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Weak Values Technique for Velocity Measurements

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In a recent letter, Brunner and Simon propose an interferometric scheme using imaginary weak values with a frequency-domain analysis to outperform standard interferometry in longitudinal phase shifts [N. Brunner and C. Simon, Phys. Rev. Lett **105** (2010)]. Here we demonstrate an interferometric scheme combined with a time-domain analysis to measure longitudinal velocities. The technique employs the near-destructive interference of non-Fourier limited pulses, one Doppler shifted due to a moving mirror, in a Michelson interferometer. We achieve a velocity measurement of 400 fm/s and show our estimator to be efficient by reaching its Cramér-Rao bound.

Introduction.– In information theory, the Cramér-Rao bound (CRB) [1, 2] is the fundamental limit in the minimum uncertainty for parameter estimation. Measurements of phase [3–5], beam deflection [6, 7], pulse arrival time [8], Doppler shift [9, 10] and velocity [11–14] are all fundamentally bounded by a CRB. If a measurement technique reaches the CRB, its estimator is said to be efficient.

Spurred by fundamental studies of quantum phenomena [15] and by developments in precision measurements [3, 5, 7, 16], the field of weak values [17–21] has become a powerful tool for parameter estimation [22–25]. The precision measurements inspired by weak values are not necessarily new or quantum. For example Zernike's phase contrast imaging [26], awarded the 1953 Nobel prize, can be classified as a weak value technique. An important aspect of the weak-values framework is that it provides a methodology for mitigating technical noise and amplifying an effect in one domain that is technologically difficult to observe in the conjugate domain.

Recently, Simon and Brunner showed that a weak-values technique allows us to observe a large spectral shift induced by a small temporal shift. They also showed their technique outperforms standard interferometry limited by technical noise [5]. Here we consider the opposite regime, where a small spectral shift causes a large temporal shift. The spectral shift in our experiment is a Doppler frequency shift produced by a moving mirror. Using established interferometry to measure velocities arriving at the CRB is difficult but achievable as seen in Ref [2]. Our technique is comparable to standard interferometry, but allows us to reach its' CRB with a relatively simple method in a regime where $1/f$ noise typically dominates. In this letter we show a weak-value optical technique to measure sub pm/s velocities. The protocal reaches the predicted CRB, averting technical noise and experimental imperfections.

Theoretical description.– The protocol, shown in Fig. 1, uses a non-Fourier limited Gaussian pulse (i.e., $c\tau \gg$ coherence length of the laser where τ is the length of the pulse). The pulse, with initial intensity profile $I_{in}(t) = I_0 \exp\left(-t^2/2\tau^2\right)$, is sent through a Michelson interferometer with a slowly moving mirror in one arm. The interferometer is tuned slightly off destructive interference by an amount 2ϕ, such that the output signal takes the form

$$I_{out}(t) \propto I_{in}(t) \left|1 - \exp\left(i2\phi + i2kx(t)\right)\right|^2$$
$$\propto I_0 \exp\left(-t^2/2\tau^2\right) \sin^2\phi \left|\frac{\sin\left(\phi + kvt\right)}{\sin\phi}\right|^2, \quad (1)$$

where $k = 2\pi/\lambda$, $x(t) = vt$ and v is the velocity of the mirror. Assuming $kv\tau \ll \phi$, making a small angle approximation of ϕ and re-exponentiating the output intensity, we obtain

$$I_{out}(t) \approx \left(I_0 \sin^2\phi\right) \exp\left[-\frac{1}{2\tau^2}\left(t - \frac{2kv\tau^2}{\phi}\right)^2\right]. \quad (2)$$

Near-destructive interference reduces the peak intensity of the pulse by a factor $\sin^2\phi$, which is the probability for a single photon passing through the interferometer to reach the detector. Importantly, a time shift in the peak output intensity, $\delta t = 2kv\tau^2/\phi$, has been induced with respect to the input. The velocity v can be obtained from measurements of the time shift δt.

We can rewrite the time shift in Eq. (2) in terms of the spectral shift $\delta t = 2kv\tau^2/\phi = 2\pi f_d\tau^2/\phi$ where the spectral shift, $f_d = 2v/\lambda$, of the pulse is proportional to velocity v. Instead of a direct spectral measurement, we obtain the velocity by measuring the induced time shift of the non-Fourier limited pulses. The time shift is amplified in the measurement of v which is accompanied by a decrease in the measured intensity. These two results are well-known properties of the interferometric weak value amplification technique. In fact, a full weak value description, using coherent Fourier limited pulses, can be formulated obtaining an identical result to Eq. (2). In our case, the use of non-Fourier limited pulses allows us to produce large time shifts regardless of the laser linewidth.

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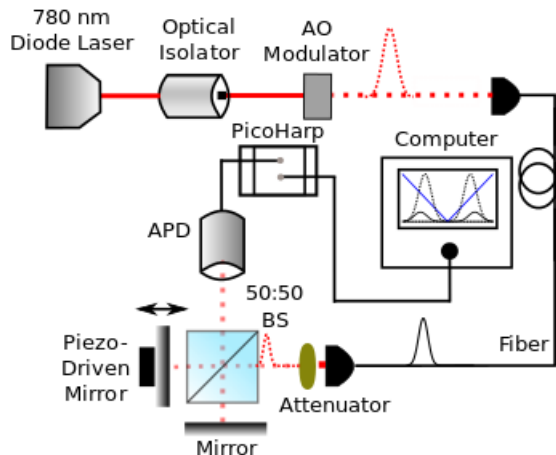† jrmartir@pas.rochester.edu

FIG. 1. An optical modulator generates a non-Fourier limited Gaussian shaped pulse. We couple the pulse to a fiber and launch it to a Michelson interferometer where one mirror is moving with constant speed v. The interference is controlled by inducing a phase offset 2ϕ with the piezoelectric mirror. Photons exiting the interferometer are coupled into a fiber (not shown) and the arrival time of single photons are measured with an avalanche photo diode (APD) and a photon counting module.

We now consider the fundamental limitations of our velocity measurement set by the CRB. The CRB is equal to the inverse of the Fisher information, the amount of information a random variable (arrival time of photons) provides about a parameter of interest (velocity). Assume that N photons are sent through the interferometer. We want to determine the shift δt from the set of $N\sin^2\phi$ independent measurements of photon arrival times. Such measurements follow the distribution $P(t;\delta t) = \left(2\pi\tau^2\right)^{-1/2}\exp\left[-(t-\delta t)^2/2\tau^2\right]$. The Fisher information is

$$F(\delta t) = N\sin^2\phi \int dt\, P(t;\delta t)\left[\frac{d}{d\,\delta t}\ln P(t;\delta t)\right]^2 \approx \frac{N\phi^2}{\tau^2}. \tag{3}$$

The CRB, F^{-1}, is the minimum variance [27, 28] of an unbiased estimation of δt. The sensitivity in the determination of δt is therefore bounded by $\Delta\left(\delta t\right) \geq \tau/\phi\sqrt{N}$. The error in the estimation of v is then bounded by

$$\Delta v_{CRB} = \frac{\Delta\left(\delta t\right)\phi}{2\,k\,\tau^2} = \frac{1}{2k\tau\sqrt{N}}. \tag{4}$$

Note that this minimum uncertainty is independent of the actual value of v measured. This also determines the smallest resolvable velocity, when the signal-to-noise ratio is unity. The signal-to-noise ratio is

$$\mathcal{SNR} = \frac{\delta t}{\tau}\phi\sqrt{N} = \frac{v}{\Delta v} = \frac{f_d}{\Delta f_d}. \tag{5}$$

Experiment.– We use a grating feedback laser with $\lambda \approx 780$ nm. An acoustic optical modulator creates Gaussian pulses of length τ which we couple into a fiber. We launch them through the 50:50 beam splitter (BS) of the interferometer. The piezoelectric actuated mirror is driven by a triangle function with frequency f_m and peak-to-peak voltage V_{pp}. The pulse length is smaller than half the oscillating mirror period, so that a pulse experiences a single, constant velocity. An opposite constant velocity is observed for each sequential pulse because the sign depends on whether the mirror moves toward the BS, or recedes away (see Fig. 1). The piezoelectric response α is calibrated by varying the voltage to change the dark port to a bright port. The piezo response was found to be $\alpha \approx 27$ pm/mV for a low frequency-voltage product. The arm lengths (beam splitter-mirror distances) are approximately 1 mm (not including the BS size) to ensure long term phase stability. Photon arrival times are recorded with an avalanche photon diode (APD) and a photon counting module (PicoQuant PicoHarp 300). The detector collects arrival times with 350 ps resolution.

To calibrate the experiment we record the number of detected photons entering the interferometer, N. Then, the piezo-driven mirror is biased near destructive interference and fed a triangle signal. We calculate the mean and error of the arrival time of the N_ϕ detected photons for each set of pulses. The mean of the Gaussian determines the time shift δt from which the velocity is extracted, and the angle $\phi \approx \sqrt{N_\phi/N}$ is calculated. Lastly, to reach the CRB, we attenuate the peak of the pulses to about a million photons a second.

Results.– We present velocity measurements v as a function of the pulse width τ for different amplitudes on the moving mirror in Fig. 2. The lines are the theoretical predictions, $v = 2f_m V_{pp}\alpha$, where $2f_m = 1/6\tau$. The mirror voltages are $V_{pp} = \{105, 52.5, 26.25, 10.5\}$ mV and angle is $\phi = 0.31 \pm 0.02$ rad. The results agree well with the theoretical predictions. The smallest measurement of velocity in Fig. 2 is $v = 60 \pm 11$ pm/s. The angle $\phi = 0.31$ might seem large; however, comparing the exact form in Eq (1), $|\sin(\phi+kvt)/\sin(\phi)|$, to the approximation, $|\exp(kvt/\phi)|$, shows a discrepancy less than 1% for the experimental parameters.

The uncertainties of the measurements in Fig. 2 are plotted separately in Fig. 3 and compared to the CRB Eq. (4). The error matches the CRB, thus the estimator is efficient and no other estimator can produce smaller uncertainties. This technique did not require noise filters or frequency locking to reach the fundamental uncertainty in the mean arrival time of the photons. In addition, the fluctuations in the post selection angle ϕ are negligible. Therefore, our velocity measurement is fundamentally bounded by its CRB.

It is important to note our CRB is scaled by the maximum number of detected photons N. The collection-detection efficiency is about 20% due to the 50:50 BS (not shown in Fig.1) located before the APD used for alignment of the dark port, the efficiencies of the APD
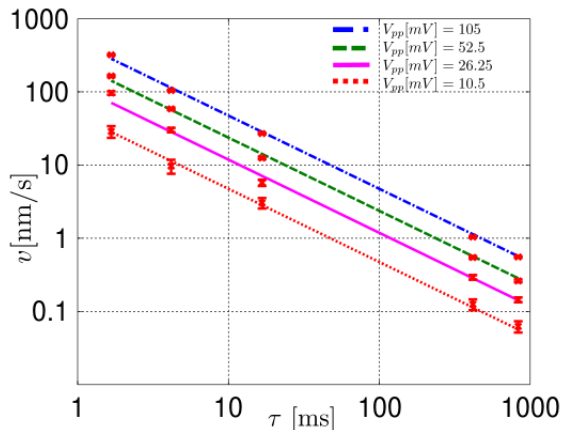
FIG. 2. (color online). The Doppler shift, f_d, is plotted as a function of $\tau = \{1.67, 4.17, 16.7, 417 \text{ and } 833\}$ ms. The phase offset angle is $\phi = 0.31$ radians. The points are the experimental results and the lines are the theoretical predictions for different voltages. Signal-to-noise ratios are 54, 27.4, 14.7, and 5.7 for $Vpp = \{105, 52.5, 26.25, 10.5\}$ mV respectively.

FIG. 3. Experimental error in Fig. 2 as a function of $\tau\sqrt{N}$. The solid line is the CRB as in Eq. (4) for $N \approx 54 \times 10^6$ photons. Note there are no error bars.

and the fiber coupling. Our calculations do not take the collection-detection efficiency into account.

The results show precise and accurate detection of velocity measurements in the pm/s range. Results from Fig. 2 show smaller velocities can be measured with longer pulses.

Now we seek to achieve the smallest velocities without the concern of reaching the CRB. Consider the temporal shift, advance or delay, of the pulse exiting the interferometer. Since the peak of the pulse is sufficient to detect a the shift, we require a small region around the peak to determine the shift. This allows the use of effectively large values of τ without requiring long term interferometric stability. Since the pulses are non-Fourier transform limited it is not necessary to use an entire Gaussian pulse. We truncate the Gaussian pulse to a width of τ, that is $2f_m = 1/\tau$. In other words, the light intensity into the interferometer never drops below the 88% of the peak intensity and there is 12% peak to peak intensity variation following the peak of the Gaussian profile.

We show velocities in the sub pm/s range using truncated pulses in Table I. The mirror frequency was set to 10 mHz, which corresponds to $\tau = 50$ s, and data was taken for voltages peak to peak, $V_{pp} = \{2, 1, 0.5\}$ mV, for the piezo driving the mirror. Data was collected in intervals of 10 minutes (due to drift instability in intensity), and 13 sets of data were taken for each voltage. We did a Gaussian fit for each 10 minute interval. The time shift and its error were found as the mean and standard deviation respectfully of the 13 time shifts obtained. The time shift was in the 10 s of millisecond and corresponds to small Doppler shifts in the mircoHz range. This leads to the best technical noise limited measurement of (400 ± 400) fm/s. Nevertheless, both accuracy and precision are lost due to numerically fitting the trun-
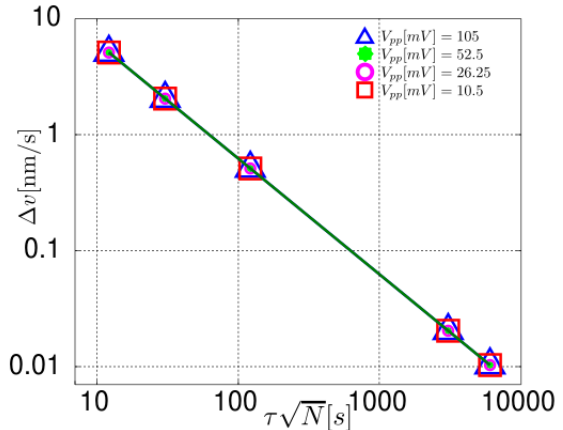
cated distributions. Note that the measurements are all relative velocities because of the oscillating mirror. In one period there would be two pulses each with opposite but equal speeds.

The results remain consistent with the full Gaussian picture theory, Eq. (2), but not with the CRB theory in Eq. (4). Calculating the mean arrival time of the photons is not a good estimator of the time shift because we lack the full Gaussian pulse profile. Therefore we numerically fit the data to a unnormalized function $A \exp\left[-(t - \delta t)^2/2\tau^2\right]$, the shift δt is extracted and the velocity, v, is backed out.

V_{pp} [mV]	ϕ [± 0.002 rad]	f_d [μHz]	v [pm/s]
2.0	0.275	3.6 ± 1.2	1.4 ± 0.5
1.0	0.276	1.6 ± 1.1	0.6 ± 0.4
0.5	0.279	1 ± 1	0.4 ± 0.4

TABLE I. Results of the cut Gaussian profile with $\tau = 50$ s and $N \approx 66 \times 10^9$. The collection-detection efficiency is about 20%. The error is from the statistics of numerically fitting each run. Integration time was about two hours worth of data.

Conclusion.– In this letter, we show using non-Fourier limited pulses and standard interferometry inspired by weak values, sub pm/s velocities can be measured. Using a Michelson interferometer tuned near a dark port we measure velocities as low as 400 ± 400 fm/s. We accomplished sub pm/s velocity detection by bypassing the technical noise that flood intensity detectors to reach the CRB. The uncertainty of the phase measurement is negligible when compared to the uncertainty of v for our parameter values so our uncertainty is the fundamental limit. Finally the error in our measurement of v matches the predicted CRB making this estimator efficient and the ultimate limit in uncertainty for velocity measure-

ments.

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