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Totally Multicolored Rado Numbers For the Equation x_1 + x_2 + x_3 + ... + x_(m−1) = x_m

Skylar Halverson
South Dakota State University, shalverson1996@gmail.com

TOTALLY MULTICOLORED RADO NUMBERS FOR THE EQUATION

$x_1 + x_2 + x_3 + ... + x_{m-1} = x_m$

BY

SKYLAR HALVERSON

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This thesis is approved as a creditable and independent investigation by a candidate for the master's degree and is acceptable for meeting the thesis requirements for this degree. Acceptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Daniel Schaal
Advisor Date

Kurt Cogswell
Department Head Date

Nicole Lounsbery, PhD
Director, Graduate School Date

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CONTENTS

LIST OF TABLES

ABSTRACT

TOTALLY MULTICOLORED RADO NUMBERS FOR THE EQUATION

$x_1 + x_2 + x_3 + ... + x_{m-1} = x_m$

SKYLAR HALVERSON

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A set is called Totally Multicolored (TMC) if no elements in the set are colored the same. For all natural numbers t, m, let $R(t, m)$ be the least natural number n such that for every t-coloring of the set $\{1, 2, 3, ..., R(t, m)\}$ there exist a solution set $\{x_i\}_{i=1}^{m}$ to $L(m)$, $x_1 + x_2 + x_3 + ... + x_{m-1} = x_m$ such that $x_i \neq x_j$ for all $i \neq j$, that avoids being Totally Multicolored. For all natural numbers $t \geq 1$ and $m \geq 3$ let $h = \sum_{i=1}^{m-2}(i)$, $j = \sum_{i=1}^{m-3}(i)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, and $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$. This paper shows that,

$$R(t,m) = \begin{cases} \frac{(m-1)m}{2} & \text{if } t < m \\ \frac{(m-1)m}{2} + 2 & \text{if } t = m \\ \frac{6t+5}{5} & \text{if } t > m,\ m = 3, \text{ and } t \equiv 0 mod(5) \\ \frac{6t+9}{5} & \text{if } t > m,\ m = 3, \text{ and } t \equiv 1 mod(5) \\ \frac{6t+3}{5} & \text{if } t > m,\ m = 3, \text{ and } t \equiv 2 mod(5) \\ \frac{6t+7}{5} & \text{if } t > m,\ m = 3, \text{ and } t \equiv 3 mod(5) \\ \frac{6t+6}{5} & \text{if } t > m,\ m = 3, \text{ and } t \equiv 4 mod(5) \\ max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\} & \text{if } t > m \text{ and } m \geq 4. \end{cases}$$

INTRODUCTION

Defintion 1. Let $[a, b]$ denote $\{n \in \mathbb{N} : a \leq n \leq b\}$. A *t-coloring* is a function $\Delta : [1, n] \longrightarrow [1, t]$.

Defintion 2. For every natural number m, let $L(m)$ represent the system consisting of

$$x_1 + x_2 + x_3 + ... + x_{m-1} = x_m$$

such that $x_i \neq x_j$ for all $i \neq j$.

Defintion 3. A solution $\{x_i\}_{i=1}^{m}$ to $L(m)$ is *monochromatic* if and only if

$$\Delta(x_1) = \Delta(x_2) = ... = \Delta(x_m).$$

Shur [19], in 1916, proved that for every $t \geq 2$ there exists a least natural number $n = S(t)$ such that for every t-coloring of the set $[1, n]$, there exists a monochromatic solution to $B(3)$ which is the same as $L(m)$ without the condition that $x_i \neq x_j$ for all $i \neq j$. We call the natural numbers $S(t)$ *Shur numbers* of which only $S(2) = 5$, $S(3) = 14$, $S(4) = 45$, and $S(5) = 160$ are known [20].

R. Rado, in 1933, expanded this by generalizing the Shur numbers to arbitrary systems of linear equations. Rado found conditions to determine if an arbitrary systems of linear equations admits a monochromatic solution for every t-coloring of $[1, n]$ [11,12,13]. For a linear system L, if a least natural number n exist such that for every t-coloring of $[1, n]$ there exist a monochromatic solution to L then n is call the t-color *Rado number* for the linear system L. If n does not exist, then the t-color Rado number for the linear system L is infinite. Since then the Rado numbers for many families of equations have been found and this paper looks to add another to that growing list [2,3,4,7,8,9,10,14,15]. One of these Families of equations is the Beutelspacher equation $B(m)$. In 1982, Beutelspacher found the 2-color monochromatic Rado number for $B(m)$ as $m^2 - m - 1$ for all $m \geq 3$ [1]. The main results of this paper uses $L(m)$ and we will show the t-color Totally Multicolored Rado number for $t \geq 2$ and $m \geq 3$ using the definitions below.

Defintion 4. A solution set $\{x_i\}_i^m$ to $L(m)$ is *Totally Multicolored* (*TMC*) if $\Delta(x_i) \neq \Delta(x_j)$ for all $i \neq j$.

Defintion 5. For every natural number m, let $R(t, m)$ represent the t-color *TMC-Rado* number for $L(m)$. That is, $R(t, m)$ is the smallest natural number such that for all $\Delta : [1, R(t, m)] \longrightarrow [1, t]$ there exist a solution set in $[1, R(t, m)]$ to $L(m)$ that is not TMC.

Defintion 6. The set $[a, b]_{odd}$ is equivalent to $\{x \in [a, b] : x \in \mathbb{Z} \ is \ odd\}$.

BACKGROUND

Here we will prove what Beutelspacher found. That the 2-color monochromatic Rado number for $B(m)$ is $m^2 - m - 1$ for all $m \geq 3$.

Lower Bound:

Let $\Delta : [1, m^2 - m - 2] \longrightarrow [1, 2]$ be defined by

$$\Delta(x) = \begin{cases} 1 & \text{if } x \in [1, m-2] \cup [m^2 - 2m + 1, m^2 - m - 2] \\ 2 & \text{if } x \in [m-1, m^2 - 2m]. \end{cases}$$

We will show $\Delta : [1, m^2 - m - 2] \longrightarrow [1, 2]$ contains no monochromatic solution sets to $B(m)$.

For any set $\{x_i\}_{i=1}^{m}$ such that $\Delta(x_i) = 2$ for all $i = 1, 2, ..., m$ we have $x_i \in [m-1, m^2 - 2m]$ for all $i = 1, 2, ..., m$ and,

$$\begin{aligned} x_1 + x_2 + x_3 + ... + x_{m-1} &\geq (m-1) + (m-1) + ... + (m-1) \\ &= (m-1)(m-1) = m^2 - 2m + 1 > m^2 - 2m \geq x_m. \end{aligned}$$

Thus $\{x_i\}_{i=1}^{m}$ is not a solution to $B(m)$.

Suppose there exist a set $\{x_i\}_{i=1}^{m}$ that is a solution to $B(m)$ such that $\Delta(x_i) = 1$ for all $i = 1, 2, ..., m$ we have $x_i \in [1, m-2]$ or $x_i \in [m^2 - 2m + 1, m^2 - m - 2]$ for all $i = 1, 2, ..., m$.

If for all $i = 1, 2, ..., m$ we have $x_i \in [1, m-2]$, then we have,

$$x_m = x_1 + x_2 + x_3 + ... + x_{m-1} \geq 1 + 1 + ... + 1 = m - 1 > m - 2$$

and $x_m \notin [1, m-2]$ which gives us a contradiction. So let at least one of the elements $x_i \in [m^2 - 2m + 1, m^2 - m - 2]$ for some $i = 1, 2, ..., m$.

If only $x_m \in [m^2 - 2m + 1, m^2 - m - 2]$ then,

$$x_m = x_1 + x_2 + x_3 + ... + x_{m-1} \leq (m-2) + (m-2) + ... + (m-2)$$
$$= (m-1)(m-2) = m^2 - 3m + 2 \leq m^2 - 2m - 1 < m^2 - 2m + 1.$$

Thus $x_m \notin [m^2 - 2m + 1, m^2 - m - 2]$ however this gives us a contradiction.

If there exist another $x_i \in [m^2 - 2m + 1, m^2 - m - 2]$ then,

$$x_m = x_1 + x_2 + x_3 + ... + x_{m-1} \geq (m^2 - 2m + 1) + 1 + 1 + ... + 1 =$$
$$(m^2 - 2m + 1) + (m - 2) = m^2 - m - 1 > m^2 - m - 2.$$

Thus $x_m \notin [m^2 - 2m + 1, m^2 - m - 2]$ which contradicts what we had. Therefore $\Delta : [1, m^2 - m - 2] \longrightarrow [1, 2]$ as defined contains no monochromatic solution sets to $B(m)$.

Upper Bound:

First let $\Delta : [1, m^2 - m - 1] \longrightarrow [1, 2]$ be an arbitrary coloring. We will show there exist a monochromatic solution to $B(m)$.

Arbitrarily let $\Delta(1) = 1$. If $\Delta(m-1) = 1$ then $\{1, 1, ..., 1, m-1\}$, with $m-1$ many 1's, is a monochromatic solution to $B(m)$. So let $\Delta(m-1) = 2$. If $\Delta(m^2 - 2m + 1) = 2$, then $\{m-1, m-1, ..., m-1, m^2 - 2m + 1\}$, with $m-1$ many $(m-1)$'s, is a monochromatic solution to $B(m)$.

So let $\Delta(m^2 - 2m + 1) = 1$. If $\Delta(m) = 1$, then $\{1, m, m, ..., m, m^2 - 2m + 1\}$, with $m-2$ many m's, is a monochromatic solution to $B(m)$. So let $\Delta(m)=2$. If $\Delta(m^2-m-1)=2$, then $\{m-1,m,m,...,m,m^2-m-1\}$, with $m-2$ many m's, is a monochromatic solution to $B(m)$. If $\Delta(m^2-m-1)=1$, then $\{m^2-2m+1,1,1,...,1,m^2-m-1\}$, with $m-2$ many 1's, is a monochromatic solution to $B(m)$.

Therefore the 2-color monochromatic Rado number for $B(m)$ is m^2-m-1 for all $m \geq 3$.

MAIN RESULTS

Lemma 1. *For* $m = 3$ *and* $\Delta : [1, n] \longrightarrow [1, t]$ *that has only TMC solution sets to* $L(m)$*, if there exist* $a, b \in [1, n]$ *such that* $a < b$ *and* $\Delta(a) = \Delta(b)$ *then* b *is even and* $a = b/2$.

Proof. Suppose $m = 3$ and $\Delta : [1, n] \longrightarrow [1, t]$, there exist $a, b \in [1, n]$ such that $a < b$ and $\Delta(a) = \Delta(b)$. Let $c = b - a$. If $c \neq a$ then $a + c = b$ and $\{a, c, b\}$ is not TMC. If $c = a$ then $2a = b$ or $a = \frac{b}{2}$. Which implies that b is even. □

Lemma 2. *For* $\Delta : [1, n] \longrightarrow [1, t]$ *that is TMC, if there exist* $a, b \in [1, n]$ *such that* $a < b$ *and* b *is odd, then* $\Delta(a) \neq \Delta(b)$.

Proof. Suppose for a TMC $\Delta : [1, n] \longrightarrow [1, t]$, there exist $a, b \in [1, n]$ such that $a < b$ and b is odd. If $\Delta(a) = \Delta(b)$ then by Lemma 1 b is even. However, we assumed b to be odd so we have a contradiction. □

Lemma 3. *For* $\Delta : [1, n] \longrightarrow [1, t]$*, if there exist* $a, b \in [1, n]$ *such that* $b > a \geq m$*,* $\Delta(a) = \Delta(b)$*, and* $b - a \geq \sum_{i=1}^{m-2} i = h$ *then* Δ *has a solution set to* $L(m)$ *that avoids being TMC.*

Proof. Let $\Delta : [1, n] \longrightarrow [1, t]$, such that there exist $a, b \in [1, n]$ with $b > a \geq m$, $b - a \geq \sum_{i=1}^{m-2} i$, and $\Delta(a) = \Delta(b)$. Also, let $c_1 = b - a - \sum_{i=1}^{m-3} i$. Then,

$$b > c_1 = b - a - \sum_{i=1}^{m-3} i \geq \sum_{i=1}^{m-2} i - \sum_{i=1}^{m-3} i = m - 2 > m - 3 > ... > 2 > 1.$$

If $c_1 \neq a$ then,

$$\sum_{i=1}^{m-3} i + c_1 + a = \sum_{i=1}^{m-3} i + b - a - \sum_{i=1}^{m-3} i + a = b.$$

Thus $\{1, 2, ..., m-4, m-3, c_1, a, b\} \subseteq [1, n]$ is not TMC.

If $a = c_1 = b - a - \sum_{i=1}^{m-3} i$ let $c_2 = a - 1 \geq m - 1$. Then $b = 2a + \sum_{i=1}^{m-3} i$ and,

$$\begin{aligned} a + c_2 + 1 + 2 + ... + (m-5) + (m-4) + (m-2) &= \\ a + (a-1) + 1 + 2 + ... + (m-5) + (m-4) + (m-2) &= \\ 2a + 1 + 2 + ... + (m-5) + (m-4) + (m-3) &= \\ 2a + \sum_{i=1}^{m-3} i = b. \end{aligned}$$

Thus $\{1, 2, ..., m-5, m-4, m-2, c_2, a, b\} \subseteq [1, n]$ is not TMC.

Therefore in both cases $\Delta : [1, n] \longrightarrow [1, t]$ has a solution set to $L(m)$ that is not TMC. □

Lemma 4. *For* $m \geq 4$, $t > m$, $h = \sum_{i=1}^{m-2}(i)$, $j = \sum_{i=1}^{m-3}(i)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, *and* $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$ *we have* $m - 1 \leq \underline{s} \leq \overline{s}$.

Proof. Now with $h = \sum_{i=1}^{m-2}(i)$, $j = \sum_{i=1}^{m-3}(i)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, and $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$ we have $h \geq 6$, $t \geq m + 1$ and,

$$\begin{aligned} h < h + \frac{h-3}{m} = \frac{(m+1)h-3}{m} &\leq \frac{th-3}{m} \\ j + (m-2) + hm - h = h + hm - h = hm &< th - 3 \\ (m-1)(1+h) = m - 1 + hm - h &< th - j - 2 \\ m - 1 &< \frac{th-j-2}{1+h}. \end{aligned}$$

Further with $m - 1 \in \mathbb{Z}$ this implies that $m - 1 \leq \underline{s} \leq \overline{s}$. □

Theorem 1. *For all integers $t \geq 1$ and $m \geq 3$ let $h = \sum_{i=1}^{m-2}(i)$, $j = \sum_{i=1}^{m-3}(i)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, and $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$. Then we have*

$$R(t,m) = \begin{cases} \frac{(m-1)m}{2} & \text{if } t < m \\ \frac{(m-1)m}{2} + 2 & \text{if } t = m \\ \frac{6t+5}{5} & \text{if } t > m,\ m = 3,\ \text{and } t \equiv 0 mod(5) \\ \frac{6t+9}{5} & \text{if } t > m,\ m = 3,\ \text{and } t \equiv 1 mod(5) \\ \frac{6t+3}{5} & \text{if } t > m,\ m = 3,\ \text{and } t \equiv 2 mod(5) \\ \frac{6t+7}{5} & \text{if } t > m,\ m = 3,\ \text{and } t \equiv 3 mod(5) \\ \frac{6t+6}{5} & \text{if } t > m,\ m = 3,\ \text{and } t \equiv 4 mod(5) \\ max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\} & \text{if } t > m \text{ and } m \geq 4. \end{cases}$$

Proof. Let t and m be integers such that $t \geq 1$ and $m \geq 3$. Also, let $h = \sum_{i=1}^{m-2}(i)$, $j = \sum_{i=1}^{m-3}(i)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, and $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$.

Case 1: Suppose $t < m$, we will show that $R(t,m) = \frac{(m-1)m}{2}$.

Lower Bound:

Let $\Delta : [1, \sum_{i=1}^{m-1}(i) - 1] \longrightarrow [1,t]$. Then for any solution set $\{x_i\}_{i=1}^{m} \subseteq [1, \sum_{i=1}^{m-1}(i) - 1]$ we have $x_m = \sum_{i=1}^{m-1} x_i \geq \sum_{i=1}^{m-1}(i) - 1$ and $x_m \notin [1, \sum_{i=1}^{m-1}(i) - 1]$. Therefore $\Delta : [1, \sum_{i=1}^{m-1}(i) - 1] \longrightarrow [1,t]$ has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a coloring $\Delta : [1, \sum_{i=1}^{m-1}(i)] \longrightarrow [1,t]$ that has only TMC solution sets to $L(m)$. Since $t < m$ there exist $a, b \in \{1, 2, ..., m-2, m-1, \sum_{i=1}^{m-1}(i)\}$ such that $\Delta(a) = \Delta(b)$. Thus $\Delta : [1, \sum_{i=1}^{m-1}(i)] \longrightarrow [1,t]$ has a solution set $\{1, 2, ...n-2, n-1, \sum_{k=1}^{n-1} k\}$ to $L(m)$ that is not TMC. Therefore if $t < m$ then $R(t,m) = \frac{(m-1)m}{2}$.

Case 2: Suppose $t = m$, we will show that $R(t,m) = \frac{(m-1)m}{2} + 2$.

Lower Bound:

Let $\Delta : [1, \sum_{i=1}^{m-1}(i)+1] \longrightarrow [1,t]$ be defined by

$$\Delta(x) = \begin{cases} x & \text{if } x < m-1 \\ t-1 & \text{if } x = m-1, m \\ t & \text{if } x > m. \end{cases}$$

Now for if $\Delta(a) = \Delta(b)$ then $a, b \in [m-1, \sum_{i=1}^{m-1}(i)+1]$. Then for any $\{x_i\}_{i=1}^{n-1} \subseteq [1, \sum_{i=1}^{m-1}(i)+1]$ such that $a, b \in \{x_i\}_{i=1}^{m-1}$ we have,

$$x_m = \sum_{i=1}^{m-1} x_i \geq \sum_{i=1}^{m-3}(i) + a + b \geq \sum_{i=1}^{m-3}(i) + (m-1) + m$$
$$= \sum_{i=1}^{m-3}(i) + (m-2) + (m-1) + 2 = \sum_{i=1}^{m-1}(i) + 2 > \sum_{i=1}^{m-1}(i) + 1.$$

Thus $\Delta : [1, \sum_{i=1}^{m-1}(i)+1] \longrightarrow [1,t]$ only has TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist $\Delta : [1, \sum_{i=1}^{m-1}(i)+2] \longrightarrow [1,t]$ that only has TMC solution sets to $L(m)$. With $\sum_{i=1}^{m-1}(i) + 2 > m = t$, we have that there exist $a, b \in [1, \sum_{i=1}^{m-1}(i)+2]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Let $a, b \in [1, m]$. If $b = m$ and $a \leq m-1$ let $x_{m-1} = b$, $x_{m-2} = a$, and $x_i = i$ for $i = 1, 2, ..., m-3$. If $b \neq m$ let $x_i = i$ for $i = 1, 2, ..., m-1$. Then,

$$x_m = \sum_{i=1}^{m-1}(x_i) \leq \sum_{i=1}^{m-3}(i) + m - 1 + m = \sum_{i=1}^{m-1}(i) + 2.$$

Thus, $x_m \in [1, \sum_{i=1}^{m-1}(i)+2]$ and $\{x_i\}_{i=1}^{m}$ is not TMC. Then for all $a, b \in [1, m]$ we have $\Delta(a) \neq \Delta(b)$. Further, with $t = m$ for all $b \in [m+1, \sum_{i=1}^{m-1}(i)+2]$ there exactly one $a \in [1, m]$ such that $\Delta(a) = \Delta(b)$.

Consider $b = \sum_{i=1}^{m-1}(i) + 2$. Then $\sum_{i=1}^{m-3}(i) + (m-1) + m = \sum_{i=1}^{m-1}(i) + 2$. This implies that $\Delta(b) \notin \{\{i\}_{i=1}^{m-3}, m-1, m\}$ and $\Delta(b) = m-2$.

Now $\sum_{i=1}^{m-2}(i) + m + 1 = \sum_{i=1}^{m-1}(i) + 2$. Thus $\{\{i\}_{i=1}^{m-2}, m+1, b\}$ is not TMC. Therefore $\Delta : [1, \sum_{i=1}^{m-1}(i) + 2] \longrightarrow [1, t]$ has a solution to $L(m)$ that is not TMC.

Case 3: Let $n = 3$ and $t > m$.

Sub-Case 3a: Suppose $t \equiv 0 mod(5)$.

Lower Bound:

Let $\Delta : [1, \frac{6t}{5}] \longrightarrow [1, t]$ be defined by

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \frac{4t+5}{5} \\ \frac{x}{2} & \text{if } x > \frac{4t+5}{5} \text{ and } x \text{ is even} \\ \frac{5x+4t+5}{5} & \text{if } x > \frac{4t+5}{5} \text{ and } x \text{ is odd.} \end{cases}$$

By the definition of Δ there does not exist $a, b \in [1, \frac{4t+5}{5}]$ such that $a \neq b$ and $\Delta(a) = \Delta(b)$. Let $a, b \in [\frac{4t+10}{5}, \frac{6t}{5}]$ then,

$$c = a + b \geq \frac{4t+10}{5} + \frac{4t+15}{5} = \frac{8t+25}{5} > \frac{6t}{5}.$$

Thus $c \neq [1, \frac{6t}{5}]$.

Let $a \in [1, \frac{4t+5}{5}]$ and $b \in [\frac{4t+10}{5}, \frac{6t}{5}]$. If b is even then $b = 2a$ and,

$$c = a + b \geq \frac{4t+10}{10} + \frac{4t+10}{5} = \frac{12t+30}{10} = \frac{6t+15}{5} > \frac{6t}{5}.$$

Thus $c \neq [1, \frac{6t}{5}]$. If b is odd. Then, $\Delta(b) = \frac{5x+4t+5}{5} > \frac{4t+5}{5} \geq \Delta(a)$. Therefore $\Delta : [1, \frac{6t}{5}] \longrightarrow [1, t]$ as defined above has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a $\Delta : [1, \frac{6t+5}{5}] \longrightarrow [1, t]$ that has only TMC solution sets to $L(m)$. Since $t < \frac{6t+5}{5}$ there exist $a, b \in [1, \frac{6t+5}{5}]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Part 1: Suppose $a, b \in [1, \frac{4t}{5}]$. By Lemma 1 $a = \frac{b}{2}$, then

$$c = a + b \leq \frac{4t}{10} + \frac{4t}{5} = \frac{12t}{10} = \frac{6t}{5} < \frac{6t+5}{5}.$$

This implies there exist $c \in [1, \frac{6t+5}{5}]$ such that $a + b = c$ and $\{a, b, c\}$ is not TMC. Arbitrarily let $\Delta(x) = x$ for $x \in [1, \frac{4t}{5}]$.

Part 2: Consider $a, b \in [\frac{4t+5}{5}, \frac{6t+5}{5}]_{odd}$ and $c \in [1, \frac{4t}{5}]$ such that $c < a < b$. By Lemma 2, since a, b are odd we have $\Delta(a) \neq \Delta(b) \neq \Delta(c)$. From (Part 1) above, $\Delta(a), \Delta(b) \notin [1, \frac{4t}{5}]$ and $\Delta(a), \Delta(b) \in [\frac{4t+5}{5}, t]$.

Now there are $t - \frac{4t+5}{5} = \frac{t-5}{5}$ colors in $[\frac{4t+5}{5}, t]$. Further there are $\frac{\frac{6t+5}{5} - \frac{4t+5}{5}}{2} + 1 = \frac{t+5}{5}$ many elements in $[\frac{4t+5}{5}, \frac{6t+5}{5}]_{odd}$. So we have $\frac{t-5}{5}$ many colors to color $\frac{t+5}{5}$ many elements. By the PHP, with $\frac{t-5}{5} < \frac{t+5}{5}$ there exist $a, b \in [\frac{4t+5}{5}, \frac{6t+5}{5}]_{odd}$ such that $\Delta(a) = \Delta(b)$. This contradiction Lemma 2 and $\Delta : [1, \frac{6t+5}{5}] \longrightarrow [1, t]$ has a solution set to $L(m)$ that avoids being TMC.

Sub-Case 3b: Suppose $t \equiv 1 mod(5)$.

Lower Bound:

Let $\Delta : [1, \frac{6t+4}{5}] \longrightarrow [1, t]$ be defined by

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \frac{4t+1}{5} \\ \frac{x}{2} & \text{if } x > \frac{4t+1}{5} \text{ and } x \text{ is even} \\ \frac{5x+4t+1}{5} & \text{if } x > \frac{4t+1}{5} \text{ and } x \text{ is odd.} \end{cases}$$

By the definition of Δ there does not exist $a, b \in [1, \frac{4t+1}{5}]$ such that $a \neq b$ and $\Delta(a) = \Delta(b)$. Let $a, b \in [\frac{4t+6}{5}, \frac{6t+4}{5}]$ then,

$$c = a + b \geq \frac{4t+6}{5} + \frac{4t+11}{5} = \frac{8t+17}{5} > \frac{6t+4}{5}.$$

Thus $c \neq [1, \frac{6t+4}{5}]$.

Let $a \in [1, \frac{4t+1}{5}]$ and $b \in [\frac{4t+6}{5}, \frac{6t+4}{5}]$. If b is even then $b = 2a$ and,

$$c = a + b \geq \frac{4t+6}{10} + \frac{4t+6}{5} = \frac{12t+18}{10} = \frac{6t+9}{5} > \frac{6t+4}{5}.$$

Thus $c \neq [1, \frac{6t+4}{5}]$. If b is odd. Then, $\Delta(b) = \frac{5x+4t+1}{5} > \frac{4t+1}{5} \geq \Delta(a)$. Therefore $\Delta : [1, \frac{6t+4}{5}] \longrightarrow [1, t]$ as defined above have only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a $\Delta : [1, \frac{6t+9}{5}] \longrightarrow [1, t]$ that has only TMC solution sets to $L(m)$. Since $t < \frac{6t+9}{5}$ there exist $a, b \in [1, \frac{6t+9}{5}]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Part 1: Suppose $a, b \in [1, \frac{4t+6}{5}]$. By Lemma 1 $a = \frac{b}{2}$, then

$$c = a + b \leq \frac{4t+6}{10} + \frac{4t+6}{5} = \frac{12t+18}{10} = \frac{6t+9}{5}.$$

This implies there exist $c \in [1, \frac{6t+9}{5}]$ such that $a + b = c$ and $\{a, b, c\}$ is not TMC. Arbitrarily let $\Delta(x) = x$ for $x \in [1, \frac{4t+6}{5}]$.

Part 2: Consider $a, b \in [\frac{4t+11}{5}, \frac{6t+9}{5}]_{odd}$ and $c \in [1, \frac{4t+6}{5}]$ such that $c < a < b$. By Lemma 2, since a, b are odd we have $\Delta(a) \neq \Delta(b) \neq \Delta(c)$. From (Part 1) above, $\Delta(a), \Delta(b) \notin [1, \frac{4t+6}{5}]$ and $\Delta(a), \Delta(b) \in [\frac{4t+11}{5}, t]$.

Now there are $t - \frac{4t+6}{5} = \frac{t-6}{5}$ colors in $[\frac{4t+11}{5}, t]$. Further there are $\frac{\frac{6t+9}{5} - \frac{4t+11}{5}}{2} + 1 = \frac{t+4}{5}$ many elements in $[\frac{4t+11}{5}, \frac{6t+9}{5}]_{odd}$. So we have $\frac{t-6}{5}$ many colors to color $\frac{t+4}{5}$ many elements. By the PHP, with $\frac{t-6}{5} < \frac{t+4}{5}$ there exist $a, b \in [\frac{4t+11}{5}, \frac{6t+9}{5}]_{odd}$ such that $\Delta(a) = \Delta(b)$. This contradiction Lemma 2 and $\Delta : [1, \frac{6t+9}{5}] \longrightarrow [1, t]$ has a solution set to $L(m)$ that avoids being TMC.

Sub-Case 3c: Suppose $t \equiv 2 mod(5)$.

Lower Bound:

Let $\Delta : [1, \frac{6t-2}{5}] \longrightarrow [1, t]$ be defined defined as below.

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \frac{4t-3}{5} \\ \frac{x}{2} & \text{if } x > \frac{4t-3}{5} \text{ and } x \text{ is even} \\ \frac{5x+4t-3}{5} & \text{if } x > \frac{4t-3}{5} \text{ and } x \text{ is odd} \end{cases}$$

By the definition of Δ there does not exist $a, b \in [1, \frac{4t-3}{5}]$ such that $a \neq b$ and $\Delta(a) = \Delta(b)$. Let $a, b \in [\frac{4t+2}{5}, \frac{6t-2}{5}]$ then,

$$c = a + b \geq \frac{4t+2}{5} + \frac{4t+7}{5} = \frac{8t+9}{5} = \frac{8t-1}{5} + 2 > \frac{6t-2}{5}.$$

Thus $c \neq [1, \frac{6t-2}{5}]$.

Let $a \in [1, \frac{4t-3}{5}]$ and $b \in [\frac{4t+2}{5}, \frac{6t-2}{5}]$. If b is even then $b = 2a$ and,

$$c = a + b \geq \frac{4t+2}{10} + \frac{4t+2}{5} = \frac{12t+6}{10} = \frac{6t+3}{5} > \frac{6t-2}{5}.$$

Thus $c \neq [1, \frac{6t-2}{5}]$. If b is odd. Then, $\Delta(b) = \frac{5x+4t-3}{5} > \frac{4t-3}{5} \geq \Delta(a)$. Therefore $\Delta : [1, \frac{6t-2}{5}] \longrightarrow [1, t]$ as defined above has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a $\Delta : [1, \frac{6t+3}{5}] \longrightarrow [1, t]$ that has only TMC solution sets to $L(m)$. Since $t < \frac{6t+3}{5}$ there exist $a, b \in [1, \frac{6t+3}{5}]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Part 1: Suppose $a, b \in [1, \frac{4t+2}{5}]$. By Lemma 1 $a = \frac{b}{2}$, then

$$c = a + b \leq \frac{4t+2}{10} + \frac{4t+2}{5} = \frac{12t+6}{10} = \frac{6t+3}{5}.$$

This implies there exist $c \in [1, \frac{6t+3}{5}]$ such that $a + b = c$ and $\{a, b, c\}$ is not TMC. Arbitrarily let $\Delta(x) = x$ for $x \in [1, \frac{4t+2}{5}]$.

Part 2: Consider $a, b \in [\frac{4t+7}{5}, \frac{6t+3}{5}]_{odd}$ and $c \in [1, \frac{4t+2}{5}]$ such that $c < a < b$. By

Lemma 2, since a, b are odd we have $\Delta(a) \neq \Delta(b) \neq \Delta(c)$. From (Part 1) above, $\Delta(a), \Delta(b) \notin [1, \frac{4t+2}{5}]$ and $\Delta(a), \Delta(b) \in [\frac{4t+7}{5}, t]$.

Now there are $t - \frac{4t+7}{5} = \frac{t-7}{5}$ colors in $[\frac{4t+7}{5}, t]$. Further there are $\frac{\frac{6t+3}{5} - \frac{4t+7}{5}}{2} + 1 = \frac{t+1}{5}$ many elements in $[\frac{4t+7}{5}, \frac{6t+3}{5}]_{odd}$. So we have $\frac{t-7}{5}$ many colors to color $\frac{t+1}{5}$ many elements. By the PHP, with $\frac{t-7}{5} < \frac{t+1}{5}$ there exist $a, b \in [\frac{4t+7}{5}, \frac{6t+3}{5}]_{odd}$ such that $\Delta(a) = \Delta(b)$. This contradiction Lemma 2 and $\Delta : [1, \frac{6t+3}{5}] \longrightarrow [1, t]$ has a solution set to $L(m)$ that avoids being TMC.

Sub-Case 3d: Suppose $t \equiv 3 mod(5)$.

Lower Bound:

Let $\Delta : [1, \frac{6t+2}{5}] \longrightarrow [1, t]$ be defined as below.

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \frac{4t+3}{5} \\ \frac{x}{2} & \text{if } x > \frac{4t+3}{5} \text{ and } x \text{ is even} \\ \frac{5x+4t+3}{5} & \text{if } x > \frac{4t+3}{5} \text{ and } x \text{ is odd} \end{cases}$$

By the definition of Δ there does not exist $a, b \in [1, \frac{4t+3}{5}]$ such that $a \neq b$ and $\Delta(a) = \Delta(b)$. Let $a, b \in [\frac{4t+8}{5}, \frac{6t+2}{5}]$ then,

$$c = a + b \geq \frac{4t+8}{5} + \frac{4t+13}{5} = \frac{8t+21}{5} > \frac{6t+2}{5}.$$

Thus $c \neq [1, \frac{6t+2}{5}]$.

Let $a \in [1, \frac{4t+3}{5}]$ and $b \in [\frac{4t+8}{5}, \frac{6t+2}{5}]$. If b is even then $b = 2a$ and,

$$c = a + b \geq \frac{4t+8}{10} + \frac{4t+8}{5} = \frac{12t+24}{10} = \frac{6t+12}{5} > \frac{6t+2}{5}.$$

Thus $c \neq [1, \frac{6t+2}{5}]$. If b is odd. Then, $\Delta(b) = \frac{5x+4t+3}{5} > \frac{4t+3}{5} \geq \Delta(a)$. Therefore $\Delta : [1, \frac{6t+2}{5}] \longrightarrow [1, t]$ as defined above has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a $\Delta[1, \frac{6t+7}{5}] \longrightarrow [1, t]$ that is TMC. Since $t < \frac{6t+7}{5}$ there exist $a, b \in [1, \frac{6t+7}{5}]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Part 1: Suppose $a, b \in [1, \frac{4t+3}{5}]$. By Lemma 1 $a = \frac{b}{2}$, then

$$c = a + b \leq \frac{4t+3}{10} + \frac{4t+3}{5} = \frac{12t+9}{10} < \frac{6t+5}{5} < \frac{6t+7}{5}.$$

This implies there exist $c \in [1, \frac{6t+7}{5}]$ such that $a + b = c$ and $\{a, b, c\}$ is not TMC. Arbitrarily let $\Delta(x) = x$ for $x \in [1, \frac{4t+3}{5}]$.

Part 2: Consider $a, b \in [\frac{4t+8}{5}, \frac{6t+7}{5}]_{odd}$ and $c \in [1, \frac{4t+3}{5}]$ such that $c < a < b$. By Lemma 2, since a, b are odd we have $\Delta(a) \neq \Delta(b) \neq \Delta(c)$. From (Part 1) above, $\Delta(a), \Delta(b) \notin [1, \frac{4t+3}{5}]$ and $\Delta(a), \Delta(b) \in [\frac{4t+8}{5}, t]$.

Now there are $t - \frac{4t+8}{5} = \frac{t-8}{5}$ colors in $[\frac{4t+5}{5}, t]$. Further there are $\frac{\frac{6t+7}{5} - \left(\frac{4t+8}{5} - 1\right)}{2} = \frac{t+2}{5}$ many elements in $[\frac{4t+8}{5}, \frac{6t+7}{5}]_{odd}$. So we have $\frac{t-8}{5}$ many colors to color $\frac{t+2}{5}$ many elements. By the PHP, with $\frac{t-8}{5} < \frac{t+2}{5}$ there exist $a, b \in [\frac{4t+8}{5}, \frac{6t+7}{5}]_{odd}$ such that $\Delta(a) = \Delta(b)$. This contradiction Lemma 2 and $\Delta : [1, \frac{6t+7}{5}] \longrightarrow [1, t]$ has a solution set to $L(m)$ that avoids being TMC.

Sub-Case 3e: Suppose $t \equiv 4 mod(5)$.

Lower Bound:

Let $\Delta : [1, \frac{6t+1}{5}] \longrightarrow [1, t]$ be defined as below.

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \frac{4t-1}{5} \\ \frac{x}{2} & \text{if } x > \frac{4t-1}{5} \text{ and } x \text{ is even} \\ \frac{5x+4t-1}{5} & \text{if } x > \frac{4t-1}{5} \text{ and } x \text{ is odd} \end{cases}$$

By the definition of Δ there does not exist $a, b \in [1, \frac{4t-1}{5}]$ such that $a \neq b$ and

$\Delta(a) = \Delta(b)$. Let $a, b \in [\frac{4t+4}{5}, \frac{6t+1}{5}]$ then,

$$c = a + b \geq \frac{4t+4}{5} + \frac{4t+9}{5} = \frac{8t+13}{5} > \frac{6t+1}{5}.$$

Thus $c \neq [1, \frac{6t+1}{5}]$.

Let $a \in [1, \frac{4t-1}{5}]$ and $b \in [\frac{4t+4}{5}, \frac{6t+1}{5}]$. If b is even then $b = 2a$ and,

$$c = a + b \geq \frac{4t+4}{10} + \frac{4t+4}{5} = \frac{12t+12}{10} = \frac{6t+6}{5} > \frac{6t+1}{5}.$$

Thus $c \neq [1, \frac{6t+1}{5}]$. If b is odd. Then, $\Delta(b) = \frac{5x+4t-1}{5} > \frac{4t-1}{5} \geq \Delta(a)$. Therefore $\Delta : [1, \frac{6t+1}{5}] \longrightarrow [1, t]$ as defined above has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a $\Delta : [1, \frac{6t+7}{5}] \longrightarrow [1, t]$ that is TMC. Since $t < \frac{6t+7}{5}$ there exist $a, b \in [1, \frac{6t+5}{5}]$ such that $a < b$ and $\Delta(a) = \Delta(b)$.

Part 1: Suppose $a, b \in [1, \frac{4t+4}{5}]$. By Lemma 1 $a = \frac{b}{2}$, then

$$c = a + b \leq \frac{4t+4}{10} + \frac{4t+4}{5} = \frac{12t+12}{10} = \frac{6t+6}{5} < \frac{6t+7}{5}$$

This implies there exist $c \in [1, \frac{6t+7}{5}]$ such that $a + b = c$ and $\{a, b, c\}$ is not TMC. Arbitrarily let $\Delta(x) = x$ for $x \in [1, \frac{4t+4}{5}]$.

Part 2: Consider $a, b \in [\frac{4t+9}{5}, \frac{6t+7}{5}]_{odd}$ and $c \in [1, \frac{4t=4}{5}]$ such that $c < a < b$. By Lemma 2, since a, b are odd we have $\Delta(a) \neq \Delta(b) \neq \Delta(c)$. From (Part 1) above, $\Delta(a), \Delta(b) \notin [1, \frac{4t+4}{5}]$ and $\Delta(a), \Delta(b) \in [\frac{4t+9}{5}, t]$.

Now there are $t - \frac{4t+9}{5} = \frac{t-9}{5}$ colors in $[\frac{4t+9}{5}, t]$. Further there are $\frac{\frac{6t+7}{5} - \left(\frac{4t+9}{5} - 1\right)}{2} = \frac{t+1}{5}$ many elements in $[\frac{4t+9}{5}, \frac{6t+7}{5}]_{odd}$. So we have $\frac{t-9}{5}$ many colors to color $\frac{t+1}{5}$ many elements. By the PHP, with $\frac{t-9}{5} < \frac{t+1}{5}$ there exist $a, b \in [\frac{4t+9}{5}, \frac{6t+7}{5}]_{odd}$ such that $\Delta(a) = \Delta(b)$. This contradiction Lemma 2 and $\Delta : [1, \frac{6t+7}{5}] \longrightarrow [1, t]$ has a solution set to $L(m)$ that avoids being TMC.

Case 4: Suppose $t > m$ and $m \geq 4$. Then let $h = \sum_{k=1}^{n-2}(k)$, $j = \sum_{k=1}^{n-3}(k)$, $\underline{s} = \lfloor \frac{th-j-2}{1+h} \rfloor$, and $\overline{s} = \lceil \frac{th-j-2}{1+h} \rceil$ and we will show that $R(t,m) = max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\}$.

Sub-Case 4a: Suppose $j + 2\underline{s} + 3 = max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\}$ or $\underline{s} = \overline{s}$. Then $j + 2\underline{s} + 3 \geq \overline{s} + (t - \overline{s})h + 1$.

Lower Bound:

Let $\Delta : [1, j + 2\underline{s} + 2] \longrightarrow [1, t]$ be defined as below.

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \underline{s} \\ \lceil \frac{x-\underline{s}}{h} \rceil + \underline{s} & \text{if } x > \underline{s} \end{cases}$$

Suppose Δ avoids having only TMC solutions to $L(m)$. Then there exist $a, b \in [1, j + 2\underline{s} + 2]$ such that $a < b$, $\Delta(a) = \Delta(b)$, and $a, b \in \{x_i\}_{i=1}^{m} \subseteq [1, j + 2\underline{s} + 2]$ such that $\sum_{i=1}^{m-1}(x_i) = x_m$.

If $a, b \in [1, \underline{s}]$ then $\Delta(a) = a < b = \Delta(b)$. So $\Delta(a) \neq \Delta(b)$. If $a \in [1, \underline{s}]$ and $b \in [\underline{s} + 1, j + 2\underline{s} + 2]$ then $\Delta(a) = a \leq \underline{s} < \lceil \frac{b-\underline{s}}{h} \rceil + \underline{s} = \Delta(b)$. So $\Delta(a) \neq \Delta(b)$. If $a, b \in [\underline{s} + 1, j + 2\underline{s} + 2]$ and $b - a < h$ then for any $\{x_i\}_{i=1}^{m-2} \subseteq [1, j + 2\underline{s} + 2]$ such that $a \notin \{x_i\}_{i=1}^{m-2}$ we have $\sum_{i=1}^{m-2}(x_i) + a \geq \sum_{i=1}^{m-2}(i) + a = h + a > b$, so $x_m \neq b$. Also, for any $\{x_i\}_{i=1}^{m-3} \subseteq [1, j + 2\underline{s} + 2]$ such that $a, b \notin \{x_i\}_{i=1}^{m-2}$ we have $\sum_{i=1}^{m-3}(x_i) + a + b \geq \sum_{i=1}^{m-3}(i) + (\underline{s} + 1) + (\underline{s} + 2) > j + 2\underline{s} + 2$, so $x_m \notin [1, j + 2\underline{s} + 2]$. If $a, b \in [\underline{s} + 1, j + 2\underline{s} + 2]$ and $b - a \geq h$ then,

$$\Delta(a) = \left\lceil \frac{a - \underline{s}}{h} \right\rceil + \underline{s} \leq \left\lceil \frac{(b - h) - \underline{s}}{h} \right\rceil + \underline{s} = \left\lceil \frac{b - \underline{s}}{h} - 1 \right\rceil + \underline{s}$$
$$\left\lceil \frac{b - \underline{s}}{h} \right\rceil - 1 + \underline{s} < \left\lceil \frac{b - \underline{s}}{h} \right\rceil + \underline{s} = \Delta(b).$$

So $\Delta(a) \neq \Delta(b)$. Thus $\Delta : [1, j + 2\underline{s} + 2] \longrightarrow [1, t]$ has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a coloring $\Delta : [1, j+2\underline{s}+3] \longrightarrow [1,t]$ that has only TMC solution sets to $L(m)$.

Part 1: Consider $a, b \in [1, \underline{s}+2]$.

If $a, b \in [1, m-1]$ let $\{x_i\}_{i=1}^{m-1} = \{i\}_{i=1}^{m-1}$. If $a \in [1, m-1]$ and $b \notin [1, m-1]$ let $\{x_i\}_{i=1}^{m-2} = \{i\}_{i=1}^{m-2}$ and $b = x_{m-1}$. If $a, b \notin [1, m-1]$ let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $a = x_{m-2}$, and $b = x_{m-1}$. In any of these cases we have,

$$x_m = \sum_{i=1}^{m-1}(x_i) \leq \sum_{i=1}^{m-3}(i) + (\underline{s}+1) + (\underline{s}+2) = j + 2\underline{s} + 3$$

and $x_m \in [1, j+2\underline{s}+3]$. Thus for Δ to have only TMC solution sets to $L(m)$ it has to be true that $\Delta(a) \neq \Delta(b)$. For $x \in [1, \underline{s}+2]$ arbitrarily let $\Delta(x) = x$.

Part 2: Consider $a \in [1, \underline{s}+2]$ and $b \in [\underline{s}+2+h, j+2\underline{s}+3]$.

If $a \leq m-2$ then let $\{x_i\}_{i=1}^{m-2} = \{i\}_{i=1}^{m-2}$ and $x_{m-1} = b-h \geq \underline{s}+2 > m-2$. Then $\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-2}(i) + (b-h) = b$. If $a = m-1$ then,

$$\begin{aligned} b-(m-1) \geq \underline{s}+2+h-(m-1) &= \underline{s}+1+\sum_{i=1}^{m-2}(i) - (m-2) \\ &= \underline{s}+1+\sum_{i=1}^{m-3}(i) > \underline{s} + \sum_{i=1}^{m-3}(i). \end{aligned}$$

Which implies $b-a-\sum_{i=1}^{m-3}(i) > \underline{s} \geq m-1 = a$. So let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $x_{m-2} = a$, $x_{m-1} = c_1 = b-a-\sum_{i=1}^{m-3}(i)$, and $x_m = b$. Then it follows that,

$$\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-3}(i) + (b-a-\sum_{i=1}^{m-3}(i)) + a = b.$$

If $a \geq m$ and $a \neq c_1 = b-a-\sum_{i=1}^{m-3}(i)$ then let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $x_{m-2} = c_1$, $x_{m-1} = a$, and $x_m = b$. Then $\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-3}(i) + (b-a-\sum_{i=1}^{m-3}(i)) + a = b$. If $a \geq m$ and $a = c_1 = b-a-\sum_{i=1}^{m-3}(i)$ then $b = 2a + \sum_{i=1}^{m-3}(i)$. Now let $\{x_i\}_{i=1}^{m-4} = \{i\}_{i=1}^{m-4}$, $x_{m-3} = m-2$, $x_{m-2} = a-1 \geq m-1$, $x_{m-1} = a$, and $x_m = b$.

Then it follows that,

$$\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-4}(i) + (m-2) + (a-1) + a = \sum_{i=1}^{m-3}(i) + 2a = b.$$

Thus if $a \in [1, \underline{s}+2]$ and $b \in [\underline{s}+2+h, j+2\underline{s}+3]$ we can find a set with $a, b \in \{x_i\}_{i=1}^{m} \in [1, j+2\underline{s}+3]$ such that $\sum_{i=1}^{m-1}(x_i) = x_m$. Therefor for Δ to have only TMC solution sets to $L(m)$ if $a \in [1, \underline{s}+2]$ and $b \in [\underline{s}+2+h, j+2\underline{s}+3]$ then $\Delta(a) \neq \Delta(b)$.

Part 3: If $\underline{s} = \overline{s}$ then $\underline{s} = \frac{th-j-2}{1+h} > \frac{th-j-2-h}{1+h}$ and,

$$\begin{aligned}
\underline{s}(1+h) &> th - j - 2 - h \\
\underline{s} + \underline{s}h &> th - j - 2 - h \\
j + \underline{s} - h + 2 &> th - \underline{s}h - 2h \\
\frac{j + \underline{s} - h + 2}{t - (\underline{s}+2)} &> h.
\end{aligned}$$

If $\underline{s} \neq \overline{s}$ then $\underline{s} + 1 = \overline{s}$, then again it follows that,

$$\begin{aligned}
j + 2\underline{s} + 3 &\geq \overline{s} + (t - \overline{s})h + 1 \\
j + 2\underline{s} + 3 &\geq (\underline{s}+1) + (t - (\underline{s}+1))h + 1 \\
j + \underline{s} - h + 1 &\geq (t - (\underline{s}+1))h - h \\
j + \underline{s} - h + 2 &> (t - (\underline{s}+2))h \\
\frac{j + \underline{s} - h + 2}{t - (\underline{s}+2)} &> h.
\end{aligned}$$

Now in $[\underline{s}+2+h, j+2\underline{s}+3]$ there exist $j+2\underline{s}+3-(\underline{s}+2+h)+1 = j+\underline{s}-h+2$ many elements with only $t-(\underline{s}+2)$ many colors to color those elements (Parts 1 & 2 of Sub-Case 4a). By the PHP there exist at least $\frac{j+\underline{s}-h+2}{t-(\underline{s}+2)} > h$ many elements in $[\underline{s}+2+h, j+2\underline{s}+3]$ that share the same color. So at least $a_0, a_1, a_2, ..., a_h \in [\underline{s}+2+h, j+2\underline{s}+3]$ such that $a_0 < a_1 < a_2 < ... < a_h$ and $\Delta(a_0) = \Delta(a_h)$.

By Lemma 3, we have $a_h > a_0 \geq \underline{s} + 2 + h > m$ (Lemma 4), $a_h - a_0 \geq h$, and $\Delta(a_0) = \Delta(a_h)$ which implies that Δ avoids having only TMC solution sets to $L(m)$. Therefore if $j + 2\underline{s} + 3 = max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\}$ or $\underline{s} = \overline{s}$ then $R(t, n) = j + 2\underline{s} + 3$.

Sub-Case 4b: Suppose $\overline{s} + (t - \overline{s})h + 1 = max\{j + 2\underline{s} + 3, \overline{s} + (t - \overline{s})h + 1\}$ and $\underline{s} \neq \overline{s}$. Then $j + 2\underline{s} + 3 \leq \overline{s} + (t - \overline{s})h + 1$, $\underline{s} + 1 = \overline{s}$, and $j + 2\overline{s} + 2 \geq \overline{s} + (t - \overline{s})h$.

Lower Bound:

Let $\Delta : [1, \overline{s} + (t - \overline{s})h] \longrightarrow [1, t]$ be defined as below.

$$\Delta(x) = \begin{cases} x & \text{if } x \leq \overline{s} \\ \left\lceil \frac{x - \overline{s}}{h} \right\rceil + \overline{s} & \text{if } x > \overline{s} \end{cases}$$

Suppose Δ avoids having only TMC solutions to $L(m)$. Then there exist $a, b \in [1, \overline{s} + (t - \overline{s})h]$ such that $a < b$, $\Delta(a) = \Delta(b)$, and $a, b \in \{x_i\}_{i=1}^{m} \subseteq [1, \overline{s} + (t - \overline{s})h]$ such that $\sum_{i=1}^{m-1}(x_i) = x_m$.

If $a, b \in [1, \overline{s}]$ then $\Delta(a) = a < b = \Delta(b)$. So $\Delta(a) \neq \Delta(b)$. If $a \in [1, \overline{s}]$ and $b \in [\overline{s} + 1, \overline{s} + (t - \overline{s})h]$ then $\Delta(a) = a < \overline{s} < \lceil \frac{b - \overline{s}}{h} \rceil + \overline{s} = \Delta(b)$. So $\Delta(a) \neq \Delta(b)$. If $a, b \in [\overline{s} + 1, \overline{s} + (t - \overline{s})h]$ and $b - a < h$ then for any $\{x_i\}_{i=1}^{m-2} \subseteq [1, \overline{s} + (t - \overline{s})h]$ such that $a \notin \{x_i\}_{i=1}^{m-2}$ we have $\sum_{i=1}^{m-2}(x_i) + a \geq \sum_{i=1}^{m-2}(i) + a = h + a > b$, so $x_m \neq b$. Also, for any $\{x_i\}_{i=1}^{m-3} \subseteq [1, \overline{s} + (t - \overline{s})h]$ such that $a, b \notin \{x_i\}_{i=1}^{m-3}$ we have $\sum_{i=1}^{m-3}(x_i) + a + b \geq \sum_{i=1}^{m-3}(i) + (\overline{s} + 1) + (\overline{s} + 2) > j + 2\overline{s} + 2 \geq \overline{s} + (t - \overline{s})h$, so $x_m \notin [1, \overline{s} + (t - \overline{s})h]$. If $a, b \in [\overline{s} + 1, \overline{s} + (t - \overline{s})h]$ and $b - a \geq h$ then,

$$\begin{aligned} \Delta(a) = \left\lceil \frac{a - \overline{s}}{h} \right\rceil + \overline{s} \leq \left\lceil \frac{(b - h) - \overline{s}}{h} \right\rceil + \overline{s} = \left\lceil \frac{b - \overline{s}}{h} - 1 \right\rceil + \overline{s} \\ = \left\lceil \frac{b - \overline{s}}{h} \right\rceil - 1 + \overline{s} < \left\lceil \frac{b - \overline{s}}{h} \right\rceil + \overline{s} = \Delta(b). \end{aligned}$$

So $\Delta(a) \neq \Delta(b)$. Thus $\Delta : [1, \overline{s} + (t - \overline{s})h] \longrightarrow [1, t]$ has only TMC solution sets to $L(m)$.

Upper Bound:

Suppose there exist a coloring $\Delta : [1, \overline{s} + (t - \overline{s})h + 1] \longrightarrow [1, t]$ that has only TMC solution sets to $L(m)$.

Part 1: Consider $a, b \in [1, \overline{s} + 1]$.

If $a, b \in [1, m-1]$ let $\{x_i\}_{i=1}^{m-1} = \{i\}_{i=1}^{m-1}$. If $a \in [1, m-1]$ and $b \notin [1, m-1]$ let $\{x_i\}_{i=1}^{m-2} = \{i\}_{i=1}^{m-2}$ and $b = x_{m-1}$. If $a, b \notin [1, m-1]$ let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $a = x_{m-2}$, and $b = x_{m-1}$. In any of these cases we have,

$$x_m = \sum_{i=1}^{m-1}(x_i) \leq \sum_{i=1}^{m-3}(i) + (\overline{s}) + (\overline{s} + 1) = j + 2\overline{s} + 1$$

$$= j + 2(\underline{s} + 1) + 1 = j + 2\underline{s} + 3 \leq \overline{s} + (t - \overline{s})h + 1$$

and $x_m \in [1, \overline{s} + (t - \overline{s})h + 1]$. Thus for Δ to have only TMC solution sets to $L(m)$ it has to be true that $\Delta(a) \neq \Delta(b)$. For $x \in [1, \overline{s} + 1]$ arbitrarily let $\Delta_{t,n}(x) = x$.

Part 2: Consider $a \in [1, \overline{s} + 1]$ and $b \in [\overline{s} + 1 + h, \overline{s} + (t - \overline{s})h + 1]$.

If $a \leq m-2$ then let $\{x_i\}_{i=1}^{m-2} = \{i\}_{i=1}^{m-2}$ and $x_{m-1} = b - h \geq \overline{s} + 1 = \underline{s} + 2 \geq m - 2$. Then $\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-2}(i) + (b - h) = b$. If $a = m - 1$ then,

$$b - (m-1) \geq \overline{s} + 1 + h - (m-1) = \overline{s} + \sum_{i=1}^{m-2}(i) - (m-2) = \overline{s} + \sum_{i=1}^{m-3}(i)$$

which implies $b - a - \sum_{i=1}^{m-3}(i) \geq \overline{s} > \underline{s} \geq m - 1 = a$. So let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $x_{m-2} = a$, $x_{m-1} = c_1 = b - a - \sum_{i=1}^{m-3}(i)$, and $x_m = b$. Then
$\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-3}(i) + (b - a - \sum_{i=1}^{m-3}(i)) + a = b$.

If $a \geq m$ and $a \neq c_1 = b - a - \sum_{i=1}^{m-3}(i)$ then let $\{x_i\}_{i=1}^{m-3} = \{i\}_{i=1}^{m-3}$, $x_{m-2} = c_1$, $x_{m-1} = a$, and $x_m = b$. Then $\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-3}(i) + (b - a - \sum_{i=1}^{m-3}(i)) + a = b$. If $a \geq m$ and $a = c_1 = b - a - \sum_{i=1}^{m-3}(i)$ then $b = 2a + \sum_{i=1}^{m-3}(i)$. Now let
$\{x_i\}_{i=1}^{m-4} = \{i\}_{i=1}^{m-4}$, $x_{m-3} = m - 2$, $x_{m-2} = a - 1 \geq m - 1$, $x_{m-1} = a$, and $x_m = b$.

Then we have,

$$\sum_{i=1}^{m-1}(x_i) = \sum_{i=1}^{m-4}(i) + (m-2) + (a-1) + a = \sum_{i=1}^{m-3}(i) + 2a = b.$$

Thus if $a \in [1, \overline{s}+1]$ and $b \in [\overline{s}+1+h, \overline{s}+(t-\overline{s})h+1]$ we can find a set with $a, b \in \{x_i\}_{i=1}^{m} \in [1, \overline{s}+(t-\overline{s})h+1]$ such that $\sum_{i=1}^{m-1}(x_i) = x_m$. Therefore for Δ to have only TMC solution sets to $L(m)$ if $a \in [1, \overline{s}+1]$ and $b \in [\overline{s}+1+h, \overline{s}+(t-\overline{s})h+1]$ then $\Delta(a) \neq \Delta(b)$.

Part 3: Now in $[\overline{s}+1+h, \overline{s}+(t-\overline{s})h+1]$ there exist $\overline{s}+(t-\overline{s})h+1-(\overline{s}+1+h)+1 = (t-(\overline{s}+1))h+1$ many elements with only $t-(\overline{s}+1)$ many colors to color those elements (Parts 1 & 2). By the PHP there exist at least,

$$\frac{(t-(\overline{s}+1))h+1}{t-(\overline{s}+1)} = h + \frac{1}{t-(\overline{s}+1)} > h$$

many elements in $[\overline{s}+1+h, \overline{s}+(t-\overline{s})h+1]$ that share the same color. So at least $a_0, a_1, a_2, ..., a_h \in [\overline{s}+1+h, \overline{s}+(t-\overline{s})h+1]$ such that $a_0 < a_1 < a_2 < ... < a_h$ and $\Delta(a_0) = \Delta(a_h)$. By Lemma 3, we have $a_h > a_0 \geq \overline{s}+1+h > m$ (Lemma 4) $a_h - a_0 \geq h$, and $\Delta(a_0) = \Delta(a_h)$ which implies that Δ avoids having only TMC solution sets to $L(m)$.

Thus if $\overline{s}+(t-\overline{s})h+1 = max\{j+2\underline{s}+3, \overline{s}+(t-\overline{s})h+1\}$ and $\underline{s} \neq \overline{s}$ then $R(t,n) = \overline{s}+(t-\overline{s})h+1$. □

Here is a table of the TMC-Rado numbers with t rows for t colors and m columns for equation $L(m)$. So, $R(7,5)$ is found in position $(7,5)$ which is 16. The cells highlighted in green are where Sub-case 4b occur.

Table 1. TMC-Rado Numbers

t\m	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45
2	3	6	10	15	21	28	36	45
3	5	6	10	15	21	28	36	45
4	6	8	10	15	21	28	36	45
5	7	10	12	15	21	28	36	45
6	9	11	14	17	21	28	36	45
7	9	12	16	19	23	28	36	45
8	11	14	18	21	25	30	36	45
9	12	16	20	23	27	32	38	45
10	13	17	21	25	29	34	40	47
11	15	18	22	27	31	36	42	49
12	15	20	24	29	33	38	44	51
13	17	22	26	31	35	40	46	53
14	18	23	28	33	37	42	48	55
15	19	24	30	34	39	44	50	57
16	21	26	32	35	41	46	52	59
17	21	28	33	37	43	48	54	61
18	23	29	34	39	45	50	56	63
19	24	30	36	41	47	52	58	65
20	25	32	38	43	49	54	60	67
21	27	34	40	45	50	56	62	69
22	27	35	42	47	51	58	64	71
23	29	36	44	49	53	60	66	73
24	30	38	45	51	55	62	68	75
25	31	40	46	53	57	64	70	77
26	33	41	48	54	59	66	72	79
27	33	42	50	55	61	68	74	81
28	35	44	52	57	63	69	76	83
29	36	46	54	59	65	70	78	85
30	37	47	56	61	67	72	80	87
31	39	48	57	63	69	74	82	89
32	39	50	58	65	71	76	84	91
33	41	52	60	67	73	78	86	93
34	42	53	62	69	75	80	88	95
35	43	54	64	71	77	82	90	97
36	45	56	66	73	79	84	91	99
37	45	58	68	74	80	86	92	101
38	47	59	69	75	81	88	94	103
39	48	60	70	77	83	90	96	105
40	49	62	72	79	85	92	98	107
41	51	64	74	81	87	94	100	109
42	51	65	76	83	89	96	102	111
43	53	66	78	85	91	98	104	113
44	54	68	80	87	93	100	106	115
45	55	70	81	89	95	102	108	116
46	57	71	82	91	97	104	110	117
47	57	72	84	93	99	106	112	119
48	59	74	86	94	101	108	114	121
49	60	76	88	95	103	110	116	123
50	61	77	90	97	105	111	118	125

SUGGESTIONS FOR FURTHER RESEARCH

This problem could be expanded by adding a content integer c to the left side of the $L(m)$ equation. From what I saw a non-negative c may not be too difficult, however when c is negative $R(t, m)$ becomes less clear. Another change may be instead using TMC we could use Totally Dualcolored or TDC, where a set $\{x_i\}_i^m$ is TDC if and only if for any x_i and x_j such that $\Delta(x_i) = \Delta(x_j)$ then $\Delta(x_i) \neq \Delta(x_k)$ for all $k \neq i, j$.

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