# Generalized Smagorinsky model for anisotropic grids

Alberto Scotti and Charles Meneveau
*Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, Maryland 21218*
Douglas K. Lilly
*Center for the Analysis and Prediction of Storms, University of Oklahoma, Norman, Oklahoma 73019-0628*

The Smagorinsky subgrid model is revised to properly account for grid anisotropy, using energy equilibrium considerations in isotropic turbulence. For moderate resolution anisotropies, Deardorff's estimate involving an equivalent grid scale $\Delta_{eq}=(\Delta_1\Delta_2\Delta_3)^{1/3}$ is given a rigorous basis. For more general grid anisotropies, the Smagorinsky eddy viscosity is recast as $\nu_T=[c_s\Delta_{eq}f(a_1,a_2)]^2|\widetilde{S}|$, where $f(a_1,a_2)$ is a function of the grid aspect ratios $a_1$ and $a_2$, and $|S|$ is the resolved strain rate magnitude. The asymptotic behavior of $\nu_T$ at several limits of the aspect ratios are examined. Approximation formulas are developed so that $f(a_1,a_2)$ can easily be evaluated in practice, for arbitrary values of $a_1$ and $a_2$. It is argued that these results should be used in conjunction with the dynamic model of Germano *et al.* whenever the anisotropy of the test-filter differs significantly from that of the basic grid.

In this Brief Communication we consider the issue of proper subgrid-scale modeling for a large eddy simulation that employs a rectangular grid with mesh sides of arbitrary lengths $(\Delta_1,\Delta_2,\Delta_3)=\mathbf{\Delta}$. In the Smagorinsky model, the deviatoric part of the subgrid-scale stress tensor is modeled as

$$\tau_{ij}=-2[L(\mathbf{\Delta})]^2\,[2\widetilde{S}_{mn}\widetilde{S}_{mn}]^{1/2}\,\widetilde{S}_{ij}, \tag{1}$$

where $\widetilde{S}_{ij}$ is the resolved rate-of-strain tensor and $L(\mathbf{\Delta})$ is a factor that has units of length. Also, $L(\mathbf{\Delta})$ depends on the size of the computational mesh and on its anisotropy. If the spacings in all directions are equal $(\Delta)$, dimensional considerations lead to

$$L(\mathbf{\Delta})=c_s\Delta, \tag{2}$$

where $c_s$ is the Smagorinsky constant (usually taken between 0.1 and 0.2). When the LES grid has unequal sides $\Delta_i$, the common practice is to follow Deardorff[1] in employing Eq. (2) using an equivalent length scale

$$\Delta_{eq}=(\Delta_1\Delta_2\Delta_3)^{1/3}. \tag{3}$$

In the case of a cubic mesh, we recall that if the assumption is made that $\Delta$ lies in the inertial range and that the universal Kolmogorov spectrum applies, then the condition that the SGS dissipation be equal to $\langle\epsilon\rangle$,

$$\langle\epsilon\rangle=-\langle\tau_{ij}\widetilde{S}_{ij}\rangle \tag{4}$$

can be used to estimate $c_s$.[2] In the case of an anisotropic grid, moments of the resolved rate-of-strain depend on the grid anisotropy. However, the correct value for $\langle\epsilon\rangle$ must still be generated. Therefore, $L(\mathbf{\Delta})$ can also be obtained from the energy-flux equilibrium condition, namely

$$\langle\epsilon\rangle=2[L(\mathbf{\Delta})]^2\langle[2\widetilde{S}_{mn}\widetilde{S}_{mn}]^{1/2}\,\widetilde{S}_{ij}\widetilde{S}_{ij}\rangle. \tag{5}$$

A similar approach was used some time ago by Schumann[3] to evaluate model constants for the control volume and control surface formulation. Here, we wish to revisit this issue for several reasons. The first is to arrive at an expression that can be implemented in practice more easily than that arising from Schumann's original formulation. The second motivation is to give a more rigorous foundation to Deardorff's expression [Eq. (3)] and to quantify its limits of validity. Yet another motivation is the need to elucidate the influence of anisotropic filtering during statistical *a priori* testing.[4] Lastly, we believe that when the Smagorinsky model is employed in conjunction with the dynamic model,[5] the grid anisotropy needs to be taken into account separately from the model constant. This is of current importance, given the need to extend LES to complex three-dimensional (3-D) geometries where nonuniform grids are usually employed. We shall elaborate on this particular point at the end of this communication.

We return to Eq. (5), which shows that the dependence of the resolved rate of strain statistics on the grid geometry must be canceled through an appropriate expression for $L(\mathbf{\Delta})$ in order to generate the correct, grid independent $\langle\epsilon\rangle$. Next, the right-hand side (rhs) of Eq. (5) is approximated in terms of second-order moments as follows

$$\langle\epsilon\rangle\simeq[L(\mathbf{\Delta})]^2(\langle 2\widetilde{S}_{ij}\widetilde{S}_{ij}\rangle)^{3/2}. \tag{6}$$

This assumption neglects intermittency effects, which will be considered in more detail in future work. Presently, we proceed by utilizing second-order information only. The evaluation of $\langle\widetilde{S}_{ij}\widetilde{S}_{ij}\rangle$ is in principle straightforward if the energy spectral tensor $Q_{ij}(\mathbf{k})$ is prescribed. Assuming that all lengths $\Delta_i$ of the grid belong to the inertial range, we have

$$Q_{ij}(\mathbf{k})=\frac{1}{4\pi k^2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)E(k), \tag{7}$$

where $E(k)=C_k\epsilon^{2/3}k^{-5/3}$. As in Lilly,[2] Schumann,[3] etc., we assume that the variables resolved at the grid scale are the physical variables convolved with a spatial filter $F(\mathbf{x})$ with characteristic scales in each direction equal to $\Delta_i$. It follows that

$$\langle\widetilde{S}_{ij}\widetilde{S}_{ij}\rangle=\epsilon^{2/3}\frac{C_k}{2\pi}\int|F(\mathbf{k})|^2k^{-5/3}\,d\mathbf{k}. \tag{8}$$

Substitution into Eq. (6) yields

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$$L(\Delta)=\left(\frac{C_k}{2\pi}\int |F(\mathbf{k})|^2 k^{-5/3}\,d\mathbf{k}\right)^{-3/4}. \tag{9}$$

The correspondence between the filter type (e.g., cutoff, top hat, etc.) and the actual LES numerical method employed (spectral, finite differences, etc.) is not precise. Since the approach is thus of an approximate nature, the cutoff filter is employed, for which the calculations are done most easily. This filter is defined in Fourier space as the indicator function of the region $B$

$$B=\{|k_1|<\pi/\Delta_1,\ |k_2|<\pi/\Delta_2, |k_3|<\pi/\Delta_3\}. \tag{10}$$

For definiteness, we now select the largest edge of the grid (or filter) to be $\Delta_{max}=\max\{\Delta_1,\Delta_2,\Delta_3\}$ and define $a_1=\Delta_i/\Delta_{max}$, $a_2=\Delta_k/\Delta_{max}$ as the aspect ratios of the other two sides of the filter. Also the angles $\beta_1=\arctan(a_1/a_2)$, $\beta_2=\arctan(a_2/a_1)$ will be employed.

It is then convenient to work in spherical coordinates, where the elevation angle $\theta$ is measured from the axis aligned with the direction of $\Delta_{max}$. Performing all integrations that can be done analytically one obtains

$$\int_B k^{-5/3}\,d\mathbf{k}=6\left(\frac{\pi}{\Delta_{max}}\right)^{4/3}\left(\int_0^{\beta_1} q_1(\phi)d\phi + \int_0^{\beta_2} q_2(\phi)d\phi\right), \tag{11}$$

where

$$q_i(\phi)=3(\{1+(a_i\cos\phi)^{-2}\}^{1/6}-1) + \frac{1}{(a_i\cos\phi)^{1/3}}\int_{\arctan[1/(a_i\cos\phi)]}^{\pi/2}[\sin\phi]^{-1/3}\,d\theta. \tag{12}$$

The behavior of highly anisotropic grids is as follows. The case of pencil-like grids with $\Delta_1\sim\Delta_2\sim\Delta_{min}\ll\Delta_{max}$ (while maintaining the Kolmogorov scale $\eta\ll\Delta_i$) corresponds to $\beta_1\sim\beta_2\sim(\pi/4)$; $a_1\sim a_2\ll 1$. Thus the integral diverges like $(\Delta_{min}/\Delta_{max})^{-1/3}$. From Eq. (9) we obtain

$$L\sim\Delta_{max}\,(\Delta_{min}/\Delta_{max})^{1/4}\sim\Delta_{eq}\,a_1^{-5/12}. \tag{13}$$

In the case of a sheetlike grid $\Delta_1=\Delta_{min}\ll\Delta_2\sim\Delta_{max}$ the integral approaches a constant. This results because in Fourier-space one integrates $k^{-5/3}$ over a long but approximately one-dimensional (1-D) domain, and this yields integrals that do not diverge at high $k$. In other words, the two-dimensional filtering effectively dampens most of the fine scales in the third direction as well. Therefore, $L$ tends to a constant, namely $0.0844\,\Delta_{max}=0.0844\,\Delta_{eq}\,a_1^{-1/3}$, where we have assumed the Kolmogorov constant to be $C_k=1.6$. Therefore, Eq. (3) underestimates the Smagorinsky constant for meshes with small aspect ratios.

We conclude that in general the Smagorinsky–Deardorff model should be corrected by introducing a function $f(a_1,a_2)$, according to

$$\tau_{ij}=-2[c_s\Delta_{eq}f(a_1,a_2)]^2\,[2\widetilde{S}_{mn}\widetilde{S}_{mn}]^{1/2}\,\widetilde{S}_{ij}. \tag{14}$$

In such a formulation, the grid anisotropy is properly isolated from the model constant through $\Delta_{eq}$ and the correction function $f(a_1,a_2)$ which has the property that $f(1,1)=1$.

Next, the function $f(a_1,a_2)$ is computed by evaluating Eq. (11) numerically (using a mixed Gauss–Gauss–Kronrod rule) and by substitution into Eq. (9):

$$f(a_1,a_2)=(a_1a_2)^{-1/3}\,[L(\Delta)/L(\Delta,\Delta,\Delta)], \tag{15}$$

In Fig. 1 we show $f(a_1,a_2)$ for $10^{-4}<a_1,a_2<1$. For instance, it can be seen that $f(a_1,a_2)$ becomes larger than 1.3 (an error of 30% in Deardorff's formula) for aspect ratios smaller than about 1/5. We also point out that when $a_1=a_2=1$, the numerical result for $L(\Delta)$ is $L=0.13\Delta$, which implies that $c_s=0.13$. This value is slightly smaller than the value obtained by Lilly[2] ($c_s\simeq 0.16$) due to the fact that the latter is obtained integrating in the sphere inscribed in $B$ only.

To permit easier evaluation of the function $f(a_1,a_2)$ an empirical formula is developed that reproduces its behavior over a wide range of values of $a_1$ and $a_2$, as well as the proper asymptotics. It is obtained by expanding the integrand in (11) in Taylor series and performing the inner integral analytically. Further approximation of the remaining integrals with polynomials by means of curvilinear regression (keeping three terms only) yields the formula

$$\begin{aligned} f(a_1,a_2) &\simeq 1.736(a_1a_2)^{-1/3}[4P_1(\beta_1)a_1^{-1/3}+0.222P_2(\beta_1)a_1^{5/3} \\ &\quad +0.077P_3(\beta_1)a_1^{11/3}-3\beta_1+4P_1(\beta_2)a_2^{-1/3} \\ &\quad +0.222P_2(\beta_2)a_2^{5/3}+0.077P_3(\beta_2)a_2^{11/3}-3\beta_2]^{-3/4}, \end{aligned} \tag{16}$$

where the polynomials $P_i$ are defined as follows:

$$\begin{aligned} P_1(z)&=2.5P_2(z)-1.5(\cos z)^{2/3}\sin z, \\ P_2(z)&=0.986z+0.073z^2-0.418z^3+0.120z^4, \\ P_3(z)&=+0.976z+0.188z^2-1.169z^3+0.755z^4-0.151z^5. \end{aligned} \tag{17}$$

The maximum error of this approximation is smaller than 4% over the entire range of $a_1$ and $a_2$.

Next, we wish to gain a better understanding of the behavior of $f(a_1,a_2)$ near the isotropic limit $a_1=a_2=1$. This is best accomplished by using an ellipsoidal domain $B''$ in Fourier space, instead of the rectangular one which led to Eqs. (11) and (12). In spherical coordinates, we write

$$\int_{B''} k^{-5/3}\,d\mathbf{k}=\int_0^{2\pi}\int_0^{\pi}\int_0^{k_b} k^{1/3}\sin\theta\,dk\,d\theta\,d\phi,$$

Here $k_b$ is the distance from the origin to the surface of an ellipsoid with major axes $\pi/\Delta_{max}$ (in the axial direction $\theta=0$), $\pi/a_1\Delta_{max}$ and $\pi/a_2\Delta_{max}$,

$$k_b=\frac{\pi}{\Delta_{max}}(\cos^2\theta+a_1^2\cos^2\phi\sin^2\theta+a_2^2\sin^2\phi\sin^2\theta)^{-1/2}.$$

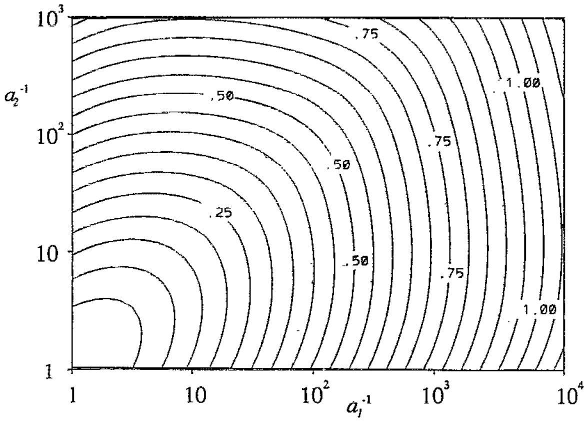


FIG. 1. Numerically computed contour plots of the grid-anisotropy correction function $f(a_1,a_2)$. Contours are values of $\log_{10} f(a_1,a_2)$, separated by 0.05 starting from 0 at the origin $(a_1,a_2)=(1,1)$.

Performing the radial integration, substituting into Eqs. (9), and comparing with Eq. (14) one recognizes that

$$f(a_1,a_2)^{-4/3}=(1/2\pi)(a_1a_2)^{4/9} \times \int_0^{2\pi}\int_0^{\pi/2} (\cos^2\theta+a_1^2\cos^2\phi\sin^2\theta + a_2^2\sin^2\phi\sin^2\theta)^{-2/3}\sin\theta\, d\theta\, d\phi.$$

Also, for this choice of limits, $c_s=(3C_k/2)^{-3/4}\pi^{-1}\approx 0.16$, the spherically symmetric value. The expression for $f(a_1,a_2)$ is easy to expand in Taylor series around $a_1=a_2=1$, and it is most instructive to do this, with respect to the logarithmic variables $\ln a_1$ and $\ln a_2$. Up to second order in these variables one can write

$$f\approx 1+\tfrac{2}{27}[(\ln a_1)^2-\ln a_1 \ln a_2+(\ln a_2)^2]. \tag{18}$$

The fact that there are no linear terms can be viewed as further justification for the Deardorff approximation for moderate anisotropies.

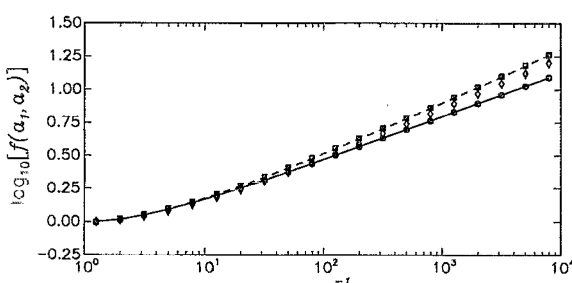


FIG. 2. Comparison between different expressions for the correction function $f(a_1,a_2)$. Solid line: exact (numerical) value for $a_2=1$; dashed line; exact (numerical) value for $a_2=a_1$. Approximation using Eq. (16), for $a_2=1$ (circles) and for $a_1=a_2$ (squares). Diamonds: simple fit using the cosh function of Eq. (19) (same curve is obtained for $a_2=1$ and $a_1=a_2$).

Returning to the question of finding practically useful fits for $f(a_1,a_2)$, we recall that the asymptotic limits were $f\sim a_1^{-1/3}$ if $a_2=1$, and $f\sim a_1^{-5/12}$ if $a_1=a_2$. Therefore, by writing

$$f\approx\cosh\sqrt{\tfrac{4}{27}[(\ln a_1)^2-\ln a_1\ln a_2+(\ln a_2)^2]}, \tag{19}$$

one simultaneously obtains the small anisotropy limit of Eq. (18), and an asymptotic power law with exponent equal to $-\sqrt{\tfrac{4}{27}}$ at large anisotropies. Although this exponent is the same in the different directions of the $(a_1,a_2)$ plane, its value is between the two exact exponents of $-\tfrac{1}{3}$ and $-\tfrac{5}{12}$. Therefore, the expression (19) provides a reasonable approximation even at large anisotropies. Figure 2 shows a comparison between the different expressions for $f(a_1,a_2)$, along the lines $a_2=1$ and $a_1=a_2$. The series expansions of Eq. (16) can be used if good approximations are needed at very high anisotropies, but it is evident that Eq. (19) provides sufficient accuracy over most practically relevant ranges of anisotropy.

Finally, we elaborate on this formulation in conjunction with the dynamic model of Germano *et al.*[5] There, the assumption is made that the Smagorinsky model is valid with the same model constant at the grid scale, as well as at the scale of the test filter. This is not entirely consistent with present results if the anisotropy of the grid at the scale of the test filter is different from that of the fundamental grid. This is easily remedied by rewriting the proposed generalized Smagorinsky model of Eq. (14) using the function $f(a_1,a_2)$ at both the grid- and test-filter levels. Then the dynamic model is used to find the model constant $c_s$, independently of grid anisotropy.

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