Some Decomposition Theorems for the Vector Space of Matrix Summability Operators

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Consider the set $\tau = \{ (a_{ij}) \mid a_{ij} \text{ are real} \}$ of matrix summability operators on the set B of bounded sequences of real numbers. For $T \epsilon \tau$, $\xi \epsilon B$,

$$T(\xi) = (a_{ij})(\xi) = \left\{ \sum_j a_{ij} \xi_j \right\} = \left\{ t_i(\xi) \right\},$$

for $i = 1, 2, 3, \ldots$. With the usual definitions of matrix addition and multiplication by a scalar, τ becomes a vector space. Some subsets of τ are characterized by the following table, as in Hardy [1], where $C \subset B$ and $\mathbf{Z} \subset B$ denote the sets of convergent and null sequences respectively.

	τ_c	τ_r	τ_b	τ_z	τ_θ	τ_o
$\delta_j = \lim_i a_{ij}$	δ_j	$\delta_j = 0$	δ_j	$\delta_j = 0$	$\delta_j = 0$	$\delta_j = 0$
$\delta = \lim_i \sum_j a_{ij}$	δ	$\delta = 1$	$\delta = \sum_j \delta_j$	δ	$\delta = 0$	$\delta = 0$
Mapping relations on subspaces of B	$C \to C$	$C \to C$	$B \to C$	$C \to C$ $Z \to Z$	$C \to Z$	$B \to Z$

For each of the subsets characterized in the above table, $\sum_j |a_{ij}| < M$, for every i. For τ_b and τ_o, the uniform convergence with respect to i of $\sum_j |a_{ij}|$ is required. For τ_r, $\lim_i t_i(\xi) = \lim \xi_j$. The sets τ_c and τ_r are recognized as the sets of conservative and regular matrix summability operators, respectively. All of the subsets of τ listed above except τ_r form subspaces of τ. τ_r forms a convex set in τ, actually a coset of τ_θ in τ_c. Inclusion relations for these subsets of τ are shown by the following diagram:

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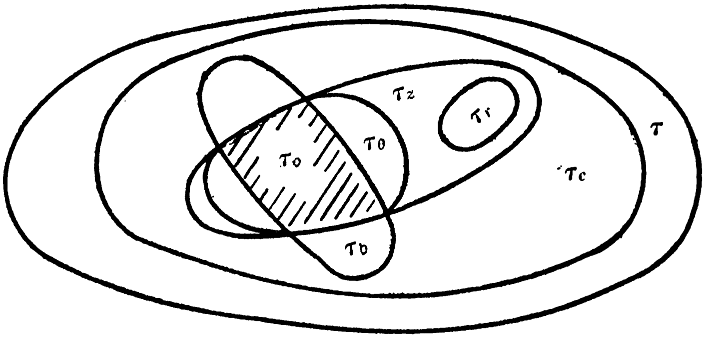

The following theorems yield some decompositions of τ:

LEMMA. $\tau_b \cap \tau_z = \tau_o$.

If $T \in \tau_b \cap \tau_z$, then $\delta_j = 0$. Since $T \in \tau_b$,

$$\delta = \lim_i \sum_j a_{ij} = \sum_j \lim a_{ij} = \sum_j \delta_j = 0, \text{ so } T \in \tau_o.$$

Therefore $\tau_b \cap \tau_z \subset \tau_o$.
If $T \in \tau_o$, then $T(B) \subset Z$, so $T \in \tau_b$.
Since $T(B) \subset Z, T(C) \subset Z, T \in \tau_z$.
Therefore $T \in \tau_b \cap \tau_z$, implying $\tau_o = \tau_b \cap \tau_z$.

THEOREM 1. $\tau_c/\tau_o = \tau_c/\tau_o \oplus \tau_b/\tau_o$.
Clearly $\tau_b + \tau_z \subset \tau_c$.

Let $T \in \tau_c$, $\xi \in C$. *Since* $\sum_j |a_{ij}| < M$ for every i, $\sum_{j-1}^{n} |a_{ij}| < M$ for any n.
Taking the limit with respect to i yields

$$\sum_{j-1}^{n} |\delta_j| < M, \text{ for every } n.$$

Therefore $\sum_j |\delta_j| < M$.

Hence,

$$t_i(\xi) = \sum_j a_{ij}\xi_j = \sum_j (a_{ij} - \delta_j)\xi_j + \sum_j \delta_j \xi_j$$

and T is expressed as the sum of
$(a_{ij} - \delta_j) \in \tau_z$ and the matrix (b_{ij}), where $b_{ij} = \delta_j$ for every i.

Therefore $\tau_c = \tau_b + \tau_z$.
Suppose $\{ t_i(\xi) \}$ also has the representation

$$t_i(\xi) = \sum_j \beta_{ij}\xi_j + \sum_j \gamma_{ij}\xi_j,$$

where $(\beta_{ij}) \in \tau_b$, $(\gamma_{ij}) \in \tau_z$.

Then $\sum_j (a_{ij} - \delta_j)\xi_j - \sum_j \gamma_{ij}\xi_j = \sum_j \delta_j\xi_j - \sum \beta_{ij}\xi_j$,

for every i.

But $(a_{ij} - \delta_j - \gamma_{ij}) \in \tau_z$ and $(\delta_j - \beta_{ij}) \in \tau_b$, so each matrix belongs to $\tau_o = \tau_z \cap \tau_b$. Therefore the two representations for T differ by a matrix $\in \tau_o$. Taking coset decomposition with respect to τ_o, one has the direct sum

$$\tau_c/\tau_o = \tau_b/\tau_o \oplus \tau_z/\tau_o$$

Now consider a decomposition of τ_z. Let $T_z \in \tau_z$ and A be any matrix summability operator in τ_r. Then if $\delta \neq O$,

(1) $$T_z = \delta(1/\delta T_z - A) + \delta A$$
$$= T_\theta + \delta A,$$

where $T_\theta = \delta(1/\delta T_z - A) \in \tau_\theta$. If $\delta = O$, then $T_z \in \tau_O$ and the equation (1) is still valid with $T_\theta = T_z$. Now suppose T_z admits another decomposition of the form

$$T_z = T'_\theta + \delta' A,$$

where $T'_\theta \in \tau_\theta$ and $\delta' \neq O$. Then $T_\theta + \delta A = T'_\theta + \delta' A$ or $T_\theta - T'_\theta = (\delta' - \delta)A$. Since $T_\theta - T'_\theta \in \tau_O$, $(\delta' - \delta)A \in \tau_O$ which implies $\delta = \delta'$ and thus $T_\theta = T'_\theta$. Therefore the decomposition given in (1) is unique. This yields the following direct sum decomposition:

$$\tau_z = \tau_\theta \oplus \{ \delta A \mid \delta \text{ a real number} \} .$$

For a coset decomposition of τ_z with respect to τ_o, first notice that $\tau_o \subset \tau_\theta$. If $\tau_A = \{ \delta A \mid \delta \text{ a real number} \} + \tau_o$, the coset decomposition of τ_z with respect to τ_o is given by

(2) $$\tau_z/\tau_o = \tau_\theta/\tau_o \oplus \tau_A.$$

Combining (2) with Theorem 1 yields

THEOREM 2. τ_c can be decomposed into a direct sum of the form

$$\tau_c/\tau_o = \tau_b/\tau_o \oplus \tau_\theta/\tau_o \oplus \tau_A.$$

REFERENCE

[1] Hardy, G. H. *Divergent Series.* Oxford University Press, London, 1956.