{for a.a. } \boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}.$$

This implies that almost all systems in $\Sigma_{\mathtt{n}}$ are impulse-uncontrollable, *i.e.*, $\Sigma_{\mathtt{n}}$ is essentially impulse-uncontrollable. This concludes the proof as no other cases are possible. □

Remark 3.32. Theorem 3.31 states that the classes of switched DAEs with fixed mode sequences fall into four disjoint categories: 1) strongly impulse-controllable, 2) essentially (but not strongly) impulse-controllable, 3) essentially (but not strongly) impulse-uncontrollable, 4) strongly impulse-uncontrollable. Interestingly, there are only *three* categories for the notions of observability and controllability for switched systems with a fixed mode sequences (cf. [117] for observability, which by the duality arguments of [69] also carry over to controllability). The underlying reason is that the characterization of impulse-controllability is expressed in terms of sums *and* intersections of certain subspaces (see the forthcoming discussion) which can result in a singular dimension drop as well as a singular dimension increase in the involved duration-dependent subspaces; this in contrast to the observability (reachability) subspaces, which only involve intersections (sums).

In order to further investigate the different notions of impulse-controllability for the system class $\Sigma_{\mathtt{n}}$, we again consider the sequences defined in (3.2). For each switched DAE $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_{\mathtt{n}}$ with corresponding mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$ and mode sequence $\boldsymbol{q} = [\,0\ 1\ \ldots\ \mathtt{n}\,]$ we can write

$$\begin{aligned} &\mathcal{K}_{\mathtt{n}}^{\boldsymbol{\tau}} := \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}, \\ &\mathcal{K}_{i-1}^{\boldsymbol{\tau}} := \Big(\operatorname{im} \Pi_{i-1} \cap (e^{-A_{i-1}^{\mathrm{diff}} \tau_{i-1}} \mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1})\Big) \oplus \mathcal{D}_{i-1}^{\mathrm{imp}}, \\ &\qquad i = \mathtt{n}, \mathtt{n}-1, \ldots, 1. \end{aligned} \tag{3.14}$$

where

$$\mathcal{D}_{i-1}^{\mathrm{imp}} = \langle E_{i-1}^{\mathrm{imp}} \mid \operatorname{im} B_{i-1}^{\mathrm{imp}} \rangle + \ker E_{i-1}.$$

In view of invertibility of each exponential term $e^{-A^{\text{diff}}_{i-1}\tau_{i-1}}$ in (3.14) and A^{diff}_{i-1}-invariance of the subspaces $\operatorname{im}\Pi_{i-1}$ and $\mathcal{R}_{i-1}$, it follows that the recursive definition (3.14) can equivalently be written as

$$\mathcal{K}^{\boldsymbol{\tau}}_{i-1} = e^{-A^{\text{diff}}_{i-1}\tau_{i-1}} \left(\operatorname{im}\Pi_{i-1} \cap (\mathcal{K}^{\boldsymbol{\tau}}_{i} + \mathcal{R}_{i-1})\right) \oplus \mathcal{D}^{\text{imp}}_{i-1}.$$

An obvious characterization of strong impulse-(un)controllability of the system class $\overline{\Sigma}_{\mathtt{n}}$ is therefore the condition that (3.9) does (not) hold for all $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$. However, this characterization is not very insightful and impracticable because uncountably many subspaces need to be calculated. We can obtain more practible (sufficient) conditions for strong impulse-(un-)controllability by using the fact that for any subspace $\mathcal{S}$, any matrix A and any $t \in \mathbb{R}$ we have

$$\langle \mathcal{S} \mid A \rangle \subseteq e^{At}\mathcal{S} \subseteq \langle A \mid \mathcal{S} \rangle, \tag{3.15}$$

where $\langle \mathcal{S} \mid A \rangle$ denotes the largest A-invariant subspace contained in $\mathcal{S}$ and $\langle A \mid \mathcal{S} \rangle$ denotes the smallest A-invariant subspace containing $\mathcal{S}$. In fact, we can construct an over- and underestimation of $\mathcal{K}^{\boldsymbol{\tau}}_{i}$ as follows:

$$\overline{\mathcal{K}}_{i-1} := \left\langle A^{\text{diff}}_{i-1} \mid \operatorname{im}\Pi_{i-1} \cap (\overline{\mathcal{K}}_{i} + \mathcal{R}_{i-1}) \right\rangle \oplus \mathcal{D}^{\text{imp}}_{i-1}, \tag{3.16}$$

$$\underline{\mathcal{K}}_{i-1} := \left\langle \operatorname{im}\Pi_{i-1} \cap (\underline{\mathcal{K}}_{i} + \mathcal{R}_{i-1}) \mid A^{\text{diff}}_{i-1} \right\rangle \oplus \mathcal{D}^{\text{imp}}_{i-1}, \tag{3.17}$$

each for $i = \mathtt{n}, \mathtt{n}-1, \ldots, 1$ and with $\overline{\mathcal{K}}_{\mathtt{n}} = \underline{\mathcal{K}}_{\mathtt{n}} = \mathcal{C}^{\text{imp}}_{\mathtt{n}}$. By construction we have $\underline{\mathcal{K}}_{i} \subseteq \mathcal{K}^{\boldsymbol{\tau}}_{i} \subseteq \overline{\mathcal{K}}_{i}$, which immediately leads to the following sufficient condition for strong impulse-(un-)controllability.

Corollary 3.33. *The system class $\Sigma_{\mathtt{n}}$ is strongly impulse-controllable if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \underline{\mathcal{K}}_0$$

and it is strongly impulse-uncontrollable if

$$\mathcal{V}_{(E_0,A_0,B_0)} \not\subseteq \overline{\mathcal{K}}_0.$$

Remark 3.34. It is also possible to obtain under- and overestimation of $\mathcal{K}^{\boldsymbol{\tau}}_{i}$ by using (3.15) directly in (3.14), however it turns out that this leads to smaller underestimations and bigger overestimations and hence leads to more conservative sufficient conditions.

3.2.3 (Quasi)-causal impulse-controllability of Σ_{n}

So far we have presented several sufficient conditions for strong impulse-controllability, which is concerned with the existence of an input (depending on the initial value) which results in an impulse-free solution. Clearly, this "impulse-avoiding" input in general depends on the switching signal and in particular for the system class Σ_{n} with known mode sequence it is not clear whether an impulse-avoiding input can be constructed *independently* of the (unknown) mode durations. The following example shows, that indeed the impulse-avoiding input may depend on future mode durations.

Example 3.35 (Non-causal impulse-controllability)**.** Consider the class Σ_2 of switched systems with fixed mode sequence $\mathbf{q} = (0, 1, 2)$ and with modes given by

$$\begin{aligned}
(E_0, A_0, B_0) &= (I, 0, \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]),\\
(E_1, A_1, B_1) &= (I, \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], 0),\\
(E_2, A_2, B_2) &= (\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], 0).
\end{aligned}$$

For a given switching signal with mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1) \in \mathbb{R}^2_{>0}$ the sequence (3.14) is given by

$$\begin{aligned}
\mathcal{K}_2^{\boldsymbol{\tau}} &= \mathcal{C}_2^{\mathrm{imp}} = \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right],\\
\mathcal{K}_1^{\boldsymbol{\tau}} &= \operatorname{span}\left\{ e^{A_1 \tau_1} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] \right\} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ e^{-\tau_1} \end{smallmatrix}\right] \right\},\\
\mathcal{K}_0^{\boldsymbol{\tau}} &= \mathcal{K}_1^{\boldsymbol{\tau}} + \mathcal{R}_0 = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ e^{-\tau_1} \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \right\} = \mathbb{R}^2.
\end{aligned}$$

Hence the system class is strongly impulse-controllable. However, for two mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1)$ and $\overline{\boldsymbol{\tau}} = (\overline{\tau}_0, \overline{\tau}_1)$ with $\tau_1 \neq \overline{\tau}_1$ we have that

$$\mathcal{K}_1^{\boldsymbol{\tau}} \cap \mathcal{K}_1^{\overline{\boldsymbol{\tau}}} = \{0\}.$$

Since the first mode is not null-controllable, this means that the value of the state $x(t_1^-)$ explicitly depends on the future mode duration in order to guarantee impulse-freeness. For example, for the (consistent) initial condition $x(0^+) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, it follows for the first state component that $x_1(t_2^-) = 1$ as $\dot{x}_1 = 0$ in the zeroth and first mode. Hence in order to ensure an impulse-free solution it is required that the second state component satisfies $x_2(t_2^-) = -1$. This is achieved if and only if $x_2(t_1^-) = e^{-\tau_1}$. Consequently, the control on the interval $(0, t_1)$ needs to ensure that $x_2(t_1^-) = e^{-\tau_1}$ and therefore necessarily depends on the future mode duration τ_1. $\diamond$

quasi-causality

In some applications it may be the case that the current mode duration is known once the mode is activated, but the mode durations of the future modes are not known yet;

for example, if a switch is induced by shutting down or decoupling components for scheduled maintenance whose duration is known upfront. In this case causality of the input means that it should be independent from the future mode durations, but it can utilize the knowledge when the next switch happens. This somewhat weaker notion of causal impulse-controllability is called *quasi-causal impulse-controllability* and is defined in terms of the existence of a family of input-defining maps

$$\mathcal{U}_t : (\sigma_{(t_0,t)}, x_0) \mapsto u_{(t_0,t)},$$

such that for all $\sigma \in \mathcal{S}_\mathtt{n}$ and all initial values $x_0 \in \mathcal{V}_{\left(E_{\sigma(t_0)}, A_{\sigma(t_0)}, B_{\sigma(t_0)}\right)}$ the corresponding solution $(x, u)_{(t_0,t)}$ of $(E_\sigma, A_\sigma, B_\sigma)$ on (t_0, t) satisfying $x(t_0^+) = x_0$ is impulse-free. Additionally, we have to require that the map $\mathcal{U}_t$ is itself quasi-causal, *i.e.*, for all switching times t_i and $s > t_i$ the following holds

$$\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0) = \mathcal{U}_s(\sigma_{(t_0,s)}, x_0)_{(t_0,t_i)}. \tag{3.18}$$

Observe that for two switching signals $\sigma, \bar{\sigma} \in \mathcal{S}_\mathtt{n}$ satisfying $\sigma_{(t_0,s)} = \bar{\sigma}_{(t_0,s)}$ for some $s \in (t_i, t_{i+1})$ it may occur that $\mathcal{U}_s(\sigma_{(t_0,s)}, x_0) \neq \mathcal{U}_s(\bar{\sigma}_{(t_0,s)}, x_0)$.

Before presenting conditions for quasi-causal impulse-controllability we will present the following lemma, which is required in the proofs to come.

Lemma 3.36. *For all $p \in \{0, 1, \ldots, , \mathtt{n}-1\}$ and $\underline{\mathcal{K}}_p$ as in* (3.17) *we have*

$$\underline{\mathcal{K}}_p = \left\{ x_p \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \forall \tau > 0 \;\exists \text{ impulse-free solution } (x, u) \\ \text{on } [t_p, t_p + \tau) \text{ of } E_p\dot{x} = A_p x + B_p u, \\ \text{with } x(t_p^-) = x_p \text{ and } x((t_p + \tau)^-) \in \underline{\mathcal{K}}_{p+1} \end{array} \right\},$$

i.e., the subspace $\underline{\mathcal{K}}_p$ consists of all initial states for mode p which can be controlled impulse-freely into the subspace $\underline{\mathcal{K}}_{p+1}$ within a given time duration $\tau > 0$.

Before providing the proof we want to highlight that in the statement above the impulse avoiding input in general depends on τ, *i.e.*, on the mode duration of the current mode, whereas the subspaces given by (3.17) are independent from the mode duration (but depend on the mode sequence).

Proof. Let $x_p \in \underline{\mathcal{K}}_p$. Then $x_p = w + v$ for some $w \in \langle \operatorname{im} \Pi_p \cap \left(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\right) \mid A_p^{\text{diff}} \rangle$ and $v \in \mathcal{D}_p^{\text{imp}}$. Recall that any $v \in \mathcal{D}_p^{\text{imp}}$ can be impulse-freely controlled to zero with a smooth input for any given time duration $\tau > 0$. Hence, in view of linearity, it suffices to consider the case $x_p \in \langle \operatorname{im} \Pi_p \cap \left(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\right) \mid A_p^{\text{diff}} \rangle$. It follows then from A_p^{diff}-invariance that for $\tau \in \mathbb{R}$

$$e^{A_p^{\text{diff}}\tau} \Pi_p x_p = k_{p+1}^\tau + \eta^\tau,$$

for some $k_{p+1}^{\tau} \in \underline{\mathcal{K}}_{p+1}$ and $\eta^{\tau} \in \mathcal{R}_p$. In particular, there exists a smooth input u defined on $[t_p, t_p + \tau)$ which stears the state x from zero to $-\eta^{\tau}$. Applying the same input for the initial value $x(t_p^-) = x_p$ results in

$$\begin{aligned} x_u((t_p+\tau)^-, x_p) &= e^{A_p^{\text{diff}}\tau}\Pi_p x_p - \eta^{\tau} \\ &= k_{p+1}^{\tau} + \eta^{\tau} - \eta^{\tau} \\ &= k_{p+1}^{\tau}, \end{aligned}$$

as desired.

Conversely, let x_p be such that for all τ there exists an impulse-free solution (x, u) of $E_p\dot{x} = A_p x + B_p u$ with $x(t_p^-) = x_p$ and $x((t_p+\tau)^-) \in \underline{\mathcal{K}}_{p+1}$. Using the same inductive arguments as in Lemma 3.7 and utilizing A_p^{diff} invariance of $\operatorname{im}\Pi_p$, $\mathcal{R}_p$, $\mathcal{D}_p^{\text{imp}}$, it then follows for all $\tau \in \mathbb{R}$ that

$$x_p \in \operatorname{im}\Pi_p \cap \left(e^{-A_p^{\text{diff}}\tau}\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\right) \oplus \mathcal{D}_p^{\text{imp}} = e^{-A_p^{\text{diff}}\tau}\left(\operatorname{im}\Pi_p \cap \left(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\right) \oplus \mathcal{D}_p^{\text{imp}}\right).$$

As this holds for all $\tau > 0$ it follows from Lemma A.1 that

$$x_p \in \langle \operatorname{im}\Pi_p \cap (\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p) \oplus \mathcal{D}_p^{\text{imp}} \mid A_p^{\text{diff}} \rangle = \underline{\mathcal{K}}_p.$$

This concludes the proof. □

Given this result, we can present the following simple characterization of quasi-causally impulse-controllable system classes.

Theorem 3.37. *The system class $\Sigma_{\mathtt{n}}$ is quasi-causally impulse-controllable if and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \underline{\mathcal{K}}_0.$$

Proof. ($\Rightarrow$) Suppose the system class is quasi-causally impulse-controllable. Consider the solution (x, u) of (3.1) with $x(t_0^+) = x_0$ and $u_{(t_0,t_f)}$ given by $\mathcal{U}_{t_f}(\sigma_{(t_0,t_f)}, x_0)$. Then by definition, the solution (x, u) is impulse-free on (t_0, t_f), in particular, $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}} = \underline{\mathcal{K}}_{\mathtt{n}}$ for all possible switching signals.

In the following, we want to show by induction that $x(t_i^-) \in \underline{\mathcal{K}}_i$ for $i \in \{\mathtt{n}-1, \ldots, 1, 0\}$. Hence, inductively, we may assume that if (x, u) satisfies $x(t_0^+) = x_0$ and u is defined by $\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0)$, then $x(t_i^-) \in \underline{\mathcal{K}}_i$ for all switching signals. We want to show that $x(t_{i-1}^-) \in \underline{\mathcal{K}}_{i-1}$ for any solution (x, u) of (3.1) with $x(t_0^+) = x_0$ and u given by $\mathcal{U}_{t_{i-1}}(\sigma_{(t_0,t_{i-1})}, x_0)$. For any $\tau > 0$, consider the switching signal $\bar{\sigma}$ with $\bar{\sigma}_{(t_0,t_{i-1})} = \sigma_{(t_0,t_{i-1})}$ and $\bar{t}_i = \bar{t}_{i-1} + \tau = t_{i-1} + \tau$. Let $\bar{u}$ be given by $\mathcal{U}_{\bar{t}_i}(\bar{\sigma}_{(t_0,\bar{t}_i)}, x_0)$, then the corresponding solution $(\bar{x}, \bar{u})$ is impulse-free and by induction assumption satisfies $\bar{x}(\bar{t}_i) \in \underline{\mathcal{K}}_i$. Since $\tau > 0$ was arbitary, Lemma 3.36 yields that $\bar{x}(t_{i-1}^-) \in \underline{\mathcal{K}}_{i-1}$ By causality, $u_{(t_0,t_{i-1})} = \bar{u}_{(t_0,t_{i-1})}$ and hence $x(t_{i-1}^-) = \bar{x}(t_{i-1}^-)$ which concludes the inductive proof. Since for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists an impulse-free solution (x, u) satisfying $x(t_0^+) = x(t_0^-) = x_0$ we can conclude that $x_0 \in \underline{\mathcal{K}}_0$ and hence

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \underline{\mathcal{K}}_0.$$

($\Leftarrow$) Let $\sigma \in \mathcal{S}_{\mathtt{n}}$. Recall that by definition for all $\sigma \in \mathcal{S}_{\mathtt{n}}$, for each mode $p \in \{0, 1, ..., \mathtt{n}-1\}$ and each $x_p \in \underline{\mathcal{K}}_p$ there exists an input $u^p(\cdot, x_p)$ on $[t_p, t_{p+1})$ such that the solution x of mode p satisfies $x(t_p^-) = x_p$ and $x(t_{p+1}^-) \in \underline{\mathcal{K}}_{p+1}$. Now, concatenate these inputs inductively as follows: $u(t) := u^0(t, x_0)$ for $t \in [t_0, t_1)$ and $u(t) := u^p(t, x(t_p^-))$ for $t \in [t_p, t_{p+1})$ where $x(t_p^-)$ is the value of the solution x corresponding to the already defined input u on $[t_0, t_p)$. Finally, by assumption $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$, hence the input u can be extended on $[t_{\mathtt{n}}, \infty)$ in such a way that the solution remains impulse-free. Altogether we can define $\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0) := u_{(t_0,t_i)}$ which satisfies the quasi-causality properties for all switching signals and all x_0. Hence the system class is quasi-causally impulse-controllable. $\square$

Causal impulse-controllability with dwell time

Knowledge of the current mode duration cannot always be assumed, hence we want to provide in this subsection a characterization of a more strict causality notion. To that extent we will study the notion of *causal impulse-controllability*. Here causality can be considered with respect to the whole switching signal (*i.e.*, the impulse-avoiding input should not depend on the future mode sequence as well as on the future mode durations) or only with respect to the future mode duration (*i.e.*, the future mode sequence is assumed to be known and can be used in the construction of the impulse-avoiding input).

In both cases causality can be defined in terms of the existence of a family of input-defining maps

$$\mathcal{U}_t : (\sigma_{[t_0,t)}, x_0) \mapsto u_{[t_0,t)},$$

such that for all $\sigma \in \mathcal{S}_{\mathtt{n}}$ (or $\overline{\mathcal{S}}_{\mathtt{n}}$) and all $x_0 \in \mathcal{V}_{(E_{\sigma(t_0)},A_{\sigma(t_0)},B_{\sigma(t_0)})}$ the corresponding solution $x_{[t_0,t)}$ of $(E_\sigma, A_\sigma, B_\sigma)$ on $[t_0, t)$ for the input $u_{[t_0,t)} = \mathcal{U}_t(\sigma_{[t_0,t)}, x_0)$ is impulse-free; additionally, we have to require that this map $\mathcal{U}_t$ is itself causal, *i.e.*, for all $s_1 < s_2$ it holds that

$$\mathcal{U}_{s_1}(\sigma_{[t_0,s_1)}, x_0) = \mathcal{U}_{s_2}(\sigma_{[t_0,s_2)}, x_0)_{[t_0,s_1)}. \tag{3.19}$$

Furthermore, we need to restrict the class of switching signals by requiring a dwell time condition, *i.e.*, we have to assume a lower bound for the mode duration. Without such a bound the control input needs to steer the state into a "safe" subspace within any given time-interval, but then the causality property of $\mathcal{U}_t$ cannot hold: Consider the situation that for some switching times $t_i < t_{i+1}$ we have $t_i < s_1 < s_2 := t_{i+1}$, the input produced by $\mathcal{U}_{s_1}$ must result in a state value $x(s_1^-)$ in a suitable subspace (to allow for a possible switch at s_1) while $\mathcal{U}_{s_2}$ has more time to achieve that $x(s_2^-)$ is in a suitable subspace, consequently, without the dwell time condition, if s_1 is approaching the switching time t_i the input needs to be more and more aggressive and hence $\mathcal{U}_{s_1}$ cannot remain equal to the initial part of $\mathcal{U}_{s_2}$.

In order to derive a necessary and sufficient condition for causal impulse-controllability for the system class $\Sigma_{\mathtt{n}}$ with a dwell time, we consider the following sequence:

$$\underline{\mathcal{C}}_{i-1} := \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1} + \ker E_{i-1}, \tag{3.20}$$

for $i = \{\mathtt{n}, \mathtt{n}-1, ..., 1\}$ and with $\underline{\mathcal{C}}_{\mathtt{n}} = \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$.

We can now provide a simple necessary and sufficient condition for causal impulse-controllability in terms of the sequence $\underline{\mathcal{C}}_i$ given by (3.17).

Theorem 3.38. *The system class $\Sigma_{\mathtt{n}}$ with some dwell time $d > 0$ is causally impulse-controllable if and only if $\mathcal{V}_{(E_0,A_0,A_0)} \subseteq \underline{\mathcal{C}}_0^{\mathrm{imp}}$.*

Proof. $(\Rightarrow)$ Suppose the system is causal with a dwell time $d > 0$. Then given a switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$ there exists an impulse-free solution (x, u) where $u_{[t_0,t)} = \mathcal{U}_t(\sigma_{[t_0,t)}, x_0)$.

We will proof by induction that $x(t_i^-) \in \underline{\mathcal{C}}_i$ for all $i \in \{\mathtt{n}, \mathtt{n}-1, ..., 1\}$. Since (x, u) is impulse-free, it follows that $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}} = \underline{\mathcal{C}}_{\mathtt{n}}$. Hence we assume that the statement holds for i and continue to proof the statement for $i-1$.

Consider now another switching signal $\tilde{\sigma} \in \mathcal{S}_{\mathtt{n}}$ (with dwell time $d > 0$) such that $\sigma_{(t_0,t_i)} = \tilde{\sigma}_{(t_0,t_i)}$ (in particular, $\tilde{t}_i \geqslant t_i$) and with corresponding impulse-free solution $(\tilde{x}, \tilde{u})$, where $\tilde{u}_{[t_0,t)} = \mathcal{U}_t(\tilde{\sigma}_{[t_0,t)}, x_0)$ By the inductive assumption we have $\tilde{x}(\tilde{t}_i^-) \in \underline{\mathcal{C}}_i$. Consequently, we can always find an input $\tilde{u}$ on $[t_i, \tilde{t}_i)$ which ensures that the trajectory $\tilde{x}$ which starts at $x(t_i^-) \in \underline{\mathcal{C}}_i$ stays in the same subspace for arbitrary $\tilde{t}_i > t_i$ under the dynamics of $E_{i-1}\dot{x} = A_{i-1}x + B_{i-1}u$. Consequently, $x(t_i^-)$ must be contained in the largest controlled invariant subspace within $\underline{\mathcal{C}}_i$, *i.e.*, $x(t_i^-) \in \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle$.

Consequently, it follows from the solution formula for differential algebraic equations that

$$e^{A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \Pi_{i-1} x(t_{i-1}^-) \in \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1}.$$

By Lemma 2.48 we have

$$A_{i-1}^{\mathrm{diff}} \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle \subseteq \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1} \tag{3.21}$$

and hence

$$\begin{aligned} &\Pi_{i-1} x(t_{i-1}^-) \\ &\in e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \left(\langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1}\right) \\ &\subseteq \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1}. \end{aligned}$$

Since (x, u) is impulse-free it follows that $x(t_{i-1}^-) \in \mathcal{C}_{i-1}^{\text{imp}}$ and hence $(I - \Pi_{i-1})x(t_{i-1}^-) \in \mathcal{R}_{i-1} + \ker E_{i-1}$. Altogether, we conclude the inductive proof by observing that

$$\begin{aligned} x(t_{i-1}^-) &= \Pi_{i-1}x(t_{i-1}^-) + (I - \Pi_{i-1})x(t_{i-1}^-) \\ &\in \langle \underline{\mathcal{C}}_i \mid E_{i-1}, A_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1} + \ker E_{i-1} \\ &= \underline{\mathcal{C}}_{i-1}. \end{aligned}$$

Now we can conclude that $x_0 \in \underline{\mathcal{C}}_0$ and since this holds for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ we have shown the necessity part of the statement.

($\Leftarrow$) Let $x_i \in \underline{\mathcal{C}}_i$. Then then $x_i = c_{i+1} + \eta_i + z_i$ with $c_{i+1} \in \langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle$, $\eta_i \in \mathcal{R}_i$ and $z_i \in \ker E_i$. Consequently, by Lemma 2.48 it follows that

$$A_i^{\text{diff}} \langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle \subseteq \langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i$$

and hence we have

$$\begin{aligned} e^{A_i^{\text{diff}} d} x_i &= e^{A_i^{\text{diff}} d}(c_{i+1} + \eta_i + z_i) \\ &\in e^{A_i^{\text{diff}} d} \left(\langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i + \ker E_i \right) \\ &\subseteq \langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i + \ker E_i. \end{aligned}$$

Hence multiplying by Π_i and using the result of Lemma 2.49 we obtain

$$\begin{aligned} \Pi_i e^{A_i^{\text{diff}} d} x_i &= e^{A_i^{\text{diff}} d} \Pi_i x_i \\ &\in \Pi_i \left(\langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i + \ker E_i \right) \\ &\subseteq \langle \underline{\mathcal{C}}_{i+1} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i. \end{aligned}$$

Consequently, $e^{A_i^{\text{diff}} d} \Pi_i x_i = c_{i+1} + \eta_i$ for some $c_{i+1} \in \underline{\mathcal{C}}_{i+1}$ and $\eta_i \in \mathcal{R}_i$. Let u_d^i be a smooth input such that $x[t_i] = 0$ and $x_u(d, 0) = -\eta_i$. Then the solution (x, u) on $[t_i, t_{i+1})$ with $x(t_i^-) = x_i$ is impulse-free and satisfies $x(t_{i+1}^-) = c_{i+1} \in \underline{\mathcal{C}}_{i+1}$; furthermore, due to the controlled invariance of $\underline{\mathcal{C}}_{i+1}$ it is possible to extend $u_d^i(\cdot, x_i)$ onto $[t_i, t_{i+1})$ such that the corresponding solution satisfies $x(t^-) \in \overline{\mathcal{C}}_{i+1}^{\text{imp}}$ for all $t \in [t_i + d, t_{i+1})$. Now, concatenate these inputs inductively as follows: $u(t) := u_d^0(t, x_0)$ for $t \in [t_0, t_1)$ and $u(t) := u_d^i(t, x(t_i^-))$ for $t \in [t_i, t_{i+1})$ where $x(t_i^-)$ is the value of the solution x corresponding to the already defined input u on $[t_0, t_i)$. Finally, by assumption $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}}$, hence the input u can by extended also on $[t_{\mathtt{n}}, \infty)$ in such a way that the solution remains impulse-free. Altogether, we can define $\mathcal{U}(\sigma_{[t_0,t)}, x_0) := u_{[t_0,t)}$ which satisfies the causality properties with a dwell time for all switching signals and all x_0. $\square$

3.3 Concluding remarks

In this chapter we have studied impulse-free solutions of switched differential algebraic equations with a fixed and known switching signal. In particular, a characterization of impulse-controllable switched DAEs with finitely many switches has been given based on an algorithm that runs backward in time. In the case of infinitely many switches sufficient conditions for impulse-controllability are given in terms of an algorithm that runs forward in time.

Next, impulse-controllability of system classes of switched DAEs has been investigated. System classes generated by arbitrary switching signals that are strongly impulse-controllable are characterized. For system classes generated by switching signals inducing the same mode sequence it was shown that there are several notions of impulse-controllabilty. Finally, necessary and sufficient conditions for quasi-causal impulse-controllability and causal impulse-controllability given a dwell time of Σ_n have been given.

If a switched system generated by some matrix triplets is impulse-controllable for some arbitrary switching signal, with a probability one the system class is essentially impulse-controllable, it remains to give necessary conditions for essential impulse-controllability of the system class Σ_n. The same holds for essential impulse-uncontrollability. Furthermore, a natural direction of research is to consider quasi causal and causal impulse-controllability given some dwell time of general system classes $\overline{\Sigma}_n$. Finally, it remains an open question under what conditions system classes are causally impulse-controllable without a dwell time.

4 | Stabilizability

In this chapter we are interested in the concept of stabilizability for switched differential algebraic equations. In particular, we investigate to what extent the solutions can be influenced by means of applying an input such that the system is stabilized while guaranteeing the absence of Dirac impulses. In the previous chapter we showed that a switched DAE is impulse-controllable if Dirac impulses in the state can be avoided for any initial trajectory. We call a system stabilizable if any initial trajectory can be steered asymptotically to the origin as time tends to infinity. However, for a switched DAE that is both stabilizable and impulse-controllable not necessarily every initial trajectory can be stabilized in an impulse-free manner. An example of such a system is given by the electrical circuit given below.

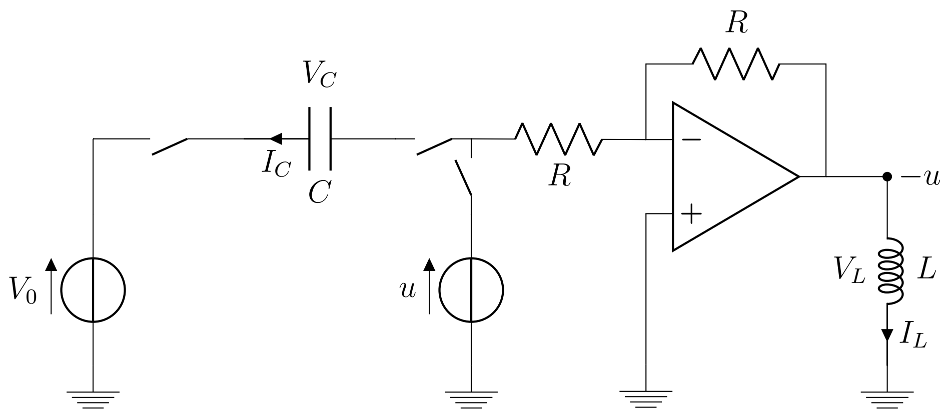

Figure 4.1: An example of an electrical circuit that is stabilizable and impulse-controllable, but not stabilizable without Dirac impulses.

Consider the circuit in Figure 4.1. Assume that for maintenance reasons the capacitor and the component consisting of the operational amplifier combined with an inductor are disconnected at $t = t_1$. In order to keep the network to which this circuit is connected operational, the voltage source V_0 needs to remain constant. However, there is another controllable voltage source u available. Since the system is operational at $t = t_0$ it is assumed that the state at t_0 is consistent. The equations describing the dynamics of the inductor and the capacitor are given by

$$L\dot{I}_L = V_L, \qquad \text{and} \qquad C\dot{V}_C = I_C,$$

respectively. Note that the voltage over the capacitor is given by $V_C = u - V_0$. Assuming that $R_1 = R_2$ and that the operational amplifier is ideal, we have $V_{\mathrm{OA}}^- = V_{\mathrm{OA}}^+ = 0$ and no current flows in or out of the operational amplifier. Consequently, $I_L = I_{R_1} = I_{R_2}$.

Since the current through the resistors is given by $I_{R_1} = I_{R_2} = -\frac{u}{R}$ and it follows that $RI_L = -u$.

Defining the state as $x = [\, V_L \; I_L \; V_C, I_C, V_0 \,]$, we obtain that for $t \in [t_0, t_1)$ the system is described by equation (4.1).

$$\begin{bmatrix} 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u, \tag{4.1}$$

$$\begin{bmatrix} 0 & L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x. \tag{4.2}$$

After opening the switch, the voltage over the resistor is zero and thus $I_{R_1} = I_{R_2} = I_L = 0$. Furthermore, since we are only interested in stabilizing the states of the components and V_0 will not affect the charge on the voltage once the switch is opened, we will assume $V_0 = 0$ for $t \in [t_1, \infty)$. Consequently, the system can be described on this interval by equation (4.2). Hence for a non-zero input u at t_1^-, we obtain that I_L jumps to zero at t_1^+ and consequently a Dirac impulse occurs in $V_L = L\dot{I}_L$. However, if the input is brought to zero smoothly, no Dirac impulses occur and hence the system is impulse-controllable.

Since the amount of charge stored on the capacitor is given by $q = C(V_0 - u)$, we have for every input u with $u(t_1^-) \neq V_0$, that the capacitor is charged and is unable to discharge, since the current $I_C = 0$. The capacitor can be discharged before t_1, but that requires a nonzero u at t_1^-, which produces a Dirac impulse yet stabilizes the state of the components. Hence we have an example of a system which is impulse-controlollable and stabilizable, but not stabilizable with an impulse-free trajectory.

Motivated by this example, this chapter considers stabilization of switched DAEs where Dirac impulses are to be avoided, so called impulse-free stabilization.

4.1 Stabilizability concepts

In this section we will introduce the concept of stabilizability for switched differential algebraic equations. To that extent, consider the system

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{4.3}$$

with a switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$. Since the switching signal is assumed to be fixed a priori, we assume that the switching times $t_1 < ... < t_{\mathtt{n}}$ are known. The index $\mathtt{n} \in \mathbb{N}$ is arbitrary and hence the switching signal possibly induces infinitely many switches. We will consider (4.3) on the interval $[t_0, \infty)$ for some $t_0 \in (-\infty, t_1)$ and hence we assume that the system was already active for $t \in (-\infty, t_0)$. Any initial trajectory is thus implicitly assumed to be a solution of (3.1), which implies that $x(t_0^-) = x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$.

Roughly speaking, in classical literature on non-switched systems, a linear system is called stabilizable if every trajectory can be steered to zero as time tends to infinity. This concept can elegantly be defined formally for non-switched systems in terms of its behavior. Given the behavior of a switched differential algebraic equations, we can extend this definition of behavioral stabilizability for non-switched systems readily to switched DAEs. To do so, recall the definition of the behavior of (4.3):

Definition 4.1. Consider the system (4.3) for some switching signal $\sigma \in \mathcal{S}_{\mathrm{n}}$. The behavior $\mathfrak{B}_\sigma$ is the set of all distribution pairs (x, u) that solve (4.3), *i.e.,*

$$\mathfrak{B}_\sigma := \{(x, u) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n+m} \mid E_\sigma \dot{x} = A_\sigma x + B_\sigma u\}.$$

Given the behavior of (4.3) we are able to define behavioral stabilizability for switched DAEs. As we will mainly focus on finding conditions such that a system can be stabilized in an impulse-free way, the definitions that follow define the impulse-free variant if the term *(impulse-free)* is read without parentheses.

Definition 4.2 ((Impulse-free) Stabilizability)**.** The switched DAE (4.3) with a switching signal $\sigma \in \mathcal{S}_{\mathrm{n}}$ is *(impulse-free) stabilizable* if for every solution $(x, u) \in \mathfrak{B}_\sigma$ there exists a solution $(x^*, u^*) \in \mathfrak{B}_\sigma$, (which is impulse-free on $[t_0, \infty)$), satisfying

$$(x^*, u^*)_{(-\infty,t_0)} = (x, u)_{(-\infty,t_0)} \quad \text{and} \quad \lim_{t\to\infty} \big(x^*(t^+), u^*(t^+)\big) = 0.$$

For many applications it is not sufficient to asymptotically steer the state of the system to the origin, but it is necessary to control the state to zero in finite time. Once the state of the system is zero, the system will remain at the origin if the input is switched off. Hence this phenomenon can be regarded as a special form of (impulse-free) stabilizability. If the system can be steered to zero on some bounded interval (t_0, t_f) we will call it time-t_f (impulse-free) null-controllable. For non-switched systems there exist various other definitions of controllability, which can be proven to be equivalent to null-controllability. These definitions define controllability *e.g.,* as the possibility to connect solutions or as the possibility to reach states from the origin. The latter definition is often referred to as reachability. These concepts can readily be extended to switched systems and impulse-free solutions. A formal definition of these concepts in terms of the systems behavior is given as follows.

Definition 4.3. Given some $t_f \in \mathbb{R}$, the switched DAE (4.3) with a switching signal $\sigma \in \mathcal{S}_{\mathrm{n}}$ is called

$i)$ *time-t_f (impulse-free) behaviorally controllable* if for all solutions $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_\sigma$ there exists a solution (x, u), (which is impulse-free on $[t_0, t_f)$) satisfying

$$(x, u) = \begin{cases} (x_1, u_1) & -\infty < t < t_0, \\ (x_2, u_2) & t_f < t < \infty, \end{cases} \tag{4.4}$$

$ii)$ *time-t_f (impulse-free) behaviorally null-controllable* if $i)$ holds for $(x_2, u_2) = (0, 0)$,

$iii)$ *time-t_f (impulse-free) behaviorally reachable* if $i)$ holds for $(x_1, u_1) = (0, 0)$.

Since we consider linear switched DAEs, the sum of solutions is again a solution. Consequently, we can proof the following equivalence between the controllability notions if Dirac impulses in the state are allowed.

Lemma 4.4. *Consider the system* (4.3). *The following statements are equivalent:*

$i)$ *The system is behaviorally null-controllable,*

$ii)$ *The system is behaviorally reachable,*

$iii)$ *The system is behaviorally controllable.*

Proof. $i) \Rightarrow ii)$ As the system is time-t_f null-controllable, there existss for every solution $(x, u) \in \mathfrak{B}_\sigma$ an solution (x^*, u^*), satisfying $(x, u)_{(-\infty, t_0)} = (x^*, u^*)_{(-\infty, t_0)}$ and $x^*(t_f^-) = 0$. Let $(\tilde{x}, \tilde{u})$ be an solution satisfying $\tilde{x}(t_0^-) = x(t_0^-) = x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}$. Then we obtain by linearity of solutions that $(\bar{x}, \bar{u})$ defined as $\bar{x} = x^* - \tilde{x}$ and $\bar{u} = u^* - \tilde{u}$ satisfies $\bar{x}(t_0^-) = 0$ and $\bar{x}(t_f^-) = \tilde{x}(t_f^-)$. Hence we can conclude that the system is time-t_f reachable.

$ii) \Rightarrow iii)$ Let $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_\sigma$ be solutions. As the system is reachable there exist solutions $(x_3, u_3), (x_4, u_4)$ satisfying $x_3(t_0^-) = x_4(t_0^-) = 0$ and $(x_3, u_3)_{(t_f, \infty)} = (x_1, u_1)_{(t_f, \infty)}$, $(x_4, u_4)_{(t_f, \infty)} = (x_2, u_2)_{(t_f, \infty)}$. Then by linearity of solutions we obtain that $(\bar{x}, \bar{u})$ defined by $\bar{x} = x_1 - x_3 + x_4$ and $\bar{u} = u_1 - u_3 + u_4$ satisfy

$$(\bar{x}, \bar{u}) = \begin{cases} (x_1, u_1) & -\infty < t \leqslant t_0, \\ (x_2, u_2) & t_f < t < \infty. \end{cases}$$

Since $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_\sigma$ were arbitrary solutions, it holds for all $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_\sigma$. Hence the system is controllable.

$iii) \Rightarrow i)$ Since the zero distribution is contained in $\mathfrak{B}_\sigma$, it follows that if the system is controllable, there exists for every solution $(x, u) \in \mathfrak{B}_\sigma$ an solution (x^*, u^*) satisfying $(x, u)_{(-\infty, t_0)} = (x^*, u^*)_{(-\infty, t_0)}$ and $x^*(t_f^-) = 0$. Hence the system is null-controllable. $\square$

Since the various controllability concepts of (null-)controllabillity and reachability are equivalent, we will simply refer to these concepts as controllability (at t_f). A similar statement in terms of the impulse-free versions of the controllabillity concepts does not hold as the following example shows. Hence we need to make a clear distinction between the concepts when impulse-freeness has to be guaranteed.

Example 4.5. Consider the switched DAE given by

$$\Sigma := \begin{cases} \dot{x} &= 0 \qquad t_0 \leqslant t < t_1, \\ \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \dot{x} &= x \qquad t_1 \leqslant t < t_2, \\ \dot{x} &= u \qquad t_2 \leqslant t < \infty. \end{cases}$$

As the mode on the interval $[t_2, \infty)$ is controllable, the system is clearly time-t_f impulse-free reachable for all $t_f \geqslant t_2$. Futhermore, the system is time-t_f null-controllable for any $t_f \geqslant t_1$ as every state will jump to zero at t_1. However, in the case $x_2(t_0^-) \neq 0$ a Dirac impulse will occur inevitably at t_1 and consequently the system is not time-t_f impulse-free null-controllable. ◇

Finding conditions on the existence of a t_f such that the system is (impulse-free) controllable in a finite amount of steps might become cumbersome as a switching signal possibly induces an large number of switches. The same problem arises when it comes to finding conditions for (impulse-free) stabilizability. However, for some cases, *e.g.*, for switched DAEs with a periodic switching signal, it is still possible to conclude controllability and stabilizability based on some finite interval. If all initial values can be contracted sufficiently over a period of the switching signal by means of applying a suitable input on that time interval, any initial value can be steered asymptotically to zero by applying a concatenation of such inputs. Hence we would be able to conclude existence of an input that stabilizes the system. To formalize this notion of contraction on a bounded interval to systems with a general switching signal we introduce the following definition.

Definition 4.6 ((Impulse-free) interval-stabilizability)**.** The switched DAE (4.3) is called (t_0, t_f)-*(impulse-free) stabilizable* for a given switching signal σ, if for every solution $(x, u) \in \mathfrak{B}_\sigma$ there exists a class $\mathcal{KL}$ function[1] $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with

$$\beta(r, t_f - t_0) < r, \quad \forall r > 0 \tag{4.5}$$

and an (impulse-free) solution $(x^*, u^*) \in \mathfrak{B}_\sigma$ satisfying $(x, u)_{(-\infty, t_0)} = (x^*, u^*)_{(-\infty, t_0)}$ and

$$|x^*(t^+)| \leqslant \beta(|x_0|, t - t_0), \quad \forall t \in (t_0, t_f). \tag{4.6}$$

[1]A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is called a class $\mathcal{KL}$ function if 1) for each $t \geqslant 0$, $\beta(\cdot, t)$ is continuous, strictly increasing, with $\beta(0, t) = 0$; 2) for each $r \geqslant 0$, $\beta(r, \cdot)$ is decreasing and converging to zero as $t \to \infty$.

One should note that a local solution on some interval is not necessarily a part of a global solution on a larger interval. Consequently, stabilizability does not always imply that all local solutions are interval-stabilizable. The switched system $0 = x$ on $[0, t_1)$ and $\dot{x} = 0$ on $[t_1, \infty)$ is obviously stabilizable, since the only global solution is the zero solution. However, on the interval $[t_1, s)$ there are nonzero local solutions which do not converge towards zero.

Furthermore according to Definition 4.6 it is required that the norm of the state is smaller at the end of an interval. This means that (impulse-free) interval-stability could depend on the length of the considered interval instead of the asymptotic behavior of the system. If for example some states converge to zero, whereas some grow unbounded regardless of the choice of input, it may occur that on some interval the system is (impulse-free) interval-stable as the norm of the state initially decreases. However, as time tends to infinity the norm of the state grows unbounded. Spefcifically, the state does not converge to zero and hence the system is not (impulse-free) stabilizable. However, under the following uniformity assumption on the switched DAE we can conclude global stabilizability.

Assumption 4.7 (Uniform interval-stabilizability)**.** Consider the switched system (4.3) with switching signal σ. Let $\tau_0 := t_0$ and assume that there exists an unbounded, strictly increasing sequence $\tau_i \in (t_0, \infty)$, $i \in \mathbb{N} \setminus \{0\}$, of non-switching times such that the system is (impulse-free) (τ_{i-1}, τ_i) -stabilizable with $\mathcal{KL}$ function β_i for which additionally it holds that

$$\beta_i(r, \tau_i - \tau_{i-1}) \leqslant \alpha r, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0}, \tag{4.7}$$

$$\beta_i(r, 0) \leqslant Mr, \quad \forall r > 0, \forall i \in \mathbb{N}_{>0}, \tag{4.8}$$

for some $\alpha \in (0, 1)$ and $M \geqslant 1$.

Proposition 4.8. *For systems of the form* (4.3) *satisfying Assumption 4.7, (impulse-free) interval-stabilizability implies (impulse-free) stabilizability.*

Proof. For uniformly interval-stabilizable systems (4.3), there exists an input u for every $x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}$ such that the solution (x, u) with $x(t_0^-) = x_0$ satisfies

$$x(t^+) \leqslant \beta_i(|x(t_{p_i}^-)|, t - t_{p_i}), \tag{4.9}$$

for $t \in [t_{p_i}, t_{q_i})$ and $i \in \mathbb{N}$. Let

$$\hat{\beta}_i(r, t - t_{p_i}) = Mr - (t - t_{p_i}) \frac{Mr(1 - \alpha)}{t_{q_i} - t_{p_i}},$$

that is, for each $r > 0$ the function $\hat{\beta}_i(r, \cdot)$ is linear on $[t_{p_i}, t_{q_i})$ and decreasing from Mr towards αMr. Let

$$\beta(r, t) := \max\{\beta_i(\alpha^i r, t - t_{p_i}), \hat{\beta}_i(\alpha^i r, t - t_{p_i})\},$$

where $i \in \mathbb{N}$ is such that $t \in [t_{p_i}, t_{q_i})$. For fixed t the function $\beta(\cdot, t)$ is continuous and strictly increasing. From (4.7) and $M \geqslant 1$ it follows that

$$\beta(r, t_{q_i}^-) = \max\{\beta_i(\alpha^i r, t_{q_i} - t_{p_i}), M\alpha^i r\} = M\alpha^i r$$

and invoking (4.8),

$$\beta(r, t_{p_i}^-) = \max\{\beta_i(\alpha^i r, 0), M\alpha^i r\} = M\alpha^i r.$$

Because $q_i = p_{i+1}$, continuity of $\beta(r, \cdot)$ with fixed $r > 0$ follows. Furthermore, on each interval $[t_{p_i}, t_{q_i})$ the function $\beta(r, \cdot)$ is strictly decreasing as a maximum of two strictly decreasing functions. Furthermore, $\beta(r, t_{p_i}) = M\alpha^i r$ with $\alpha \in (0, 1)$ implies that $\beta(r, t)$ converges to zero as $t \to \infty$. Hence β is a $\mathcal{KL}$-function and it remains to be shown that $|x(t^+)| \leqslant \beta(|x(t_0^-)|, t)$.

First observe that by (4.7) and continuity of β_i it follows that

$$|x(t_{p+1}^-| = |x(t_{q_i}^-| \leqslant \beta_i(|x(t_{p_i}^-)|, t_{q_i} - t_{p_i}) \leqslant \alpha |x(t_{p_i}^-|,$$

and thus $|x(t_{p_i}^-| \leqslant \alpha^i |x(t_0^-)|$. Therefore,

$$|x(t^+)| \leqslant \beta_i(|x(t_{p_i}|, t_0 - t_{p_i}) \leqslant \beta_i(\alpha^i |x(t_0^-)|, t - t_{p_i}) \leqslant \beta(|x(t_0^-)|, t).$$

Hence we can conclude that $\lim_{t\to\infty} x(t) = 0$. Consequently, the system is stabilizable. □

Assumption 4.7 is readily satisfied for certain classes of systems, such as the class of systems with a periodic switching signal and the class of systems with a finite amount of modes. Therefore we turn our attention to finding necessary and sufficient conditions for interval-stabilizability. The methods used to analyze impulse-free interval-stabilizabilty of the system (4.3) will also lead to a characterization of impulse-free controllability. The conditions that need to be verified are based on sequences that run forward in time and hence are suitable to verify for systems with a large amount of modes.

4.2 Impulse-free interval-stabilizability

As shown in the previous section, a switched DAE which is impulse-controllable and stabilizable is not necessarily impulse-free stabilizable. However, if there does not exist an impulse-free solution (x, u) satisfying $x(t_0^-) \in \mathcal{V}_{(E_0, A_0, B_0)}$, there is no hope of finding an impulse-free solution (x^*, u^*) satisfying $x^*(t_0^-) = x_0$ and (4.6) for some class $\mathcal{KL}$ function β satisfying (4.5). Obviously, in order to stabilize a state on a bounded interval in an impulse-free way, there needs to exist an impulse-free solution in the first place. To that extent, we will make the following standing assumption throughout the rest of this section:

Assumption 4.9. Consider the system (4.3) and assume it is impulse-controllable. Furthermore, any initial trajectory x^0 is assumed to satisfy $x^0(t_0^-) = x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$.

Under this assumption, we will derive necessary and sufficient conditions for impulse-free stabilizability. The approach taken is as follows. First we consider the space of points that can be reached in an impulse-free way from an initial value x_0. It will then be shown that this space is an affine subspace. We then consider an element of this affine subspace with minimal norm; if this norm is smaller than the norm of the corresponding initial value, we can conclude interval-stabilizability.

4.2.1 Impulse-free stabilizability

We will start our investigation by considering the space of points that can be reached from an initial condition $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. The reason that we do not consider the spaces (3.7) in our analysis, is because although for each $\xi \in \mathcal{W}_i^{\tau}$ there exists an initial value x_0 and an impulse-free solution (x, u) satisfying $x(t_0^-) = x_0$, the converse of this statement is not necessarily true. That is, not for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exist an element $\xi \in \mathcal{W}_i^{\tau}$ and a solution (x, u) satisfying $x(t_0^-) = x_0$ and $x(t_i) = \xi$. Hence we consider the following sequence of (affine) subspaces (defined *forward* in time), given some $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$:

$$\begin{aligned} \mathcal{W}_0^{\tau}(x_0) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \mathcal{R}_0, \\ \mathcal{W}_i^{\tau}(x_0) &= e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i(\mathcal{W}_{i-1}^{\tau}(x_0) \cap \mathcal{C}_i^{\text{imp}}) + \mathcal{R}_i, \ i > 0. \end{aligned} \tag{4.10}$$

For $x_0 = 0$ we drop the dependency on x_0 and adopt the following notation:

$$\overline{\mathcal{W}_i^{\tau}} := \mathcal{W}_i^{\tau}(0),$$

for notational convenience.

Remark 4.10. The space $\overline{\mathcal{W}_i^{\tau}}$ defined above is different from $\mathcal{W}_i^{\tau}$ in (3.7); the latter is defined as the space of all points that can be reached in an impulse-free way.

The intuition behind the sequence is as follows: $\mathcal{W}_0^{\tau}(x_0)$ contains all values x_{t_1} for which there exists an impulse-free solution (x, u) satisfying $x(t_0^-) = x_0$ and $x(t_1^-) = x_{t_1}$. Now, inductively, we calculate the set $\mathcal{W}_i^{\tau}(x_0)$ of points which can be reached just before the switching time t_{i+1} by first considering the points $\mathcal{W}_{i-1}^{\tau}(x_0)$ which can be reached in an impulse-free way just before t_i. Then we pick those which can be continued in mode i impulse-freely by intersecting them with $\mathcal{C}_i^{\text{imp}}$ and propagate this set forward according to the evolution operator. Finally the reachable space of mode i is added. This intuition is verified by the following lemma.

Lemma 4.11. *Consider the switched system* (4.3) *on some bounded interval* (t_0, t_f) *with the switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *Then for all* $i = 0, 1, \ldots, \mathtt{n}$ *and* $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$

$$\mathcal{W}_i^{\tau}(x_0) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ an impulse-free solution } (x,u) \\ \text{of (4.3) on } (t_0, t_{i+1}) \text{ s.t.} \\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi \end{array} \right\}.$$

Proof. First we will show that $x(t_{i+1}^-)$ is contained in $\mathcal{W}_i^{\tau}(x_0)$ if (x,u) is an impulse-free solution on (t_0, t_f). To that extent, consider an impulse-free solution (x,u) of (4.3) on (t_0, t_1), which by definition satisfies the solution formula (2.15), i.e.,

$$\begin{aligned} x(t_1^-) &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \int_{t_0}^{t} e^{A_0^{\text{diff}}(t-\tau)} B_0^{\text{diff}} u(\tau)\,\mathrm{d}\tau - \sum_{i=0}^{\nu-1} (E_0^{\text{imp}})^i B_0^{\text{imp}} u^{(i)}(t^-) \\ &= e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0, \end{aligned}$$

for some $\eta_0 \in \mathcal{R}_0$ and $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$. This shows that $x(t_1^-) \in \mathcal{W}_0^{\tau}(x_0)$. We proceed inductively by assuming that the statement holds for $i > 0$ and prove the statement for $i+1$.

Let (x,u) be an impulse-free solution on (t_0, t_{i+1}). Then we have that $x(t_{i+1}^-)$ is of the form

$$x(t_{i+1}^-) = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i \xi_{i-1} + \eta_i,$$

for some $\eta_i \in \mathcal{R}_i$ and $\xi_{i-1} \in \mathcal{C}_i^{\text{imp}}$. Furthermore, since (x,u) is impulse-free on (t_0, t_{i+1}), it follows that ξ_i can be reached impulse-freely from x_0 and hence $\xi_{i-1} \in \mathcal{W}_{i-1}^{\tau}(x_0)$. This proves that $x(t_{i+1}^-) \in \mathcal{W}_i^{\tau}(x_0)$.

In the following we will prove that for all elements of $\mathcal{W}_i^{\tau}(x_0)$ there exists an impulse-free solution (x,u) with initial condition $x(t_0^+) = x_0$. We will again prove this inductively. Therefore, consider $\xi_0 \in \mathcal{W}_0^{\tau}(x_0)$. Then for some $\eta_0 \in \mathcal{R}_0$ we have

$$\xi_0 = e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0 x_0 + \eta_0.$$

Since $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)} \subset \mathcal{C}^{\text{imp}}$, we have that there exists a $\tilde{u}$ such that the solution $(\tilde{x}, \tilde{u})$ satisfying $\tilde{x}(t_0^-) = x_0$ is impulse-free on $[t_0, t_1)$. Then it follows from the solution formula (2.15) that

$$\tilde{x}(t_1^-) = e^{A^{\text{diff}} t_1}\Pi x_0 + \tilde{\eta}_0,$$

for some $\tilde{\eta}_0 \in \mathcal{R}_0$. Since $\eta_0 \in \mathcal{R}_0$, there exists a smooth input $\hat{u}$ such that the solution $(\hat{x}, \hat{u})$ with $\hat{x}(t_0^-) = 0$ satisfies $\hat{x}(t_1^-) = \eta_0 - \tilde{\eta}_0$ and is impulse-free on $[t_0, t_1)$.

If we define $u = \hat{u} + \tilde{u}$ it then follows from linearity of solutions that (x,u) with $x(t_0^-) = x_0$ satisfies $x(t_1^-) = \xi_0$ and is impulse-free on (t_0, t_1). Assuming that the statement holds for $i > 0$ we continue by proving the statement for $i+1$.

Let $\xi_i \in \mathcal{K}^{\tau}_{i+1}(x_0)$, then we have for some $\xi_{i-1} \in \mathcal{W}^{\tau}_i(x_0) \cap \mathcal{C}^{\text{imp}}_{i-1}$ that

$$\xi_i = e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i\xi_{i-1} + \eta_i.$$

It follows from the induction assumption that there exists an impulse-free solution (x, u) on (t_0, t_i) with $x(t_i^-) = \xi_{i-1}$, because $\xi_{i-1} \in \mathcal{W}^{\tau}_i(x_0)$. Furthermore, $\xi_{i-1} \in \mathcal{C}^{\text{imp}}_{i-1}$ and $\eta_i \in \mathcal{R}_i$ implies that the impulse-free input u can be altered on the interval $[t_i, t_{i+1})$ such that $x(t_{i+1}^-) = \xi_i$ and $x(\cdot)$ is impulse-free. □

Remark 4.12. If the system is not impulse-controlollable, then there exist x_0 for which $\mathcal{W}^{\tau}_i(x_0) = \varnothing$ as follows from the definition. This also follows from the subspace algorithm because $\mathcal{W}^{\tau}_{i-1}(x_0) \cap \mathcal{C}^{\text{imp}}_i$ would be empty for some mode i and the sum of an empty set and a subspace is empty.

Lemma 4.11 gives rise to another characterization of impulse-controlollability, which follows as a corollary.

Corollary 4.13. *Consider the switched system* (4.3) *on some interval* (t_0, t_f) *with the switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$ *and the sequence of affine subspaces* $\mathcal{W}^{\tau}_i(x_0)$ *given by* (4.10). *Then* (4.3) *is impulse-controlollable on* (t_0, t_f) *if and only if*

$$\forall x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}: \quad \mathcal{W}^{\tau}_{\mathtt{n}}(x_0) \neq \varnothing.$$

Proof. If the system is impulse-contrololable, then for every initial condition x_0 there exists an impulse-free solution (x, u) on (t_0, t_f). Therefore $x(t_f^-) \in \mathcal{W}^{\tau}_{\mathtt{n}+1}(x_0)$ (recall the convention that $t_{\mathtt{n}+1} := t_f$) and hence $\mathcal{W}^{\tau}_{\mathtt{n}+1}(x_0) \neq \varnothing$. Conversely, if $\mathcal{W}^{\tau}_{\mathtt{n}+1}(x_0) \neq \varnothing$, then let $\xi \in \mathcal{W}^{\tau}_{\mathtt{n}+1}(x_0)$. By definition there exists an impulse-free solution (x, u) on (t_0, t_f) with $x(t_0^-) = x_0$ and $x(t_f^-) = \xi$. This holds for every $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ and hence (4.3) is impulse-contrololable. □

In the following we will show that $\mathcal{W}^{\tau}_i(x_0)$ is an affine shift of $\overline{\mathcal{W}}^{\tau}_i$ and hence $\mathcal{W}^{\tau}_i(x_0)$ is an affine subspace. In proving this statement, we will use some general results which can be found in Appendix C.

Lemma 4.14. *Consider the switched system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$ *and assume it is impulse-controllable. The impulse-free-reachable space from* x_0 *at* t_i *is an affine shift from the impulse-free reachable space, in particular, there exists a matrix* M_i*, such that*

$$\mathcal{W}^{\tau}_i(x_0) = \{M_i x_0\} + \overline{\mathcal{W}}^{\tau}_i. \tag{4.11}$$

Proof. First we simplify the notation by introducing the following shorthand notation $Y_i := e^{A_i^{\text{diff}}(t_{i+1}-t_i)}\Pi_i$. Then we prove the statement inductively. The statement holds trivially for $n = 0$, for $\mathcal{W}^{\tau}_0(x_0) = \{Y_0 x_0\} + \mathcal{R}_0$ and $\mathcal{R}_0 = \overline{\mathcal{W}}^{\tau}_0$. Hence we assume that the statement holds for n. Since we assumed that the system is impulse-controlollable, we have that $\mathcal{W}_i^\tau(x_0) \cap \mathcal{C}_{i+1}^{\text{imp}} \neq \varnothing$ for all x_0. Then for $n+1$ we obtain that

$$\begin{aligned}
\mathcal{W}_{i+1}^\tau(x_0) &= Y_{i+1}(\mathcal{W}_i^\tau(x_0) \cap \mathcal{C}_{i+1}^{\text{imp}}) + \mathcal{R}_{i+1} \\
&\overset{*}{=} Y_{i+1}((\{M_i x_0\} + \mathcal{W}_i^\tau) \cap \mathcal{C}_{i+1}^{\text{imp}}) + \mathcal{R}_{i+1} \\
&\overset{**}{=} Y_{i+1}(\{N_i M_i x_0\} + (\mathcal{W}_i^\tau \cap \mathcal{C}_{i+1}^{\text{imp}})) + \mathcal{R}_{i+1} \\
&= \{Y_{i+1} N_i M_i x_0\} + \overline{\mathcal{W}}_{i+1}^\tau,
\end{aligned}$$

for some matrix N_i, $i \in \{0, 1, ..., \mathtt{n}\}$, where $(*)$ follows from the induction step and $(**)$ follows from Proposition C.1 in Appendix C. Defining $M_{i+1} = Y_{i+1} N_i M_i$ yields the result. □

Note that the matrix M_i in (4.11) exists only in the case of an impulse-controllable system, otherwise M_i would also need to map to the empty set. In the case M_i does exist, this matrix can be chosen independently of x_0. It is however not necessarily unique, because M_{i+1} is dependent on N_i obtained from Proposition C.1 in a nonunique way. It follows from Lemma C.2 from Appendix C that N_i can be any matrix for which

$$\begin{aligned}
&1. \quad \operatorname{im}(N_i - I)M_i \subseteq \mathcal{R}_i, \\
&2. \quad \operatorname{im} N_i M_i \subseteq \mathcal{C}_{i+1}^{\text{imp}}.
\end{aligned} \tag{4.12}$$

Thus, from the proof of Lemma 4.14 together with Lemma C.2 from the Appendix the following constructive result can be obtained.

Corollary 4.15. *Consider the switched system* (4.3) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$ and assume it is impulse-controllable. Let $M_0 = e^{A_0^{\text{diff}}(t_1 - t_0)}\Pi_0$. Then for any choice of N_i satisfying* (4.12)*, a matrix M_{i+1} satisfying* (4.11) *can be calculated sequentially as follows:*

$$M_{i+1} = e^{A_{i+1}^{\text{diff}}(t_{i+2} - t_{i+1})\Pi_{i+1}} N_i M_i.$$

Remark 4.16. In order to compute an N_i that satisfies (4.12) we can invoke Lemma C.3 from the Appendix. This means that given projectors onto $\mathcal{R}_i$ and $\mathcal{C}_{i+1}^{\text{imp}}$, an N_i that satisfies the conditions (4.12) can be constructed by solving

$$(I - \Pi_{\mathcal{R}_i})\Pi_{\mathcal{C}_{i+1}^{\text{imp}}} Q_i M_i = (I - \Pi_{\mathcal{R}_i})M_i, \tag{4.13}$$

for Q_i and defining $N_i := \Pi_{\mathcal{C}_{i+1}^{\text{imp}}} Q_i$. Since the existence of a solution of (4.13) is guaranteed by the assumption of impulse-controllability, cf. Lemma 4.14, such a matrix equation can be solved using a linear programming solver.

Since $\mathcal{W}_i^\tau(x_0)$ contains all the states that can be reached from x_0 in an impulse-free way, it follows that the norm of the state with minimal norm is given by the distance

$\mathrm{dist}(\mathcal{W}_i^\tau(x_0), 0)$. The computation of this distance is straightforward, because $\mathcal{W}_i^\tau(x_0)$ is an affine subspace. It follows from elementary linear algebra that the distance between an affine subspace and the origin, is equal to the norm of any element projected to the orthogonal complement of the vector space associated with the affine subspace. In the case of $\mathcal{W}_i^\tau(x_0)$ we would need to project onto $(\overline{\mathcal{W}}_i^\tau)^\perp$. To find such a projector, let $\Pi_{\overline{\mathcal{W}}_i^\tau}$ be an orthogonal projector onto $\overline{\mathcal{W}}_i^\tau$, then $(I - \Pi_{\overline{\mathcal{W}}_i^\tau})$ is projector onto $(\overline{\mathcal{W}}_i^\tau)^\perp$. An important property of these projectors is that their restriction to the corresponding augmented consistency space is well-defined.

Lemma 4.17. *Consider the DAE* (4.3) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$. For any $i \in \{0, 1, \dots, \mathtt{n}\}$ let $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$, then*

$$(I - \Pi_{\overline{\mathcal{W}}_i^\tau})\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Proof. From $\xi \in \mathcal{V}_{(E_i,A_i,B_i)}$ and $(I - \Pi_{\overline{\mathcal{W}}_i^\tau}) + \Pi_{\overline{\mathcal{W}}_i^\tau} = I$, it follows that

$$(I - \Pi_{\overline{\mathcal{W}}_i^\tau})\xi + \Pi_{\overline{\mathcal{W}}_i^\tau}\xi \in \mathcal{V}_{(E_i,A_i,B_i)}.$$

Since $\operatorname{im} \Pi_{\overline{\mathcal{W}}_i^\tau} = \overline{\mathcal{W}}_i^\tau$ and $\overline{\mathcal{W}}_i^\tau \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ we obtain

$$(I - \Pi_{\overline{\mathcal{W}}_i^\tau})\xi \in \mathcal{V}_{(E_i,A_i,B_i)} - \Pi_{\overline{\mathcal{W}}_i^\tau}\xi \subseteq \mathcal{V}_{(E_i,A_i,B_i)}.$$

as was to be shown. □

Consequently, the following result follows.

Lemma 4.18. *Consider the DAE* (4.3) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$ and assume it is impulse-controllable. For any M_i satisfying* (4.11) *we have that*

$$\min_{x \in \mathcal{W}_i^\tau(x_0)} |x| = |(I - \Pi_{\overline{\mathcal{W}}_i^\tau})M_i x_0|.$$

It follows that we can consider $(I - \Pi_{\overline{\mathcal{W}}_i^\tau})M_i$ as a linear map from the initial condition x_0 to the state with minimal norm in $\mathcal{W}_i^\tau(x_0)$. This allows us to formulate the following characterization of impulse-free stabilizability, which is independent of the initial condition x_0 and independent of any coordinate system.

Theorem 4.19. *Consider the switched DAE* (4.3) *with switching signal $\sigma \in \mathcal{S}_{\mathtt{n}}$ and assume it is impulse-contrololLable. Then the system is impulse-free interval-stabilizable on (t_0, t_f) if and only if*

$$\|(I - \Pi_{\overline{\mathcal{W}}_{\mathtt{n}}^\tau})M_{\mathtt{n}}\|_2 = \sup_{x \neq 0} \frac{|(I - \Pi_{\overline{\mathcal{W}}_{\mathtt{n}}^\tau})M_{\mathtt{n}} x|_2}{|x|_2} < 1.$$

Proof. It follows from Lemma 4.18 that $(I - \Pi_{\overline{\mathcal{W}}_{\mathrm{n}}^{\tau}})M_{\mathrm{n}}$ is the linear operator that maps x_0 to the element in $\mathcal{W}_{\mathrm{n}}^{\tau}(x_0)$ with minimal norm. Therefore we see that if $\|(I - \Pi_{\overline{\mathcal{W}}_{i}^{\tau}})M_i\|_2 < 1$ that for all x_0 there exists an input u such that

$$|x_u(t_f, x_0)| = |(I - \Pi_{\overline{\mathcal{W}}_{i}^{\tau}})M_i x_0| < |x_0|.$$

From this we can conclude that there exists a class $\mathcal{KL}$ function $\beta(|x_0|, t_f - t_0)$ such that the system is impulse-free interval-stabilizable in the sense of Definition 8.

Conversely, if the system is impulse-free interval-stabilizable, then there exists a trajectory for each initial condition $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ such that $|x_u(t_f^-, x_0)| \leqslant \beta_i(|x_0|, t_f - t_0) < |x_0|$. This means that for the operator $\Pi_{(\mathcal{K}_{\mathrm{n}}^{\tau})^{\perp}} M_{\mathrm{n}}$ that maps $|x_0|$ to the element with minimal norm that can be reached in an impulse-free way it must hold that

$$\|(I - \Pi_{\overline{\mathcal{W}}_{i}^{\tau}})M_{\mathrm{n}}\|_2 = \sup_{x \neq 0} \frac{|(I - \Pi_{\overline{\mathcal{W}}_{i}^{\tau}})M_{\mathrm{n}} x|_2}{|x|_2} < 1,$$

which proves the result. □

4.2.2 Impulse-free controllability

Thus far we have only focused on impulse-free interval-stabilizability. However, the methods used in the above are well suited to characterize impulse-free controllability. We start by proving the following.

Theorem 4.20. *Consider the system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathrm{n}}$. *There exists an impulse-free solution* $(x, u) \in \mathfrak{B}_\sigma$ *satisfying* $x(t_0^-) = x_0$ *and* $x(t_f^-) = 0$ *if and only if for some* $i \geqslant 0$

$$\mathcal{W}_i^{\tau}(x_0) \subseteq \overline{\mathcal{W}}_i^{\tau}.$$

Proof. If an initial condition x_0 is impulse-free null-controllable, there exists an input u such that (x, u) with $x(t_0^-) = x_0$ and $x(t_f^-) = 0$ is impulse-free. This means that $0 \in \mathcal{W}_{\mathrm{n}+1}^{\tau}(x_0)$. Consequently

$$\{0\} \subset \{M_{\mathrm{n}+1} x_0\} + \overline{\mathcal{W}}_{\mathrm{n}+1}^{\tau},$$

from which it follows that $M_{\mathrm{n}+1} x_0 \in \overline{\mathcal{W}}_{\mathrm{n}+1}^{\tau}$ and therefore $\mathcal{W}_{\mathrm{n}+1}^{\tau}(x_0) \subseteq \overline{\mathcal{W}}_{\mathrm{n}+1}^{\tau}$.

Conversely, assume that for some $i \geqslant 0$ we have $\mathcal{W}_i^{\tau}(x_0) \subseteq \overline{\mathcal{W}}_i^{\tau}$. Then it follows that $M_i x_0 \in \overline{\mathcal{W}}_i^{\tau}$ and thus $\{0\} \in \{M_i x_0\} + \overline{\mathcal{W}}_i^{\tau} = \mathcal{W}_i^{\tau}(x_0)$, which means that x_0 is impulse-free null-controllable. □

As a direct consequence we can state the following result.

Corollary 4.21. *Consider the switched system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$ *and assume it is impulse-controlollable. Then the system is impulse-free null-controllable on* (t_0, t_f) *if, and only if, for some* $i \in \{0, 1, \ldots, \mathtt{n}\}$

$$(I - \Pi_{\overline{\mathcal{W}}_i^{\tau}})M_i = 0.$$

Proof. If the system is impulse-null-controllable, we have that $\mathcal{W}_i^{\tau}(x_0) \subset \overline{\mathcal{W}}_i^{\tau}$ for all x_0. Then it follows that

$$M_i x_0 + \overline{\mathcal{W}}_i^{\tau} \subset \overline{\mathcal{W}}_i^{\tau},$$

for all x_0 and hence $\operatorname{im} M_i \subset \overline{\mathcal{W}}_i^{\tau}$. The result then follows.

Conversely, if $(I - \Pi_{\overline{\mathcal{W}}_i^{\tau}})M_i = 0$, then $(I - \Pi_{\overline{\mathcal{W}}_i^{\tau}})\mathcal{W}_i^{\tau}(x_0) = 0$ for all x_0, which implies that $\mathcal{W}_i^{\tau}(x_0) \subset \overline{\mathcal{W}}_i^{\tau}$ for all x_0. □

$\overline{\mathcal{W}}_i^{\tau}$ and M_i can both be computed sequentially forward in time. This means that it might not be necessary to have knowledge of all the modes of the switched system. According to Corollary 4.21 we can conclude impulse-free null-controllability already if the conditions are satisfied for some $i \in \mathbb{N}$.

4.3 Impulsive interval-stabilizability

For some applications the occurrence of Dirac impulses is irrelevant in the stabilization process. Example applications are those where only a part of the state is of interest or a certain output is to be stabilized, and any possible Dirac impulses will not be visible in the output. For such cases it suffices for a system to be stabilizable or controllable rather than impulse-free stabilizable or impulse-free controllable. When impulse-controllable systems are considered, stabilizability and controllability are implied by the impulse-free versions of these properties. The converse however is not necessarily true as was shown in the at the beginning of this chapter.

In the case that Dirac impulses are allowed in the trajectory similar results as in the previous section can be formulated. The crucial condition for impulse-free trajectories is that the state is in the impulse-controlollable space of the next mode at each switching instance. If this condition is dropped, a similar lemma as Lemma 4.11 can be formulated after considering the following sequence of sets

$$\begin{aligned} \widetilde{\mathcal{W}}_0^{\tau}(x_0) &= e^{A_0^{\text{diff}}(t_1 - t_0)}\Pi_0 x_0 + \mathcal{R}_0, \\ \widetilde{\mathcal{W}}_i^{\tau}(x_0) &= e^{A_i^{\text{diff}}(t_{i+1} - t_i)}\Pi_i \widetilde{\mathcal{W}}_{i-1}^{\tau}(x_0) + \mathcal{R}_i,\ i > 0, \end{aligned} \tag{4.14}$$

For $x_0 = 0$ we drop the dependency on x_0, *i.e.*,

$$\widetilde{\mathcal{W}}_i^{\tau} := \widetilde{\mathcal{W}}_i^{\tau}(0).$$

Lemma 4.22. *Consider the switched system* (4.3) *on some bounded interval* (t_0, t_f) *with the switching signal given by* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *Then for all* $i = 0, 1, \ldots, \mathtt{n}$

$$\widetilde{\mathcal{W}}_i^{\tau}(x_0) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ a solution } (x,u) \\ \text{of (4.3) on } (t_0, t_{i+1}) \text{ s.t.} \\ x(t_0^+) = x_0 \wedge x(t_{i+1}^-) = \xi \end{array} \right\}.$$

Proof. The proof is along similar lines as the proof of Lemma 4.11 when $\mathcal{C}_i^{\mathrm{imp}}$ is replaced by $\mathbb{R}^n$ for all $i \in \{1, 2, ..., \mathtt{n}\}$. □

It follows directly that $\widetilde{\mathcal{W}}_i^{\tau}(x_0)$ is an affine shift from $\widetilde{\mathcal{W}}_i^{\tau}$, whether the system is impulse-controlollable or not. This is formalized in the next lemma.

Lemma 4.23. *Consider the switched system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *Then* $\widetilde{\mathcal{W}}_i^{\tau}(x_0)$ *is an affine shift of* $\widetilde{\mathcal{W}}_i^{\tau}$, *i.e. for all* i *there exists a matrix* $\tilde{M}_i$ *such that*

$$\widetilde{\mathcal{W}}_i^{\tau}(x_0) = \{\tilde{M}_i x_0\} + \widetilde{\mathcal{W}}_i^{\tau}. \tag{4.15}$$

Proof. Denote $Y_i = e^{A_i^{\mathrm{diff}}(t_{i+1}-t_i)}\Pi_i$ for shorthand notation. Then for $i = 0$ we have $\tilde{M}_0 = Y_0$ satisfies (4.15). Hence assume the statement holds for i. Then if we define $\tilde{M}_{i+1} = Y_i\tilde{M}_i$ for $i+1$ we have that

$$\begin{aligned} \widetilde{\mathcal{W}}_{i+1}^{\tau}(x_0) &= Y_{i+1}\widetilde{\mathcal{W}}_i^{\tau}(x_0) + \mathcal{R}_i \\ &= Y_i(\{\tilde{M}_i x_0\} + \widetilde{\mathcal{W}}_i^{\tau}) + \mathcal{R}_i \\ &= \{Y_i\tilde{M}_i x_0\} + \widetilde{\mathcal{W}}_{i+1}^{\tau} \\ &= \{\tilde{M}_{i+1}x_0\} + \widetilde{\mathcal{W}}_{i+1}^{\tau}, \end{aligned}$$

which proves the statement. □

Lemma 4.24. *Consider the DAE* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *For any* $\tilde{M}_i$ *satisfying* (4.15) *we have that*

$$\min_{x \in \widetilde{\mathcal{W}}_i^{\tau}(x_0)} |x| = |(I - \Pi_{\widetilde{\mathcal{W}}_i^{\tau}})\tilde{M}_i x_0|.$$

Proof. The proof of Lemma 4.24 is analogous to the proof of Lemma 4.18. □

Theorem 4.25. *Consider the switched DAE* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$. *Then the system is stabilizable if and only if for any* $\tilde{M}_{\mathtt{n}}$ *satisfying* (4.15)

$$\|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\tau}})\tilde{M}_{\mathtt{n}}\|_2 = \sup_{x \neq 0} \frac{|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\tau}})\tilde{M}_{\mathtt{n}} x|_2}{|x|_2} < 1.$$

Proof. The proof follows the proof of Theorem 4.19 analogously. □

As was already shown at the beginning of this chapter, not every stabilizable system that is also impulse-controllable, is automatically impulse-free stabilizable. This can be explained by viewing $\mathcal{W}_i^{\boldsymbol{\tau}}(x_0)$ and $\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$ as affine subspaces. Note that since every state that can be reached impulse-free from x_0 is by definition also an element of $\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. This leads to the following result.

Lemma 4.26. *Consider the switched system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$ *and assume the system is impulse-controllable. Then*

$$\mathcal{W}_i^{\boldsymbol{\tau}}(x_0) \subseteq \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0).$$

Proof. This follows immediately from Lemma 4.11 and 4.22. □

As a consequence, we can state the following corollary.

Corollary 4.27. *Consider the system* (4.3) *with switching signal* $\sigma \in \mathcal{S}_{\mathtt{n}}$ *and assume it is impulse-controllable. Then for any* M_i *satisfying* (4.11) *we have*

$$\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0) = \{M_i x_0\} + \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}},$$

i.e. M_i *satisfies* (4.15).

Proof. Let $y \in \mathcal{W}_i^{\boldsymbol{\tau}}(x_0) \subseteq \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$ and consider an arbitrary element $x \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. Then we have that $x - y \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$. This means that $x = y + \tilde{\eta}$ for some $\tilde{\eta} \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$. However, since the system is impulse-controllable, by Lemma 4.14 there exists an M_i such that $y = M_i x_0 + \eta$ for some $\eta \in \overline{\mathcal{W}}_i^{\boldsymbol{\tau}}$. This means that for any $x \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$ we obtain that $x = M_i x_0 + \eta + \tilde{\eta} \subset M_i x_0 + \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. This proves that $\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0) \subseteq \{M_i x_0\} + \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$.

Consider $\alpha = M_i x_0 + \tilde{\eta}$ for some $\tilde{\eta} \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$. Then since $\mathcal{W}_i^{\boldsymbol{\tau}} \subseteq \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$ there exists an $\bar{\eta} \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$ and an $\eta \in \mathcal{W}_i^{\boldsymbol{\tau}}$ such that $\tilde{\eta} = \bar{\eta} + \eta$. Hence we obtain that $\alpha = M_i x_0 + \bar{\eta} + \eta = \beta + \eta$ for some $\beta \in \mathcal{W}_i^{\boldsymbol{\tau}}(x_0) \subset \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. But this means that for some $\tilde{M}_i$ satisfying (4.15) and $\hat{\eta} \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$ that $\alpha = \tilde{M}_i x_0 + \hat{\eta} + \eta$. Because $\hat{\eta} + \eta \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$ we have that $\alpha \in \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. Since α was chosen arbitrary, it follows that $\{M_i x_0\} + \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}} \subseteq \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}(x_0)$. □

Given that a system is impulse-controllable and stabilizable, we have that there exists an M_i satisfying (4.11) and we know that $\|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}}) M_{\mathtt{n}}\|_2 < 1$. However, the system is impulse-free stabilizable if and only if $\|(I - \Pi_{\overline{\mathcal{W}}_i^{\boldsymbol{\tau}}}) M_{\mathtt{n}}\|_2 < 1$. This is however not implied by the statement that $\|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}}) M_{\mathtt{n}}\|_2 < 1$. Indeed, since $\overline{\mathcal{W}}_i^{\boldsymbol{\tau}} \subseteq \widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}$ we have that $\operatorname{im}(I - \Pi_{\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}}) \subseteq \operatorname{im}(I - \Pi_{\overline{\mathcal{W}}_i^{\boldsymbol{\tau}}})$, which means that it could happen that there exists an initial condition $x_0 \neq 0$ for which

$$\frac{|(I - \Pi_{\overline{\mathcal{W}}_i^{\boldsymbol{\tau}}}) M_{\mathtt{n}} x_0|}{|x_0|} \geqslant 1, \quad \text{and} \quad \frac{|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\boldsymbol{\tau}}}) M_{\mathtt{n}} x_0|}{|x_0|} < 1.$$

Example 4.28. Again consider the example given in the at the beginning of this chapter. Consider the system on the interval $(0, t_f)$ with a switch at $t = t_1$. The matrices (E_0, A_0, B_0) correspond the system matrices given in (4.1) and (E_1, A_1, B_1) are the system matrices given in (4.2). Then it follows from the algorithm (4.14) that the reachable space of the switched system $\widetilde{\mathcal{W}}_1^{\tau}$, a suitable matrix $\tilde{M}_1$ and $I - \Pi_{\widetilde{\mathcal{W}}_1^{\tau}}$ are given respectively by

$$\widetilde{\mathcal{W}}_1^{\tau} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \ \tilde{M}_1 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}, (I - \Pi_{\widetilde{\mathcal{W}}_i^{\tau}}) = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}.$$

From which it follows that $\|(I - \Pi_{\widetilde{\mathcal{W}}_i^{\tau}})\tilde{M}_1\| = 0 < 1$ and hence the system is stabilizable. Moreover, it is controllable. However, the impulse-free reachable space $\mathcal{K}_1^{\tau}$ can be calculated from (4.10) and is given by

$$\overline{\mathcal{W}_1}^{\tau} = 0, \ (I - \Pi_{\overline{\mathcal{W}}_i^{\tau}}) = I, \ M_1 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} = (I - \Pi_{\overline{\mathcal{W}}_i^{\tau}})M_1,$$

from which it follows that $\|(I - \Pi_{\overline{\mathcal{W}}_i^{\tau}})\tilde{M}_1\| = 1$ and hence the system is not impulse-free stabilizable.

Remark 4.29. In the case V_0 becomes a control input after the switch the system would be null-controllable, but not impulse-free null-controllable. Furthermore, since the state of the initial condition can be reduced via an impulse-free trajectory, the system would also become impulse-free (interval) stabilizable. However, since there is no way of discharging the capacitor, it follows that there exists no input such that $\lim_{t \to \infty} x(t) = 0$.

Remark 4.30. All the results on stabilizability in this paper can be applied to switched ordinary differential equations (ODEs) without difficulty. In the case of a switched ODE we have $E_i = I$, $\Pi_i = I$, $B_i^{\text{diff}} = B_i$ and $A_i^{\text{diff}} = A_i$. Note that all solutions are trivially impulse-free, hence, impulse-free stabilizability is equivalent to stabilizability.

4.4 Concluding remarks

In this chapter we considered stabilizability of switched differential algebraic equations. It was shown that controllability, reachability and null-controllability are equivalent concepts for switched DAEs. Furthermore, we have introduced the notion of interval-stabilizability. Necessary and sufficient conditions for a DAE to be impulse-free interval-stabilizable have been presented. These conditions lead naturally to a novel characterization of impulse-free controllability of switched DAEs.

A natural future direction of research would be the investigation of controllers achieving interval-stabilizability for switched systems. The theory established in this thesis could be used as starting point in the search (for feedback) controllers. Furthermore, a natural extension would be to consider stabilizability properties of switched systems with unknown switching signals.

5 | The linear quadratic optimal control

In the previous chapters we have been concerned with control problems that require the controlled system to satisfy specific qualitative properties, such as impulse-controllability, (impulse-free) controllability and stabilizability. In the present chapter we will take into account quantitative aspects. Given a control system we will express the performance of the controlled system in terms of a cost functional. The control problem will then be to find all optimal controllers *i.e.*, all controllers that minimize the cost functional. Such controllers lead to a controlled system with optimal performance.

5.1 The linear quadratic regulator problem

Consider the switched differential algebraic system

$$\Sigma_\sigma \begin{cases} E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \\ \quad y = C_\sigma x + D_\sigma u, \end{cases} \qquad x(t_0^-) = x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}, \tag{5.1}$$

where $\sigma \in \mathcal{S}_{\mathrm{n}}$ is the switching signal, $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, $C_p \in \mathbb{R}^{q\times n}$ and $D_p \in \mathbb{R}^{q\times m}$, $n, m, p, q \in \mathbb{N}$. As we consider distributional solutions $(x, u) \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n+m}$, it follows that the output $y \in \mathbb{D}^q_{\mathrm{pw}\mathcal{C}^\infty}$. We assume that the switching signal $\sigma \in \mathcal{S}_{\mathrm{n}}$ is fixed and known a priori. That is, the order in which the modes appear and the switching times are assumed to be known. Hence the switched differential algebraic system (5.1) can be regarded as a particular time-varying system.

For many applications it is desired to keep all components of the output as "small" as possible. In the case of an initial value $x_0 \in \ker E_0$, applying a zero input will cause the state of the system to jump to the zero distribution $x = 0$ at $t = t_0$ and evolve around this stationary distribution. Consequently the output y will be the zero distribution as well. However, in the case that the initial value $x_0 \notin \ker E$, the output y will generally be some non-zero distribution, even if no input is applied. However, we need to be careful when we say that a distribution is "small".

In the case that the solution space consists of locally integrable functions, the $\mathcal{L}_2$ norm squared of the output can be considered as a measure of the output. The requirement of having the output as small as possible can then be expressed by requiring the $\mathcal{L}_2$ norm of the output to be as small as possible by means of applying a suitable input. In the case of non-switched DAEs and ODEs this optimal control problem is the well known linear quadratic regulator problem for DAEs and ODEs. Since we adopt the piecewise-smooth distributional framework as a solution space for switched DAEs, we cannot simply take the $\mathcal{L}_2$ norm of the output as a measure. However, for switched DAEs we would like to formulate a similar problem within the distributional solution framework. Specifically, as a distribution is not a function in the classical sense but a map from a function space to the real numbers, the space $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ cannot simply be equipped with the $\mathcal{L}_2$ norm. Instead, the $\mathcal{L}_2$ induced norm of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ will be considered. To do so, the space of test-functions $\mathcal{C}_0^\infty$ needs to be equipped with the $\mathcal{L}_2$ norm, such that the $\mathcal{L}_2$ induced norm of a linear functional $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is then given by

$$\|D\|_2 = \sup_{\substack{\varphi \in \mathcal{C}_0^\infty \\ \|\varphi\|_2 = 1}} |D(\varphi)|. \tag{5.2}$$

As solutions (x, u) of (5.12) generally contain Dirac impulses and Dirac impulses are not induced by locally integrable functions, we need specify how to regard the $\mathcal{L}_2$ induced norm of the Dirac delta distribution, *i.e.*, $\|\delta\|_2$. The following result shows that the Dirac delta is an unbounded distribution.

Lemma 5.1. *Consider $\mathcal{C}_0^\infty$ equipped with the $\mathcal{L}_2$ norm. Then a distribution $D \in \mathbb{D}$ satisfies*

$$\|D\|_2 = \begin{cases} \|f\|_{\mathcal{L}_2}, & D \in \{f_{\mathbb{D}} \mid f \in \mathcal{L}_2\}, \\ \infty, & D \notin \{f_{\mathbb{D}} \mid f \in \mathcal{L}_2\}. \end{cases}$$

Proof. First we will show that $\|D\|_2 = \|f\|_{\mathcal{L}_2}$ if D is induced by an square integrable function, *i.e.*, $D = f_{\mathbb{D}}$ for some $f \in \mathcal{L}_2$. Note that it follows from the Cauchy-Schwartz inequality that

$$f_{\mathbb{D}}(\varphi) = \int_0^\infty f(t)\varphi(t)\, dt \leqslant \left(\int_0^\infty f(t)^2\, dt\right)^{\frac{1}{2}} \left(\int_0^\infty \varphi(t)^2\, dt\right)^{\frac{1}{2}} = \|f\|_{\mathcal{L}_2} \|\varphi\|_{\mathcal{L}_2},$$

which shows that $\|f_{\mathbb{D}}\|_\infty \leqslant \|f\|_{\mathcal{L}_2}$. It remains to show that this upperbound is the smallest upperbound. However, as $\mathcal{C}_0^\infty$ is dense in $\mathcal{L}_2$ [110, Theorem 3.14] there exists a sequence $\{\varphi_n\} \in \mathcal{C}_0^\infty$ that converges to $f \in \mathcal{L}_2$. By continuity of $f_{\mathbb{D}}$ we have that

$$\lim_{n\to\infty} f_{\mathbb{D}}\left(\frac{\varphi_n}{\|f\|_{\mathcal{L}_2}}\right) = \lim_{n\to\infty} \int_0^\infty \frac{f(t)\varphi_n(t)}{\|f\|_{\mathcal{L}_2}}\, \mathrm{d}t = \left(\int_0^\infty f(t)^2\, \mathrm{d}t\right)^{\frac{1}{2}} = \|f\|_{\mathcal{L}_2},$$

which proves the result.

Next assume that D is not induced by a square integrable function. Recall that as $\mathcal{C}_0^\infty$ is dense in $\mathcal{L}_2$, their dual spaces are equal (see Lemma D.1 in Appendix D). Hence every bounded linear map from $\mathcal{C}_0^\infty$ can be represented as $f_{\mathbb{D}}$ for some $f \in \mathcal{L}_2$. Since D is not induced by some $f \in \mathcal{L}_2$, it follows that

$$\|D(\varphi)\|_2 = \infty,$$

which concludes the proof. $\square$

Remark 5.2. According to Lemma 5.1 the Dirac delta distribution is unbounded over $\mathcal{C}_0^\infty$ with respect to the induced $\mathcal{L}_2$ norm. Consequently we can conclude that the Dirac delta is not a continuous linear functional [109, Theorem 1.32]. This might seem contradictory as $\delta \in \mathbb{D}$, *i.e.*, the space of linear and continuous functionals on $\mathcal{C}_0^\infty$. However, continuity of distributions $D \in \mathbb{D}$ is not defined in terms of the induced $\mathcal{L}_2$ norm, but with respect to a different topology.

According to Lemma 5.1 it thus makes sense to consider the induced $\mathcal{L}_2$ norm of the output y as a measure. Keeping the output as small as possible can thus be understood as finding an input to (5.1) such that $\|y\|_2$ is as small as possible. As a direct consequence of Lemma 5.1 we can conclude that if the $\|y\|_2$ is bounded for a given input, the output is impulse-free.

Corollary 5.3. *If there exists a distributional solution* $(x, u) \in (\mathbb{D}_{pw\mathcal{C}^\infty})^{n+m}$ *of* (5.1) *such that* $\|y\|_2 < \infty$ *then* y *is impulse-free. Moreover,* y *is generated by some piecewise-smooth function.*

As the output is required to be impulse-free in order for an optimal control to exist we assume that for each $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ there exists a solution (x, u) satisfying $x(t_0^-) = x_0$ such that the corresponding output is impulse-free. Abusing notation slightly, we can write

$$\|y\|_2 = \int_0^\infty \|y(t)\|^2 \, \mathrm{d}t, \tag{5.3}$$

where $\|\cdot\|$ denotes the Euclidean norm.

In the case the piecewise-smooth distributions are considered as solutions of (5.1), the solution x is uniquely determined by the input u and the initial value $x(t_0^-)$. Hence there is mathematically no problem considering the whole space $\left(\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}\right)^m$ as the input space. However, from a practical point of view impulsive inputs are often undesirable as Dirac impulses are difficult to generate. Therefore, we would like to focus on finding an impulse-free input that minimizes (5.3). In that case the optimal control problem for switched DAEs within the distributional framework can be formulated as follows.

Problem 5.4. Consider the system (5.1). Find an impulse-free input $u \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^m$ that solves the following problem:

$$\begin{aligned} \min \qquad J(x_0, u) &= \int_{t_0}^\infty \|y(t)\|^2 \, \mathrm{d}t, \\ \text{s.t.} \qquad E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \\ y &= C_\sigma x + D_\sigma u, \\ x(t_0^-) &= x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}. \end{aligned} \tag{5.4}$$

Here $y(t)$ denotes the output generated by the solution (x, u) satisfying $x(t_0^-) = x_0$. A solution (x, u) for which (5.2) is minimal for all $u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$ will be referred to as an optimal solution and we will call x and u the optimal trajectory and optimal control respectively.

Since the switched differential algebraic system (5.1) is linear and the integrand in the cost functional is a quadratic function of y, the problem is called *linear quadratic*. Of course $\|y\|^2 = x^\top C_\sigma^\top C_\sigma x + 2u^\top D_\sigma^\top C_\sigma x + u^\top D_\sigma^\top D_\sigma u$, so the integrand can also be considered as a quadratic functional of (x, u). As an infinite time horizon is considered in this problem, it is referred to as the *infinite horizon* problem.

Due to the quadratic nature of the cost functional and the linearity of the constraints Problem 5.4 we are also able to prove the following necessary results. Namely, if there exists an input that solves Problem 5.4, there exists a linear map between the optimal input and the optimal trajectory. To prove this result, we first define the value function $V(x, t)$ as follows.

Definition 5.5. Consider Problem 5.4. The value function $V(x, t)$ is defined as

$$V(x_0, t) = \inf_u J(x_0, u) = \inf_u \int_t^\infty \|C_\sigma x(t) + D_\sigma u(t)\|^2 \,\mathrm{d}t, \tag{5.5}$$

where (x, u) is a local solution of (5.1) on $[t, \infty)$ satisfying $x(t^-) = x_0$.

The proof of the next results is along similar lines as the proof of Clements and Anderson in [19], but for the sake of completeness the proof is included here.

Lemma 5.6. *If there exists an input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ that solves Problem 5.4 then $u(t) = F(t)x(t)$ for some $F : \mathbb{R} \to \mathbb{R}^{m\times n}$.*

Proof. First we will show that the map $x_0 \mapsto u$ is linear, where $x(t_0^-) = x_0$ and u solves Problem 5.4; in particular, we will show that λu is the optimal control for any initial value λx_0 and that for any optimal inputs u_x, u_z corresponding to any initial values $x_0, z_0 \in \mathbb{R}^n$ the input $u_x + u_z$ is optimal for any initial value $y_0 = x_0 + z_0$.

To that extent, let $V(x_0, t)$ be the value function as defined in Definition 5.5. Applying the input λu to an initial condition λx_0 results in a trajectory λx, due to the linearity of solutions of the switched DAE. This means that $J(\lambda x_0, \lambda u) = \lambda^2 J(x_0, u)$ for any $\lambda \in \mathbb{R}$ and we can conclude that

$$\lambda^2 V(x_0, t_0) = \lambda^2 J(x_0, u) = J(\lambda x_0, \lambda u) = V(\lambda x_0, t_0). \tag{5.6}$$

Hence we can conclude that if u is the optimal input for x_0 that λu is the optimal input for λx_0. In the following we will prove if u and w are the optimal inputs for x_0 and z_0

respectively, that $u + w$ is the optimal input for $x_0 + z_0$. To do so, it will be shown that $V(x_0 + z_0, t_0) + V(x_0 - z_0, t_0) = 2V(x_0, t_0) + 2V(z_0, t_0)$. Observe that

$$\begin{aligned}
&\|C_\sigma(x+z) + D_\sigma(u+w)\|^2 + \|C_\sigma(x-z) + D_\sigma(u-w)\|^2 \\
&\qquad = (C_\sigma(x+z) + D_\sigma(u+w))^\top (C_\sigma(x+z) + D_\sigma(u+w)) \\
&\qquad\qquad + (C_\sigma(x-z) + D_\sigma(u-w))^\top (C_\sigma(x-z) + D_\sigma(u-w)) \\
&\qquad = 2(C_\sigma x + D\sigma u)^\top (C_\sigma x + D_\sigma u) + 2(C_\sigma z + D_\sigma w)^\top (C_\sigma z + D_\sigma w) \\
&\qquad = 2\|C_\sigma x + D_\sigma u\|^2 + 2\|C_\sigma z + D_\sigma w\|,
\end{aligned}$$

from which we can conclude that

$$J(x_0 + z_0, u + w) + J(x_0 - z_0, v - w) = 2J(x_0, u) + 2J(z_0, w). \tag{5.7}$$

Hence for all input u and w (and thus not necessarily the optimal ones) we obtain

$$\begin{aligned}
V(x_0 + z_0, t_0) + V(x_0 - z_0, t_0) &\leqslant J(x_0 + z_0, u + w) + J(x_0 - z_0, u - w) \\
&= 2J(x_0, u) + 2J(z_0, w),
\end{aligned}$$

which means that $V(x_0 + z_0, t_0) + V(x_0 - z_0, t_0) \leqslant 2V(x_0, t_0) + 2V(z_0, t_0)$. Conversely

$$\begin{aligned}
2V(x_0, t_0) + 2V(z_0, t_0) &\leqslant 2J(x_0, u) + 2J(z_0, w) \\
&= J(x_0 + z_0, u + w) + J(x_0 - z_0, u - w),
\end{aligned}$$

from which we can conclude that $2V(x_0, t_0) + 2V(z_0, t_0) \leqslant V(x_0 + z_0, t_0) + V(x_0 - z_0, t_0)$ and therefore the equality $V(x_0 + z_0, t_0) + V(x_0 - z_0, t_0) = 2V(x_0, t_0) + 2V(z_0, t_0)$ follows. Furthermore, if v_x is the optimal input for x and w_z is the optimal input for z then

$$\begin{aligned}
V(x_0 - z_0, t_0) + V(x_0 + z_0, t_0) &= 2V(x_0, t_0) + 2V(z_0, t_0) \\
&= 2J(x_0, u_x) + 2J(z_0, w_z) \\
&= J(x_0 + z_0, u_x + w_z) + J(x_0 - z_0, u_x - w_z).
\end{aligned}$$

Since $V(x_0 + z_0, t_0) \leqslant J(x_0 + z_0, u_x + w_z)$ and similarly $V(x_0 - z_0, t_0) \leqslant J(x_0 - z_0, u_x - w_z)$, it follows that

$$0 \leqslant J(x_0 + z_0, u_x + w_z) - V(x_0 + z_0, t_0) = V(x_0 - z_0, t_0) - J(x_0 - z_0, u_x - w_z) \leqslant 0,$$

and thus

$$V(x_0 + z_0, t_0) = J(x_0 + z_0, u_x + w_z),$$

which also shows that $u_x + w_z$ is optimal for $x_0 + z_0$.

Hence there exists a linear map between the optimal trajectory and the optimal input. In particular, the map $x(t_0^-) \mapsto u(t_0^+)$ is linear, *i.e.*, there exists a matrix $F(t_0) \in \mathbb{R}^{m\times n}$ such that $u(t_0) = F(t_0)x(t_0^-)$.

From the dynamic programming principle [4,12] it follows that $u_{[\tau,t_f)}$ is the optimal control for the cost function in Problem 5.4 considered on the interval $[\tau, t_f)$ for any $\tau \in [t_0, t_f)$, hence by replacing the initial time t_0 in the above argumentation by $\tau \in [t_0, t_f)$ we can conclude that for every $\tau \in [t_0, t_f)$ a matrix $F(\tau) \in \mathbb{R}^{m\times n}$ exists such that the optimal control satisfies $u(\tau) = F(\tau)x(\tau^-)$. □

Given this result, it follows as a corollary that if Problem 5.4 is solvable, the optimal cost is a quadratic function of the initial value. That is, if u is an input that solves Problem 5.4 and $J(x_0, u)$ is the corresponding optimal cost, we obtain the following result.

Corollary 5.7. *If there exists an input that solves Problem 5.4 then the optimal cost $J(x_0, u)$ is quadratic in $x(t_0^-)$, i.e.,*

$$J(x_0, u) = x(t_0^-)^\top K(t_0)x(t_0^-),$$

for some $K : \mathbb{R} \to \mathbb{R}^{n\times n}$.

Proof. Given the optimal feedback $u = K(t)x$ it follows that the value function is given by

$$V(x_0, t_0) = \int_{t_0}^{\infty} \|y(t)\|^2 \,\mathrm{d}t = \int_{t_0}^{\infty} \|(C_\sigma + D_\sigma F(t))x(t)\|^2 \,\mathrm{d}t.$$

Observe that the value function only depends on x_0 as the trajectory x is uniquely determined by this initial value and the optimal input. Clearly $V(\lambda x_0, t_0) = \lambda^2 V(x_0, t_0)$. Now consider the function defined as follows for a fixed t_0

$$\overline{V}(x_0, z_0, t_0) := V(x_0 + z_0, t_0) - V(x_0, t_0) - V(z_0, t_0).$$

Then by noting that the optimal control is linear in the state it follows that

$$\begin{aligned}\frac{1}{2}\overline{V}(x_0, z_0, t_0) &= \frac{1}{2}\Big(J(x_0 + z_0, F(x + z)) - J(x_0, Fx) - J(z_0, Fz)\Big)\\ &= \int_{t_0}^{\infty} ((C_\sigma + D_\sigma F(t))x(t))^\top ((C_\sigma + D_\sigma F(t))z(t)) \,\mathrm{d}t.\end{aligned}$$

Therefore, $\overline{V}(\alpha x_0, z_0, t_0) = \overline{V}(x_0, \alpha z_0, t_0) = \alpha\overline{V}(x_0, z_0, t_0)$ for $\alpha \in \mathbb{R}$, which shows that $\overline{V}(x_0, z_0, t_0)$ is bilinear in x_0 and z_0. This proves that $V(x_0, t_0)$ is quadratic in x_0, which means there exists a $K(t) \in \mathbb{R}^{n\times n}$ such that

$$V(x_0, t_0) = x_0^\top K(t_0)\bar{x}_0,$$

which proves the desired result. □

Remark 5.8. Although we consider optimal control of switched DAEs with a fixed switching signal, *i.e.*, the order in which the modes appear and the switching times are known a priori, Corollary 5.7 has several consequences for the case that the switching signal is considered an input. If chattering behavior is excluded and thus $\sigma \in \mathcal{S}_{\mathtt{n}}$, it follows that if there exists a solution (x, u, σ) for which $\|y\|_2$ is minimal, it follows from Corollary 5.7 that for this particular switching signal

$$J(x_0, u, \sigma) = x(t_0^-)^\top K(t_0) x(t_0^-).$$

Hence if there exists an optimal sequence in which the modes appear together with an optimal set of mode durations the optimal cost will be quadratic in the initial value.

The fact that we can regard the switched system (5.1) as a piecewise continous time-varying system allows for a dynamic programming approach to Problem 5.4. The approach relies on the principle of optimality as formulated by Bellman [4]. For the sake of completeness we state this principle with Problem 5.4 in mind.

Lemma 5.9. *Consider Problem 5.4. For all $\Delta t \in (t_0, \infty)$ the value function satisfies*

$$V(x_0, t) = \inf_{u_{[t_0, t_0+\Delta t)}} \left\{ \int_{t_0}^{t_0+\Delta t} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \, \mathrm{d}t + V\left(t_0 + \Delta t, x(t_0 + \Delta t^-)\right) \right\}.$$

Proof. Let

$$\overline{V}(x_0, t) = \inf_{u_{[t_0, t_0+\Delta t)}} \left\{ \int_{t_0}^{t_0+\Delta t} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \, \mathrm{d}t + V\left(t_0 + \Delta t, x(t_0 + \Delta t^-)\right) \right\},$$

where x is a local solution of (5.1) satisfying $x(t^-) = x_0$. By definition of the infimum there exists for every $\varepsilon > 0$ an input u_ε on $[t, \infty)$ such that $V(x_0, t) + \varepsilon = J(x_0, u_\varepsilon)$. Consequently we have

$$\begin{aligned} J(x_0, u_\varepsilon) &= \int_{t_0}^{t_0+\Delta t} \|C_\sigma x_\varepsilon(t) + D_\sigma u_\varepsilon(t)\|^2 \, \mathrm{d}t + J(x(t + \Delta t), u_\varepsilon) \\ &\geqslant \int_{t_0}^{t_0+\Delta t} \|C_\sigma x_\varepsilon(t) + D_\sigma u_\varepsilon(t)\|^2 \, \mathrm{d}t + V(x(t + \Delta t), t + \Delta t) \\ &\geqslant \overline{V}(x_0, t). \end{aligned}$$

Since ε can be chosen arbitrarily it follows that $\overline{V}(x_0, t) \leqslant V(x_0, t)$.

Conversely, we have

$$
\begin{aligned}
V(x_0,t) &= \inf_u \int_t^{\infty} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t \\
&= \inf_u \int_t^{t+\Delta t} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t + \int_{t+\Delta t}^{\infty} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t \\
&\leqslant \inf_u \int_t^{t+\Delta t} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t + V(x(t+\Delta t), t+\Delta t) \\
&\leqslant \inf_{u_{[t,t+\Delta t)}} \left\{ \int_t^{t+\Delta t} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t + V(x(t+\Delta t), t+\Delta t) \right\} \\
&= \overline{V}(x_0,t)
\end{aligned}
$$

and hence we can conclude that $\overline{V}(x_0,t) = V(x_0,t)$ □

It follows from Lemma 5.9 that (x,u) is optimal on $[t_0,\infty)$ if and only if $(x,u)_{[t,\infty)}$ is optimal for all $t \in [t_0,\infty)$. Hence if there exists optimal control on the interval $[t_{\mathtt{n}},\infty)$ and the resulting optimal cost is given by $x(t_{\mathtt{n}}^-)^\top P_{\mathtt{n}} x(t_{\mathtt{n}}^-)$ for some $P_{\mathtt{n}} \in \mathbb{R}^{n\times n}$, the problem of minimizing $J(x_0,u)$ in (5.4) on the interval $[t_0,\infty)$ reduces to the optimization of

$$
\begin{aligned}
J(x_0,u,t_{\mathtt{n}}) \;&= \int_{t_0}^{t_{\mathtt{n}}} \|y(t)\|^2 \,\mathrm{d}t + \int_{t_{\mathtt{n}}}^{\infty} \|y(t)\|^2 \,\mathrm{d}t \\
&= \int_{t_0}^{t_{\mathtt{n}}} \|y(t)\|^2 \,\mathrm{d}t + x(t_{\mathtt{n}}^-)^\top P_{\mathtt{n}} x(t_{\mathtt{n}}^-). \qquad (5.8)
\end{aligned}
$$

Regarding the cost functional (5.8) several observations can be made. First of all, as at $t = t_{\mathtt{n}}$ the $\mathtt{n}^{\text{th}}$ mode is already active, it remains to minimize (5.8) on the half open interval $[t_0,t_{\mathtt{n}})$. Due to the algebraic state variables of the mode active on $[t_{\mathtt{n}-1},t_{\mathtt{n}})$ the state $x(t_{\mathtt{n}}^-)$ is not necessarily equal to $x(t_{\mathtt{n}})$. Moreover, it might not even be well-defined.

The second observation is that the terminal cost matrix $P_{\mathtt{n}}$ in (5.8) can only be assumed to be positive semi-definite, as the $\mathtt{n}^{\text{th}}$ mode is not necessarily structurally related to the $(\mathtt{n}-1)^{\text{st}}$ mode. Hence even if each mode of (5.1) would be index-1 (and hence impulse-free for all impulse-free inputs), an optimal control might fail to exist as is illustrated in the next example.

Example 5.10. Consider the switched DAE with $\sigma \in \mathcal{S}_{\mathtt{n}}$ generated by the matrices

$$
\begin{aligned}
E_0 &= \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] & A_0 &= \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right], & B_0 &= \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right], & C_0 &= I, & D_0 &= 1, \\
E_1 &= \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], & A_1 &= \left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right], & B_1 &= \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], & C_1 &= 2\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right], & D_1 &= 1.
\end{aligned}
$$

i.e., we consider the switched DAE given by

$$
\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\dot{x} = \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]x + \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]u, \qquad t_0 \leqslant t < t_1, \qquad (5.9a)
$$

$$
\dot{x} = \left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]x, \qquad t_1 \leqslant t < \infty, \qquad (5.9b)
$$

with $x(t_0^-) = \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$ for some $x_0, y_0 \in \mathbb{R}$ and the cost functional

$$\begin{aligned} J(x_0, u) &= \int_{t_0}^{\infty} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \,\mathrm{d}t \\ &= \int_{t_0}^{t_1} (x_1(t)^2 + x_2(t)^2 + u(t)^2) \,\mathrm{d}t + \int_{t_1}^{\infty} (x_1(t) + x_2(t))^2 + u(t)^2 \,\mathrm{d}t. \end{aligned}$$

Note that the input does not affect the system on $[t_1, \infty)$. Since a nonzero input yields a nonzero cost resulting from this interval, we can conclude that the optimal input on $[t_1, \infty)$ satisfies $u_{[t_0,\infty)} = 0$ and hence the state x on the interval $[t_1, \infty)$ is given by $x(t) = e^{-t}x(t_1^-)$. Consequently the cost resulting from this interval is given by

$$\begin{aligned} \int_{t_1}^{\infty} 2(x_1(t) + x_2(t))^2 + u(t)^2 \,\mathrm{d}t &= \int_{t_1}^{\infty} x(t)^\top \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] x(t) \,\mathrm{d}t \\ &= x(t_1^-)^\top P_1 x(t_1^-) \int_{t_1}^{\infty} e^{-t} \,\mathrm{d}t \\ &= x(t_1^-)^\top P_1 x(t_1^-), \end{aligned}$$

where $P_1 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Note that this result can also be computed via the standard theory on the LQR problem with infinite horizon for ordinary differential equations, see *e.g.* [127].

Hence if there exists an input to (5.9) that minimizes $J(x_0, u)$ then there exists an input to (5.9a) that minimizes

$$J(x_0, u) = \int_{t_0}^{t_1} (x_1(t)^2 + x_2(t)^2 + u(t)^2)\mathrm{d}t + x(t_1^-)^\top P_1 x(t_1^-),$$

in which a finite horizon LQR problem for DAEs can be recognized.

Considering $t_0 = 0$ and $t_1 = 1$ and observing that (5.9a) states that $x_2(t) = -u(t)$ as well as $x_1(t) = x_0$ on $[0, 1)$ leads, after substitution, to

$$J(x_0, u) = x_0^2 + 2\int_0^1 u(\tau)^2 d\tau + (x_0 + u(1^-))^2 \geqslant x_0^2.$$

By choosing $u(t) = 0$ on $[0, 1-\varepsilon)$ for some $\varepsilon > 0$ and $u(t) = -x_0$ on $[1-\varepsilon, 1)$, we obtain

$$J(x_0, u) = x_0^2 + 2\int_{1-\varepsilon}^1 u(t)^2 = (1 + 2\varepsilon)x_0^2.$$

This shows that $\inf_u J(x_0, u) = x_0^2$. However for any input for which $u(1^-) = -x_0 \neq 0$ we have that $\inf_u J(x_0, u) < J(x_0, u)$, because $\int_0^1 u(t)^2 \,\mathrm{d}t \geqslant 0$. Hence there does not exist an optimal control. ◇

Remark 5.11. In the case of a structural relationship between the mode of interest and the corresponding terminal cost matrix, the consideration of $x(t)$ or $x(t^-)$ in the terminal cost might be indifferent. When for example the terminal cost matrix P_n is of the form $P_{\mathtt{n}} = E_{\mathtt{n}-1}^\top \bar{P}_{\mathtt{n}} E_{\mathtt{n}-1}$ for some positive semi-definite $\bar{P}_{\mathtt{n}}$ and $E_{\mathtt{n}-1}x$ is a continuous distribution on $[t_{\mathtt{n}-1}, t_{\mathtt{n}})$, it follows that

$$x(t_{\mathtt{n}}^-)^\top P_{\mathtt{n}} x(t_{\mathtt{n}}^-) = x(t_{\mathtt{n}})^\top P_{\mathtt{n}} x(t_{\mathtt{n}}).$$

and hence there is no difference in considering $x(t_{\mathtt{n}}^-)$ or $x(t_{\mathtt{n}})$ in the terminal cost. Indeed, if $E_{n-1}x$ is continuous, it follows that $E_{\mathtt{n}-1}x$ is induced by a continuous function and hence $E_{\mathtt{n}-1}x(t^-) = E_{\mathtt{n}-1}x(t)$. Hence the first and the second observation are closely related.

Although it might seem that a terminal cost of the form $E_{\mathtt{n}-1}^\top \bar{P}_{\mathtt{n}} E_{\mathtt{n}-1}$ is necessary, this is not the case. The following example shows that this is indeed only a sufficient condition.

Example 5.12. Consider the switched DAE generated by the matrices

$$\begin{aligned}
E_0 &= \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] & A_0 &= \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], & B_0 &= \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right], & C_0 &= I, & D_0 &= 1, \\
E_1 &= \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], & A_1 &= \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right], & B_1 &= \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], & C_1 &= I, & D_1 &= 1.
\end{aligned}$$

That is, we consider the switched DAE given by

$$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \dot{x} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right] u, \qquad t_0 \leqslant t < t_1, \tag{5.10a}$$

$$\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \dot{x} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] u, \qquad t_1 \leqslant t < \infty, \tag{5.10b}$$

with $x(t_0^-) = \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]$ for some $x_0, y_0 \in \mathbb{R}$ and the cost functional

$$\begin{aligned}
J(x_0, u) &= \int_{t_0}^{\infty} \|C_\sigma x(t) + D_\sigma u(t)\|^2 \mathrm{d}t \\
&= \int_{t_0}^{t_1} 2(x_1(t)^2 + x_2(t)^2 + u(t)^2)\ \mathrm{d}t + \int_{t_1}^{\infty} x_1(t)^2 + x_2(t)^2 + u(t)^2 \mathrm{d}t.
\end{aligned}$$

For an input with $u = 0$ on $[t_1, \infty)$ the solution on (t_1, ∞) is given by $e^{-t}\left[\begin{smallmatrix} 0 \\ x_2(t_1^-) \end{smallmatrix}\right]$ and thus the cost resulting form this interval is given by $x_2(t_1^-)^2$. Clearly this is minimal on the interval $[t_1, \infty)$ and thus it remains to minimize

$$J(x_0, u) = \int_{t_0}^{t_1} 2(x_1^2 + x_2^2 + u^2) + x_2(t_1^-)^2. \tag{5.11}$$

subject to (5.10a). The input given by $u(t) = 0$ for all $t \in [0, 1)$ results in $J(x_0, u) = x_1(t_0^-)^2$, which is clearly minimal. The terminal cost matrix in (5.11) is not of the form $E_0^\top P E_0$ and thus this example shows that a terminal cost weight matrix of this form is not necessary for existence of an optimal control. $\diamond$

The third observation to be made regarding the cost functional (5.8), is that $x(t_{\mathtt{n}}^-)$ is necessarily such that the output is impulse-free on $[t_{\mathtt{n}}, \infty)$. In general, there does not exist a solution (x, u) satisfying $x(t_{\mathtt{n}}^-) = x_{\mathtt{n}}$ for which the output $y(t)$ is impulse-free on $[t_{\mathtt{n}}, \infty)$ for all $x_{\mathtt{n}} \in \mathbb{R}^n$. However it is not difficult to show that the values for which there exists a solution (x, u) with $x(t_{\mathtt{n}}^-) = x_{\mathtt{n}}$ generating an impulse-free output form a subspace $\mathcal{V}^{\text{end}}$. Consequently, if we aim to solve Problem 5.4 via a dynamic programming approach, we have to optimize over all inputs that transfer the initial value to a state $x(t_f^-) \in \mathcal{V}^{\text{end}}$. These observations together give rise to the following *finite horizon* problem.

Problem 5.13. Consider the system (5.1). Find an impulse-free input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$ that solves the following problem:

$$\begin{aligned}
\min \qquad J(x_0, u, t_f) \ &= \int_{t_0}^{t_f} \|y(t)\|^2 \, \mathrm{d}t + x(t_f^-) P x(t_f^-), \\
\text{s.t.} \qquad\qquad E_\sigma \dot{x} \ &= A_\sigma x + B_\sigma u, \\
y \ &= C_\sigma x + D_\sigma u, \\
x(t_0^-) \ &= x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}, \\
x(t_f^-) \ &\in \mathcal{V}^{\text{end}},
\end{aligned}$$

where $\mathcal{V}^{\text{end}} \subseteq \mathbb{R}^n$ is a subspace. Here $y(t)$ denotes the output generated by the solution (x, u) satisfying $x(t_0^-) = x_0$. A solution (x, u) for which (5.2) is minimal for all $u \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}^m$ will be referred to as an optimal solution and we will call x and u the optimal trajectory and optimal control respectively.

Lemma 5.14. *If there exists an input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ that solves Problem 5.13 then $u(t) = F(t)x(t)$ for some $F : \mathbb{R} \to \mathbb{R}^{m \times n}$.*

Proof. Let u be an input such that the solution (x, u) with $x(t_0^-) = x_0$ satisfies $x(t_f^-) \in \mathcal{V}^{\text{end}}$ and let $\bar{u}$ be such that a solution $(\bar{x}, \bar{u})$ with $\bar{x}(t_0^-) = \bar{x}_0$ satisfies $\bar{x}(t_f^-) \in \mathcal{V}^{\text{end}}$. Then by linearity of solutions, it follows that the input $\tilde{u} = u + \bar{u}$ results in a solution $(\tilde{x}, \tilde{u})$ satisfying $\tilde{x}(t_f^-) \in \mathcal{V}^{\text{end}}$ if $\tilde{x}(t_0^-) = \tilde{x}_0 = x_0 + \bar{x}_0$. Given, this observation, the proof is analogous to the proof of Lemma 5.6, which is given in Appendix D. □

Similar to Corollary 5.7 we can also state the following result.

Corollary 5.15. *If there exists an input that solves Problem 5.13 then the optimal cost $J(x_0, u)$ is quadratic in $x(t_0^-)$, i.e.,*

$$J(x_0, u) = x(t_0^-)^\top K(t_0) x(t_0^-),$$

for some $K : \mathbb{R} \to \mathbb{R}^{n \times n}$.

The proof is analogous to the proof of Corollary 5.7, which is given in Appendix D. In order to solve the problem, we can first solve the finite horizon problem given a switched system on $[t_0, t_f)$ by solving $\mathtt{n}$ finite horizon problems for non switched differential algebraic equations, each on $[t_i, t_{i+1})$, $i \in [0, 1, ..., \mathtt{n}-1)$.

In the following section we will present necessary and sufficient conditions for solvability of Problem 5.13 for some system with a constant switching signal, *i.e.*, a non-switched DAE. These conditions will enable us to formulate conditions for solvabvility of the general case as we can regard the general case as a repeated non-switched DAE optimal control problem.

5.2 Finite horizon optimal control

In this section we will consider the finite horizon optimal control problem for non-switched differential algebraic equations on a finite horizon. Hence we consider the following system:

$$\Sigma := \begin{cases} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{cases} \tag{5.12}$$

together with the following cost functional

$$J(x_0, u, t_f) = \int_{t_0}^{t_f} \|y(t)\|^2 \, \mathrm{d}t + x(t_f^-)^\top P x(t_f^-). \tag{5.13}$$

We study this problem within the framework of switched DAEs. Consequently we have to assume that the initial trajectory is a solution of the previous mode and thus the initial value is not necessarily consistent. As a consequence, we have to consider arbitrary initial values $x(t_0^-) = x_0 \in \mathbb{R}^n$.

However, in order for (5.13) to be finite, it is necessary that the output is impulse-free on $[t_0, t_f)$ and at t_0 in particular. There exists an impulse-free input u such that the solution (x, u) satisfying $x(t_0^-) = x_0$ results in an impulse-free output if and only if x_0 contained in a particular subspace.

Lemma 5.16. *There exists an impulse-free input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ such that for the solution (x, u) satisfying $x(t_0^-) = x_0$ of* (5.12) *the output is impulse-free at t_0, i.e., $y[t_0] = Cx[t_0] + Du[t_0] = 0$, if and only if $x_0 \in \mathcal{C}^{\mathrm{imp}} + \mathcal{O}^{\mathrm{imp}}$ where*

$$\mathcal{O}^{\mathrm{imp}} := \ker \begin{bmatrix} CE^{\mathrm{imp}} \\ C(E^{\mathrm{imp}})^2 \\ \vdots \\ C(E^{\mathrm{imp}})^{\nu-1} \end{bmatrix}, \tag{5.14}$$

and ν is the index of nilpotency of E^{imp}.

Proof. ($\Rightarrow$) Suppose that there exists an impulse-free input such that $y[t] = 0$. Then since the input u is impulse-free, *i.e.*, $u[t] = 0$, it follows that $y[t] = Cx[t] + Du[t] = Cx[t]$. Consequently, the output is impulse-free for a given impulse-free input if and only if $x[t] \in \ker C$. In the case $u[t] = 0$ then it follows from the solution formula (2.14) that

$$Cx[t] = -C\sum_{i=0}^{\nu-1}(E^{\text{imp}})^i(I-\Pi)(x_0 - x(t_0^+))\delta^{(i)} = 0.$$

Consequently $(I-\Pi)(x_0 - x(t_0^+)) \in \ker C(E^{\text{imp}})^i$, for $i \in \{1, 2, ..., \nu - 1\}$. Hence we can conclude that $(I-\Pi)(x_0 - x(t_0^+)) \in \mathcal{O}^{\text{imp}}$. Since $(I-\Pi)x(t_0^+) \in \mathcal{C}^{\text{imp}}$ it follows that $(I-\Pi)x_0 \in \mathcal{O}^{\text{imp}} + \mathcal{C}^{\text{imp}}$. Finally, by recalling that $\operatorname{im}\Pi \subseteq \mathcal{C}^{\text{imp}}$ we can conclude that

$$\begin{aligned} x_0 &= \Pi x_0 + (I-\Pi)x_0 \\ &\in \mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}}, \end{aligned}$$

which proves the desired result.

($\Leftarrow$). Let $x_0 = p_0 + q_0$ for some $p_0 \in \mathcal{C}^{\text{imp}}$ and $q_0 \in \mathcal{O}^{\text{imp}}$. Then by definition of $\mathcal{C}^{\text{imp}}$ there exists an impulse-free input u such that (p, u) satisfying $p(t_0^-) = p_0$ is impulse-free, *i.e.*, $p[t] = 0$ for all $t \geqslant t_0$. Furthermore, as $E^{\text{imp}}(I-\Pi) = E^{\text{imp}}$ the solution $(q, 0)$ with $q(t_0^-) = q_0$ will satisfy

$$\begin{aligned} Cq[t_0] &= -C\sum_{i=0}^{\nu-1}\left(E^{\text{imp}}\right)^{i+1}(I-\Pi)\,q_0\delta^{(i)} \\ &= -C\sum_{i=0}^{\nu-1}\left(E^{\text{imp}}\right)^{i+1}q_0\delta^{(i)} \\ &= 0. \end{aligned}$$

Hence the solution $(q, 0)$ with $q(t_0^-) = q_0$ will only generate a Dirac impulse at t_0, which will not appear in the output y. By linearity of solutions, (x, u) with $x(t_0^-) = x_0$ will satisfy $x(t) = p(t) + q(t)$ and hence

$$\begin{aligned} y[t] &= Cx[t] + Du[t] \\ &= C(p[t] + q[t]) \\ &= Cq[t] \\ &= 0. \end{aligned}$$

Hence u is an impulse-free input such that (x, u) with $x(t_0^-) = x_0$ ensures $y[t] = 0$. $\square$

As the condition $x_0 \in \mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}}$ is necessary and sufficient for the existence of an impulse-free output, it is a necessary condition for the existence of an impulse-free input that minimizes (5.13), subject to (5.12). However, it suffices to consider initial values contained in $\mathcal{C}^{\text{imp}}$ only. Indeed, let $c_1, ..., c_p$ be a basis for $\mathcal{C}^{\text{imp}}$ and let $c_{p+1}, ..., c_j$ be vectors that are orthogonal to $c_1, ..., c_p$ and are such that $c_1, ..., c_j$ is a basis for $\mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}}$. Then clearly, $\text{span}\{c_{p+1}, ..., c_j\} \subseteq (\mathcal{C}^{\text{imp}})^{\perp}$, but equality does not hold in general. A solution $(x, \bar{u})$ with $x(t_0^-) \in \text{span}\{c_{p+1}, ..., c_j\}$ and $\bar{u} = 0$ will thus satisfy $x(t^-) = 0$ for $t \in (t_0, \infty)$ and a Dirac impulse will occur at t_0, although the output will remain impulse-free. Consequently $y = 0$ and the input is clearly optimal. Hence it remains to find an optimal input u for initial values $x(t_0^-) \in \mathcal{C}^{\text{imp}}$.

As the DAE (5.12) is assumed to be part of a switched DAE, we assume that the terminal cost matrix $P \in \mathbb{R}^{n\times n}$ is some arbitrary positive semi-definite matrix resulting from the cost on the interval $[t_f, \infty)$. Furthermore, we assume that in order to prevent Dirac impulses from occuring in the output at t_f, *i.e.*, to ensure $y[t_f] = 0$, it is required that $x(t_f^-) \in \mathcal{V}^{\text{end}}$ for some subspace $\mathcal{V}^{\text{end}} \subseteq \mathbb{R}^n$. Hence we restrict our attention to those inputs which transfer an initial condition $x_0 \in \mathcal{C}^{\text{imp}}$ to some state $x(t_f^-) \in \mathcal{V}^{\text{end}}$.

For this particular case Problem 5.13 can be reformulated as follows.

Problem 5.17. Consider the system (5.12). Find an impulse-free input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$ that solves the following problem:

$$\begin{aligned}
\text{min} \qquad J(x_0, u, t_f) \; &= \int_{t_0}^{t_f} \|y(t)\|^2 \,\mathrm{d}t + x(t_f^-)Px(t_f^-),\\
\text{s.t.} \qquad E\dot{x} \; &= Ax + Bu,\\
y \; &= Cx + Du,\\
x(t_0^-) \; &= x_0 \in \mathcal{C}^{\text{imp}},\\
x(t_f^-) \; &\in \mathcal{V}^{\text{end}},
\end{aligned}$$

where $\mathcal{V}^{\text{end}} \subset \mathbb{R}^n$ is some subspace. Here $y(t)$ denotes the output generated by the solution (x, u) satisfying $x(t_0^-) = x_0$.

5.2.1 Reformulation of the problem

In the following we will show that although (5.12) might be of higher index, solvability of Problem 5.17 is equivalent to an optimal control problem concerning an index-1 DAE. Because the initial values are considered to be in the impulse controllable space, *i.e.*, $x_0 \in \mathcal{C}^{\text{imp}}$ the input output behavior of (5.12) is equivalent to an impulse-controllable DAE. Hence we can instead of (5.12) itself, equivalently consider the impulse-controllable representation of (5.12) as a constraint in (5.17). That is, we consider

$$EW\dot{x} = Ax + Bv, \tag{5.15}$$

$$y = Cx + Dv, \tag{5.16}$$

where W is given as in (2.18). Recall that by Corollary 2.45 the DAE (5.15) is impulse-controllable, and hence by Lemma 2.41 we can apply a preliminary feedback of the form $u = Lx + v$ such that we obtain the system

$$\Sigma^{\text{aux}} : \begin{cases} EW\dot{x} &= (A + BL)x + Bv, \\ \quad y &= (C + DL)x + Dv, \end{cases} \tag{5.17}$$

which is of index-1. The following result shows that instead of trying to find an optimal input u, we can try to find an optimal input v such that the input $u = Lx + v$ is optimal for Problem 5.17.

Lemma 5.18. *Consider Problem 5.17 and let $\mathcal{C}^{\text{imp}}$ be the impulse-controllable space corresponding to the DAE (E, A, B). There exists an input $u \in \mathbb{D}_{pw\mathcal{C}^\infty}$ that solves Problem 5.17 if and only if there exists an input $v \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ that solves*

$$\begin{array}{lrl} \textit{min} & \bar{J}(x_0, v) & = \int_{t_0}^{t_f} \|(\bar{y}(t)\|^2 \ \mathrm{d}t + x(t_f^-)^\top P x(t_f^-), \\ & & \\ \textit{s.t.} & EW\dot{x} & = (A + BL)x + Bv, \\ & \bar{y} & = (C + DL)x + Dv, \\ & x(t_0^-) & = x_0 \in \mathcal{C}^{\text{imp}}, \\ & x(t_f^-) & \in \mathcal{V}^{\text{end}}, \end{array} \tag{5.18}$$

where W is given as in (2.18)*. Furthermore, the optimal inputs satisfies $u = Lx + v$, where (x, v) is the optimal solution of* (5.18)*.*

Proof. As $x_0 \in \mathcal{C}^{\text{imp}}$ it follows form Corollary 2.45 that the solution (x, u) solves (5.12) if and only if it solves (5.15). Hence we will consider solutions of (5.15). Applying a feedback to (5.15) can be regarded as a change of coordinates

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}. \tag{5.19}$$

Writing (5.15) as

$$\begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

enables us to write

$$\begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{x}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} = \begin{bmatrix} (A + BL) & B \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}.$$

Hence (x, u) solves (5.15) if and only if, $(\bar{x}, v)$ satisfying (5.19) solves (5.17). Furthermore, it follows naturally from that if $u = Lx + v$ that $J(x_{,0}\, u) = \bar{J}(\bar{x}_0, v)$. □

In the remainder of this section we assume that the index-reducing feedback has already been applied, *i.e.*, we assume that (EW, A) is of index-1, where W is given as in (2.18). Under this assumption we will show that given the index-1 system (5.12) Problem 5.17 can be reformulated as an equivalent optimal control problem for ordinary differential equations. This problem will not be a standard LQR problem for ODEs as the value of the input at t_f^- will be penalized by the terminal cost. However, this equivalent problem will be easier to analyze. Given that the DAE (5.12) is index-1, let the $(\Pi, \Pi^{\text{diff}}, \Pi^{\text{imp}})$ be the matrices resulting from the Wong sequences based on (EW, A) and define

$$A^{\text{diff}} = \Pi^{\text{diff}} A, \quad , B^{\text{diff}} = \Pi^{\text{diff}} B, \quad B^{\text{imp}} = \Pi^{\text{imp}} B.$$

Then we can decompose the state as

$$\begin{aligned} x &= x^{\text{diff}} + x^{\text{imp}} \\ &= x^{\text{diff}} - B^{\text{imp}} u \end{aligned} \tag{5.20}$$

and substitute (5.20) in the output y of (5.12). The following ODE system, which has the same input-output behavior is then obtained:

$$\overline{\Sigma} := \begin{cases} \dot{x}^{\text{diff}} &= A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u, \\ \bar{y} &= \bar{C} x^{\text{diff}} + \bar{D} u, \end{cases} \qquad x^{\text{diff}}(t_0^-) = \Pi x_0, \tag{5.21}$$

where $\bar{C} = C\Pi$ and $\bar{D} = (D - CB^{\text{imp}})$.

Remark 5.19. Any solution (x^{diff}, u) of (5.21) satisfying $x^{\text{diff}}(t_0^-) \in \operatorname{im}\Pi$ satisfies $\Pi x^{\text{diff}}(t) = x^{\text{diff}}(t)$ for all $t > t_0$. This follows from the fact that $\Pi A^{\text{diff}} = A^{\text{diff}}$ and $\Pi B^{\text{diff}} = B^{\text{diff}}$. Consequently the solution x^{diff} will satisfy $Cx^{\text{diff}} = C\Pi x^{\text{diff}}$, which shows that there is in fact no difference in defining $\bar{C} = C$ or $\bar{C} = C\Pi$. However, for computational reasons which will become apparent later, it is of more convenience to define $\bar{C} = C\Pi$.

To find necessary and sufficient conditions for solvability of Problem 5.17 we can thus either approach it with the DAE dynamics directly or, after some rewriting, with the ODE dynamics (5.21) and the cost functional

$$\bar{J}(x_0, u) = \int_{t_0}^{t_f} \|\bar{y}(t)\|^2 \, \mathrm{d}t + \left(x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-)\right)^\top P \left(x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-)\right).$$

This leads to the following problem formulation which is equivalent to Problem 5.17:

Problem 5.20. Consider the system (5.21). Find an impulse-free input $u \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})$ that solves the following problem:

$$\begin{aligned}
\min \qquad \bar{J}(x_0, u, t_f) &= \int_{t_0}^{t_f} \|\bar{y}(t)\|^2 \, \mathrm{d}t \\
&\qquad + ((x^{\mathrm{diff}} - B^{\mathrm{imp}}u)(t_f^-))^\top P((x^{\mathrm{diff}} - B^{\mathrm{imp}}u)(t_f^-)), \\
\text{s.t.} \qquad \dot{x}^{\mathrm{diff}} &= A^{\mathrm{diff}} x^{\mathrm{diff}} + B^{\mathrm{diff}} u, \\
\bar{y} &= \bar{C} x^{\mathrm{diff}} + \bar{D} u, \\
x^{\mathrm{diff}}(t_0^-) &= \Pi x_0, \\
x(t_f^-) &= x^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}} u(t_f^-) \in \mathcal{V}^{\mathrm{end}},
\end{aligned} \tag{5.22}$$

where A^{diff}, B^{diff} and B^{imp} are the matrices resulting from the Wong-sequences based on (EW, A). Furthermore $\bar{C} = C\Pi$ and $\bar{D} = D - CB^{\mathrm{imp}}$.

Lemma 5.21. *Consider the DAE* (5.12) *with the corresponding impulse-controllable space $\mathcal{C}^{\mathrm{imp}}$. If the matrix pair (EW, A) where W is given as in* (2.18) *is of index-1, then an input u solves Problem 5.17 if and only if it solves Problem 5.20.*

Recall that the Dirac impulse is an unbounded distribution with respect to the $\|\cdot\|_2$ norm. Hence to ensure that the optimal control is generated by an $\mathcal{L}_2$ integrable function, *i.e.*, impulse-free, it is often assumed that any Dirac impulse in the input, results in a Dirac impulse in the output. This is the case for the output in Problem 5.20 if the feedforward term $\bar{D}$ has full column rank.

Lemma 5.22. *If $\bar{D}$ has full column rank, then $\bar{y}[t] = 0$ implies $u[t] = 0$.*

Proof. We will proof the statement by contradiction. Suppose that $\bar{y}[t] = 0$, but the input $u[t] = \sum_{i=0}^{j} c_i \delta_t^{(i)}$ for some real constants $c_0, c_1, ..., c_j \in \mathbb{R}$ and some $j \in \mathbb{N}$. We will show that $c_i = 0$ for all $i \in \{1, 2, ..., j\}$. Note that

$$\bar{y}[t] = \bar{C} x^{\mathrm{diff}}[t] + \bar{D} u[t].$$

However, as x^{diff} is the solution of $\dot{x}^{\mathrm{diff}} = A^{\mathrm{diff}} x^{\mathrm{diff}} + B^{\mathrm{diff}} u$, it follows that the component $x^{\mathrm{diff}}[t] = \sum_{i=0}^{j-1} \xi_i \delta_t^{(i)}$ where the coefficient vectors ξ_i are generated by the backward recursion

$$\xi_j := 0, \quad \xi_{i-1} = A^{\mathrm{diff}} \xi_i + B c_i, \quad (i = j, ...0),$$

see [127, Thrm 8.5] and thus $x^{\mathrm{diff}}[t]$ is of one order less then $u[t]$. As the Dirac impulse and its derivatives are linearly independent, the statement $\bar{y}[t] = 0$ thus implies $\bar{D} c_j \delta^{(j)} = 0$. This can only be the case if $c_j = 0$. Consequently, we can write $u = \sum_{i=0}^{j-1} c_i \delta_t^{(i)}$ and by repeating the argument above it follows that $c_{j-1} = 0$. Repeating this argument j times, lead to the conclusion that $c_i = 0$ for all $i \in \{1, 2, ..., j\}$ which implies $u[t] = 0$. $\square$

In order to ensure that the columns of $\bar{D}$ are linearly independent, we make the following assumption.

Assumption 5.23. The output matrices of the system (5.12) are assumed to satisfy

$$\operatorname{rank}\begin{bmatrix} C\Pi_{\ker E} & D \end{bmatrix} = m,$$

where $\Pi_{\ker E}$ is a projector onto $\ker E$.

Remark 5.24. In the literature on optimal control on DAEs it is often assumed that the matrix D has full column rank, in addition to some rank condition on the matrix C, see *e.g.*, [8]. However, since we only require $\bar{D}$ to have full column rank, this assumption is too strict. It was also observed in [104] invertibility of $D^\top D$ is an artificial assumption in optimal control of differential algebraic equations.

Lemma 5.25. *Let Assumption 5.23 hold. Then $\bar{D} := D - CB^{\mathrm{imp}}$ has full column rank.*

Proof. Let $v \in \mathbb{R}^m$ be some vector. Then as $B^{\mathrm{imp}} \subseteq \ker E$ it follows that $B^{\mathrm{imp}} = \Pi_{\ker E} B^{\mathrm{imp}}$ for any projector $\Pi_{\ker E}$. Consequently it follows from Assumption 5.23 that

$$\begin{aligned}
\bar{D}v &= (D - CB^{\mathrm{imp}})v \\
&= \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} -B^{\mathrm{imp}} \\ I \end{bmatrix} v \\
&= \begin{bmatrix} C\Pi_{\ker E} & D \end{bmatrix} \begin{bmatrix} -B^{\mathrm{imp}} \\ I \end{bmatrix} v \\
&= 0,
\end{aligned}$$

if and only if $v = 0$. This proves that $\bar{D}$ has full column rank. □

Remark 5.26. For any feedback $u = Lx + v$ that reduces the index of the impulse-controllable representation of (5.12) the matrix $D - CB^{\mathrm{imp}} - DLB^{\mathrm{imp}}$ has full rank. The proof is analoguous to the proof of Lemma 5.25 once noted that for some $v \in \mathbb{R}^m$

$$(D - CB^{\mathrm{imp}} - DLB^{\mathrm{imp}})v = \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} B^{\mathrm{imp}} \\ I \end{bmatrix} v.$$

Given Assumption 5.23 it thus follows that if there exists an optimal control u that solves Problem 5.17, it is impulse-free. However, as an additional consequence of Assumption 5.23, the entire optimal solution (x, u) is required to be impulse-free.

Lemma 5.27. *Let Assumption 5.23 hold. The output satisfies $y[t] = 0$ if and only if $(x, u)[t] = 0$.*

Proof. If (x,u) is impulse-free it follows trivially that $y = Cx + Du$ is impulse-free.

Conversely, let $y = Cx + Du$ be impulse-free. Then by Lemma 5.22 the input u is impulse-free. Hence it remains to show that x is impulse-free. Assume for the sake of contradiction that $x[t] \neq 0$. Since $u[t] = 0$ it follows that $\Pi x[t] = x^{\text{diff}}[t] = 0$. Consequently, it remains to prove that $x^{\text{imp}}[t] = 0$. However, observe that as $u[t] = 0$ and $x^{\text{diff}}[t] = 0$ we have

$$\begin{aligned} y[t] &= Cx[t] + Du[t] \\ &= C(I-\Pi)x[t] + C\Pi x[t] + Du[t] \\ &= C(I-\Pi)x[t] + Cx^{\text{diff}}[t] + Du[t] \\ &= C(I-\Pi)x[t]. \end{aligned}$$

By assumption $\operatorname{rank}\left[\, C\Pi_{\ker E} \;\; D \,\right] = m$ and by definition $\operatorname{im}(I-\Pi) \subseteq \ker E$. Consequently

$$\begin{aligned} y[t] &= C(I-\Pi)x[t] \\ &= C\Pi_{\ker E}(I-\Pi)x[t] \\ &= \begin{bmatrix} C\Pi_{\ker E} & D \end{bmatrix} \begin{bmatrix} (I-\Pi)x[t] \\ 0 \end{bmatrix} = 0. \end{aligned}$$

implies $(I-\Pi)x[t] = 0$. □

Since we assume (EW, A) is index-1, we will restrict our attention in the remainder of this section to finding an input that solves Problem 5.20. If we find such an input, it follows from Lemma 5.21 that the input also solves Problem 5.17 if the DAE considered is of index-1. As non-switched ODEs are a special form of switched DAEs, it follows that Problem 5.20 is a special type of Problem 5.13. Consequently, it follows that if there exists an optimal control it is a feedback and that the optimal cost is a quadratic function of x_0, *i.e.*, $\bar{J}(x_0,u) = x_0^\top K x_0$. Hence in the remainder of this chapter, we aim to find the optimal feedback.

To illustrate the results of this section we introduce the following example, which demonstrates how to obtain the index-1 representation from an impulse-controllable DAE.

Example 5.28. Consider the following optimal control problem

$$\begin{array}{lrl} \text{min} & J(x_0,u) & = \int_0^1 \|y(t)\|^2 \, dt + \|x(t_f^-)\|^2, \\ \text{s.t.} & \begin{bmatrix} 1&0&0\\0&0&1\\0&0&0 \end{bmatrix} \dot{x} & = -x + \begin{bmatrix} 1\\0\\1 \end{bmatrix} u, \\ & y & = x + \begin{bmatrix} 1\\0\\0 \end{bmatrix} u, \\ & x(0^-) & = x_0, \\ & x(1^-) & \in \operatorname{span}\left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix} \right\}. \end{array} \tag{5.23}$$

As $\operatorname{im} E + A \ker E \subset \mathbb{R}^3$, it follows from Lemma 2.41 the system is not index-1. However, $\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^3$ and thus the system is impulse-controllable. Hence in order to find an optimal control, we need to apply a preliminary feedback. Any index-reducing feedback can be applied and therefore we choose $u = Fx = [\,0\ 1\ 1\,]\, x + v$. After calculating Π, Π^{diff} and Π^{imp} from the Wong-sequences based on $(E, A + BF)$ we can compute

$$\begin{aligned}
A^{\text{diff}} &= \Pi^{\text{diff}}(A+BF) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}, & \bar{C} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \\
B^{\text{diff}} &= \Pi^{\text{diff}} B = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, & \bar{D} &= D - CB^{\text{imp}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \\
B^{\text{imp}} &= \Pi^{\text{imp}} B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Finally decomposing the state as $x = x^{\text{diff}} - B^{\text{imp}} u$, we can write $\bar{C} = C$ and $\bar{D} = D - CB^{\text{imp}}$, which allows to rewrite the problem equivalently as follows. .

$$\begin{aligned}
\min \qquad & \bar{J}(x_0, v) &&= \int_0^1 \|\bar{y}(t)\|^2\, dt + \|x^{\text{diff}}(t_f^-) - B^{\text{imp}} v(t_f^-)\|^2, \\
\text{s.t.} \qquad & \dot{x}^{\text{diff}} &&= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix} x^{\text{diff}} + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} v, \\
& \bar{y} &&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} x^{\text{diff}} +, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v, \\
& x^{\text{diff}}(0^-) &&= \Pi x_0, \\
& x^{\text{diff}}(1^-) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v(t_f^-) &&\in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.
\end{aligned} \tag{5.24}$$

Note that in the reformulated problem the initial value is contained in the augmented consistency space corresponding to the original system, *i.e.*, $\Pi x_0 \in \mathcal{V}_{(E,A,B)}$. ◇

5.2.2 Regarding the terminal cost

Observe that the terminal cost term in Problem 5.20 penalizes the value of the input u at t_f^-. Furthermore, recall that the choice of input is free in the sense that it is allowed to be any piece-wise smooth distribution, and thus any value $u(t_f^-)$ can be ensured. Consequently, an input u with a value $u(t_f^-)$ that minimizes the terminal cost with respect to the resulting $x^{\text{diff}}(t_f^-)$ can be chosen. However, as the terminal cost penalizes the value of u at t_f^- from the left and this value needs to be well-defined, the input u needs to be continuous on at least $[t_f - \varepsilon, t_f)$ for some $\varepsilon > 0$. Therefore altering a solution (x^{diff}, u) such the output has a desired value at t_f^- will in general influence the running-cost. As a result, we can not optimize the running-cost and the terminal cost independently of each other. However, the following result shows that the value of the optimal input $u(t_f^-)$ minimizes the terminal cost with respect to the value $x^{\text{diff}}(t_f^-) \in \operatorname{im} \Pi$.

Lemma 5.29. *Let u be an input that solves Problem 5.20 and let x^{diff} be the corresponding optimal trajectory. Denote $u(t_f^-) = \psi \in \mathbb{R}^m$ and $x^{\text{diff}}(t_f^-) = \zeta \in \operatorname{im} \Pi$. Then ψ is a minimizer of* *the following problem.*

$$\begin{aligned} \min_{\psi\in\mathbb{R}^m} \quad & (\zeta - B^{\mathrm{imp}}\psi)^\top P(\zeta - B^{\mathrm{imp}}\psi), \\ \textit{s.t.} \quad & \zeta - B^{\mathrm{imp}}\psi \in \mathcal{V}^{\mathrm{end}}. \end{aligned} \tag{5.25}$$

Proof. Assume that u solves Problem 5.20. Let x^{diff} be the corresponding optimal trajectory on $[t_0, t_f)$. Denote $u(t_f^-) = \psi \in \mathbb{R}^m$ and $x^{\mathrm{diff}}(t_f^-) = \zeta \in \operatorname{im}\Pi$. Suppose there exists a value w for which $\zeta - B^{\mathrm{imp}}w \in \mathcal{V}^{\mathrm{end}}$ and

$$(\zeta - B^{\mathrm{imp}}w)^\top P(\zeta - B^{\mathrm{imp}}w) = (\zeta - B^{\mathrm{imp}}\psi)^\top P(\zeta - B^{\mathrm{imp}}\psi) - M,$$

for some $M > 0$. Consider the solution (x_s, u_s) where $u_s = u + \bar{u}_s$ be an input where $\bar{u}_s$ is defined as

$$\bar{u}_s = \begin{cases} 0, & t_0 \leqslant t < t_f - s, \\ \alpha e^{-A^{\mathrm{diff}}\frac{s}{2}}, & t_f - s \leqslant t < t_f - \frac{s}{2}, \\ -\alpha, & t_f - \frac{s}{2}, \leqslant t < t_f \end{cases}$$

and $\alpha = w - \psi \in \mathbb{R}^m$ is constant. Note that it follows from Lemma D.2 that this solution $(x_s^{\mathrm{diff}}, u_s)$ satisfies $x_s^{\mathrm{diff}}(t_f^-) = x^{\mathrm{diff}}(t_f^-)$. Furthermore, it follows from Lemma D.3 that for any $\varepsilon > 0$ there exists a u_s such that the output $\bar{y}_s$ resulting from the solution $(x_s^{\mathrm{diff}}, u_s)$ satisfies

$$\begin{aligned} \int_{t_0}^{t_f} \|\bar{y}_s(t)\|^2 \,\mathrm{d}t &= \int_{t_0}^{t_f - s} \|\bar{y}_s(t)\|^2 \,\mathrm{d}t + \int_{t_f - s}^{t_f} \|\bar{y}_s(t)\|^2 \,\mathrm{d}t \\ &= \int_{t_0}^{t_f - s} \|\bar{y}(t)\|^2 \,\mathrm{d}t + \int_{t_f - s}^{t_f} \|\bar{y}_s(t)\|^2 \,\mathrm{d}t \\ &\leqslant \int_{t_0}^{t_f} \|\bar{y}(t)\|^2 \,\mathrm{d}t + \varepsilon, \end{aligned}$$

Furthermore, $u_s(t_f^-) = u(t_f^-) + \bar{u}_s(t_f^-) = w$ and thus $x_s^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}}u_s(t_f^-) \in \mathcal{V}^{\mathrm{end}}$ and

$$\bar{J}(x_0, u_s) = \bar{J}(x_0, u) + \varepsilon - M.$$

Hence for $\varepsilon < M$ there exists a s such that the solution $(x_s^{\mathrm{diff}}, u_s)$ satisfies $\bar{J}(x_0, u_s) < \bar{J}(x_0, u)$, which contradicts the optimality of (x^{diff}, u). Hence the result follows. □

In general, there does not exist a $\psi \in \mathbb{R}^m$ for every $\zeta \in \operatorname{im}\Pi$ that solves the optimization problem (5.25). Necessary and sufficient conditions for solvability are given in the following lemma.

Lemma 5.30. *Given $\zeta \in \operatorname{im}\Pi$, there exists a vector ψ that solves the problem*

$$\begin{aligned} \min_{\psi\in\mathbb{R}^m} \quad & (\zeta - B^{\mathrm{imp}}\psi)^\top P(\zeta - B^{\mathrm{imp}}\psi), \\ \textit{s.t.} \quad & \zeta - B^{\mathrm{imp}}\psi \in \mathcal{V}^{\mathrm{end}}, \end{aligned} \tag{5.26}$$

if and only if

$$\zeta \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}.$$

Proof. Assume that $\psi \in \mathbb{R}^m$ solves (5.26). Then $\zeta - B^{\text{imp}}\psi \in \mathcal{V}^{\text{end}}$. Since $\zeta \in \operatorname{im}\Pi$ the equality $\zeta = \Pi\zeta$ holds and consequently $\Pi\zeta \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$.

Conversely, assume $\Pi\zeta \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$. Then since $\zeta \in \operatorname{im}\Pi$ we have $\Pi\zeta = \zeta$. Hence there exists a ψ_1 such that $\zeta - B^{\text{imp}}\psi_1 \in \mathcal{V}^{\text{end}}$. If ψ_1 is unique, then the problem is solvable. If ψ_1 is not unique, it follows that for any ψ_2 satisfying $\zeta - B^{\text{imp}}\psi_2 \in \mathcal{V}^{\text{end}}$ we can write

$$B^{\text{imp}}(\psi_1 - \psi_2) = (x^{\text{diff}} - B^{\text{imp}}\psi_2) - (x^{\text{diff}} - B^{\text{imp}}\psi_1) \in \mathcal{V}^{\text{end}}.$$

Clearly $B^{\text{imp}}(\psi_1 - \psi_2) \in \operatorname{im} B^{\text{imp}}$ and thus we can conclude $B^{\text{imp}}\psi_1 = B^{\text{imp}}\psi_2 + \eta$ for some $\eta \in \mathcal{V}^{\text{end}} \cap \operatorname{im} B^{\text{imp}}$. Consequently, (5.26) is solvable if and only if given $\bar{\zeta} := \zeta - B^{\text{imp}}\psi_1$ the following problem is solvable.

$$\begin{aligned} \min_{\eta\in\mathbb{R}^n} \quad & (\bar{\zeta} + \eta)^\top P(\bar{\zeta} + \eta), \\ \text{s.t.} \quad & \eta \in \mathcal{V}^{\text{end}} \cap B^{\text{imp}}. \end{aligned} \tag{5.27}$$

It follows from Lemma D.4 this problem is solvable. □

Corollary 5.31. *If there exists an input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ that solves Problem 5.20, then the corresponding optimal solution (x^{diff}, u) satisfies $x^{\text{diff}}(t_f^-) \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$.*

Given the result of Lemma 5.30 we know that if a trajectory x^{diff} satisfies $x^{\text{diff}}(t_f^-) = \zeta \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$, there exists a value ψ that solves (5.25). Moreover, we can determine all the possible values $\psi \in \mathbb{R}^m$ that solve (5.25).

Lemma 5.32. *For a given $\zeta \in \left(\mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}\right) \cap \operatorname{im}\Pi$ the vector $\psi \in \mathbb{R}^m$ solves*

$$\begin{aligned} \min_{\psi\in\mathbb{R}^m} \quad & (\zeta - B^{\text{imp}}\psi)^\top P(\zeta - B^{\text{imp}}\psi), \\ \text{s.t.} \quad & \zeta - B^{\text{imp}}\psi \in \mathcal{V}^{\text{end}}, \end{aligned} \tag{5.28}$$

if and only if $\zeta \in [\,0\ 0\ I\,] \ker \mathcal{H}$ and $\psi \in [\,I\ 0\ 0\,] \ker \mathcal{H}$, where

$$\mathcal{H} := \begin{bmatrix} B^{\text{imp}\top} P B^{\text{imp}} & B^{\text{imp}\top}(I - \Pi_{\mathcal{V}^{\text{end}}})^\top & -B^{\text{imp}\top} P \Pi \\ (I - \Pi_{\mathcal{V}^{\text{end}}})B^{\text{imp}} & 0 & -(I - \Pi_{\mathcal{V}^{\text{end}}})\Pi \end{bmatrix} \tag{5.29}$$

and $\Pi_{\mathcal{V}^{\text{end}}}$ is any projector onto $\mathcal{V}^{\text{end}}$.

Proof. Note that the terminal cost function

$$(\zeta - B^{\text{imp}}\psi)^\top P(\zeta - B^{\text{imp}}\psi), \tag{5.30}$$

for a given $\zeta \in \operatorname{im}\Pi$ is a convex function of $\psi \in \mathbb{R}^m$. Furthermore $\psi \in \mathbb{R}^m$ minimizes (5.30) if and only if ψ minimizes

$$\psi^\top B^{\mathrm{imp}\top} P B^{\mathrm{imp}} \psi - 2\zeta^\top P B^{\mathrm{imp}} \psi.$$

The constraint $\zeta - B^{\mathrm{imp}}\psi \in \mathcal{V}^{\mathrm{end}}$ is satisfied if and only if $(I - \Pi_{\mathcal{V}^{\mathrm{end}}})(\zeta - B^{\mathrm{imp}}\psi) = 0$, where $\Pi_{\mathcal{V}^{\mathrm{end}}}$ is a projector onto $\mathcal{V}^{\mathrm{end}}$. This condition can be written equivalently as

$$(I - \Pi_{\mathcal{V}^{\mathrm{end}}}) B^{\mathrm{imp}} \psi = (I - \Pi_{\mathcal{V}^{\mathrm{end}}})\zeta.$$

As this constraint is a convex function and P is positive semi-definite, it follows that this optimization problem is a convex problem. Hence any local minimizer is a global minimizer. The first order necessary conditions are thus also sufficient. Hence ψ is a minimizer that satisfies the constraints if and only if there exists a Lagrange multiplier λ such that

$$\begin{bmatrix} B^{\mathrm{imp}\top} P B^{\mathrm{imp}} & B^{\mathrm{imp}\top}(I - \Pi_{\mathcal{V}^{\mathrm{end}}})^\top \\ (I - \Pi_{\mathcal{V}^{\mathrm{end}}}) B^{\mathrm{imp}} & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \lambda \end{bmatrix} = \begin{bmatrix} B^{\mathrm{imp}\top} P \\ (I - \Pi_{\mathcal{V}^{\mathrm{end}}}) \end{bmatrix} \zeta.$$

This can equivalently be written as $\mathcal{H}\xi = 0$ where

$$\mathcal{H} := \begin{bmatrix} B^{\mathrm{imp}\top} P B^{\mathrm{imp}} & B^{\mathrm{imp}\top}(I - \Pi_{\mathcal{V}^{\mathrm{end}}})^\top & -B^{\mathrm{imp}\top} P \\ (I - \Pi_{\mathcal{V}^{\mathrm{end}}}) B^{\mathrm{imp}} & 0 & -(I - \Pi_{\mathcal{V}^{\mathrm{end}}}) \end{bmatrix} \tag{5.31}$$

and $\xi^\top = \left[\begin{smallmatrix} \psi^\top & \lambda^\top & \zeta^\top \end{smallmatrix}\right]^\top$. Since $\zeta \in \operatorname{im}\Pi$ and hence $\zeta = \Pi\zeta$ the result follows. □

Given the result of Lemma 5.32, we can compute which states $\zeta \in \operatorname{im}\Pi$ are possibly an endpoint of an optimal trajectory. Moreover, for each endpoint $\zeta \in \operatorname{im}\Pi$ a value of ψ that solves (5.28) can be computed. Consequenlty, for a given optimal solution (x^{diff}, u) where $x^{\mathrm{diff}}(t_f^-) = \zeta$, we are able to express the terminal cost of this solution in terms of $x^{\mathrm{diff}}(t_f^-)$ only.

Corollary 5.33. *If there exists an input u that solves Problem 5.20 then the optimal terminal cost satisfies*

$$\left(x^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}} u(t_f^-)\right)^\top P \left(x^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}} u(t_f^-)\right) = x^{\mathrm{diff}}(t_f^-)^\top \Psi^\top P \Psi x^{\mathrm{diff}}(t_f^-),$$

where $\Psi = (I - B^{\mathrm{imp}} N)\Pi$, for any N satisfying $\left[\begin{smallmatrix} I & 0 & -N \end{smallmatrix}\right] \ker \mathcal{H} = 0$.

Although the minimum of the objective function in (5.28) is uniquely given for a particular $x^{\mathrm{diff}} \in \mathbb{R}^n$, a minimizer $u \in \mathbb{R}^m$ is not necessarily unique. However, the following result can still be concluded regarding an optimal input.

Corollary 5.34. *If an input u solves Problem 5.20 then $u(t_f^-) = N x^{\mathrm{diff}}(t_f^-)$ for some N satisfying $\left[\begin{smallmatrix} I & 0 & -N \end{smallmatrix}\right] \ker \mathcal{H} = 0$.*

Given the result of Corollary 5.33, we are able to express the terminal cost in terms of x^{diff}. Consequently, Problem 5.20 is closely related to the following optimal control problem, which only penalizes x^{diff} in the terminal cost. Particularly we can show that if an input solves the following problem and satisfies certain conditions at t_f^-, we can conclude that it solves Problem 5.20 and vice-versa.

Problem 5.35. Consider the system (5.21). Find an input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})$ that solves the following problem:

$$\begin{aligned} \min \qquad \bar{J}_\Psi(x_0, u, t_f) &= \textstyle\int_{t_0}^{t_f} \|\bar{y}(t)\|^2 \,\mathrm{d}t + x^{\text{diff}}(t_f^-)\Psi^\top P \Psi x^{\text{diff}}(t_f^-), \\ \text{s.t.} \qquad \dot{x}^{\text{diff}} &= A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u, \\ \bar{y} &= \bar{C} x^{\text{diff}} + \bar{D} u, \\ x^{\text{diff}}(t_0^-) &= \Pi x_0 \in \mathcal{V}_{(E_0, A_0, B_0)}, \\ x(t_f^-) &= x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-) \in \mathcal{V}^{\text{end}}, \end{aligned} \tag{5.32}$$

where $A^{\text{diff}}, B^{\text{diff}}$ and B^{imp} are the matrices resulting from the Wong-sequences based on (EW, A). Furthermore $\bar{C} = C\Pi$, $\bar{D} = D - CB^{\text{imp}}$ and $\Psi = (I - B^{\text{imp}}N)\Pi$ for some N satisfying $\left[\, (I \;\, 0 \;\, N) \,\right] \ker \mathcal{H} = 0$, where $\mathcal{H}$ is given by (5.31).

Lemma 5.36. *An input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ solves Problem 5.20 if and only if u solves Problem 5.35 and $u(t_f^-) = N x^{\text{diff}}(t_f^-)$ for some N satisfying $\left[\, I \;\, 0 \; -N \,\right] \ker \mathcal{H} = 0$.*

Proof. Let $\mathcal{U}(\Sigma, x_0)$ be the class of inputs for which Problem 5.20 is feasible. That is, if $u \in \mathcal{U}(\Sigma, x_0)$ then the output $\bar{y}$ corresponding to the solution (x^{diff}, u) satisfying $x^{\text{diff}}(t_0^-) = x_0$ results in a cost $\bar{J}(x_0, u, t_f)$ in (5.22) which is finite and $x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-) \in \mathcal{V}^{\text{end}}$. Similarly, let $\mathcal{U}_\psi(\Sigma, x_0)$ be the class of inputs for which Problem (5.35) is feasible.

($\Leftarrow$) Suppose $u \in \mathcal{U}(\Sigma, x_0)$ solves Problem 5.20. Then it follows from Corollary 5.34 that $u(t_f^-) = N x^{\text{diff}}(t_f^-)$ for some N satisfying $\left[\, I \;\, 0 \; -N \,\right] \ker \mathcal{H} = 0$ and $\mathcal{H}$ is defined as in (5.31). Furthermore, it follows from Corollary 5.33 that

$$(x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-))^\top P (x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-)) = x^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi x^{\text{diff}}(t_f^-).$$

Consequently $\bar{J}_\Psi(x_0, u) = \bar{J}(x_0, u)$. Hence we can conclude

$$\inf_{u \in \mathcal{U}_\Psi(\Sigma, x_0)} \bar{J}_\Psi(x_0, u) \leqslant \inf_{u \in \mathcal{U}(\Sigma, x_0)} \bar{J}(x_0, u). \tag{5.33}$$

We will show that equality holds for (5.33). To do so, assume for the sake of contradiction that this inequality is strict. Then there exists an input $\bar{u} \in \mathcal{U}_\Psi(\Sigma, x_0)$ such that

$$\bar{J}_\Psi(x_0, \bar{u}) < \bar{J}_\Psi(x_0, u) = \bar{J}(x_0, u).$$

Then it follows that $\bar{J}(x_0, u) - J_\Psi(x_0, \bar{u}) = M$ for some $M > 0$.

Consider the solution $(\tilde{x}_s, \tilde{u}_s)$ where $\tilde{u}_s = \bar{u} + u_s^*$ be an input where u_s^* is defined as

$$u_\delta^* = \begin{cases} 0, & t_0 \leqslant t < t_f - s, \\ \alpha e^{-A^{\text{diff}} \frac{s}{2}}, & t_f - s \leqslant t < t_f - \frac{s}{2}, \\ -\alpha, & t_f - \frac{s}{2} \leqslant t < t_f \end{cases}$$

and $\alpha = N\bar{x}^{\text{diff}}(t_f^-) - \bar{u}(t_f^-) \in \mathbb{R}^m$ is constant. Then it follows from Lemma D.2 that for any $s > 0$ we have $\tilde{x}_s^{\text{diff}}(t_f^-) = \bar{x}^{\text{diff}}(t_f^-) \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$. Furthermore, the input $\tilde{u}_s$ satisfies

$$\begin{aligned} \tilde{u}_s(t_f^-) &= u(t_f^-) + N\tilde{x}_s^{\text{diff}}(t_f^-) - \bar{u}(t_f^-) \\ &= N\tilde{x}_s^{\text{diff}} \\ &= N\bar{x}^{\text{diff}}(t_f^-). \end{aligned}$$

and thus $\tilde{x}_s^{\text{end}}(t_f^-) - B^{\text{imp}}\tilde{u}_s(t_f^-) \in \mathcal{V}^{\text{end}}$. Furthermore it follows that

$$\begin{aligned} (\tilde{x}_s^{\text{diff}}(t_f^-) - B^{\text{imp}}\tilde{u}_s(t_f^-))^\top P(\tilde{x}_s^{\text{diff}}(t_f^-) - B^{\text{imp}}\tilde{u}_s(t_f^-)) \\ = \tilde{x}_s^{\text{diff}}(t_f^-)^\top (I - B^{\text{imp}}N)^\top P (I - B^{\text{imp}}N)\tilde{x}_s^{\text{diff}}(t_f^-) \\ = \tilde{x}_s^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi \tilde{x}_s^{\text{diff}}(t_f^-) \\ = \bar{x}^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi \bar{x}^{\text{diff}}(t_f^-). \end{aligned}$$

From Lemma D.3 we can conclude that for any $\varepsilon > 0$ there exists a s such that

$$\int_{t_0}^{t_f} \|\tilde{y}(t)\| \, dt = \int_{t_0}^{t_f} \|\bar{y}(t)\| \, dt + \varepsilon.$$

Combining these results yields

$$\bar{J}(x_0, \tilde{u}) = J_\Psi(x_0, \bar{u}) + \varepsilon = \bar{J}(x_0, u) + \varepsilon - M$$

and thus by choosing s such that ε is sufficiently small, we obtain $\bar{J}(x_0, \tilde{u}_s) < \bar{J}(x_0, u)$ which contradicts the optimality of (x^{diff}, u). Hence equality in (5.33) holds.

($\Leftarrow$) Suppose $u \in \mathcal{U}_\Psi(\Sigma, x_0)$ solves Problem 5.35 and satisfies $u(t_f^-) = Nx^{\text{diff}}(t_f^-)$. Then for this particular input we obtain

$$\bar{J}_\Psi(x_0, u) = \bar{J}(x_0, u).$$

Suppose there exists an input $\bar{u} \in \mathcal{U}(\Sigma, x_0)$ for which $\bar{J}(x_0, \bar{u}) < \bar{J}(x_0, u)$. Then

$$\bar{x}^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi \bar{x}^{\text{diff}}(t_f^-) \leqslant (\bar{x}^{\text{diff}}(t_f^-) - B^{\text{imp}}\bar{u}(t_f^-))^\top P(\bar{x}^{\text{diff}}(t_f^-) - B^{\text{imp}}\bar{u}(t_f^-))$$

and consequently

$$J_\Psi(x_0, \bar{u}) \leqslant \bar{J}(x_0, \bar{u}) < \bar{J}(x_0, u) = J_\Psi(x_0, u),$$

which contradicts the assumption that (x^{diff}, u) solves Problem (5.35). Hence u solves Problem 5.20. $\square$

As a consequence of Lemma 5.36 we can focus on finding conditions under which an input $u \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^m$ solves Problem 5.35 and satisfies $u(t_f^-) = N x^{\mathrm{diff}}(t_f^-)$. As the terminal cost in Problem 5.35 only penalizes the state x^{diff} and not the input u, this problem is easier to analyze as standard tools for optimal control can be used. Hence in the following, we will focus on solving Problem 5.35.

Example 5.37 (Example 5.28 Continued)**.** Observe that for the optimal control problem defined in (5.24) we have that

$$\Pi_{\mathcal{V}^{\mathrm{end}}} = \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}$$

and thus we can compute

$$\mathcal{H} = \begin{bmatrix} B^{\mathrm{imp}\top} P B^{\mathrm{imp}} & B^{\mathrm{imp}\top}(I-\Pi_{\mathcal{V}^{\mathrm{end}}})^\top & -B^{\mathrm{imp}\top} P \Pi \\ (I-\Pi_{\mathcal{V}^{\mathrm{end}}})B^{\mathrm{imp}} & 0 & -(I-\Pi_{\mathcal{V}^{\mathrm{end}}})\Pi \end{bmatrix} = \begin{bmatrix} 1&0&-1&0&0&0&-2\\0&0&0&0&0&0&0\\1&0&0&0&0&0&-2\\0&0&0&0&0&0&1 \end{bmatrix}.$$

Such that *e.g.*, $N = \begin{bmatrix} 0&0&0 \end{bmatrix}$ satisfies $\begin{bmatrix} I&0&-N \end{bmatrix} \ker \mathcal{H} = 0$. Note that this N is not uniquely determined. Given this particular N we compute $\Psi = I$. Thus the solvability of (5.24) is equivalent to

$$\begin{aligned}
\min \qquad & \bar{J}_\Psi(x_0, v) &&= \textstyle\int_0^1 \|\bar{y}(t)\|^2 \, dt + x^{\mathrm{diff}}(t_f^-)^\top x^{\mathrm{diff}}(t_f^-), \\
\text{s.t.} \qquad & \dot{x}^{\mathrm{diff}} &&= \begin{bmatrix} -1&0&1\\0&0&-4\\0&0&2 \end{bmatrix} x^{\mathrm{diff}} + \begin{bmatrix} 0\\2\\-1 \end{bmatrix} v, \\
& \bar{y} &&= \begin{bmatrix} 1&0&1\\0&0&-2\\0&0&1 \end{bmatrix} x^{\mathrm{diff}} + \begin{bmatrix} 0\\1\\0 \end{bmatrix} v, \\
& x^{\mathrm{diff}}(t_0^-) &&= \Pi x_0, \\
& x^{\mathrm{diff}}(t_f^-) + \begin{bmatrix} 0\\1\\0 \end{bmatrix} v(t_f^-) &&\in \operatorname{span}\left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix} \right\}.
\end{aligned} \tag{5.34}$$

Observe that the optimal control problem (5.34) does not penalize the terminal value of the input anymore. ◇

Regarding the running cost

Whereas the previous section focused on the terminal cost and the value of the optimal control at the end of the interval $[t_0, t_f)$, we will now turn our attention to the running cost and the optimal control given in the interval of interest. To that extent we will write

$$\|\bar{y}(t)\|^2 = \begin{bmatrix} x^{\mathrm{diff}\top} & u^\top \end{bmatrix}^\top \begin{bmatrix} \bar{C}^\top \\ \bar{D}^\top \end{bmatrix}^\top \begin{bmatrix} \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix} = \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}.$$

As mentioned previously, the optimal cost is a quadratic function of x_0. In particular, the minimum of the integral

$$\int_{t_0}^{t_f} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix} \, \mathrm{d}t$$

is a quadratic function $x_0^\top K(t_0)x_0$. We assume that $K(t)$ is any symmetric-matrix-valued continuously differentiable function, defined on $[t_0, t_f)$. Considering the difference $J_\Psi(x^{\text{diff}}, u) - x_0^\top K(t_0)x_0$ yields under this assumption

$$J_\Psi(x^{\text{diff}}, u) - x_0^\top K(t_0)x_0 = \int_{t_0}^{t_f} \left[\begin{smallmatrix} x^{\text{diff}} \\ u \end{smallmatrix}\right]^\top \left[\begin{smallmatrix} \bar{Q} & \bar{S}^\top \\ \bar{S} & \bar{R} \end{smallmatrix}\right] \left[\begin{smallmatrix} x^{\text{diff}} \\ u \end{smallmatrix}\right] + \frac{d}{dt} x^{\text{diff}}(t)^\top K(t) x^{\text{diff}}(t)\ \mathrm{d}t \\ + x^{\text{diff}}(t_f^-)^\top \left(\Psi^\top P \Psi - K(t_f^-)\right) x^{\text{diff}}(t_f^-).$$

Taking the two integrals together and computing the second integral using the differential equation and the completion of the squares formula, we obtain while omitting the dependence on t:

$$\begin{aligned} x^{\text{diff}\top} \bar{Q} x^{\text{diff}} &+ 2x^{\text{diff}\top} \bar{S}^\top u + u^\top \bar{R} u + \frac{d}{dt} x^{\text{diff}\top} K x^{\text{diff}} \\ &= x^{\text{diff}\top}(\bar{Q} + A^{\text{diff}\top} K + K A^{\text{diff}} + \dot{K}) x^{\text{diff}\top} + 2u^\top (B^{\text{diff}} K^\top + \bar{S}) x^{\text{diff}} \\ &\quad + u^\top \bar{R} u \\ &= x^{\text{diff}\top} K B^{\text{diff}\top} \bar{R}^{-1} B^{\text{diff}} K x^{\text{diff}} + 2u^\top (B^{\text{diff}} K^\top + \bar{S}^\top) x^{\text{diff}} \\ &\quad + u^\top \bar{R} u + x^{\text{diff}\top} W x^{\text{diff}} \\ &= \|\bar{R} u + (B^{\text{diff}\top} K + \bar{S}^\top) x^{\text{diff}}\|^2 + x^{\text{diff}\top} W x^{\text{diff}}, \end{aligned}$$

where

$$W := \dot{K} + A^{\text{diff}\top} K + K A^{\text{diff}} - (\bar{S} + K^\top B^{\text{diff}}) \bar{R}^{-1} (B^{\text{diff}\top} K + \bar{S}^\top) + \bar{Q}.$$

Consequently, we can rewrite the cost in Problem 5.35 as

$$J_\Psi(x_0, u) = x_0^\top K(t_0^-) x_0 + \int_{t_0}^{t_f} \|\bar{R} u(t) + (B^{\text{diff}\top} K(t) + \bar{S}^\top) x^{\text{diff}}(t)\|^2 \\ + x^{\text{diff}}(t)^\top W(t) x^{\text{diff}}(t)\ \mathrm{d}t + x^{\text{diff}}(t_f^-)^\top \left(\Psi^\top P \Psi - K(t_f^-)\right) x^{\text{diff}}(t_f^-).$$

This expression will play a crucial role in our analysis of the optimal control problem. Choosing $K(t)$ such that $W = 0$ and $K(t_f^-) = \Psi^\top P \Psi$ we obtain that the cost $J_\Psi(x^{\text{diff}}, u)$ can be expressed as

$$J_\Psi(x_0, u) = x_0^\top K(t_0^-) x_0 + \int_{t_0}^{t_f} \|\bar{R} u(t) + (B^{\text{diff}\top} K(t) + \bar{S}^\top) x^{\text{diff}}(t)\|^2\ \mathrm{d}t. \tag{5.35}$$

Clearly without the constraint $x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-)$ it follows that $J_\Psi(x_0, u)$ is minimized if $u = -\bar{R}^{-1}(B^{\text{diff}\top} K + \bar{S}^\top) x^{\text{diff}}$. Using this observation, one can show that there always exists a function K satisfying $K(t_f^-) = \Psi^\top P \Psi$ such that $W = 0$, as $K(t)$ is then a solution to the well known Riccati differential equation

$$\dot{K} = -A^{\text{diff}\top} K - K A^{\text{diff}} + (\bar{S} + K^\top B^{\text{diff}}) \bar{R}^{-1} (B^{\text{diff}\top} K + \bar{S}^\top) - \bar{Q}, \\ K(t_f^-) = \Psi^\top P \Psi. \tag{5.36}$$

However, the following result shows that if we assume there exists a solution to Problem 5.20, the input also solves the optimal control problem without the constraint $x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) \in \mathcal{V}^{\text{end}}$.

Lemma 5.38. *If an input $u \in \mathcal{U}(\overline{\Sigma}, x_0)$ minimizes $J_\Psi(x^{\text{diff}}, u)$ then*

$$u = -\bar{R}^{-1}\left(B^{\text{diff}\top}K(t) + \bar{S}^\top\right)x^{\text{diff}}(t). \tag{5.37}$$

Proof. Let (x^{diff}, u) be a solution for the input defined as (5.37) satisfying $x^{\text{diff}}(t_0^-) = x_0$.

Case 1: $x^{\text{diff}}(t_f^-) \in \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$

Then consider the input $u_\delta = u + \bar{u}_\delta$ where u_δ is defined as

$$u_\delta = \begin{cases} 0, & t_0 \leqslant t < t_f - \delta, \\ \alpha e^{-A^{\text{diff}}\frac{s}{2}}, & t_f - \delta \leqslant t < t_f - \frac{\delta}{2}, \\ -\alpha, & t_f - \frac{\delta}{2} \leqslant t < t_f, \end{cases}$$

where $\alpha = Nx^{\text{diff}}(t_f^-) - u(t_f^-) \in \mathbb{R}^m$, for some N satisfying $[\,I\ 0\ -N\,]\ker\mathcal{H}$ is constant. Then the solution $(x_\delta^{\text{diff}}, u_\delta)$ with $x_\delta^{\text{diff}}(t_0^-) = x^{\text{diff}}(t_0^-) = x_0$ satisfies $x_\delta^{\text{diff}}(t_f^-) = x^{\text{diff}}(t_f^-) = q$. Note that

$$\begin{aligned} x_\delta^{\text{diff}}(t_f^-) - B^{\text{imp}}u_\delta(t_f) &= x^{\text{diff}}(t_f^-) - B^{\text{imp}}Nx_\delta^{\text{diff}}(t_f^-) \\ &= x^{\text{diff}}(t_f^-) - B^{\text{imp}}Nx^{\text{diff}}(t_f^-) \in \mathcal{V}^{\text{end}} \end{aligned}$$

and thus u_δ is a feasible input. It follows from (5.35) that for every $\varepsilon > 0$ there exists a solution $(x_\delta^{\text{diff}}, u_\delta)$ such that

$$J_\Psi(x_\delta^{\text{diff}}, u_\delta) = x_0^\top K(t_0^-)x_0 + \varepsilon$$

and thus we can conclude

$$\inf_{u \in \mathcal{U}(\Sigma, x_0)} J_\Psi(x^{\text{diff}}, u) = x_0^\top K(t_0^-)x_0.$$

However, the infimum is attained if and only if the input is given by (5.37). Hence if there exists an input $u \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$ that solves Problem 5.35 then (5.37) holds.

Case 2: $x^{\text{diff}}(t_f^-) \notin \mathcal{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$

We will prove that there does not exist an optimal control. For the sake of contradiction, assume that the optimal control is given by $\tilde{u} \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$. Then the solution $(\tilde{x}^{\text{diff}}, \tilde{u})$ must satisfy $\tilde{x}^{\text{diff}}(t_f^-) = q$ for some $q \in [\,0\ 0\ \Pi\,]\ker\mathcal{H}$, as the input at t_f^- needs to be such that terminal cost must be minimal with respect to $\tilde{x}^{\text{diff}}(t_f^-)$.

Let $y_0 \in \mathbb{R}^n$, $y_0 \neq x_0$ be an initial value such that the solution (y^{diff}, u) with $y^{\text{diff}}(t_0^-) = y_0$ satisfies $y^{\text{diff}}(t_f^-) = q$. Recall, that by Lemma 5.14, the optimal control is a feedback. As a consequence of the linearity of the optimal control in the state, it must hold that $v = u - \tilde{u}$

is the optimal control for the initial value $z_0 = y_0 - x_0$. However, by linearity of solutions, the solution (z^{diff}, v) satisfies

$$\begin{aligned} z^{\text{diff}}(t^-) &= e^{A^{\text{diff}} t - t_t} z_0 + \int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} v(\tau) \,\mathrm{d}\tau \\ &= e^{A^{\text{diff}} t - t_t} x_0 + \int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} \tilde{u}(\tau) \,\mathrm{d}\tau \\ &\quad - e^{A^{\text{diff}} t - t_t} y_0 - \int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)} B^{\text{diff}} u(\tau) \,\mathrm{d}\tau \\ &= \tilde{x}^{\text{diff}}(t^-) - y^{\text{diff}}(t^-) \end{aligned}$$

and consequently $z^{\text{diff}}(t_f^-) = 0$. However, this implies that $z_0 = 0$, as a feedback can not control an initial condition to zero, unless it is zero. Hence we can conclude that $x_0 = y_0$, which yields a contradiction. Hence there does not exist an optimal control for x_0. $\square$

Corollary 5.39. *If an input $u \in (\mathbb{D}_{pw\mathcal{C}^\infty})^m$ solves Problem 5.20 then*

$$u(t) = -\bar{R}^{-1} \left(B^{\text{diff}\top} K(t) + \bar{S}^\top \right) x^{\text{diff}}(t), \tag{5.38}$$

where K solves (5.36) *with terminal condition $K(t_f^-) = \Psi^\top P \Psi$.*

5.2.3 Combining the results

Thus far we have only been concerned with necessary conditions for solvability of Problem 5.20. The reason that the conditions in Corollary 5.39 are not sufficient is that a feedback of the form (5.38) does not necessarily ensure that all the constraints are satisfied. That is, a solution (x^{diff}, u) with u given by (5.38) and $x^{\text{diff}}(t_0^-) = x_0 \in \operatorname{im} \Pi$ does not necessarily satisfy

$$x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-) \in \mathcal{V}^{\text{end}},$$

nor

$$(x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-))^\top P (x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-)) = x^{\text{diff}}(t_f^-) \Psi^\top P \Psi x^{\text{diff}}(t_f^-),$$

for any N for which $\left[\begin{smallmatrix} I & 0 \\ 0 & -N \end{smallmatrix}\right] \ker \mathcal{H} = 0$. Both these conditions can be rewritten equivalently as

$$(I - \Pi_{\mathcal{V}^{\text{end}}})(I - B^{\text{imp}} \Lambda) x^{\text{diff}}(t_f^-) = 0 \tag{5.39}$$

and

$$(I - B^{\text{imp}} \Lambda)^\top P ((I - B^{\text{imp}} \Lambda) - \Psi^\top P \Psi) x^{\text{diff}} = 0, \tag{5.40}$$

where we have written

$$\Lambda := -\bar{R}^{-1}\left(B^{\mathrm{diff}\top}\Psi^\top P\Psi + \bar{S}^\top\right),$$

for notational convenience. However, if a solution (x^{diff}, u) with $x^{\mathrm{diff}}(t_0^-) = x_0$ and u satisfies (5.38) is such that (5.39) and (5.40) are satisfied the input is optimal. To proof this, we will first introduce the backwards state-transition matrix, defined similarly to [56] or [128] and which also appears in [8].

Definition 5.40. The backwards state transition matrix for the closed loop time-varying differential equation

$$\dot{x}^{\mathrm{diff}} = \left(A^{\mathrm{diff}} - B^{\mathrm{diff}}\bar{R}^{-1}(B^{\mathrm{diff}\top}K + \bar{S}^\top)\right)x^{\mathrm{diff}},$$

is given by $\Omega(t, t_f)$, where K is a solution to (5.36) with terminal condition $K(t_f^-) = \Psi^\top P\Psi$. Hence $x^{\mathrm{diff}}(t) = \Omega(t, t_f)x^{\mathrm{diff}}(t_f^-)$.

Theorem 5.41. *Problem 5.20 is solvable if and only if*

$$x_0 \in \mathcal{V}^{\mathrm{init}} := \Omega(t_0, t_f)\ker\begin{bmatrix}(I - \Pi_{\mathcal{V}^{\mathrm{end}}})(I - B^{\mathrm{imp}}\Lambda)\\ (I - B^{\mathrm{imp}}\Lambda)^\top P(I - B^{\mathrm{imp}}\Lambda) - \Psi^\top P\Psi\end{bmatrix}\Pi, \tag{5.41}$$

where $\Omega(t_0, t_f)$ is the backward state transition matrix as defined in Definition 5.40 and the optimal control is given by

$$u(t) = -\bar{R}^{-1}(B^{\mathrm{diff}\top}K(t) + \bar{S}^\top)x^{\mathrm{diff}}(t), \tag{5.42}$$

where K is a solution to (5.36) *with terminal condition $K(t_f^-) = \Psi^\top P\Psi$.*

Proof. $(\Rightarrow)$ It follows from Corollary 5.39 that if an input $u \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^m$ solves Problem 5.20 then u is given by (5.42). Particularly, it follows that $u(t_f^-) = \Lambda x^{\mathrm{diff}}(t_f^-)$. Furthermore, it follows from Corollary 5.33 that the terminal cost resulting from the the optimal solution (x^{diff}, u) satisfies

$$\begin{aligned}(x^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}}u(t_f^-))^\top P(x^{\mathrm{diff}}(t_f^-) - B^{\mathrm{imp}}u(t_f^-))&\\ = x^{\mathrm{diff}}(t_f^-)^\top(I - B^{\mathrm{imp}}\Lambda)^\top P(I - B^{\mathrm{imp}}\Lambda)x^{\mathrm{diff}}(t_f^-)&\\ = x^{\mathrm{diff}}(t_f^-)\Psi^\top P\Psi x^{\mathrm{diff}}(t_f^-).&\end{aligned}$$

This implies

$$\|\left((I - B^{\mathrm{imp}}\Lambda)^\top P(I - B^{\mathrm{imp}}\Lambda) - \Psi^\top P\Psi\right)^{\frac{1}{2}}x^{\mathrm{diff}}(t_f^-)\|^2 = 0,$$

from which we can conclude

$$x^{\mathrm{diff}}(t_f^-) \in \ker\left((I - B^{\mathrm{imp}}\Lambda)^\top P(I - B^{\mathrm{imp}}\Lambda) - \Psi^\top P\Psi\right).$$

As the input is assumed to be such that all constraints in Problem 5.20 are satisfied, it follows that

$$x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) = (I - B^{\text{imp}})\Lambda x^{\text{diff}}(t_f^-) \in \mathcal{V}^{\text{end}}$$

and consequently given any projector $\Pi_{\mathcal{V}^{\text{end}}}$ onto $\mathcal{V}^{\text{end}}$ we can conclude that

$$(I - \Pi_{\mathcal{V}^{\text{end}}})(I + B^{\text{imp}}\Lambda)x^{\text{diff}}(t_f^-) = 0.$$

Combining these observations leads to the conclusion that

$$x^{\text{diff}}(t_f^-) \in \ker \begin{bmatrix} (I - \Pi_{\mathcal{V}^{\text{end}}})(I - B^{\text{imp}}\Lambda) \\ (I - B^{\text{imp}}\Lambda)^\top P(I - B^{\text{imp}}\Lambda) - \Psi^\top P\Psi \end{bmatrix}.$$

Finally, recall that $\Pi x^{\text{diff}}(t_f^-) = x^{\text{diff}}(t_f^-)$ and $x^{\text{diff}}(t_0^-) = \Omega(t_0, t_f)x^{\text{diff}}(t_f^-)$ by definition. Since $x^{\text{diff}}(t_0^-) = \Pi x_0$ the result follows.

($\Leftarrow$) Suppose $\Pi x_0 \in \mathcal{V}^{\text{init}}$. Then by definition the input (5.42) ensures that a solution (x^{diff}, u) with $x^{\text{diff}}(t_0^-) = \Pi x_0$ satisfies

$$x^{\text{diff}}(t_f^-) \in \ker \begin{bmatrix} (I - \Pi_{\mathcal{V}^{\text{end}}})(I - B^{\text{imp}}\Lambda) \\ (I - B^{\text{imp}}\Lambda)^\top P(I - B^{\text{imp}}\Lambda) - \Psi^\top P\Psi \end{bmatrix} \Pi.$$

Then since $x^{\text{diff}}(t_0^-) = \Pi x_0$ it follows that $x^{\text{diff}}(t) \in \operatorname{im} \Pi$ for all $t \geqslant t_0$. Consequently

$$\begin{aligned}(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-))^\top P(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-)) \\ = x^{\text{diff}}(t_f^-)^\top (I - B^{\text{imp}}N)^\top P(I - B^{\text{imp}}N)x^{\text{diff}}(t_f^-) \\ = x^{\text{diff}}(t_f^-)^\top \Psi^\top P\Psi x^{\text{diff}}(t_f^-).\end{aligned}$$

Consequently for this input we have

$$J(x_0, u) = J_\Psi(x_0, u) = x_0^\top K(t_0^-)x_0.$$

Note that as $(I - \Pi_{\mathcal{V}^{\text{end}}})(I - B^{\text{imp}}\Lambda)x^{\text{diff}}(t_f^-) = 0$ and $u(t_f^-) = \Lambda x^{\text{diff}}(t_f^-)$ we can conclude

$$x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) \in \mathcal{V}^{\text{end}}$$

and thus all the constraints are satisfied. Hence by Lemma 5.36 the input u solves Problem 5.20. □

The equivalence between solvability of Problem 5.20 and 5.17 leads to the following corollary.

Corollary 5.42. *Problem 5.17 is solvable if and only if*

$$x_0 \in \mathcal{V}^{\mathrm{init}} \cap \mathcal{C}^{\mathrm{imp}}.$$

The optimal control is given by

$$u(t) = -\bar{R}^{-1}(B^{\mathrm{diff}\top} K(t) + \bar{S}^\top)\Pi x(t), \tag{5.43}$$

where K is a solution to (5.36) *with terminal condition $K(t_f^-) = \Psi^\top P \Psi$.*

Proof. It follows from the constraints of Problem 5.17 that $x_0 \in \mathcal{C}^{\mathrm{imp}}$ and from Lemma 5.21 that the problem is solvable if and only if Problem 5.20 is solvable. The latter is the case if and only if $x_0 \in \mathcal{V}^{\mathrm{init}}$. Consequently, the result follows.

The optimal input that solves Problem 5.20 also solves Problem 5.17. Observing that $x^{\mathrm{diff}}(t) = \Pi x(t)$ leads to te conclusion. □

Given the result of Theorem 5.41 we can verify given an initial value $x_0 \in \operatorname{im} \mathcal{C}^{\mathrm{imp}}$ whether Problem 5.20 is solvable and compute the optimal control if it exists. Since Problem 5.20 was the result of a reformulation of Problem 5.17, an overview of how to verify the existence of an input that solves Problem 5.17 and how to compute it, is presented in Algorithm 1.

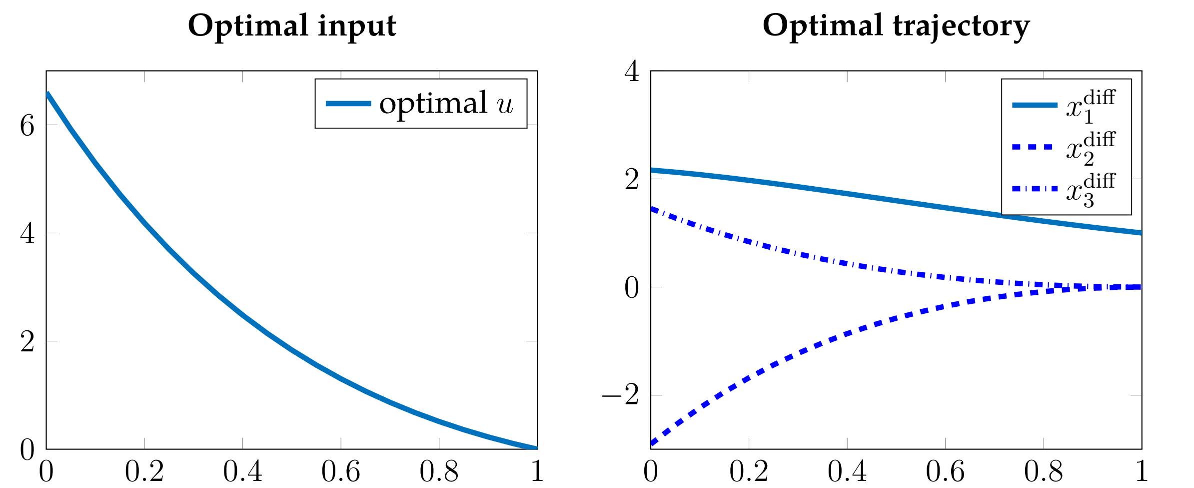

Figure 5.1: The optimal input $u(t)$ that solves the optimal control problem given in Example 5.28 and the corresponding optimal trajectory $x^{\mathrm{diff}}(t)$ (5.24) with initial value $x_0 = \begin{bmatrix} 2.163 & -2.906 & 1.453 \end{bmatrix}^\top$.

Example 5.43 (Example 5.28 continued)**.** Recall that for the optimal control problem (5.34) we computed $\Psi = \Pi$ and $N = 0$. Consequently we obtain

$$\Lambda = -\bar{R}^{-1}(B^{\mathrm{diff}\top} \Psi P \Psi + \bar{S}^\top) = \begin{bmatrix} 0 & -2 & 3 \end{bmatrix}.$$

Algorithm 1 LQR with subspace constraint

Input : $E, A, B, C, D, P, t_0, t_f, \mathcal{V}^{\text{end}}, \mathcal{C}^{\text{imp}}$

Step 1: Preconditioning
Compute W as in (2.18)
Compute L such that $\operatorname{rank}\left[\begin{smallmatrix} EW & A+BL \end{smallmatrix}\right] = n$
Compute projectors $\Pi, \Pi^{\text{diff}}, \Pi^{\text{imp}}$ from the Wong sequences based on $(EW, A+BL)$.
Define $A^{\text{diff}} = \Pi^{\text{diff}}(A+BL), B^{\text{diff}} = \Pi^{\text{diff}}B$, $B^{\text{imp}} = \Pi^{\text{imp}}B$.
Define $\bar{C} = (C+DL)\Pi$, $\bar{D} = D - (C+DL)B^{\text{imp}}$
Define $Q = \bar{C}^\top \bar{C}$, $S = \bar{C}^\top D$, $R = \bar{D}^\top \bar{D}$.

Step 2: Terminal cost matrix
Compute orthogonal projector $\Pi_{\mathcal{V}^{\text{end}}}$ onto $\mathcal{V}^{\text{end}}$
Define $\mathcal{H}$ according to (5.31) and compute $\ker \mathcal{H}$.
Compute N such that $\left[\begin{smallmatrix} I & 0 & -N \end{smallmatrix}\right] \ker H = 0$.
Define $\Psi = (I - B^{\text{imp}}N)\Pi$ and $X_{t_f} = \Psi^\top P \Psi$.
Solve Riccati differential equation (5.36) on $[t_0, t_f]$ with terminal condition $X(t_f) = X_{t_f}$.

Step 3: Verify solvability
Compute backward state transition matrix $\Omega(t_0, t_f)$ for $\dot{x}^{\text{diff}} = A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u$ with $u(t) = -R^{-1}(B^{\text{diff}\top} X(t) + S^\top) x^{\text{diff}}(t)$.
Set $\Lambda = -R^{-1}(B^{\text{diff}\top} \Psi^\top P \Psi + \bar{S}^\top)$
Compute $\ker \left[\begin{smallmatrix} (I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda) \\ (I-B^{\text{imp}}\Lambda)^\top P (I-B^{\text{imp}}\Lambda) - \Psi P \Psi \end{smallmatrix}\right]$
Compute $\mathcal{V}^{\text{init}} = \Omega(t_0, t_f) \ker \left[\begin{smallmatrix} (I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda) \\ (I-B^{\text{imp}}\Lambda)^\top P (I-B^{\text{imp}}\Lambda) - \Psi P \Psi \end{smallmatrix}\right]$
if $x_0 \in \mathcal{V}^{\text{init}} \cap \mathcal{C}^{\text{imp}}$ **then**
  return Optimal feeback matrix $K(t) = -\bar{R}^{-1}(B^{\text{diff}\top} X + S^\top)$
else
  return There does not exist an optimal control
end if

Based on this matrix, we compute

$$
\begin{aligned}
(I - \Pi_{\mathcal{V}^{\text{end}}})(I - B^{\text{imp}}\Lambda)\Pi &= \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{smallmatrix}\right], \\
(I - B^{\text{imp}}\Lambda)^\top P (I - B^{\text{imp}}\Lambda) - \Psi^\top P \Psi \Pi &= \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 21 \end{smallmatrix}\right]
\end{aligned}
$$

and hence

$$\ker \begin{bmatrix} (I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda) \\ (I-B^{\text{imp}}\Lambda)^\top P(I-B^{\text{imp}}\Lambda)-\Psi^\top P\Psi \end{bmatrix} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \right\}.$$

Finally, computing the backwards state transition matrix $\Omega(t_0, t_f)$ from Definition 5.40 given the solution of the Riccati equation (5.36) with terminal condition $X(t_f) = \Psi P \Psi$ yields

$$\Omega(t_0, t_f) := \left[\begin{smallmatrix} 2.163 & 0 & -6.2782 \\ -2.906 & 1 & -15.466 \\ 1.453 & 0 & 8.733 \end{smallmatrix}\right].$$

According to Theorem 5.41 we can conclude that

$$\mathcal{V}^{\text{init}} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 2.163 \\ -2.906 \\ 1.453 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \right\}$$

and that the optimal input is given by $u = -\bar{R}(B^{\text{diff}\top}X + \bar{S}^\top)x^{\text{diff}}$. Applying this input to the initial condition $x^{\text{diff}}(t_0^-) = \left[\begin{smallmatrix} 2.163 & -2.906 & 1.453 \end{smallmatrix}\right]^\top$ yields the optimal input $u(t)$ and $x^{\text{diff}}(t)$ as given in Figure 5.1. Observe that the solution indeed satisfies $x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \in \mathcal{V}^{\text{end}}$. Furthermore, the cost corresponding to this solution satisfies

$$\bar{J}(x_0, u) = 9.2352 = x^{\text{diff}}(t_0^-)^\top X(t_0) x^{\text{diff}}(t_0^-)$$

and is indeed quadratic in the inital value. ◇

The result of Corollary 5.42 shows that Problem 5.17 is generally not solvable for arbitrary initial values $x_0 \in \mathcal{C}^{\text{imp}}$. Hence if the initial value of a system is unknown, implementing an optimal control might be complicated. Secondly, if the system is perturbed at some time $t \in (t_0, t_f)$, the subspace endpoint constraint will be violated in general. Hence we can conclude that the optimal input is loosely speaking not very robust with respect to disturbances. However, in some special cases the problem is solvable for any x_0 and the control problem will be robust with respect to disturbances. Indeed, as the optimal control input u is a feedback, a perturbation in the state will lead to a perturbation in the input. To that extent we present the following result.

Lemma 5.44. *Problem 5.20 is solvable for all initial values $x_0 \in \mathcal{V}_{(E,A,B)}$ if and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \ker \begin{bmatrix} (I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda) \\ (I-B^{\text{imp}}\Lambda)^\top P(I-B^{\text{imp}}\Lambda)-\Psi^\top P\Psi \end{bmatrix} \Pi. \tag{5.44}$$

Proof. It follows from Theorem 5.41 that Problem 5.20 is solvable, if and only if

$$x_0 \in \Omega(t_0, t_f) \ker \begin{bmatrix} (I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda) \\ (I-B^{\text{imp}}\Lambda)^\top P(I-B^{\text{imp}}\Lambda)-\Psi^\top P\Psi \end{bmatrix} \Pi.$$

Hence, this holds for any $\mathcal{V}_{(E,A,B)}$ if and only if

$$\mathcal{V}_{(E,A,B)} \subseteq \Omega(t_0,t_f)\ker\begin{bmatrix}(I-\Pi_{\mathcal{V}^{\text{end}}})(I-B^{\text{imp}}\Lambda)\\(I-B^{\text{imp}}\Lambda)^\top P(I-B^{\text{imp}}\Lambda)-\Psi^\top P\Psi\end{bmatrix}\Pi$$

Observing that the matrix $\Omega(t_0,t_f)$ is invertible and $\Omega(t_0,t_f)^{-1}\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A,B)}$ leads to the conclusion that Problem 5.20 is solvable for all $x_0 \in \mathcal{V}_{(E,A,B)}$ if and only if (5.44) holds. □

In the case that the constraint $x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) \in \mathcal{V}^{\text{end}}$ is relaxed by assuming $\mathcal{V}^{\text{end}} = \mathbb{R}^n$ the conditions of Lemma 5.44 can be simplified. Note that $\mathcal{V}^{\text{end}} = \mathbb{R}^n$ occur in the case that *e.g.*, the next mode in the switched DAE is impulse-controllable or yields an ODE.

Lemma 5.45. *Consider Problem 5.17 and assume $\mathcal{V}^{\text{end}} = \mathbb{R}^n$. There exists an optimal control for any $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ if and only if*

$$B^{\text{imp}\top}P\Pi = B^{\text{imp}\top}PB^{\text{imp}}R^{-1}(B^{\text{diff}\top}\Psi^\top P\Psi + S^\top)\Pi. \tag{5.45}$$

Proof. ($\Rightarrow$) If there exists an optimal control that solves Problem 5.17 with $\mathcal{V}^{\text{end}} = \mathbb{R}^n$ for any x_0 then it follows from Lemma 5.39 that u is given by

$$u(t) = -R^{-1}(B^{\text{diff}\top}K(t) + S^\top)x^{\text{diff}}(t),$$

where K solves (5.36) with terminal condition $K(t_f^-) = \Psi^\top P\Psi$. Furthermore, it follows from Lemma 5.32 that $u(t_f^-)$ minimizes the terminal cost with respect to $x^{\text{diff}}(t_f^-)$ if and only if $\left[\begin{smallmatrix} u(t_f^-)^\top & 0 & x^{\text{diff}}(t_f^-)^\top \end{smallmatrix}\right]^\top \in \ker\mathcal{H}$ where now $\ker\mathcal{H}$ is given by those $\left[\begin{smallmatrix}\alpha & \beta & \gamma\end{smallmatrix}\right]$ for which

$$B^{\text{imp}\top}P\Pi\alpha = B^{\text{imp}\top}PB^{\text{imp}}\gamma.$$

Substituting u and noting that it needs to hold for all $x^{\text{diff}}(t_f^-) \in \operatorname{im}\Pi$ yields

$$B^{\text{imp}\top}P\Pi = B^{\text{imp}\top}PB^{\text{imp}}R^{-1}(B^{\text{diff}\top}\Psi^\top P\Psi + \bar{S}^\top)\Pi.$$

($\Leftarrow$) If (5.45) holds then the input defined as

$$u(t) = -R^{-1}(B^{\text{diff}\top}K(t) + \bar{S}^\top)x^{\text{diff}}(t)$$

where K solves (5.36) with $K(t_f^-) = \Psi^\top P\Psi$ ensures

$$\begin{aligned}(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-))^\top P(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-))&\\ \leqslant (x^{\text{diff}}(t_f^-) - B^{\text{imp}}w)^\top P(x^{\text{diff}}(t_f^-) - B^{\text{imp}}w),&\end{aligned}$$

for all $w \in \mathbb{R}^m$. Consequently

$$(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-))^\top P(x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-)) = x^{\text{diff}}(t_f^-)\Psi^\top P\Psi x^{\text{diff}}(t_f^-).$$

Therefore, it follows that $J(x^{\text{diff}}, v) = x^{\text{diff}}(t_0^-)^\top K(t_0)x^{\text{diff}}(t_0^-)$ and thus the infimum is attained. We can thus conclude that u is an optimal control. □

Finally, we can observe from Lemma 5.45 that if the terminal cost matrix $P = E^\top \tilde{P} E$ for some $\tilde{P} \in \mathbb{R}^{n\times n}$ there always exists an optimal solution under the assumption that $\mathcal{V}^{\text{end}} = \mathbb{R}^n$. In fact, the condition that $PB^{\text{imp}} = 0$ is sufficient for the existence of an optimal control. Under this assumption the standard linear quadratic optimal control problem for differential algebraic equations is obtained. Hence the following result should not be surprising.

Corollary 5.46. *Consider Problem 5.17 and assume $\mathcal{V}^{\text{end}} = \mathbb{R}^n$ and $PB^{\text{imp}} = 0$. Then there exists an optimal control for any $x_0 \in \mathcal{C}^{\text{imp}}$.*

Given the results on solvability of Problem 5.17 under the assumption that $x_0 \in \mathcal{C}^{\text{imp}}$ we can now restrict our attention to more general initial values. Recall from Lemma 5.16 that there exists an impulse-free input that ensures an impulse-free output if and only if $x_0 \in \mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}}$. Consider the decomposition

$$\begin{aligned}
\mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}} &= \mathcal{C}^{\text{imp}} + (W - (I - W))\mathcal{O}^{\text{imp}} \\
&= \mathcal{C}^{\text{imp}} + W\mathcal{O}^{\text{imp}} + (I - W)\mathcal{O}^{\text{imp}} \\
&= \mathcal{C}^{\text{imp}} + (I - W)\mathcal{O}^{\text{imp}}.
\end{aligned}$$

For initial values $x_0 \in \mathcal{C}^{\text{imp}}$ we have already stated results. For initial values $x_0 \in (I - W)\mathcal{O}^{\text{imp}}$ the optimal control is given by $u = 0$. Although the state trajectory will contain a Dirac impulse at t_0 and be zero elsewhere as a consequence of this input and initial value, the corresponding output $y = 0$, which yields an optimum. Observe that as $x_0 \in (I - W)\mathcal{O}^{\text{imp}}$ we can write

$$0 = u(t) = -\bar{R}^{-1}(B^{\text{diff}\top} X(t) + \bar{S}^\top)\Pi x(t).$$

This leads to the following corollary

Corollary 5.47. *Problem 5.17 is solvable if and only if*

$$x_0 \in \mathcal{V}^{\text{init}} \cap (\mathcal{C}^{\text{imp}} + \mathcal{O}^{\text{imp}})$$

and the corresponding optimal input is given by

$$u(t) = -\bar{R}^{-1}(B^{\text{diff}\top} X(t) + \bar{S}^\top)\Pi x(t).$$

where $X(t)$ solves the Riccati differential equation (5.36) *with terminal condition $X(t_f^-) = \Psi^\top P \Psi$.*

5.3 LQR for the switched case

Given the necessary and sufficient conditions for solvability of Problem 5.17 for general initial values we can now return to Problem 5.13 regarding the linear quadratic optimal control problem for switched DAEs on a bounded interval. Similar to the assumption in the previous section that the DAE (5.12) was index-1, we will assume that each mode of the switched DAE (5.1) is index-1. We can consider the impulse-controllable representation of each mode and assume a preliminary index-reducing feedback of the form $u = F_\sigma x$ has been applied. However, in order to guarantee that if there exists an optimal input, it is impulse-free, we make the following assumption.

Assumption 5.48. The output matrices of the system (5.1) are assumed to satisfy

$$\operatorname{rank}\begin{bmatrix} C_i \Pi_{\ker E_i} & D_i \end{bmatrix} = m, \qquad \forall i \in \{0, 1, ..., \mathtt{n}\},$$

where each $\Pi_{\ker E_i}$ is a projector onto $\ker E_i$.

As a consequence of this assumption, it follows that the output to (5.1) is impulse-fee if and only if (x, u) is impulse-free.

Lemma 5.49. *Consider the switched DAE* (5.1) *and let Assumption 5.48 hold. The output satisfies* $y[t] = 0$ *if and only if* $(x, u)[t] = 0$.

Proof. It follows from Lemma 5.27 that the output y is impulse-free on $[t_i, t_{i+1})$ for $i \in \{0, 1, ..., \mathtt{n} - 1\}$ if and only if (x, u) is impulse-free on $[t_i, t_{i+1})$. Hence if y is impulse-free on $[t_0, t_\mathtt{n})$ it follows that (x, u) is impulse-free on $[t_0, t_\mathtt{n})$. Conversely, if (x, u) is impulse-free on $[t_0, t_\mathtt{n})$ then y is impulse-free. □

Lemma 5.50. *Consider the switched DAE* (5.1) *and let Assumption 5.48 hold. If* $y[t_i] = 0$ *for* $i \in \{0, 1, ..., \mathtt{n}\}$ *then*

$$x(t_i^-) \in \mathcal{C}_i^{\text{imp}} + \mathcal{O}_i^{\text{imp}}.$$

Given these results we can construct a sequence of subspaces from which solvability of Problem 5.13 can be concluded. Indeed consider the following sequence of subspaces

$$\begin{aligned} \mathcal{V}_\mathtt{n}^{\text{end}} &= \mathcal{V}^{\text{end}}, \\ \mathcal{V}_{i-1}^{\text{end}} &= \mathcal{V}_i^{\text{init}} \cap (\mathcal{C}_i^{\text{imp}} + \mathcal{O}_i), \end{aligned} \qquad i = \mathtt{n}, \mathtt{n} - 1, ..., 0, \tag{5.46}$$

where $\mathcal{V}_i^{\text{init}}$ is defined according to Theorem 5.41 on the interval $[t_i, t_{i+1})$ w.r.t. $\mathcal{V}_i^{\text{end}}$. Then we can state the following result regarding solvability of Problem 5.13.

Theorem 5.51. *Consider the sequences* (5.46). *Problem 5.13 is solvable if and only if*

$$x_0 \in \mathcal{V}_0^{\text{init}} \cap (\mathcal{C}_0^{\text{imp}} + \mathcal{O}_0^{\text{imp}}).$$

Furthermore, if the problem is solvable, the optimal input is given by

$$u(t) = -\bar{R}_\sigma^{-1} (B_\sigma^{\text{diff}\top} K(t) + \bar{S}_\sigma^\top) \Pi_\sigma x(t),$$

where $\bar{R}_\sigma = \bar{D}_\sigma^\top \bar{D}_\sigma$, $\bar{S}_\sigma = \bar{C}_\sigma^\top \bar{D}_\sigma$ and Π_i is a projector resulting from the Wong sequence based on $(E_i W_i, A_i)$ and W_i an orthogonal projector onto $\mathcal{C}_i^{\mathrm{imp}}$, $i \in \{0, 1, ..., \mathtt{n}\}$. Finally, $K(t)$ is a solution to the switched Riccati differential equation

$$\dot{K} = -A_\sigma^{\mathrm{diff}\top} K - K A_\sigma^{\mathrm{diff}} + (\bar{S}_\sigma + K^\top B_\sigma^{\mathrm{diff}}) \bar{R}_\sigma^{-1} (B_\sigma^{\mathrm{diff}\top} K + \bar{S}_\sigma^\top) - \bar{Q}_\sigma,$$

where $Q_\sigma = \bar{C}_\sigma^\top \bar{C}_\sigma$ and time conditions

$$\begin{aligned} K(t_{i+1}^-) &= \Psi_i^\top K(t_{i+1}^+) \Psi_i, \qquad i \in \{0, 1, ..., \mathtt{n}-1\}, \\ K(t_f^-) &= \Psi_{\mathtt{n}}^\top P \Psi_{\mathtt{n}}. \end{aligned}$$

The optimal cost is given by

$$\min_u J(x_0, u) = x_0^\top K(t_0) x_0.$$

As a consequence of Theorem 5.51 it follows that there generally does not exist an impulse-free input that solves Problem 5.13. However, if the problem is solvable, it is generally solved by a distribution u which is only piecewise continuous and hence contains jumps. The same holds for the corresponding optimal state trajectory x. The following example illustrates this observation.

Example 5.52. Consider the switched DAE given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, && 0 \leqslant t < 1, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} &= -x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u, && 1 \leqslant t < 2, \\ \dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, && 2 \leqslant t < 3, \end{aligned}$$

together with the output

$$y = x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u.$$

If the terminal state is not penalized at $t^- = 3^-$, but it is required that $x(3^-) \in \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} := \mathcal{V}^{\mathrm{end}}$, we can compute the optimal feedback matrix on each interval $[t_i, t_{i+1})$, $i \in \{0, 1, 2\}$ by solving

$$\begin{aligned} \dot{K}_i = -A_i^{\mathrm{diff}\top} K_i - K_i A_i^{\mathrm{diff}} + (\bar{S} + K_i^\top B_i^{\mathrm{diff}}) \bar{R}^{-1} (B_i^{\mathrm{diff}\top} K + \bar{S}^\top) - \bar{Q}, \\ K_i(t_i^-) = \Psi_i^\top K_{i+1}(t_i^+) \Psi_i, \end{aligned}$$

where $\Psi_i = (I - B_i^{\mathrm{imp}} N_i) \Pi_i$ for some N_i which satisfies $[\, I \;\; 0 \;\; -N \,] \ker \mathcal{H}_i = 0.$ and $K(t_3^-) = 0$. We obtain from the calculations that

$$\begin{aligned} \mathcal{V}_2^{\mathrm{init}} &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, & K(2^-) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathcal{V}_1^{\mathrm{init}} &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0.54 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, & K(1^-) &= \begin{bmatrix} 0.39 & 0 & 0.38 \\ 0 & 0 & 0 \\ 0.38 & 0 & 2.40 \end{bmatrix}, \\ \mathcal{V}_0^{\mathrm{init}} &= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0.49 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0.06 \end{bmatrix} \right\}, & K(0^-) &= \begin{bmatrix} 0.21 & -0.03 & 0.07 \\ -0.03 & 0.03 & -0.19 \\ 0.07 & -0.19 & 1.59 \end{bmatrix}. \end{aligned}$$

Given the solution K we can compute the optimal input and optimal state trajectory, which are shown in Figure 5.2. As can be seen, both the optimal input and the optimal trajectory are piecewise continuous and contain jumps. ◇

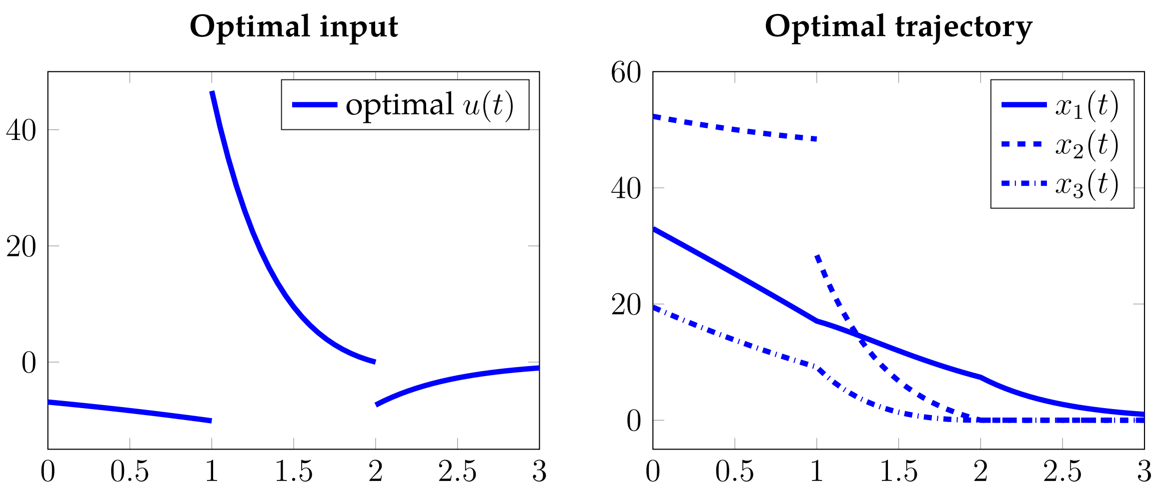

Figure 5.2: The optimal input $u(t)$ that solves Problem 5.20 and the corresponding optimal trajectory $x(t)$ (5.24) with initial value $x_0 = \begin{bmatrix} 32.98 & 52.30 & 19.46 \end{bmatrix}^\top$.

5.4 Concluding remarks

In this chapter we have considered the linear quadratic regulator problem for switched differential algebraic equations. We showed that if there exists an input that minimizes the cost functional subject to switched DAE constraints, the input is linear in the state. Furthermore, if there exists an optimal input, the optimal cost is quadratic in the initial value. Consequently, by taking a dynamic programming approach, the infinite horizon case can be reduced to a repeated finite horizon optimal control problem for non-switched DAEs with subspace endpoint constraints and a general quadratic terminal cost. It was shown that generally there does not exist an input that solves the problem and necessary and sufficient conditions for the existence of an optimal solution were given. Furthermore, it was shown how to compute the optimal control if it exists.

Given the conditions for solvability of the constraint optimal control problem for non-switched DAEs on a finite horizon, it was shown how to use these results to obtain conditions for solvability of the optimal control problem for switched DAEs.

Although the linear quadratic regulator problem for non-switched ODEs where the integrand of the cost functional is a possibly indefinite quadratic function of the state and input variable has been studied by *e.g.*, by Willems [134], it remains a future direction of research in the field of switched DAEs. Moreover, as the distributional framework is considered, a natural future direction of research is singular optimal control. That is, the case where impulsive inputs are allowed and the state can contain Dirac impulses. If the cost matrix is still quadratic in the state and the input, allowing Dirac impulses lead to singular linear quadratic optimal control and cheap control problems. For ODEs such problems have been studied by [19,31,35,36,72,126].

In the current chapter the results are stated in terms of a Riccati differential equation in terms of the matrices A^{diff} and B^{diff}, which in turn are computed based on the matrices E, A and B. An interesting question would be whether the results could be formulated in terms of a Riccati differential equation given in terms of the matrices E, A and B only. Such generalized Riccati equations have been studied for the non-switched case by *e.g.*, [51,63]. Furthermore, the techniques developed so far do not take any constraints on the state or input into account, besides the subspace endpoint constraint. In practical applications often only bounded inputs can be applied and it is desired to keep the state within certain bounds. Hence it remains to investigate the optimal control problem given such additional constraints.

Whereas we have studied the optimal control problem for switched DAEs only for switched systems with a fixed switching signal, it also remains to investigate the problem for systems where the switching signal is not fixed, but is considered to be an input. Since impulse-controllability of a switched systems generally depends on the switching times, it is likely that there will not exists an optimal control for all switched DAEs generated by the system matrices (E_p, A_p, B_p). Alternatively, the optimal control problem could also be considered in the context of switched systems where the switching depends on the state of the systems instead of time.

6 | Conclusions

In this thesis fundamental control theoretical properties of switched differential algebraic equations such as impulse-controllability and stabilizability have been considered. Furthermore, the linear quadratic optimal control problem has been investigated. The switched DAEs were assumed to have a known and a priori fixed switching signal. The precise type of systems have been introduced in Chapter 2 alongside the necessary mathematical preliminaries to analyze such systems.

In Chapter 3 we have studied impulse-free solutions of switched DAEs. Moreover, for systems with a fixed switching signal that induces finitely many mode changes a characterization of impulse-controllability was presented. The characterization is based on an algorithm that runs backward in time. A sufficient conditions for impulse-controllability of systems for which the switching signal induces infinitely many switches has been presented as well. This condition was based on an algorithm that runs forward in time and hence can be applied in real time.

Next, the notion of system classes of switched DAEs generated by a set of matrix triplets and a class of switching signals has been introduced. Strong impulse-controllability of such system classes has been defined. For the system class generated by some matrix triplets and the calss of arbitrary switching signals a characterization has been presented. In the case the order in which the modes are induced is fixed it turns out to be much more difficult to characterize strong impulse-controllability. Furthermore, it was shown that either all, or almost all systems in such system classes are impulse-controllable or uncontrollable. A sufficient condition for strong impulse-(un)controllability was given. Even in the case all systems in the system class are impulse-controllable and the property is thus independent of the switching signal, the controller that achieves impulse-freeness of the system might still depend on the switching signal. To that extent the concepts of quasi-causal impulse-controllability and causal impulse-controllability given a dwell time were introduced an characterized.

In Chapter 4 stabilizability of switched DAEs was studied. It was shown that controllability, reachability and null-controllability are equivalent concepts for switched DAEs in the behavioral sense. Furthermore, we have introduced the notion of interval-stabilizability. Necessary and sufficient conditions for a DAE to be impulse-free interval-stabilizable have been presented. These conditions lead naturally to a novel characterization of impulse-free controllability of switched DAEs.

Finally, in Chapter 5 the linear quadratic regulator problem for switched DAEs has been studied. We showed that if there exists an input that minimizes the cost functional subject to switched DAE constraints, the input is linear in the state. Furthermore, if there exists an optimal input, the optimal cost is quadratic in the initial value. Consequently, by taking a dynamic programming approach, the infinite horizon case can be reduced to a repeated finite horizon optimal control problem for non switched DAEs with subspace endpoint constraints and a general quadratic terminal cost. It was shown that generally there does not exist an input that solves the problem and necessary and sufficient conditions for the existence of an optimal solution were given. Furthermore, it was shown how to compute the optimal control if it exists.

Given the conditions for solvability of the constraint optimal control problem for non-switched DAEs on a finite horizon, it was shown how to use these results to obtain conditions for solvability of the optimal control problem for switched DAEs.

Future direction of research

Although the properties such as impulse-controllabilty and stabilizability have been discussed and characterizations have been presented, it remains a future direction of research to design algorithms to compute controllers that achieve stability and guarantee impulse-free solutions. It seems likely that impulse-freeness is generally not achievable by means of feedback control and hence other control tools need to be developed.

Furthermore, it remains to find necessary conditions for essential impulse-controllability and impulse-uncontrollability. So far only sufficient conditions for the strong variants have been given and hence a systematic way to determine whether all or almost all systems in a system class are impulse-controllable. Similarly, it remains an open question for future research how to characterize causal impulse-controllability of system classes without a dwell time. If a dwell time is not considered then an interesting problem is how to deal with the limiting case where all the switching times accumulate and how we should interpret such results. The concepts of quasi-causal and causal impulse-controllability have thus far only been considered for the special system class where the mode sequence is the same for all systems. Hence it also remains to investigate these properties for general system classes.

As for the optimal control problem, we have investigated the LQR problem in terms of the Riccati differential equation. The matrices involved in this differential equation depend on the transformation matrices computed based on the Wong-sequences of the matrix pair (E_pW_p, A_p). Hence it remains a future direction of research to investigate whether an equivalent result in terms of the original matrices can be obtained. Alternatively, it might be possible to obtain results in terms of a generalized Kalman-Yakubovich-Popov lemma. Furthermore, the techniques developed so far do not take any constraints on the state or input into account, besides the subspace endpoint constraint. In practical applications often only bounded inputs can be applied and it is desired to keep the state within certain bounds. Hence it remains to investigate the optimal control problem given such additional constraints.

Whereas we have studied the optimal control problem for switched DAEs with a fixed switching signal, it also remains to investigate the problem for systems where the switching signal is not fixed, but is considered to be an input. Since impulse-controllability of a switched systems generally depends on the switching times, it is likely that there will not exists an optimal control for all switched DAEs generated by the system matrices (E_p, A_p, B_p). Alternatively, the optimal control problem could also be considered in the context of switched systems where the switching depends on the state of the systems instead of time.

A | Appendix to Chapter 2

Lemma A.1. *Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace and let $A \in \mathbb{R}^{n\times n}$. Then $e^{At}x_0 \in \mathcal{V}$ for all $t > 0$ if and only if $x_0 \in \langle \mathcal{V} \mid A\rangle$.*

Proof. Let $x_0 \in \langle \mathcal{V} \mid A\rangle$. Then $Ax_0 \in \mathcal{V}$ and consequently $e^{At}x_0 \in \mathcal{V}$ for all $t > 0$. Conversely, let $e^{At}x_0 \in \mathcal{V}$ for all $t > 0$. Then for any $t_1 > t_0$ we have

$$\frac{1}{t_1 - t_0}(e^{At_0} - e^{At_1})x_0 \in \mathcal{V}.$$

Being a subspace of $\mathbb{R}^n$, $\mathcal{V}$ is closed in the Euclidean topology. Hence taking the limit of $t_1 \to t_0$ gives

$$Ae^{At_0}x_0 \in \mathcal{V}.$$

Consequently, since $e^{At_0}x_0 \in \mathcal{V}$ and $Ae^{At_0}x_0 \in \mathcal{V}$, it follows that $e^{At_0}x_0 \in \langle \mathcal{V} \mid A\rangle$. Hence we obtain

$$x_0 \in e^{-At_0}\langle \mathcal{V} \mid A\rangle \subseteq \langle \mathcal{V} \mid A\rangle,$$

which proves the desired result. $\square$

B | Appendix to Chapter 3

The proof of Theorem 3.31 relies on utilizing properties of analytic functions, which are recalled first.

Definition B.1. A function $f : \mathbb{R}^p \to \mathbb{R}$ is called *analytic* if for each $x \in \mathbb{R}^p$ the function f may be represented by a convergent power series in some neighborhood of x.

A useful property of analytic functions is the following well known result.

Lemma B.2 (Cf. [43, Cor I.A.10])**.** *The zero-set of a non-trivial analytic function $f : \mathbb{R}^p \to \mathbb{R}$ has (Lebesgue) measure zero.*

The notion of analycity can be extended to matrix-valued function as follows.

Definition B.3. The matrix valued function $M : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ is called an *analytic matrix* if each entry $m_{ij} : \mathbb{R}^p \to \mathbb{R}$ of M is an analytic function.

Definition B.4. A analytic matrix $M : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ is called *generically full rank* if either $\det\big(M(\boldsymbol{\tau})^\top M(\boldsymbol{\tau})\big) \neq 0$ for almost all[1] $\boldsymbol{\tau} \in \mathbb{R}^p$ or $\det\big(M(\boldsymbol{\tau})M(\boldsymbol{\tau})^\top\big) \neq 0$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.

Lemma B.5. *Let $A \in \mathbb{R}^{n\times n}$, $W : \mathbb{R}^p \to \mathbb{R}^{n\times k}$ a generically full rank analytic matrix and $\mathcal{R} \subseteq \mathbb{R}^n$ some subspace. Then $N : \mathbb{R}^{p+1} \to \mathbb{R}^{n\times q}$ given by*

$$\operatorname{im} N(\tau_0, \boldsymbol{\tau}) = e^{A\tau_0} \operatorname{im} W(\boldsymbol{\tau}) + \mathcal{R} \tag{B.1}$$

is a generically full rank analytic matrix.

Proof. We use $\mathcal{N}_{\tau_0,\boldsymbol{\tau}} \subseteq \mathbb{R}^n$ as short hand notation for the right-hand side of (B.1) in the following. Pick any $(\overline{\tau}_0, \overline{\boldsymbol{\tau}}) \in \mathbb{R}^{p+1}$ such that $\dim \mathcal{N}_{\overline{\tau}_0,\overline{\boldsymbol{\tau}}} = \max_{(\tau_0,\boldsymbol{\tau})} \dim \mathcal{N}_{\tau_0,\boldsymbol{\tau}} =: q$ and let $r_1, ..., r_l \in \mathbb{R}^n$ be a basis of $\mathcal{R}$. Choose $B_W \in \mathbb{R}^{k\times(q-l)}$ such that $[\overline{w}_1, \ldots, \overline{w}_{q-l}] = W(\overline{\boldsymbol{\tau}})B_W$ yields a basis

$$r_1, \ldots, r_l, e^{A\overline{\tau}_0}\overline{w}_1, \ldots, e^{A\overline{\tau}_0}\overline{w}_{q-l}$$

of $\mathcal{N}_{\overline{\tau}_0,\overline{\boldsymbol{\tau}}}$. Consider now the matrix valued function $N : \mathbb{R}^{p+1} \to \mathbb{R}^{n\times q}$ defined by

$$N(\tau_0, \boldsymbol{\tau}) := \left[r_1, \ldots, r_l, e^{A\tau_0} W(\boldsymbol{\tau}) B_W\right].$$

This matrix is analytic because the matrix exponential is analytic and the product of two analytic matrices is again analytic. By construction

$$\det\big(N(\overline{\tau}_0, \overline{\tau})^\top N(\overline{\tau}_0, \overline{\tau})\big) \neq 0,$$

[1] A property $P(\boldsymbol{\tau})$ is said to hold for almost all (a.a.) $\boldsymbol{\tau} \in \mathbb{R}^p$, if there exists $S \subseteq \mathbb{R}^p$ of Lebesgue measure zero, such that $P(\boldsymbol{\tau})$ holds for all $\boldsymbol{\tau} \in \mathbb{R}^p \setminus S$.

and hence the analytic function $(\tau_0, \boldsymbol{\tau}) \mapsto \det\left(N(\tau_0, \boldsymbol{\tau})^\top N(\tau_0, \boldsymbol{\tau})\right)$ is not identically zero. In view of Lemma B.2 it therefore follows that N is generically full rank.

It remains to be shown that (B.1) holds. By construction, $\operatorname{im} N(\tau_0, \boldsymbol{\tau}) \subseteq \mathcal{N}_{\tau_0,\boldsymbol{\tau}}$ for all $(\tau_0, \boldsymbol{\tau}) \in \mathbb{R}^{p+1}$. Furthermore, since $\dim \mathcal{N}_{\tau_0,\boldsymbol{\tau}} \leqslant q$ and $\dim \operatorname{im} N(\tau_0, \boldsymbol{\tau}) = q$ for a.a. $(\tau_0, \boldsymbol{\tau}) \in \mathbb{R}^{p+1}$ the claim follows. $\square$

Remark B.6. It is indeed possible that for some specific $(\tau_0, \boldsymbol{\tau})$ we have $\operatorname{im} N(\tau_0, \boldsymbol{\tau}) \subsetneq N_{\tau_0,\boldsymbol{\tau}}$. As an example consider for $\alpha > 0$

$$\begin{aligned} W(\tau_1) &= \operatorname{span}\left\{\left[\begin{smallmatrix} e^{\tau_1} \\ e^{\tau_1} - e^{\alpha} \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ e^{\tau_1} \end{smallmatrix}\right]\right\} := \operatorname{span}\{w_1(\tau_1), w_2(\tau_2)\}, \\ \mathcal{R} &= \operatorname{span}\left\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right\} := \operatorname{span}\{r_1\}, \quad A = 0. \end{aligned}$$

Then clearly, $e^{A\tau_0} W(\tau_1) + \mathcal{R} = \mathbb{R}^2$ for all $(\tau_0, \tau_1) \in \mathbb{R}^2$. However, while the choice $N(\tau_0, \tau_1) := [r_1, w_1(\tau_1)]$ satisfies

$$\operatorname{im} N(\tau_0, \tau_1) = e^{A\tau_0} W(\tau_1) + \mathcal{R} = \mathbb{R}^2 \quad \text{for a.a. } (\tau_0, \tau_1),$$

for $\tau_1 = \alpha$ we have

$$\operatorname{im} N(\tau_0, \alpha) = \operatorname{span}\{r_1\} \neq \mathbb{R}^2.$$

Lemma B.7. *Let $W : \mathbb{R}^p \to \mathbb{R}^{q \times n}$, $n > q$, be an analytic matrix with generically full rank. Then there exists an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n \times (n-q)}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \ker W(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.*

Proof. By considering the field of meromorphic functions (i.e. fractions of scalar-valued analytic functions), we can apply Gauss-Jordan eliminations on $W(\boldsymbol{\tau})$ to obtain a reduced row echolon form (RREF), which contains meromorphic entries and whose kernel for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$ equals $\ker W(\boldsymbol{\tau})$. Identically as for constant matrices, a full rank matrix $\overline{N}(\boldsymbol{\tau}) \in \mathbb{R}^{n \times (n-q)}$, can be easily constructed from the (meromorphic) entries of the obtained RREF such that $W(\boldsymbol{\tau})\overline{N}(\boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau}$ for which $\overline{N}(\boldsymbol{\tau})$ is well-defined. As a final step, let $N(\boldsymbol{\tau}) = \overline{N}(\boldsymbol{\tau}) \begin{bmatrix} \alpha_1(\boldsymbol{\tau}) & & \\ & \ddots & \\ & & \alpha_{n-q}(\boldsymbol{\tau}) \end{bmatrix}$, where $\alpha_i(\boldsymbol{\tau})$ is the product of all denominators of the entries in the i-th column of $\overline{N}(\boldsymbol{\tau})$. Then $M(\boldsymbol{\tau})N(\boldsymbol{\tau}) = 0$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$ and $\boldsymbol{\tau} \mapsto N(\boldsymbol{\tau})$ is an analytic matrix and has generically the same rank as $\overline{N}$, i.e. N is generically full rank. $\square$

Lemma B.8. *Let $W : \mathbb{R}^p \to \mathbb{R}^{n \times k}$, $k \leqslant n$, be an analytic matrix with generically full rank. Then for any $\Pi \in \mathbb{R}^{n \times n}$ there exists an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n \times m}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.*

Proof. By Lemma B.7 there exists an analytic matrix $\overline{N} : \mathbb{R}^p \to \mathbb{R}^{n\times q}$ with generically full rank and $\operatorname{im} \overline{N}(\boldsymbol{\tau}) = \ker W(\boldsymbol{\tau})^\top$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$. Consequently,

$$\begin{aligned}(\operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau}))^\perp &= \ker \Pi^\top + \ker W(\boldsymbol{\tau})^\top, \\ &= \ker \Pi^\top + \operatorname{im} \overline{N}(\boldsymbol{\tau}).\end{aligned}$$

Applying Lemma B.5 for $\mathcal{R} = \ker \Pi^\top$ and $A = 0$, we find an analytic matrix $\widetilde{N} : \mathbb{R}^p \to \mathbb{R}^{n\times \widetilde{q}}$ with generically full rank such that $\operatorname{im} \widetilde{N}(\boldsymbol{\tau}) = \ker \Pi^\top + \operatorname{im} \overline{M}(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau}$. Finally, using Lemma B.7 again we can find an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n\times q}$, $q = n - \widetilde{q}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \ker \widetilde{N}(\boldsymbol{\tau})^\top$ for a.a. $\boldsymbol{\tau}$. Altogether, we have for a.a. $\boldsymbol{\tau}$

$$\operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau}) = \left(\operatorname{im} \widetilde{N}(\boldsymbol{\tau})\right)^\perp = \ker \widetilde{N}(\boldsymbol{\tau})^\top = \operatorname{im} N(\boldsymbol{\tau}).$$

$\square$

C | Appendix to Chapter 4

Proposition C.1. *Let $\mathcal{V}$ and $\mathcal{S}$ be subspaces of $\mathbb{R}^n$ and let $M \in \mathbb{R}^{n\times n}$. If $(\{Mx_0\}+\mathcal{S})\cap\mathcal{V} \neq \varnothing$ for all $x_0 \in \mathbb{R}^n$, then there exists a matrix $N \in \mathbb{R}^{n\times n}$ such that for all x_0*

$$(\{Mx_0\}+\mathcal{S})\cap\mathcal{V} = \{NMx_0\}+\mathcal{S}\cap\mathcal{V}. \tag{C.1}$$

Proof. Let $m_1, m_2, ..., m_p$ be a basis for the image of M. Then the statement is proven if we can prove that

$$(\{m_i\}+\mathcal{S})\cap\mathcal{V} = \{Nm_i\}+(\mathcal{S}\cap\mathcal{V}), \qquad \forall i \in \{1,2,...,p\},$$

for some matrix N. Since we have that $(\{m_i\}+\mathcal{S})\cap\mathcal{V} \neq \varnothing$ it follows that for all i there exists an $\eta_i \in \mathcal{S}$ such that $m_i+\eta_i \in \mathcal{V}$. Let $\hat{N}$ be a linear map such that

$$\hat{N}m_i = \eta_i.$$

Then if we define $N = I+\hat{N}$ we have that

$$\begin{aligned} Nm_i &= m_i + \hat{N}m_i, \\ &= m_i + \eta_i, \\ &\in \mathcal{V}\cap(\{m_i\}+\mathcal{S}). \end{aligned}$$

Since subspaces are closed under addition, it follows that for all $\bar{\eta} \in \mathcal{S}\cap\mathcal{V} \subseteq \mathcal{V}$ we have that

$$Nm_i + \bar{\eta} = m_i + \eta_i + \bar{\eta} \in \mathcal{V}.$$

and

$$m_i + \eta_i + \bar{\eta} = m_i + \hat{\eta} \in \{m_i\}+\mathcal{S},$$

for some $\eta_i + \bar{\eta} = \hat{\eta} \in \mathcal{S}$, which proves that N is such that $\{Nm_i\}+\mathcal{S}\cap\mathcal{V} \subseteq (m_i+\mathcal{S})\cap\mathcal{V}$.

Conversely, we have for $\xi \in (\{m_i\}+\mathcal{S})\cap\mathcal{V}$ and for some $\beta \in \mathcal{S}$ that $\xi = m_i+\beta \in \mathcal{V}$. Let $\beta = \hat{N}m_i+\gamma$, for some $\gamma \in \mathcal{S}$. Then we obtain

$$\begin{aligned} \xi &= m_i + \beta \\ &= m_i + \hat{N}m_i + \gamma \\ &= Nm_i + \gamma \\ &\in (\{m_i\}+\mathcal{S})\cap\mathcal{V}. \end{aligned}$$

It remains to prove that $\gamma \in \mathcal{S}\cap\mathcal{V}$. Since $Nm_i \in (\{m_i\}+\mathcal{S})\cap\mathcal{V} \subseteq \mathcal{V}$ by definition, we have that $\xi - Nm_i = \gamma \in \mathcal{V}$. Furthermore, by definition, we had $\gamma \in \mathcal{S}$ and hence $\gamma \in \mathcal{S}\cap\mathcal{V}$. Hence we have proven that $(m_i+\mathcal{S})\cap\mathcal{V} \subseteq \{Nm_i\}+\mathcal{S}\cap\mathcal{V}$. With the inclusion in both direction proven, the equality follows. □

It follows from Proposition C.1 that if the intersection $(Mx_0 + \mathcal{S}) \cap \mathcal{V} \neq \varnothing$ for all x_0, that this matrix N is not unique. In fact, this observation results in the next lemma.

Lemma C.2. *With the same notation as in Proposition C.1 we have that $N \in \mathbb{R}^{n\times n}$ satisfies* (C.1) *if and only if*

1. $\operatorname{im}(N - I)M \subseteq \mathcal{S}$,
2. $\operatorname{im} NM \subseteq \mathcal{V}$.

Proof. Assume that N satisfies $\operatorname{im}(N - I) \subseteq \mathcal{S}$ and $\operatorname{im} NM \subseteq \mathcal{V}$. This means that $\operatorname{im}(N - I)M \subseteq \mathcal{S}$. Hence $NMx_0 \in \mathcal{S} + Mx_0$ for arbitrary $x_0 \in \mathbb{R}^n$. Furthermore, by assumption we had that $NMx_0 \in \operatorname{im} N \subseteq \mathcal{V}$ and hence $NMx_0 \in (Mx_0 + \mathcal{S}) \cap \mathcal{V}$. Hence it follows that $NMx_0 + \mathcal{S} \cap \mathcal{V} \subseteq (Mx_0 + \mathcal{S}) \cap \mathcal{V}$.

On the otherhand, let $\xi \in (Mx_0 + \mathcal{S}) \cap \mathcal{V}$. Then $\xi = Mx_0 + \eta$ for some $\eta \in \mathcal{S}$ and $\xi \in \mathcal{V}$. Since $NMx_0 \in \mathcal{V}$ we have that $NMx_0 - \xi \in \mathcal{V}$. From which it follows that $(N - I)Mx_0 \in \mathcal{V}$ and also $(N - I)Mx_0 \in \mathcal{S}$. Thus we have that $NMx_0 - \xi \in \mathcal{S} \cap \mathcal{V}$. From this it follows that $\xi \in NMx_0 + \mathcal{S} \cap \mathcal{V}$ and thus it is proven that under the assumptions (4.11) holds.

Next assume that (4.11) holds. Then it follows that

$$\begin{aligned} NMx_0 &\in (Mx_0 + \mathcal{S}) \cap \mathcal{V} + \mathcal{S} \cap \mathcal{V} \\ &= (Mx_0 + \mathcal{S}) \cap \mathcal{V}. \end{aligned}$$

Since this holds for all x_0 it follows that $\operatorname{im} NM \subseteq \mathcal{V}$. Furthermore, it follows that $NMx_0 \in Mx_0 + \mathcal{S}$, from which it follows that $(N - I)Mx_0 \in \mathcal{S}$ for all x_0, and thus $\operatorname{im}(N - I)M \subset \mathcal{S}$. Which proves the result. □

Given the subspaces $\mathcal{V}$, $\mathcal{S}$ and the matrix M, a matrix N satisfying the conditions of Lemma C.2 can constructively be computed.

Lemma C.3. *Let $\Pi_{\mathcal{V}}$ and $\Pi_{\mathcal{S}}$ be projectors onto $\mathcal{V}$ and $\mathcal{S}$ respectively. For any Q that solves*

$$(I - \Pi_{\mathcal{S}})\Pi_{\mathcal{V}} QM = (I - \Pi_{\mathcal{S}})M,$$

the matrix $N = \Pi_{\mathcal{V}} Q$ satisfies (C.1).

Proof. Since $\operatorname{im} N \subset \operatorname{im} \Pi_{\mathcal{V}} = \mathcal{V}$ the condition $\operatorname{im} NM \subset \mathcal{V}$ is satisfied. Furthermore, we have that

$$\begin{aligned} \operatorname{im}(N - I)M &= \operatorname{im}(\Pi_{\mathcal{V}} Q - I)M \\ &= \operatorname{im}(\Pi_{\mathcal{S}} + (I - \Pi_{\mathcal{S}}))(\Pi_{\mathcal{V}} Q - I)M \\ &\subseteq \mathcal{S} + \operatorname{im}(I - \Pi_{\mathcal{S}})(\Pi_{\mathcal{V}} Q - I)M \\ &= \mathcal{S} + \operatorname{im}\left((I - \Pi_{\mathcal{S}})M - (I - \Pi_{\mathcal{S}})M\right) = \mathcal{S}. \end{aligned}$$

Hence N satisfies the conditions of Lemma C.2, which proves the result. □

D | Appendix to Chapter 5

Lemma D.1. *Let Y be a Banach space and let $X \subseteq Y$ be a dense subspace. Then the dual space X^* of X is isometrically isomorphic to the dual space Y^* of Y.*

Proof. By the Hahn Banach theorem there exists for any bounded linear functional $f \in X^*$ a bounded linear functional $\bar{f} \in Y^*$ satisfying $\bar{f}|_X = f$ and $\|\bar{f}\|_{Y^*} = \|f\|_{X^*}$. [109, Theorem 3.2].

Suppose there exist linear functionals $\bar{f}_1, \bar{f}_2$ on Y such that $\bar{f}_1|_X = \bar{f}_2|_X = f_1$ for some $f_1 \in X^*$. Let $\bar{f}_3 = \bar{f}_1 - \bar{f}_2$. Then $\bar{f}_3|_X = 0$. Let $\{x_n\}$ be a sequence in X converging to $x \in Y$. Then since $\bar{f}_3|_X$ is continuous, it follows that $\bar{f}_3(x) = 0$. Consequently $\bar{f}_3 = 0$. As $x \in Y$ was arbitrary, we can conclude that it holds for all $x \in Y$. Hence $\bar{f}_3$ is bounded and thus for every bounded linear functional $f \in X^*$ there exists a unique bounded linear functional $\bar{f} \in Y^*$ satisfying $\bar{f}|_X = f$. Hence there exists an isomorphism between X^* and Y^*.

Observe that as $\|\bar{f}\|_{Y^*} = \|f\|_{X^*}$ and as both $\bar{f}$ and f are linear functionals, it follows that for any $\bar{f}_3 \in Y^*$ satisfying $\bar{f}_3 = \bar{f}_2 - \bar{f}_1$ for some $\bar{f}_1, \bar{f}_2 \in Y^*$

$$\|\bar{f}_3\|_{Y^*} = \|f_3\|_{X^*} = \|\bar{f}_2 - \bar{f}_1\|_{Y^*} = \|f_2 - f_1\|_{X^*},$$

which proves that the isomorphism is in fact isometric. $\square$

Lemma D.2. *Consider the system $\dot{x} = Ax + Bu$ on the interval $[t_0, t_f)$ and a solution (x_1, u_1) satisfying $x_1(t_0^-) = x_0$. The solution (x_2, u_2) with $x_2(t_0^-) = x_0$ and $u_2 = u_1 + u_3$ where*

$$u_3 = \begin{cases} 0, & t_0 \leqslant t < t_f - t_s, \\ \alpha e^{-A^{\mathrm{diff}} \frac{t_s}{2}}, & t_f - t_s \leqslant t < t_f - \frac{t_s}{2}, \\ -\alpha, & t_f - \frac{t_s}{2} \leqslant t < t_f, \end{cases}$$

for some $t_s \in \mathbb{R}$, $t_0 < t_s < t_f$ satisfies $x_2(t_f^-) = x_1(t_f^-)$.

Proof. The general solution formula that given an initial value x_0 the solution to $\dot{x} = Ax + Bu$ is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau.$$

Observe that with the change of variables $\bar{\tau} = \tau + \frac{t_s}{2}$ we have $\mathrm{d}\bar{\tau} = \mathrm{d}\tau$ and

$$\int_{t_f - t_s}^{t_f - \frac{t_s}{2}} e^{A(t_f - \frac{t_s}{2} - \tau)}B\alpha\,\mathrm{d}\tau = \int_{t_f - \frac{t_s}{2}}^{t_f} e^{A(t_f - \bar{\tau})}B\alpha\,\mathrm{d}\bar{\tau}.$$

Consequently, it follows that

$$\begin{aligned}
x_1(t_f^-) &= e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)}Bu_1(\tau)\,\mathrm{d}\tau \\
&= e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)}Bu_1(\tau)\,\mathrm{d}\tau + \int_{t_f-t_s}^{t_f-\frac{t_s}{2}} e^{A(t_f-\tau)}B\alpha e^{-A^{\mathrm{diff}}\frac{t_s}{2}}\,\mathrm{d}\tau \\
&\quad - \int_{t_f-\frac{t_s}{2}}^{t_f} e^{A(t_f-\tau)}B\alpha\,\mathrm{d}\tau \\
&= e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)}Bu_1(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t_f} e^{A(t_f-\tau)}Bu_3(\tau)\,\mathrm{d}\tau \\
&= e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)}B(u_1(\tau)+u_3(\tau))\,\mathrm{d}\tau \\
&= e^{A(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A(t_f-\tau)}Bu_2(\tau)\,\mathrm{d}\tau \\
&= x_2(t_f^-).
\end{aligned}$$

This proves the result. □

Lemma D.3. *Consider the system $\dot{x} = Ax + Bu$ together with the output $y = Cx + Bu$ and let a cost functional be given by*

$$J(x_0, u) = \int_{t_0}^{t_f} \|y(t)\|\,\mathrm{d}t. \tag{D.1}$$

Let (x_1, u_1) satisfying $x_1(t_0^-) = x_0$ be a solution for which (D.1) *is finite, i.e. $J(x_0, u_1) = M_1$ for some $M_1 \geqslant 0$. Let u_ε be an input depending on $\varepsilon \in (t_0, t_f)$ defined by $u_\varepsilon = u_1 + \bar{u}_\varepsilon$ where*

$$\bar{u}_\varepsilon = \begin{cases} 0, & t_0 \leqslant t < t_f - \varepsilon, \\ \alpha e^{-A\frac{\varepsilon}{2}}, & t_f - \varepsilon \leqslant t < t_f - \frac{\varepsilon}{2}, \\ -\alpha, & t_f - \frac{\varepsilon}{2} \leqslant t < t_f. \end{cases}$$

Let $(x_\varepsilon, u_\varepsilon)$ be a solution satisfying $x_\varepsilon(t_0^-) = x_0$. Then there exists an M such that for any $\varepsilon \in (t_0, t_f)$

$$J(x_0, u_\varepsilon) \leqslant J(x_0, u_1) + \varepsilon M.$$

Proof. By linearity of solutions we have $x_\varepsilon = x_1 + \bar{x}_\varepsilon$. Consequently

$$\begin{aligned}
y_\varepsilon &= Cx_\varepsilon + Du_\varepsilon \\
&= C(\bar{x}_\varepsilon + x_1) + D(\bar{u}_\varepsilon + u_1) \\
&= Cx_1 + Du_1 + C\bar{x}\varepsilon + D\bar{u}_\varepsilon \\
&= y_1 + \bar{y}_\varepsilon
\end{aligned}$$

and hence the Minkowski inequality we have

$$\begin{aligned}\left(\int_{t_0}^{t_f} \|y_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} &= \left(\int_{t_0}^{t_f} \|y_1(t) + \bar{y}_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_{t_0}^{t_f} \|\bar{y}_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &= \left(\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_{t_f - t_s}^{t_f} \|\bar{y}_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}.\end{aligned}$$

Next observe that

$$\bar{x}_\varepsilon(t) = \int_{t_0}^{t_f} e^{A(t_f - \tau)} B \bar{u}_\varepsilon(\tau) \,\mathrm{d}\tau$$

and hence $\bar{x}_\varepsilon$ is a continuous function of $\bar{u}_\varepsilon$. Consquently, $\sup_{t\in[t_0,t_f)} \|\bar{x}_\varepsilon(t)\|$ is bounded on the bounded interval $[t_0, t_f)$ as $\sup_{t\in[t_0,t_f)} \|u_\varepsilon(t)\| \leqslant \alpha$. As $\bar{y}_\varepsilon$ is linear in $\bar{x}_\varepsilon$ and $\bar{u}_\varepsilon$, the output is also a conitnuous function thus it follows that $\bar{y}_\varepsilon$ is bounded as well on $[t_0, t_f)$. Moreover, there exists an $\bar{M} > 0$ such for any $\varepsilon \in [t_0, t_f)$ we have $\sup_{[t_0,t_f)} \|\bar{y}_\varepsilon(t)\| \leqslant \bar{M}$ for all $t \in [t_0, t_f)$. Hence we can conclude that

$$\begin{aligned}\left(\int_{t_0}^{t_f} \|y_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} &\leqslant \left(\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_{t_f - t_s}^{t_f} \|\bar{y}_\varepsilon(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} + \varepsilon M.\end{aligned}$$

Since we assume that $\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t = M_1$ and $\varepsilon \leqslant t_f$ it follows that

$$\begin{aligned}\int_{t_0}^{t_f} \|y_\varepsilon(t)\|^2 \,\mathrm{d}t &\leqslant \int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t + 2\varepsilon\bar{M}\left(\int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t\right) + (\varepsilon\bar{M})^2 \\ &= \int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t + 2\varepsilon M\bar{M}_1 + (\varepsilon\bar{M})^2 \\ &= \int_{t_0}^{t_f} \|y_1(t)\|^2 \,\mathrm{d}t + \varepsilon M,\end{aligned}$$

where $M = 2M_1\bar{M} + t_f\bar{M}^2$. □

Lemma D.4. *Let $P = P^\top$ be a positive semi-definite matrix. For all $x \in \mathcal{V}^{\mathrm{end}}$ there exists an $u \in \mathbb{R}^n$ that solves*

$$\begin{aligned} \textit{min} \qquad & (x - B^{\mathrm{imp}}u)^\top P(x - B^{\mathrm{imp}}u), \\ \textit{s.t.} \qquad & B^{\mathrm{imp}}u \in \mathcal{V}^{\mathrm{end}}. \end{aligned} \tag{D.2}$$

Proof. Since x is fixed and

$$(x - B^{\mathrm{imp}}u)^\top P(x - B^{\mathrm{imp}}u) = x^\top Px - 2x^\top PB^{\mathrm{imp}}u + u^\top B^{\mathrm{imp}\top} PB^{\mathrm{imp}}u,$$

it follows after denoting $B^{\mathrm{imp}}u = y$ for some $y \in \operatorname{im} B^{\mathrm{imp}}$, solvability of (D.2) is equivalent to solvability of

$$\begin{aligned} \min \quad & y^\top Py + c^\top y, \\ \text{s.t.} \quad & y \in \mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}}, \end{aligned} \tag{D.3}$$

where $c := -2Px$. Suppose that $\mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}} = \mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}} \cap \ker P$. Then any $y \in \mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}}$ solves the problem and the minimum is given by 0.

Next, suppose that $\mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}} \cap \ker P \subset \mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}}$. Let $y_1, ..., y_k$ be a basis for $\mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}} \cap \ker P$, and let $y_{k+1}, ..., y_p$ be such that $y_1, ..., y_p$ is a basis for $\mathcal{V}^{\mathrm{end}} \cap B^{\mathrm{imp}}$. Let Y be a matrix defined as

$$Y = \begin{bmatrix} y_{k+1} & \cdots & y_p \end{bmatrix}.$$

Observe that for any $y \in \mathcal{V}^{\mathrm{end}} \cap \operatorname{im} B^{\mathrm{imp}} \cap \ker P$ the objective function equals zero. Hence we will focus on $y \in \operatorname{im} Y$. Hence we rewrite the optimization problem as follows

$$\begin{aligned} \min \quad & x^\top Y^\top PYx + \bar{c}^\top x, \\ \text{s.t.} \quad & x \in \mathbb{R}^{k-p+1}, \end{aligned} \tag{D.4}$$

where $\bar{c} = Yc$. Note that $\operatorname{im} Y \cap \ker P = 0$ and hence $x^\top Y^\top PYx > 0$ for all $x \in \mathbb{R}^{k-p+1}$ and hence $Y^\top PY$ is positive definite. Hence we can write

$$\begin{aligned} x^\top Y^\top PYx + \bar{c}^\top x &= x^\top Y^\top PYx + \bar{c}^\top x + \frac{1}{4}\bar{c}^\top c - \frac{1}{4}\bar{c}^\top \bar{c} \\ &= \|Y^\top PYx + \frac{1}{2}\bar{c}\|^2 - \frac{1}{4}\bar{c}^\top \bar{c}. \end{aligned}$$

Consequently, the minimum is given by

$$x = -\frac{1}{2}(Y^\top PY)^{-1}\bar{c}.$$

Hence (D.3) is solvable and the optimal value is given by $\min\{0, -\frac{1}{4}\bar{c}^\top (Y^\top PY)^{-1}\bar{c}\}$. Note that the minimizer is not unique in general. $\square$

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Summary

This thesis is concerned with the study of a particular subclass of hybrid systems, namely switched systems. A switched system is a dynamical system that consists of a finite number of subsystems, referred to as modes and a logical rule that orchestrates the switching between these subsystems. The main property of switched systems is that these systems switch among a finite number of subsystems and the discrete events interacting with the subsystems are governed by a piecewise continuous function called the switching signal. In the case where each subsystem is given by a linear differential algebraic equation (DAE) a switched DAE is obtained.

In contrast to (switched) ordinary differential equations and non-switched DAEs, switched DAEs have gained little attention in the literature, despite their many applications. In the case of *e.g.,* modeling dynamical systems that undergo abrupt structural changes or component failure, switched DAE models are naturally obtained. Solutions of switched DAEs generally contain jumps and Dirac impulses, which may exclude classical solutions from existence. These phenomena are not only mathematical artifacts, but can be observed in practice as well. Thus far the literature has mainly focused on the study of qualitative properties of switched DAEs without taking Dirac impulses into account or Dirac impulses are exploited to obtain additional information about the state of a system. However, Dirac impulses are undesired in general as they can cause damage to the system or cause hazardous situations. Furthermore, no quantitative properties have been studied yet. Quantitative properties such as optimal control aim to quantify the performance of a system, which is necessary in many applications. Therefore, this thesis is concerned with impulse-free properties and optimal control of switched DAEs

Regarding the qualitative properties, impulse-free solutions of switched DAEs are investigated in particular. Systems for which impulse-free solutions can be ensured via a suitable choice of control input are characterized. Regarding a switched DAE as generated by a set of matrix triplets and a class of switching signals gives rise to the concept of system classes of switched DAEs. Several impulse-controllability concepts regarding system classes are presented and characterized. Interestingly, the conditions simplify significantly in the case the system class is generated by arbitrary switching signals instead of switching signals that induce the same mode sequence. It is shown that although all systems in a system class are impulse-controllable and hence the property can be regarded as independent of the switching signal, the control required to ensure impulse-free solutions is in general not independent of the switching signal.

Besides impulse-free solutions, the concept of impulse-free stabilizability is also studied, where a system is said to be impulse-free stabilizable if any initial trajectory can be

steered asymptotically to the origin by means of a suitable choice of control input. In order to deal with systems with an infinite number of switches, the concept op impulse-free interval stabilizability is introduced. Under some mild assumption global stabilizability can be concluded from interval-stabilizability. Necessary an sufficient conditions for impulse-free interval-stabilizability are given, followed by a novel characterization of impulse-free controllability.

The final chapter deals with a quantitative property, namely optimal control of switched DAEs. It is shown that if a quadratic cost functional is considered, the optimal control is a feedback and the optimal cost is a function that is quadratic in the initial value. It is shown that these results give rise to a dynamic programming approach and it is shown that the optimal control problem for switched DAEs can be regarded as a repeated optimal control problem for non-switched DAEs. However, for each problem for non-switched DAEs additional subspace endpoint constraints need to be imposed and a general terminal cost matrix needs to be considered. Necessary and sufficient conditions for solvability of the constraint optimal control problem for DAEs are presented and it is shown how these results lead to results for the switched case.

Samenvatting

In dit proefschrift staat een bepaalde klasse van hybride system centraal, namelijk schakelsystemen. Een schakelsysteem is een dynamisch systeem dat bestaat uit een aantal deelsystemen, ook wel modes genoemd, en een logische regel die het schakelen tussen de deelsystemen orchestreert. De voornaamste eigenschap van schakelsystemen is dat deze systemen schakelen tussen een eindig aantal deelsystemen en de discrete interactie tussen de systemen wordt geregeld door een stuksgewijs continue functie, ook wel schakelsignaal genoemd. Indien ieder deelsysteem gegeven wordt door een liniaire differentiaal algebraische vergelijking (DAE), heeft men te maken met een schakel-DAE.

In tegenstelling tot (schakel) differentiaal vergelijkingen en niet geschakelde DAEs, hebben schakel-DAEs ondanks hun vele toepassingen weinig aandacht gekregen in de literatuur. Bij het modelleren van dynamische systemen die abrupte structurele veranderingen ondergaan, of systemen waarbij onderdelen kapot gaan, krijgt men te maken met schakel-DAEs. Oplossingen van schakel-DAEs zijn in de regel genomen discontinue en bevatten Dirac impulsen, waardoor klassieke oplossingen vaak niet bestaan. Deze phenomenen zijn niet alleen een wiskundig artifact, maar komen in de praktijk ook voor. Tot nu toe heeft de literatuur haar aandacht vooral gefocuset op kwalitatieve eigenschappen van schakel-DAEs waarbij Dirac impulsen niet in acht worden genomen, dan wel gebruikt worden om de toestand van het systeem te acherhalen. Dirac impulsen zijn echter dikwijls ongewenst, aangezien ze het systeem kunnen beschadigen of een gevaarlijk situatie kunnen veroorzaken. Ook kwantitatieve eigenschappen zijn nog niet bestudeerd. Kwantitatieve eigenschapenp zoals optimale regeling hebben als doel om de prestatie van een systeem te kwantificeren, wat nodig is voor menig applicatie. Daarom staan impulse-vrije eigenschappen en optimale regeling van schakel-DAEs centraal in dit proefschrift.

Wat beteft de kwalitatieve eigenschappen, worden imulse-vrije oplossingen van schakel-DAEs in thet bijzonder onderzocht. Systemen waarvoor een impulse-vrije oplossing kan worden gegarandeerd door een juist regel signaal toe te passen ongeacht de beginconditie worden gekarakterizeerd. Door schakel-DAEs te beschouwen als gegenereerd door een set van matrix triplets en een klasse van schakelsignalen, kan een systeemklasse van schakel-DAEs worden gedefinieerd. Diverse concepten van impulse-regelbaarheid worden geïntroduceerd en gekarakterizeerd. Interessant genoeg versimpelen de voorwaarden in het geval de systeemklasse gegenereerd door willekeurige schakelsignalen beschouwt worden in plaats van schakelsignalen die dezelfde volgorde van modes induceren. Het wordt aangetoond dat hoewel ieder systeem in de systeemklasse impulse-regelbaar kan zijn en dat de eigenschap dus onafhankelijk van het schakelsignaal genoemd kan worden, dit niet impliceert dat het regelsignaal dat een impulse-vrije oplossing garandeert onafhankelijk is van het schakelsignaal.

Naast impulse-vrije oplossing, wordt het concept van impulse-vrije stabilizering ook bestudeerd, waar een systeem impulse-vrij stabilizeerbaar wordt genoemd indien iedere beginoplossing asymptotisch naar de oorsprong gestuurd kan worden door het juiste regelsignaal te kiezen. Om iets te kunnen zeggen over systemen met een oneindig aantal schakelingen, wordt het idee van impulse-vrije stabilizatie op een interval geintroduceerd. Onder een milde aanname kan globale stabilizeerbaarheid aangetoond worden op basis van stabilizeerbaarheid op een interval. Nodige en voldoende voorwaarde voor impulse-vrij interva-stabilizeerbaarheid worden gepresenteerd en een nieuwe karakterizatie van impulse-vrije regelbaarheid volgt als een gevolg.

Het laatste hoofdstuk heeft te maken met een kwantitatieve eigenschap, namelijk die van optimale regeling van schakel-DAEs. Er wordt aangetoond dat indien de kostenfunctionaal kwadratisch is, het optimale regelsignaal een terugkoppeling is en de optimale kost een kwadratische functie is van de beginwaarde. Deze observatie geeft aanleiding om een dynamische programmeringsaanpak te nemen waardoor het optimale regelprobleem voor schakel-DAEs herschreven kan worden als een repeterend optimaal regelprobleem voor niet geschakelde DAEs. Voor het niet geschakelde probleem moeten echter wel extra voorwaarde op de terminale toestand worden gesteld en een algemene terminale kost moet in acht genomen worden. Nodige en voldoende voorwarde voor het bestaan van een oplossing voor het probleem met extra voorwaarden worden gepresenteerd en het wordt aangetoond hoe deze resultaten leiden tot het resultaat voor het geschakelde geval.