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Propositional and Predicate Logics of Incomplete Information

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Abstract

One of the most common scenarios of handling incomplete information occurs in relational databases. They describe incomplete knowledge with three truth values, using Kleene's logic for propositional formulae and a rather peculiar extension to predicate calculus. This design by a committee from several decades ago is now part of the standard adopted by vendors of database management systems. But is it really the right way to handle incompleteness in propositional and predicate logics?

Our goal is to answer this question. Using an epistemic approach, we first characterize possible levels of partial knowledge about propositions, which leads to six truth values. We impose rationality conditions on the semantics of the connectives of the propositional logic, and prove that Kleene's logic is the maximal sublogic to which the standard optimization rules apply, thereby justifying this design choice. For extensions to predicate logic, however, we show that the additional truth values are not necessary: every many-valued extension of first-order logic over databases with incomplete information represented by null values is no more powerful than the usual two-valued logic with the standard Boolean interpretation of the connectives. We use this observation to analyze the logic underlying SQL query evaluation, and conclude that the many-valued extension for handling incompleteness does not add any expressiveness to it.

Keywords: Many-valued logics, Incomplete information, SQL

1. Introduction

Incomplete information is ubiquitous in applications that involve querying and reasoning about data. It is one of the oldest topics in database research [1], and is essential in many applications the combine

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techniques from data management and AI. Such applications include data integration [2], data exchange [3], inconsistent databases [4], and ontology-based data access [5]. It is very common for them to reduce the problem at hand to a setting where one issues a query against a relational database.

This is problematic, however, as relational database management systems (DBMSs) use a rather simplistic way of representing incomplete information (nulls) combined with a rather convoluted method of handling it, based on *many-valued logics*. Specifically, every relational DBMS uses a three-valued logic for handling incomplete information, namely Kleene's logic [6]. This was the design choice of SQL, the language of relational DBMSs, which is now written into the SQL Standard [7], presented in all database textbooks, and implemented in all database products. It leads to many well documented cases of unexpected behavior [8]. For example, given a relation `S` and a relation `R` with attributes `A` and `B`, the SQL query

```
SELECT * FROM S WHERE NOT EXISTS (SELECT * FROM R WHERE R.B=R.B)
```

seemingly returns relation `S` iff $\mathtt{R} = \varnothing$, i.e., it does not have a single tuple satisfying the tautological condition `R.B=R.B`. However, if $\mathtt{R} = \{(1, \mathtt{null})\}$, this query always returns `S`. If `null` means that a value is missing, then in every possible world for relation `R` where we know the value of that `null`, the query will return the empty set since $\mathtt{R} \neq \varnothing$. That is, when evaluated on the original database, the query returns *false positives.* This makes it hard to trust results produced by relational DBMSs, especially in AI-motivated applications that rely on the database technology.

The reason for the unexpected behavior of the above query is the use of a many valued-logic; in particular, the seemingly tautological condition `R.B=R.B` becomes `null=null` which evaluates neither to true nor to false but rather to the *unknown* truth value of Kleene's three-valued logic. The use of Kleene's logic was first proposed by [1], but it is far from the only logic to have been considered for representing incomplete information, and many others appeared afterward. [9] looked at a four-valued logic, but in the end argued against it due to the additional complexity. Nonetheless, well-documented problems with incomplete information [10, 8] led to the search of more appropriate logics for handling incompleteness. For example, [11] revisited four-valued logics, while [12] considered logics with four, five, and seven values, and showed how to encode them with three. A different kind of four-valued logics for missing data was studied by [13], while [14] suggested dropping nulls altogether and go back to the usual Boolean two-valued logic.

There is also no shortage of many-valued logics that have been proposed in closely related contexts. For example, a variety of many-valued logics were used in the study of default reasoning [15] or in reasoning about inconsistency [16]. Those are typically based on the notion of bilattices, providing truth and knowledge orderings on the truth values [17, 18]. A common one is Belnap's bilattice with four truth values [19, 20], which also found database applications [21]; but others exist as well, e.g., many generalizations of Kleene's logic based on numerical intervals describing the degree of being true [22]. A many-valued propositional logic must also provide an interpretation of propositional connectives. To make the general picture even muddier, for different sets of truth values, different semantics of propositional connectives exist, sometimes even non-deterministic ones [23].

Thus, we are far from having a clear picture of what to use as a logic of incomplete information in data management applications. Choices are numerous, and there is no final argument as to why the approach of DBMSs that use Kleene's logic is the right one. Hence, the first question we address is:

1) *What is the right many-valued propositional logic for handling incomplete information?*

Now suppose we have a propositional logic that correctly accounts for truth values of statements about incomplete information, and for operations on them. In querying data, however, we use *predicate* logics. Indeed, the core of SQL is essentially a programming syntax for relational calculus, which is another name for first-order (FO) predicate logic.

Of course we know how to lift the semantics of propositional logic to the full predicate calculus by treating existential and universal quantifiers as disjunctions and conjunctions over all elements of the universe. What we do not know is how different choices of propositional logic for incomplete information affect the power of predicate calculus. As one example, consider the version of FO that underlies SQL and is based on Kleene's logic. What extra power does it possess over FO under the usual two-valued Boolean interpretation of the connectives? It was recently argued, by means of rewriting SQL queries, that FO based on Kleene's logic can be encoded in the usual Boolean FO [24]. But is there a general result in logic that underlies such a translation, and what is so special about Kleene's logic that makes it work?

Even more generally, the second question we would like to address is:

2) *How does the choice of a propositional logic for incomplete information affect predicate logic?*

Finally, we would like to understand how these theoretical considerations relate to the practice of incomplete data in relational databases. A rough approximation of the core of SQL – the way it is presented in many database textbooks – is first-order logic. But as soon as incomplete information enters the picture, this becomes a many-valued FO. And yet there is even more to it: in SQL queries, answer tuples are split into *true* ones that need to be returned, and others that are not returned, thus collapsing a three-valued logic to two-valued. This leads to our last question:

3) *What is the logic that underlies real-life handling of incomplete information in relational databases (i.e., SQL's logic), and how much more power than the usual two-valued* FO *does it possess?*

The goal of this paper is to address these three questions. Below we outline our main contributions.

Propositional logic. To understand what a proper propositional logic for reasoning about incomplete information is, we need to define its truth values, and truth tables for its connectives (we shall concentrate on the standard ones, i.e., $\wedge$, $\vee$, and $\neg$, although we shall see others as well). We follow the approach of [18] to turn partial knowledge about the truth of a proposition into truth values. If we have a set W of worlds, and two of its subsets T and F in which a proposition is true and false, respectively, this produces a description (T, F, W). It is possible that $T \cup F \neq W$, i.e., we may have partial knowledge about the truth or falsity of a proposition. We require however that $T \cap F = \varnothing$, as here we do not consider inconsistent descriptions.

Taking those descriptions (T, F, W) directly as truth values, however, is not satisfactory: we shall have too many of them. Instead, we want to take as truth values *what we know* about such descriptions.

We abstract this knowledge as *epistemic theories* of such descriptions: they say what is known about a proposition being possibly or certainly true or false. Then, as truth values we take maximally consistent epistemic theories. We show that there are only six such theories, resulting in a six-valued logic $\mathbb{L}_{6v}$. Its truth tables are again very naturally derived from epistemic theories of partial knowledge about truth of propositions.

As a final step, we then look at what makes a many-valued logic database friendly. It needs to be a sublogic of $\mathbb{L}_{6v}$ and yet satisfy some basic equivalences we expect to hold to be able to perform query evaluation and optimization. We then show that the maximal sublogic of $\mathbb{L}_{6v}$ that satisfies those equivalences is $\mathbb{L}_{3v}$, the three-valued logic of Kleene used in all commercial DBMSs. Thus, we justify the choice that was made by SQL designers and standards committees in choosing $\mathbb{L}_{3v}$ as the logic to be implemented in all database products.

Predicate logic. We have justified Kleene's logic $\mathbb{L}_{3v}$ as the right choice for handling incompleteness in database contexts. But database languages are not propositional: they are based on FO instead. Thus, we next look at variants of FO based on propositional many-valued logics such as $\mathbb{L}_{3v}$ and $\mathbb{L}_{6v}$, and compare their power with that of the usual Boolean FO (denoted by BFO from now on), based on just two values $\mathbf{t}$ and $\mathbf{f}$. Our main result is that when added to FO, these many-valued propositional logics add no power: FO based on $\mathbb{L}_{3v}$, or on $\mathbb{L}_{6v}$, or on any other many-valued logic (under some mild restrictions on the connectives) has no more power than BFO.

The logic of SQL. We finally apply the above observation to SQL's logic. We explain that it corresponds to a $\mathbb{L}_{3v}$-based FO with an extra connectives that allows one to collapse truth values $\mathbf{f}$ and $\mathbf{u}$ into one, but it still has no more power than BFO. Thus, even though SQL designers were justified in choosing Kleene's logic as the propositional logic for reasoning about incomplete information, they overlooked the fact that, when considered within FO, such a logic does not add any expressive power.

To sum up, our investigation validates the choice of Kleene's logic by the designers of SQL, but at the same time asks whether it was really necessary and opens up a possibility for future languages that handle incomplete information to avoid the recourse to many-valued logics. Notice that much of the criticism of SQL concentrated on its propositional logic. However we showed that it was very reasonable: a six-valued logic would have been more refined, but the three-valued logic is better at handling computational aspects. For predicate logics, our results say that these many-valued logics could have been avoided altogether. However, the price for this is a different way of expressing logical queries, and thus this result is of more interest for future language design rather than changing the current choices.

Organization. The paper is structured around three main themes: propositional logics, predicate logics, and the logic of SQL, followed by conclusions and future work. The proofs of the results on propositional logic are in Appendix A and the proofs of the results on predicate logic are in Appendix B.

This is an extended version of a paper of the same title [25] presented at the 16th International Conference on Principles of Knowledge Representation and Reasoning (KR-18), where it was awarded the Ray Reiter Best Paper Prize. In addition to including full proofs, the current version includes the following new material:

- By refining the notion of sublogic, we strenghtened the result (Theorem 3) that justifies the use of Kleene's logic ($\mathbb{L}_{3v}$) for handling incomplete information at the propositional level. Indeed, the new definition of sublogic is less strict and captures more logics, including a four-valued logic that was introduced in [13].

- We show that, when we are interested – as is the case in SQL – only in answers that evaluate to true, going from 3 to 2 truth values does not incur a blow up in the size of the formula. Indeed, we exhibit a linear rewriting that preserves equivalence of the true answers (Theorem 6). This tells us that, not only is SQL's logic unjustified w.r.t. expressiveness, but also from a *succinctness* point of view.

2. Propositional Logics

Our study of logics for incomplete information starts at the propositional level. The goal of this section is to define a propositional logic for handling incompleteness, with a special regard to applications that deal with incomplete data, including relational databases query languages.

To this end, we first need to formally define propositional formulae. We assume a countably infinite set of symbols, referred to as *propositional atoms*. For a set Ω of connectives with associated (positive) arities, the *propositional language* $\mathcal{L}$ over Ω is defined inductively as follows: every propositional atom is a formula of $\mathcal{L}$; if ω is an n-ary connective in Ω and $\alpha_1, \ldots, \alpha_n$ are formulae of $\mathcal{L}$, then so is $\omega(\alpha_1, \ldots, \alpha_n)$; nothing else is in $\mathcal{L}$. We assume that the binary connectives $\wedge$ and $\vee$, for which we use the infix notation, and the unary connective $\neg$ are always present. As will be relevant in the next section, this general definition allows for the inclusion of additional connectives in the language.

The standard way of evaluating propositional formulae is to associate atoms with *truth values*, which are then propagated through the connectives by means of *truth tables*. We define a *(propositional) logic* $\mathbb{L}$ as a pair $(\mathbf{T}, \Omega)$, where $\mathbf{T}$ is the set of truth values and Ω is the set of truth tables, which are functions

$\wedge$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$

$\vee$	$\mathbf{t}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$

Figure 1: The truth tables of $\mathbb{L}_{\mathrm{Bool}}$.

$\wedge$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{u}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{u}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{u}$	$\mathbf{u}$	$\mathbf{f}$	$\mathbf{u}$

$\vee$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{u}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$\mathbf{u}$
$\mathbf{u}$	$\mathbf{t}$	$\mathbf{u}$	$\mathbf{u}$

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{u}$	$\mathbf{u}$

Figure 2: The truth tables of $\mathbb{L}_{3\mathrm{v}}$.

$\omega\colon \mathbf{T}^n \to \mathbf{T}$, of appropriate arities, associated with the connectives. We say that $\mathbb{L}$ is a logic for a language $\mathcal{L}$ if $\mathbb{L}$ defines truth tables for every connective of $\mathcal{L}$. With a deliberate abuse of notation, we denoted by Ω both the connectives of $\mathcal{L}$ and the truth tables associated with them in $\mathbb{L}$. When it is not clear from the context, we use $\omega^{\mathbb{L}}$ to explicitly denote the truth table of $\mathbb{L}$ for the connective ω.

Given a logic $\mathbb{L} = (\mathbf{T}, \Omega)$ for a language $\mathcal{L}$, and a mapping μ from propositional atoms to the truth values in $\mathbf{T}$, the evaluation of a formula $\alpha \in \mathcal{L}$ under μ in $\mathbb{L}$ is the truth value $\mathsf{tv}_{\mathbb{L}}(\alpha, \mu)$ in $\mathbf{T}$ defined inductively as follows:

$$\mathsf{tv}_{\mathbb{L}}(\alpha, \mu) = \mu(\alpha) \quad \text{if } \alpha \text{ is a propositional atom,}$$
$$\mathsf{tv}_{\mathbb{L}}\big(\omega(\alpha_1, \dots, \alpha_n), \mu\big) = \omega^{\mathbb{L}}\big(\mathsf{tv}_{\mathbb{L}}(\alpha_1, \mu), \dots, \mathsf{tv}_{\mathbb{L}}(\alpha_n, \mu)\big),$$

for every $\alpha, \alpha_1, \dots, \alpha_n \in \mathcal{L}$ and every n-ary connective ω.

For $\Omega = \{\wedge, \vee, \neg\}$, the standard Boolean logic $\mathbb{L}_{\mathrm{Bool}}$ has truth values $\{\mathbf{t}, \mathbf{f}\}$ and truth tables as in Figure 1, while SQL uses Kleene's three-valued logic, denoted by $\mathbb{L}_{3\mathrm{v}}$, with truth values $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ and truth tables as in Figure 2. But is $\mathbb{L}_{3\mathrm{v}}$ the right propositional logic to deal with incomplete information in relational databases? To answer this question, we first need an appropriate model of incompleteness; then, we must define what kind of information truth values represent in this model, and how many of them are needed; finally, we need to define truth tables for $\wedge$, $\vee$ and $\neg$ that propagate information in a consistent way.

2.1. Model of Incompleteness

In many data management applications, especially those involving knowledge representation and reasoning, the veracity of data is a common problem. This results in dealing with two main sources of incomplete information: first, queries must be evaluated over incomplete data, i.e., multiple interpretations are possible, and, second, we may be able to evaluate a query only partially, e.g., due to computational constraints. In our logical framework, we represent the first type of incompleteness by allowing sets of possible worlds, i.e., multiple possible evaluations of formulae. We capture the second type of incompleteness by allowing partial evaluation functions, i.e., the evaluation a formula may not be defined in every world.

In the literature on many-valued logics, the approach of [18] accounts for both these sources of incomplete information, and we follow it here as a basis for our model. As we shall discuss later on in this section, our approach deviates from the [18] with respect to what truth values are and represent.

A *propositional interpretation* $\mathcal{I}$ is a triple (t, f, W), where W is a non-empty set of *worlds*, and t and f are functions from $\mathcal{L}$ to the powerset of W such that, for every $\alpha, \beta \in \mathcal{L}$, all of the following hold:

$$\begin{aligned} t(\alpha) \cap f(\alpha) &= \varnothing \quad ; \\ f(\neg\alpha) &= t(\alpha) \; ; \\ t(\neg\alpha) &= f(\alpha) \; ; \end{aligned} \qquad \begin{aligned} t(\alpha \wedge \beta) &= t(\alpha) \cap t(\beta) \; ; \\ t(\alpha \vee \beta) &= t(\alpha) \cup t(\beta) \; ; \\ f(\alpha \wedge \beta) &= f(\alpha) \cup f(\beta) \; ; \\ f(\alpha \vee \beta) &= f(\alpha) \cap f(\beta) \; . \end{aligned}$$

Intuitively, t tells us on which worlds a given formula is true, while f indicates where it is false. When a world w is neither in $f(\alpha)$ nor in $t(\alpha)$, the formula α is said to be *undefined* in w. Observe that propositional interpretations capture the two types of incompleteness in our model: sets of possible worlds capture the multiple possible interpretations of a formula, and undefined formulae capture the possibly incomplete evaluation.

In [18], objects similar to propositional interpretations defined above are used as truth values for formulae. This approach produces infinitely many truth values, each of which representing a possible evaluation of a formula over a set possible worlds. In this framework, the truth value of a formula φ being true in a world w and false in w' is different from the truth value of a formula ψ being false in w and true in w'. Such a fine grained description, however, is incompatible with the standard evaluation of formulae we defined earlier. Instead, we want to collate the information provided by propositional interpretations and abstract it as truth values. In other words, we want to conclude that φ and ψ have the same truth value representing the fact that they are true somewhere and false somewhere else. To formalize this intuition, we make use of a modal formalism suitable to define what is known in propositional interpretations.

Given a propositional language $\mathcal{L}$, the language $\mathcal{L}^{\mathbf{KP}}$ of epistemic formulae is defined inductively as follows:

- $\mathbf{K}\alpha$ and $\mathbf{P}\alpha$ are in $\mathcal{L}^{\mathbf{KP}}$, for every $\alpha \in \mathcal{L}$;
- if φ and ψ are in $\mathcal{L}^{\mathbf{KP}}$, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\neg\varphi$;
- nothing else is in $\mathcal{L}^{\mathbf{KP}}$.

The semantics of epistemic formulae is given with respect to a propositional interpretation $\mathcal{I} = (t, f, W)$. Whether $\mathcal{I}$ satisfies $\varphi \in \mathcal{L}^{\mathbf{KP}}$, written $\mathcal{I}, w \models \varphi$, is inductively defined as follows:

- $\mathcal{I} \models \mathbf{K}\alpha$ if $w \in t(\alpha)$, for every $w \in W$;
- $\mathcal{I} \models \mathbf{P}\alpha$ if $w \in t(\alpha)$, for some $w \in W$;
- $\mathcal{I} \models \neg\varphi$ if $\mathcal{I} \not\models \varphi$;
- $\mathcal{I} \models \varphi \wedge \psi$ if $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$;
- $\mathcal{I} \models \varphi \vee \psi$ if $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$.

We denote by $\mathsf{Mod}(\varphi)$ the set of all models of φ, i.e., propositional interpretations that satisfy φ. We say that φ is satisfiable whenever $\mathsf{Mod}(\varphi)$ is non-empty.

Intuitively, the fact that $\mathcal{I}$ satisfies $\mathbf{K}\alpha$ means α is true in all the possible worlds $\mathcal{I}$, while $\mathcal{I} \models \mathbf{P}\alpha$ means that α is true in at least one of the possible worlds of $\mathcal{I}$. Following the approach of Hintikka [26], we interpret the former as "α is known in $\mathcal{I}$" and the latter as "α is possible in $\mathcal{I}$". However, the logic $\mathcal{L}^{\mathbf{KP}}$ differs from standard modal logics in two main respects. First, $\mathcal{L}^{\mathbf{KP}}$ is not concerned with introspection, i.e., we do not allow nesting of modal operators. Second, unlike the standard operators $\Box$ and $\Diamond$ in classical modal logic, $\mathbf{K}$ and $\mathbf{P}$ here are not dual: while $\mathbf{K}\varphi$ implies $\neg\mathbf{P}\neg\varphi$, the converse is not necessarily true. To see this, consider a propositional formula α and the interpretation $\mathcal{I} = (t, f, \{w_1, w_2\})$ such that $t(\alpha) = \{w_1\}$ and $f(\alpha) = \varnothing$; then, it is easy to verify that $\mathcal{I}$ satisfies $\neg\mathbf{P}\neg\alpha$ but not $\mathbf{K}\alpha$, because $w_2 \notin t(\alpha)$.

2.2. Truth Values

We need to understand what it means for a propositional formula to be true or false in a propositional interpretation. To do that, we resort to the notion of modalities.

Given a propositional formula α, the modalities of α are the modal formulae $\mathbf{K}\alpha$, $\mathbf{P}\alpha$, and their negation. Intuitively, the modalities of α describe the way α is true on a given propositional interpretation. To define truth values, then, we will look at the modalities of propositional formulae and their negations.

More formally, for a propositional formula α, we denote by $\mathcal{M}(\alpha)$ the set consisting of all modalities of α and $\neg\alpha$. A subset M of $\mathcal{M}(\alpha)$ is called *consistent* if there exists at least one propositional interpretation $\mathcal{I}$ for which every formula in M is satisfied. A subset of $\mathcal{M}(\alpha)$ is *maximally consistent* if, in addition, none of its proper supersets is a consistent subset of $\mathcal{M}(\alpha)$.

Intuitively, every maximally consistent subset of $\mathcal{M}(\alpha)$ defines a possible way in which a propositional formula can be evaluated on a propositional interpretation. Thus, to capture all possibilities, we need as many truth values as there are maximally consistent subsets of $\mathcal{M}(\alpha)$. The following shows that our propositional logic must be six-valued.

Theorem 1. *For every propositional formula α, there are at most 6 maximally consistent subsets of $\mathcal{M}(\alpha)$. These are:*

$$\{\ \mathbf{K}\alpha,\ \ \mathbf{P}\alpha, \neg\mathbf{K}\neg\alpha, \neg\mathbf{P}\neg\alpha\,\} \tag{1}$$

$$\{\neg\mathbf{K}\alpha, \neg\mathbf{P}\alpha,\ \ \mathbf{K}\neg\alpha,\ \ \mathbf{P}\neg\alpha\,\} \tag{2}$$

$$\{\neg\mathbf{K}\alpha,\ \ \mathbf{P}\alpha, \neg\mathbf{K}\neg\alpha,\ \ \mathbf{P}\neg\alpha\,\} \tag{3}$$

$$\{\neg\mathbf{K}\alpha,\ \ \mathbf{P}\alpha, \neg\mathbf{K}\neg\alpha, \neg\mathbf{P}\neg\alpha\,\} \tag{4}$$

$$\{\neg\mathbf{K}\alpha, \neg\mathbf{P}\alpha, \neg\mathbf{K}\neg\alpha,\ \ \mathbf{P}\neg\alpha\,\} \tag{5}$$

$$\{\neg\mathbf{K}\alpha, \neg\mathbf{P}\alpha, \neg\mathbf{K}\neg\alpha, \neg\mathbf{P}\neg\alpha\,\} \tag{6}$$

Proof. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation. If $\mathcal{I}$ satisfies $\mathbf{K}\alpha$, then by the assumption that $W \neq \varnothing$ it also satisfies $\mathbf{P}\alpha$, $\neg\mathbf{K}\neg\alpha$ and $\neg\mathbf{P}\neg\alpha$. Thus, we get (1).

Otherwise, when $\mathcal{I} \not\models \mathbf{K}\alpha$, $\mathcal{I}$ may or may not satisfy $\mathbf{P}\alpha$. If it does, then $\mathcal{I} \not\models \mathbf{K}\neg\alpha$. Under this assumption, we have two possibilities: either $\mathcal{I}$ satisfies $\mathbf{P}\neg\alpha$, in which case we get the set (3), or not, and we get (4).

Suppose now $\mathcal{I} \not\models \mathbf{K}\alpha$ and $\mathcal{I} \not\models \mathbf{P}\alpha$. If $\mathcal{I}$ satisfies $\mathbf{K}\neg\alpha$, then by the assumption that $W \neq \varnothing$ it also satisfies $\mathbf{P}\neg\alpha$. Thus, we get the set (2).

Finally, if $\mathcal{I} \not\models \mathbf{K}\neg\alpha$, then $\mathcal{I}$ may or may not satisfy $\mathbf{P}\neg\alpha$. Thus, we get the sets (5) and (6), respectively. □

We now analyze the information each of the above sets gives us for an arbitrary propositional formula α, and abstract it as a truth value, referring to the six maximally consistent sets in Theorem 1.

(1) We know that α is *true in all worlds* ($\mathbf{K}\alpha$). We abstract this as the truth value **t** (*always true*).

(2) We know that $\neg\alpha$ is true in all worlds ($\mathbf{K}\neg\alpha$), hence α is *false in all worlds*. We abstract this as the truth value **f** (*always false*).

(3) We know that there exists a world w in which α is true ($\mathbf{P}\alpha$) and there exists a world w' in which its negation is true ($\mathbf{P}\neg\alpha$). Since α cannot be both true and false in the same world, we have $w \neq w'$. We abstract this as the truth value **s** (*sometimes true and sometimes false*).

(4) We know that there is a world in which α is true ($\mathbf{P}\alpha$) but we do not know whether there is a (distinct) world in which its negation is true ($\neg\mathbf{P}\neg\alpha$). Thus, α could be true in all worlds, but we do not know that ($\neg\mathbf{K}\alpha$). We abstract this as the truth value **st** (*sometimes true*).

(5) We know that there is a world in which the negation of α is true ($\mathbf{P}\neg\alpha$) and where α is then false, but we do not know whether there is a (distinct) world in which α is true ($\neg\mathbf{P}\alpha$). Thus, α could be false in all worlds, but we do not know that ($\neg\mathbf{K}\neg\alpha$). We abstract this as the truth value **sf** (*sometimes false*).

(6) We do not know whether there exists a world in which α is true ($\neg\mathbf{P}\alpha$) nor whether there is one where its negation is true ($\neg\mathbf{P}\neg\alpha$). That is, we have no information at all, and we abstract this as the truth value $\mathbf{u}$ (*unknown*).

Thus, our set of truth values is $\mathbf{T}_{6\mathrm{v}} = \{\mathbf{t}, \mathbf{f}, \mathbf{s}, \mathbf{st}, \mathbf{sf}, \mathbf{u}\}$.

With each truth value τ and each propositional formula α, we associate the epistemic formula χ_α^τ given by the conjunction of all formulae in the maximally consistent subset of $\mathcal{M}(\alpha)$ corresponding to τ. So, for example, $\chi_\alpha^\mathbf{s}$ is the conjunction of all formulae in (3), that is, $\neg\mathbf{K}\alpha \wedge \mathbf{P}\alpha \wedge \neg\mathbf{K}\neg\alpha \wedge \mathbf{P}\neg\alpha$. Intuitively, the satisfiability of χ_α^τ tells us whether it is possible for α to evaluate to the truth value τ.

2.3. Truth Tables

With the set of truth values in place, we now look at how the truth tables for the connectives are defined. Starting from the fact that truth values correspond to maximally consistent sets of modalities, we will argue that the truth tables must satisfy two reasonable requirements: *consistency* and *generality*.

Consistency. Let us first consider the unary connective $\neg$; given a truth value τ, which truth value should $\neg\tau$ denote? If τ is $\mathbf{t}$, intuition tells us that $\neg\tau$ should not be $\mathbf{t}$. Indeed, such a situation cannot occur, in the sense that for every propositional formula α there exists no interpretation $\mathcal{I}$ that satisfies both $\chi_\alpha^\mathbf{t}$ and $\chi_{\neg\alpha}^\mathbf{t}$.

For binary connectives, the situation is similar; for example, $\mathbf{t} \wedge \mathbf{t}$ should not be $\mathbf{f}$, as it cannot happen that for propositional formulae α and β there exists an interpretation $\mathcal{I}$ that satisfies $\chi_\alpha^\mathbf{t}$, $\chi_\beta^\mathbf{t}$ and $\chi_{\alpha\wedge\beta}^\mathbf{f}$.

Thus, we require that each entry in a truth table be consistent in the following sense.

Definition 1. *Let τ_1, τ_2, and τ be truth values in $\mathbf{T}_{6\mathrm{v}}$, and let ω be a binary connective. We say that τ is* consistent *with ω on τ_1 and τ_2 if there exist propositional formulae α and β such that $\chi_\alpha^{\tau_1} \wedge \chi_\beta^{\tau_2} \wedge \chi_{\omega(\alpha,\beta)}^\tau$ is satisfiable. Similarly, τ is consistent with $\neg$ on τ_1 if there exists a propositional formula α such that $\chi_\alpha^{\tau_1} \wedge \chi_{\neg\alpha}^\tau$ is satisfiable.*

The notion of consistency directly yields the truth table of $\neg$ shown in Figure 3c, due to the following:

Proposition 1. *For every $\tau \in \mathbf{T}_{6\mathrm{v}}$ there exists one and only one truth value in $\mathbf{T}_{6\mathrm{v}}$ that is consistent with $\neg$ on τ.*

However, this is not the case for binary connectives: there are combinations of truth values that admit more than one consistent truth value, so consistency alone does not suffice to univocally define the truth tables for $\wedge$ and $\vee$. For example, both $\mathbf{f}$ and $\mathbf{sf}$ are consistent with $\mathbf{sf} \wedge \mathbf{sf}$, and both $\mathbf{t}$ and $\mathbf{st}$ are consistent with $\mathbf{st} \vee \mathbf{st}$. In such cases, how do we choose a suitable truth value? This is what we answer next.

Generality. When there is more than one truth value that is consistent with a binary connective, we should pick the *most general* among them. To illustrate this point, let us consider the case of two propositional formulae, α and β, whose truth values are both **sf**. The formula $\alpha \wedge \beta$ admits two consistent truth values: **sf** and **f**. Since both α and β are false in some of the possible worlds, we can safely conclude that also $\alpha \wedge \beta$ is false in some of these worlds. Observe, however, that, due to our current incomplete knowledge on α and β, it may still be the case that $\alpha \wedge \beta$ is true in some world. Choosing **f** as truth value for $\alpha \wedge \beta$ would preclude this possibility altogether; on the other hand, **sf** allows for this possibility without losing the information that the formula is certain false in some world. We will make this intuition more precise in what follows.

For a propositional formula α and propositional interpretations $\mathcal{I} = (t, f, W)$ and $\mathcal{I}' = (t', f', W')$, we say that $\mathcal{I}$ is *more general* than $\mathcal{I}'$ w.r.t. α (and write $\mathcal{I}' \preceq_\alpha \mathcal{I}$), if there exists a surjective mapping $h \colon W \to W'$ such that all of the following hold:

- $w \in t(\alpha)$ implies $h(w) \in t'(\alpha)$, and
- $w \in f(\alpha)$ implies $h(w) \in f'(\alpha)$.

Intuitively, $\mathcal{I}$ is more general than $\mathcal{I}'$ w.r.t. α if it has more worlds where α is not known to be true or false – that is, worlds that do not belong to either $t(\alpha)$ nor $f(\alpha)$ – but $\mathcal{I}$ agrees with $\mathcal{I}'$ on all the worlds for which this information is present.

Using this notion, we can define a partial ordering on epistemic formulae as follows: we say that φ is *more general* than ψ w.r.t. $\alpha \in \mathcal{L}$ (and write $\psi \preceq_\alpha \varphi$) if for every model $\mathcal{I}$ of ψ there exists a model $\mathcal{I}'$ of φ such that $\mathcal{I} \preceq_\alpha \mathcal{I}'$.

Finally, we can use generality to define a preference criterion for choosing a truth value over another when more than one are consistent with a connective.

Definition 2. *Let τ and τ' be truth values that are consistent with ω on τ_1 and τ_2. Then, τ' is* preferable *to τ with respect to $\omega(\tau_1, \tau_2)$ if*

$$\chi_\alpha^{\tau_1} \wedge \chi_\beta^{\tau_2} \wedge \chi_{\omega(\alpha,\beta)}^{\tau} \quad \preceq_{\omega(\tau_1,\tau_2)} \quad \chi_\alpha^{\tau_1} \wedge \chi_\beta^{\tau_2} \wedge \chi_{\omega(\alpha,\beta)}^{\tau'}$$

for all propositional formulae α and β such that both $\chi_\alpha^{\tau_1} \wedge \chi_\beta^{\tau_2} \wedge \chi_{\omega(\alpha,\beta)}^{\tau}$ and $\chi_\alpha^{\tau_1} \wedge \chi_\beta^{\tau_2} \wedge \chi_{\omega(\alpha,\beta)}^{\tau'}$ are satisfiable.

Of course, the above still leaves open the possibility that, among the truth values that are consistent with a binary connective, there might not be one that is preferable to all others. Below, we show that this is not the case.

Theorem 2. *Let $\omega \in \{\wedge, \vee\}$, let $\tau_1, \tau_2 \in \mathbf{T}_{6\mathrm{v}}$, and let $\mathbf{C}$ be the subset of truth values in $\mathbf{T}_{6\mathrm{v}}$ that are consistent with ω on τ_1 and τ_2. Then, there exists a unique $\tau \in \mathbf{C}$ such that, for every $\tau' \in \mathbf{C}$, τ is preferable to τ' with respect to $\omega(\tau_1, \tau_2)$.*

$\wedge$	**t**	**f**	**s**	**st**	**sf**	**u**
t	**t**	**f**	**s**	**st**	**sf**	**u**
f	**f**	**f**	**f**	**f**	**f**	**f**
s	**s**	**f**	**sf**	**sf**	**sf**	**sf**
st	**st**	**f**	**sf**	**u**	**sf**	**u**
sf	**sf**	**f**	**sf**	**sf**	**sf**	**sf**
u	**u**	**f**	**sf**	**u**	**sf**	**u**

(a)

$\vee$	**t**	**f**	**s**	**st**	**sf**	**u**
t	**t**	**t**	**t**	**t**	**t**	**t**
f	**t**	**f**	**s**	**st**	**sf**	**u**
s	**t**	**s**	**st**	**st**	**st**	**st**
st	**t**	**st**	**st**	**st**	**st**	**st**
sf	**t**	**sf**	**st**	**st**	**u**	**u**
u	**t**	**u**	**st**	**st**	**u**	**u**

(b)

	$\neg$
t	**f**
f	**t**
s	**s**
st	**sf**
sf	**st**
u	**u**

(c)

Figure 3: The truth tables of $\mathbb{L}_{6\text{v}}$ for $\wedge$, $\vee$ and $\neg$.

Thus, to define the truth table of a binary connective ω, for each combination of truth values τ_1 and τ_2 in $\mathbf{T}_{6\text{v}}$ we assign to $\omega(\tau_1, \tau_2)$ the most preferable truth value that is consistent with ω on τ_1 and τ_2. This yields the truth tables for $\wedge$ and $\vee$ shown in Figure 3a and 3b, respectively. Finally, we call $\mathbb{L}_{6\text{v}}$ the propositional logic consisting of the truth values in $\mathbf{T}_{6\text{v}}$ and the truth tables in Figure 3.

Coming back to the example of **sf**$\wedge$**sf** mentioned earlier, we now illustrate intuitively why the requirement of generality is indeed reasonable. Suppose that two non-equivalent propositional formulae α and β are both assigned the truth value **sf**. If the evaluation is correct, then for every propositional interpretation there exists a world in which α is false and a world (not necessarily the same) in which β is false. Both **sf** and **f** are consistent with **sf** $\wedge$ **sf**, so what truth value should $\alpha \wedge \beta$ evaluate to? The truth value **f** would indicate that $\alpha \wedge \beta$ is false in all worlds of every interpretation for which both α and β result in **sf**. Clearly, there are interpretations for which this happens, for example $(t, f, \{w_1, w_2\})$ with $f(\alpha) = \{w_1\}$, $f(\beta) = \{w_2\}$ and $t(\alpha) = t(\beta) = \varnothing$. However, there are also interpretations where this is not the case, for instance $(t', f', \{w_1, w_2\})$ with $t'(\alpha) = t'(\beta) = \varnothing$ and $f'(\alpha) = f'(\beta) = \{w_1\}$. The truth value **sf** is general enough to correctly capture the outcome of **sf** $\wedge$ **sf** in all situations, including those mentioned above, while **f** may be incorrect in some cases.

2.4. SQL's Propositional Logic

The propositional logic $\mathbb{L}_{6\text{v}} = \big(\mathbf{T}_{6\text{v}}, \{\wedge, \vee, \neg\}\big)$ can express many nuances of the truth value of a propositional formula in the case of incomplete information. But can this logic be used in practice?

The query optimization engines of modern relational database management systems are based on decades of research that relies on a well established set of assumptions on the logic underlying the evaluation. Among these assumptions, there are three crucial properties of the connectives, see [27, 28]:

- *idempotency*, that is, $\tau \vee \tau = \tau$ and $\tau \wedge \tau = \tau$; and
- *distributivity*, that is, $\tau \wedge (\tau' \vee \tau'') = (\tau \wedge \tau') \vee (\tau \wedge \tau'')$ and similarly with $\wedge$ and $\vee$ swapped;

- *double-negation elimination*, that is, $\neg\neg\tau = \tau$.

These properties are used in all RDBMS query optimizers to transform redundant expressions into equivalent non-redundant ones, in order to reduce the number of superfluous operations to be executed during query evaluation.

While $\mathbb{L}_{6\mathrm{v}}$ has the double-negation elimination property, it fails idempotency and distributivity. Indeed, $\mathbf{s} \wedge \mathbf{s}$ and $\mathbf{s} \vee \mathbf{s}$ give $\mathbf{sf}$ and $\mathbf{st}$, respectively, rather than $\mathbf{s}$. Moreover, $\wedge$ does not distribute over $\vee$:

$$\underbrace{\mathbf{s} \wedge \overbrace{(\mathbf{s} \vee \mathbf{s})}^{\mathbf{st}}}_{\mathbf{sf}} \quad \neq \quad \underbrace{\overbrace{(\mathbf{s} \wedge \mathbf{s})}^{\mathbf{sf}} \vee \overbrace{(\mathbf{s} \wedge \mathbf{s})}^{\mathbf{sf}}}_{\mathbf{u}}$$

and $\vee$ does not distribute over $\wedge$:

$$\underbrace{\mathbf{s} \vee \overbrace{(\mathbf{s} \wedge \mathbf{s})}^{\mathbf{sf}}}_{\mathbf{st}} \quad \neq \quad \underbrace{\overbrace{(\mathbf{s} \vee \mathbf{s})}^{\mathbf{st}} \wedge \overbrace{(\mathbf{s} \vee \mathbf{s})}^{\mathbf{st}}}_{\mathbf{u}}$$

Observe that the binary connectives in $\mathbb{L}_{6\mathrm{v}}$ are *weakly idempotent*, i.e., for every truth value $\tau \in \mathbf{T}_{6\mathrm{v}}$ we have $\tau \wedge \tau \wedge \tau = \tau \wedge \tau$, and likewise for $\vee$. However, due to the lack of idempotency and distributivity, $\mathbb{L}_{6\mathrm{v}}$ is unlikely to be implemented in real systems for query evaluation. To overcome this, we look for *sublogics* of $\mathbb{L}_{6\mathrm{v}}$ with the desired properties.

To this end, we formalize the notion of sublogic as follows. Given two logics $\mathbb{L} = (\mathbf{T}, \Omega)$ and $\mathbb{L}' = (\mathbf{T}', \Omega)$, with $\mathbf{T}' \subseteq \mathbf{T}$ and over the same set of connectives, we say that $\mathbb{L}'$ *is a sublogic of* $\mathbb{L}$ if there is a mapping $h\colon \mathbf{T} \to \mathbf{T}'$ such that, for every n-ary connective $\omega \in \Omega$ and every n-tuple $\bar{\tau}$ of truth values from $\mathbf{T}'$, we have $h\big(\omega^{\mathbb{L}}(\bar{\tau})\big) = \omega^{\mathbb{L}'}\big(h(\bar{\tau})\big)$. If h is a bijection, we say that $\mathbb{L}'$ *is equivalent to* $\mathbb{L}$ (i.e., the same up to renaming of truth values). Also, if h is the identity over a set $\mathbf{T}'' \subseteq \mathbf{T}$, we say that $\mathbb{L}'$ *preserves* $\mathbf{T}''$.

Intuitively, if $\mathbb{L}'$ is a sublogic of $\mathbb{L}$, the truth tables of $\mathbb{L}'$ behave consistently with those of $\mathbb{L}$, over a more refined set of truth values. This definition of sublogic captures several interesting cases, e.g., it can be shown that the four-valued logic presented in [13] is a sublogic of $\mathbb{L}_{6\mathrm{v}}$ that preserves $\{\mathbf{t}, \mathbf{f}\}$. The same holds for $\mathbb{L}_{3\mathrm{v}}$, due to the following mapping: $h(\tau) = \tau$, if $\tau \in \{\mathbf{t}, \mathbf{f}\}$, $h(\tau) = \mathbf{u}$, otherwise.

To handle incomplete information in practice, we want a logic that preserves, as much as possible, the behavior of $\mathbb{L}_{6\mathrm{v}}$. A sublogic $\mathbb{L}'$ of $\mathbb{L}$ is *maximal with respect to a property* P if it has P and every sublogic $\mathbb{L}''$ of $\mathbb{L}$ with property P is also a sublogic of $\mathbb{L}'$. For practical purposes, we want a sublogic of $\mathbb{L}_{6\mathrm{v}}$ that is maximal with respect to distributivity, idempotency, and double-negation elimination. In general, such a logic need not be unique. A sublogic $\mathbb{L}'$ of $\mathbb{L}$, maximal with respect to P, is *unique* (up to renaming of truth values) if every sublogic $\mathbb{L}''$ of $\mathbb{L}$ that is maximal with respect to P is equivalent to $\mathbb{L}'$.

A sublogic that is maximal w.r.t. the above properties, however, is not yet enough for practical applications. To answer database queries, we need a logic that can at least distinguish between true and false answers. For this reason, we thus require a sublogic of $\mathbb{L}_{6\mathrm{v}}$ that preserves the truth values $\{\mathbf{t}, \mathbf{f}\}$ and that is maximal with respect to distributivity, idempotency, and double-negation elimination.

Theorem 3. $\mathbb{L}_{3v}$ *is the unique, up to renaming of truth values, sublogic of* $\mathbb{L}_{6v}$ *that preserves* $\{\mathbf{t}, \mathbf{f}\}$*, and that is maximal with respect to distributivity, idempotency, and double-negation elimination.*

Therefore, when it comes to balancing expressiveness and practicality, the much criticized three-valued logic used by SQL is in fact a good choice for dealing with incomplete information in relational databases, at least for the propositional case.

We next examine extensions of propositional logics such as $\mathbb{L}_{6v}$ and $\mathbb{L}_{3v}$ to predicate logics.

3. Predicate Logics

As already explained, the need to consider a predicate logic of incomplete information arises most commonly in querying incomplete databases, where special values – commonly referred to as *nulls* – indicate incompleteness of some sort. When atomic formulae may involve nulls – e.g., comparing a null with another value, or checking whether a tuple with nulls belongs to a relation – the standard approach is not to follow the Boolean semantics of FO, but instead to look for a many-valued semantics that will properly lift a propositional logic to all of FO. Such a semantics is by no means unique; we shall see three common versions later in this section.

We now define incomplete relational databases (which are in fact two-sorted relational structures), and consider many-valued FO logics on them, based on particular propositional logic. While propagating truth values through connectives and quantifiers simply follows the truth tables of the propositional logic, assigning them to atoms is not unique. We consider three commonly occurring ways:

- one uses the Boolean semantics [6],
- one adopts the approach of SQL [10],
- and yet another is based on tuple unification, to achieve query answers with certainty guarantees [29].

As our main result, we show that in the context of many-valued FO, the exact choice of semantics of atoms, or truth values, or propositional connectives, does not matter: whatever combination of these one chooses, the resulting logic never exceeds the power of Boolean FO and can be naturally encoded into it.

3.1. Incomplete Relational Structures (Databases)

As is standard in the database field and many applications of incomplete information, elements of relational structures (or relational databases; these terms are used interchangeably) come from two disjoint sets. One is the set Const of *constants*, i.e., known values that are stored in databases. The other is the set Null of *nulls* that represent unknown values. We always assume that Const is countably infinite. For the set Null, some options exist, of which the most common are the following.

- Null too is a countably infinite set. This corresponds to the model of *marked* nulls used both in relational databases and their many applications, such as data exchange [3], data integration [2] and ontology-based data access [5].

- Null is a singleton set containing one element denoted by **N**. This is the approach of SQL and implementations of relational DBMSs, where there is just one single null value.

A relational *vocabulary* σ (which is usually called *schema* in the database context) is a set $\{R_1, \ldots, R_n, =\}$ consisting of relation names $R_1, \ldots, R_n$, each with an associated arity, plus a binary relation symbol "=" for equality. A *structure* $\mathfrak{A}$ of this vocabulary is a tuple $\langle A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}}, =^{\mathfrak{A}} \rangle$, where:

- A is a finite subset of $\mathsf{Const} \cup \mathsf{Null}$,

- $R_i^{\mathfrak{A}} \subseteq A^k$ for every $i \in \{1, \ldots, n\}$, and

- $=^{\mathfrak{A}}$ is the binary relation defined as $\{(a, a) \mid a \in A\}$.

3.2. Many-valued Predicate Logics

A many-valued *predicate* logic $\langle \mathrm{FO}(\mathbb{L}), [\![\,]\!] \rangle$ is based on a many-valued propositional logic $\mathbb{L}$ with a set $\mathbf{T}$ of truth values and Ω of propositional connectives; the extra element here is the semantics $[\![\,]\!]$ of its formulae. We now define these. Throughout the section, we make the following assumptions:

- $\mathbb{L}$ has connectives $\vee, \wedge$ which are commutative and associative (this is necessary to define quantifiers); other connectives are arbitrary.

- Truth values $\mathbf{t}$ and $\mathbf{f}$ are always included in $\mathbf{T}$, and the connectives $\vee, \wedge, \neg$ restricted to them follow the rules of Boolean logic (in other words, we do not re-define true and false).

Syntax and semantics of $\mathrm{FO}(\mathbb{L})$. Given a propositional logic $\mathbb{L}$ with truth values $\mathbf{T}$ and connectives Ω, formulae of $\mathrm{FO}(\mathbb{L})$ are defined by the following rules.

- Atomic formulae:

  - if R is a k-ary vocabulary symbol, and $x_1, \ldots, x_k$ are variables, then $R(x_1, \ldots, x_k)$ is an atomic formula; we shall also write the more common $x_1 = x_2$ in place of $=(x_1, x_2)$;

  - $\mathsf{const}(x)$ and $\mathsf{null}(x)$ are atomic formulae.

- If $\omega \in \Omega$ is a k-ary connective, and $\varphi_1, \ldots, \varphi_k$ are formulae, then $\omega(\varphi_1, \ldots, \varphi_k)$ is a formula.

- If φ is a formula and x is a variable, then $\exists x\, \varphi$ and $\forall x\, \varphi$ are formulae.

The notion of free variables is defined in the usual way.

The semantics of a formula φ is given with respect to a structure $\mathfrak{A}$ with universe A and an assignment ν of values in A to free variables of φ (i.e., ν is a partial function that is defined on all free variables of φ and takes values in A). This semantics will be denoted by $[\![\varphi]\!]_{\mathfrak{A},\nu}$, and it is a value in $\mathbf{T}$. In other words, $[\![\,]\!]$ assigns a truth value in $\mathbf{T}$ to φ in a structure $\mathfrak{A}$ under assignment ν.

The semantics of atoms const and null is as follows:

$$[\![\mathsf{const}(x)]\!]_{\mathfrak{A},\nu} = \begin{cases} \mathbf{t} & \text{if } \nu(x) \in \mathsf{Const}, \\ \mathbf{f} & \text{if } \nu(x) \in \mathsf{Null}. \end{cases}$$

$$[\![\mathsf{null}(x)]\!]_{\mathfrak{A},\nu} = \begin{cases} \mathbf{t} & \text{if } \nu(x) \in \mathsf{Null}, \\ \mathbf{f} & \text{if } \nu(x) \in \mathsf{Const}. \end{cases}$$

For propositional connectives and quantifiers, the semantics is defined with the *standard lifting rules*:

$$[\![\omega(\varphi_1,\ldots,\varphi_k)]\!]_{\mathfrak{A},\nu} = \omega([\![\varphi_1]\!]_{\mathfrak{A},\nu},\ldots,[\![\varphi_k]\!]_{\mathfrak{A},\nu})\,,$$

$$[\![\exists x\,\varphi]\!]_{\mathfrak{A},\nu} = \bigvee_{a\in A}[\![\varphi]\!]_{\mathfrak{A},\nu[a/x]}\,,$$

$$[\![\forall x\,\varphi]\!]_{\mathfrak{A},\nu} = \bigwedge_{a\in A}[\![\varphi]\!]_{\mathfrak{A},\nu[a/x]}\,,$$

where $\nu[a/x]$ is the same as ν except that it assigns a to x. The last two rules rely on the fact that $\vee$ and $\wedge$ are commutative and associative.

For atomic formulae $R(\bar{x})$, with $R \in \sigma$, there are several options, which we now consider, when the underlying logic is either $\mathbb{L}_{\text{bool}}$ or $\mathbb{L}_{\text{3v}}$.

Boolean semantics. This is the standard two-valued FO semantics, with only $\mathbf{t}$ and $\mathbf{f}$ as truth values, and it is given by

$$[\![R(\bar{x})]\!]^{\text{bool}}_{\mathfrak{A},\nu} \;=\; \begin{cases} \mathbf{t} & \text{if } \nu(\bar{x}) \in R^{\mathfrak{A}}, \\ \mathbf{f} & \text{if } \nu(\bar{x}) \notin R^{\mathfrak{A}}, \end{cases}$$

for every R in the vocabulary σ (which, recall, includes $=$). It is then extended to all of FO with the above rules, resulting in the semantics $[\![\,]\!]^{\text{bool}}$ defined for all FO formulae. When $[\![\varphi]\!]^{\text{bool}}_{\mathfrak{A},\nu} = \mathbf{t}$ we also write the more customary $\mathfrak{A},\nu \models \varphi$.

The logic BFO, or *Boolean* FO, is now formally defined as FO($\mathbb{L}_{\text{bool}}$) interpreted under $[\![\,]\!]^{\text{bool}}$; it is the standard FO with only $\mathbf{t}$ and $\mathbf{f}$ as truth values.

Null-free semantics. A tuple $\bar{a}$ is *null-free* if all of its values are from Const. The null-free semantics of FO($\mathbb{L}_{\text{3v}}$) is the same as the Boolean semantics for tuples of constants; if any nulls are present, it produces the truth value $\mathbf{u}$:

$$[\![R(\bar{x})]\!]^{\mathrm{nf}}_{\mathfrak{A},\nu} = \begin{cases} \mathbf{t} & \text{if } \nu(\bar{x}) \in R^{\mathfrak{A}} \text{ and } \nu(\bar{x}) \text{ is null-free,} \\ \mathbf{f} & \text{if } \nu(\bar{x}) \notin R^{\mathfrak{A}} \text{ and } \nu(\bar{x}) \text{ is null-free,} \\ \mathbf{u} & \text{if } \nu(\bar{x}) \text{ contains a null,} \end{cases}$$

for every R in the vocabulary σ (which, recall, includes =). In particular, for the equality predicate =, this is exactly the semantics used by SQL [10].

Unification semantics. A semantics based on the notion of tuple unification was proposed by [29] to enforce certainty guarantees for query answers. We say that two tuples $\bar{a}$ and $\bar{b}$ *unify* if there is a map $h\colon \mathsf{Const} \cup \mathsf{Null} \to \mathsf{Const}$ that is the identity on constants and such that $h(\bar{a}) = h(\bar{b})$. Then, for every relation symbol R in the vocabulary σ, the unification semantics is defined by

$$[\![R(\bar{x})]\!]^{\Uparrow}_{\mathfrak{A},\nu} = \begin{cases} \mathbf{t} & \text{if } \nu(\bar{x}) \in R^{\mathfrak{A}}, \\ \mathbf{f} & \text{if } \nexists \bar{a} \in R^{\mathfrak{A}} \text{ s.t. } \nu(\bar{x}) \text{ and } \bar{a} \text{ unify,} \\ \mathbf{u} & \text{otherwise.} \end{cases}$$

The semantics $[\![\,]\!]^{\Uparrow}$ is then lifted to all of FO by the standard lifting rules.

The reason this semantics was introduced is that it ensures certainty of answers to FO queries: if $[\![\varphi(\bar{x})]\!]^{\Uparrow}_{\mathfrak{A},\nu} = \mathbf{t}$, then the tuple $\bar{u} = \nu(\bar{x})$ is what is known as a certain answer to φ, i.e., $h(\mathfrak{A}) \models \varphi\big(h(\bar{u})\big)$ for every map $h : \mathsf{Const} \cup \mathsf{Null} \to \mathsf{Const}$ that is the identity on constants. Here $h(\mathfrak{A})$ is obtained from $\mathfrak{A}$ by replacing every value v in its domain by $h(v)$.

Mixed semantics. There is a priori no reason to apply the same semantics on each relation symbol $R \in \sigma$; instead we can freely mix them. A mixed semantics $[\![\,]\!]^s$ is then given by a function $s : \sigma \to \{\mathrm{bool}, \Uparrow, \mathrm{nf}\}$ so that $[\![R(\bar{x})]\!]^s_{\mathfrak{A},\nu} = [\![R(\bar{x})]\!]^{s(R)}_{\mathfrak{A},\nu}$. This generalizes Boolean, unification, and null-free semantics.

3.3. Boolean FO Captures Many-valued FO

We now show that in most cases, many-valued predicate logics do not give any extra power compared to BFO, i.e., the usual FO under the standard Boolean interpretation of connectives and the Boolean semantics of atomic formulae. The notion of capturing a many-valued FO logic by BFO needs to account is defined as follows.

Definition 3. *A formula φ of* FO($\mathbb{L}$) *over a many-valued propositional logic $\mathbb{L}$ with truth values $\mathbf{T}$ is* captured *by* BFO *under semantics $[\![\,]\!]$ if there exist* BFO *formulae φ_τ for each $\tau \in \mathbf{T}$ such that for every structure $\mathfrak{A}$ and assignment ν of free variables of φ we have*

$$[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau \;\Leftrightarrow\; \mathfrak{A}, \nu \models \varphi_\tau \,.$$

FO($\mathbb{L}$) *is captured by* BFO *if each of its formulae is.*

Usually we are interested in formulae that are true in a given structure, i.e., $[\![\varphi]\!]_{\mathfrak{A},\nu} = \mathbf{t}$. If a formula is captured by BFO, this tells us that we do not need many-valued semantics, and instead can simply check whether $\mathfrak{A}, \nu \models \varphi_{\mathbf{t}}$ under the usual Boolean semantics.

To capture a many-valued FO by BFO we need very few assumptions. Recall that $\mathbb{L} = \langle \mathbf{T}, \Omega \rangle$ is given by a set of truth values and truth tables for connectives in Ω, which we assume to contain at least $\vee, \wedge$ to define quantifiers. In logics such as $\mathbb{L}_{\text{bool}}$ and $\mathbb{L}_{\text{3v}}$, these connectives are *idempotent*, i.e., $\tau \wedge \tau = \tau \vee \tau = \tau$ for every $\tau \in \mathbf{T}$. In $\mathbb{L}_{\text{6v}}$, they are *weakly idempotent*: $\tau \wedge \tau \wedge \tau = \tau \wedge \tau$ and likewise for $\vee$. Notice that idempotency implies weak idempotency. This is the only condition we need to impose to be able to lift capturing formulae by Boolean FO from atoms to arbitrary formulae.

Theorem 4. *Let $\mathbb{L}$ be a propositional many-valued logic in which connectives $\wedge$ and $\vee$ are weakly idempotent. Assume that every relational atom $R(\bar{x})$, for $R \in \sigma$, is captured by* BFO *under* $[\![\,]\!]$. *Then every* FO($\mathbb{L}$) *formula over vocabulary σ is captured by* BFO *under* $[\![\,]\!]$.

To apply this result to the previously considered semantics, we need to capture atomic formulae, under different semantics, in BFO. This is possible for all of them.

Proposition 2. *Relational atoms are captured by* BFO *under Boolean, unification, and null-free semantics.*

Finally, this tells us that any mixed semantics (including its pure versions, i.e., Boolean, unification, null-free) coupled with any propositional many-valued logic like $\mathbb{L}_{\text{3v}}$ or $\mathbb{L}_{\text{6v}}$ (as long as it has weakly idempotent conjunction and disjunction) is no more powerful than the standard semantics over two truth values $\mathbf{t}$ and $\mathbf{f}$.

Corollary 1. *Let $\mathbb{L}$ be a propositional many-valued logic whose truth values include $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$, with an arbitrary set of connectives where $\vee$ and $\wedge$ are (weakly) idempotent. Then for every vocabulary σ, every function s defining a mixed semantics, and every formula φ of* FO($\mathbb{L}$) *there is a formula φ' of* BFO *such that $[\![\varphi]\!]^s_{\mathfrak{A},\nu} = \mathbf{t}$ iff $\mathfrak{A}, \nu \models \varphi'$.*

Using this result, we can clarify, in the next section, the question of the power of the logic that underlies real-life database applications that use incomplete information.

4. The Logic of SQL

Most database texts will claim that the core of SQL, the main relational database query language, is first-order logic FO. This was certainly true in the early stages of SQL design, as it grew out of relational calculus, which is just another name for FO. But then the language gained many features, in particular null values, leading to more complex underlying logics.

These logics are still not well understood, as the formalization of SQL mainly took a different route via *relational algebra*, which is the procedural counterpart of FO. Several attempts to provide a theoretical language behind SQL looked at relational algebra translations of the language [30, 31] or presented semantics of various fragments of the language, often under the simplifying assumption that no nulls are present and no three-valued logic is used [32, 33]. An attempt to find a logic underlying SQL concentrated on its features that go beyond FO (i.e., aggregation) rather than nulls [34]. More recent work [24], while providing a direct semantics of SQL, accounted for null values and three-valued logic, and even gave a translation of SQL queries that, similarly in spirit to the results in the previous section, showed how to evaluate them without ever producing the unknown truth value **u**. This was done, however, at the level of SQL queries. We now analyze the power of SQL and the need for three truth values at a purely logical level.

We start with the basic fragment of relational languages that has the power of FO, or – equivalently – the basic operations of relational algebra, or SQL's select-from-where queries without aggregation. These operate on databases whose values come from Const. Recall that SQL uses a single null denoted here by $\mathbf{N}$. Now we add it; how should the logic change to capture this extension? It depends on who is asked to produce such an extension.

A logician's approach. If the domain is extended by a single constant, we simply consider FO over $\mathsf{Const}\cup\{\mathbf{N}\}$ with a unary predicate null( ) that is only true in $\mathbf{N}$ (to keep the vocabulary relational; alternatively a constant symbol could be added). The interpretation of $=$ is simply $\{(c,c) \mid c \in \mathsf{Const}\} \cup \{(\mathbf{N},\mathbf{N})\}$, i.e., *syntactic equality*: $\mathbf{N}$ is equal to itself, and not equal to any element of Const. In other words, the logic is the usual BFO, with all the atoms interpreted under the Boolean semantics $[\![\,]\!]^{\text{bool}}$.

It would thus be seen, by a logician, as an overkill to introduce a many-valued logic to deal with just one extra element of the domain. Nonetheless, this is what SQL did.

SQL approach: a textbook version. The usual explanation of the logic behind SQL is that it adds a new truth value **u** to account for any comparisons involving nulls. In other words, the logic is $\text{FO}(\mathbb{L}_{3\text{v}})$, and the semantics $[\![\,]\!]^{\text{sql}}$ is mixed, combining Boolean and null-free semantics:

- for relational atoms, $[\![R(\bar{x})]\!]^{\text{sql}}_{\mathfrak{A},\nu} = [\![R(\bar{x})]\!]^{\text{bool}}_{\mathfrak{A},\nu}$;
- for equality, $[\![x = y]\!]^{\text{sql}}_{\mathfrak{A},\nu} = [\![x = y]\!]^{\text{nf}}_{\mathfrak{A},\nu}$.

SQL approach: what really happens. While the textbook approach comes close to describing the logic of SQL, it misses one important feature of such logic. In essence, we can think of SQL queries as expressions

$$\begin{array}{ll}\textsf{select} & \bar{x}\\ \textsf{from} & Q_1,\ldots,Q_n\\ \textsf{where} & \theta(\bar{x}_1,\ldots,\bar{x}_n)\end{array}$$

where $Q_1, \ldots, Q_n$ are either queries or relations, $\bar{x}_i$ is a tuple of variables returned by Q_i, and θ is a condition composed of equalities of variables and constants, or statements $Q'(\bar{y})$, where Q' is another query, or statements $Q' \neq \varnothing$, combined using $\wedge$, $\vee$, and $\neg$.

Note that in SQL query evaluation, it is the conditions θ that are evaluated in $\mathbb{L}_{3\text{v}}$; once the evaluation of the where θ clause is finished, only tuples that evaluated to $\mathbf{t}$ are kept. To capture this in logic, we need a propositional operator that collapses $\mathbf{f}$ and $\mathbf{u}$ into $\mathbf{f}$. Such an operator does exist in propositional many-valued logics [35] and is known as an *assertion* operator: $\uparrow p$ for a proposition p evaluates to $\mathbf{t}$ if p evaluates to $\mathbf{t}$, and to $\mathbf{f}$ otherwise. Let $\mathbb{L}_{3\text{v}}^{\uparrow}$ be the extension of $\mathbb{L}_{3\text{v}}$ with this operator.

The basic SQL query can then be expressed in $\text{FO}(\mathbb{L}_{3\text{v}}^{\uparrow})$:

$$Q(\bar{x}) \;=\; \exists \bar{y} \bigwedge_{i=1}^{n} Q_i(\bar{x}_i) \wedge \uparrow\theta(\bar{x}_1, \ldots, \bar{x}_n)\,,$$

where $\bar{y}$ lists the variables present in $\bar{x}_1, \ldots, \bar{x}_n$ but not in $\bar{x}$. Thus, the many-valued predicate logic capturing SQL's behavior is $\text{FO}(\mathbb{L}_{3\text{v}}^{\uparrow})$ under $[\![\,]\!]^{\text{sql}}$.

To sum up, there are three choices of a logic capturing SQL's behavior:

1) Boolean predicate logic BFO;

2) FO based on Kleene's logic under the $[\![\,]\!]^{\text{sql}}$ semantics;

3) FO based on Kleene's logic with the assertion operator under the $[\![\,]\!]^{\text{sql}}$ semantics.

These logics use different sets of truth values. However, it only matters when formulae evaluate to true, as this determines the output of queries. Thus, to compare logics with different sets of truth values, we say that two logics, $\text{FO}(\mathbb{L}_1)$ under $[\![\,]\!]^1$, and $\text{FO}(\mathbb{L}_2)$ under $[\![\,]\!]^2$, are *true-equivalent* if the models of $\mathbf{t}$ are the same in both. That is, for every formula φ_1 of $\text{FO}(\mathbb{L}_1)$ there is a formula φ_2 of $\text{FO}(\mathbb{L}_2)$ such that

$$[\![\varphi_1]\!]^1_{\mathfrak{A},\nu} = \mathbf{t} \quad \Leftrightarrow \quad [\![\varphi_2]\!]^2_{\mathfrak{A},\nu} = \mathbf{t}$$

for every $\mathfrak{A}, \nu$, and vice versa, for each φ_2 of $\text{FO}(\mathbb{L}_2)$ there is a formula φ_1 of $\text{FO}(\mathbb{L}_1)$ such that the above condition holds.

Then, with respect to the truth value $\mathbf{t}$, there is no difference between the logics that attempt to model SQL's behavior.

Theorem 5. *The logics* $\text{FO}(\mathbb{L}_{3\text{v}})$ *and* $\text{FO}(\mathbb{L}_{3\text{v}}^{\uparrow})$*, both under* $[\![\,]\!]^{\text{sql}}$*, and* BFO*, are all true-equivalent.*

Therefore, the use of a many-valued logic to handle incomplete information adds no extra expressiveness. However, one may still wonder whether many-valued logics may give an advantage in terms of succinctness of formulae. To prove that this is not the case, we first defined the size $|\varphi|$ of a formula φ.

- $|R(x)| = |(x_1 = x_2)| = 1$;
- $|\varphi \wedge \psi| = |\varphi \vee \psi| = |\varphi| + |\psi|$;
- $|\neg\varphi| = |\forall x.\varphi| = |\exists x.\varphi| = 1 + |\varphi|$.

The use of BFO to express properties in $\mathrm{FO}(\mathbb{L}_{3\mathrm{v}}^{\uparrow})$ does not have a dramatic impact on the size of the formulae, as the following theorem shows.

Theorem 6. *There is $c \in \mathbb{N}$ such that, for each formula $\varphi_1 \in \mathrm{FO}(\mathbb{L}_{3\mathrm{v}}^{\uparrow})$, there exists a formula $\varphi_2 \in \mathrm{BFO}$ for which $|\varphi_2| = c \cdot |\varphi_1|$ and $\mathfrak{A}, \nu \models \varphi_2 \;\Leftrightarrow\; [\![\varphi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$.*

Thus, the more natural logical approach to adding a null value to the language does not miss any fundamental characteristic of the approaches based on many-valued logics.

5. Conclusions

To conclude, let us revisit history. Handling incomplete information by logical languages is an important topic, especially in data management. All commercial database systems that speak SQL offer a solution based on a three-valued propositional logic that is lifted then to full predicate logic. This solution was heavily criticized in the literature, but at the level of the chosen propositional logic.

We proposed a principled approach to justifying a proper logic for handling incomplete information, which resulted in a six-valued logic $\mathbb{L}_{6\mathrm{v}}$. However, taking into account the needs of SQL query evaluation (e.g., distributivity laws), the largest fragment of $\mathbb{L}_{6\mathrm{v}}$ that does not break traditional evaluation and optimization strategies is Kleene's logic $\mathbb{L}_{3\mathrm{v}}$, precisely the one chosen by SQL.

However, even though the SQL designers were justified in their choice of Kleene's logic, they neglected to consider the impact that lifting it to full predicate logic would have. We showed that it leads to no increase in expressive power; had this been known to the SQL designers, perhaps other choices would have been considered too.

But does this mean that we should abandon many-valued logics of incomplete information? Most likely not: while the theoretical complexity of formulae that result from eliminating many-valuedness is the same as that of original many-valued formulae, their *practical* complexity (i.e., if implemented as real life database queries) is likely to be different. This is mainly due to the fact that 40 years of research on query evaluation and optimization had one particular model in mind, and that model used a many-valued logic. However, the observations we made here might have an impact on the design of new languages, since avoiding many-valued logics for handling incompleteness is now an option.

Regarding future directions, we would like to extend the propositional setup with bilattice orderings as is often done [17, 18], and understand the right orderings for logics like $\mathbb{L}_{6\mathrm{v}}$. Yet another direction is to drop the restriction $t(\alpha) \cap f(\alpha) = \varnothing$ for every propositional formula α. Such restrictions have been lifted in the study of paraconsistent logics [23, 16], and in fact the question of looking for the right many-valued logic for reasoning about inconsistency has been raised [36]. Our focus would be slightly different, as we want to extend the current study to handle the most common case of inconsistency in data management, namely inconsistency with respect to integrity constraints [37, 4].

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Appendix A. Proofs of Results on Propositional Logic

In this section, we present the proofs of Proposition 1 and Theorem 2. In what follows, we use the following notation. Given a propositional interpretation $\mathcal{I} = (t, f, W)$, we use $u_{\mathcal{I}}$ to denote the function from $\mathcal{L}$ to the powerset of W such that $u_{\mathcal{I}}(\alpha) = W \setminus (t(\alpha) \cup f(\alpha))$, for every $\alpha \in \mathcal{L}$. Intutively, $u_{\mathcal{I}}$ denotes the set of worlds of W where α is undefined. As customary, given two (propositional or modal) formulae φ and ψ, we use $\varphi \to \psi$ for $\neg\varphi \vee \psi$ and $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \psi)$. A modal formula φ is a *tautology* if it is satisfied by every propositional interpretation. In our proofs, we use tautologies that represent fundamental properties of our modal formalism. We present these properties now, starting from *Weak Duality* (WD). Intuitively, Weak Duality characterizes the interaction between the modal operators.

Proposition 3. *Let α and β be propositional formulae. The following modal formulae are tautologies:*

- *(WD1)* $\mathbf{K}\alpha \to (\mathbf{P}\beta \leftrightarrow \mathbf{P}(\alpha \wedge \beta))$.

- *(WD2)* $\neg\mathbf{K}\alpha \to (\neg\mathbf{P}\beta \to \neg\mathbf{K}(\alpha \vee \beta))$
- *(WD3)* $\mathbf{K}\alpha \to \neg\mathbf{P}\neg\alpha$

Proof. Assume a propositional interpretation $\mathcal{I} = (t, f, W)$. We prove each statement separately.

(WD1). If $\mathcal{I} \models \mathbf{P}(\alpha \wedge \beta)$, then there exists $w \in t(\alpha) \cap t(\beta)$. Trivially, then, $w \in t(\beta)$, i.e., $\mathcal{I} \models \mathbf{P}\beta$. We can conclude that the formula $\mathbf{P}(\alpha \wedge \beta) \to \mathbf{P}(\alpha)$ is a tautology. Assume now $\mathcal{I} \models \mathbf{K}\alpha$, i.e., $t(\alpha) = W$, we prove that $\mathcal{I}$ satisfies $(\mathbf{P}\beta \to \mathbf{P}(\alpha \wedge \beta))$. If $\mathcal{I} \models \mathbf{P}\beta$, then there exists $w \in t(\beta)$. Therefore, $w \in t(\alpha) \cap t(\beta)$, and we can conclude that $w \in t(\alpha \wedge \beta)$, i.e., $\mathcal{I} \models \mathbf{P}(\alpha \wedge \beta)$.

(WD2). Assume $\mathcal{I} \models \neg\mathbf{K}\alpha$, i.e., $t(\alpha) \neq W$. We prove that $\mathcal{I} \models (\neg\mathbf{P}\beta \to \neg\mathbf{K}(\alpha \vee \beta))$. Suppose $\mathcal{I} \models \neg\mathbf{P}\beta$, i.e., $t(\beta) = \varnothing$, and let $w \in W$ such that $w \notin t(\alpha)$. Therefore, $w \notin t(\alpha) \cup t(\beta)$, proving $\mathcal{I} \models \neg\mathbf{K}(\alpha \vee \beta)$.

(WD3). Assume $\mathcal{I} \models \mathbf{K}\alpha$, i.e., $t(\alpha) = W$. By definition, $t(\alpha) \cap t(\neg\alpha) = \varnothing$. Therefore, $t(\neg\alpha) = \varnothing$, i.e., $\mathcal{I} \models \neg\mathbf{P}\neg\alpha$.

□

The following tautologies represents additional relevant properties of the modal operators.

Proposition 4. *Let α and β be propositional formulae. The following modal formulae are tautologies:*

- *(Completeness of $\mathbf{P}$)* $\neg\mathbf{P}\alpha \to \neg\mathbf{P}(\alpha \wedge \beta)$
- *(Distributivity of $\mathbf{K}$ over $\wedge$)* $\mathbf{K}(\alpha \wedge \beta) \leftrightarrow \mathbf{K}\alpha \wedge \mathbf{K}\beta$
- *(Distributivity of $\mathbf{P}$ over $\vee$)* $\mathbf{P}(\alpha \vee \beta) \leftrightarrow \mathbf{P}\alpha \vee \mathbf{P}\beta$

Proof. Assume a propositional interpretation $\mathcal{I} = (t, f, W)$. We prove each statement separately.

(Completeness of $\mathbf{P}$). If $\mathcal{I} \models \neg\mathbf{P}\alpha$, then $t(\alpha) = \varnothing$. From the definition of propositional interpretations, it follows that $t(\alpha \wedge \beta) = \varnothing$, in turn proving $\mathcal{I} \models \neg\mathbf{P}(\alpha \wedge \beta)$.

(Distributivity of $\mathbf{K}$ over $\wedge$). We prove the two claims separately. ($\Rightarrow$) If $\mathcal{I} \models \mathbf{K}(\alpha \wedge \beta)$, then $W = t(\alpha \wedge \beta)$. From the definition of propositional interpretations, it follows $t(\alpha \wedge \beta) = t(\alpha) \cap t(\beta)$. Therefore, $W = t(\alpha) = t(\beta)$, and we can conclude $\mathcal{I} \models (\mathbf{K}\alpha \wedge \mathbf{K}\beta)$. ($\Leftarrow$) If $\mathcal{I} \models (\mathbf{K}\alpha \wedge \mathbf{K}\beta)$, then $W = t(\alpha) =\in t(\beta)$. From the definition of propositional interpretations, it follows that $W = t(\alpha \wedge \beta)$, and we can conclude $\mathcal{I} \models \mathbf{K}(\alpha \wedge \beta)$.

(Distributivity of $\mathbf{P}$ *over* $\vee$*).* We prove the two claims separately. ($\Rightarrow$) If $\mathcal{I} \models \mathbf{P}(\alpha \vee \beta)$, then there exists $w \in W$ such that $w \in t(\alpha \vee \beta)$. From the definition of propositional interpretations, it follows that either $w \in t(\alpha)$ or $w \in t(\beta)$. In turn, the latter proves $\mathcal{I} \models (\mathbf{P}\alpha \vee \mathbf{P}\beta)$. ($\Leftarrow$) If $\mathcal{I} \models (\mathbf{P}\alpha \vee \mathbf{P}\beta)$, then there exists $w \in W$ such that either $w \in t(\alpha)$ or $w \in t(\beta)$. From the definition of propositional interpretations, it follows that $w \in t(\alpha \vee \beta)$, in turn proving $\mathcal{I} \models \mathbf{P}(\alpha \vee \beta)$. □

Finally, we prove the following statements on the interpretation of propositional formulae in proprositional interpretations.

Proposition 5. *Let* α *and* β *be two propositional formulae, and let* $\mathcal{I} = (t, f, W)$ *be a propositional interpretation. For every* $w \in W$*, the following holds.*

- $(P1)$*: if* $w \in t(\alpha)$ *then* $w \in t(\alpha \vee \beta)$
- $(P2)$*: if* $w \in t(\neg\alpha)$ *then* $w \in t(\neg(\alpha \wedge \beta))$

Proof. The proof follows straightforwardly from the definition of propositional interpretations. □

Truth Table of Conjunction

We now analyze the truth table of conjunction (Figure 3a). First, we show that, for some combination of truth values τ, τ', only one truth value σ is consistent with $\tau \wedge \tau'$.

Lemma 1. *The following claims hold.*

- *(*$\mathbf{f} \wedge \tau = \mathbf{f}$*) For every* $\tau \in T$*, the only truth value that is consistent* $\wedge$ *on* $\mathbf{f}$ *and* τ *is* $\mathbf{f}$*.*
- *(*$\mathbf{t} \wedge \mathbf{t} = \mathbf{t}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{t}$ *and* $\mathbf{t}$ *is* $\mathbf{f}$*.*
- *(*$\mathbf{t} \wedge \mathbf{s} = \mathbf{s}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{t}$ *and* $\mathbf{s}$ *is* $\mathbf{s}$*.*
- *(*$\mathbf{t} \wedge \mathbf{st} = \mathbf{st}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{t}$ *and* $\mathbf{st}$*is* $\mathbf{st}$*.*
- *(*$\mathbf{t} \wedge \mathbf{sf} = \mathbf{sf}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{t}$ *and* $\mathbf{sf}$ *is* $\mathbf{sf}$*.*
- *(*$\mathbf{t} \wedge \mathbf{u} = \mathbf{u}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{t}$ *and* $\mathbf{u}$ *is* $\mathbf{u}$*.*
- *(*$\mathbf{s} \wedge \mathbf{u} = \mathbf{sf}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{s}$ *and* $\mathbf{u}$ *is* $\mathbf{sf}$*.*
- *(*$\mathbf{st} \wedge \mathbf{sf} = \mathbf{sf}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{st}$ *and* $\mathbf{sf}$ *is* $\mathbf{sf}$*.*
- *(*$\mathbf{st} \wedge \mathbf{u} = \mathbf{u}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{st}$ *and* $\mathbf{u}$ *is* $\mathbf{u}$*.*
- *(*$\mathbf{sf} \wedge \mathbf{u} = \mathbf{sf}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{sf}$ *and* $\mathbf{u}$ *is* $\mathbf{sf}$*.*
- *(*$\mathbf{u} \wedge \mathbf{u} = \mathbf{u}$*) The only truth value that is consistent with* $\wedge$ *on* $\mathbf{u}$ *and* $\mathbf{u}$ *is* $\mathbf{u}$*.*

Proof. We prove each claim separately.

($\mathbf{f} \wedge \tau = \mathbf{f}$). We prove that $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\tau} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{f}}$ is a tautology. From $\mathbf{K}\neg\alpha$ and $(P2)$ we can derive $\chi_\alpha^{\mathbf{f}} \rightarrow \mathbf{K}\neg(\alpha \wedge \beta)$. From $\neg\mathbf{P}\alpha$ and completeness of $\mathbf{P}$ we derive $\chi_\alpha^{\mathbf{f}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$. From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive $\chi_\alpha^{\mathbf{f}} \rightarrow \mathbf{P}\neg(\alpha\wedge\beta)$. Finally, using $\neg\mathbf{K}\alpha$ and the distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{f}} \rightarrow \neg\mathbf{K}(\alpha\wedge\beta)$.

($\mathbf{t} \wedge \mathbf{t} = \mathbf{t}$). We prove that $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{t}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{t}}$ is a tautology. From $\mathbf{K}\alpha \wedge \mathbf{K}\beta$, using distributivity of $\mathbf{K}$ over $\wedge$ we can derive $(a) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{t}} \rightarrow \mathbf{K}(\alpha \wedge \beta)$. Using $(WD3)$ and (a) we can now derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{t}} \rightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. From $\mathbf{K}\alpha \wedge \mathbf{P}\beta$, using $(WD1)$ we can derive $(b) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{t}} \rightarrow \mathbf{P}(\alpha \wedge \beta)$. Using $(WD3)$ and (b) we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{t}} \rightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{t} \wedge \mathbf{s} = \mathbf{s}$). We prove that $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{s}}$ is a tautology. From $\mathbf{K}\alpha \wedge \mathbf{P}\beta$ and $(WD1)$ we can derive $(a) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \mathbf{P}(\alpha \wedge \beta)$. From (a) and $(WD3)$ we derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$. From $\mathbf{P}\neg\beta$ and $(P2)$ $(b) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (b) and $(WD3)$ we derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$.

($\mathbf{t} \wedge \mathbf{st} = \mathbf{st}$). We prove that $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{st}}$ is a tautology. From $\mathbf{K}\alpha \wedge \mathbf{P}\beta$ and $(WD1)$ we can derive $(a) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \mathbf{P}(\alpha \wedge \beta)$. From (a) and $(WD3)$ we derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$. From $\neg\mathbf{P}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \neg\mathbf{P}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. Finally, from $\neg\mathbf{K}\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$.

($\mathbf{t} \wedge \mathbf{sf} = \mathbf{sf}$). We prove that $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{sf}}$ is a tautology. From $\mathbf{P}\neg\beta$ and $(P2)$ we can derive $(a) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. From $\neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$. Finally, from $\neg\mathbf{K}\neg\beta \wedge \neg\mathbf{P}\neg\alpha$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{t} \wedge \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{P}\alpha$ and completeness of $\mathbf{P}$ we can derive $(a) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\alpha\wedge\beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\alpha\wedge\beta)$. From $\neg\mathbf{P}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $(b) : \chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. From (b) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{t}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{s} \wedge \mathbf{u} = \mathbf{sf}$). We prove that $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{sf}}$ is a tautology. From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive $(a) : \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. Moreover, from $\neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$. Finally from $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{st} \wedge \mathbf{sf} = \mathbf{sf}$). We prove that $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{sf}}$ is a tautology. From $\mathbf{P}\neg\beta$ and $(P2)$ we can derive $(a) : \chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. From $\neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\neg\mathbf{P}(\alpha \wedge \beta)$. Finally from $\neg\mathbf{K}\neg\beta \wedge \neg\mathbf{P}\neg\alpha$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{st} \wedge \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$. From $\neg\mathbf{P}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. From $\neg\mathbf{K}\alpha \wedge \neg\mathbf{K}\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. Finally from $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{sf} \wedge \mathbf{u} = \mathbf{sf}$). We prove that $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{sf}}$ is a tautology. From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive (a) : $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. From $\neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg(\alpha \wedge \beta)$. Finally from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$.

($\mathbf{u} \wedge \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \chi_{\alpha\wedge\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{P}\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$. From distributivity of $\mathbf{K}$ over $\wedge$ and $\neg\mathbf{K}\alpha \wedge \neg\mathbf{K}\beta$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. From $\neg\mathbf{P}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{P}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. Finally, from $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \rightarrow \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$. □

For the remaining combinations of truth values , we have more than one possible consistent choice. In the following set of lemmas, we show that only one of these choices is the most preferable.

Lemma 2 **($\mathbf{s} \wedge \mathbf{s} = \mathbf{sf}$).** *The truth value* $\mathbf{sf}$ *is consistent with* $\wedge$ *on* $\mathbf{s}$ *and* $\mathbf{s}$*. Moreover,* $\mathbf{sf}$ *is preferable to* τ*, for every truth value* τ *consistent with* $\wedge$ *on* $\mathbf{s}$ *and* $\mathbf{s}$*.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\wedge$ on $\mathbf{s}$ and $\mathbf{s}$ are three: $\mathbf{sf}$, $\mathbf{s}$, and $\mathbf{f}$. Second, we prove that $\mathbf{sf}$ is preferable to both $\mathbf{s}$ and $\mathbf{f}$ w.r.t. $\mathbf{s} \wedge \mathbf{s}$.

We start by proving that the following formula is a tautology: $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta) \wedge \neg\mathbf{K}(\alpha \wedge \beta)$. From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive (a) : $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{s}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. From the above result, we can conclude that, if $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{s}} \wedge \chi_{\alpha\wedge\beta}^{\tau}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{s}, \mathbf{f}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{s}} \wedge \chi_{\alpha\wedge\beta}^{\tau}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Truth Value $\mathbf{sf}$*).* Assume that W can be partitioned into three non-empty subsets, namely W', W'', and W''', having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq f(\beta)$;
- $W'' \subseteq f(\alpha)$ and $W' \subseteq t(\beta)$;
- $W''' \in u_{\mathcal{I}}(\alpha) \cap u_{\mathcal{I}}(\beta)$.

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta$. Moreover, $t(\alpha \wedge \beta) = \varnothing$ due the definition of propositional interpretatinons, and $f(\alpha \wedge \beta) \neq W$, due to W'''. We can conclude that $\mathcal{I}$ satisfies both $\neg\mathbf{P}(\alpha \wedge \beta)$ and $\neg\mathbf{K}\neg(\alpha \wedge \beta)$.

(Truth Value **s***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$ and $W' \subseteq f(\beta)$;
- $W'' \subseteq t(\alpha)$ and $W'' \subseteq t(\beta)$.

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta$. Moreover, $W' \subseteq f(\alpha \wedge \beta)$ and $W'' \subseteq t(\alpha \wedge \beta)$. We can conclude that $\mathcal{I}$ satisfies both $\mathbf{P}(\alpha \wedge \beta)$ and $\neg\mathbf{K}\neg(\alpha \wedge \beta)$.

(Truth Value **f***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq f(\beta)$;
- $W'' \subseteq f(\alpha)$ and $W'' \subseteq t(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta$. Moreover, $t(\alpha \wedge \beta) = \varnothing$, due to the definition of propositional interpretations. We can conclude that $\mathcal{I}$ satisfies both $\neg\mathbf{P}(\alpha \wedge \beta)$ and $\mathbf{K}\neg(\alpha \wedge \beta)$.

(The Most Preferable Truth Value Is **sf***).* We now prove that **sf** is preferable to both **s** and **f** w.r.t. $\mathbf{s} \wedge \mathbf{s}$. Given propositional formulae α and β, we use $\mathcal{X}^{\mathbf{f}}$, $\mathcal{X}^{\mathbf{s}}$, and $\mathcal{X}^{\mathbf{sf}}$, to denote, respectively, the formulae $\chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta \wedge \chi^{\mathbf{f}}_{\alpha\wedge\beta}$, $\chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta \wedge \chi^{\mathbf{s}}_{\alpha\wedge\beta}$ and $\chi^{\mathbf{s}}_\alpha \wedge \chi^{\mathbf{s}}_\beta \wedge \chi^{\mathbf{sf}}_{\alpha\wedge\beta}$. Assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{s}}$ and $\mathcal{X}^{\mathbf{sf}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{s}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{sf}}$; and
- for each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{sf}}$ satisfiable. Let $g : W' \to W$ be a mapping such that g is the identity over W, and $g(w') \in f(\alpha \wedge \beta)$, for each $w' \in f(\alpha \wedge \beta)$. The mapping g proves $\mathcal{I} \preceq_{\alpha\wedge\beta} \mathcal{I}'$. We can conclude that **sf** is preferable to **s** w.r.t. $\mathbf{s} \wedge \mathbf{s}$.

Similarly, assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{f}}$ and $\mathcal{X}^{\mathbf{sf}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{f}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{sf}}$; and
- For each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{sf}}$ satisfiable. Moreover, the function $g : W' \to W$ defined above proves that $\mathcal{I}' \preceq_{\alpha\wedge\beta} \mathcal{I}$. We can conclude that **sf** is preferable to **f** w.r.t. $\mathbf{s} \wedge \mathbf{s}$. □

Lemma 3 **($\mathbf{st} \wedge \mathbf{st} = \mathbf{u}$).** *The truth value* **u** *is consistent with* $\wedge$ *on* **st** *and* **st**. *Moreover,* **u** *is preferable to* τ*, for every truth value* τ *consistent with* $\wedge$ *on* **s** *and* **s**.

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\wedge$ on **s** and **s** are two: **st** and **u**. Second, we prove that **u** is preferable to **st** w.r.t. $\mathbf{st} \wedge \mathbf{st}$.

We start by proving that the following formula is a tautology: $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}(\alpha \wedge \beta) \wedge \neg\mathbf{P}\neg(\alpha \wedge \beta) \wedge \neg\mathbf{K}\neg(\alpha \wedge \beta)$. From $\neg\mathbf{P}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{P}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \wedge \beta)$. From $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}(\neg\alpha \vee \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \wedge \beta)$. Finally, from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{K}\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}(\alpha \wedge \beta)$. From the above result, we can conclude that, if $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{s}\}$. We proceed to show that , for each such τ, there exist propositional formulae α and β such that $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Truth Value **st***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'' having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq t(\beta)$;
- $W'' \subseteq u_{\mathcal{I}}(\alpha)$ and $W'' \subseteq u_{\mathcal{I}}(\beta)$.

Clearly, $\mathcal{I} \models \chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta}$. Moreover, $W' \subseteq t(\alpha \wedge \beta)$, and we can conclude that $\mathcal{I}$ satisfies $\mathbf{P}(\alpha \wedge \beta)$.

(Truth Value **u***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq u_{\mathcal{I}}(\beta)$;
- $W'' \subseteq u_{\mathcal{I}}(\alpha)$ and $W'' \subseteq t(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta}$. Moreover, $t(\alpha \wedge \beta) = \varnothing$ due to the definition of propositional interpretations. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{P}(\alpha \wedge \beta)$.

(The Most Preferable Truth Value Is **u***).* We proceed to prove that **u** is preferable to **st** w.r.t. $\mathbf{st} \wedge \mathbf{st}$. Given propositional formulae α and β, we use $\mathcal{X}^{\mathbf{u}}$, and $\mathcal{X}^{\mathbf{st}}$, to denote, respectively, the formulae $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\mathbf{u}}_{\alpha\wedge\beta}$ and $\chi^{\mathbf{st}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\mathbf{st}}_{\alpha\wedge\beta}$. Assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{st}}$ and $\mathcal{X}^{\mathbf{u}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{st}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{u}}$; and

- for each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{u}}$ satisfiable. Let $g : W' \to W$ be the identity over W. The mapping g proves $\mathcal{I}' \preceq_{\alpha\wedge\beta} \mathcal{I}$. We can conclude that $\mathbf{u}$ is preferable to $\mathbf{st}$ w.r.t. $\mathbf{st} \wedge \mathbf{st}$. □

Lemma 4 **($\mathbf{s} \wedge \mathbf{st} = \mathbf{sf}$).** *The truth value* $\mathbf{sf}$ *is consistent with* $\wedge$ *on* $\mathbf{s}$ *and* $\mathbf{st}$*. Moreover,* $\mathbf{sf}$ *is preferable to* τ*, for every truth value* τ *consistent with* $\wedge$ *on* $\mathbf{s}$ *and* $\mathbf{st}$*.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\wedge$ on $\mathbf{s}$ and $\mathbf{st}$ are two: $\mathbf{sf}$ and $\mathbf{s}$. Second, we prove that $\mathbf{sf}$ is preferable to $\mathbf{s}$ w.r.t. $\mathbf{s} \wedge \mathbf{st}$.

We start by proving that the following formula is a tautology: $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}(\alpha \wedge \beta) \wedge \neg\mathbf{K}\neg(\alpha \wedge \beta) \wedge \mathbf{P}\neg(\alpha\wedge\beta)$ From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive $(a) : \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \mathbf{P}\neg(\alpha\wedge\beta)$. From (a) and $(WD3)$ we can derive $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}(\alpha \wedge \beta)$. Moreover, from $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{P}\neg\beta$ and $(WD2)$ we can derive $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}\neg(\alpha \wedge \beta)$. From the above result, we can conclude that, if $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{s}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Truth Value $\mathbf{sf}$*).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq u_{\mathcal{I}}(\beta)$;
- $W'' \subseteq f(\alpha)$ and $W'' \subseteq t(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta}$. Moreover, $W' \subseteq u_{\mathcal{I}}(\alpha \wedge \beta)$ and $W' \subseteq f(\alpha \wedge \beta)$. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{P}(\alpha \wedge \beta)$.

(Truth Value $\mathbf{s}$*).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$ and $W' \subseteq t(\beta)$;
- $W'' \subseteq f(\alpha)$ and $W'' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta}$. Moreover, $W' \subseteq t(\alpha \wedge \beta)$. We can conclude that $\mathcal{I}$ satisfies $\mathbf{P}(\alpha \wedge \beta)$.

(The Most Preferable Truth Value Is $\mathbf{sf}$*).* We now prove that $\mathbf{sf}$ is preferable to $\mathbf{s}$ w.r.t. $\mathbf{s} \wedge \mathbf{st}$. Given propositional formulae α and β, we use $\mathcal{X}^{\mathbf{s}}$, and $\mathcal{X}^{\mathbf{sf}}$, to denote, respectively, the formulae $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\mathbf{s}}_{\alpha\wedge\beta}$, and $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\mathbf{sf}}_{\alpha\wedge\beta}$. Assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{s}}$ and $\mathcal{X}^{\mathbf{sf}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{st}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{sf}}$; and

- for each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{sf}}$ satisfiable. Let $g : W' \to W$ be the mapping such that g identity over W and $g(w') \in f(\alpha \wedge \beta)$, for each $w' \in f'(\alpha \wedge \beta)$. The mapping g proves $\mathcal{I}' \preceq_{\alpha\wedge\beta} \mathcal{I}$. We can conclude that **sf** is preferable to **s** w.r.t. $\mathbf{s} \wedge \mathbf{st}$. □

Lemma 5 ($\mathbf{sf} \wedge \mathbf{sf} = \mathbf{sf}$)**.** *The truth value* **sf** *is consistent with* $\wedge$ *on* **sf** *and* **sf**. *Moreover,* **sf** *is preferable to* τ*, for every truth value* τ *consistent with* $\wedge$ *on* **sf** *and* **sf**.

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\wedge$ on **sf** and **sf** are two: **sf**, and **f**. Second, we prove that **sf** is preferable **f** w.r.t. $\mathbf{sf} \wedge \mathbf{sf}$.

We start by proving that the following formula is a tautology: $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \to \neg\mathbf{K}(\alpha\wedge\beta) \wedge \neg\mathbf{P}(\alpha\wedge\beta) \wedge \mathbf{P}\neg(\alpha\wedge\beta)$ From $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive $(a) : \chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \to \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \to \neg\mathbf{K}(\alpha \wedge \beta)$. Finally from $\neg\mathbf{P}\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \to \neg\mathbf{P}(\alpha \wedge \beta)$.

From the above result, we can conclude that, if $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{f}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \wedge \chi^{\tau}_{\alpha\wedge\beta}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Truth Value **sf***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' f(\alpha)$, and $W' \in f(\beta)$;

- $W' u_{\mathcal{I}}(\alpha)$, and $W' \in u_{\mathcal{I}}(\beta)$.

Clearly $\mathcal{I} \models \chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta}$. Moreover, $W \neq f(\alpha \wedge \beta)$, and we can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{K}\neg(\alpha \wedge \beta)$.

(Truth Value **f***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$;

- $W' \subseteq u_{\mathcal{I}}(\alpha)$, and $W' \subseteq f(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta}$. Moreover, $W = f(\alpha \wedge \beta)$, from the definition of propositional interpretations. We can conclude that $\mathcal{I}$ satisfies $\mathbf{K}\neg(\alpha \wedge \beta)$.

(The Most Preferable Truth Value Is **sf***).* A construction similar to the one used in the prove Lemma 2 proves that **sf** is preferable to **f** with respect to $\mathbf{sf} \wedge \mathbf{sf}$. □

Lemma 6 ($\mathbf{s} \wedge \mathbf{sf} = \mathbf{sf}$)**.** *The truth value* **sf** *is consistent with* $\wedge$ *on* **s** *and* **sf***. Moreover,* **sf** *is preferable to* τ*, for every truth value* τ *consistent with* $\wedge$ *on* **s** *and* **sf***.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\wedge$ on **s** and **sf** are two: **sf**, and **f**. Second, we prove that **sf** is preferable **f** w.r.t. $\mathbf{sf} \wedge \mathbf{sf}$.

We start by proving that the following is a tautology: $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta) \wedge \neg\mathbf{P}(\alpha \wedge \beta) \wedge \mathbf{P}\neg(\alpha \wedge \beta)$ Form $\mathbf{P}\neg\alpha$ and $(P2)$ we can derive (a) : $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \wedge \beta)$. Moreover, from $\neg\mathbf{P}\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{P}(\alpha \wedge \beta)$.

From the above result, we can conclude that, if $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha\wedge\beta}^{\tau}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{f}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha\wedge\beta}^{\tau}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value **sf***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$;
- $W'' \subseteq f(\alpha)$, and $W'' \subseteq f(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}}$. Moreover, $F(\alpha \wedge \beta) \neq \varnothing$ due to W'. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{K}\neg(\alpha \wedge \beta)$.

(Case of truth value **f***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq f(\beta)$;
- $W' \subseteq f(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}}$. Moreover, the formula $W = f(\alpha \wedge \beta)$, due to the definition of propositional interpretations. We can conclude that $\mathcal{I}$ satisfies $\mathbf{K}\neg(\alpha \wedge \beta)$.

(The Most Preferable Truth Value Is **sf***).* To prove that **sf** is the most preferable truth value in this case, we can use a construction similar to the one used in the proof of Lemma 2. □

Truth Table of Disjunction

We now analyze the truth table of disjunction (Figure 3b). First, we show that, for some combination of truth values τ, τ', only one truth value σ is consistent with $\tau \vee \tau'$.

Lemma 7. *The following claims hold.*

- *($\mathbf{t} \vee \tau = \mathbf{t}$) For every $\tau \in T$, the only truth value that is consistent with $\vee$ on $\mathbf{t}$ and τ is $\mathbf{t}$.*
- *($\mathbf{f} \vee \mathbf{f} = \mathbf{f}$) The only truth value that is consistent with $\vee$ on $\mathbf{f}$ and $\mathbf{f}$ is $\mathbf{f}$.*
- *($\mathbf{f} \vee \mathbf{s} = \mathbf{s}$) The only truth value that is consistent with $\vee$ on $\mathbf{f}$ and $\mathbf{s}$ is $\mathbf{s}$.*
- *($\mathbf{f} \vee \mathbf{st} = \mathbf{st}$) The only truth value that is consistent with $\vee$ on $\mathbf{f}$ and $\mathbf{st}$ is $\mathbf{st}$.*
- *($\mathbf{f} \vee \mathbf{sf} = \mathbf{sf}$) The only truth value that is consistent with $\vee$ on $\mathbf{f}$ and $\mathbf{sf}$ is $\mathbf{sf}$.*
- *($\mathbf{f} \vee \mathbf{u} = \mathbf{u}$) The only truth value that is consistent with $\vee$ on $\mathbf{f}$ and $\mathbf{u}$ is $\mathbf{u}$.*
- *($\mathbf{s} \vee \mathbf{u} = \mathbf{st}$) The only truth value that is consistent with $\vee$ on $\mathbf{s}$ and $\mathbf{u}$ is $\mathbf{st}$.*
- *($\mathbf{u} \vee \mathbf{u} = \mathbf{u}$) The only truth value that is consistent with $\vee$ on $\mathbf{u}$ and $\mathbf{u}$ is $\mathbf{u}$.*
- *($\mathbf{st} \vee \mathbf{sf} = \mathbf{st}$) The only truth value that is consistent with $\vee$ on $\mathbf{st}$ and $\mathbf{sf}$ is $\mathbf{st}$.*
- *($\mathbf{st} \vee \mathbf{u} = \mathbf{st}$) The only truth value that is consistent with $\vee$ on $\mathbf{st}$ and $\mathbf{u}$ is $\mathbf{st}$.*
- *($\mathbf{sf} \vee \mathbf{u} = \mathbf{u}$) The only truth value that is consistent with $\vee$ on $\mathbf{sf}$ and $\mathbf{u}$ is $\mathbf{u}$.*

Proof. We prove each claim separately.

($\mathbf{t} \vee \tau = \mathbf{t}$). We prove that $\chi^{\mathbf{t}}_{\alpha} \wedge \chi^{\tau}_{\beta} \to \chi^{\mathbf{t}}_{\alpha\vee\beta}$ is a tautology. From $\mathbf{K}\alpha$ and $(P1)$ we can derive (a) : $\chi^{\mathbf{t}}_{\alpha} \to \mathbf{K}(\alpha \vee \beta)$. From (a) and $(WD3)$ we derive $\chi^{\mathbf{t}}_{\alpha} \to \neg\mathbf{P}\neg(\alpha \vee \beta)$. From $\mathbf{P}\alpha$ and $(P1)$ we derive $(b) : \chi^{\mathbf{t}}_{\alpha} \to \mathbf{P}(\alpha \vee \beta)$. Finally, from (b) and the we can derive $(b) : \chi^{\mathbf{f}}_{\alpha} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$.

($\mathbf{f} \vee \mathbf{f} = \mathbf{f}$). We prove that $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{f}}_{\beta} \to \chi^{\mathbf{f}}_{\alpha\vee\beta}$ is a tautology. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{f}}_{\beta} \to \neg\mathbf{P}(\alpha \vee \beta)$. From $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{f}}_{\beta} \to \neg\mathbf{K}(\alpha \vee \beta)$. From $\mathbf{K}\neg\alpha \wedge \mathbf{K}\neg\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{f}}_{\beta} \to \mathbf{K}(\neg\alpha \wedge \neg\beta) \leftrightarrow \mathbf{K}\neg(\alpha \vee \beta)$. Finally, from $\mathbf{K}\neg\alpha \wedge \mathbf{P}\neg\beta$ and $(WD1)$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{f}}_{\beta} \to \mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \mathbf{P}\neg(\alpha \vee \beta)$.

($\mathbf{f} \vee \mathbf{s} = \mathbf{s}$). We prove that $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \to \chi^{\mathbf{s}}_{\alpha\vee\beta}$ is a tautology. From $\mathbf{K}\neg\alpha \wedge \mathbf{P}\neg\beta$ and $(WD1)$ we can derive $(a) : \chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \to \mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \mathbf{P}\neg(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \to \neg\mathbf{K}(\alpha \vee \beta)$. From $\mathbf{P}\beta$ and $(P1)$ we can derive $(b) : \chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \to \mathbf{P}(\alpha \vee \beta)$. Finally, from (b) and $(WD3)$ we can derive $\chi^{\mathbf{f}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$.

($\mathbf{f} \vee \mathbf{st} = \mathbf{st}$). We prove that $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{st}} \to \chi_{\alpha\vee\beta}^{\mathbf{st}}$ is a tautology. From $\mathbf{P}\beta$ and $(P1)$ we can derive (a) : $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{st}} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{st}} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\mathbf{K}\neg\alpha \wedge \mathbf{P}\neg\beta$ and $(WD1)$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{st}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. Finally from $\neg\mathbf{K}\beta \wedge \neg\mathbf{P}\alpha$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{st}} \to \neg\mathbf{K}(\alpha \vee \beta)$.

($\mathbf{f} \vee \mathbf{sf} = \mathbf{sf}$). We prove that $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{sf}} \to \chi_{\alpha\vee\beta}^{\mathbf{sf}}$ is a tautology. From $\mathbf{K}\neg\alpha \wedge \mathbf{P}\neg\beta$ and $(WD1)$ we can derive (a) : $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{sf}} \to \mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \mathbf{P}\neg(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}(\alpha \vee \beta)$. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{P}(\alpha \vee \beta)$. Finally from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{K}\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$.

($\mathbf{f} \vee \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{u}} \to \chi_{\alpha\vee\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{K}\neg\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\alpha \vee \beta)$. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\alpha \vee \beta)$. Finally from $\neg\mathbf{P}\neg\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{f}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$.

($\mathbf{s} \vee \mathbf{u} = \mathbf{st}$). We prove that $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \to \chi_{\alpha\vee\beta}^{\mathbf{st}}$ is a tautology. From $\mathbf{P}\alpha$ and $(P1)$ we can derive (a) : $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{P}\neg\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. Finally, from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\alpha \vee \beta)$.

($\mathbf{u} \vee \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \to \chi_{\alpha\vee\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. From $\neg\mathbf{K}\neg\alpha$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\alpha \vee \beta)$. Finally, from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{u}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\alpha \vee \beta)$.

($\mathbf{st} \vee \mathbf{sf} = \mathbf{st}$). We prove that $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \to \chi_{\alpha\vee\beta}^{\mathbf{st}}$ is a tautology. From $\mathbf{P}\alpha$ and $(P1)$ we can derive (a) : $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{P}\neg\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. Finally from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}(\alpha \vee \beta)$.

($\mathbf{st} \vee \mathbf{u} = \mathbf{st}$). We prove that $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \to \chi_{\alpha\vee\beta}^{\mathbf{st}}$ is a tautology. From $\mathbf{P}\alpha$ and $(P1)$ we can derive (a) : $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{P}\neg\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. Finally from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\alpha \vee \beta)$.

($\mathbf{sf} \vee \mathbf{u} = \mathbf{u}$). We prove that $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \to \chi_{\alpha\vee\beta}^{\mathbf{u}}$ is a tautology. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\vee$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\alpha \vee \beta)$. From $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}(\alpha \vee \beta)$. From $\neg\mathbf{P}\neg\beta$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. Finally, from $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{K}\neg\alpha$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{u}} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. □

For the remaining combinations of truth values, we have more than one possible compatible choice. In the following set of lemmas, we show that only one of these choices is the most preferable.

Lemma 8 ($\mathbf{s} \vee \mathbf{s} = \mathbf{st}$)**.** *The truth value* $\mathbf{st}$ *is consistent with* $\vee$ *on* $\mathbf{s}$ *and* $\mathbf{s}$*. Moreover,* $\mathbf{st}$ *is preferable to* τ*, for every truth value* τ *consistent with* $\vee$ *on* $\mathbf{s}$ *and* $\mathbf{s}$*.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\vee$ on $\mathbf{s}$ and $\mathbf{s}$ are three: $\mathbf{st}$, $\mathbf{s}$, and $\mathbf{t}$. Second, we prove that $\mathbf{st}$ is preferable to both $\mathbf{s}$ and $\mathbf{t}$ w.r.t. $\mathbf{s} \vee \mathbf{s}$.

We start by proving that the following formula is a tautology: $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \rightarrow \mathbf{P}(\alpha \vee \beta) \wedge \neg\mathbf{K}\neg(\alpha \vee \beta)$ is a tautology. From $\mathbf{P}\alpha$ and $(P1)$ we can derive $(a) : \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \rightarrow \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \rightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$. From the above result, we can conclude that, if $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\tau}_{\alpha\vee\beta}$ is satisfiable, then $\tau \in \{\mathbf{st}, \mathbf{s}, \mathbf{t}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\tau}_{\alpha\vee\beta}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value $\mathbf{st}$*).* Assume that W can be partitioned into three non-empty subsets, namely W', W'', and W''', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq f(\beta)$;
- $W'' \subseteq f(\alpha)$, and $W'' \subseteq t(\beta)$;
- $W''' \subseteq u_{\mathcal{I}}(\alpha)$, and $W''' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta}$. Moreover, the formula $u_{\mathcal{I}}(\alpha \vee \beta) \neq \varnothing$ due to W'''. Moreover, due to the definition of propositional interpretations, $f(\alpha \vee \beta) = \varnothing$. We can conclude that $\mathcal{I}$ satisfies both $\neg\mathbf{P}\neg(\alpha \vee \beta)$ and $\neg\mathbf{K}(\alpha \vee \beta)$.

(Case of truth value $\mathbf{s}$*).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq t(\beta)$;
- $W'' \subseteq f(\alpha)$, and $W'' \subseteq f(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta}$. Moreover, $t(\alpha \vee \beta) \neq \varnothing$ due to W', and $f(\alpha \vee \beta) \neq \varnothing$ and true in W''. We can conclude that $\mathcal{I}$ satisfies both $\mathbf{P}(\alpha \vee \beta)$ and $\neg\mathbf{K}(\alpha \vee \beta)$.

(Case of truth value $\mathbf{t}$*).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq f(\beta)$;

- $W'' \subseteq f(\alpha)$, and $W'' \subseteq t(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta}$. Moreover, due to the definition of propositional formulae,the formula $t(\alpha \vee \beta) = W$. We can conclude that $\mathcal{I}$ satisfies both $\neg\mathbf{P}\neg(\alpha \vee \beta)$ and $\mathbf{K}(\alpha \vee \beta)$.

(The Most Preferable Truth Value Is $\mathbf{st}$*).* We now prove that $\mathbf{st}$ is preferable to both $\mathbf{s}$ and $\mathbf{t}$ w.r.t. $\mathbf{s} \vee \mathbf{s}$. Given propositional formulae α and β, we use $\mathcal{X}^{\mathbf{t}}$, $\mathcal{X}^{\mathbf{s}}$, and $\mathcal{X}^{\mathbf{st}}$, to denote, respectively, the formulae $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\mathbf{t}}_{\alpha \vee \beta}$, $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\mathbf{s}}_{\alpha \vee \beta}$ and $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\mathbf{st}}_{\alpha \vee \beta}$. Assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{s}}$ and $\mathcal{X}^{\mathbf{st}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{s}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{st}}$; and

- for each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{st}}$ satisfiable. Let $g : W' \to W$ be a mapping such that g is the identity over W, and $g(w') \in t(\alpha \vee \beta)$, for each $w' \in t(\alpha \vee \beta)$. The mapping g proves $\mathcal{I} \preceq_{\alpha \vee \beta} \mathcal{I}'$. We can conclude that $\mathbf{st}$ is preferable to $\mathbf{s}$ w.r.t. $\mathbf{s} \wedge \mathbf{s}$.

Similarly, assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{t}}$ and $\mathcal{X}^{\mathbf{st}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{t}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{st}}$; and

- For each $w \in W$, $w \in$ W' and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists being $\mathcal{X}^{\mathbf{st}}$ satisfiable. Moreover, the function $g :$ $W' \to W$ defined above proves that $\mathcal{I}' \preceq_{\alpha \vee \beta} \mathcal{I}$. We can conclude that $\mathbf{st}$ is preferable to $\mathbf{t}$ w.r.t. $\mathbf{s} \vee \mathbf{s}$. □

Lemma 9 ($\mathbf{s} \vee \mathbf{st} = \mathbf{st}$)**.** *The truth value* $\mathbf{st}$ *is consistent with* $\vee$ *on* $\mathbf{s}$ *and* $\mathbf{st}$*. Moreover,* $\mathbf{st}$ *is preferable to* τ*, for every truth value* τ *consistent with* $\vee$ *on* $\mathbf{s}$ *and* $\mathbf{s}$*.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\vee$ on $\mathbf{s}$ and $\mathbf{st}$ are two: $\mathbf{st}$ and $\mathbf{t}$. Second, we prove that $\mathbf{st}$ is preferable to $\mathbf{t}$ w.r.t. $\mathbf{s} \vee \mathbf{st}$.

We start by proving that the following formula is a tautology: $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \mathbf{P}(\alpha \vee \beta) \wedge \neg\mathbf{K}\neg(\alpha \vee \beta) \wedge \neg\mathbf{P}\neg(\alpha \vee \beta)$. From $\mathbf{P}\alpha$ and $(P1)$ we can derive $(a) : \chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. Finally, from $\neg\mathbf{P}\neg\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \to \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. From the above result, we can conclude that, if $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{st}}_{\beta} \wedge \chi^{\tau}_{\alpha \vee \beta}$ is satisfiable, then $\tau \in \{\mathbf{st}, \mathbf{t}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi^{\mathbf{s}}_{\alpha} \wedge \chi^{\mathbf{s}}_{\beta} \wedge \chi^{\tau}_{\alpha \vee \beta}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value **st***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, $W' \subseteq t(\beta)$;
- $W'' \subseteq f(\alpha)$, $W'' \subseteq u_{\mathcal{I}}(\beta)$.

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{st}}$. Moreover, the formula $u_{\mathcal{I}}(\alpha \vee \beta) \neq \varnothing$ due to W''. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{K}(\alpha \vee \beta)$.

(Case of truth value **t***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$, $W' \subseteq t(\beta)$;
- $W'' \subseteq t(\alpha)$, $W'' \subseteq u_{\mathcal{I}}(\beta)$.

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{st}}$. Moreover, due to the definition of propositional interpretations, $t(\alpha \vee \beta) = W$. We can conclude that $\mathcal{I}$ satisfies $\mathbf{K}(\alpha \vee \beta)$.

(The Most Preferable Truth Value Is **st***).* To prove that **st** is the most preferable truth value in this case, we can use a construction similar to the one used in the proof of Lemma 8. □

Lemma 10 ($\mathbf{s} \vee \mathbf{sf} = \mathbf{st}$)**.** *The truth value* **st** *is consistent with* $\vee$ *on* **s** *and* **sf***. Moreover,* **st** *is preferable to* τ*, for every truth value* τ *consistent with* $\vee$ *on* **s** *and* **sf***.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\vee$ on **s** and **sf** are two: **st** and **s**. Second, we prove that **st** is preferable to **s** w.r.t. $\mathbf{s} \vee \mathbf{st}$.

We start by proving that the following formula is a tautology: $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}(\alpha \vee \beta) \wedge \mathbf{P}(\alpha \vee \beta) \wedge \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\mathbf{P}\alpha$ and $(P1)$ we can derive $(a) : \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \to \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}\neg(\alpha \vee \beta)$. Finally, from $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \to \neg\mathbf{K}(\alpha \vee \beta)$. From the above result, we can conclude that, if $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha \vee \beta}^{\tau}$ is satisfiable, then $\tau \in \{\mathbf{st}, \mathbf{s}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha \vee \beta}^{\tau}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value **s***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$, and $W' \subseteq f(\beta)$;
- $W'' \subseteq t(\alpha)$, and $W'' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}}$. Moreover, the formula $f(\alpha \vee \beta) \neq \varnothing$ due to W''. We can conclude that $\mathcal{I}$ satisfies $\mathbf{P}\neg(\alpha \vee \beta)$.

(Case of truth value **st***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq f(\beta)$;
- $W' \subseteq f(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$.

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{s}} \wedge \chi_\beta^{\mathbf{sf}}$. Moreover, $t(\alpha \vee \beta) \neq \varnothing$ due to W', and $u_{\mathcal{I}}(\alpha \vee \beta) \neq \varnothing$ due to W''. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{P}\neg(\alpha \vee \beta)$.

(The Most Preferable Truth Value Is **st***).* To prove that **st** is the most preferable truth value in this case, we can use a construction similar to the one used in the proof of ☐

Lemma 11 (**st** $\vee$ **st** $=$ **st**)**.** *The truth value* **st** *is consistent with* $\vee$ *on* **st** *and* **st**. *Moreover,* **st** *is preferable to* τ*, for every truth value* τ *consistent with* $\vee$ *on* **st** *and* **st**.

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\vee$ on **st** and **st** are two: **st** and **t**. Second, we prove that **st** is preferable to **t** w.r.t. **st** $\vee$ **st**.

We start by proving that the following formula is a tautology: $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \mathbf{P}(\alpha \vee \beta) \wedge \neg\mathbf{P}\neg(\alpha \vee \beta) \wedge \neg\mathbf{K}\neg(\alpha \vee \beta)$ is a tautology. From $\mathbf{P}\alpha$ and $(P1)$ we can derive $(a) : \chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \mathbf{P}(\alpha \vee \beta)$. From (a) and $(WD3)$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$. Finally, from $\neg\mathbf{P}\neg\alpha$ and completeness of $\mathbf{P}$ we can derive $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \rightarrow \neg\mathbf{P}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{P}\neg(\alpha \vee \beta)$. From the above result, we can conclude that, if $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \wedge \chi_{\alpha\vee\beta}^{\tau}$ is satisfiable, then $\tau \in \{\mathbf{st}, \mathbf{t}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}} \wedge \chi_{\alpha\vee\beta}^{\tau}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value **t***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$;
- $W'' \subseteq u_{\mathcal{I}}(\alpha)$, and $W'' \subseteq t(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}}$. Moreover, due to the definition of propositional interpretations, $t(\alpha \vee \beta) = W$. We can conclude that $\mathcal{I}$ satisfies $\mathbf{K}(\alpha \vee \beta)$.

(Case of truth value **st***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq t(\alpha)$, and $W' \subseteq t(\beta)$;
- $W'' \subseteq u_{\mathcal{I}}(\alpha)$, and $W'' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{st}} \wedge \chi_\beta^{\mathbf{st}}$. Moreover, $u_{\mathcal{I}}(\alpha \vee \beta) \neq \varnothing$ due to W''. We can conclude that $\mathcal{I}$ satisfies $\models \neg\mathbf{K}(\alpha \vee \beta)$.

(The Most Preferable Truth Value Is **st***).* To prove that **st** is the most preferable truth value in this case, we can use a construction similar to the one used in the proof of Lemma 8. □

Lemma 12 **(sf** $\vee$ **sf = u).** *The truth value* **u** *is consistent with* $\vee$ *on* **sf** *and* **sf***. Moreover,* **u** *is preferable to* τ*, for every truth value* τ *consistent with* $\vee$ *on* **sf** *and* **sf***.*

Proof. We prove the claim in two steps. First, we prove that the truth values consistent with $\vee$ on **sf** and **sf** are two: **sf** and **u**. Second, we prove that **u** is preferable to **sf** w.r.t. **sf** $\vee$ **sf**.

We start by proving that the following formula is a tautology: $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \vee \beta) \wedge \neg\mathbf{P}(\alpha \vee \beta) \wedge \neg\mathbf{K}\neg(\alpha \vee \beta)$. From $\neg\mathbf{K}\alpha \wedge \neg\mathbf{P}\beta$ and $(WD2)$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\alpha \vee \beta)$. From $\neg\mathbf{P}\alpha \wedge \neg\mathbf{P}\beta$ and distributivity of $\mathbf{P}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{P}(\alpha \vee \beta)$. From $\neg\mathbf{K}\neg\alpha \wedge \neg\mathbf{K}\neg\beta$ and distributivity of $\mathbf{K}$ over $\wedge$ we can derive $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \rightarrow \neg\mathbf{K}(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\mathbf{K}\neg(\alpha \vee \beta)$.

From the above result, we can conclude that, if $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha\vee\beta}^{\tau}$ is satisfiable, then $\tau \in \{\mathbf{sf}, \mathbf{u}\}$. We proceed to show that, for each such τ, there exist propositional formulae α and β such that $\chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}} \wedge \chi_{\alpha\vee\beta}^{\tau}$ is satisfiable. In what follows, we assume propositional formulae α, and β, and a propositional interpretation $\mathcal{I} = (t, f, W)$.

(Case of truth value **u***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$, and $W' \subseteq f(\beta)$;
- $W'' \subseteq u_{\mathcal{I}}(\alpha)$, and $W'' \subseteq u_{\mathcal{I}}(\beta)$;

Clearly, $\mathcal{I} \models \chi_\alpha^{\mathbf{sf}} \wedge \chi_\beta^{\mathbf{sf}}$. Moreover, the formula $f(\alpha \vee \beta) \neq \varnothing$ due to W'. We can conclude that $\mathcal{I}$ satisfies $\mathbf{P}\neg(\alpha \vee \beta)$.

(Case of truth value **sf***).* Assume that W can be partitioned into two non-empty subsets, namely W' and W'', having the following properties.

- $W' \subseteq f(\alpha)$, and $W' \subseteq u_{\mathcal{I}}(\beta)$;

- $W'' \subseteq u_{\mathcal{I}}(\alpha)$, and $W'' \subseteq f(\beta)$;

Clearly, $\mathcal{I} \models \chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta}$. Moreover, $u_{\mathcal{I}}(\alpha \vee \beta) = W$ due to the definition of propositional interpretations. We can conclude that $\mathcal{I}$ satisfies $\neg\mathbf{P}\neg(\alpha \vee \beta)$.

(The Most Preferable Truth Value Is $\mathbf{u}$*).* We now show that $\mathbf{u}$ is preferable to $\mathbf{sf}$ w.r.t. $\mathbf{sf} \vee \mathbf{sf}$.

Given propositional formulae α and β, we use $\mathcal{X}^{\mathbf{sf}}$ and $\mathcal{X}^{\mathbf{u}}$, to denote, respectively, the formulae $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \wedge \chi^{\mathbf{sf}}_{\alpha\vee\beta}$ and $\chi^{\mathbf{sf}}_{\alpha} \wedge \chi^{\mathbf{sf}}_{\beta} \wedge \chi^{\mathbf{u}}_{\alpha\vee\beta}$. Assume propositional formulae α and β such that $\mathcal{X}^{\mathbf{sf}}$ and $\mathcal{X}^{\mathbf{u}}$ are satisfiable. Let $\mathcal{I} = (t, f, W)$ be a propositional interpretation that satisfies $\mathcal{X}^{\mathbf{sf}}$, we define $\mathcal{I}' = (t', f', W')$ as follows.

- $\mathcal{I}' \models \mathcal{X}^{\mathbf{u}}$; and
- for each $w \in W$, $w \in W'$ and $w \in u_{\mathcal{I}'}(\gamma)$, for each $\gamma \in \mathcal{L}$.

Such $\mathcal{I}'$ exists, being $\mathcal{X}^{\mathbf{u}}$ satisfiable. Let $g : W' \rightarrow W$ be the identity over W. The mapping g proves $\mathcal{I} \preceq_{\alpha\vee\beta} \mathcal{I}'$. We can conclude that $\mathbf{u}$ is preferable to $\mathbf{sf}$ w.r.t. $\mathbf{s} \wedge \mathbf{s}$.

□

Truth Table of Negation

We finally analyze the truth table of negation (Figure 3c). As stated in Proposition 1, we only have one compatible truth value in each case.

Lemma 13. *The following claims hold.*

- *(*$\neg\mathbf{t} = \mathbf{f}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{t}$ *is* $\mathbf{f}$.
- *(*$\neg\mathbf{f} = \mathbf{t}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{f}$ *is* $\mathbf{t}$.
- *(*$\neg\mathbf{s} = \mathbf{s}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{s}$ *is* $\mathbf{s}$.
- *(*$\neg\mathbf{st} = \mathbf{sf}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{st}$ *is* $\mathbf{sf}$.
- *(*$\neg\mathbf{sf} = \mathbf{st}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{sf}$ *is* $\mathbf{st}$.
- *(*$\neg\mathbf{u} = \mathbf{u}$*) The only truth value that is consistent with* $\neg$ *on* $\mathbf{u}$ *is* $\mathbf{u}$.

Proof. To prove the claims, we observe the following. Assume a propositional formula α. The formula $\chi^{\tau}_{\neg\alpha}$ is equivalent to $\bigwedge_{\varphi\in\mathcal{M}(\alpha)} \neg\varphi$. The claim follows straightforwardly. □

Proof of Theorem 3

The proof is by inspection of all sublogics of $\mathbb{L}_{6\mathrm{v}}$. To this end, we devised a Python script that automatically enumerates these logics and checks whether they satisfy the desired properties.

Appendix B. Proofs of Results on Predicate Logic

Proof of Theorem 4

We prove that for every formula φ of FO($\mathbb{L}$) and every $\tau \in \mathbf{T}$, there exists a formula $tr(\varphi, \tau)$ such that for every structure $\mathfrak{A}$, and assignment ν of free variables of φ we have $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau \;\Leftrightarrow\; \mathfrak{A}, \nu \models tr(\varphi, \tau)$. This formula $tr(\varphi, \tau)$ is what we call earlier φ_τ; the new notation is used in the proof for readability, to ensure that multiple indexes do not clash.

The claim is proved by induction on the structure of formulae of $\varphi \in$ FO($\mathbb{L}$).

Atomic formulae. By the assumption of the theorem, if φ is an atomic formula then it is captured by BFO.

Logical connectives. Assume $\varphi = \omega(\varphi_1, \ldots, \varphi_n)$, where $\omega \in \Omega$ is an n-ary connective.

Let $T_{\tau,\omega}$ denote the set of all the n-tuples $\bar{\tau}$ of truth values such that $\omega^{\mathbb{L}}(\bar{\tau}) = \tau$. More formally, $T_{\tau,\omega} = \{(\tau_1, \ldots, \tau_n) \in \mathbf{T}^n \mid \omega^{\mathbb{L}}(\tau_1, \ldots, \tau_n) = \tau\}$.

By the induction hypothesis, for $i = 1 \ldots n$ there exist formulae $tr(\varphi_i, \tau_i) \in$ BFO such that $[\![\varphi_i]\!]_{\mathfrak{A},\nu} = \tau_i$ iff $\mathfrak{A}, \nu \models tr(\varphi_i, \tau_i)$. From these formulae, we can define $tr(\varphi, \tau) \in$ BFO as follows:

$$tr(\varphi, \tau) = \bigvee_{(\tau_1,\ldots,\tau_n)\in T_{\tau,\omega}} (tr(\varphi_1, \tau_1) \wedge \cdots \wedge tr(\varphi_n, \tau_n)) \tag{B.1}$$

Suppose $\mathfrak{A}, \nu \models tr(\varphi, \tau)$. If this is the case, at least one of the disjuncts of $tr(\varphi, \tau)$ is satisfied by $\mathfrak{A}, \nu$, which in turn proves (due to the inductive hypothesis) that for some $(\tau_1, \ldots, \tau_n) \in T_{\tau,\omega}$ we have $[\![\varphi_i]\!]_{\mathfrak{A},\nu} = \tau_i$ for every $i = 1 \ldots n$. Then, from the definition of $T_{\tau,\omega}$ it follows that $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau$.

Suppose now that $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau$. Then for some $(\tau_1, \ldots, \tau_n) \in T_{\tau,\omega}$ we have that $[\![\varphi_i]\!]_{\mathfrak{A},\nu} = \tau_i$ for every $i = 1 \ldots n$. By the inductive hypothesis then, one of the disjuncts of $tr(\varphi, \tau)$ is satisfied by $\mathfrak{A}, \nu$, proving $\mathfrak{A}, \nu \models tr(\varphi, \tau)$.

Existential quantification. Assume that $\varphi(\bar{x}) = \exists y.\psi(\bar{x}, y)$. The semantics of existential quantifiers says that for every structure $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}}, \mathsf{Eq}^{\mathfrak{A}} \rangle$ and assignment ν for $\bar{x}$ the following equality holds: $[\![\exists y.\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu} = [\![\bigvee_{a\in A} \psi(\bar{x}, a)]\!]_{\mathfrak{A},\nu[a/y]}$.

To define $tr(\varphi, \tau) \in$ BFO that captures $\exists y.\psi(\bar{x}, y)$, we will make use of this equality. First, notice that the number of disjuncts in $\bigvee_{a\in A} \psi(\bar{x}, a)$ depends on the size of the domain of $\mathfrak{A}$. For this reason, we cannot straightforwardly apply the argument used to prove the case of general connectives. However, due to the assumptions of commutativity, associativity, and weak idempotence of the disjunction operation, in defining $tr(\varphi, \tau)$ we need to take into account only a limited number of combinations of truth values.

To define these combinations, we start by observing the following. Let $\bar{t}$ be a tuple of truth values such that $\bigvee^{\mathbb{L}} \bar{t} = \tau$, and let $|\bar{t}|_t$ denote the number of occurrences of truth value t in $\bar{t}$. Due to the assumptions on $\vee$ in $\mathbb{L}$, in order to determine the truth value of $\bigvee^{\mathbb{L}} \bar{t}$, we only need to know if τ occurs once, twice, or none at all in $\bar{t}$, for each truth value $\tau \in \mathbf{T}$. Indeed, $\vee$ is commutative and associtive in $\mathbb{L}$, and from weak idempotency it follows that $t \vee t \vee t \vee \ldots \vee t = t \vee t$ whenever the size of the disjunction is at least two. In other words, the truth value of $\bigvee^{\mathbb{L}} \bar{t}$ is fully determined by the function function $f : \mathbf{T} \to \{0,1,2\}$ where $f(t) = 0$ or $f(t) = 1$ indicate that t occurs zero or once in $\bar{t}$, respectively, and $f(t) = 2$ indicates that t occurs at least twice. If the following condition holds, the function f witnesses $\bigvee^{\mathbb{L}} \bar{t} = \tau$

$$\Big(\bigvee^{\mathbb{L}}_{\tau' \in f^{-1}(1)} \tau' \Big) \vee^{\mathbb{L}} \Big(\bigvee^{\mathbb{L}}_{\tau'' \in f^{-1}(2)} (\tau'' \vee \tau'') \Big) \;=\; \tau \tag{B.2}$$

Intuitively, the functions defined above represent multisets of truth values whose disjunction yields τ. Moreover, every tuple of truth values whose disjunction yields τ can be represented by one of these multisets. We now discuss how we can encode these multisets into BFO formulae.

Let $F^{\mathbf{t}}_{\vee}$ be the set of functions $f : \mathbf{T} \to \{0,1,2\}$ having the property (B.2) for truth value τ. We define $tr(\exists y.\psi(\bar{x}, y), \tau)$ as

$$tr(\exists y.\psi(\bar{x}, y), \tau) \;=\; \bigvee_{f \in F^{\tau}_{\vee}} \Big(\bigwedge_{t_0 \in f^{-1}(0)} \mathcal{Zero}(\psi, t_0) \wedge \bigwedge_{t_1 \in f^{-1}(1)} \mathcal{One}(\psi, t_1) \wedge \bigwedge_{t_2 \in f^{-1}(2)} \mathcal{Two}(\psi, t_2) \Big) \tag{B.3}$$

Where $\mathcal{Zero}$, $\mathcal{One}$, and $\mathcal{Two}$ are formulae defined as follows:

$$\mathcal{Zero}(\varphi, t) = \forall y \neg tr(\varphi(\bar{x}, y), t) \tag{B.4}$$

$$\mathcal{One}(\varphi, t) = \exists y. tr(\varphi(\bar{x}, y), t) \wedge \forall z. (y \neq z) \to \neg tr(\varphi(\bar{x}, z), t) \tag{B.5}$$

$$\mathcal{Two} = \exists y. \exists z. (y \neq z) \wedge tr(\varphi(\bar{x}, y), t) \wedge tr(\varphi(\bar{x}, z), t) \tag{B.6}$$

We proceed to prove that $\mathfrak{A}, \nu \models tr(\varphi, \tau)$ if and only if $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau$. Suppose $\mathfrak{A}, \nu \models tr(\varphi, \tau)$, then at least one of its disjuncts is satisfied by $\mathfrak{A}, \nu$. Let $f \in F^{\tau}_{\vee}$ be the function defining this disjunct. Applying the inductive hypothesis to $tr(\psi, t)$, we can see that

- There is no element $a \in A$ such that $[\![\psi(\bar{x}, a)]\!]_{\mathfrak{A},\nu} = t_0$, for every $t_0 \in f^{-1}(0)$;
- There is exactly one element $a \in A$ such that $[\![\psi(\bar{x}, a)]\!]_{\mathfrak{A},\nu} = t_1$, for every $t_1 \in f^{-1}(1)$;
- There are at least two elements $a, a' \in \mathcal{A}$ such that $[\![\psi(\bar{x}, a) \vee \psi(\bar{x}, a')]\!]_{\mathfrak{A},\nu} = t_2$, for every $t_2 \in f^{-1}(2)$.

Then from (B.2) it follows that $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau$.

Suppose now that $[\![\varphi]\!]_{\mathfrak{A},\nu} = \tau$. For every $a \in A$, let t_a denote the truth value such that $[\![\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[a/y]} = t_a$. By the definition of existential quantification, we know that $\bigvee^{\mathbb{L}}_{a \in A} t_a = \tau$. As defined above, there exists a function $f \in F^{\tau}_{\vee}$ such that all of the following hold:

- f satisfies condition (B.2);

- $f(t_a) = 0$ if there exists no $b \in A$ such that $[\![\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[b/y]} = t_a$;
- $f(t_a) = 1$ if there exists exactly one $b \in A$ such that $[\![\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[b/y]} = t_a$;
- $f(t_a) = 2$ if there exist two distinct $b, c \in A$ such that $[\![\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[b/y]} = t_a$ and $[\![\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[c/y]} = t_a$.

The above together with the inductive hypothesis prove that one of the disjuncts of $tr(\varphi, \tau)$ is satisfied by $\mathfrak{A}, \nu$, and hence the claim follows.

Universal quantification. For universal quantification, we can use an argument similar to the one used to prove the case of existential quantification. As stated before in the paper, for every $\varphi = \forall y.\psi(\bar{x}, y)$, structure $\mathfrak{A}$, and assignment ν for $\bar{x}$ the following equality holds: $[\![\forall y.\psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu} = [\![\bigwedge_{a \in A} \psi(\bar{x}, y)]\!]_{\mathfrak{A},\nu[a/y]}$.

Using weak idempotency of $\wedge$ in $\mathbb{L}$, we can define $tr(\forall y.\psi(\bar{x}, y)$ in the same way as we did for existential quantification, using a set $F^{\tau}_{\wedge}$ of functions $f : \mathbf{T} \to \{0, 1, 2\}$ having the following property:

$$\Big(\bigwedge^{\mathbb{L}}_{\tau' \in f^{-1}(1)} \tau' \Big) \wedge^{\mathbb{L}} \Big(\bigwedge^{\mathbb{L}}_{\tau'' \in f^{-1}(2)} (\tau'' \wedge \tau'') \Big) \;=\; \tau\,. \tag{B.7}$$

Now using (B.7) in place of (B.2), we conclude the proof in exact same way as for existential quantification.

Proof of Proposition 2

We discuss each of the three cases separately. In what follows, we will say that a formula φ captures the truth value τ of a formula ψ under semantics s if for every structure $\mathfrak{A}$ and assignment ν for the free variables of ψ we have that $\mathfrak{A}, \nu \models \varphi$ if and only if $[\![\psi]\!]^s_{\mathfrak{A},\nu} = \tau$. By Theorem 4, BFO captures atomic formulae under semantics s if and only if for every atomic formula α and truth value τ there exists a formula $tr^s(\alpha, \tau)$ that captures the truth value τ of α. We will use this observation to prove the claim of Proposition 2.

Boolean semantics. Trivially, for every atomic formula $R(\bar{x})$, formula $R(\bar{x})$ captures truth value $\mathbf{t}$ of $R(\bar{x})$ under boolean semantics, and $\neg R(\bar{x})$ captures truth value $\mathbf{f}$ of $R(\bar{x})$ under boolean semantics.

Null-free semantics. For a given tuple of variables $x_1, \ldots, x_n$, we define the formula $\mathcal{N}(\bar{x})$ as follows: $\mathcal{N}(\bar{x}) = \mathsf{const}(x_1) \wedge \cdots \wedge \mathsf{const}(x_n)$.

Case $[\![R(\bar{x})]\!]^{\mathrm{nf}}_{\mathfrak{A},\nu} = \mathbf{t}$. Define $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{t}) = R(\bar{x}) \wedge \mathcal{N}(\bar{x})$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{t})$ if and only if $\nu(\bar{x})$ contains no nulls and $R(\nu(\bar{x})) \in R^{\mathfrak{A}}$. In turn, this proves that $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{t})$ captures truth value $\mathbf{t}$ of $R(\bar{x})$ under null-free semantics, for every atomic formula $R(\bar{x})$.

Case $[\![R(\bar{x})]\!]^{\mathrm{nf}}_{\mathfrak{A},\nu} = \mathbf{f}$. Define $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{f}) = \neg R(\bar{x}) \wedge \mathcal{N}(\bar{x})$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{t})$ if and only if $\nu(\bar{x})$ contains no nulls and $R(\nu(\bar{x})) \notin R^{\mathfrak{A}}$. This in turn proves that $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{f})$ captures truth value $\mathbf{f}$ of $R(\bar{x})$ under null-free semantics, for every atomic formula $R(\bar{x})$.

Case $[\![R(\bar{x})]\!]^{\mathrm{nf}}_{\mathfrak{A},\nu} = \mathbf{u}$. Define $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{u}) = \neg\mathcal{N}(\bar{x})$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{t})$ if and only if $\nu(\bar{x})$ contains at least one null. This in turn proves that $tr^{\mathrm{nf}}(R(\bar{x}), \mathbf{u})$ captures truth value $\mathbf{u}$ of $R(\bar{x})$ under null-free semantics, for every atomic formula $R(\bar{x})$.

Unification semantics. To show that BFO captures an atomic formula $R(x_1, \dots, x_n)$ under unification semantics we will make use of a formula that encodes the notion of unification. The intuition behind this formula is the following.

Let $\bar{x}$ and $\bar{y}$ be two n-tuples of variables, not necessarily distinct. By x_i and y_i, we will denote the variable in position i of $\bar{x}$ and $\bar{y}$ respectively, and by X we will denote the set of variables in $\bar{x}$ and $\bar{y}$ (i.e., $X = \{x_1, \dots, x_n, y_1, \dots, y_n\}$). Assume now a structure $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, \dots, R_n^{\mathfrak{A}}, \mathsf{Eq}^{\mathfrak{A}} \rangle$, and suppose that for some substitution $\nu : X \to A$ we have that $\nu(\bar{x})$ unifies with $\nu(\bar{y})$.

Let h denote a mapping from the elements of $\nu(\bar{x})$ and $\nu(\bar{y})$ to the set A, assume that the image of h is the set $Im(h) = \{a_1, \dots, a_m\}$, and let B_j denote the set of all those elements of A that are mapped into a_j by h, i.e., $B_j = h^{-1}(a_j)$. In order to be a unifier for $\nu(\bar{x})$ and $\nu(\bar{y})$, h needs to enjoy the following properties: h is the identity on the constants of $\nu(\bar{x})$ and $\nu(\bar{y})$, and $h(\nu(\bar{x}) = h(\nu(\bar{y}))$. In other words, for $j = 1, \dots, m$ each set B_j must contain at most one constant, and for each $i = 1, \dots, n$ variables x_i and y_i must belong to the same set B. We now show how the existence of such mapping can be tested by a BFO formula.

Let Π denote the set of all the partitions of $X = \{x_1, \dots, x_n, y_1, \dots, y_n\}$. In light of what we said above, a unifier for $\nu(\bar{x})$ and $\nu(\bar{y})$ exists if and only if there exists a partition $\pi \in \Pi$ with the following properties.

1) For each $B \in \pi$, and each $u, v \in B$, $\nu(u) = \nu(v)$,

2) for each $B, B' \in \pi$ with $B \neq B'$, and each $u \in B$ and $v \in B'$, $\nu(u) \neq \nu(v)$,

3) there exists a set $B \in \pi$ such that $x_i, y_i \in B$ for each $i = 1, \dots, n$,

4) for each $B \in \pi$, and $u, v \in B$, if $\nu(u)$ and $\nu(v)$ are constant then they are the same.

From these considerations, a formula $\mathcal{U}(\bar{x}, \bar{y})$ such that $\mathfrak{A}, \nu \models \mathcal{U}(\bar{x}, \bar{y})$ if and only if $\nu(\bar{x})$ unifies with $\nu(\bar{y})$ can be defined as follows. First, for $\pi \in \Pi$ define $\alpha_\pi(\bar{x}, \bar{y})$ as follows.

$$\begin{aligned} \alpha_\pi(\bar{x}, \bar{y}) = & \bigwedge_{B \in \pi} \Big(\bigwedge_{u,v \in B} \big(u = v\big)\Big) \wedge \\ & \bigwedge_{B, B' \in \pi,\ B \neq B'} \Big(\bigwedge_{u \in B, v \in B'} \big(u \neq v\big)\Big) \wedge \\ & \bigwedge_{B \in \pi} \Big(\neg \bigvee_{u,v \in B} \big(\mathsf{const}(u) \wedge \mathsf{const}(v) \wedge u \neq v\big)\Big) \end{aligned} \tag{B.8}$$

We are now ready to define $\mathcal{U}(\bar{x}, \bar{y})$. Let P be the subset of Π such that for each $p \in P$ there exists $B \in p$ such that $x_i, y_i \in B$ for each $i = 1, \dots, n$. The formula $\mathcal{U}(\bar{x}, \bar{y})$ is defined as follows.

$$\mathcal{U}(\bar{x}, \bar{y}) = \bigvee_{p \in P} \alpha_p(\bar{x}, \bar{y}) \tag{B.9}$$

For what stated above, for every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$ and $\bar{y}$ we have that $\mathfrak{A}, \nu \models \mathcal{U}(\bar{x}, \bar{y})$ if and only if $\nu(\bar{x})$ unifies with $\nu(\bar{y})$.

With formula $\mathcal{U}(\bar{x}, \bar{y})$ in place, we are now ready to show that BFO captures atomic formulae under unification semantics.

Case $\llbracket R(\bar{x}) \rrbracket^{\Uparrow}_{\mathfrak{A},\nu} = \mathbf{t}$. Define $tr^{\Uparrow}(R(\bar{x}), \mathbf{t}) = R(\bar{x})$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\Uparrow}(R(\bar{x}), \mathbf{t})$ if and only if $R(\nu(\bar{x})) \in R^{\mathfrak{A}}$. This in turn proves that $tr^{\Uparrow}(R(\bar{x}), \mathbf{t})$ captures truth value $\mathbf{t}$ of $R(\bar{x})$ under unification semantics, for every atomic formula $R(\bar{x})$.

Case $\llbracket R(\bar{x}) \rrbracket^{\Uparrow}_{\mathfrak{A},\nu} = \mathbf{f}$. Define $tr^{\Uparrow}(R(\bar{x}), \mathbf{t}) = \forall \bar{y} R(\bar{y}) \rightarrow \neg \mathcal{U}(\bar{x}, \bar{y})$, where $\bar{y}$ is an n-tuple of different variables, not appearing in $\bar{x}$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\Uparrow}(R(\bar{x}), \mathbf{t})$ if and only if $\nu(\bar{x})$ does not unify with any $\bar{t} \in R^{\mathfrak{A}}$. This in turn proves that $tr^{\Uparrow}(R(\bar{x}), \mathbf{f})$ captures truth value $\mathbf{f}$ of $R(\bar{x})$ under unification semantics, for every atomic formula $R(\bar{x})$.

Case $\llbracket R(\bar{x}) \rrbracket^{\Uparrow}_{\mathfrak{A},\nu} = \mathbf{u}$. Define $tr^{\Uparrow}(R(\bar{x}), \mathbf{u}) = \exists \bar{y}. R(\bar{y}) \wedge \mathcal{U}(\bar{x}, \bar{y}) \wedge \neg(\bar{x} = \bar{y})$, where $\bar{y}$ is an n-tuple of different variables, not appearing in $\bar{x}$. For every structure $\mathfrak{A}$ and assignment ν for $\bar{x}$, $\mathfrak{A}, \nu \models tr^{\Uparrow}(R(\bar{x}), \mathbf{t})$ if and only if $\nu(\bar{x})$ unifies with at least one $\bar{t} \in R^{\mathfrak{A}}$. This in turn proves that $tr^{\Uparrow}(R(\bar{x}), \mathbf{u})$ captures truth value $\mathbf{u}$ of $R(\bar{x})$ under unification semantics, for every atomic formula $R(\bar{x})$.

Proof of Theorem 5 and Theorem 6

To ease the presentation, we assume that $\mathbb{L}^{\uparrow}_{3v}$ contains also the unary operator $\downarrow$, as a shorthand for $\neg \uparrow$. With $\downarrow$ in the language, we can assume that formulae in $\mathbb{L}^{\uparrow}_{3v}$ are given in negation normal form, i.e., with negation appearing only in front of the atoms. It is easy to see that, if $\varphi \in \mathbb{L}^{\uparrow}_{3v}$ is not in negation normal form, there exists a formula $\psi \in \mathbb{L}^{\uparrow}_{3v}$, equivalent to φ and in negation normal form, such that $|\psi|$ is bounded by a linear in the size of φ.

Given $\varphi \in \mathbb{L}^{\uparrow}_{3v}$ in negation normal form, we define $\varphi^{\mathbf{t}}$ inductively as follows.

- $(R(\bar{x}))^{\mathbf{t}} = R(\bar{x})$;
- $(\neg R(\bar{x}))^{\mathbf{t}} = \neg R(\bar{x})$;
- $(x_1 = x_2)^{\mathbf{t}} = (x_1 = x_2) \wedge \mathsf{const}(x_1) \wedge \mathsf{const}(x_2)$;
- $(\neg(x_1 = x_2))^{\mathbf{t}} = \neg(x_1 = x_2) \wedge \mathsf{const}(x_1) \wedge \mathsf{const}(x_2)$;
- $(\varphi \wedge \psi)^{\mathbf{t}} = \varphi^{\mathbf{t}} \wedge \psi^{\mathbf{t}}$;
- $(\varphi \vee \psi)^{\mathbf{t}} = \varphi^{\mathbf{t}} \vee \psi^{\mathbf{t}}$;
- $(\uparrow \varphi)^{\mathbf{t}} = (\varphi)^{\mathbf{t}}$;
- $(\downarrow \varphi)^{\mathbf{t}} = \neg(\varphi)^{\mathbf{t}}$;
- $(\exists x. \varphi(x))^{\mathbf{t}} = \exists x. (\varphi(x))^{\mathbf{t}}$;
- $(\forall x. \varphi(x))^{\mathbf{t}} = \forall x. (\varphi(x))^{\mathbf{t}}$.

Clearly, the size of $\varphi^{\mathbf{t}}$ grows linearly with respect to the size of φ. More precisely, there exists a constant $c \in \mathbb{N}$ such that $|\varphi| = c \cdot |\varphi^{\mathbf{t}}|$. We can prove this by induction on φ. For $\varphi = (x_1 = x_2)$ and $\varphi = \neg(x_1 = x_2)$, $|\varphi^{\mathbf{t}}| \leq 4|\varphi|$, otherwise $|\varphi^{\mathbf{t}}| = |\varphi|$. It follows that $|\varphi^{\mathbf{t}}| \leq 4 \cdot |\varphi|$.

To conclude, we prove by induction that $\varphi^{\mathbf{t}}$ captures the truth value $\mathbf{t}$ of φ, that is, for every $\varphi \in \mathrm{FO}(\mathbb{L}_{3\mathrm{v}}^{\uparrow})$, $[\![\varphi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ if and only if $\mathfrak{A}, \nu \models \varphi^{\mathbf{t}}$.

Base case. For $R(\bar{x})$ and $\neg R(\bar{x})$, the claim follows straightforwardly from the definition of $[\![]\!]^{\mathrm{sql}}$. For $(x_1 = x_2)$, observe that $[\![(x_1 = x_2)]\!]^{\mathrm{sql}} = \mathbf{t}$ if and only if x_1 is equal to x_2 and they are both constants. Similarly, observe that $[\![\neg(x_1 = x_2)]\!]^{\mathrm{sql}} = \mathbf{t}$ if and only if $[\![(x_1 = x_2)]\!]^{\mathrm{sql}} = \mathbf{f}$. In turn, this is the case if and only if x_1 is not equal to x_2 and they are both constants.

Case $\varphi = (\psi_1 \wedge \psi_2)$. Assume $[\![\psi_1 \wedge \psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. By definition, $[\![\psi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ and $[\![\psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. By inductive hypothesis then, $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}}$ and $\mathfrak{A}, \nu \models \psi_2^{\mathbf{t}}$, proving $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}} \wedge \psi_2^{\mathbf{t}}$. Assume now $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}} \wedge \psi_2^{\mathbf{t}}$, then, by definition, $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}}$ and $\mathfrak{A}, \nu \models \psi_2^{\mathbf{t}}$. Applying the inductive hypothesis, we obtain $[\![\psi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ and $[\![\psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. In turn, this implies $[\![\psi_1 \wedge \psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = t$.

$\varphi = (\psi_1 \vee \psi_2)$. Assume $[\![\psi_1 \vee \psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. By definition, either $[\![\psi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ or $[\![\psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. Suppose, w.l.o.g., that $[\![\psi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, then, by inductive hypothesis, $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}}$. In turn, this proves $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}} \vee \psi_2^{\mathbf{t}}$. Assume now $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}} \vee \psi_2^{\mathbf{t}}$, then, by definition, either $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}}$ or $\mathfrak{A}, \nu \models \psi_2^{\mathbf{t}}$. Suppose, w.l.o.g., that $\mathfrak{A}, \nu \models \psi_1^{\mathbf{t}}$. Then, by inductive hypothesis, $[\![\psi_1]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. In turn, this implies $[\![\psi_1 \vee \psi_2]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$.

$\varphi = (\uparrow \psi)^{\mathbf{t}}$. Assume $[\![\uparrow \psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, then, by definition, $[\![\psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. Applying the inductive hypothesis, we obtain $\mathfrak{A}, \nu \models \psi^{\mathbf{t}}$ and the claim follows. Assume now $\mathfrak{A}, \nu \models \psi^{\mathbf{t}}$. By inductive hypothesis, $[\![\psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. In turn, this implies $[\![\uparrow \psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$.

$\varphi = (\downarrow \psi)^{\mathbf{t}}$. Assume $[\![\downarrow \psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. By definition, $[\![\psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu}$ is either $\mathbf{f}$ or $\mathbf{u}$. By inductive hypothesis then, $\mathfrak{A}, \nu \not\models \psi^{\mathbf{t}}$. In turn, this implies that $\mathfrak{A}, \nu \models \neg\psi^{\mathbf{t}}$ and the claim follows. Assume now $\mathfrak{A}, \nu \models \neg\psi^{\mathbf{t}}$. In turn, this implies $\mathfrak{A}, \nu \not\models \psi^{\mathbf{t}}$. By inductive hypothesis then, $[\![\psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} \neq \mathbf{t}$ and then, by definition, $[\![\downarrow \psi]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, proving the claim.

$\varphi = (\exists x.\psi(x))^{\mathbf{t}}$. Assume $[\![\exists x.\psi(x)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ then, by definition, $[\![\bigvee_{c \in dom(\mathfrak{A})} \psi(c)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. In turn, this implies that $[\![\psi(a)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, for some $a \in dom(\mathfrak{A})$. By inductive hypothesis, the latter implies $\mathfrak{A}, \nu \models \psi(a)^{\mathbf{t}}$, proving $\mathfrak{A}, \nu \models \exists x.(\psi(x)^{\mathbf{t}})$. Assume now $\mathfrak{A}, \nu \models \exists x.(\psi(x)^{\mathbf{t}})$. By definition, there exists $a \in dom(\mathfrak{A})$ such that $\mathfrak{A}, \nu \models (\psi(a)^{\mathbf{t}})$. Applying the inductive hypothesis, we obtain $[\![\psi(a)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ which, in turn, proves $[\![\bigvee_{c \in dom(\mathfrak{A})} \psi(c)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ and the claim follows.

$\varphi = (\forall x.\psi(x))^{\mathbf{t}}$. Assume $[\![\forall x.\psi(x)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ then, by definition, $[\![\bigwedge_{c \in dom(\mathfrak{A})} \psi(c)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$. In turn, this implies that $[\![\psi(a)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, for every $a \in dom(\mathfrak{A})$. By inductive hypothesis then, $\mathfrak{A}, \nu \models \psi(a)^{\mathbf{t}}$ for every $a \in dom(\mathfrak{A})$ which, in turn, proves $\mathfrak{A}, \nu \models \forall x.(\psi(x)^{\mathbf{t}})$. Assume now $\mathfrak{A}, \nu \models \forall x.(\psi(x)^{\mathbf{t}})$. By definition, for every $a \in dom(\mathfrak{A})$ we have $\mathfrak{A}, \nu \models (\psi(a)^{\mathbf{t}})$. By inductive hypothesis then, $[\![\psi(a)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$, for every $a \in dom(\mathfrak{A})$. In turn, this proves $[\![\bigwedge_{c \in dom(\mathfrak{A})} \psi(c)]\!]^{\mathrm{sql}}_{\mathfrak{A},\nu} = \mathbf{t}$ and the claim follows.