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EQUISEGMENTARY LINES OF A TRIANGLE IN THE ISOTROPIC PLANE

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Abstract. In this paper we introduce the concept of equisegmentary lines in the isotropic plane. We derive the equations of equisegmentary lines for a standard triangle and prove that the angle between them is equal to the Brocard angle of a standard triangle. We study the dual Brocard circle, the circle whose tangents are equisegmentary lines, as well as the inertial axis and the Steiner axis. Some interesting properties of this circle are also investigated.

1. Introduction

The isotropic plane is a projective–metric plane, where the absolute consists of a line, the absolute line ω, and a point on that line, the absolute point Ω. Lines through the point Ω are isotropic lines and points on the line ω are isotropic points.

The *distance* between two points $P_i = (x_i, y_i)$ $(i = 1, 2)$ in the isotropic plane is defined by $d(P_1, P_2) = x_2 - x_1$ and if $x_1 = x_2$ we say that P_1 and P_2 are *parallel*. For two parallel points P_1, P_2 we define their *span* by $s(P_1, P_2) = y_2 - y_1$. The *angle* of two lines with equations $y = k_i x + l_i$ $(i = 1, 2)$ is $k_2 - k_1$ and if $k_1 = k_2$ we say that they are *parallel*. Any isotropic line is perpendicular to any non-isotropic line. Facts about the isotropic plane can be found in [11, 12].

We say that a triangle is *allowable* if none of its sides is isotropic. If we choose the coordinate system in such a way the circumscribed circle of an allowable triangle ABC has the equation $y = x^2$ and therefore its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$, while $a + b + c = 0$, we say that the triangle ABC is in *standard position* or shorter triangle ABC is a standard triangle. Its sides BC, CA, and AB have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. In order to prove geometric

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facts for any allowable triangle, it suffices to prove it for a standard triangle [7].

Denoting $p = abc$ and $q = bc + ca + ab$, the authors proved a number of useful equalities in [7], e.g. $a^2 + b^2 + c^2 = -2q$, $(b - c)^2 = -(q + 3bc)$, $a^2 = bc - q$, $(c - a)(a - b) = 2q - 3bc$. We proved also the following identities

$$p_1 = \frac{1}{3}(bc^2 + ca^2 + ab^2), \quad p_2 = \frac{1}{3}(b^2c + c^2a + a^2b),$$

$$p_1 + p_2 = \frac{1}{3}\left((b + c)(c + a)(a + b) - 2abc\right) = \frac{1}{3}(-abc - 2abc) = -p,$$

i.e.

$$p + p_1 + p_2 = 0,$$

$$p_1 - p_2 = \frac{1}{3}(bc^2 + ca^2 + ab^2 - b^2c - c^2a - a^2b) = \frac{1}{3}(b - c)(c - a)(a - b) = -q\omega$$

and

$$p_1^2 + p_1p_2 + p_2^2 = -\frac{q^3}{9}, \quad p^2 + pp_1 + p_1^2 = -\frac{q^3}{9}, \quad p^2 + pp_2 + p_2^2 = -\frac{q^3}{9}.$$

In the isotropic plane we have the following formula for Brocard angle of standard triangle: $\omega = -\frac{1}{3q}(b - c)(c - a)(a - b)$ (see [3]).

2. Equisegmentary lines of a triangle

The motivation for this consideration are two Ocagne's papers [9, p. 131], and [10, p. 265]. In this section we consider the equisegmentary lines of a standard triangle ABC. According to [7], standard triangle ABC has the centroid $G = \left(0, -\frac{2}{3}q\right)$ and the inertial axis with the equation $y = -\frac{2}{3}q$.

Theorem 2.1. *Let D_1, E_1, F_1 and D_2, E_2, F_2 be points on the lines BC, CA, and AB such that*

$$d(D_1, C) = d(E_1, A) = d(F_1, B) = d(B, D_2) = d(C, E_2) = d(A, F_2) = u.$$

For variable u the centroids G_1 and G_2 of the triangles $D_1E_1F_1$ and $D_2E_2F_2$ lie on the inertial axis of the triangle ABC. The points G_1 and G_2 are symmetric with respect to the centroid G of that triangle (Figure 1).

Proof. Points D_1 and D_2 have abscissae $c - u$ and $b + u$ respectively. As they lie on the line BC with the equation $y = -ax - bc$, their ordinates are

$$-a(c - u) - bc = c^2 + au,$$
$$-a(b + u) - bc = b^2 - au.$$

Because of that we have

$$D_1 = (c - u, c^2 + au), \quad E_1 = (a - u, a^2 + bu), \quad F_1 = (b - u, b^2 + cu), \tag{2.1}$$

$$D_2 = (b + u, b^2 - au), \quad E_2 = (c + u, c^2 - bu), \quad F_2 = (a + u, a^2 - cu). \tag{2.2}$$

The triangles $D_1E_1F_1$ and $D_2E_2F_2$ have centroids

$$G_1 = \left(-u, -\frac{2}{3}q\right) \text{ and } G_2 = \left(u, -\frac{2}{3}q\right),$$

which lie on the inertial axis with equation $y = -\frac{2}{3}q$ and its midpoint is the point $G = \left(0, -\frac{2}{3}q\right)$. ∎

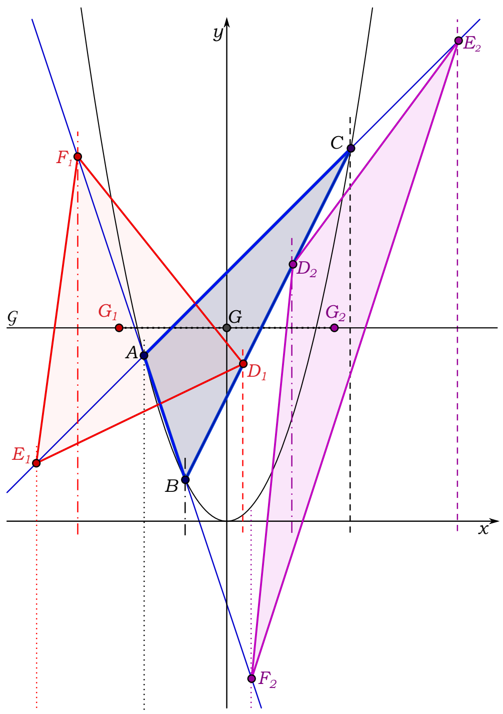

FIGURE 1. Visualization of the statement of Theorem 2.1.

THEOREM 2.2. *The points D_1, E_1, F_1 and respectively D_2, E_2, F_2 from Theorem 2.1, lie on one of the lines $\mathcal{P}_1$ and $\mathcal{P}_2$ if and only if $u = \omega$, where ω is the Brocard angle of the triangle ABC (Figure 2).*

PROOF. The required conditions for collinearity of points (2.1) and (2.2) are

$$\begin{aligned} 0 &= \begin{vmatrix} c-u & c^2+au & 1 \\ a-u & a^2+bu & 1 \\ b-u & b^2+cu & 1 \end{vmatrix} = \begin{vmatrix} c & c^2 & 1 \\ a & a^2 & 1 \\ b & b^2 & 1 \end{vmatrix} + u \cdot \begin{vmatrix} c & a & 1 \\ a & b & 1 \\ b & c & 1 \end{vmatrix} \\ &= (b-c)(c-a)(a-b) + u(bc+ca+ab-a^2-b^2-c^2) \\ &= -3q\omega + u \cdot 3q = 3q(u-\omega), \end{aligned}$$

and

$$
\begin{aligned}
0 &= \begin{vmatrix} b+u & b^2-au & 1 \\ c+u & c^2-bu & 1 \\ a+u & a^2-cu & 1 \end{vmatrix} = \begin{vmatrix} b & b^2 & 1 \\ c & c^2 & 1 \\ a & a^2 & 1 \end{vmatrix} - u \cdot \begin{vmatrix} b & a & 1 \\ c & b & 1 \\ a & c & 1 \end{vmatrix} \\
&= (b-c)(c-a)(a-b) - u(a^2+b^2+c^2-bc-ca-ab) \\
&= -3q\omega - u\cdot(-3q) = 3q(u-\omega),
\end{aligned}
$$

respectively, i.e. $u=\omega$. ∎

Corollary 2.3. *Let D_1, E_1, F_1 and D_2, E_2, F_2 be points on the lines BC, CA, and AB such that*

$$d(D_1,C) = d(E_1,A) = d(F_1,B) = d(B,D_2) = d(C,E_2) = d(A,F_2) = \omega,$$

where ω is the Brocard angle of the triangle ABC. The points D_1, E_1, F_1 and D_2, E_2, F_2 respectively, lie on one of the lines $\mathcal{P}_1$ and $\mathcal{P}_2$. If ABC is a standard triangle, then

$$D_1 = (c-\omega, c^2+a\omega), \quad E_1 = (a-\omega, a^2+b\omega), \quad F_1 = (b-\omega, b^2+c\omega), \tag{2.3}$$

$$D_2 = (b+\omega, b^2-a\omega), \quad E_2 = (c+\omega, c^2-b\omega), \quad F_2 = (a+\omega, a^2-c\omega). \tag{2.4}$$

The triples D_1, E_1, F_1 and D_2, E_2, F_2 have the centroids

$$G_1 = \left(-\omega, -\frac{2}{3}q\right) \text{ and } G_2 = \left(\omega, -\frac{2}{3}q\right). \tag{2.5}$$

We say that lines $\mathcal{P}_1$ and $\mathcal{P}_2$ from Corollary 2.3 are *equisegmentary lines* of the triangle ABC. These lines are reciprocal with respect to the triangle ABC.

In [9] and [10] d'Ocagne considered the equisegmentary lines in Euclidean geometry and obtained two pairs of reciprocal equisegmentary lines.

Theorem 2.4. *Equisegmentary lines $\mathcal{P}_1$ and $\mathcal{P}_2$ of a standard triangle ABC have equations*

$$\mathcal{P}_1 \;\cdots\; y = \frac{p_1-p}{q}x - \frac{7}{9}q - \frac{3p_1^2}{q^2}, \tag{2.6a}$$

$$\mathcal{P}_2 \;\cdots\; y = \frac{p_2-p}{q}x - \frac{7}{9}q - \frac{3p_2^2}{q^2}. \tag{2.6b}$$

Proof. From (2.3) and (2.4) for the slopes of lines E_1F_1 and E_2F_2 we get

$$
\begin{aligned}
\frac{b^2-a^2+c\omega-b\omega}{b-a} &= a+b+\frac{b-c}{a-b}\omega = -c - \frac{1}{3q}(b-c)^2(c-a) \\
&= -c + \frac{c-a}{3q}(q+3bc) = -c + \frac{1}{3}(c-a) + \frac{1}{q}bc(c-a)
\end{aligned}
$$

and

$$\begin{aligned}\frac{a^2-c^2-c\omega+b\omega}{a-c}&=a+c-\frac{b-c}{c-a}\omega=-b+\frac{1}{3q}(b-c)^2(a-b)\\&=-b-\frac{a-b}{3q}(q+3bc)=-b-\frac{1}{3}(a-b)-\frac{1}{q}bc(a-b)\end{aligned}$$

respectively, and analogously lines F_1D_1, D_1E_1 and F_2D_2, D_2E_2 have slopes

$$-a+\frac{1}{3}(a-b)+\frac{1}{q}ca(a-b),\quad -b+\frac{1}{3}(b-c)+\frac{1}{q}ab(b-c)$$

and

$$-c-\frac{1}{3}(b-c)-\frac{1}{q}ca(b-c),\quad -a-\frac{1}{3}(c-a)-\frac{1}{q}ab(c-a)$$

respectively. After adding and dividing by 3 we get the slopes k_1 and k_2 of lines $\mathcal{P}_1$ and $\mathcal{P}_2$

$$\begin{aligned}k_1&=\frac{1}{3q}[bc(c-a)+ca(a-b)+ab(b-c)]=\frac{1}{3q}(bc^2+ca^2+ab^2-3abc)\\&=\frac{1}{3q}(3p_1-3p)=\frac{p_1-p}{q},\\k_2&=-\frac{1}{3q}[bc(a-b)+ca(b-c)+ab(c-a)]=\frac{1}{3q}(b^2c+c^2a+a^2b-3abc)\\&=\frac{1}{3q}(3p_2-3p)=\frac{p_2-p}{q}.\end{aligned}$$

It remains to prove that lines $\mathcal{P}_1$ and $\mathcal{P}_2$ with equations (2.6a) and (2.6b) pass through the points G_1 and G_2 from (2.5). For the point G_1 and line $\mathcal{P}_1$ we get

$$\begin{aligned}-\frac{p_1-p}{q}\omega-\frac{7}{9}q-\frac{3p_1^2}{q^2}+\frac{2}{3}q&=\frac{(2p_1+p_2)(p_1-p_2)}{q^2}-\frac{3p_1^2}{q^2}-\frac{q}{9}\\&=-\frac{1}{q^2}(p_1^2+p_1p_2+p_2^2)-\frac{q}{9}=\frac{1}{q^2}\cdot\frac{q^3}{9}-\frac{q}{9}=0.\end{aligned}$$

The proof for the point G_2 and line $\mathcal{P}_2$ is obtained by switching the indices 1 and 2. □

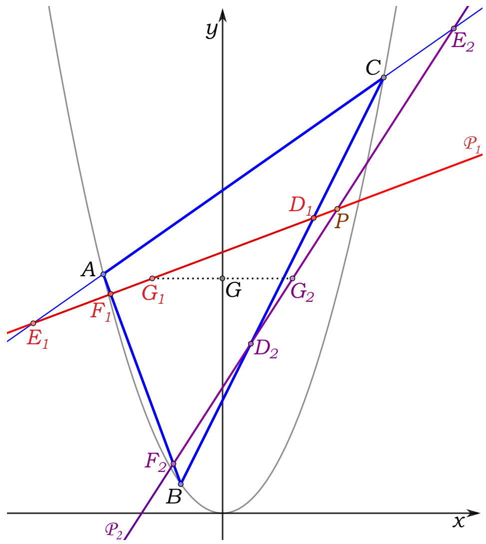

FIGURE 2. Equisegmentary lines of the triangle ABC.

THEOREM 2.5. *The intersection of lines $\mathcal{P}_1$ and $\mathcal{P}_2$ from Theorem 2.4 is the point*

$$P = \left(-\frac{3p}{q}, \frac{6p^2}{q^2} - \frac{4}{9}q\right) \tag{2.7}$$

and the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ is equal to the Brocard angle of the triangle ABC (Figure 3).

PROOF. The point P from (2.7) lies on the line $\mathcal{P}_1$ with equation (2.6a) because

$$\frac{p_1 - p}{q}\left(-\frac{3p}{q}\right) - \frac{7}{9}q - \frac{3p_1^2}{q^2} - \frac{6p^2}{q^2} + \frac{4}{9}q = -\frac{3}{q^2}(p^2 + pp_1 + p_1^2) - \frac{q}{3} = \frac{3}{q^2}\cdot\frac{q^3}{9} - \frac{q}{3} = 0$$

and similarly for the line $\mathcal{P}_2$.
The angle of lines $\mathcal{P}_1$ and $\mathcal{P}_2$ equals

$$\angle(\mathcal{P}_1, \mathcal{P}_2) = \frac{p_2 - p}{q} - \frac{p_1 - p}{q} = \frac{p_2 - p_1}{q} = \omega. \qquad \square$$

THEOREM 2.6. *Using the notation from Corollary 2.3 the lines* E_1F_2, F_1D_2, *and* D_1E_2 *have equations*

$$E_1F_2 \ \dots \ y = \frac{a}{2}x + \frac{a^2}{2} + \frac{\omega}{2}(b-c) \tag{2.8a}$$

$$F_1D_2 \ \dots \ y = \frac{b}{2}x + \frac{b^2}{2} + \frac{\omega}{2}(c-a) \tag{2.8b}$$

$$D_1E_2 \ \dots \ y = \frac{c}{2}x + \frac{c^2}{2} + \frac{\omega}{2}(a-b) \tag{2.8c}$$

and determine the triangle with vertices $D_1' = F_1D_2 \cap D_1E_2$, $E_1' = D_1E_2 \cap E_1F_2$, *and* $F_1' = E_1F_2 \cap F_1D_2$ *(Figure 3) given by*

$$\begin{aligned} D_1' &= \left(\frac{3p}{q} - a, \frac{bc}{2} - \frac{2}{3}q\right), \\ E_1' &= \left(\frac{3p}{q} - b, \frac{ca}{2} - \frac{2}{3}q\right), \\ F_1' &= \left(\frac{3p}{q} - c, \frac{ab}{2} - \frac{2}{3}q\right). \end{aligned} \tag{2.9}$$

PROOF. The points E_1 and F_2 from (2.3) and (2.4) lie on the line (2.8a) because

$$\frac{a}{2}(a-\omega) + \frac{a^2}{2} + \frac{\omega}{2}(b-c) = a^2 + \frac{\omega}{2}(b-c-a) = a^2 + b\omega,$$

$$\frac{a}{2}(a+\omega) + \frac{a^2}{2} + \frac{\omega}{2}(b-c) = a^2 + \frac{\omega}{2}(a+b-c) = a^2 - c\omega,$$

and this is the line E_1F_2, and analogously for lines F_1D_2 and D_1E_2. For the abscissa x of the intersection F_1D_2 and D_1E_2, from (2.8b) and (2.8c), after multiplication by 2, we obtain the equation $(b-c)x + b^2 - c^2 + \omega(b+c-2a) = 0$ with the solution

$$x = -(b+c) + \frac{3a\omega}{b-c} = a - \frac{a}{q}(c-a)(a-b) = a - \frac{a}{q}(2q - 3bc) = \frac{3p}{q} - a,$$

and by (2.8b) this x gives

$$\begin{aligned} y &= \frac{b}{2}\left(\frac{3p}{q} - a\right) + \frac{b^2}{2} + \frac{\omega}{2}(c-a) \\ &= \frac{bc}{2} - \frac{b}{2}(c+a-b) + \frac{3}{2q}ca \cdot b^2 - \frac{1}{6q}(c-a)^2(a-b)(b-c) \\ &= \frac{bc}{2} + b^2 + \frac{1}{6q}\big(9ca(ca-q) + (q+3ca)(2q-3ca)\big) \\ &= \frac{bc}{2} + ca - q + \frac{1}{6q}(2q^2 - 6caq) = \frac{bc}{2} - \frac{2}{3}q, \end{aligned}$$

and this intersection is the point D_1' from (2.9). ∎

THEOREM 2.7. *The triangle $D_1'E_1'F_1'$ from Theorem 2.6 is homologous to the triangle ABC. The center of homology is the point*

$$P_1' = \left(\frac{6p}{q}, -\frac{5}{6}q\right), \tag{2.10}$$

and the axis of homology is the line $\mathcal{P}_1'$ with equation

$$\mathcal{P}_1' \ \ldots \ y = \frac{3p}{q}x - \frac{3p^2}{q^2} - \frac{q}{9}. \tag{2.11}$$

PROOF. The line with equation

$$y = -\frac{q}{6a}x + bc - \frac{5}{6}q$$

passes through the point $A = (a, a^2)$ and the point D_1' from (2.9) because

$$-\frac{q}{6} + bc - \frac{5}{6}q = bc - q = a^2,$$

$$-\frac{q}{6a}\left(\frac{3p}{q} - a\right) + bc - \frac{5}{6}q = -\frac{bc}{2} + \frac{q}{6} + bc - \frac{5}{6}q = \frac{bc}{2} - \frac{2}{3}q,$$

and it is the line AD_1'. This line passes also through the point P_1' from (2.10) because

$$-\frac{q}{6a} \cdot \frac{6p}{q} + bc - \frac{5}{6}q = -\frac{5}{6}q,$$

and the same is also valid for lines BE_1' and CF_1'. For the abscissa of the intersection of lines E_1F_2, with equation (2.8a), and BC, given by $y = -ax - bc$, we get the equation

$$\frac{3}{2}ax + \frac{a^2}{2} + bc + \frac{\omega}{2}(b - c) = 0$$

with the solution

$$\begin{aligned} x &= -\frac{1}{3a}\big(a^2 + 2bc + \omega(b-c)\big) = -\frac{1}{3a}\Big(3bc - q - \frac{1}{3q}(b-c)^2(c-a)(a-b)\Big) \\ &= -\frac{1}{9aq}\big(3q(3bc - q) + (q + 3bc)(2q - 3bc)\big) = \frac{1}{9aq}(q^2 - 12bcq + 9b^2c^2), \end{aligned}$$

and for the abscissa of the intersection of the line BC with the line $\mathcal{P}_1'$ given by (2.11) we get the equation

$$\left(a + \frac{3p}{q}\right)x = \frac{3p^2}{q^2} + \frac{q}{9} - bc$$

whose solution is

$$\begin{aligned}
x &= \frac{q}{aq+3p} \cdot \frac{1}{9q^2}\big(q^3 - 9bcq^2 + 27b^2c^2(bc-q)\big) \\
&= \frac{1}{9aq(q+3bc)}(q^3 - 9bcq^2 - 27b^2c^2q + 27b^3c^3) \\
&= \frac{1}{9aq}(q^2 - 12bcq + 9b^2c^2).
\end{aligned}$$

The equality of these two abscissae means that the point $E_1F_2 \cap BC$ lies on the line $\mathcal{P}_1'$, and the same is also valid for points $F_1D_2 \cap CA$ and $D_1E_2 \cap AB$. $\square$

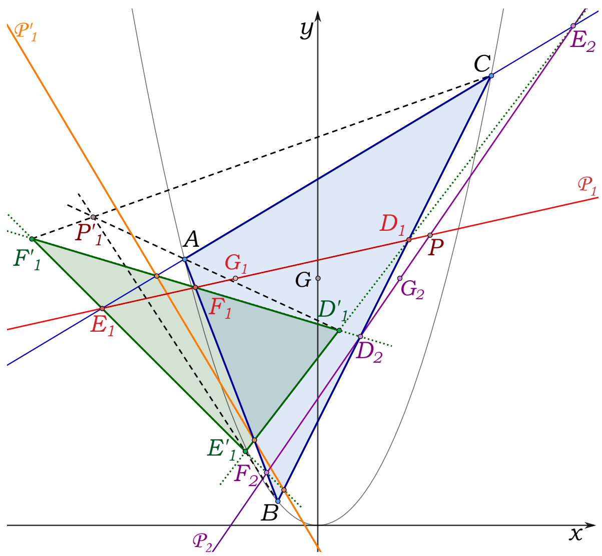

FIGURE 3. Visualization of the statements of Theorems 2.6 and 2.7.

According to [5] the bisectors of angles A, B, and C determine the triangle $A_sB_sC_s$, the so called *symmetral triangle* of the triangle ABC, with vertices

$$A_s = \left(a, -\frac{bc}{2}\right), \quad B_s = \left(b, -\frac{ca}{2}\right), \quad C_s = \left(c, -\frac{ab}{2}\right),$$

and by [4] the triangle ABC has the *symmedian center* $K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$. Comparing with the equalities (2.9) we directly get $A_s + D_1' = 2K$, $B_s + E_1' = 2K$, $C_s + F_1' = 2K$, hence the point K is the midpoint of A_sD_1', B_sE_1', C_sF_1', i.e. we have:

COROLLARY 2.8. *The triangle $D_1'E_1'F_1'$ from Theorem 2.6 is symmetrical to the symmetral triangle $A_sB_sC_s$ of the triangle ABC with respect to its symmedian center K (Figure 4).*

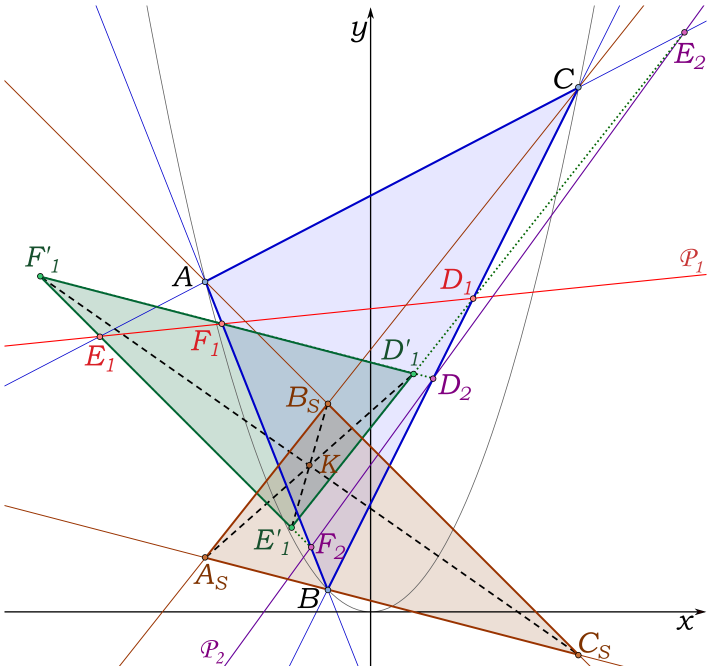

FIGURE 4. Visualization of the statement of Corollary 2.8.

THEOREM 2.9. *Using the notations from Corollary 2.3 the lines E_2F_1, F_2D_1, and D_2E_1 meet the lines BC, CA, and AB at three points D_2', E_2', and F_2', which lie on the line $\mathcal{P}_2'$ (Figure 5) given by*

$$\mathcal{P}_2' \ldots\ y = -\frac{3p}{2q}x - \frac{3p^2}{2q^2} - \frac{8}{9}q. \tag{2.12}$$

PROOF. Let $D_2' = (x, y)$, given by $(a-c)E_2 + (a-b)F_1 = 3aD_2'$, be a point on line E_2F_1. From (2.3) and (2.4) we obtain

$$\begin{aligned} 3ax &= (a-c)(c+\omega) + (a-b)(b-\omega) = ca - b^2 + ab - c^2 + (b-c)\omega \\ &= 2q - \frac{1}{3q}(b-c)^2(c-a)(a-b) = \frac{1}{3q}[6q^2 + (q+3bc)(2q-3bc)] \\ &= \frac{1}{3q}(8q^2 + 3bcq - 9b^2c^2), \end{aligned}$$

$$
\begin{aligned}
3ay &= (a-c)(c^2-b\omega)+(a-b)(b^2+c\omega) = a(b^2+c^2)-(b^3+c^3)-a(b-c)\omega \\
&= a(-q-bc)-(b+c)(b^2-bc+c^2)+\frac{a}{3q}(b-c)^2(c-a)(a-b) \\
&= -a(q+bc)+a(-q-2bc)-\frac{a}{3q}(q+3bc)(2q-3bc) \\
&= -\frac{a}{3q}[3q(2q+3bc)+(q+3bc)(2q-3bc)] = -\frac{a}{3q}(8q^2+12bcq-9b^2c^2),
\end{aligned}
$$

i.e.

$$
x = \frac{1}{9aq}(8q^2+3bcq-9b^2c^2), \quad y = -\frac{1}{9q}(8q^2+12bcq-9b^2c^2).
$$

This implies $y+ax=-bc$, so this point lies on the line BC, i.e. we have $D_2' = E_2F_1 \cap BC$. However, this point also lies on the line $\mathcal{P}_2'$ from (2.12) because

$$
\begin{aligned}
&-\frac{1}{9q}(8q^2+12bcq-9b^2c^2)+\frac{3p}{2q}\cdot\frac{1}{9aq}(8q^2+3bcq-9b^2c^2)+\frac{3p^2}{2q^2}+\frac{8}{9}q \\
&= -\frac{8}{9}q+\frac{bc}{3q}(3bc-4q)+\frac{bc}{6q^2}(8q^2+3bcq-9b^2c^2)+\frac{3p^2}{2q^2}+\frac{8}{9}q \\
&= \frac{bc}{6q^2}(9bcq-9b^2c^2)+\frac{3p^2}{2q^2} = \frac{3b^2c^2}{2q^2}(q-bc+a^2) = 0.
\end{aligned}
$$

The same is also valid for points $E_2' = F_2D_1 \cap CA$ and $F_2' = D_2E_1 \cap AB$. $\square$

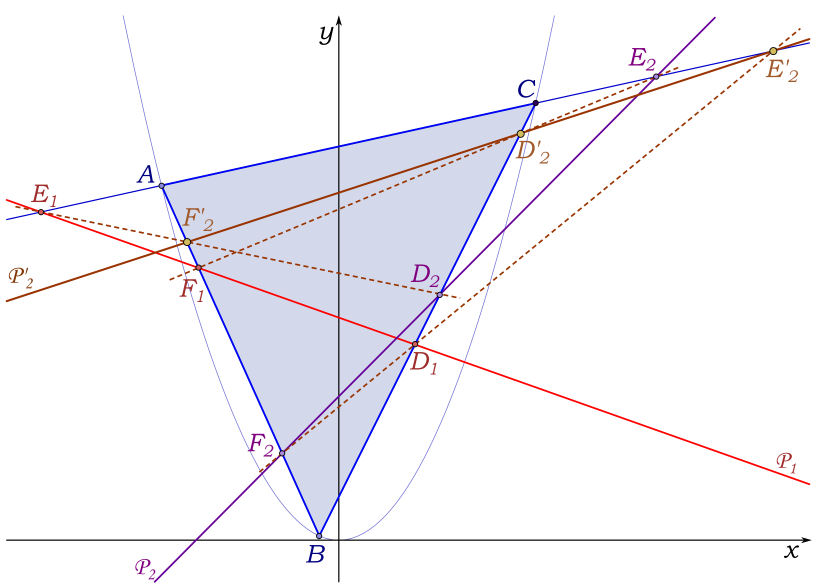

FIGURE 5. Visualization of the statement of Theorem 2.9.

The equality $(a-c)E_2+(a-b)F_1 = 3aD_2'$ can be written as $(a-c)(E_2 - D_2') = (b-a)(F_1 - D_2')$, i.e. $d(C,A)\cdot d(D_2',E_2) = d(A,B)\cdot d(D_2',F_1)$ or $d(D_2',E_2) : d(D_2',F_1) = d(A,B) : d(C,A)$, because $3a = a-c+a-b$. So we have:

Corollary 2.10. *The points D_2', E_2' and F_2' in Theorem 2.9 divide the segments E_2F_1, F_2D_1, and D_2E_1 in ratios $d(A,B) : d(C,A)$, $d(B,C) : d(A,B)$ and $d(C,A) : d(B,C)$ respectively.*

Theorem 2.11. *Equisegmentary lines and the inertial axis of an allowable triangle ABC, and its Steiner axis touch a circle (Figure 6), which in the case of a standard triangle ABC has the equation*

$$y = \frac{1}{8}x^2 - \frac{3p}{4q}x + \frac{9p^2}{8q^2} - \frac{2}{3}q. \tag{2.13}$$

Proof. According to [7], the inertial axis has the equation $y = -\frac{2}{3}q$, and this equation and (2.13) imply

$$\frac{1}{8}\left(x^2 - \frac{6p}{q}x + \frac{9p^2}{q^2}\right) = 0$$

with double solution $x = \frac{3p}{q}$, and inertial axis touches the circle (2.13) at the point $G' = \left(\frac{3p}{q}, -\frac{2}{3}q\right)$. By [14] the Steiner axis $\mathcal{S}$ has the equation

$$y = -\frac{3p}{2q}x - \frac{2}{3}q, \tag{2.14}$$

and from (2.13) and (2.14) we get the equation

$$\frac{1}{8}\left(x^2 + \frac{6p}{q}x + \frac{9p^2}{q^2}\right) = 0$$

with double solution $x = -\frac{3p}{q}$. For this x, from (2.14) we obtain $y = \frac{9p^2}{2q^2} - \frac{2}{3}q$. Therefore the line $\mathcal{S}$ touches the circle (2.13) at the point

$$G'' = \left(-\frac{3p}{q}, \frac{9p^2}{2q^2} - \frac{2}{3}q\right). \tag{2.15}$$

Equation (2.6a) of the equisegmentary line $\mathcal{P}_1$ and equation (2.13) give the following equation for the abscissa x

$$\frac{1}{8}x^2 - \frac{3p}{4q}x + \frac{9p^2}{8q^2} - \frac{2}{3}q = \frac{p_1 - p}{q}x - \frac{7}{9}q - \frac{3p_1^2}{q^2},$$

which, because of

$$\frac{p-p_1}{q} - \frac{3p}{4q} = \frac{p-4p_1}{4q} = -\frac{5p_1+p_2}{4q},$$

$$\frac{q}{9}+\frac{9p^2}{8q^2}+\frac{3p_1^2}{q^2}=-\frac{1}{q^2}(p_1^2+p_1p_2+p_2^2)+\frac{9}{8q^2}(p_1+p_2)^2+\frac{3p_1^2}{q^2}$$
$$=\frac{1}{8q^2}(25p_1^2+10p_1p_2+p_2^2)=\frac{(5p_1+p_2)^2}{8q^2},$$

after multiplication by 8, gets the form

$$x^2-\frac{2}{q}(5p_1+p_2)x+\frac{1}{q^2}(5p_1+p_2)^2=0$$

having the double solution $x=\frac{1}{q}(5p_1+p_2)$. For this x from (2.6a) we get

$$\begin{aligned}y&=\frac{p_1-p}{q}\cdot\frac{4p_1-p}{q}-\frac{7}{9}q-\frac{3p_1^2}{q^2}\\&=\frac{1}{q^2}(p_1-p)(4p_1-p)-\frac{3p_1^2}{q^2}+\frac{7}{q^2}(p^2+pp_1+p_1^2)\\&=\frac{1}{q^2}(8p^2+2pp_1+8p_1^2).\end{aligned}$$

Hence the circle (2.13) touches the line $\mathcal{P}_1$ at the first of two analogous points

$$\begin{aligned}P_1&=\left(\frac{5p_1+p_2}{q},\frac{1}{q^2}(8p^2+2pp_1+8p_1^2)\right),\\P_2&=\left(\frac{p_1+5p_2}{q},\frac{1}{q^2}(8p^2+2pp_2+8p_2^2)\right),\end{aligned}\tag{2.16}$$

while the other point is the point of tangency of this circle with the line $\mathcal{P}_2$.

□

According to [2] and [14], Gergonne and Steiner points of a standard triangle ABC are the points

$$\Gamma=\left(-\frac{3p}{q},-\frac{4}{3}q\right),\quad S=\left(-\frac{3p}{q},\frac{9p^2}{q^2}\right)$$

and its midpoint is the point G'' from (2.15). The centroid $G=\left(0,-\frac{2}{3}q\right)$ of the triangle ABC and the point G' from previous proof have the midpoint $\left(\frac{3p}{2q},-\frac{2}{3}q\right)$. It lies on the inertial axis and the Brocard diameter of the triangle ABC, which by [4], has the equation $x=\frac{3p}{2q}$. The midpoint of P_1 and P_2 from (2.16) has coordinates

$$x=\frac{1}{2q}(6p_1+6p_2)=-\frac{3p}{q},$$

$$\begin{aligned}y&=\frac{1}{q^2}(8p^2+pp_1+pp_2+4p_1^2+4p_2^2)=\frac{1}{q^2}\left(7p^2+4(p_1^2+p_1p_2+p_2^2)-4p_1p_2\right)\\&=\frac{1}{q^2}\left(7p^2-\frac{4}{9}q^3-4\left(p^2+\frac{q^3}{9}\right)\right)=\frac{3p^2}{q^2}-\frac{8}{9}q,\end{aligned}$$

and it is the point

$$P_0 = \left(-\frac{3p}{q}, \frac{3p^2}{q^2} - \frac{8}{9}q\right),$$

which lies on the same isotropic line with points Γ, S, and G''. We just proved:

Theorem 2.12. *The circle from Theorem 2.11 touches the inertial axis of triangle ABC at the point symmetrical to its centroid with respect to the intersection of this inertial axis with the Brocard diameter of triangle ABC. This circle touches Steiner axis of this triangle at the midpoint of its Gergonne and its Steiner point. The isotropic line through the last three points is the bisector of the points of tangency of the considered circle with the equisegmentary lines of the triangle ABC (Figure 6).*

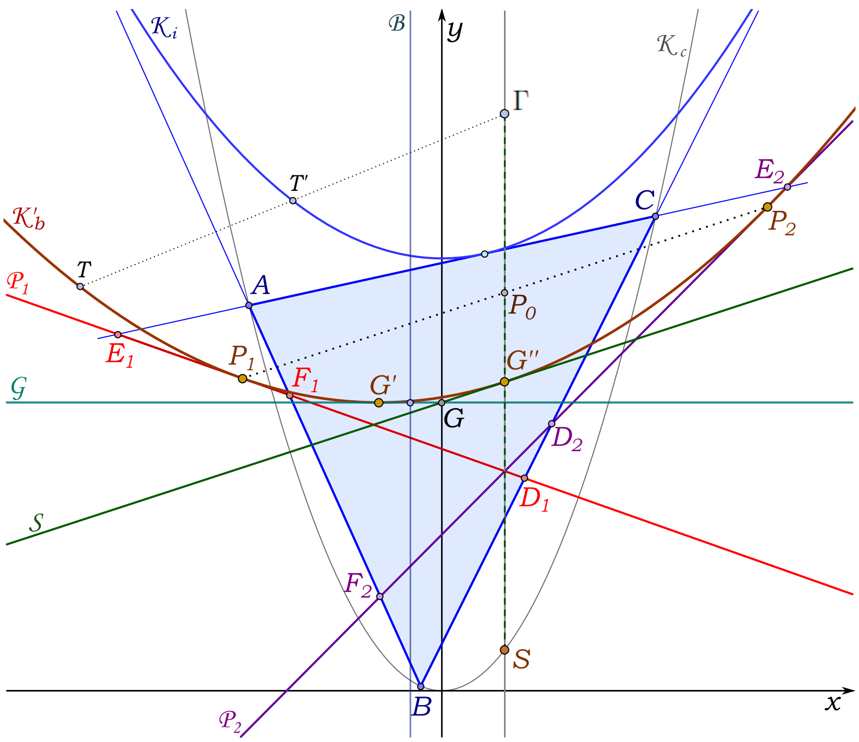

Figure 6. Centroid G, Steiner point S, Gergonne point Γ, inertial axis $\mathcal{G}$, Steiner axis $\mathcal{S}$, equisegmentary lines $\mathcal{P}_1$ and $\mathcal{P}_2$, Brocard diameter $\mathcal{B}$, circumscribed circle $\mathcal{K}_c$, inscribed circle $\mathcal{K}_i$ and the dual Brocard circle $\mathcal{K}'_b$ of the triangle ABC (Visualization of the statements of Theorems 2.11, 2.12 and 2.13.)

Equisegmentary lines of a triangle are dual to the concept of Crelle-Brocard points of that triangle considered in [13], and the circle from Theorems 2.11 and 2.12 is then dual to the Brocard circle of the triangle ABC from [6]. And for this reason we call the circle from Theorems 2.11 and 2.12 the *dual* Brocard circle of triangle ABC.

THEOREM 2.13. *Dual Brocard circle of an allowable triangle ABC is the image of its inscribed circle by homothety $(\Gamma, 2)$, where Γ is the Gergonne point of that triangle (Figure 6).*

PROOF. Let $T' = (x', y')$ be the midpoint of $\Gamma = \left(-\frac{3p}{q}, -\frac{4}{3}q\right)$ and $T = (x, y)$. If the point $T' = (\frac{x}{2} - \frac{3p}{2q}, \frac{y}{2} - \frac{2}{3}q)$ lies on the inscribed circle of triangle ABC, then by [1] we get $y' = \frac{1}{4}x'^2 - q$, and then, because of previous equalities we obtain

$$\frac{y}{2} - \frac{2}{3}q = \frac{1}{4}\left(\frac{x}{2} - \frac{3p}{2q}\right)^2 - q,$$

which by multiplication by 2 and rearranging gets the form (2.13). ∎

At the end we will mention one more interesting property of equisegmentary lines.

THEOREM 2.14. *Equisegmentary lines of an allowable triangle ABC meet at the point on the line BC if and only if $d(B,C)^2 + d(C,A) \cdot d(A,B) = 0$.*

PROOF. The point P from (2.7) lies on the line BC under the condition

$$\frac{6p^2}{q^2} - \frac{4}{9}q = \frac{3ap}{q} - bc,$$

which after multiplication by $9q^2$ becomes

$$54b^2c^2(bc - q) - 4q^3 = 27bcq(bc - q) - 9bcq^2,$$

and after rearrangement gets the form $4q^3 - 36bcq^2 + 81b^2c^2q - 54b^3c^3 = 0$, or finally $(2q - 3bc)^2(q - 6bc) = 0$. As $2q - 3bc = (c - a)(a - b) \neq 0$, the final condition is $q - 6bc = 0$. On the other hand we have

$$\begin{aligned} d(B,C)^2 + d(C,A) \cdot d(A,B) &= (b - c)^2 + (c - a)(a - b) \\ &= -(q + 3bc) + 2q - 3bc = q - 6bc. \end{aligned}$$

∎

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Ekvisegmentarni pravci trokuta u izotropnoj ravnini

Ružica Kolar–Šuper

Sažetak. U ovom radu uvodi se pojam ekvisegmentarnih pravaca u izotropnoj ravnini. Izvode se jednadžbe ekvisegmentarnih pravaca za standardni trokut i dokazuje da je kut između njih jednak Brocardovom kutu standardnog trokuta. Proučava se dualna Brocardova kružnica, kružnica čije su tangente ekvisegmentarni pravci, kao i inercijalna os i Steinerova os. Istražuju se i neka zanimljiva svojstva ove kružnice.

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