3-12-2021

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Recommended Citation

Alpay, D., Colombo, F., Pinton, S. *et al.* Infinite-order Differential Operators Acting on Entire Hyperholomorphic Functions. *J Geom Anal* **31**, 9768–9799 (2021). https://doi.org/10.1007/s12220-021-00627-y

Infinite order differential operators acting on entire hyperholomorphic functions

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October 13, 2020

Abstract

Infinite order differential operators appear in different fields of Mathematics and Physics and in the last decades they turned out to be of fundamental importance in the study of the evolution of superoscillations as initial datum for Schrödinger equation. Inspired by the operators arising in quantum mechanics, in this paper we investigate the continuity of a class of infinite order differential operators acting on spaces of entire hyperholomorphic functions. The two classes of hyperholomorphic functions, that constitute a natural extension of functions of one complex variable to functions of paravector variables are illustrated by the Fueter-Sce-Qian mapping theorem. We show that, even though the two notions of hyperholomorphic functions are quite different from each other, entire hyperholomorphic functions with exponential bounds play a crucial role in the continuity of infinite order differential operators acting on these two classes of entire hyperholomorphic functions. We point out the remarkable fact that the exponential function of a paravector variable is not in the kernel of the Dirac operator but entire monogenic functions with exponential bounds play an important role in the theory.

AMS Classification: 32A15, 32A10, 47B38.

Key words: Infinite order differential operators, Slice hyperholomorphic functions, functions in the kernel of the Dirac operator, entire functions with growth conditions, spaces of entire functions.

1 Introduction

Infinite order differential operators turned out to be of fundamental importance in the study of the evolution of superoscillations as initial datum for Schrödinger equation. To study the evolution of superoscillatory functions under Schrödinger equation is highly nontrivial and a natural functional setting is the space of entire functions with growth conditions, for more details see the monograph [8] and [20]. In fact, the Cauchy problem for Schrödinger equation with superoscillatory initial datum leads to infinite order differential operators of the type

$$\mathcal{U}(t, x; D_x) = \sum_{m=1}^{\infty} u_m(t, x) D_x^m,$$

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where the coefficients $u_m(t,x)$ depend on the Green function of the Schrödinger equation with the potential V, and t and x are the time and the space variables. According to the structure of the Green function the coefficients $u_m(t,x)$ satisfy given growth conditions. For some potentials V we are forced to consider infinite order differential operators $\mathcal{P}(t,x;D_\xi)$ depending on an auxiliary complex variable ξ

$$\mathcal{P}(t,x;D_\xi)=\sum_{m=1}^{\infty}u_m(t,x)D_\xi^m,$$

with coefficients $u_m(t,x)$ that depend on V. The continuity properties of the operators $\mathcal{U}(t,x;D_x)$ or $\mathcal{P}(t,x;D_\xi)$ acting on the spaces of entire functions with exponential bounds are the heart of the study of the evolution of superoscillatory initial datum in quantum mechanics. For $p\geq 1$ the natural spaces on which such the operators $\mathcal{U}(t,x;D_x)$ and $\mathcal{P}(t,x;D_\xi)$ act are the spaces of entire functions with either order lower than p or order equal to p and finite type. They consist of entire functions f for which there exist constants $B,C>0$ such that $|f(z)|\leq Ce^{B|z|^p}$.

This paper is devoted to a double audience: for researchers working in complex and hypercomplex analysis and for experts working in the area of infinite order differential operators. More precisely, we investigate the continuity of a class of infinite order differential operators acting on spaces of entire hyperholomorphic functions. There are two main classes of hyperholomorphic functions that constitute the natural extension of functions of one complex variable to functions of paravector variables as illustrated by the Fueter-Sce-Qian mapping theorem, as it is recalled in the last section of this paper. From this theorem naturally emerge the slice hyperholomorphic functions and the functions in the kernel of the Dirac operator that are called monogenic functions. We show that, even though the two notions of hyperholomorphic functions are quite different from each other, and the exponential function is not in the kernel of the Dirac operator, hyperholomorphic functions with exponential bounds play a crucial role in the continuity of a class of infinite order differential operators in the hypercomplex settings. The complex version of these results where studied in the paper [17].

In the following we denote by $\mathbb{R}_n$ the real Clifford algebra over n imaginary units $e_1,\dots,e_n$ satisfying the relations $e_\ell e_m+e_m e_\ell=0$, $\ell\neq m$, $e_\ell^2=-1$. An element $(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}$ will be identified with the element $x=x_0+\underline{x}=x_0+\sum_{\ell=1}^n x_\ell e_\ell\in\mathbb{R}_n$ which is called paravector. We denote by $\mathbb{S}$ the sphere $\mathbb{S}=\{\underline{x}=e_1x_1+\ldots+e_nx_n \mid x_1^2+\ldots+x_n^2=1\}$, we observe that for $\mathbf{j}\in\mathbb{S}$ we obviously have $\mathbf{j}^2=-1$, with the imaginary unit $\mathbf{j}$ we obtain the complex plane $C_{\mathbf{j}}$ whose elements are of the form $u+\mathbf{j}v$ for u, $v\in\mathbb{R}$. The first class of hyperholomorphic functions we consider are called slice hyperholomorphic (or slice monogenic functions) and are defined as follows. Let $U\subseteq\mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U}=\{(u,v)\in\mathbb{R}^2: u+\mathbb{S}v\subset U\}$. A function $f:U\to\mathbb{R}_n$ is called a left slice function, if it is of the form

$$f(q)=f_0(u,v)+\mathbf{j}f_1(u,v)\qquad\text{for}\qquad(u,v)\in\mathcal{U}$$

where $q=u+\mathbf{j}v$ and the two functions $f_0,f_1:\mathcal{U}\to\mathbb{R}_n$ satisfy the compatibility conditions

$$f_0(u,-v)=f_0(u,v),\qquad f_1(u,-v)=-f_1(u,v).\tag{1}$$

If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equations then f is called left slice monogenic functions. A similar notion is given in the sequel for right slice monogenic functions.

The second class of hyperholomorphic functions that we consider consists of the monogenic functions. Left monogenic functions are define as those functions $f:U\subseteq\mathbb{R}^{n+1}\to\mathbb{R}_n$ that are

C^1 and that are in the kernel of the Dirac operator $\mathcal{D}$ defined as:

$$\mathcal{D} := \frac{\partial}{\partial_{x_0}} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial_{x_i}},$$

that is, $\mathcal{D}f(x) = 0$. Also in this case there exists the notion of right monogenic functions.

Let us point out the main differences between the two function theories in order to appreciate the analogies with respect to infinite order differential operators.

(A) The pointwise product of two hyperholomorphic functions, in general, is not hyperholomorphic, so we need to define the product in a way that preserves the hyperholomorphicity. Given two entire left slice monogenic functions f and g, then their star-product (or slice hyperholomorphic product) is defined by

$$(f \star_L g)(x) = \sum_{\ell=0}^{+\infty} x^{\ell} \sum_{k=0}^{\ell} a_k b_{\ell-k}. \tag{2}$$

where $f(x) = \sum_{k=0}^{+\infty} x^k a_k$ and $g(x) = \sum_{k=0}^{+\infty} x^k b_k$.

When we deal with monogenic functions the Fueter's polynomials $V_k(x)$ defined by

$$V_k(x) := \frac{k!}{|k|!} \sum_{\sigma \in perm(k)} z_{j_{\sigma(1)}} z_{j_{\sigma(2)}} \dots z_{j_{\sigma(|k|)}},$$

where here k is a multi-index, play the same role as the monomials x^k, for $k \in \mathbb{N}_0$, of the paravector variable x for slice monogenic functions. The CK-product of two left entire monogenic f and g is defined by

$$f \odot_L g := \sum_{|k|=0}^{+\infty} \sum_{|j|=0}^{+\infty} V_{k+j}(x) f_k g_j,$$

where $f(x) = \sum_{|k|=0}^{\infty} V_k(x) f_k$, and $g(x) = \sum_{|k|=0}^{\infty} V_k(x) g_k$ are given in terms of $V_k(x)$.

(B) It is also possible to define slice hyperholomorphic functions, as functions in the kernel of the first order linear differential operator, introduced in [33], and defined by

$$\mathcal{G}f = \Big(|\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \Big) f = 0,$$

where $\underline{x} = x_1 e_1 + \ldots + x_n e_n$. The interesting observe that the operator $\mathcal{G}$ is linear nonconstant coefficients differential operator while the Dirac operator is linear but with constant coefficients.

(C) The contour integral, in the Cauchy formula of slice monogenic functions and their derivatives, is computed on a the complex plane $C_{\mathbf{j}}$ in $\mathbb{R}^{n+1}$ (for $\mathbf{j} \in \mathbb{S}$). Such contour is the boundary of $U \cap C_{\mathbf{j}}$, where the regular domain U is contained in $\mathbb{R}^{n+1}$ and is contained in a set where f is slice monogenic. For the monogenic case the integral, in the Cauchy formula for monogenic functions and their derivatives, is computed on the boundary of $U \subset \mathbb{R}^{n+1}$ where $\overline{U}$ is contained in the set of monogenicity of f.

(D) For slice monogenic functions there exists two different Cauchy kernels according to left and right slice hyperholomorphicity, while left and right monogenic functions have same the Cauchy kernel.

The main results are summarized as follows. In the sections 2 and 4 we collect the preliminary results on function spaces of entire slice monogenic functions and of entire monogenic functions with growth conditions, respectively. These results are of crucial importance in order to study the continuity properties of a class of infinite order differential operators acting on entire slice monogenic and monogenic functions, that are treated in sections 3 and 5, respectively. We conclude this section with an overview of some of the main results.

(I) Consider the formal infinite order differential operator

$$U_L(x, \partial_{x_0})f(x) := \sum_{m=0}^{\infty} u_m(x) \star_L \partial_{x_0}^m f(x),$$

defined on entire left slice monogenic functions f, where $\star_L$ denotes the hyperholomorphic product. Suppose that $(u_m)_{m\in\mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ is a sequence of entire left slice monogenic functions. Assume that $(u_m)_{m\in\mathbb{N}_0}$ satisfy the condition such that there exists a constant $B > 0$ so that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ for which

$$|u_m(x)| \leq C_\varepsilon \frac{\varepsilon^m}{(m!)^{1/q}} \exp(B|x|^p), \quad \text{for all} \quad m \in \mathbb{N}_0, \tag{3}$$

where $1/p + 1/q = 1$ and $1/q = 0$ when $p = 1$. Then in Theorem 3.2 we show that for $p \geq 1$ the operator $U_L(x, \partial_{x_0})$ acts continuously on the space of entire left slice monogenic functions with the growth condition $|f(x)| \leq Ce^{B|x|^p}$. In the same theorem we also considered right slice monogenic functions.

(II) For monogenic functions we let $p \geq 1$ and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(u_m)_{m\in(\mathbb{N}_0)^n} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be left entire monogenic functions such that there exists a constant $B > 0$ such that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ for which

$$|u_m(x)| \leq C_\varepsilon \frac{\varepsilon^{|m|}}{(|m|!)^{1/q}} \exp(B|x|^p), \quad \text{for all} \quad m \in (\mathbb{N}_0)^n, \tag{4}$$

where $1/p+1/q = 1$ and $1/q = 0$ when $p = 1$, and we observe that in this case m is a multi-index. We define the formal infinite order differential operator

$$U_L(x, \partial_x)f(x) := \sum_{|m|=0}^{\infty} u_m(x) \odot_L \partial_x^m f(x),$$

for left entire monogenic functions f where $\partial_x^m := \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n}$ and $\odot_L$ denotes the CK-product. Then for $p \geq 1$, in Theorem 5.3, we prove that the operator $U_L(x, \partial_x)$ acts continuously on the space of left monogenic functions with the condition $|f(x)| \leq Ce^{B|x|^p}$.

Even though the two classes of hyperholomorphic functions have very different Taylor series expansions they have strong similarities with respect to the action of infinite order differential operators when we assume similar growth conditions on the coefficients of the operators. The results are even more surprising because of the exponential bounds $|f(x)| \leq Ce^{B|x|^p}$ is used for both classes of functions even though the function $f(x) = e^{Bx}$, for $B \in \mathbb{R}$, is slice monogenic but it is not monogenic.

2 Function spaces of entire slice monogenic functions

In this section we recall some results on slice monogenic functions (see Chapter 2 in [37]) and we prove some important properties of entire slice monogenic functions that appear here for the first time. We recall that $\mathbb{R}_n$ is the real Clifford algebra over n imaginary units $e_1, \dots, e_n$. The element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the paravector $x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^{n} x_\ell e_\ell$ and the real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. An element in $\mathbb{R}_n$, called a *Clifford number*, can be written as

$$a = a_0 + a_1 e_1 + \dots + a_n e_n + a_{12} e_1 e_2 + \dots + a_{123} e_1 e_2 e_3 + \dots + a_{12\dots n} e_1 e_2 \dots e_n.$$

Denote by A an element in the power set $P(1, \dots, n)$. If $A = i_1 \dots i_r$, then the element $e_{i_1} \dots e_{i_r}$ can be written as $e_{i_1 \dots i_r}$ or, in short, e_A. Thus, in a more compact form, we can write a Clifford number as

$$a = \sum_A a_A e_A.$$

Possibly using the defining relations, we will order the indices in A as $i_1 < \dots < i_r$. When $A = \emptyset$ we set $e_\emptyset = 1$. The Euclidean norm of an element $y \in \mathbb{R}_n$ is given by $|y|^2 = \sum_A |y_A|^2$, in particular the norm of the paravector $x \in \mathbb{R}^{n+1}$ is $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^{n} x_\ell e_\ell$. Recall that $\mathbb{S}$ is the sphere

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \dots + e_n x_n \mid x_1^2 + \dots + x_n^2 = 1\};$$

so for $\mathbf{j} \in \mathbb{S}$ we have $\mathbf{j}^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us define $\mathbf{j}_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} \ : \ y = x_0 + \mathbf{j}|\underline{x}|, \ \mathbf{j} \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R} + \mathbf{j}\mathbb{R}$ passing through 1 and $\mathbf{j} \in \mathbb{S}$ will be denoted by $\mathbb{C}_\mathbf{j}$ and an element belonging to $\mathbb{C}_\mathbf{j}$ will be indicated by $u + \mathbf{j}v$, for $u, v \in \mathbb{R}$. With an abuse of notation we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector x. We say that $U \subseteq \mathbb{R}^{n+1}$ is axially symmetric if $[x] \subset U$ for any $x \in U$.

Definition 2.1 (Slice hyperholomorphic functions with values in $\mathbb{R}_n$ (or slice monogenic functions)). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{R}_n$ is called a left slice function, if it is of the form*

$$f(q) = f_0(u, v) + \mathbf{j} f_1(u, v) \qquad \text{for } q = u + \mathbf{j}v \in U$$

where the two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ satisfy the compatibility conditions

$$f_0(u, -v) = f_0(u, v), \qquad f_1(u, -v) = -f_1(u, v). \tag{5}$$

If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equations

$$\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0 \tag{6}$$

$$\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0, \tag{7}$$

then f is called left slice hyperholomorphic (or left slice monogenic). A function $f : U \to \mathbb{R}_n$ is called a right slice function if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)\mathbf{j} \qquad \textit{for } q = u + \mathbf{j}v \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy (5). If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equation, then f is called right slice hyperholomorphic (or right slice monogenic).

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic. We denote the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SM}_L(U)$ and $\mathcal{SM}_R(U)$, respectively. When we do not distinguish between left of right we indicate the space $\mathcal{SM}(U)$.

Definition 2.2. *Let $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ and let $x = u + \mathbf{j}v \in U$. If x is not real, then we say that f admits left slice derivative in x if*

$$\partial_S f(x) := \lim_{p\to x,\, p\in\mathbb{C}_\mathbf{j}} (p-x)^{-1}(f_\mathbf{j}(p) - f_\mathbf{j}(x)) \tag{8}$$

exists and is finite. If x is real, then we say that f admits left slice derivative in x if (8) exists for any $\mathbf{j} \in \mathbb{S}$. Similarly, we say that f admits right slice derivative in a nonreal point $x = u + \mathbf{j}v \in U$ if

$$\partial_S f(x) := \lim_{p\to x,\, p\in\mathbb{C}_\mathbf{j}} (f_\mathbf{j}(p) - f_\mathbf{j}(x))(p-x)^{-1} \tag{9}$$

exists and is finite, and we say that f admits right slice derivative in a real point $x \in U$ if (9) exists and is finite, for any $\mathbf{j} \in \mathbb{S}$.

Remark 2.3. Observe that $\partial_S f(x)$ is uniquely defined and independent of the choice of $\mathbf{j} \in \mathbb{S}$ even if x is real. If f admits slice derivative, then $f_\mathbf{j}$ is $\mathbb{C}_\mathbf{j}$-complex left resp. right differentiable and we find

$$\partial_S f(x) = f_\mathbf{j}'(x) = \frac{\partial}{\partial u} f_\mathbf{j}(x) = \frac{\partial}{\partial u} f(x), \quad x = u + \mathbf{j}v. \tag{10}$$

Theorem 2.4. *Let $a \in \mathbb{R}$, let $r > 0$ and let $B_r(a) = \{x \in \mathbb{H} : |x - a| < r\}$. If $f \in \mathcal{SM}_L(B_r(a))$, then*

$$f(x) = \sum_{k=0}^{+\infty} (x-a)^k \frac{1}{k!} \partial_S^k f(a) \qquad \forall x = u + \mathbf{j}v \in B_r(a). \tag{11}$$

If on the other hand $f \in \mathcal{SM}_R(B_r(a))$, then

$$f(x) = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\partial_S^k f(a)\right) (x-a)^k \qquad \forall x = u + \mathbf{j}v \in B_r(a).$$

We now recall the natural product that preserves slice monogenicity of functions admitting power series expansion as shown by Theorem 2.4.

Definition 2.5. *Let $f(x) = \sum_{k=0}^{+\infty} x^k a_k$ and $g(x) = \sum_{k=0}^{+\infty} x^k b_k$ be two left slice monogenic power series, the left-star product, denoted by $\star_L$, is defined by*

$$(f \star_L g)(x) = \sum_{\ell=0}^{+\infty} x^\ell \left(\sum_{k=0}^{\ell} a_k b_{\ell-k} \right). \tag{12}$$

Similarly, for right slice monogenic power series $f(x) = \sum_{k=0}^{+\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{+\infty} b_k x^k$ the right-star product, denoted by $\star_R$, is defined by

$$(f \star_R g)(x) = \sum_{\ell=0}^{+\infty} \left(\sum_{k=0}^{\ell} a_k b_{\ell-k} \right) x^{\ell}. \tag{13}$$

The Cauchy formula of slice monogenic functions has two different Cauchy kernels according to left or right slice monogenicity. Let $x, s \in \mathbb{R}^{n+1}$, with $x \notin [s]$, be paravectors then the slice monogenic Cauchy kernels are defined by

$$S_L^{-1}(s,x) := -(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}(x - \overline{s}),$$

and

$$S_R^{-1}(s,x) := -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}.$$

Theorem 2.6 (The Cauchy formulas for slice monogenic functions)**.** *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $\partial(U \cap \mathbb{C}_{\mathbf{j}})$ is a finite union of continuously differentiable Jordan curves for every $\mathbf{j} \in \mathbb{S}$ and set $ds_{\mathbf{j}} = -ds\mathbf{j}$ for $\mathbf{j} \in \mathbb{S}$. Let f be a slice monogenic function on an open set that contains $\overline{U}$ and set $x = x_0 + \underline{x}$, $s = s_0 + \underline{s}$. Then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} S_L^{-1}(s,x)\, ds_{\mathbf{j}}\, f(s), \qquad \textit{for any} \quad x \in U. \tag{14}$$

If f is a right slice monogenic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} f(s)\, ds_{\mathbf{j}}\, S_R^{-1}(s,x), \qquad \textit{for any} \quad x \in U. \tag{15}$$

Moreover, the integrals depend neither on U nor on the imaginary unit $\mathbf{j} \in \mathbb{S}$.

Theorem 2.7 (Derivatives of slice monogenic functions)**.** *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $\partial(U \cap \mathbb{C}_{\mathbf{j}})$ is a finite union of continuously differentiable Jordan curves for every $\mathbf{j} \in \mathbb{S}$ and set $ds_{\mathbf{j}} = -ds\mathbf{j}$ for $\mathbf{j} \in \mathbb{S}$. Let f be a left slice monogenic function on an open set that contains $\overline{U}$ and set $x = x_0 + \underline{x}$, $s = s_0 + \underline{s}$. Then the slice derivatives $\partial_{x_0}^k f(x)$ are given by*

$$\partial_{x_0}^k f(x) = \frac{k!}{2\pi} \int_{\partial(U \cap \mathbb{C}_{\mathbf{j}})} (x^2 - 2s_0 x + |s|^2)^{-k-1} (x - \overline{s})^{*(k+1)} ds_{\mathbf{j}} f(s), \tag{16}$$

where

$$(x - \overline{s})^{*k} = \sum_{m=0}^{k} \frac{k!}{(k-m)!m!} x^{k-m} \overline{s}^m, \tag{17}$$

Moreover, the integral depends neither on U nor on the imaginary unit $\mathbf{j} \in \mathbb{S}$.

A similar formula holds also for right slice monogenic functions.

After the basic facts on slice monogenic functions we can introduce some function spaces of entire slice monogenic functions in the spirit or the quaternionic version introduced in the book [35]. Let f be a non-constant entire monogenic function. We define

$$M_{f_{\mathbf{j}}}(r) = \max_{|z|=r, x \in \mathbb{C}_{\mathbf{j}}} |f(x)|, \quad \text{for} \;\; r \geq 0$$

and

$$M_f(r) = \max_{|x|=r} |f(x)|, \quad \text{for} \;\; r \geq 0.$$

Then, see Chapter 5 [35], we have for intrinsic functions that $M_{f_{\mathbf{j}}}(r) = M_f(r)$.

Definition 2.8. *Let f be an entire slice monogenic function. Then we say that f is of finite order if there exists $\kappa > 0$ such that*

$$M_f(r) < e^{r^\kappa}$$

for sufficiently large r. The greatest lower bound ρ of such numbers κ is called order of f. Equivalently, we can define the order as

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}.$$

Definition 2.9. *Let f be an entire slice monogenic function of order ρ and let $A > 0$ be such that for sufficiently large values of r we have*

$$M_f(r) < e^{Ar^\rho}.$$

We say that f of order ρ is of type σ if σ is the greatest lower bound of such numbers and we have

$$\sigma = \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}.$$

Moreover:

- *When $\sigma = 0$ we say that f is of minimal type.*
- *When $\sigma = \infty$ we say that f is of maximal type.*
- *When $\sigma \in (0, \infty)$ we say that f is of normal type.*

The constant functions are said to be of minimal type of order zero.

Definition 2.10. *Let $p \geq 1$. We denote by $\mathcal{SM}^p$ the space of entire slice monogenic functions with either order lower than p or order equal to p and finite type. It consists of those functions $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$, for which there exist constants $B, C > 0$ such that*

$$|f(x)| \leq Ce^{B|x|^p}. \tag{18}$$

Let $(f_m)_{m\in\mathbb{N}}$, $f_0 \in \mathcal{SM}^p$. Then $f_m \to f_0$ in $\mathcal{SM}^p$ if there exists some $B > 0$ such that

$$\lim_{m\to\infty} \sup_{x\in\mathbb{R}^{n+1}} \left| (f_m(x) - f_0(x))e^{-B|x|^p} \right| = 0. \tag{19}$$

Functions in $\mathcal{SM}^p$ that are left slice monogenic will be denoted by $\mathcal{SM}^p_L$, while right slice monogenic will be denoted by $\mathcal{SM}^p_R$.

We now give a characterization of functions in $\mathcal{SM}^p$ in terms of their Taylor coefficients. In order to prove our results we need some very well known estimates on the Gamma function Γ and on the binomial. We collect them in the following lemma.

Lemma 2.11. *We have the following estimates:*

(I) For j, $k \in \mathbb{N}$, the we have $(j + k)! \leq 2^{j+k} j!k!$.

(II) For n, $k \in \mathbb{N}$, the we have $\Gamma(n + 1)\Gamma(k + 1) \leq \Gamma(n + k + 2)$.

(III) For $q \in [1, \infty)$ and $n \in \mathbb{N}$ we have $\Gamma\left(\frac{n}{q} + 1\right) \leq (n!)^{1/q}$.

(IV) $(a+b)^p \le 2^p(a^p+b^p), \quad a>0, \quad b>0, \quad p>0.$

Lemma 2.12. *Let $x \in \mathbb{R}^{n+1}$ then the Mittag-Leffler function*

$$E_{\alpha,\beta}(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}$$

is an entire slice monogenic function of order $1/\alpha$ (and of type 1) for $\alpha>0$ and $Re(\beta)>0$.

Proof. The proof follows the same line as in complex case. □

We are now ready to prove a crucial result which is the slice monogenic version of the complex version proved in [17].

Lemma 2.13. *Let $p \ge 1$. A function*

$$f(x) = \sum_{k=0}^{\infty} x^k \alpha_k$$

belongs to $\mathcal{SM}_L^p$ if and only if there exist constants $C_f, b_f > 0$ such that

$$|\alpha_k| \le C_f \frac{b_f^k}{\Gamma(\frac{k}{p}+1)}. \tag{20}$$

Furthermore, a sequence f_m in $\mathcal{SM}_L^p$ tends to zero if and only if $C_{f_m} \to 0$ and $b_{f_m} < b$ for some $b>0$.

Proof. We consider the case of left slice monogenic functions the prove for right slice monogenic functions follows the same lines. We first prove that if $f \in \mathcal{SM}_L^p$ we have the estimates (20) on the coefficients α_k, for $k \in \mathbb{N}_0$. Observe that the kernel

$$(x,s) \mapsto (x^2 - 2s_0x + |s|^2)^{-k-1}(x-\overline{s})^{*(k+1)}$$

in formula (16) can be decomposed by the Representation formula. Moreover, the zeros of the function $x \mapsto x^2 - 2s_0x + |s|^2$ consist of real points or of a 2-sphere. In fact, on $\mathbb{C}_{\mathbf{j}_x}$ we find only the point x as a singularity and the result follows from the Cauchy formula on the plane $\mathbb{C}_{\mathbf{j}_x}$. In the complex plane $\mathbb{C}_{\mathbf{j}}$ for $\mathbf{j} \neq \mathbf{j}_x$ if the singularities are real we obtain again the Cauchy formula of complex analysis. It the zeros are not real and $\mathbf{j} \neq \mathbf{j}_x$ then on any complex plane $C_{\mathbf{j}}$ we find the two zeros $s_{1,2} = x_0 \pm \mathbf{j}|\underline{x}|$ in this case using the representation formula we have the decomposition:

$$(x^2 - 2s_0x + |s|^2)^{-k-1}(x-\overline{s})^{*(k+1)} = \frac{1-\mathbf{ij}}{2}\ \frac{1}{(s-w)^{k+1}} + \frac{1+\mathbf{ij}}{2}\ \frac{1}{(s-\overline{w})^{k+1}}$$

for $x = u + \mathbf{i}v$ and $w = u + \mathbf{j}v$. So the integral representation of the derivatives becomes

$$\partial_{x_0}^k f(x) = \frac{k!}{2\pi}\int_{\partial(U\cap\mathbb{C}_{\mathbf{j}})} \Big(\frac{1-\mathbf{ij}}{2}\ \frac{1}{(s-w)^{k+1}} + \frac{1+\mathbf{ij}}{2}\ \frac{1}{(s-\overline{w})^{k+1}}\Big) ds_{\mathbf{j}} f(s)$$

and also

$$\begin{aligned}\partial_{x_0}^k f(x) &= \frac{1-\mathbf{ij}}{2}\ \frac{k!}{2\pi}\int_{\partial(U_1\cap\mathbb{C}_{\mathbf{j}})} \frac{1}{(s-w)^{k+1}} ds_{\mathbf{j}} f(s) \\ &+ \frac{1+\mathbf{ij}}{2}\ \frac{k!}{2\pi}\int_{\partial(U_2\cap\mathbb{C}_{\mathbf{j}})} \frac{1}{(s-\overline{w})^{k+1}} ds_{\mathbf{j}} f(s),\end{aligned} \tag{21}$$

where $\partial(U_1 \cap \mathbb{C}_{\mathbf{j}})$ is the path of integration in $\mathbb{C}_{\mathbf{j}}$ that contains the point $s_1 = x_0 + \mathbf{j}|\underline{x}|$ and $\partial(U_2 \cap \mathbb{C}_{\mathbf{j}})$ is the path of integration in $\mathbb{C}_{\mathbf{j}}$ that contains the point $s_2 = x_0 - \mathbf{j}|\underline{x}|$. Now we suppose that the above paths of integration are the two circles $|s - s_1| = \tau|s_1|$ and $|s - s_2| = \tau|s_2|$ where $\tau > 0$ is a parameter. Now, we estimate the two terms

$$f^{(k)}(w) := \frac{k!}{2\pi} \int_{\partial(U_1 \cap \mathbb{C}_{\mathbf{j}})} \frac{1}{(s - w)^{k+1}} ds_{\mathbf{j}} f(s)$$

and

$$f^{(k)}(\overline{w}) := \frac{k!}{2\pi} \int_{\partial(U_2 \cap \mathbb{C}_{\mathbf{j}})} \frac{1}{(s - \overline{w})^{k+1}} ds_{\mathbf{j}} f(s)$$

using the Cauchy formula for the derivatives in the complex plane $\mathbb{C}_{\mathbf{j}}$. Recalling that we assume the growth condition $|f(x)| \leq C_f e^{B|x|^p}$, we obtain:

$$|f^{(k)}(w)| \leq \frac{k!}{(\tau|w|)^j} \max_{|w-z|=\tau|z|} |f(w)| \leq \frac{C_f k!}{(\tau|w|)^k} \exp(B(1+\tau)^p |w|^p)$$

and similarly for $f^{(k)}(\overline{w})$

$$|f^{(k)}(\overline{w})| \leq \frac{C_f k!}{(\tau|\overline{w}|)^k} \exp(B(1+\tau)^p |\overline{w}|^p)$$

so we conclude that

$$|f^{(k)}(w)| + |f^{(k)}(\overline{w})| \leq 2\frac{C_f k!}{(\tau|w|)^k} \exp(B(1+\tau)^p |w|^p)$$

for all $\tau > 0$, where we have used the fact that $f \in A_p$ and $|w| \leq (1+s)|z|$. Now we can estimate the slice derivative from the formula (21), precisely

$$\partial_{x_0}^k f(x) = \frac{1 - \mathbf{ij}}{2}\ f^{(k)}(w) + \frac{1 + \mathbf{ij}}{2}\ f^{(k)}(\overline{w})$$

gives

$$|\partial_{x_0}^k f(x)| \leq |\frac{1 - \mathbf{ij}}{2}|\ |f^{(k)}(w)| + |\frac{1 + \mathbf{ij}}{2}|\ |f^{(k)}(\overline{w})| \leq |f^{(k)}(w)| + |f^{(k)}(\overline{w})|.$$

The well known estimate (IV) in Lemma 2.11 gives $(1+\tau)^p \leq 2^p(\tau^p + 1)$ for all $\tau > 0$. Hence we have

$$|f^{(k)}(w)| + |f^{(k)}(\overline{w})| \leq 2C_f \frac{k!}{(\tau|w|)^k} \exp(B \cdot 2^p \tau^p |w|^p) \exp(B \cdot 2^p |w|^p) \tag{22}$$

for all $w \in \mathbb{C}_{\mathbf{j}}$ and $\tau > 0$. Now we observe that the point $\tau_{\min} = \left(\frac{k}{2^p Bp}\right)^{1/p} \frac{1}{|w|}$ for $w \neq 0$ is the minimum of the function

$$\tau \mapsto \frac{1}{(\tau|w|)^k} \exp(B \cdot 2^p \tau^p |w|^p)$$

which is the right-hand side of (22), so that we obtain

$$|\partial_{x_0}^k f(x)| \leq |f^{(k)}(w)| + |f^{(k)}(\overline{w})| \leq 2C_f\ k! \Big(\frac{2^p Bp}{k}\Big)^{k/p} e^{k/p} \exp(A 2^p |w|^p).$$

If we set

$$b := (2^p Bpe)^{1/p}$$

we deduce the estimate

$$|\partial_{x_0}^k f(x)| \leq |f^{(k)}(w)| + |f^{(k)}(\overline{w})| \leq 2C_f k! \frac{b^k}{k^{k/p}} \exp(B \cdot 2^p |w|^p)$$

for all $w \in \mathbb{C}_{\mathbf{j}}$. Since

$$\alpha_k = \frac{\partial_{x_0}^k(0)}{k!}$$

we have, by the maximum modules principle applied in a disc centered at the origin and with radius $\epsilon > 0$ sufficiently small, in the complex plane $\mathbb{C}_{\mathbf{j}}$

$$|\alpha_k| \leq C_f \ \frac{b^k}{k^{k/p}} \exp(B \cdot 2^p \epsilon^p) \leq 2C_f \ \frac{b^k}{k^{k/p}} = C'_f \ \frac{b^k}{(k!)^{1/p}} \leq C'_f \ \frac{b^k}{\Gamma(\frac{k}{p}+1)}.$$

The other direction follows form the properties of the Mittag-Leffler function because it is of order $1/\alpha$ (and of type 1) for $\alpha > 0$ and $Re(\beta) > 0$, so, in our case, f is entire of order p. The fact that f_m in $\mathcal{SM}^p$ tends to zero if and only if $C_{f_m} \to 0$ and $b_{f_m} < b$ for some $b > 0$ is a consequence of the estimate on the α_k.

□

3 Infinite order differential operators on slice monogenic functions

In this section we study a class of infinite order differential operators acting on spaces of entire slice monogenic functions. The definition of these infinite order differential operators preserve the slice monogenicity. In fact, we consider the star-product of the coefficients $(u_m)_{m\in\mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, of the operator and the slice derivative $\partial_{x_0}^m f(x)$ of the slice monogenic function $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$. Precisely we have:

Definition 3.1. *Let $p \geq 1$.*

- *Let $(u_m)_{m\in\mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be entire functions in $\mathcal{SM}_L$. We define the set $\mathbf{D}_{p,0}^L$ of formal operators defined by*

$$U_L(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} u_m(x) \star_L \partial_{x_0}^m f(x),$$

  for entire functions f in $\mathcal{SM}_L$.

- *Let $(u_m)_{m\in\mathbb{N}_0} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be entire functions in $\mathcal{SM}_R$. We define the set $\mathbf{D}_{p,0}^R$ of formal operators defined by*

$$U_R(x, \partial_{x_0}) f(x) := \sum_{m=0}^{\infty} u_m(x) \star_R \partial_{x_0}^m f(x)$$

  for entire functions f in $\mathcal{SM}_R$.

The entire functions $(u_m)_{m\in\mathbb{N}_0}$ in $\mathcal{SM}_L$ (resp. in $\mathcal{SM}_R$) satisfy the additional condition: There exists a constant $B > 0$ such that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ for which

$$|u_m(x)| \leq C_\varepsilon \frac{\varepsilon^m}{(m!)^{1/q}} \exp(B|x|^p), \quad \text{for all} \quad m \in \mathbb{N}_0, \tag{23}$$

where $1/p + 1/q = 1$ and $1/q = 0$ when $p = 1$.

We are now in the position to state and proof the main result of this section.

Theorem 3.2. *Let $p \geq 1$ and let $\mathbf{D}^L_{p,0}$ and $\mathbf{D}^R_{p,0}$ be the sets of formal operators in Definition 3.1.*

(I) Let $U_L(x, \partial_{x_0}) \in \mathbf{D}^L_{p,0}$ and let $f \in \mathcal{SM}^p_L$, then $U_L(x, \partial_{x_0})f \in \mathcal{SM}^p_L$ and the operator $U_L(x, \partial_{x_0})$ acts continuously on $\mathcal{SM}^p_L$, i.e., if $f_m \in \mathcal{SM}^p_L$ and $f_m \to 0$ in $\mathcal{SM}^p_L$ then $U_L(x, \partial_{x_0})f_m \to 0$ in $\mathcal{SM}^p_L$.

(II) Let $U_L(x, \partial_{x_0}) \in \mathbf{D}^R_{p,0}$ and let $f \in \mathcal{SM}^p_R$, then $U_R(x, \partial_{x_0})f \in \mathcal{SM}^p_R$ and the operator $U_R(x, \partial_{x_0})$ acts continuously on $\mathcal{SM}^p_R$, i.e., if $f_m \in \mathcal{SM}^p_R$ and $f_m \to 0$ in $\mathcal{SM}^p_R$ then $U_R(x, \partial_{x_0})f_m \to 0$ in $\mathcal{SM}^p_R$.

Proof. Let us prove case (I), since case (II) follows with similar computations. We apply operator $U_L(x, \partial_{x_0}) \in \mathbf{D}^L_{p,0}$ (see Definition 3.1) to a function $f \in \mathcal{SM}^p_L$,

$$\begin{aligned} U_L(x, \partial_{x_0})f(x) &= \sum_{m=0}^{\infty} u_m(x) \star_L \partial^m_{x_0} \sum_{j=0}^{\infty} \alpha_j x^j \\ &= \sum_{m=0}^{\infty} u_m(x) \star_L \sum_{j=0}^{\infty} \alpha_j \partial^m_{x_0} x^j \\ &= \sum_{m=0}^{\infty} u_m(x) \star_L \sum_{j=m}^{\infty} \alpha_j \frac{j!}{(j-m)!} x^{j-m} \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{m+k} u_m(x) \frac{(k+m)!}{k!} x^k. \end{aligned}$$

Now we observe that

$$|U_L(x, \partial_{x_0})f(x)| \leq \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |\alpha_{m+k}||u_m(x)| \frac{(k+m)!}{k!} |x|^k$$

and we recall that since $U_L(x, \partial_{x_0}) \in \mathbf{D}^L_{p,0}$ the coefficients $u_m(x)$ of the operator satisfy estimate (23) and since $f \in \mathcal{SM}^p_L$, the coefficients $|\alpha_k|$ of f satisfy estimate (20) so we have

$$|U_L(x, \partial_{x_0})f(x)| \leq C_f C_\varepsilon \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varepsilon^m}{(m!)^{1/q}} \exp(B|x|^p) \quad \frac{b^{m+k}}{\Gamma\Big(\frac{m+k}{p}+1\Big)} \frac{(k+m)!}{k!} |x|^k.$$

We now use estimates (I) and (III) in Lemma 2.11 to get the estimates

$$\begin{aligned} |U_L(x, \partial_{x_0})f(x)| &\leq C_f C_\varepsilon \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\varepsilon^m}{\Gamma\Big(\frac{m}{q}+1\Big)} \quad \frac{b^{m+k}}{\Gamma\Big(\frac{m+k}{p}+1\Big)} \frac{2^{k+m} k! m!}{k!} |x|^k \exp(B|x|^p) \\ &\leq C_f C_\varepsilon \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} (2b)^k (2\varepsilon b)^m \frac{1}{\Gamma\Big(\frac{m}{q}+1\Big)} \quad \frac{m!}{\Gamma\Big(\frac{m+k}{p}+1\Big)} |x|^k \exp(B|x|^p), \end{aligned} \tag{24}$$

from (III) in Lemma 2.11 it follows that $\Gamma\Big(\frac{k+m}{p}+1\Big) \geq \Gamma\Big(\frac{k}{p}+\frac{1}{2}\Big)\Gamma\Big(\frac{m}{p}+\frac{1}{2}\Big)$ and so we can write (24) as

$$|U_L(x, \partial_{x_0})f(x)| \leq \beta(p, q, b, \varepsilon) C_f C_\varepsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\Big(\frac{k}{p}+\frac{1}{2}\Big)} |x|^k \exp(B|x|^p)$$

with the position

$$\beta(p,q,b,\varepsilon) := \sum_{m=0}^{\infty} (2\varepsilon b)^m \frac{m!}{\Gamma\Big(\frac{m}{p}+\frac{1}{2}\Big)\Gamma\Big(\frac{m}{q}+1\Big)} \tag{25}$$

where we can show that the series (25) is convergent, for ε is arbitrary small the series converges, using the asymptotic expansion of the Gamma function

$$(2\varepsilon b)^m \frac{m!}{\Gamma\Big(\frac{m}{p}+\frac{1}{2}\Big)\Gamma\Big(\frac{m}{q}+1\Big)} \sim \frac{m^m (2\varepsilon b)^m}{\Big(\frac{m}{p}\Big)^{m/p}\Big(\frac{m}{q}\Big)^{m/q}} = (2\varepsilon b)^m [p^{1/p} q^{1/q}]^m.$$

We finally obtain

$$|U_L(x,\partial_{x_0})f(x)| \leq \beta(p,q,b,\varepsilon) C_f C_\varepsilon \sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\Big(\frac{k}{p}+\frac{1}{2}\Big)} |x|^k \exp(B|x|^p)$$

and, by the properties of the Mittag-Leffler function, we have

$$\sum_{k=0}^{\infty} \frac{(2b)^k}{\Gamma\Big(\frac{k}{p}+\frac{1}{2}\Big)} |x|^k \leq C' \exp(B'|x|^p).$$

We conclude that there exists $B'' > 0$ such that

$$|U_L(x,\partial_{x_0})f(x)| \leq \beta(p,q,b,\varepsilon) C_f C_\varepsilon \exp(B''|x|^p)$$

that is $U_L(x,\partial_{x_0})f(x) \in \mathcal{SM}_L^p$ and for $C_{f_m} \to 0$ the same estimate proves the continuity, i.e. $|U_L(x,\partial_{x_0})f_m(x)| \to 0$ when $f_m(x) \to 0$. □

4 Function spaces of entire monogenic functions

We recall the in the sequel we work in the real Clifford algebra $\mathbb{R}_n$, so for the definition and the notation we refer the reader to Section 2, for more more details on monogenic functions see the book of [26]. We start with some definitions. The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by:

$$\mathcal{D} := \frac{\partial}{\partial_{x_0}} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial_{x_i}}.$$

Definition 4.1 (Left and right Monogenic Functions)**.** *Let $U \subseteq \mathbb{R}^{n+1}$ be an open subset. A function $f : U \to \mathbb{R}_n$, of class C^1, is called left monogenic if*

$$\mathcal{D}f = \frac{\partial}{\partial_{x_0}} f + \sum_{i=1}^{n} e_i \frac{\partial}{\partial_{x_i}} f = 0.$$

A function $g : U \to \mathbb{R}_n$, of class C^1, is called right monogenic if

$$f\mathcal{D} = \frac{\partial}{\partial_{x_0}} f + \sum_{i=1}^{n} \frac{\partial}{\partial_{x_i}} f e_i = 0.$$

The set of left monogenic functions (resp. right monogenic functions) will be denoted by $\mathcal{M}_L(U)$ (resp. $\mathcal{M}_R(U)$); if $U = \mathbb{R}^{n+1}$ we simply denote it by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$)

Definition 4.2 (Fueter's homogeneous polynomials). *Given a multi–index $k = (k_1, ..., k_n)$ where $k_i \geq 0$, we set $|k| = \sum_{i=1}^n k_i$ and $k! = \prod_{i=1}^n k_i!$.*

(I) For a multi–index k with at least one negative component we set

$$P_k(x) := 0$$

for $0 = (0, ..., 0)$ we set

$$P_0(x) := 1.$$

(II) For a multi-index k with $|k| > 0$ we define $P_k(x)$ as follows: for each k consider the sequence of indices $j_1, j_2, \ldots, j_{|k|}$ be given such that the first k_1 indices equals 1, the next indices k_2 equals 2 and, finally, the last k_n equals n. We define $z_i = x_i e_0 - x_0 e_i$ for any $i = 1, \ldots, n$ and $z = (z_1, \ldots, z_n)$. We set

$$z^k := z_{j_1} z_{j_2} \ldots z_{j_{|k|}} = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$$

and

$$|z|^k = |z_1|^{k_1} \cdots |z_n|^{k_n}$$

these products contains z_1 exactly k_1-times, z_2 exactly k_2-times and so on. We define

$$P_k(x) = \frac{1}{|k|!} \sum_{\sigma \in perm(k)} \sigma(z^k) := \frac{1}{|k|!} \sum_{\sigma \in perm(k)} z_{j_{\sigma(1)}} z_{j_{\sigma(2)}} \ldots z_{j_{\sigma(|k|)}},$$

where $perm(k)$ is the permutation group with $|k|$ elements. When we multiply by $k!$ the Fueter's polynomial $P_k(x)$ we will denote it by $V_k(x)$ i.e.

$$V_k(x) := k! P_k(x) = \frac{k!}{|k|!} \sum_{\sigma \in perm(k)} z_{j_{\sigma(1)}} z_{j_{\sigma(2)}} \ldots z_{j_{\sigma(|k|)}}$$

These polynomials play an important role in the monogenic function theory and we collect some of their properties in the next proposition (see Theorem 6.2 in [46]):

Theorem 4.3. *Consider the Fueter polynomials $P_k(x)$ defined above. Then the following facts hold:.*

(I) the recursion formula

$$k P_k(x) = \sum_{i=1}^m k_i P_{k-\varepsilon_i}(x) z_i = \sum_{i=1}^m k_i z_i P_{k-\varepsilon_i}(x),$$

and also

$$\sum_{i=1}^m k_i P_{k-\varepsilon_i}(x) e_i = \sum_{i=1}^m k_i e_i P_{k-\varepsilon_i}(x),$$

where $\varepsilon_i = (0, ..., 0, 1, 0, ..., 0)$ with 1 in the position i.

(II) The derivatives ∂_{x_j} for $j = 1, ..., n$, are given by

$$\partial_{x_j} P_k(x) = k_j P_{k-\varepsilon_j}(x).$$

(III) The Fueter Polynomials $P_k(x)$ are both left and right monogenic.

(IV) The following estimates holds

$$|P_k(x)| \leq |x|^{|k|}.$$

(V) (Binomial formula) For all paravectors x and y, and for the multi–index k, j and i

$$P_k(x+y) = \sum_{i+j=k} \frac{k!}{i!j!} P_i(x) P_j(y).$$

We introduce the Cauchy kernel function.

Definition 4.4. *The Cauchy kernel $\mathcal{G}(x)$ is defined by*

$$\mathcal{G}(x) = \frac{1}{\sigma_n} \frac{\overline{x}}{|x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\}, \qquad \sigma_n := 2\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

Moreover, we define for any multi–index $k = (k_1, \dots, k_n)$

$$\mathcal{G}_k(x) = \frac{\partial^{|k|}}{\partial x^k} \mathcal{G}(x).$$

Monogenic functions satisfy a generalized integral Cauchy formula (see Theorem 7.12 in [46]).

Theorem 4.5 (The Cauchy formula)**.** *Let U be a bounded domain in $\mathbb{R}^{n+1}$ with smooth boundary ∂G so that the normal unit vector is orientated outwards. For the left monogenic functions f, defined on an open set that contains $\overline{U}$, we have*

$$f(x) = \int_{\partial G} \mathcal{G}(y-x) Dy f(y), \quad x \in G.$$

where

$$Dy = \sum_{j=0}^{n} (-1)^j e_j \, dy_0 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n.$$

Moreover, for any $x \in G$ and for any multi–index $k = (k_1, \dots, k_n)$, we also have

$$\frac{\partial^{|k|}}{\partial x^k} f(x) = (-1)^{|k|} \int_{\partial G} \mathcal{G}_k(y-x) Dy f(y).$$

If f is left monogenic in a ball centered at the origin and of radius R then for any $|x| < r$ with $0 < r < R$ we have

$$f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k,$$

where the a_k's are Clifford numbers defined by

$$a_k := \frac{(-1)^{|k|}}{k!} \int_{|y|=r} \mathcal{G}_k(y) Dy f(y).$$

By the estimate

$$|\mathcal{G}_k(x)| \leq \frac{n(n+1)\cdots(n+|k|-1)}{|x|^{n+|k|}} \tag{26}$$

the following sharp estimate holds to be true (see [41])

$$|a_k| \leq M_g(r) \frac{c(n,k)}{r^{|k|}}. \tag{27}$$

where

$$c(n,k) := \frac{n(n+1)\cdots(n+|k|-1)}{k!} = \frac{(n+|k|-1)!}{(n-1)!\,k!}. \tag{28}$$

The first important property that concerns the number $c(n,k)$ is contained in the following lemma (see Lemma 1 in [40]).

Lemma 4.6. *For all multi–indexes $k \in (\mathbb{N}_0)^n \setminus \{0\}$ and for all positive integers n we have*

$$\limsup_{p\to+\infty} \left(\sum_{|k|=p} c(n,k) \right)^{\frac{1}{p}} = n.$$

Definition 4.7. *Let f be an entire left monogenic function. Then we say that f is of finite order if there exists $\kappa > 0$ such that*

$$M_f(r) < e^{r^k}$$

for sufficiently large r. The greatest lower bound ρ of such numbers κ is called order of f. Equivalently, we can define the order as

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M_f(r)}{\ln r}.$$

Definition 4.8. *Let f be an entire left monogenic function of order ρ and let $A > 0$ be such that for sufficiently large values of r we have*

$$M_f(r) < e^{Ar^\rho}.$$

We say that f of order ρ is of type σ if σ is the greatest lower bound of such numbers and we have

$$\sigma = \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}.$$

Moreover, we have

- *When $\sigma = 0$ we say that f is of minimal type.*
- *When $\sigma = \infty$ we say that f is of maximal type.*
- *When $\sigma \in (0,\infty)$ we say that f is of normal type.*

The constant functions are said to be of minimal type of order zero. The next two theorems are the generalizations of the Lindelöf and Pringsheim theorems on the growth of the Taylor coefficients of an entire holomorphic function to the case of an entire monogenic function (see Theorem 1 in [40] and Theorem 1 in [39]).

Theorem 4.9. *For an entire monogenic function $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ with a Taylor series representation of the form $f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k$ set*

$$\Pi = \limsup_{|k|\to+\infty} \frac{|k| \log |k|}{-\log |\frac{1}{c(n,k)} a_k|},$$

then $\rho(f) = \Pi$ where $c(n,k)$ is the number defined in (28).

Theorem 4.10. *For an entire monogenic function $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ with a Taylor series representation of the form $f(x) = \sum_{|k|=0}^{+\infty} V_k(x) a_k$ with order ρ $(0 < \rho < +\infty)$ and set*

$$\Pi = \limsup_{|k| \to +\infty} |k| \left(|a_k| \right)^{\frac{\rho}{|k|}},$$

then

$$\sigma(f) = \frac{\Pi}{e\rho(f)}.$$

The next lemma will be useful in the proof of Lemma 4.12 and it is a generalization of Lemma 2.11 (III).

Lemma 4.11. *Given a multi–index $k \in (\mathbb{N}_0)^n$, then for any $q \geq 1$ we have*

$$\Gamma\left(\frac{|k|}{nq} + 1\right)^{nq} \leq k!.$$

Proof. It is sufficient to observe that

$$\Gamma\left(\frac{|k|}{q} + 1\right) = \int_0^{+\infty} e^{-t} t^{\frac{|k|}{nq}}\, dt = \int_0^{+\infty} \Pi_{i=1}^n e^{\frac{t}{n}} t^{\frac{k_i}{nq}}\, dt \overset{\text{Hölder inequality}}{\leq} \Pi_{i=1}^n \Gamma\left(\frac{k_i}{q} + 1\right)^{\frac{1}{n}}$$
$$\overset{\text{Lemma 2.11}}{\leq} (k!)^{\frac{1}{nq}}.$$

So we get the statement. □

In the next lemma we introduce the equivalent of the entire Mittag–Leffler functions in one complex variable in the context of the Clifford Analysis. These functions are defined following the way presented at p. 159 in [39] and at p. 772 in [40].

Lemma 4.12. *Let $x \in \mathbb{R}^{n+1}$ then for any α, $\beta \in \mathbb{R}_{>0}$ the Mittag-Leffler function*

$$E_{\alpha,\beta}(x) = \sum_{|k|=0}^{\infty} \frac{c(n,k) V_k(x)}{\Gamma(\alpha|k| + \beta)}$$

is an entire monogenic function of order $\frac{1}{\alpha}$ and of type $n^{\frac{1}{\alpha}}$ where $c(n,k)$ is the number defined in (28).

Proof. The ray of convergence of $E_{\alpha,\beta}(x)$ is $+\infty$. For it is sufficient to observe that

$$\sum_{|k|=0}^{\infty} \frac{c(n,k)|V_k(x)|}{\Gamma(\alpha|k| + \beta)} \overset{\text{Theorem 4.3 IV}}{\leq} \sum_{p=0}^{\infty} \left(\sum_{|k|=p}^{\infty} c(n,k) \right) \frac{|x|^p}{\Gamma(\alpha p + \beta)}$$

and the conclusion follows by the Cauchy–Hadamard's Theorem for the power series once we note that

$$\limsup_{p \to +\infty} \left(\left(\sum_{|k|=p}^{\infty} c(n,k) \right) \frac{1}{\Gamma(\alpha p + \beta)} \right)^{\frac{1}{p}} \overset{\text{Lemma 4.6}}{=} 0.$$

We prove the remaining part of the lemma for $\beta = 1$ since we can deduce the general case by observing that

$$\Gamma(\alpha x + \beta) = \Gamma\left(\alpha\left(x + \frac{\beta - 1}{\alpha}\right) + 1\right).$$

To prove that the order of $E_{\alpha,1}$ is equal to $\frac{1}{\alpha}$ we apply Theorem 4.9 with $a_k = \frac{c(n,k)}{\Gamma(\alpha|k|+1)}$ and the Stirling-De Moivre formula to obtain

$$\begin{aligned}\rho\left(E_{\alpha,1}\right) &\underset{\text{Theorem 4.9}}{=} \limsup_{|k|\to+\infty} \frac{|k|\log|k|}{-\log|\frac{1}{c(n,k)}a_k|} \\ &= \limsup_{|k|\to+\infty} \frac{|k|\log|k|}{\log\Gamma\left(\alpha|k|+1\right)} \underset{\text{Stirling-De Moivre}}{=} \frac{1}{\alpha}.\end{aligned}$$

To prove that the type of $E_{\alpha,1}$ is equal to $n^{\frac{1}{\alpha}}$ we apply Theorem 4.10 with $a_k = \frac{c(n,k)}{\Gamma(\alpha|k|+1)}$ to obtain

$$\begin{aligned}\sigma(E_{\alpha,1}) &= \frac{\alpha}{e}\limsup_{|k|\to+\infty}|k|\left(|a_k|\right)^{\frac{1}{\alpha|k|}} \\ &= \frac{\alpha}{e}\limsup_{|k|\to+\infty}|k|\left(\frac{c(n,k)}{\Gamma\left(\alpha|k|+1\right)}\right)^{\frac{1}{\alpha|k|}}.\end{aligned}$$

Since by the Lemma 4.11, we have

$$\begin{aligned}\sup_{|k|=p}\frac{c(n,k)}{\Gamma\left(\alpha|k|+1\right)} &= \sup_{|k|=p}\frac{(n+|k|-1)!}{(n-1)!\,k!\,\Gamma\left(\alpha|k|+1\right)} \\ &\leq \sup_{|k|=p}\frac{(n+|k|-1)!}{(n-1)!\,\Gamma\left(\frac{|k|}{n}+1\right)^n\Gamma\left(\alpha|k|+1\right)}\end{aligned}$$

with the equality when $|k|$ is a multiple of n and $k = \left(\frac{|k|}{n},\dots,\frac{|k|}{n}\right)$, we can conclude

$$\begin{aligned}\sigma(E_{\alpha,1}) &= \frac{\alpha}{e}\limsup_{|k|\to+\infty}|k|\left(\frac{(n+|k|-1)!}{(n-1)!\,\Gamma\left(\frac{|k|}{n}+1\right)^n\Gamma\left(\alpha|k|+1\right)}\right)^{\frac{1}{\alpha|k|}} \\ &\underset{\text{Stirling De-Moivre}}{=} \frac{\alpha}{e}\limsup_{|k|\to+\infty}|k|\left(\frac{(n+|k|-1)^{n+|k|-1}}{\left(\frac{|k|}{n}\right)^{|k|}(\alpha|k|)^{\alpha|k|}\exp(-\alpha|k|)}\right)^{\frac{1}{\alpha|k|}} = n^{\frac{1}{\alpha}},\end{aligned}$$

where in the second equality we deleted the terms that do not affect the lim sup. □

Definition 4.13. *Let $p \geq 1$. We denote by $\mathcal{M}^p$ the space of entire monogenic functions with either order lower than p or order equal to p and finite type. It consists of functions f, for which there exist constants $B, C > 0$ such that*

$$|f(x)| \leq Ce^{B|x|^p}. \tag{29}$$

Let $(f_m)_{m\in\mathbb{N}}$, $f_0 \in \mathcal{M}^p$. Then $f_m \to f_0$ in $\mathcal{M}^p$ if there exists some $B > 0$ such that

$$\lim_{m\to\infty}\sup_{x\in\mathbb{R}^{n+1}}\left|(f_m(x)-f_0(x))e^{-B|x|^p}\right| = 0. \tag{30}$$

Functions in $\mathcal{M}^p$ that are left monogenic will be denoted by $\mathcal{M}^p_L$, while right monogenic will be denoted by $\mathcal{M}^p_R$.

We extend the Lemma 2.2 in [17] to the case of the monogenic entire function.

Lemma 4.14. *Let $p \geq 1$. A function*

$$f(x) = \sum_{|k|=0}^{\infty} V_k(x) a_k$$

belongs to $\mathcal{M}^p$ if and only if there exist constants C_f, b_f, $\beta > 0$ such that

$$|a_k| \leq C_f \frac{b_f^{|k|} c(n,k)}{\Gamma\left(\frac{|k|}{p} + \beta\right)}. \tag{31}$$

Furthermore, a sequence f_m in $\mathcal{M}^p$ tends to zero if and only if $C_{f_m} \to 0$ and $b_{f_m} < b$ for some $b > 0$ where $c(n,k)$ is the number defined in (28).

Proof. ($\Rightarrow$) By the Theorem 4.5 we have

$$\partial_x^k f(x) = (-1)^{|k|} \int_{\partial B(x, s|x|)} \mathcal{G}_k(y-x) Dy f(y)$$

where $s > 0$ is a constant to be determined later. In view of the estimate (26) and since $f \in \mathcal{M}^p$, we have

$$\begin{aligned}
|\partial_x^k f(x)| &\leq k! \, \frac{c(n,k)}{(s|x|)^{|k|}} \sup_{|\zeta - x| = s|x|} |f(\zeta)| \\
&\leq k! \, \frac{c(n,k)}{(s|x|)^{|k|}} M_f((1+s)|x|) \\
&\leq C_f k! \, \frac{c(n,k)}{(s|x|)^{|k|}} \exp(B(1+s)^p |x|^p) \\
&\leq C_f k! \, \frac{c(n,k)}{(s|x|)^{|k|}} \exp(B 2^p s^p |x|^p) \exp(B 2^p |x|^p)
\end{aligned}$$

where the last inequality is due to the estimate: $(1+s)^p \leq 2^p(1+s^p)$. We define

$$g(s) := \frac{\exp(B 2^p s^p |x|^p)}{(s|x|^{|k|})}$$

and we note that this function gets its minimum at

$$s_0 := \frac{1}{2} \left(\frac{|k|}{pB}\right)^{\frac{1}{p}} \frac{1}{|x|}.$$

Thus we have

$$g(s_0) = \exp\left(\frac{|k|}{p}\right) \left(\frac{2pB}{|m|}\right)^{\frac{|m|}{p}},$$

and

$$|\partial_x^k f(x)| \leq C_f \, k! \, c(n,k) \left[(2epB)^{\frac{1}{p}}\right]^{|k|} |k|^{-\frac{|k|}{p}} \exp(B 2^p |x|^p).$$

We set $b=(2epB)^{\frac{1}{p}}$ and by the maximum modulus principle we have

$$
\begin{aligned}
|a_k| = \frac{|\partial_x^k f(0)|}{k!} &\leq \frac{\sup_{|x|=r}|\partial_x^k f(x)|}{k!} \\
&\leq C_f\, c(n,k)\, b^{|k|}\, |k|^{-\frac{|k|}{p}} \exp(B2^p r^p) \\
&\leq 2C_f\, c(n,k)\, b^{|k|}\, |k|^{-\frac{|k|}{p}} \\
&\leq 2C_f'\, c(n,k)\, b^{|k|}\, (|k|!)^{-\frac{1}{p}} \\
&\overset{\text{Lemma 2.11 (III)}}{\leq} 2C_f' c(n,k) \frac{b^{|k|}}{\Gamma\left(\frac{|k|}{p}+1\right)}.
\end{aligned}
$$

($\Leftarrow$) The other direction is a consequence of the properties of the Mittag-Leffler function described in Lemma 4.12 □

To define a class of operators that act over $\mathcal{M}_L^p$ with image in the same space, it is useful to introduce the left (resp. right) C-K product between left (resp. right) monogenic entire functions (see p. 114 in [26]).

Definition 4.15. *Let $f,g\in\mathcal{M}_L$ be entire functions (resp. $f,g\in\mathcal{M}_R$). Using their Taylor series representation*

$$
f(x)=\sum_{|k|=0}^{+\infty} V_k(x) f_k \quad (\textit{resp. } f(x)=\sum_{|k|=0}^{+\infty} f_k V_k(x)\ )
$$

and

$$
g(x)=\sum_{|k|=0}^{+\infty} V_k(x) g_k \quad (\textit{resp. } g(x)=\sum_{|k|=0}^{+\infty} g_k V_k(x)\ )
$$

we define

$$
f\odot_L g := \sum_{|k|=0}^{+\infty}\sum_{|j|=0}^{+\infty} V_{k+j}(x) f_k g_j \quad (\textit{resp. } f\odot_R g := \sum_{|k|=0}^{+\infty}\sum_{|j|=0}^{+\infty} f_k g_j V_{k+j}(x)).
$$

Remark 4.16. At p. 114 in [26], the definition of C-K product is given between Fueter polynomial. Here we adapt that definition for the polynomial $V_k(x)$ introduced in the Definition 4.2.

5 Infinite order differential operators on monogenic functions

In this section, using the definition of the C-K product of two monogenic entire functions we define suitable classes of infinite order differential operators in the monogenic setting.

Definition 5.1. *Let $p\geq 1$ and set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.*

- *Let $(u_m)_{m\in(\mathbb{N}_0)^n}:\mathbb{R}^{n+1}\to\mathbb{R}_n$ be entire functions that belong to $\mathcal{M}_L$. We define the set $\mathbf{DM}_{p,0}^L$ of formal operators defined by*

$$
U_L(x,\partial_x)f(x) := \sum_{|m|=0}^{\infty} u_m(x)\odot_L \partial_x^m f(x),
$$

for entire functions f in $\mathcal{M}_L$ where $\partial_x^m := \partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n}$ (in particular no derivatives along the x_0-direction appear).

- *Let $(u_m)_{m\in(\mathbb{N}_0)^n} : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be entire functions that belong to $\mathcal{M}_R$. We define the set $\mathbf{DM}_{p,0}^R$ of formal operators defined by*

$$U_R(x, \partial_x)f(x) := \sum_{|m|=0}^{\infty} u_m(x) \odot_R \partial_x^m f(x)$$

for entire functions f in $\mathcal{M}_R$ where $\partial_x^m := \partial_{x_1}^{m_1} \dots \partial_{x_n}^{m_n}$ (in particular no derivatives along the x_0-direction appear).

The entire functions $(u_m)_{m\in(\mathbb{N}_0)^n}$ in $\mathcal{M}_L$ (resp. in $\mathcal{M}_R$) satisfy the additional condition: There exists a constant $B > 0$ such that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ for which

$$|u_m(x)| \leq C_\varepsilon \frac{\varepsilon^{|m|}}{(|m|!)^{1/q}} \exp(B|x|^p), \quad \text{for all} \quad m \in (\mathbb{N}_0)^n, \tag{32}$$

where $1/p + 1/q = 1$ and $1/q = 0$ when $p = 1$.

Remark 5.2. If we consider the Taylor expansion of the u_m's then we can write them as

$$u_m(x) = \sum_{|j|=0}^{+\infty} V_j(x) a_j^m$$

and, thanks to the Lemma 4.14, the coefficients a_j^m's satisfy the estimate

$$|a_j^m| \leq C_\epsilon \frac{\epsilon^{|m|}(b_{u_m})^{|j|} c(n,j)}{(m!)^{\frac{1}{q}} \Gamma\left(\frac{j}{p} + 1\right)},$$

where $c(n, j)$ is the number defined in (28).

We are now in the position to state and proof the main result of this section (the analogue of the Theorem 2.4 in [17]).

Theorem 5.3. *Let $p \geq 1$ and let $\mathbf{DM}_{p,0}^L$ and $\mathbf{DM}_{p,0}^R$ be the sets of formal operators in Definition 5.1.*

(I) Let $U_L(x, \partial_x) \in \mathbf{DM}_{p,0}^L$ and let $f \in \mathcal{M}_L^p$, then $U_L(x, \partial_x)f \in \mathcal{M}_L^p$ and the operator $U_L(x, \partial_x)$ acts continuously on $\mathcal{M}_L^p$, i.e., if $f_m \in \mathcal{M}_L^p$ and $f_m \to 0$ in $\mathcal{M}_L^p$ then we have $U_L(x, \partial_x)f_m \to 0$ in $\mathcal{M}_L^p$.

(II) Let $U_R(x, \partial_x) \in \mathbf{DM}_{p,0}^R$ and let $f \in \mathcal{M}_R^p$, then $U_R(x, \partial_x)f \in \mathcal{M}_R^p$ and the operator $U_R(x, \partial_x)$ acts continuously on $\mathcal{M}_R^p$, i.e., if $f_m \in \mathcal{M}_R^p$ and $f_m \to 0$ in $\mathcal{M}_R^p$ then we have $U_R(x, \partial_x)f_m \to 0$ in $\mathcal{M}_R^p$.

Proof. We write in details only the proof of the statement (I) since the proof of the statement (II) follows from minor changes. Since u_m, $f \in \mathcal{M}_L^p$ are entire functions we can rewrite them using their Taylor expansion series

$$f(x) = \sum_{|k|=0}^{\infty} V_k(x) f_k \quad \text{and} \quad u_m(x) = \sum_{|k|=0}^{\infty} V_k(x) a_k^m$$

where $a_k^m,\ f_k \in \mathbb{R}_n$ for any multi-indexes $m,\ k \in (\mathbb{N}_0)^n$. According to the Definition 5.1, we have

$$\begin{aligned}
U_L(x,\partial_x)f(x) &= \sum_{|m|=0}^{\infty} u_m(x) \odot_L \partial_x^m f(x) \\
&= \sum_{|m|=0}^{\infty} u_m(x) \odot_L \sum_{|k|=0}^{\infty} \partial_x^m (V_k(x)) f_k \\
&= \sum_{|m|=0}^{\infty} u_m(x) \odot_L \sum_{|k|=0}^{\infty} \frac{(m+k)!}{k!} V_k(x) f_{m+k} \\
&= \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \sum_{|j|=0}^{\infty} \frac{(m+k)!}{k!} V_{k+j}(x) a_j^m f_{m+k}.
\end{aligned}$$

Since the $V_k(x)$'s are paravectors, using the classical inequality: $|xy| \leq 2^{\frac{n}{2}}|x||y|$ for any $x,\ y \in \mathbb{R}_n$, we have that

$$|U_L(x,\partial_x)f(x)| \leq 2^{\frac{n}{2}} \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \sum_{|j|=0}^{\infty} \frac{(m+k)!}{k!} |a_j^m||f_{m+k}||V_{k+j}(x)|.$$

We observe that if we define

$$\underline{y} = y_1 e_1 + \cdots + y_n e_n := \left(\sqrt{x_1^2 + x_0^2}\right) e_1 + \cdots + \left(\sqrt{x_n^2 + x_0^2}\right) e_n$$

then we have:

$$|V_k(x)| \leq V_k\left(\underline{y}\right).$$

Since

$$V_k\left(\underline{y}\right) = k!\, \Pi_{i=1}^n \left(y_i\right)^{k_i}$$

we get

$$|V_{k+j}(x)| \leq V_{k+j}(\underline{y}) = \frac{(k+j)!}{k!\, j!} V_k\left(\underline{y}\right) V_j\left(\underline{y}\right) \overset{\text{Lemma 2.11 (I)}}{\leq} 2^{|k|+|j|} V_k\left(\underline{y}\right) V_j\left(\underline{y}\right). \tag{33}$$

Moreover, since $|\underline{y}| \leq \sqrt{n}|x|$ and using the Remark 5.2 in combination with the Lemma 4.14, we have

$$\begin{aligned}
|U_L(x,\partial_x)f(x)| &\leq 2^{\frac{n}{2}} \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \left(\sum_{|j|=0}^{\infty} 2^{|j|} |a_j^m| V_j(\underline{y}) \right) \frac{2^{|k|}(m+k)!}{k!} |f_{m+k}| V_k(\underline{y}) \\
&\leq 2^{\frac{n}{2}} C_\varepsilon \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{\varepsilon^{|m|}}{(|m|!)^{1/q}} \exp(B|x|^p) \frac{2^{|k|}(m+k)!}{k!} |f_{m+k}| V_k(\underline{y}).
\end{aligned}$$

Using Lemma 4.14 and the estimates (32), we obtain

$$
\begin{aligned}
|U_L(x,\partial_x)f(x)| &\leq 2^{\frac{n}{2}} C_f C_\varepsilon \sum_{|m|=0}^{\infty}\sum_{|k|=0}^{\infty} \frac{(m+k)!\,\varepsilon^{|m|} b_f^{|m|}}{k!\,(|m|!)^{1/q}} \frac{(2b_f)^{|k|} c(n,m+k)}{\Gamma\left(\frac{|m+k|}{p}+1\right)} \exp(B|x|^p) V_k(\underline{y}) \\
&\overset{\text{Lemma 2.11 (I)+(III)}}{\leq} 2^{\frac{n}{2}} C_f C_\varepsilon \sum_{|m|=0}^{\infty}\sum_{|k|=0}^{\infty} \frac{m!\,(2\varepsilon b_f)^{|m|}}{\Gamma\left(\frac{|m|}{q}+1\right)} \frac{(4b_f)^{|k|} c(n,m+k)}{\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)\Gamma\left(\frac{|k|}{p}+\frac{1}{2}\right)} \exp(B|x|^p) V_k(\underline{y}) \\
&\leq 2^{\frac{n}{2}} C_f C_\varepsilon \sum_{|m|=0}^{\infty} \frac{(\varepsilon 2 b_f)^{|m|}\, c(n,m)\, m!\, \exp(B|x|^p)}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} \sum_{|k|=0}^{\infty} \frac{c(n,m+k)}{c(n,m)c(n,k)} \frac{(4b_f)^{|k|} c(n,k)\, V_k(\underline{y})}{\Gamma\left(\frac{|k|}{p}+\frac{1}{2}\right)} \\
&\leq 2^{\frac{3n}{2}-1}(n-1)!\, C_f C_\varepsilon \sum_{|m|=0}^{\infty} \frac{(\varepsilon 4 b_f)^{|m|}\, c(n,m)\, m!\, \exp(B|x|^p)}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} \sum_{|k|=0}^{\infty} \frac{(8b_f)^{|k|} c(n,k)\, V_k(\underline{y})}{\Gamma\left(\frac{|k|}{p}+\frac{1}{2}\right)},
\end{aligned}
\tag{34}
$$

where the last inequality is due to the following estimate:

$$
\frac{c(n,m+k)}{c(n,m)c(n,k)} = \frac{(n+|m|+|k|-1)!\,((n-1)!)^2\, k!\, m!}{(n-1)!(m+k)!(n+|m|-1)!\,(n+|k|-1)!} \leq (n-1)!\, 2^{n+|m|+|k|-1}.
$$

We observe that:

$$
\begin{aligned}
\frac{m!}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} &\leq \frac{|m|!}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} \\
&\overset{\text{Stirling-DeMoivre}}{\sim} \frac{|m|^{|m|}\sqrt{|m|}\exp(-|m|)}{\left(\frac{|m|}{q}\right)^{\frac{|m|}{q}} \left(\frac{|m|}{p}-\frac{1}{2}\right)^{\frac{|m|}{p}-\frac{1}{2}} |m| \exp(-|m|)} \lesssim \left(p^{\frac{1}{p}} q^{\frac{1}{q}}\right)^{|m|}.
\end{aligned}
$$

In particular the previous inequality implies that:

$$
\sum_{|m|=0}^{\infty} \frac{(\varepsilon 4 b_f)^{|m|}\, c(n,m)\, m!}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} \lesssim \sum_{|m|=0}^{\infty} \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \varepsilon 4 b_f\right)^{|m|} c(n,m). \tag{35}
$$

By the Lemma 4.6 and since $\epsilon > 0$ can be chosen small enough, using the Cauchy-Hadamard's Theorem for the power series, we have that the previous series converges. Thus there exists a constant $C' > 0$ such that

$$
\sum_{|m|=0}^{\infty} \frac{(\varepsilon 4 b_f)^{|m|}\, c(n,m)\, m!}{\Gamma\left(\frac{|m|}{q}+1\right)\Gamma\left(\frac{|m|}{p}+\frac{1}{2}\right)} \leq C'. \tag{36}
$$

By the Lemma 4.14 there exist two constants: $B' > 0$ and $C'' > 0$ such that:

$$
\sum_{|k|=0}^{\infty} \frac{(8b_f)^{|k|} c(n,k)}{\Gamma\left(\frac{|k|}{p}+\frac{1}{2}\right)} V_k(\underline{y}) \leq C'' \exp\left(B'|x|^p\right). \tag{37}
$$

In conclusion by the estimates (36) and (37) we have proved that

$$
|U_L(x,\partial_x)f(x)| \leq 2^{2n-1}(n-1)!\, C'' C_f C_\varepsilon C' \exp\left((B+B')|x|^p\right)
$$

which means that $U_L(x,\partial_x)f(x) \in \mathcal{M}_L^p$ and also that $U_L(x,\partial_x)$ is continuous over $\mathcal{M}_L^p$ i.e. $U_L(x,\partial_x)f(x) \to 0$ as $f \to 0$ or, equivalently, $C_f \to 0$. □

Remark 5.4. If in the Definition 5.1 we use the standard product of the Clifford Algebra instead of the C-K product to define $U_L(x, \partial_x)$ (or $U_R(x, \partial_x)$), the resulting operator does not preserve the monogenicity although it remains a continuous operator from $\mathcal{M}_L^p$ (or $\mathcal{M}_R^p$) to $C^0(\mathbb{R}^{n+1}, \mathbb{R}_n)$. This can be seen starting from the inequality

$$\left| \sum_{|m|=0}^{\infty} u_m(x) \partial_x^m f(x) \right| \leq 2^{\frac{n}{2}} \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{(m+k)!}{k!} |u_m(x)| |f_{m+k}| V_k(\underline{y})$$
$$\leq 2^{\frac{n}{2}} \sum_{|m|=0}^{\infty} \sum_{|k|=0}^{\infty} \frac{(m+k)!}{k!} \frac{\epsilon^q}{(|m|!)^{\frac{1}{q}}} \exp(B|x|^p) |f_{m+k}| V_k(\underline{y})$$

and applying to the last term in the right the same estimates used in (34), (36) and (37).

6 Concluding remarks

The two hyperholomorphic function theories have several applications both in Mathematics and in Physics. Precisely, associated with slice hyperholomorphic functions, [35, 37, 42, 44] it is possible to define the spectral theory on the S-spectrum [31, 37] that has applications in quaternionic quantum mechanics [1, 43], in fractional diffusion processes [28, 29, 30, 34], characteristic operator functions [14], the spectral theorem for quaternionic normal operators [13], the perturbations theory [27], Schur analysis [16] and other fields are under investigation. The monogenic function theory is associated with harmonic analysis in higher dimension. Moreover, there exists a functional calculus based on the Cauchy formula from which the monogenic spectrum was defined, see [47], the function theory has applications to boundary value problems, see [45].

The notion of superoscillatory functions first appears in a series of works of Y. Aharonov, M. V. Berry and co-authors, see [2, 11, 12, 21, 22, 24, 25]. In this context, there are good physical reasons for such a behavior, but the discoverers pointed out the apparently paradoxical nature of such functions, thus opening the way for a more thorough mathematical analysis of the phenomenon. In a series of recent papers there are some systematic study of superoscillations from the mathematical point of view, see [3, 4, 5, 6, 7, 9, 10, 18, 19, 32] and see also [23, 38, 48].

The theory of superoscillations appears in various questions, namely extension of positive definite functions, interpolation of polynomials and also of R-functions and have applications to signal theory and prediction theory of stationary stochastic processes. Thing in some of this area are still under investigation as one can see in the paper [15].

The relation between the two classes of hyperholomorphic function can be seen by rephrasing in modern language the Sce's theorem who generalize a theorem of Fueter for the quaternionic setting in a non trivial and original way to Clifford valued functions, see the book [36] for an overview of this profound theorem of complex and hypercomplex analysis. Precisely, let $\tilde{f}(z) = f_0(u, v) + i f_1(u, v)$ be a holomorphic function defined in a domain (open and connected) D such that $f_0(u, v) + i f_1(u, v)$ satisfy the conditions of the type (5) and let

$$\Omega_D = \{x = x_0 + \underline{x} \ \mid \ (x_0, |\underline{x}|) \in D\}$$

be the open set induced by D in $\mathbb{R}^{n+1}$.

(Step I) The map T_1, defined by: $f(x) = T_1(\tilde{f}) := f_1(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|)$ takes the holomorphic functions $\tilde{f}(z)$ and induces the Clifford-valued function $f(x)$ that is slice hyperholomorphic.

(Step II) The map $T_2 := \Delta_{n+1}^{\frac{n-1}{2}}$ where Δ_{n+1} is the laplacian in $n+1$ dimensions applied to a slice hyperholomorphic function, i.e., $\breve{f}(x) := T_2\Big(f(x)\Big)$, defines a function that is in the kernel of the Dirac operator, i.e., $D\breve{f}(x) = 0 \quad \text{on} \quad \Omega_D$, where D is the Dirac operator.

In the Sce's theorem $\tilde{f}(z) = f_0(u,v) + if_1(u,v)$ is a holomorphic function where $f_0(u,v)$ and $f_1(u,v)$ are real-valued functions, but considering slice monogenic functions, where $f_0(u,v)$ and $f_1(u,v)$ are Clifford-valued functions, and applying the map $T_2 := \Delta_{n+1}^{\frac{n-1}{2}}$, in step (II) of the Sce's construction, we get a function that is in the kernel of the Dirac operator. The case when $\frac{n-1}{2}$ is fractional has been studied by T. Qian; for the precise history of this theorem in the notes of the book [36].

References

[1] S. Adler, *Quaternionic Quantum Mechanics and Quaternionic Quantum Fields*, Volume 88 of *International Series of Monographs on Physics*. Oxford University Press, New York. 1995.

[2] Y. Aharonov, D. Albert, L. Vaidman, *How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100*, Phys. Rev. Lett. **60** (1988), 1351–1354.

[3] Y. Aharonov, J. Behrndt, F. Colombo, P. Schlosser, *Schrödinger evolution of superoscillations with δ- and δ′-potentials*, to appear in Quantum Stud. Math. Found. DOI: 10.1007/s40509-019-00215-4.

[4] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *Some mathematical properties of superoscillations*, J. Phys. A **44** (2011), 365304 (16pp).

[5] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *On the Cauchy problem for the Schrödinger equation with superoscillatory initial data*, J. Math. Pures Appl. **99** (2013), 165–173.

[6] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *Superoscillating sequences as solutions of generalized Schrodinger equations*, J. Math. Pures Appl. **103** (2015), 522–534.

[7] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *Superoscillating sequences in several variables*, J. Fourier Anal. Appl. **22** (2016), 751–767.

[8] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *The mathematics of superoscillations*, Mem. Amer. Math. Soc. **247** (2017), no. 1174, v+107 pp.

[9] Y. Aharonov, F. Colombo, I. Sabadini, D. C. Struppa,J. Tollaksen, *How superoscillating tunneling waves can overcome the step potential*, Ann. Physics, **414** (2020), 168088, 19 pp.

[10] Y. Aharonov, F. Colombo, I. Sabadini, D.C. Struppa, J. Tollaksen, *Evolution of Superoscillations in the Klein-Gordon Field*, Milan J. Math., **88** (2020), 171–189.

[11] Y. Aharonov, D. Rohrlich, *Quantum Paradoxes: Quantum Theory for the Perplexed*, Wiley-VCH Verlag, Weinheim, 2005.

[12] Y. Aharonov, L. Vaidman, *Properties of a quantum system during the time interval between two measurements*, Phys. Rev. A **41** (1990), 11–20.

[13] D. Alpay, F. Colombo, D. P. Kimsey, *The spectral theorem for for quaternionic unbounded normal operators based on the S-spectrum*, J. Math. Phys., **57**(2016), pp. 023503, 27.

[14] D. Alpay, F. Colombo, I. Sabadini, *Quaternionic de Branges spaces and characteristic operator function*, SpringerBriefs in Mathematics, Springer, Cham, 2020.

[15] D. Alpay, F. Colombo, I. Sabadini, *Superoscillations and analytic extension*, submitted to Journal of Fourier Analysis and Applications.

[16] D. Alpay, F. Colombo, I. Sabadini, *Slice Hyperholomorphic Schur Analysis*, Operator Theory: Advances and Applications, 256. Birkhäuser/Springer, Cham, 2016. xii+362 pp.

[17] T. Aoki, F. Colombo, I. Sabadini, D. C. Struppa, *Continuity theorems for a class of convolution operators and applications to superoscillations*, Ann. Mat. Pura Appl., (4) **197** (2018), no. 5, 1533–1545.

[18] T. Aoki, F. Colombo, I. Sabadini, D. C. Struppa, *Continuity of some operators arising in the theory of superoscillations*, Quantum Stud. Math. Found., **5** (2018), 463–476.

[19] J. Behrndt, F. Colombo, P. Schlosser, *Evolution of Aharonov–Berry superoscillations in Dirac δ-potential*, Quantum Stud. Math. Found., **6** (2019), 279–293.

[20] M.V. Berry et al, *Roadmap on superoscillations*, 2019, Journal of Optics **21**, 053002.

[21] M.V. Berry, *Faster than Fourier*, in Quantum Coherence and Reality; in celebration of the 60th Birthday of Yakir Aharonov ed. J.S.Anandan and J. L. Safko, World Scientific, Singapore, (1994), pp.55–65.

[22] M.V. Berry, *Evanescent and real waves in quantum billiards and Gaussian beams*, J. Phys. A., **27** (1994), 391.

[23] M.V. Berry, *Representing superoscillations and narrow Gaussians with elementary functions*, Milan J. Math., **84** (2016), 217–230.

[24] M.V. Berry, M.R. Dennis, *Natural superoscillations in monochromatic waves in D dimension*, J. Phys. A, **42** (2009), 022003.

[25] M.V. Berry, S. Popescu, *Evolution of quantum superoscillations, and optical superresolution without evanescent waves*, J. Phys. A **39** (2006), 6965–6977.

[26] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Research Notes in Mathematics, 76. Pitman (Advanced Publishing Program), Boston, MA, 1982. x+308 pp.

[27] P. Cerejeiras, F. Colombo, U. Kähler, I. Sabadini, *Perturbation of normal quaternionic operators*, Trans. Amer. Math. Soc., **372** (2019), 3257–3281.

[28] F. Colombo, D. Deniz Gonzalez, S. Pinton, *Fractional powers of vector operators with first order boundary conditions*, J. Geom. Phys., **151** (2020), 103618, 18 pp.

[29] F. Colombo, J. Gantner, *An application of the S-functional calculus to fractional diffusion processes*, Milan J. Math., **86** (2018), 225–303.

[30] F. Colombo, J. Gantner, *Quaternionic closed operators, fractional powers and fractional diffusion process*, Operator Theory: Advances and Applications, vol 274, 2019. ISBN 978-3-030-16409-6.

[31] F. Colombo, J. Gantner, D.P. Kimsey, *Spectral theory on the S-spectrum for quaternionic operators*, Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham, 2018. ix+356 pp. ISBN: 978-3-030-03073-5; 978-3-030-03074-2 47-02.

[32] F. Colombo, J. Gantner, D.C. Struppa, *Evolution by Schrödinger equation of Aharonov-Berry superoscillations in centrifugal potential*, Proc. A., **475** (2019), no. 2225, 20180390, 17 pp.

[33] F. Colombo, J. O. Gonzalez-Cervantes, I. Sabadini, *A nonconstant coefficients differential operator associated to slice monogenic functions*, Trans. Amer. Math. Soc., **365** (2013), 303–318.

[34] F. Colombo, M. Peloso, S. Pinton, *The structure of the fractional powers of the noncommutative Fourier law*, Math. Methods Appl. Sci., **42** (2019), no. 18, 6259–6276.

[35] F. Colombo, I. Sabadini, D.C. Struppa, *Entire Slice Regular Functions*, SpringerBriefs in Mathematics. Springer, Cham, 2016. v+118 pp.

[36] F. Colombo, I. Sabadini, D.C. Struppa, *Michele Sce's Works in Hypercomplex Analysis. A Translation with Commentaries*, Birkhäuser, 2020, Hardcover ISBN 978-3-030-50215-7.

[37] F. Colombo, I. Sabadini, D. C. Struppa, *Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions*, Volume 289 of *Progress in Mathematics.* Birkhäuser/Springer Basel AG, Basel. 2011.

[38] F. Colombo, I. Sabadini, D.C. Struppa, A. Yger, *Superoscillating sequences and hyperfunctions*, Publ. Res. Inst. Math. Sci., **55** (2019), 665-–688.

[39] D. Constales, R. De Almeida, R. S. Krausshar, *On the growth type of entire monogenic functions*, Arch. Math., **88** (2007), pp. 153–163.

[40] D. Constales, R. De Almeida, R. S. Krausshar, *On the relation between the growth and the Taylor coefficients of entire solutions to the higher–dimensional Cauchy–Riemann system in* $\mathbb{R}^{n+1}$, J. Math. Anal. Appl., **327** (2007), pp. 763–775.

[41] D. Constales, R. S. Krausshar, *Representation formulas for the general derivatives of the fundamental solution of the Cauchy–Riemann operator in Clifford analysis and applications*, Z. Anal. Anwendungen, **21** (2002), pp. 579–597.

[42] S. G. Gal, I. Sabadini, *Quaternionic approximation. With application to slice regular functions*, Frontiers in Mathematics. Birkhäuser/Springer, Cham, 2019. x+221 pp.

[43] J. Gantner, *On the equivalence of complex and quaternionic quantum mechanics*, Quantum Stud. Math. Found., **5** (2018), 357–390.

[44] G. Gentili, C. Stoppato, D. C. Struppa, *Regular functions of a quaternionic variable*, Springer Monographs in Mathematics. Springer, Heidelberg, 2013. x+185 pp.

[45] K. Gürlebeck, W. Sprössig, *Quaternionic analysis and elliptic boundary value problems*, International Series of Numerical Mathematics, 89. Birkhäuser Verlag, Basel, 1990. 253 pp.

[46] K. Gürlebeck, K. Habetha, W. Sprössig, *Application of holomorphic functions in two and higher dimensions*, Birkhäuser/Springer, [Cham], 2016. xv+390 pp.

[47] B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin, 2004.

[48] A. Kempf, *Four aspects of superoscillations*, Quantum Stud. Math. Found., **5** (2018), 477–484.