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DIFFERENTIAL POLYNOMIALS GENERATED BY SOLUTIONS OF SECOND ORDER NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. This paper is devoted to studying the growth and the oscillation of solutions of the second order non-homogeneous linear differential equation

$$f'' + Ae^{a_1 z} f' + B(z) e^{a_2 z} f = F(z) e^{a_1 z},$$

where A, a_1, a_2 are complex numbers, $B(z)$ ($\not\equiv 0$) and $F(z)$ ($\not\equiv 0$) are entire functions with order less than one. Moreover, we investigate the growth and the oscillation of some differential polynomials generated by solutions of the above equation.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions [13, 21]. In what follows, we give the necessary notations and basic definitions.

Definition 1.1. *[13, 18, 21] Let f be a meromorphic function. Then the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined respectively by*

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r}, \ \rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined respectively as follows

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

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$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2. *[13, 18, 21] Let f be a meromorphic function. Then the exponent of convergence of the sequence of zeros of f is defined by*

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \le r\}$. Similarly, the exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \le r\}$. The hyper convergence exponents of the zero-sequence and the distinct zeros of f are defined respectively by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r}, \ \overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

For the second order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of the equation (1.1) is an entire function, and that if f_1, f_2 are two linearly independent solutions of (1.1), then by [9], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But the equation (1.1) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ has infinite order? Many authors, Frei [9], Ozawa [17], Amemiya-Ozawa [1] and Gundersen [11], Langley [16] have studied this problem. They proved that when $B(z)$ is a non-constant polynomial or $B(z)$ is a transcendental entire function with order $\rho(B) \neq 1$, then every solution $f \not\equiv 0$ of has infinite order. In [7], Chen has considered equation (1.1) and obtained some results concerning the growth of its solutions when $\rho(B) = 1$.

Theorem 1.3. *[7] Let a, b be complex numbers such that $ab \neq 0$ and $a \neq b$, let $Q(z)$ be non-constant polynomial or $Q(z) = h(z) e^{bz}$, where $h(z)$ is non-zero polynomial. Then every solution $f(z) \not\equiv 0$ of the equation*

$$f'' + e^{az} f' + Q(z) f = 0$$

has infinite order and $\rho_2(f) = 1$.

THEOREM 1.4. *[7] Let* $b \neq -1$ *be any complex number,* $h(z)$ *be non-zero polynomial. Then every solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + e^{-z} f' + h(z) e^{bz} f = 0$$

has infinite order and $\rho_2(f) = 1$.

In [19], Wang and Laine have investigated the growth of solutions of some second order non-homogenous linear differential equations and have obtained the following result.

THEOREM 1.5. *[19] Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ *and* $H(z)$ *be entire functions with* $\max\{\rho(A_j)(j = 0, 1), \rho(H)\} < 1$, *and let* a, b *be complex constants that satisfy* $ab \neq 0$ *and* $a \neq b$. *Then every non-trivial solution* f *of the equation*

$$f'' + A_1(z) e^{az} f' + A_0(z) e^{bz} f = H \tag{1.2}$$

is of infinite order.

REMARK 1.6. If $\rho(H) = 1$, then equation (1.2) can possesses a solution of finite order. For instance the equation $f'' + e^{-iz} f' + e^{iz} f = ze^{iz} + e^{-iz}$ satisfies $\rho(H) = \rho\left(ze^{iz} + e^{-iz}\right) = 1$ and has a finite order solution $f(z) = z$.

Thus, the following question arises naturally: Whether the results similar to Theorem 1.5 can be obtained if $\rho(H) = 1$? In this paper, we give answer to the above question. In fact we will prove the following results.

THEOREM 1.7. *Let* $B(z)$ $(\not\equiv 0)$, $F(z)$ $(\not\equiv 0)$ *be entire functions with*

$$\max\{\rho(B), \rho(F)\} < 1,$$

and let A, a_1, a_2 *be complex numbers such that* $Aa_1a_2 \neq 0$, $a_1 \neq a_2$. *Then every solution* f *of the differential equation*

$$f'' + Ae^{a_1 z} f' + B(z) e^{a_2 z} f = F(z) e^{a_1 z} \tag{1.3}$$

satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty, \ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq 1.$$

COROLLARY 1.8. *Let* $b \neq -1$, $A \neq 0$ *be any complex numbers,* $B(z)$ $(\not\equiv 0)$, $F(z)$ $(\not\equiv 0)$ *be entire functions with* $\max\{\rho(B), \rho(F)\} < 1$. *Then every solution* f *of the equation*

$$f'' + Ae^{-z} f' + B(z) e^{bz} f = F(z) e^{-z}$$

satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty, \ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq 1.$$

We know that a differential equation bears a relation to all derivatives of its solutions. Hence, linear differential polynomials generated by its solutions must have special nature because of the control of differential equations.

Several authors have investigated the growth and the oscillation of differential polynomial generated by solutions of linear differential equations [3, 4, 8, 14, 18]. The second main purpose of this paper is to study the growth and the oscillation of some differential polynomials generated by solutions of second order linear differential equation (1.3). We obtain some estimates of their hyper order and fixed points.

THEOREM 1.9. *Under the assumptions of Theorem 1.7, let* $d_0(z)$, $d_1(z)$, $b(z)$ *be entire functions such that at least one of* $d_0(z), d_1(z)$ *does not vanish identically with* $\rho(d_j) < 1$ $(j = 0, 1)$, $\rho(b) < \infty$, *and let* $\varphi(z)$ *be an entire function with finite order. If* f *is a solution of the equation* (1.3), *then the differential polynomial*

$$g_f = d_1 f' + d_0 f + b \tag{1.4}$$

satisfies

$$\overline{\lambda}(f) = \lambda(f) = \overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f) = \lambda_2(f) = \overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) \le 1.$$

In particular, if f *is a solution of equation* (1.3), *then the differential polynomial* $g_f = d_1 f' + d_0 f + b$ *has infinitely many fixed points and satisfies*

$$\overline{\lambda}(g_f - z) = \lambda(g_f - z) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(g_f - z) = \lambda_2(g_f - z) = \rho_2(f) \le 1.$$

In the next, we investigate the relation between infinite order solutions of a pair non-homogeneous linear differential equations and we obtain the following result.

THEOREM 1.10. *Under the assumptions of Theorem 1.9, let* $F_1 \not\equiv 0$ *and* $F_2 \not\equiv 0$ *be entire functions such that* $\max\{\rho(F_j) : j = 1, 2\} < 1$ *and* $F_1 - KF_2 \not\equiv 0$ *for any complex constant* K, $\varphi(z)$ *is an entire function with finite order. If* f_1 *is a solution of equation*

$$f'' + Ae^{a_1 z} f' + B(z) e^{a_2 z} f = F_1(z) e^{a_1 z} \tag{1.5}$$

and f_2 *is a solution of equation*

$$f'' + Ae^{a_1 z} f' + B(z) e^{a_2 z} f = F_2(z) e^{a_1 z}, \tag{1.6}$$

then the differential polynomial $g_{f_1 - Kf_2} = d_1(f_1' - Kf_2') + d_0(f_1 - Kf_2) + b$ *satisfies*

$$\begin{aligned}\overline{\lambda}(f_1 - Kf_2) = \lambda(f_1 - Kf_2) &= \overline{\lambda}(g_{f_1 - Kf_2} - \varphi) \\ &= \lambda(g_{f_1 - Kf_2} - \varphi) = \rho(f_1 - Kf_2) = \infty\end{aligned}$$

and

$$\begin{aligned}\overline{\lambda}_2(f_1-Kf_2) &= \lambda_2(f_1-Kf_2)=\overline{\lambda}_2(g_{f_1-Kf_2}-\varphi)\\ &= \lambda_2(g_{f_1-Kf_2}-\varphi)=\rho_2(f_1-Kf_2)\le 1\end{aligned}$$

for any complex constant K.

2. Some Useful Lemmas

Lemma 2.1. *[10] Let* $P_1, P_2, ..., P_n$ $(n\geq 1)$ *be non-constant polynomials with degree* d_1, d_2, ..., d_n, *respectively, such that* $\deg(P_i-P_j)=\max\{d_i,d_j\}$ *for* $i\neq j$. *Let* $A(z)=\sum_{j=1}^{n} B_j(z)\,e^{P_j(z)}$, *where* $B_j(z)\,(\not\equiv 0)$ *are entire functions with* $\rho(B_j)<d_j$. *Then* $\rho(A)=\max_{1\le j\le n}\{d_j\}$.

Lemma 2.2. *[7] Suppose that* $P(z)=(\alpha+i\beta)\,z^n+\cdots$ *(* α,β *are real numbers,* $|\alpha|+|\beta|\neq 0$*) is a polynomial with degree* $n\geq 1$, *that* $A(z)$ $(\not\equiv 0)$ *is an entire function with* $\rho(A)<n$. *Set* $g(z)=A(z)\,e^{P(z)}$, $z=re^{i\theta}$, $\delta(P,\theta)=\alpha\cos n\theta-\beta\sin n\theta$. *Then for any given* $\varepsilon>0$, *there is a set* $E_1\subset[0,2\pi)$ *that has linear measure zero, such that for any* $\theta\in[0,2\pi)\setminus(E_1\cup E_2)$, *there is* $R>0$, *such that for* $|z|=r>R$, *we have*
(i) *If* $\delta(P,\theta)>0$, *then*

$$\exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\}\le\left|g\left(re^{i\theta}\right)\right|\le\exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\}.$$

(ii) *If* $\delta(P,\theta)<0$, *then*

$$\exp\{(1+\varepsilon)\,\delta(P,\theta)\,r^n\}\le\left|g\left(re^{i\theta}\right)\right|\le\exp\{(1-\varepsilon)\,\delta(P,\theta)\,r^n\},$$

where $E_2=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ *is a finite set.*

Lemma 2.3. *[12] Let* f *be a transcendental meromorphic function of finite order* ρ. *Let* $\varepsilon>0$ *be a constant,* k *and* j *be integers satisfying* $k>j\geq 0$. *Then the following two statements hold:*
(i) *There exists a set* $E_3\subset(1,+\infty)$ *which has finite logarithmic measure, such that for all* z *satisfying* $|z|\notin E_3\cup[0,1]$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le|z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) *There exists a set* $E_4\subset[0,2\pi)$ *which has linear measure zero, such that if* $\theta\in[0,2\pi)\setminus E_4$, *then there is a constant* $R=R(\theta)>0$ *such that* (2.1) *holds for all* z *satisfying* $\arg z=\theta$ *and* $|z|\geq R$.

Lemma 2.4. *[20] Let* f *be an entire function and suppose that*

$$G(z):=\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^{\rho}}$$

is unbounded on some ray $\arg z = \theta$ *with constant* $\rho > 0$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1, 2, ...)$, *where* $r_n \to +\infty$, *such that* $G(z_n) \to \infty$ *and*

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \leq \frac{1}{(k-j)!} (1 + o(1)) r_n^{k-j}, \ j = 0, 1, ..., k-1$$

as $n \to +\infty$.

LEMMA 2.5. *[20] Let* f *be an entire function with* $\rho(f) = \rho < +\infty$. *Suppose that there exists a set* $E_5 \subset [0, 2\pi)$ *which has linear measure zero, such that* $\log^+ \left| f\left(re^{i\theta}\right) \right| \leq M r^{\sigma}$ *for any ray* $\arg z = \theta \in [0, 2\pi) \setminus E_5$, *where* M *is a positive constant depending on* θ, *while* σ *is a positive constant independent of* θ. *Then* $\rho(f) = \rho \leq \sigma$.

LEMMA 2.6. *[2,6] Let* $A_j(z)$ $(j = 0, 1, ..., k-1)$, $F(z) \not\equiv 0$ *be finite order meromorphic functions.*
(i) *If* f *is a meromorphic solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F, \tag{2.2}$$

with $\rho(f) = +\infty$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

(ii) *If* f *is a meromorphic solution of equation* (2.2) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty, \ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

LEMMA 2.7. *[3, 5] Let* $B_1(z), B_2(z), ..., B_{k-1}(z)$, $H(z)$ *be entire functions of finite order. If* f *is a solution of the equation*

$$f^{(k)} + B_{k-1}(z) f^{(k-1)} + \cdots + B_1(z) f' + B_0(z) f = H(z),$$

then $\rho_2(f) \leq \max\{\rho(B_j) \ \ (j = 0, 1, ..., k-1), \rho(H)\}$.

3. PROOF OF THEOREM 1.7

Set $a = -a_1$ and $b = a_2 - a_1$. We can see that $ab \neq 0$ and $a \neq b$. Hence, by (1.3) we get

$$e^{az} f'' + A f' + B e^{bz} f = F. \tag{3.1}$$

First we prove that every solution f of (1.3) satisfies $\rho(f) \geq 1$. We assume that $\rho(f) < 1$. It is clear that $f \not\equiv 0$. Obviously $\rho\left(f^{(j)}\right) < 1$ $(j = 1, 2)$, $\rho(Bf) < 1$. Rewrite (3.1) as

$$f'' e^{az} + B f e^{bz} = F - A f'. \tag{3.2}$$

i) If $f'' \not\equiv 0$, then by (3.2) and the Lemma 2.1, we have

$$1 = \rho\left\{f'' e^{az} + B f e^{bz}\right\} = \rho\left\{F - A f'\right\} < 1.$$

This is a contradiction.

ii) If $f'' \equiv 0$, then by (3.2) we have

$$1 = \rho\left\{Bfe^{bz}\right\} = \rho\left\{F - Af'\right\} < 1.$$

This is a contradiction. Hence, $\rho(f) \geq 1$. Therefore f is a transcendental solution of equation (1.3).

Now, we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. Since $\rho(F) < 1$, then for any given ε $(0 < 2\varepsilon < 1 - \rho(F))$ and sufficiently large r, we have

$$|F(z)| \leq \exp\left\{r^{\rho(F)+\varepsilon}\right\}. \tag{3.3}$$

By Lemma 2.2, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi) \setminus E$, then $\delta(az, \theta) \neq 0$, $\delta(bz, \theta) \neq 0$ and $\delta(az, \theta) \neq \delta(bz, \theta)$. By Lemma 2.3, there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{2\rho},\ 0 \leq i < j \leq 2. \tag{3.4}$$

For any fixed $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, set

$$\delta_1 = \max\left\{\delta(az, \theta), \delta(bz, \theta)\right\}$$

and

$$\delta_2 = \min\left\{\delta(az, \theta), \delta(bz, \theta)\right\},$$

then $\delta_2 < \delta_1$ and $\delta_1 \neq 0$, $\delta_2 \neq 0$. We now discuss three cases separately.

Case 1: Suppose that $\delta_1 = \delta(az, \theta) > 0$, then $\delta_2 = \delta(bz, \theta)$. By Lemma 2.2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_1 - \delta_2}{\delta_1}, 1 - \rho(F)\right\}$, we obtain

$$|e^{az}| \geq \exp\left\{(1 - \varepsilon)\delta_1 r\right\}, \tag{3.5}$$

for sufficiently large r. We now prove that $\log^+|f''(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+|f''(z)| / |z|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+|f''(z_m)|}{r_m^{\rho(F)+\varepsilon}} \to +\infty, \tag{3.6}$$

$$\left|\frac{f^{(j)}(z_m)}{f''(z_m)}\right| \leq \frac{1}{(2-j)!}(1 + o(1))\, r_m^{2-j} \leq 2r_m^{2-j},\ \ (j = 0, 1) \tag{3.7}$$

for m is large enough. From (3.6) for any sufficiently large number $C > 1$ we have

$$\frac{\log^+|f''(z_m)|}{r_m^{\rho(F)+\varepsilon}} > C,\ \text{then}\ |f''(z_m)| > \exp\left\{Cr_m^{\rho(F)+\varepsilon}\right\}\ \text{as}\quad m \to +\infty. \tag{3.8}$$

From (3.3) and (3.8), we get

$$\left|\frac{F(z_m)}{f''(z_m)}\right| \leq \frac{\exp\left\{r_m^{\rho(F)+\varepsilon}\right\}}{\exp\left\{Cr_m^{\rho(F)+\varepsilon}\right\}} = \frac{1}{\exp\left\{(C-1)\,r_m^{\rho(F)+\varepsilon}\right\}} \to 0 \tag{3.9}$$

as $m \to +\infty$. From (3.1), we obtain

$$|e^{az}| \leq |A|\left|\frac{f'}{f''}\right| + \left|Be^{bz}\right|\left|\frac{f}{f''}\right| + \left|\frac{F}{f''}\right|. \tag{3.10}$$

(i) If $\delta_2 > 0$, then by Lemma 2.2, for ε as above, we obtain

$$\left|B(z)\,e^{bz}\right| \leq \exp\left\{(1+\varepsilon)\,\delta_2 r\right\}, \tag{3.11}$$

for sufficiently large r. Substituting (3.5), (3.7), (3.9) and (3.11) into (3.10), we have

$$\begin{aligned} \exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} &\leq |e^{az_m}| \\ \leq |A|\left|\frac{f'(z_m)}{f''(z_m)}\right| &+ \left|B(z_m)\,e^{bz_m}\right|\left|\frac{f(z_m)}{f''(z_m)}\right| + \left|\frac{F(z_m)}{f''(z_m)}\right| \\ &\leq 2\,|A|\,r_m + 2r_m^2\exp\left\{(1+\varepsilon)\,\delta_2 r_m\right\} + o(1) \\ &\leq C_1 r_m^2\exp\left\{(1+\varepsilon)\,\delta_2 r_m\right\}, \end{aligned} \tag{3.12}$$

where $C_1 > 0$ is some constant. By $0 < \varepsilon < \frac{\delta_1-\delta_2}{2\delta_1}$ and (3.12), we can get

$$\exp\left\{\frac{(\delta_1-\delta_2)^2}{2\delta_1}r_m\right\} \leq C_1 r_m^2,$$

which is a contradiction.

(ii) If $\delta_2 < 0$, then by Lemma 2.2, for ε as above, we obtain

$$\left|B(z)\,e^{bz}\right| \leq \exp\left\{(1-\varepsilon)\,\delta_2 r\right\} < 1 \tag{3.13}$$

for sufficiently large r. Substituting (3.5), (3.7), (3.9) and (3.13) into (3.10), we have

$$\begin{aligned} \exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} &\leq |e^{az_m}| \\ \leq |A|\left|\frac{f'(z_m)}{f''(z_m)}\right| &+ \left|B(z_m)\,e^{bz_m}\right|\left|\frac{f(z_m)}{f''(z_m)}\right| + \left|\frac{F(z_m)}{f''(z_m)}\right| \\ &\leq 2\,|A|\,r_m + 2r_m^2 + o(1) \leq C_2 r_m^2, \end{aligned}$$

where $C_2 > 0$ is some constant, which is a contradiction. Therefore,

$$\log^+|f''(z)|\,/\,|z|^{\rho(F)+\varepsilon}$$

is bounded and we have

$$|f''(z)| \leq \exp\left\{Mr^{\rho(F)+\varepsilon}\right\} \quad (M > 0)$$

on the ray $\arg z = \theta$. Hence, using the same reasoning as in the proof of Lemma 3.1 in [15], by two-fold iterated integration, along the line segment $[0, z]$, we conclude that

$$f(z) = f(0) + f'(0)\frac{z}{1!} + \int_0^z \int_0^t f''(u)\,du\,dt.$$

So, we get for a sufficiently large r

$$|f(z)| \le |f(0)| + |f'(0)|\frac{|z|}{1!} + \left| \int_0^z \int_0^t f''(u)\,du\,dt \right|$$

$$\le |f(0)| + |f'(0)|\frac{|z|}{1!} + |f''(z)|\frac{|z|^2}{2!} = \frac{1}{2}(1 + o(1))\,r^2\,|f''(z)|$$

$$\le \frac{1}{2}(1 + o(1))\,r^2 \exp\left\{Mr^{\rho(F)+\varepsilon}\right\} \le \exp\left\{Mr^{\rho(F)+2\varepsilon}\right\}$$

on the ray $\arg z = \theta$.

Case 2: Suppose that $\delta_1 = \delta(bz, \theta) > 0$, then $\delta_2 = \delta(az, \theta)$. By Lemma 2.2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_1 - \delta_2}{\delta_1}, 1 - \rho(F)\right\}$, we obtain

$$\left|B(z)\,e^{bz}\right| \ge \exp\{(1 - \varepsilon)\,\delta_1 r\} \tag{3.14}$$

for sufficiently large r. We now prove that $\log^+ |f(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+ |f(z)| / |z|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+ |f(z_m)|}{r_m^{\rho(F)+\varepsilon}} \to +\infty \tag{3.15}$$

for m is large enough. From (3.3) and (3.15), we get as in (3.9)

$$\left|\frac{F(z_m)}{f(z_m)}\right| \to 0 \tag{3.16}$$

for m is large enough. From (3.1), we obtain

$$\left|Be^{bz}\right| \le |e^{az}|\left|\frac{f''}{f}\right| + |A|\left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right|. \tag{3.17}$$

(i) If $\delta_2 > 0$, then by Lemma 2.2, for ε as above, we obtain

$$|e^{az}| \le \exp\{(1 + \varepsilon)\,\delta_2 r\} \tag{3.18}$$

for sufficiently large r. Substituting (3.4), (3.14), (3.16) and (3.18) into (3.17), we have

$$\exp\{(1 - \varepsilon)\,\delta_1 r_m\} \le \left|B(z_m)\,e^{bz_m}\right|$$

$$\leq |e^{az_m}| \left|\frac{f''(z_m)}{f(z_m)}\right| + |A| \left|\frac{f'(z_m)}{f(z_m)}\right| + \left|\frac{F(z_m)}{f(z_m)}\right|$$
$$\leq r_m^{2\rho} \exp\{(1+\varepsilon)\delta_2 r_m\} + |A| r_m^{2\rho} + o(1)$$
$$\leq C_3 r_m^{2\rho} \exp\{(1+\varepsilon)\delta_2 r_m\}, \tag{3.19}$$

where $C_3 > 0$ is some constant. By $0 < \varepsilon < \frac{\delta_1 - \delta_2}{2\delta_1}$ and (3.19), we can get

$$\exp\left\{\frac{(\delta_1 - \delta_2)^2}{2\delta_1} r_m\right\} \leq C_3 r_m^{2\rho},$$

which is a contradiction.
(ii) If $\delta_2 < 0$, then by Lemma 2.2, for ε as above, we obtain

$$|e^{az}| \leq \exp\{(1-\varepsilon)\delta_2 r\} < 1 \tag{3.20}$$

for sufficiently large r. Substituting (3.4), (3.14), (3.16) and (3.20) into (3.17), we have

$$\exp\{(1-\varepsilon)\delta_1 r_m\} \leq \left|B(z_m) e^{bz_m}\right|$$
$$\leq |e^{az_m}| \left|\frac{f''(z_m)}{f(z_m)}\right| + |A| \left|\frac{f'(z_m)}{f(z_m)}\right| + \left|\frac{F(z_m)}{f(z_m)}\right|$$
$$\leq r_m^{2\rho} + |A| r_m^{2\rho} + o(1) \leq C_4 r_m^{2\rho},$$

where $C_4 > 0$ is some constant, which is a contradiction. Therefore,

$$\log^+ |f(z)| / |z|^{\rho(F)+\varepsilon}$$

is bounded and we have

$$|f(z)| \leq \exp\left\{M r^{\rho(F)+\varepsilon}\right\} \quad (M > 0)$$

on the ray $\arg z = \theta$.

Case 3: Suppose now that $\delta_1 < 0$. From (3.1) we get

$$-1 = e^{az} \frac{f''}{Af'} + Be^{bz} \frac{f}{Af'} - \frac{F}{Af'}. \tag{3.21}$$

By Lemma 2.2, for any given ε with $0 < 2\varepsilon < 1 - \rho(F)$, we obtain

$$|e^{az}| \leq \exp\{(1-\varepsilon)\delta(az,\theta) r\} \leq \exp\{(1-\varepsilon)\delta_1 r\}, \tag{3.22}$$

$$\left|B(z) e^{bz}\right| \leq \exp\{(1-\varepsilon)\delta(bz,\theta) r\} \leq \exp\{(1-\varepsilon)\delta_1 r\} \tag{3.23}$$

for sufficiently large r. We now prove that $\log^+ |f'(z)| / |z|^{\rho(F)+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+ |f'(z)| / |z|^{\rho(F)+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4 there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+ |f'(z_m)|}{r_m^{\rho(F)+\varepsilon}} \to +\infty, \tag{3.24}$$

$$\left|\frac{f(z_m)}{f'(z_m)}\right| \leq (1+o(1))\, r_m \leq 2r_m. \tag{3.25}$$

From (3.3) and (3.24), we have

$$\left|\frac{F(z_m)}{f'(z_m)}\right| \to 0 \tag{3.26}$$

for m is large enough. Substituting (3.4), (3.22), (3.23), (3.25) and (3.26) into (3.21), we have

$$1 \leq \frac{|e^{az_m}|}{|A|}\left|\frac{f''(z_m)}{f'(z_m)}\right| + \frac{\left|B(z_m)\, e^{bz_m}\right|}{|A|}\left|\frac{f(z_m)}{f'(z_m)}\right| + \frac{1}{|A|}\left|\frac{F(z_m)}{f'(z_m)}\right|$$

$$\leq \frac{r_m^{2\rho}}{|A|}\exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} + 2\frac{r_m}{|A|}\exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} + \frac{1}{|A|}o(1). \tag{3.27}$$

By $\delta_1 < 0$, we have

$$\frac{r_m^{2\rho}}{|A|}\exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} + 2\frac{r_m}{|A|}\exp\left\{(1-\varepsilon)\,\delta_1 r_m\right\} + \frac{1}{|A|}o(1) \to 0$$

as $r_m \to +\infty$. From (3.27) we obtain $1 \leq 0$ as $r_m \to +\infty$, which is a contradiction. Therefore, $\log^+|f'(z)|\,/\,|z|^{\rho(F)+\varepsilon}$ is bounded and we have

$$|f'(z)| \leq \exp\left\{Mr^{\rho(F)+\varepsilon}\right\} \quad (M>0)$$

on the ray $\arg z = \theta$. This implies, as in Case 1, that

$$|f(z)| \leq \exp\left\{Mr^{\rho(F)+2\varepsilon}\right\}. \tag{3.28}$$

Therefore, for any given $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, we have got (3.28) on the ray $\arg z = \theta$, provided that r is large enough. Then by Lemma 2.5, we have $\rho(f) \leq \rho(F) + 2\varepsilon < 1$, which is a contradiction. Hence, every transcendental solution f of (1.3) must be of infinite order.
We have

$$\max\left\{\rho\left(Ae^{a_1 z}\right), \rho\left(B(z)\, e^{a_2 z}\right), \rho\left(F(z)\, e^{a_1 z}\right)\right\} = 1,$$

so by using Lemma 2.7, we obtain $\rho_2(f) \leq 1$.

Since $F \not\equiv 0$, then by Lemma 2.6, we get

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq 1.$$

4. Proof of Theorem 1.9

Suppose that f is a solution of equation (1.3). Then by Theorem 1.7, we have $\rho(f) = +\infty$ and $\rho_2(f) \leq 1$. First, we prove $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) \leq 1$. Differentiating both sides of expression (1.4)

$$g_f' = d_1 f'' + (d_1' + d_0) f' + d_0' f + b' \tag{4.1}$$

and replacing f'' with $f'' = F(z) e^{a_1 z} - A e^{a_1 z} f' - B(z) e^{a_2 z} f$, we obtain

$$g_f' - b' - d_1 e^{a_1 z} F = (d_1' + d_0 - A d_1 e^{a_1 z}) f' + \left(d_0' - d_1 B e^{a_2 z}\right) f. \tag{4.2}$$

Then by (1.4) and (4.2), we have

$$d_1 f' + d_0 f = g_f - b, \tag{4.3}$$

$$\alpha_1 f' + \alpha_0 f = g_f' - b' - d_1 e^{a_1 z} F, \tag{4.4}$$

where $\alpha_1 = d_1' + d_0 - A d_1 e^{a_1 z}$ and $\alpha_0 = d_0' - d_1 B e^{a_2 z}$. Set

$$\begin{aligned} h &= d_1 \alpha_0 - d_0 \alpha_1 = d_1 (d_0' - d_1 B e^{a_2 z}) - d_0 (d_1' + d_0 - A d_1 e^{a_1 z}) \\ &= d_1 d_0' - d_0 d_1' - d_0^2 - d_1^2 B e^{a_2 z} + A d_0 d_1 e^{a_1 z}. \end{aligned} \tag{4.5}$$

We prove $h \not\equiv 0$. We suppose the contrary. If $d_1 \not\equiv 0$, then by (4.5) and Lemma 2.1, we obtain

$$1 = \rho\left(d_1^2 B e^{a_2 z} - A d_0 d_1 e^{a_1 z}\right) = \rho\left(d_1 d_0' - d_0 d_1' - d_0^2\right) < 1$$

which is a contradiction. Thus $h \not\equiv 0$.

Now, if $d_1 \equiv 0$, $d_0 \not\equiv 0$, then by (4.5) we obtain $h = -d_0^2 \not\equiv 0$. Hence, $h \not\equiv 0$.

By $h \not\equiv 0$ and (4.3) – (4.5), we have

$$f = \frac{d_1 \left(g_f' - b' - d_1 e^{a_1 z} F\right) - \alpha_1 (g_f - b)}{h}. \tag{4.6}$$

If $\rho(g_f) < \infty$, then by (4.6), we get $\rho(f) < \infty$ and this is a contradiction. Hence $\rho(g_f) = \infty$.

Now, we prove that $\rho_2(g_f) = \rho_2(f)$. By (4.3), we get $\rho_2(g_f) \leq \rho_2(f)$ and by (4.6) we have $\rho_2(f) \leq \rho_2(g_f)$. This yield $\rho_2(g_f) = \rho_2(f) \leq 1$.

Set $w(z) = d_1 f' + d_0 f + b - \varphi$. Since $\rho(\varphi) < \infty$, then we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho_2(f)$. In order to prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f)$, we need to prove only $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(f)$. By $g_f = w + \varphi$, we get from (4.6)

$$f = \frac{d_1 w' - \alpha_1 w}{h} + \psi, \tag{4.7}$$

where

$$\psi = \frac{d_1 (\varphi' - b' - d_1 e^{a_1 z} F) - \alpha_1 (\varphi - b)}{h}.$$

If $d_1 \not\equiv 0$, then by substituting (4.7) into equation (1.3), we obtain

(4.8)
$$\frac{d_1}{h} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = e^{a_1 z} F - \left(\psi'' + A e^{a_1 z} \psi' + B(z) e^{a_2 z} \psi\right) = H,$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. Since $\psi(z)$ is of finite order, then it cannot be a solution of (1.3), it follows that $H \not\equiv 0$. Then by Lemma 2.6, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho_2(f)$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) \leq 1$.
If $d_1 \equiv 0$, then from (4.7) we have

$$f = \frac{-\alpha_1 w}{h} + \widetilde{\psi} = \frac{-d_0 w}{h} + \widetilde{\psi}, \tag{4.9}$$

where
$$\widetilde{\psi} = \frac{-d_0(\varphi - b)}{h}.$$

By substituting (4.9) into equation (1.3), we obtain

$$-\frac{d_0}{h} w'' + \widetilde{\phi}_1 w' + \widetilde{\phi}_0 w = e^{a_1 z} F - \left(\widetilde{\psi}'' + A e^{a_1 z}\ \widetilde{\psi}' + B(z) e^{a_2 z}\ \widetilde{\psi}\right) = \widetilde{H}, \tag{4.10}$$

where $\widetilde{\phi}_j$ $(j = 0, 1)$ are meromorphic functions with $\rho\left(\widetilde{\phi}_j\right) < \infty$ $(j = 0, 1)$. Since $\widetilde{\psi}(z)$ is of finite order, then it cannot be a solution of (1.3), it follows that $\widetilde{H} \not\equiv 0$. Then by Lemma 2.6, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(w) = \infty$, $\overline{\lambda}_2(w) = \lambda_2(w) = \rho_2(w) = \rho_2(f)$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(g_f) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(g_f) = \rho_2(f) \leq 1$.
By f is infinite order solution of equation (1.3) and Lemma 2.6 again, we have

$$\overline{\lambda}(f) = \lambda(f) = \overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f) = \lambda_2(f) = \overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) \leq 1$$

which completes the proof. If we put $\varphi(z) = z$, then we get

$$\overline{\lambda}(g_f - z) = \lambda(g_f - z) = \rho(f) = +\infty,\ \overline{\lambda}_2(g_f - z) = \lambda_2(g_f - z) = \rho_2(f) \leq 1.$$

5. Proof of Theorem 1.10

Suppose that f_1 is a solution of equation (1.5) and f_2 is a solution of equation (1.6). Set $w = f_1 - K f_2$. Then w is a solution of equation

$$w'' + A e^{a_1 z} w' + B(z) e^{a_2 z} w = (F_1 - K F_2) e^{a_1 z}.$$

By $\rho(F_1 - K F_2) < 1$, $F_1 - K F_2 \not\equiv 0$ and Theorem 1.7, we have $\rho(w) = \infty$ and $\rho_2(w) \leq 1$. Thus, by using Theorem 1.9, we obtain

$$\overline{\lambda}(w) = \lambda(w) = \overline{\lambda}(g_w - \varphi) = \lambda(g_w - \varphi) = \rho(w) = +\infty,$$
$$\overline{\lambda}_2(w) = \lambda_2(w) = \overline{\lambda}_2(g_w - \varphi) = \lambda_2(g_w - \varphi) = \rho_2(w) \leq 1,$$

that is

$$\overline{\lambda}\left(f_1-Kf_2\right)=\lambda\left(f_1-Kf_2\right)=\overline{\lambda}\left(g_{f_1-Kf_2}-\varphi\right)$$
$$=\lambda\left(g_{f_1-Kf_2}-\varphi\right)=\rho\left(f_1-Kf_2\right)=\infty$$

and

$$\overline{\lambda}_2\left(f_1-Kf_2\right)=\lambda_2\left(f_1-Kf_2\right)=\overline{\lambda}_2\left(g_{f_1-Kf_2}-\varphi\right)$$
$$=\lambda_2\left(g_{f_1-Kf_2}-\varphi\right)=\rho_2\left(f_1-Kf_2\right)\leq 1$$

for any complex constant K.

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Diferencijalni polinomi generirani rješenjima nehomogene linearne diferencijalne jednadžbe drugog reda

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Sažetak. Ovaj je članak posvećen proučavanju rasta i oscilacije rješenja nehomogene linearne diferencijalne jednadžbe drugog reda

$$f'' + Ae^{a_1 z}f' + B\,(z)\,e^{a_2 z}f = F\,(z)\,e^{a_1 z},$$

gdje su A, a_1, a_2 kompleksni brojevi, $B\,(z)$ $(\not\equiv 0)$ i $F\,(z)$ $(\not\equiv 0)$ su cijele funkcije s redom manjim od jedan. Nadalje, istražujemo rast i oscilacije nekih diferencijalnih polinoma generiranih rješenjima gornje jednadžbe.

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