# On the Estimation of Parameters when the Observations are Subject to Measurement Error

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ABSTRACT

Maximum likelihood estimation of parameters is considered in the situation where a measurement x is taken to mean "x ± d." The maximum likelihood estimator for the parameter of the exponential distribution is found for this case and compared with the usual estimator.

INTRODUCTION

In this paper, "measurement error" will refer to that caused by the graduation of a measuring instrument, whereby each observation is recorded to the nearest unit of measurement, or by any other causes of the same nature. Such a process results in a measurement $x$ meaning "$x \pm d$," where $d$ is the largest error possible, usually half the smallest graduation or the smallest unit observed.

This paper investigates the effect of taking into account the above maximum error in each observation on the estimating function (estimator) of the parameter of the exponential distribution. It is found that the estimator in this case is different from that in the usual case, but has the usual estimator as a limiting function as $d$ goes to zero.

THE EXPONENTIAL DISTRIBUTION

Given the density

$$f(x;\theta) = \theta e^{-\theta x}, x \geqq 0,$$

the likelihood function based on a sample of size n is

$$L = \prod_{i=1}^{n} \int_{x_i-d}^{x_i+d} \theta e^{-\theta t}\, dt.$$

Since $ln\, L = \sum_{i=1}^{n} ln \int_{x_i-d}^{x_i+d} \theta e^{-\theta t}\, dt = \sum_{i=1}^{n} ln\, [e^{-\theta(x_i + d)} - e^{-\theta(x_i - d)}]$,

setting $\dfrac{d\, ln\, L}{d\theta} = 0$ gives $\displaystyle\sum_{i=1}^{n} \frac{-(x_i + d)\, e^{-\theta(x_i+d)} + (x_i - d)\, e^{-\theta(x_i-d)}}{e^{-\theta(x_i+d)} - e^{-\theta(x_i-d)}}$

$$= \Sigma \frac{x_i\, (e^{\theta d} - e^{-\theta d})}{e^{-\theta d} - e^{\theta d}} - \Sigma \frac{d\, (e^{\theta d} + e^{-\theta d})}{e^{-\theta d} - e^{\theta d}} = 0,$$

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which gives

$$\Sigma x_i \,(e^{\theta d} - e^{-\theta d}) - \Sigma d\,(e^{\theta d} + e^{\theta d}) = 0,$$

$$\Sigma x_i \,(e^{2\theta d} - 1) - nd\,(e^{2\theta d} + 1) = 0,$$

or

$$e^{2\theta d}\,(\Sigma x_i - nd) - \Sigma x_i - nd = 0,$$

i.e.,

$$e^{2\theta d} = \frac{\bar{x} + d}{\bar{x} - d}$$

and

$$\hat{\theta} = \frac{1}{2d}\, ln\, \frac{\bar{x} + d}{\bar{x} - d}$$

$$= \frac{1}{2d}\,[ln\,(\bar{x} + d) - ln\,(\bar{x} - d)],$$

which, incidentally, can also be written as

$$\hat{\theta} = \frac{1}{d}\, coth^{-1} \frac{\bar{x}}{d}.$$

It will be noted that de L' Hospital's rule shows that $\lim_{d \to 0} \hat{\theta} = \frac{1}{x}$ which is the maximum likelihood estimator for $\theta$ if the error $d$ is not taken into account.

## DISCUSSION

It is conjectured that for a symmetrical distribution, estimation of some parameters under the model including $d$ will not differ from estimation not including $d$. In particular, results for the normal distribution, together with results on variances and distributions of the estimators, will be reported in a later paper.