Some Functions in the Theory of Neutron Logging

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ABSTRACT.—The solution to the one-energy-group diffusion equation for the case of a point neutron source on the axis of two concentric cylindrical media has been programmed for a digital computer. Numerical results for a wide range of values of diffusion length and diffusion coefficient for the outer medium, and for four radii of the inner cylinder are presented. The outer radius in all cases is effectively infinite.

INTRODUCTION

In order to gain insight into the response of epithermal neutron logging devices Tittle (1961) solved the single-speed neutron diffusion equation for the simulated drill-hole geometry shown in Figure 1. The epithermal neutron flux was found to be expressible as a Fourier-Bessel series, and a graphical method for determining the necessary eigenvalues was given.

Shortly after the initial paper on this theory was submitted for publication, a computer program treating the entire numerical problem of the analysis was completed (Allen, 1961). This program was written for the Remington Rand Univac 1103 computer at Southern Methodist University, and was used to make an extensive study of the numerical behavior.

In the course of this work an error was discovered in the normalization procedure, as it was described in our earlier writings. The error can be corrected by setting the quantity U_n, as defined in those papers, equal to unity. This correction has been incorporated into the work described here.

A new version of the theory, utilizing an expansion into eigenfunctions of z, has been developed and programmed for the Control Data Corporation 1604 computer at Southern Methodist University. The details of this theory will be described elsewhere.

The purpose of this paper is to present the results of a series of calculations in which the diffusion parameters and bore-hole size were parameters of variation.

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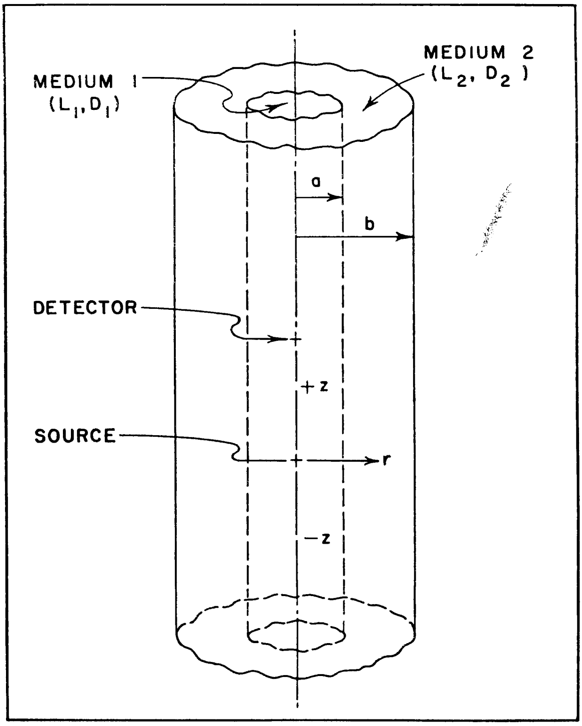

Fig. 1. Geometrical Arrangement of the Media, Source and Detector.

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RESULTS OF THE MATHEMATICAL THEORY

Since a derivation of the desired mathematical expressions has been given elsewhere (Tittle, 1961), only a listing of the important equations will be presented here. If the geometrical designations of Fig. 1 are employed, the series solution for the epithermal flux can be written as:

$$\phi_1 = \sum_n A_n J_0(C_n r) e^{-F_n z}, \qquad 0 \leq r \leq a \qquad (1)$$

or

$$\phi_2 = \sum_n G_n [J_0(P_n r) + H_n Y_0(P_n r)] e^{-F_n z}, a \leq r \leq b. \qquad (2)$$

where

$$A_n = Q/2D_1F_nM_n \tag{3}$$

$$F_n = \sqrt{C_n^2 + \frac{1}{L_1^2}} = \sqrt{P_n^2 + \frac{1}{L_2^2}} \tag{4}$$

$$G_n = A_nJ_0(C_na)/Z_0(P_na) \tag{5}$$

$$H_n = -J_0(P_nb)/Y_0(P_nb) \tag{6}$$

$$M_n = \pi a^2 \left\{ J_0^2(C_na) + J_1^2(C_na) + \frac{D_2}{D_1}\left[\frac{G_n}{A_n}\right]^2 \left[k^2Z_1^2(P_nb) - Z_1^2(P_na) - Z_0^2(P_na)\right]\right\} \tag{7}$$

$$k = b/a. \tag{8}$$

In these expressions ϕ_1 and ϕ_2 stand for the epithermal neutron flux in the inner and outer cylinders, respectively. The symbols $Z_0(x)$ and $Z_1(x)$ represent the linear combinations of Bessel functions $J_0(x) + H_nY_0(x)$ and $J_1(x) + H_nY_1(x)$, respectively. Characteristic values of C_n or P_n are determined by the transcendental equation

$$\frac{J_0(C_na)}{D_1C_nJ_1(C_na)} = \frac{J_0(P_na) + H_nY_0(P_na)}{D_2P_n[J_1(P_na) + H_nY_1(P_na)]} \tag{9}$$

Equations (1) through (9) were obtained by applying appropriate boundary and source conditions to the steady-state neutron diffusion equation

$$D\nabla^2\phi - \Sigma_r\phi = 0 \tag{10}$$

for the two regions in Fig. 1. In solving Eq. (10) it is conventional to divide the equation by the diffusion coefficient D (cm). The division of the effective removal cross section Σ_r (cm^{-1}) by the diffusion coefficient leads to the definition of a new constant, the slowing down length of epithermal neutrons. This constant is defined as

$$L_s\ (cm) = \sqrt{D/\Sigma_r} \tag{11}$$

In this paper the slowing down length is represented by the letter L with a subscript for a region designation.

THE NUMERICAL PROBLEM

The constants C_n and P_n must be known for each term which is to be included in the series expression for the flux. As pointed out, these values must be determined from Eq. (9) with the aid of Eq. (4). Unfortunately, the eigenvalues of Eq. (9) are not easily obtained. Asymptotic expressions for the Bessel functions can be used to calculate the root spacing of Eq. (9) for large n, but this information is of little use for $n < 10$, a region of vital importance. This being the case, a straightforward method for solving the equation must be employed.

A graphical method for solving Eq. (9) was presented by Tittle (1961). This method permits hand calculations to be made, but is laborious if three significant figures are desired in the values of small roots. In many cases an estimate of the third significant figure is sufficient. However, one term in the series will occasionally show considerable sensitivity to the value of the corresponding root. Although the effect of root indeterminacy may be significant for the particular term involved, the net effect on the expression for the flux is usually much less than 10%. If greater accuracy is desired, the graphical method should be abandoned in favor of a computational scheme which can be programmed for a digital computer.

As the first step in devising one such method, Eq. (9) can be rewritten as

$$xJ_o(y)\,[J_1(x)Y_o(kx) - J_o(kx)Y_1(x)]$$
$$-(D_1/D_2)yJ_1(y)[J_o(x)Y_o(kx) - J_o(kx)Y_o(x)] = 0 \qquad (12)$$

where

$$x = P_n a \qquad (13)$$

$$y = C_n a = \left(x^2 + \frac{a^2}{L_1^2} - \frac{a^2}{L_2^2}\right)^{1/2} \qquad (14)$$

Since D_1/D_2 is the only division in Eq. (12) and the Bessel functions are continuous, the left hand side of the equation must oscillate if the equation has real roots. If this portion of the equation is called $F(x,y)$, the sign of $F(x,y)$ can clearly be used as a mechanism to achieve convergence in an iterative solution for the roots of the equation. Assuming that Eq. (12) has no roots in the immediate vicinity of $x = 0$, one may compute a reference value of $F(x,y)$ for some arbitrarily small value of x, add a small increment to x and recompute $F(x,y)$, and then compare the two values of $F(x,y)$ for sign. If both values have the same sign the root obviously has not been passed; if the signs differ the root has been passed by an amount less than the magnitude of the increment and some method, such as the method of halving, can be used to locate the root. It is clear that any number of roots can be found by this procedure and that good accuracy can be achieved. Evident also is the fact that a judicious choice of the increment size is desirable to reduce computing time. Fortunately, $F(x,y)$ is nearly periodic and such a choice can readily be made. A curve showing an approximate relationship between root spacing and the diameter ratio k is shown in Fig. 2.

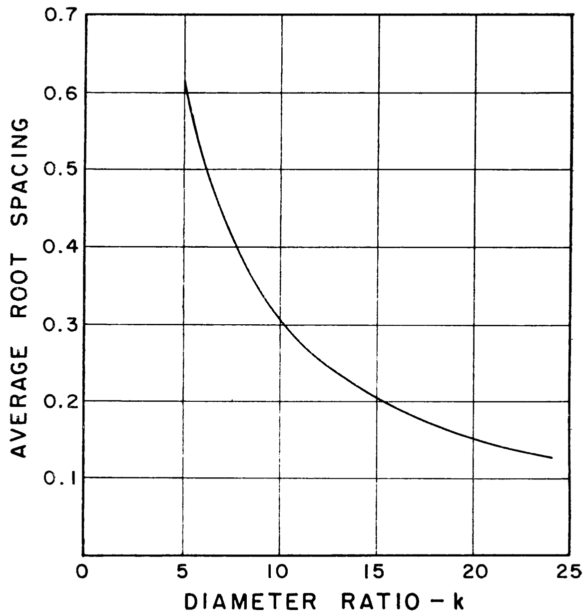

Fig. 2. Average root spacing of Eq. (12) for various values of diameter ratio k.

It was assumed earlier that Eq. (12) did not have a root in the immediate vicinity of $x = 0$. This can be shown to be true by considering the behavior of $F(x,y)$ for values of x near zero. For cases of interest in logging, L_1 is always less than L_2 if the hole is liquid filled; thus Eq. (15) demands y to be imaginary, yet finite, for values of x near zero. Even though y is imaginary, $J_o(y)$ and $yJ_1(y)$ are real; the functions $J_o(iu)$ and $-iJ_1(iu)$ are conventionally tabulated as $I_o(u)$ and $I_1(u)$. Furthermore, $J_o(y)$ is a positive real number when x is zero. Consider the following portion of Eq. (12):

$$xJ_o(y)\ [J_1(x)Y_o(kx) - J_o(kx)Y_1(x)]$$

As x approaches zero, the functions $Yo(kx)$ and $Y_1(x)$ lose their oscillatory behavior and approach minus infinity. At $x = 0$, $Y_o(kx)$ has a logarithmic pole and $Y_1(x)$ has a pole of order one. On the other hand, $J_o(kx)$ approaches unity for small x, but $J_1(x)$ goes to zero. Hence the above portion of Eq. (12) must be a positive quantity for small values of x.

Consider the remaining portion of Eq. (13):

$$-(D_1/D_2)yJ_1(y)\ [J_o(x)Y_o(kx)-J_o(kx)Y_o(x)]$$

If k is greater than unity, the preceding remarks indicate that $|J_o(x)| > |J_o(kx)|$ and $|Y_o(kx)| > |Y_o(x)|$ for small values of x. (It should be pointed out that a reasonable value of k is something like 5 or 6; $k = 1$ infers a single medium, but $k < 1$ has no meaning). Since the product $yJ_1(y)$ is negative (because y is imaginary for small x) and the subject portion of Eq. (12) is preceded by a negative sign, the entire expression is positive for values of x near zero, and there can be no root in that vicinity.

The computer program used in this work employs the scheme outlined above to determine the roots of Eq. (12). Only 10 or 12 halvings of the x-increment (usually 0.2) are required to compute a root to an accuracy of 5 parts in 10^6 and the halving can be accomplished very rapidly in the digital computer by dropping a digit from the characteristic of the floating point number. For this reason, a more refined extrapolation technique is not required in the root determination routine. Bessel functions were calculated by Tschebycheff polynomials (Werner, 1959) for values of the argument (x) which are less than 8.0, the upper range of the polynomial fit. Asymptotic series (McLachlan, 1948) were used to compute the Bessel functions which have kx as their argument for values of kx exceeding 8.0.

RESULTS OF THE COMPUTATION

The present work was undertaken to provide numerical results which can be used to obtain a flux curve for any value of the diffusion parameters or borehole size that might arise in a practical logging situation. To accomplish this end, the following values of the parameters were included in the survey:

a = 5, 6, 8, 10 cm
b = value required for medium 2 to be effectively "infinite"
L_1 = 7.00 cm
D_1 = 68.8 cm
L_2 = 7, 9, 11, 13, 15, 20, 25, 30 cm
D_2/D_1 = 0.2, 0.3, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4
z = 10, 20, 30, 40, 50, 60 cm
Q = 10^6 neutrons/sec.

Since the theory was developed for finite cylinders, it can be used readily to interpret experiments which are performed in finite models. However, if one wishes to know the flux for an outer cylinder which is effectively infinite in radius, one takes b to be so large that the flux has nearly its asymptotic value, say within 99% of that value. A method of estimating the required value of b, based on a one-cylinder theory, was given by Tittle (1961). In this computation the effect of the borehole is neglected. Our experience indicates that this is a good approximation, the error in the flux ratio resulting therefrom being less than 1 per cent in typical cases. Figure 3 presents the minimum value of b which will make the axial flux at $z = 60$ cm at least 99 per cent of the value for an infinite value of b, based on the one-cylinder theory.

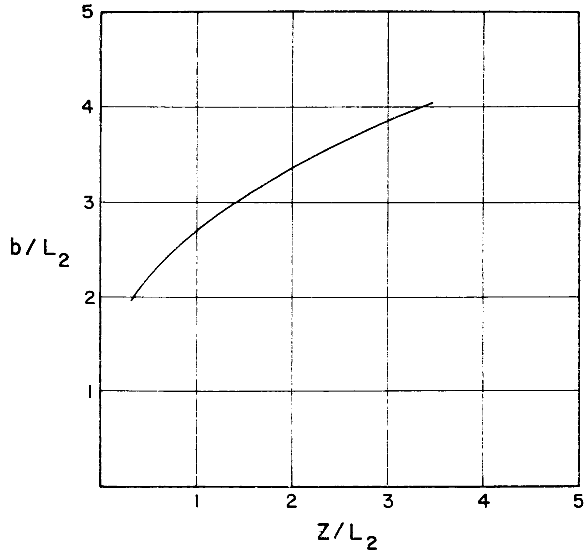

Fig. 3. Radius of Outer Cylinder Required to Give 99% of Flux at Spacing z. Based on One-Cylinder Theory.

The results of the parametric survey are presented in Tables I through VIII. In these tables the unit for the epithermal flux is neutrons/cm^2-sec-U. The symbol U stands for "unit lethargy interval" where lethargy is defined as the logarithm of the energy ratio. In all of the tabulated cases the value of b was selected to make the flux values at $z = 60$ cm approximately 99 per cent of their asymptotic values. Truncation errors are believed to be completely negligible.

In all cases the medium in the borehole is water ($L_1 = 7.0$ cm, $D_1 = 68.8$ cm). The indicated parameters for water are for Ra-Be primary neutrons, at a final energy of 1 ev. Other values of the diffusion parameters of Medium 1 can be obtained however, if a suitable transformation of the tabulated data is made. Suppose, for example, that one wishes to make a flux plot for $a = 10$ cm, $b = 150$ cm, $D_2/D_1 = 1.0$, $L_2 = 25$ cm, but, $L_1 = 8.75$ cm instead of 7.00. An examination of Eq. (12) indicates that the same eigenvalue problem is solved for the case where $a = 10$, $b = 150$, $L_1 = 8.75$, $L_2 = 25.0$, as for the case where $a = 8$, $b = 120$, $L_1 = 7.00$, $L_2 = 20.0$, if D_2/D_1 is the same in both cases. Moreover, since the product $F_n a^2$ appears in the denominator of Eq. (3), and F_n appears in the exponential function of Eq. (1), a flux value computed at a z-value of 10 cm for the case where L_1 equals 8.75 cm will agree with a flux value computed at a z-value of 12.5 cm, (10 cm multiplied by 8.75/7.00), if the former flux value is multiplied by 7.00/8.75. For precise agreement between the two cases it is clear that b, as well as a and L_2, must be increased in the same ratio as L_1. This requirement is not absolutely essential in this instance however because both values of b are effectively infinite (Fig. 3); a value of 120 cm would suffice for b in both cases. From this illustration, it can be seen that the desired fluxes for the case where $a = 10$ cm, $L_1 = 8.75$ cm, $L_2 = 25$ cm, $D_2/D_1 = 1.0$, with an effectively infinite value of b, can be obtained by multiplying the fluxes tabulated for $D_2/D_1 = 1.0$ in the table by a factor $c = 7.00/8.75$ and then identifying the resulting flux values with values of z which are $1/c$ times larger than the tabulated z's. As far as D_1 is concerned, it can be seen from Eq. (3) that this parameter simply scales the flux, the flux varying inversely with D_1 as it should. Thus the tabulated flux should be multiplied by

$$\frac{7.00}{L_1} \cdot \frac{68.8}{D_1}$$

to convert to values of L_1 and D_1 different from those employed in these computations. To summarize:

1. The flux ϕ for given values of L_1, D_1, L_2, D_2, a, b, and z is desired.
2. Compute $c = 7.0/L_1$.
3. Find the flux ϕ' for cL_2, ca, cb, ca, D_2/D_1 in the tables.
4. Multiply ϕ' by $68.8c/D_1$ to obtain ϕ.

An example is given in Table IX.

Table I; $L_2 = 7$

$D_2/D_1 = 0.2$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	78.89	64.20	46.31	37.28
20	14.09	11.86	8.576	6.505
30	2.692	2.343	1.767	1.358
40	0.5327	0.4769	0.3734	0.2933
50	0.1080	0.09898	0.08008	0.06420
60	0.02232	0.02085	0.01736	0.01418

$D_2/D_1 = 0.3$

10	63.30	53.83	41.53	34.98
20	10.35	9.128	7.103	5.691
30	1.873	1.709	1.390	1.132
40	0.3572	0.3345	0.2828	0.2357
50	0.07055	0.06738	0.05881	0.05011
60	0.01430	0.01387	0.01243	0.01080

$D_2/D_1 = 0.4$

10	52.96	46.63	38.00	33.21
20	8.086	7.368	6.067	5.087
30	1.410	1.326	1.139	0.9702
40	0.2630	0.2527	0.2249	0.1960
50	0.05124	0.05000	0.04574	0.04071
60	0.01030	0.01016	0.009500	0.008607

$D_2/D_1 = 0.6$

10	40.19	37.33	33.14	30.68
20	5.532	5.268	4.718	4.253
30	0.9222	0.8969	0.8286	0.7561
40	0.1682	0.1657	0.1568	0.1456
50	0.03239	0.03217	0.03098	0.02920
60	0.006471	0.006461	0.006301	0.006012

$D_2/D_1 = 0.8$

10	32.66	31.59	29.96	28.96
20	4.163	4.079	3.884	3.706
30	0.6761	0.6693	0.6477	0.6220
40	0.1221	0.1215	0.1190	0.1153
50	0.02343	0.02340	0.02310	0.02255
60	0.004677	0.004680	0.004644	0.004561

Table I; $L_2 = 7$ (Continued)

$D_2/D_1 = 1.0$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	27.72	27.72	27.72	27.72
20	3.321	3.321	3.321	3.321
30	0.5307	0.5307	0.5307	0.5307
40	0.09538	0.09538	0.09538	0.09538
50	0.01829	0.01829	0.01829	0.01829
60	0.003652	0.003652	0.003652	0.003652

$D_2/D_1 = 1.2$

10	24.24	24.93	26.05	26.78
20	2.757	2.801	2.917	3.036
30	0.4355	0.4383	0.4492	0.4647
40	0.07808	0.07825	0.07934	0.08132
50	0.01497	0.01497	0.01508	0.01534
60	0.002992	0.002988	0.002999	0.003035

$D_2/D_1 = 1.4$

10	21.65	22.83	24.77	26.04
20	2.354	2.422	2.613	2.817
30	0.3686	0.3727	0.3896	0.4150
40	0.06602	0.06622	0.06781	0.07092
50	0.01266	0.01266	0.01281	0.01320
60	0.002532	0.002526	0.002539	0.002591

Table II; $L_2 = 9.0$

$D_2/D_1 = 0.2$

10	85.35	68.39	48.19	38.19
20	16.84	13.72	9.477	6.965
30	3.662	3.022	2.113	1.544
40	0.8480	0.7023	0.4929	0.3597
50	0.2061	0.1701	0.1188	0.08627
60	0.05211	0.04268	0.02946	0.02120

$D_2/D_1 = 0.3$

10	69.72	58.19	43.61	36.01
20	12.99	11.01	8.079	6.213
30	2.778	2.376	1.759	1.340
40	0.6451	0.5518	0.4077	0.3090
50	0.1589	0.1351	0.09881	0.07421
60	0.04092	0.03445	0.02482	0.01840

Table II; $L_2 = 9.0$ **(Continued)**

$D_2/D_1 = 0.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	59.04	50.89	40.13	34.30
20	10.52	9.164	7.051	5.630
30	2.228	1.952	1.504	1.185
40	0.5202	0.4539	0.3473	0.2709
50	0.1296	0.1121	0.08461	0.06513
60	0.03380	0.02895	0.02147	0.01627

$D_2/D_1 = 0.6$

10	45.47	41.18	35.19	31.78
20	7.567	6.842	5.644	4.790
30	1.590	1.432	1.165	0.9646
40	0.3750	0.3348	0.2677	0.2173
50	0.09488	0.08387	0.06579	0.05236
60	0.02515	0.02203	0.01695	0.01323

$D_2/D_1 = 0.8$

10	37.24	35.03	31.86	30.01
20	5.886	5.451	4.729	4.214
30	1.234	1.128	0.9505	0.8165
40	0.2934	0.2652	0.2177	0.1816
50	0.07497	0.06710	0.05387	0.04381
60	0.02007	0.01781	0.01403	0.01116

$D_2/D_1 = 1.0$

10	31.74	30.79	29.47	28.70
20	4.809	4.528	4.089	3.794
30	1.007	0.9302	0.8029	0.7102
40	0.2410	0.2196	0.1835	0.1561
50	0.06203	0.05597	0.04563	0.03768
60	0.01671	0.01496	0.01197	0.009652

$D_2/D_1 = 1.2$

10	27.80	27.69	27.66	27.69
20	4.062	3.874	3.617	3.475
30	0.8508	0.7909	0.6953	0.6303
40	0.2046	0.1875	0.1586	0.1370
50	0.05291	0.04802	0.03959	0.03306
60	0.01431	0.01290	0.01044	0.008507

Table II; $L_2 = 9.0$ **(Continued)**

$D_2/D_1 = 1.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	24.86	25.33	26.25	26.89
20	3.515	3.387	3.254	3.224
30	0.7363	0.6878	0.6136	0.5680
40	0.1778	0.1635	0.1396	0.1222
50	0.04615	0.04207	0.03497	0.02946
60	0.01252	0.01134	0.009265	0.007607

Table III; $L_2 = 11.0$

$D_2/D_1 = 0.2$

10	89.93	71.32	49.49	38.80
20	19.02	15.16	10.15	7.299
30	4.557	3.624	2.403	1.694
40	1.192	0.9354	0.6072	0.4198
50	0.3342	0.2573	0.1619	0.1091
60	0.09917	0.07473	0.04533	0.02965

$D_2/D_1 = 0.3$

10	74.39	61.32	45.08	36.73
20	15.17	12.52	8.835	6.604
30	3.663	3.003	2.083	1.514
40	0.9836	0.7934	0.5351	0.3791
50	0.2851	0.2255	0.1469	0.1010
60	0.08749	0.06782	0.04261	0.02834

$D_2/D_1 = 0.4$

10	63.56	54.02	41.66	35.07
20	12.60	10.66	7.834	6.050
30	3.064	2.564	1.838	1.371
40	0.8390	0.6897	0.4785	0.3457
50	0.2486	0.2005	0.1342	0.09375
60	0.07784	0.06167	0.03987	0.02693

$D_2/D_1 = 0.6$

10	49.49	44.09	36.71	32.58
20	9.382	8.208	6.411	5.222
30	2.312	1.986	1.490	1.155
40	0.6499	0.5478	0.3952	0.2940
50	0.1977	0.1638	0.1141	0.08175
60	0.06336	0.05176	0.03496	0.02424

Table III; $L_2 = 11.0$ (Continued)

$D_2/D_1 = 0.8$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	40.78	37.66	33.30	30.79
20	7.465	6.674	5.450	4.633
30	1.859	1.622	1.254	1.001
40	0.5310	0.4548	0.3368	0.2558
50	0.1640	0.1384	0.09906	0.07232
60	0.05324	0.04440	0.03093	0.02188

$D_2/D_1 = 1.0$

10	34.88	33.17	30.81	29.44
20	6.197	5.625	4.758	4.192
30	1.555	1.371	1.084	0.8854
40	0.4492	0.3890	0.2935	0.2265
50	0.1401	0.1197	0.08744	0.06477
60	0.04585	0.03879	0.02765	0.01987

$D_2/D_1 = 1.2$

10	30.62	29.85	28.91	28.40
20	5.297	4.865	4.235	3.850
30	1.336	1.188	0.9545	0.7952
40	0.3893	0.3399	0.2601	0.2033
50	0.1222	0.1055	0.07823	0.05861
60	0.04023	0.03441	0.02497	0.01817

$D_2/D_1 = 1.4$

10	27.40	27.31	27.41	27.56
20	4.625	4.288	3.827	3.577
30	1.172	1.048	0.8533	0.7230
40	0.3435	0.3018	0.2335	0.1845
50	0.1084	0.09424	0.07075	0.05350
60	0.03582	0.03090	0.02273	0.01671

Table IV; $L_2 = 13.0$

$D_2/D_1 = 0.2$

10	93.32	73.47	50.42	39.23
20	20.79	16.30	10.67	7.553
30	5.364	4.152	2.648	1.816
40	1.543	1.166	0.7147	0.4741
50	0.4841	0.3555	0.2076	0.1323
60	0.1628	0.1162	0.06454	0.03935

Table IV: $L_2 = 13.0$ (Continued)

$D_2/D_1 = 0.3$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	77.93	63.66	46.16	37.26
20	16.99	13.76	9.433	6.908
30	4.492	3.573	2.365	1.662
40	1.346	1.043	0.6598	0.4449
50	0.4409	0.3327	0.2004	0.1292
60	0.1540	0.1135	0.06525	0.04028

$D_2/D_1 = 0.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	67.02	56.39	42.80	35.65
20	14.36	11.90	8.466	6.382
30	3.869	3.138	2.137	1.532
40	1.192	0.9417	0.6110	0.4179
50	0.4008	0.3092	0.1913	0.1250
60	0.1433	0.1082	0.06422	0.04023

$D_2/D_1 = 0.6$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	52.63	46.33	37.87	33.19
20	10.97	9.377	7.050	5.572
30	3.033	2.525	1.792	1.326
40	0.9671	0.7858	0.5298	0.3712
50	0.3356	0.2672	0.1726	0.1154
60	0.1230	0.09640	0.06014	0.03873

$D_2/D_1 = 0.8$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	43.58	39.72	34.41	31.39
20	8.869	7.742	6.061	4.980
30	2.497	2.114	1.544	1.171
40	0.8127	0.6728	0.4664	0.3329
50	0.2869	0.2335	0.1557	0.1062
60	0.1067	0.08568	0.05548	0.03653

$D_2/D_1 = 1.0$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	37.38	35.05	31.86	30.03
20	7.446	6.597	5.333	4.527
30	2.123	1.819	1.356	1.050
40	0.7004	0.5877	0.4159	0.3013
50	0.2500	0.2067	0.1412	0.09778
60	0.09377	0.07669	0.05104	0.03424

Table IV: $L_2 = 13.0$ **(Continued)**

$D_2/D_1 = 1.2$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	32.87	31.58	29.89	28.95
20	6.418	5.750	4.774	4.170
30	1.847	1.596	1.210	0.9522
40	0.6152	0.5214	0.3750	0.2750
50	0.2213	0.1852	0.1289	0.09038
60	0.08349	0.06921	0.04707	0.03205

$D_2/D_1 = 1.4$

10	29.45	28.89	28.34	28.08
20	5.640	5.099	4.331	3.882
30	1.634	1.422	1.092	0.8723
40	0.5483	0.4685	0.3413	0.2529
50	0.1984	0.1675	0.1183	0.08388
60	0.07517	0.06296	0.04357	0.03003

Table V: $L_2 = 15$

$D_2/D_1 = 0.2$

10	95.92	75.10	51.12	39.56
20	22.23	17.23	11.08	7.750
30	6.084	4.615	2.856	1.918
40	1.888	1.387	0.8143	0.5231
50	0.6477	0.4599	0.2543	0.1552
60	0.2401	0.1652	0.08628	0.04997

$D_2/D_1 = 0.3$

10	80.68	65.47	46.98	37.66
20	18.51	14.78	9.916	7.149
30	5.252	4.085	2.611	1.787
40	1.713	1.290	0.7786	0.5057
50	0.6163	0.4503	0.2566	0.1579
60	0.2378	0.1693	0.09168	0.05372

$D_2/D_1 = 0.4$

10	69.74	58.24	43.69	36.09
20	15.86	12.94	8.984	6.649
30	4.620	3.662	2.402	1.671
40	1.556	1.196	0.7397	0.4861
50	0.5764	0.4313	0.2527	0.1574
60	0.2275	0.1666	0.09333	0.05552

Table V; $L_2 = 15$ **(Continued)**

$D_2/D_1 = 0.6$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	55.13	48.12	38.78	33.66
20	12.34	10.38	7.583	5.861
30	3.724	3.032	2.067	1.478
40	1.305	1.034	0.6648	0.4462
50	0.4995	0.3872	0.2377	0.1516
60	0.2023	0.1542	0.09133	0.05597

$D_2/D_1 = 0.8$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	45.84	41.38	35.30	31.87
20	10.10	8.669	6.581	5.270
30	3.119	2.585	1.812	1.325
40	1.118	0.9045	0.5991	0.4094
50	0.4360	0.3463	0.2202	0.1433
60	0.1791	0.1404	0.08661	0.05441

$D_2/D_1 = 1.0$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	39.41	36.57	32.70	30.49
20	8.555	7.448	5.827	4.811
30	2.682	2.252	1.613	1.200
40	0.9764	0.8019	0.5433	0.3768
50	0.3851	0.3115	0.2035	0.1347
60	0.1597	0.1277	0.08125	0.05210

$D_2/D_1 = 1.2$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	34.71	32.98	30.69	29.40
20	7.420	6.533	5.241	4.444
30	2.353	1.995	1.453	1.098
40	0.8656	0.7192	0.4961	0.3483
50	0.3442	0.2822	0.1883	0.1264
60	0.1436	0.1166	0.07601	0.04956

$D_2/D_1 = 1.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	31.12	30.18	29.08	28.51
20	6.552	5.821	4.771	4.143
30	2.096	1.791	1.322	1.012
40	0.7770	0.6515	0.4559	0.3235
50	0.3108	0.2575	0.1748	0.1187
60	0.1303	0.1070	0.07113	0.04703

Table VI; $L_2 = 20$

$D_2/D_1 = 0.2$

z	a = 5	a = 6	a = 8	a = 10
10	100.3	77.84	52.29	40.10
20	24.88	18.90	11.80	8.093
30	7.550	5.537	3.258	2.110
40	2.680	1.882	1.029	0.6254
50	1.074	0.7245	0.3677	0.2090
60	0.4703	0.3072	0.1464	0.07830

$D_2/D_1 = 0.3$

z	a = 5	a = 6	a = 8	a = 10
10	85.42	68.55	48.37	38.32
20	21.37	16.66	10.78	7.576
30	6.845	5.135	3.098	2.028
40	2.582	1.861	1.041	0.6359
50	1.089	0.7585	0.3972	0.2272
60	0.4957	0.3363	0.1671	0.09054

$D_2/D_1 = 0.4$

z	a = 5	a = 6	a = 8	a = 10
10	74.50	61.44	45.20	36.83
20	18.73	14.90	9.932	7.128
30	6.230	4.762	2.937	1.945
40	2.441	1.798	1.031	0.6350
50	1.061	0.7592	0.4101	0.2373
60	0.4939	0.3456	0.1784	0.09839

$D_2/D_1 = 0.6$

z	a = 5	a = 6	a = 8	a = 10
10	59.59	51.26	40.37	34.48
20	15.03	12.31	8.587	6.393
30	5.246	4.125	2.640	1.785
40	2.151	1.639	0.9808	0.6158
50	0.9677	0.7209	0.4105	0.2438
60	0.4619	0.3383	0.1858	0.1061

$D_2/D_1 = 0.8$

z	a = 5	a = 6	a = 8	a = 10
10	49.90	44.34	36.87	32.70
20	12.56	10.49	7.576	5.816
30	4.515	3.622	2.384	1.643
40	1.899	1.483	0.9176	0.5866
50	0.8709	0.6676	0.3959	0.2407
60	0.4216	0.3187	0.1834	0.1078

Table VI; $L_2 = 20$ **(Continued)**

$D_2/D_1 = 1.0$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	43.10	39.32	34.21	31.31
20	10.79	9.142	6.788	5.352
30	3.956	3.222	2.168	1.518
40	1.692	1.346	0.8546	0.5550
50	0.7854	0.6149	0.3763	0.2333
60	0.3836	0.2969	0.1771	0.1065

$D_2/D_1 = 1.2$

10	38.07	35.52	32.13	30.20
20	9.457	8.104	6.158	4.971
30	3.518	2.898	1.985	1.409
40	1.522	1.227	0.7961	0.5240
50	0.7124	0.5668	0.3556	0.2242
60	0.3502	0.2759	0.1692	0.1038

$D_2/D_1 = 1.4$

10	34.20	32.55	30.45	29.28
20	8.418	7.281	5.642	4.653
30	3.165	2.631	1.830	1.314
40	1.381	1.126	0.7431	0.4948
50	0.6506	0.5242	0.3356	0.2145
60	0.3213	0.2567	0.1611	0.1004

Table VII; $L_2 = 25$

$D_2/D_1 = 0.2$

10	103.0	79.51	52.98	40.42
20	26.66	20.00	12.27	8.309
30	8.646	6.212	3.543	2.243
40	3.345	2.290	1.199	0.7045
50	1.479	0.9704	0.4693	0.2558
60	0.7183	0.4567	0.2075	0.1062

$D_2/D_1 = 0.3$

10	88.40	70.47	49.22	38.72
20	23.34	17.94	11.36	7.852
30	8.068	5.925	3.452	2.199
40	3.334	2.342	1.255	0.7387
50	1.552	1.053	0.5262	0.2887
60	0.7814	0.5167	0.2454	0.1276

Table VII: $L_2 = 25$ **(Continued)**

$D_2/D_1 = 0.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	77.53	63.46	46.13	37.29
20	20.75	16.25	10.57	7.444
30	7.490	5.605	3.334	2.143
40	3.223	2.317	1.273	0.7549
50	1.546	1.079	0.5574	0.3098
60	0.7950	0.5430	0.2684	0.1423

$D_2/D_1 = 0.6$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	62.49	53.29	41.38	34.99
20	16.97	13.67	9.278	6.752
30	6.473	4.989	3.077	2.014
40	2.919	2.178	1.251	0.7564
50	1.449	1.056	0.5772	0.3299
60	0.7634	0.5472	0.2888	0.1588

$D_2/D_1 = 0.8$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	52.58	46.28	37.88	33.23
20	14.36	11.80	8.274	6.192
30	5.661	4.458	2.830	1.884
40	2.622	2.009	1.196	0.7369
50	1.326	0.9969	0.5692	0.3336
60	0.7086	0.5254	0.2913	0.1652

$D_2/D_1 = 1.0$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	45.56	41.14	35.20	31.85
20	12.44	10.38	7.472	5.731
30	5.015	4.014	2.608	1.763
40	2.363	1.846	1.132	0.7088
50	1.210	0.9306	0.5497	0.3292
60	0.6524	0.4959	0.2856	0.1662

$D_2/D_1 = 1.2$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	40.33	37.23	33.08	30.72
20	10.98	9.262	6.818	5.345
30	4.495	3.644	2.413	1.652
40	2.144	1.701	1.067	0.6778
50	1.107	0.8665	0.5259	0.3206
60	0.6007	0.4654	0.2762	0.1641

Table VII; $L_2 = 25$ **(Continued)**

$D_2/D_1 = 1.4$

z	$a = 5$	$a = 6$	$a = 8$	$a = 10$
10	36.28	34.14	31.35	29.79
20	9.820	8.364	6.276	5.018
30	4.069	3.332	2.241	1.553
40	1.958	1.573	1.005	0.6466
50	1.018	0.8075	0.5011	0.3101
60	0.5547	0.4364	0.2654	0.1604

Table VIII; $L_2 = 30$

$D_2/D_1 = 0.2$

10	104.8	80.62	53.44	40.62
20	27.93	20.77	12.59	8.457
30	9.481	6.720	3.753	2.339
40	3.894	2.620	1.334	0.7662
50	1.841	1.187	0.5568	0.2955
60	0.9586	0.5995	0.2647	0.1319

$D_2/D_1 = 0.3$

10	90.42	71.76	49.78	38.99
20	24.76	18.84	11.76	8.042
30	9.017	6.528	3.716	2.325
40	3.964	2.740	1.428	0.8201
50	1.971	1.316	0.6388	0.3415
60	1.062	0.6917	0.3195	0.1620

$D_2/D_1 = 0.4$

10	79.62	64.84	46.76	37.59
20	22.21	17.22	11.02	7.663
30	8.481	6.258	3.634	2.289
40	3.886	2.752	1.471	0.8509
50	1.991	1.368	0.6875	0.3725
60	1.094	0.7365	0.3545	0.1835

$D_2/D_1 = 0.6$

10	64.52	54.70	42.07	35.34
20	18.41	14.68	9.776	7.007
30	7.454	5.670	3.414	2.186
40	3.584	2.637	1.476	0.8710
50	1.899	1.365	0.7272	0.4058
60	1.069	0.7557	0.3890	0.2092

Table VIII: $L_2 = 30$ **(Continued)**

$D_2/D_1 = 0.8$

z	a = 5	a = 6	a = 8	a = 10
10	54.47	47.63	38.58	33.60
20	15.71	12.77	8.783	6.462
30	6.589	5.126	3.179	2.069
40	3.255	2.462	1.431	0.8612
50	1.758	1.305	0.7273	0.4167
60	1.003	0.7343	0.3977	0.2207

$D_2/D_1 = 1.0$

z	a = 5	a = 6	a = 8	a = 10
10	47.31	42.43	35.89	32.22
20	13.69	11.30	7.977	6.006
30	5.880	4.654	2.957	1.953
40	2.956	2.283	1.368	0.8376
50	1.616	1.228	0.7096	0.4158
60	0.9302	0.6990	0.3938	0.2244

$D_2/D_1 = 1.2$

z	a = 5	a = 6	a = 8	a = 10
10	41.94	38.43	33.74	31.09
20	12.13	10.14	7.310	5.619
30	5.298	4.250	2.754	1.843
40	2.697	2.117	1.300	0.8079
50	1.487	1.151	0.6843	0.4086
60	0.8613	0.6603	0.3839	0.2236

$D_2/D_1 = 1.4$

z	a = 5	a = 6	a = 8	a = 10
10	37.77	35.27	32.00	30.15
20	10.89	9.188	6.751	5.288
30	4.816	3.905	2.572	1.742
40	2.473	1.967	1.232	0.7760
50	1.373	1.079	0.6560	0.3980
60	0.7989	0.6222	0.3711	0.2201

Table IX

Example of Transformation for

Different Values of L_1 and D_1

Flux values are desired for $a = 10$, $b = 150$, $L_1 = 8.75$, $L_2 = 25$, $D_1 = 68.8$, and $D_2 = 68.8$ (all values in cm). In the tables, values of flux for $a = 8$, $b = 120$, $L_1 = 7.00$, $L_2 = 20$, $D_2/D_1 = 1$ are found. $c = 0.8$.

cz	ϕ'	z	$\phi = 0.8\phi'$
10	34.21	12.5	27.37
20	6.788	25.	5.430
30	2.168	37.5	1.734
40	0.8546	50	0.6837
50	0.3763	62.5	0.3010
60	0.1771	75	0.1417

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