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Abstract. Let $\mathscr{T}$ be the regularity structure associated with a given system of singular stochastic PDEs. The paracontrolled representation of the $\mathsf{\Pi}$ map provides a linear parametrization of the nonlinear space of admissible models $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ on $\mathscr{T}$, in terms of the family of para-remainders used in the representation. We give an explicit description of the action of the most general class of renormalization schemes presently available on the parametrization space of the space of admissible models. The action is particularly simple for renormalization schemes associated with degree preserving preparation maps; the BHZ renormalization scheme has that property.

1 – Introduction

The systematic approach to the renormalization problem for singular stochastic partial differential equations (PDEs) was built gradually from Hairer's ad hoc construction in his groundbreaking work [17] to Bruned, Hairer and Zambotti's general setting for the BPHZ-type robust renormalization procedure [11] implemented by Chandra & Hairer in [14]. The dual action of this renormalization procedure on the equation was unveiled in Bruned, Chandra, Chevyrev and Hairer's work [10]. The specific BHZ renormalization scheme was included in [8] by Bruned in a larger class of renormalization schemes, and the dual action of schemes of this class on the equation was investigated in Bailleul & Bruned's work [3] using algebraic insights from Bruned & Manchon's work [12].

On a technical level, the setting of regularity structures disentangles the task of solving an equation from the problem of making sense of a number of ill-defined quantities that are characteristic from the singular nature of the equation. The latter are encapsulated in the notion of model over a regularity structure. It provides a finite family of reference distributions/functions which are used to give local descriptions of possible solutions to a given singular stochastic PDE around each point in its state space. The definition of a model $(\mathsf{g}, \mathsf{\Pi})$ on a given regularity structure $\mathscr{T}$ involves nonlinear operations that turn the metric space of models into a nonlinear space. Bailleul & Hoshino were able in [4, 5] to provide a parametrization of the space of models over a given regularity structure by a linear space, a product of Hölder spaces. This parametrization involves the tools of paracontrolled calculus. Having such a parametrization is useful for understanding the structure of the space of models and [4, 5] contains a number of applications. The present work tackles the question of understanding the action of a general renormalization scheme on the parametrization space of the models used for the study of systems of singular stochastic PDEs. Tapia & Zambotti had previously obtained a free transitive action of a product of Hölder spaces on the space of branched rough paths – a particular example of models over a particular regularity structure, indexed by a time interval. The action of a renormalization map on their parametrization space was investigated by Bruned in [9].

The regularity structures used for the study of singular stochastic PDEs have a particular structure described in depth in [11]. The models 'adapted' to this structure are called *admissible*. We refer the reader to Bailleul & Hoshino's *Tourist guide* [6] for a short self-contained reference on the algebraic and analytic sides of regularity structures theory and its applications to the study of singular stochastic PDEs. We need a piece of notation to describe the parametrization of the set of admissible models over a given regularity structure $\mathscr{T} = \big((\mathcal{T}, \Delta), (\mathcal{T}^+, \Delta^+)\big)$. Given $\tau \in \mathcal{T}$ write

$$\Delta\tau = \sum_{\sigma\leq\tau} \sigma \otimes (\tau/\sigma) \in \mathcal{T}\otimes\mathcal{T}^+. \tag{1.1}$$

A choice of linear basis $\mathcal{B}$ of $\mathcal{T}$ fixes uniquely this decomposition by requiring that the elements $\sigma \in \mathcal{T}$ that appear in the sum belong to $\mathcal{B}$. This notation is only used in that sense in this work. In order to stick strictly to the statements proved in [4] we formulate things in the case where the state space of the dynamics is the isotropic space $\mathbb{R}^d$, this corresponds to elliptic equations;

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a similar result holds in the anisotropic setting used for the study of parabolic equations. The bilinear operator P below stands for a paraproduct operator; its definition or properties are not needed in the present work, so we refer the reader to the first section of [4] for more information. Let $\mathscr{T}$ be the BHZ regularity structure associated with a given (elliptic) singular stochastic PDE, and $\mathcal{B}$ a basis of $\mathcal{T}$. The following statement is a particular case of Theorem 2 in [4]. We denote here and after by $\mathcal{C}^\alpha(\mathbb{R}^d)$ the usual isotropic Hölder spaces of regularity exponent $\alpha \in \mathbb{R}$.

Theorem 1 – *Given any family of distributions* $\big([\tau] \in \mathcal{C}^{\mathsf{deg}(\tau)}(\mathbb{R}^d)\big)_{\tau\in\mathcal{B},\mathsf{deg}(\tau)\leq 0}$*, there exists a unique admissible model* $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ *on* $\mathscr{T}$ *such that one has*

$$\mathsf{\Pi}\tau = \sum_{\sigma\leq\tau} \mathsf{P}_{\mathsf{g}(\tau/\sigma)}[\sigma],$$

for all $\tau \in \mathcal{B}$ *with* $\mathsf{deg}(\tau) \leq 0$.

Note the specific form of the above representation of $\mathsf{\Pi}\tau$; a different paracontrolled representation of $\mathsf{\Pi}$ involving other functions than the $\mathsf{g}(\tau/\sigma)$ has for instance no a priori reason to give rise to a parametrization of the model. It is convenient to talk of a *bracket map* $[\,\cdot\,]$ associated with the model $\mathsf{\Pi}$. The precise statement of our main result involves notations that will be introduced below. We state it here in a qualitative form and refer the reader to Theorem 9 and Theorem 12 for the full statements. Degree preserving preparation maps are defined in Definition 6 in Section 3.

Theorem 2 – *Assume that an admissible model on* $\mathscr{T}$ *is given and parametrized by the* $[\tau]$*, for* $\tau \in \mathcal{B}$ *with* $\mathsf{deg}(\tau) \leq 0$*. Let* $R : \mathcal{T} \mapsto \mathcal{T}$*, be a preparation map with associated renormalization map* $M_R : \mathcal{T} \mapsto \mathcal{T}$*, and renormalized model* $\big(\mathsf{g}^R, \mathsf{\Pi}^R\big)$.

- *If* R *is degree preserving then the map* $\mathsf{\Pi}^R$*, hence the entire admissible model, is parametrized by the* $[M_R\tau]$*, for* $\tau \in \mathcal{B}$ *with* $\mathsf{deg}(\tau) \leq 0$.
- *In the general case of a non-degree preserving preparation map* R *the bracket map* $[\,\cdot\,]^R$ *giving the parametrization of the map* $\mathsf{\Pi}^R$ *is given explicitly in terms of the bracket map* $[\,\cdot\,]$.

Emphasize here that the class of degree preserving preparation maps is much larger than the class of BHZ renormalization maps though. A number of useful results about preparation maps and their associated renormalization maps are given in Section 2. The particular case of degree preserving preparation maps is considered in Section 3 and the general case in Section 4.

We refer the reader to Hairer's review articles [18, 19], Friz & Hairer's book [15] or Bailleul & Hoshino's *Tourist guide* [6] for introductions to regularity structures and singular stochastic PDEs.

2 – Basics on preparation maps

Let $\mathscr{T} = \big((\mathcal{T}, \Delta), (\mathcal{T}^+, \Delta^+), (\mathcal{T}^-, \Delta^-)\big)$ be the BHZ regularity structure associated with a given (system of elliptic) singular stochastic PDE(s). Denote by $\delta : \mathcal{T} \to \mathcal{T}^- \otimes \mathcal{T}$ the splitting map which is part of the renormalization structure on $\mathscr{T}$. These regularity structures have special features detailed in [11, 6]. We denote by S^+ the antipode of the Hopf algebra $(\mathcal{T}^+, \Delta^+)$, and by $\mathcal{I}_a : \mathcal{T} \to \mathcal{T}$ and $\mathcal{I}_a^+ : \mathcal{T} \to \mathcal{T}^+$, for a in a finite set, the derivatives of the abstract integration operators used in the definition of $\mathscr{T}$. To lighten notations we work with only two 'non-differentiated' abstract integration operator $\mathcal{I}$ and $\mathcal{I}^+$ whose formal derivatives are given by the $\mathcal{I}_a, \mathcal{I}_a^+$; what follows works verbatim when working with several pairs of integration operators. We recall that the notion of admissibility of a model $(\mathsf{g}, \mathsf{\Pi})$ is relative to an operator K and that admissible models satisfy

$$\mathsf{\Pi}(\mathcal{I}_a\tau) = (D^aK) * (\mathsf{\Pi}\tau)$$

and

$$\mathsf{g}_x^{-1}\big(\mathcal{I}_a^+\tau\big) = -\big(D^aK * \mathsf{\Pi}_x\tau\big)(x) \tag{2.1}$$

for all x in the state space. The notation $*$ stands here for the convolution operator on the state space. The possibly multi-dimensional noise symbol in the regularity structure will be denoted by $\Xi \in \mathcal{T}$. We fix throughout a basis $\mathcal{B}$ of $\mathcal{T}$; this fixes in particular the notation in formula (1.1) describing Δ. We denote by $\mathsf{deg}:\mathcal{T}\to\mathbb{R}$, the degree map associated to the regularity structure $\mathscr{T}$. It provides the direct sum decomposition $\mathcal{T}=\bigoplus_{\beta\in A}\mathcal{T}_\beta$ of $\mathcal{T}$ into sums of vector spaces whose elements have a given degree; we denote by $\mathbf{1}$ the unique element of $\mathcal{B}$ of degree 0. Given a tree of the form $\sigma=\mathcal{I}_a(\tau)$, one has $\mathsf{deg}(\sigma)=\mathsf{deg}(\tau)+\beta-|a|$ where β corresponds to the Schauder estimate (gains in regularity) provided by the kernel associated to $\mathcal{I}$. Denote by $|\cdot|_\Xi:\mathcal{T}\to\mathbb{N}$ the map that counts the number of noises in any given tree. As a last piece of notation, we denote by $\mathcal{M}^+:\mathcal{T}^+\otimes\mathcal{T}^+\to\mathcal{T}^+$ the multiplication operator in $\mathcal{T}^+$. We recall from Bruned's work [8] the following definition.

Definition – *A* **preparation map** *is a map*

$$R:\mathcal{T}\to\mathcal{T}$$

that fixes polynomials and such that

- *for each $\tau\in\mathcal{T}$ there exist finitely many $\tau_i\in\mathcal{T}$ and constants λ_i such that*

$$R\tau=\tau+\sum_i\lambda_i\tau_i,\quad \textit{with}\quad \mathsf{deg}(\tau_i)\geq\mathsf{deg}(\tau)\quad\textit{and}\quad |\tau_i|_\Xi<|\tau|_\Xi, \tag{2.2}$$

- *one has*

$$(R\otimes\mathrm{Id})\Delta=\Delta R. \tag{2.3}$$

Example – *The archetype of a preparation map is defined from a map δ_r, with the index 'r' for 'root', defined similarly as the splitting map δ, but extracting at a time from any $\tau\in\mathcal{T}$ only one diverging subtree of τ with the same root as τ, and summing over all possible such subtrees – see Definition 4.2 in* [8]. *Given a character ℓ of the algebra $\mathcal{T}^-$, the map*

$$R_\ell:=(\ell\otimes\mathrm{Id})\delta_r$$

is a preparation map. Its associated renormalization map M_{R_ℓ}, defined below in (2.4)*, is of the type introduced in* [11]. □

We will work exclusively with preparation maps $R:\mathcal{T}\to\mathcal{T}$ such that

$$R\,\mathcal{I}_a=\mathcal{I}_a,$$

for all a. Let $M_R^\times:\mathcal{T}\to\mathcal{T}$, and $M_R:\mathcal{T}\to\mathcal{T}$, be the maps uniquely defined from R by requiring that $M_R^\times$ is *multiplicative* and satisfies

$$M_R^\times(\mathcal{I}_a\tau)=\mathcal{I}_a\big(M_R^\times(R\tau)\big)$$

and

$$M_R:=M_R^\times R. \tag{2.4}$$

The map M_R is the *renormalization map associated with the preparation map R.* While this map is *not* multiplicative, it follows from (2.3) that M_R commutes with all the integration operators $\mathcal{I}_a$. While in the setting of [11] the structure of the renormalization scheme on $\mathcal{T}$ and its induced action on $\mathcal{T}^+$ are encoded in the splitting map $\delta:\mathcal{T}\to\mathcal{T}^-\otimes\mathcal{T}$ and a character of the algebra $\mathcal{T}^-$, the algebraic structure associated with the renormalization map M_R is entirely encoded in the latter. The following result is used to describe the renormalisaed model; it was first proved in Proposition 8.36 in [17]. We give an elementary proof to be self-contained.

Lemma 3 – *The map*

$$\big(\mathrm{Id}\otimes\mathcal{M}^+\big)\,(\Delta\otimes\mathrm{Id}):\mathcal{T}\otimes\mathcal{T}^+\to\mathcal{T}\otimes\mathcal{T}^+$$

is invertible.

Proof – Writing

$$\Delta\sigma=\sum_{\sigma_1\leq\sigma}\sigma_1\otimes\sigma/\sigma_1,$$

one has for $\sigma\in\mathcal{T}$ and $\tau\in\mathcal{T}^+$

$$\big(\mathrm{Id}\otimes\mathcal{M}^+\big)\,(\Delta\otimes\mathrm{Id})(\sigma\otimes\tau)=\sum_{\sigma_1\leq\sigma}\sigma_1\otimes\big(\sigma/\sigma_1\tau\big)$$

and the only element in the previous sum whose $\mathcal{T}^+$-component has maximum degree is $\sigma\otimes\tau$. This shows the injectivity of the map from the statement. It surjectivity comes from the fact that

$$\big(\mathrm{Id}\otimes\mathcal{M}^+\big)\,(\Delta\otimes\mathrm{Id})(\sigma\otimes\tau) =: \sigma\otimes\tau + N(\sigma\otimes\tau),$$

for a nilpotent map N, so a Neumann series gives the inverse of $(\mathrm{Id}\otimes\mathcal{M}^+)\,(\Delta\otimes\mathrm{Id})$. A different representation

$$\big(\mathrm{Id}+N\big)^{-1} = (\mathrm{Id}\otimes\mathcal{M}^+)\big(\mathrm{Id}\otimes S^+\otimes\mathrm{Id}\big)\big(\Delta\otimes\mathrm{Id}\big) \tag{2.5}$$

was proved by Bruned in Lemma 3.20 of [8]. This explicit formula plays a role in the proof of Lemma 7, in Section 3. ▷

It follows from Lemma 3 that one defines inductively two maps

$$\delta_R : \mathcal{T}\to\mathcal{T}\otimes\mathcal{T}^+, \quad M_R^+ : \mathcal{T}^+\to\mathcal{T}^+,$$

setting

$$(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta\otimes\mathrm{Id})\delta_R := (M_R\otimes M_R^+)\Delta, \tag{2.6}$$

with $M_R^+ : \mathcal{T}^+\to\mathcal{T}^+$, the *multiplicative* map fixing the monomials and such that one has

$$M_R^+\Big(\mathcal{I}_a^+(\tau)\Big) = \mathcal{M}^+\big(\mathcal{I}_a^+\otimes\mathrm{Id}\big)\delta_R\tau,$$

for all $\tau\in\mathcal{T}$.

Recall now the following explicit inductive expression for the co-action Δ

$$\begin{aligned} \Delta(\bullet) &:= \bullet\otimes\mathbf{1}, \qquad \text{for } \bullet\in\big\{\mathbf{1},X_i,\Xi\big\}, \\ \Delta(\mathcal{I}_a\tau) &:= (\mathcal{I}_a\otimes\mathrm{Id})\Delta + \sum_{|\ell+m|<\mathsf{deg}(\mathcal{I}_a\tau)} \frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}\mathcal{I}^+_{a+\ell+m}(\tau). \end{aligned} \tag{2.7}$$

A similar formula for $\mathcal{I}_a^+$ holds involving only $\mathcal{I}^+$-type operators. These expressions used in Hairer's original work [17] are different from the expressions used by Bruned, Hairer and Zambotti in [11]. One moves from [17] to [11] by performing a change of basis in $\mathcal{T}^+$ and taking $\sum_{\ell\in\mathbb{N}^d}\frac{(-X)^\ell}{\ell!}\mathcal{I}^+_{a+\ell}(\tau)$ in the role of $\mathcal{I}_a^+(\tau)$. The induction rule giving the action of Δ on the abstract integration operator is more useful here in the form of relation (2.10) than in the form given in [11], identity (3.6) in [6]. The following formulas for the co-product Δ^+ and the antipode S^+ hold in that setting

$$\Delta^+(\mathcal{I}_a^+\tau) := \sum_{\ell\in\mathbb{N}^d}\left(\mathcal{I}^+_{a+\ell}\otimes\frac{(-X)^\ell}{\ell!}\right)\Delta\tau + \mathbf{1}\otimes\mathcal{I}_a^+\tau, \tag{2.8}$$

$$S^+(\mathcal{I}_a^+\tau) = -\sum_{\ell\in\mathbb{N}^d}\mathcal{M}_+\left(\mathcal{I}^+_{a+\ell}\otimes\frac{X^\ell}{\ell!}S^+\right)\Delta\tau. \tag{2.9}$$

Recall from (2.6) the definition of the map δ_R; the next statement gives a useful different representation for it. This is the very place where we take advantage of the fact that we work with renormalization maps built from a preparation map, as opposed to working with a general renormalization map as those of Section 8.3 of Hairer' seminal work [17]. Define a *multiplicative* map

$$\delta_R^\times : \mathbb{R}[\mathcal{T}]\to\mathcal{T}\otimes\mathcal{T}^+$$

setting

$$\begin{aligned} \delta_R^\times(\bullet) &:= \bullet\otimes\mathbf{1}, \qquad \text{for } \bullet\in\big\{\mathbf{1},X_i,\Xi\big\}, \\ \delta_R^\times(\mathcal{I}_a\tau) &:= (\mathcal{I}_a\otimes\mathrm{Id})\delta_R^\times(R\tau) - \sum_{|\ell|\geq\mathsf{deg}(\mathcal{I}_a\tau)}\frac{X^\ell}{\ell!}\otimes\mathcal{M}^+\big(\mathcal{I}^+_{a+\ell}\otimes\mathrm{Id}\big)\delta_R^\times(R\tau). \end{aligned} \tag{2.10}$$

Lemma 4 – *One has $\delta_R = \delta_R^\times R$.*

Proof – We proceed by induction on $\mathsf{deg}(\tau)+|\tau|_\Xi$. Using identity (2.4) to write

$$(M_R\otimes M_R^+)\Delta = (M_R^\times\otimes M_R^+)\Delta R$$

and the fact that R is invertible we are down to checking that one has

$$(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta\otimes\mathrm{Id})\delta_R^\times = (M_R^\times\otimes M_R^+)\Delta.$$

It suffices by multiplicativity to consider a tree of the form $\mathcal{I}_a(\tau)$, for which one has on the one hand

$$(M_R^\times \otimes M_R^+)\Delta\mathcal{I}_a(\tau) \overset{(2.7)}{=} (\mathcal{I}_a\otimes \mathrm{Id})\big(M_R^+\otimes M_R^+\big)\Delta\tau + \sum_{|\ell+m|<\mathsf{deg}(\mathcal{I}_a\tau)} \frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}\, M_R^+\big(\mathcal{I}^+_{a+\ell+m}(\tau)\big).$$

On the other hand we have

$$\begin{aligned}(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta\otimes\mathrm{Id})\delta_R^\times\big(\mathcal{I}_a(\tau)\big) &\overset{(2.10)}{=} (\mathcal{I}_a\otimes\mathrm{Id})(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta\otimes\mathrm{Id})\delta_R\tau \\ &+ \sum_{\ell,m\in\mathbb{N}^d}\frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}\mathcal{M}^+\big(\mathcal{I}^+_{a+\ell+m}\otimes\mathrm{Id}\big)\delta_R\tau \\ &- \sum_{|\ell+m|\geq\mathsf{deg}(\mathcal{I}_a\tau)}\frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}\mathcal{M}^+\big(\mathcal{I}^+_{a+\ell+m}\otimes\mathrm{Id}\big)\delta_R\tau\end{aligned}$$

We conclude by applying the induction hypothesis on τ. A similar proof was performed in Proposition 3.19 of [8] using the explicit formula (2.5) for $\big(\mathrm{Id}+N\big)^{-1}$.

▷

Definition – *A map $A:\mathcal{T}\to\mathcal{T}\times\mathcal{T}^+$, with $A\tau=\sum\tau_1\otimes\tau_2$, is said to be* **upper triangular** *if* $\mathsf{deg}(\tau_1)\geq\mathsf{deg}(\tau)$, *for all τ_1 in the preceding decomposition of $A\tau$.*

Lemma 5 – *The map δ_R is upper triangular.*

Proof – It suffices from the property (2.2) of preparation maps to see that $\delta_R^\times$ is upper triangular. This point is obtained from (2.10) and the multiplicativity of $\delta_R^\times$ by an elementary induction on $\mathsf{deg}(\tau)+|\tau|_\Xi$. ▷

It follows from Lemma 5 and the definition of M_R^+ that $\mathsf{deg}\big(M_R^+\sigma\big)\geq\mathsf{deg}(\sigma)$, for all $\sigma\in\mathcal{T}^+$. The last ineguality means that for $M_R^+\sigma=\sum_i\lambda_i\sigma_i$, one has for every i, $\mathsf{deg}\big(\sigma_i\big)\geq\mathsf{deg}(\sigma)$.

3 – The case of degree preserving preparation maps

We prove the first part of Theorem 2 in this section. The algebraic properties enjoyed by the renormalization maps associated with the class of degree preserving preparation maps defined below allow indeed a direct construction of renormalized models close to what is done for the BHZ models from Bruned-Hairer-Zambotti's work [11]. The construction involved in the general is not as simple; it will be detailed in Section 4.

Recall from Theorem 1 that the formula

$$\Pi\tau=\sum_{\sigma\leq\tau}\mathsf{P}_{\mathsf{g}(\tau/\sigma)}[\sigma], \tag{3.1}$$

for $\tau\in\mathcal{T}$ with $\mathsf{deg}(\tau)\leq 0$, provides a parametrization of the set of models over a large class of regularity structures containing those used for the study of singular stochastic PDEs – called hereafter 'BHZ *regularity structures*', after the initials of the authors of [11]. The class of renormalization maps used in [11] are built from specific features of BHZ regularity structures and from a character on a Hopf algebra $(\mathcal{T}^-,\Delta^-)$ that is in co-interaction with $(\mathcal{T},\Delta)$. A single feature of the fine structures involved in the definition of the preparation map associated with the BHZ renormalization map is of importance here. It singles out a large class of preparation maps for which the action of their associated renormalization maps on the parametrization space takes the simple form given in Theorem 9 below. The BHZ renormalization maps form one family of this class.

Definition 6 – *A preparation map is said to be* **degree preserving** *if for each $\tau\in\mathcal{T}$ there exists finitely many $\tau_i\in\mathcal{T}$ and constants λ_i such that*

$$R\tau=\tau+\sum_i\lambda_i\tau_i,\quad \textit{with}\quad \mathsf{deg}(\tau_i)=\mathsf{deg}(\tau)\quad\textit{and}\quad|\tau_i|_\Xi<|\tau|_\Xi. \tag{3.2}$$

The introduction in [11] of decorated trees with extended decorations allows precisely to design a setting where the splitting map associated with the renormalization procedure enjoys a similar property. One works in this setting with two degree maps deg and $\mathsf{deg}\_$, with $\mathsf{deg}\_$ not taking into account the extended decorations and involved in the definition of the Hopf algebra $(\mathcal{T}^-, \Delta^-)$ that is part of the renormalization structure on $\mathscr{T}$.

Although elementary, it is of fundamental importance that the maps $M_R^\times$ and M_R associated to a degree preserving preparation map R are also degree preserving. This is what allows to prove the next statement by induction.

Lemma 7 – *For any degree preserving preparation map R one has*

$$\delta_R\tau = (M_R\tau)\otimes\mathbf{1}, \tag{3.3}$$

and the co-interaction identity

$$\Delta M_R = \big(M_R\otimes M_R^+\big)\Delta. \tag{3.4}$$

One further has that M_R^+ commutes with the antipode S^+.

Proof – Note that the co-interaction identity (3.4) is equivalent from (2.3) to the identity

$$\Delta M_R^\times = \big(M_R^\times\otimes M_R^+\big)\Delta. \tag{3.5}$$

This identity involves only multiplicative maps on $\mathbb{R}[\mathcal{T}]$, so it suffices to prove it for elements of $\mathcal{T}$ of the form Ξ, X^k or $\mathcal{I}_a(\tau)$. It is elementary to check it for Ξ and X^k. We prove identities (3.3) and (3.5) for elements of $\mathcal{T}$ of the form $\mathcal{I}_a(\tau)$ by induction on $\mathsf{deg}(\tau)+|\tau|_\Xi$, for a generic $\tau\in\mathcal{T}$. We use the symbol $(\star)$ above an $=$ sign to emphasize the use of the induction assumption in a sequence of equalities. Write

$$M_R\tau = \tau + \sum_i c_i\sigma_i,$$

for constants c_i, with $|\sigma_i|_\Xi < |\tau|_\Xi$. As

$$\mathsf{deg}(\mathcal{I}_a\tau) > \mathsf{deg}(\tau),$$

for all $\mathcal{I}_a\tau\in\mathcal{T}$, one has

$$\mathsf{deg}(\mathcal{I}_a\tau) + |\tau|_\Xi > \mathsf{deg}(\sigma_i) + |\sigma_i|_\Xi, \qquad \forall\, i$$

from the fact that M_R is degree preserving. This justifies the use of the induction hypothesis in the $(\star)$ equality below.

$$\begin{aligned}\Delta M_R^\times(\mathcal{I}_a\tau) &= \Delta\mathcal{I}_a(M_R\tau) = (\mathcal{I}_a\otimes\mathrm{Id})\Delta(M_R\tau) + \sum_{|\ell+m|<\mathsf{deg}(\mathcal{I}_a\tau)}\frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}\mathcal{I}^+_{a+\ell+m}(M_R\tau)\\ &\overset{(\star)}{=} (\mathcal{I}_aM_R\otimes M_R^+)\Delta\tau + \sum_{|\ell+m|<\mathsf{deg}(\mathcal{I}_a\tau)}\frac{X^\ell}{\ell!}\otimes\frac{X^m}{m!}M_R^+\mathcal{I}^+_{a+\ell+m}(\tau)\\ &= \big(M_R^\times\otimes M_R^+\big)\Delta\mathcal{I}_a(\tau)\end{aligned}$$

The bound on $|\ell+m|$ in the first line comes from the degree preserving property of M_R: one has $\mathsf{deg}(\mathcal{I}_a\tau)=\mathsf{deg}(\mathcal{I}_a\sigma_i)$. We have used the induction assumption about (3.5) for the first term in the right hand side of the third equality and the induction assumption about (3.3) for the second term in the right hand side of that equality coupled with the fact that

$$\mathcal{I}_b^+(M_R\tau) = \mathcal{M}^+\big(\mathcal{I}_b^+\otimes\mathrm{Id}\big)\delta_R\tau = M_R^+\big(\mathcal{I}_b^+\tau\big), \qquad \forall\,\tau\in\mathcal{T}.$$

Identity (2.6) defining δ_R then reads

$$(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta\otimes\mathrm{Id})\delta_R\sigma = (M_R\otimes M_R^+)\Delta\sigma = \Delta(M_R\sigma),$$

and it follows from the explicit formula (2.5) that

$$\begin{aligned}\delta_R\sigma &= (\mathrm{Id}\otimes\mathcal{M}^+)\big(\mathrm{Id}\otimes S^+\otimes\mathrm{Id}\big)\big(\Delta\otimes\mathrm{Id}\big)\Delta(M_R\sigma)\\ &= (\mathrm{Id}\otimes\mathcal{M}^+)\big(\mathrm{Id}\otimes S^+\otimes\mathrm{Id}\big)\big(\mathrm{Id}\otimes\Delta^+\big)\Delta(M_R\sigma)\\ &= (\mathrm{Id}\otimes\mathbf{1}^*\mathbf{1})\Delta(M_R\sigma) = (M_R\sigma)\otimes\mathbf{1}.\end{aligned}$$

where we have used the following property of the antipode S^+

$$\mathcal{M}^+\left(S^+\otimes \mathrm{Id}\right)\Delta^+ = \mathbf{1}^*\mathbf{1}$$

the map $\mathbf{1}^* : \mathcal{T}^+ \to \mathbb{R}$ is the co-unit, it is equal to 1 on $\mathbf{1}$ and zero otherwise. One sees that M_R^+ and S^+ commute using the inductive relation

$$S^+(\mathcal{I}_a\tau) = -\sum_{\ell\in\mathbb{N}^d}\mathcal{M}^+\Big(\mathcal{I}^+_{a+\ell}\otimes\frac{X^\ell}{\ell!}\,S^+\Big)\Delta$$

for the antipode S^+ and writing

$$\begin{aligned} M_R^+S^+(\mathcal{I}_a^+\tau) &= -\sum_{\ell\in\mathbb{N}^d}\mathcal{M}^+\left(M_R^+\mathcal{I}^+_{a+\ell}\otimes\frac{X^\ell}{\ell!}M_R^+S^+\right)\Delta\tau = -\sum_{\ell\in\mathbb{N}^d}\mathcal{M}^+\left(\mathcal{I}^+_{a+\ell}M_R\otimes\frac{X^\ell}{\ell!}S^+M_R^+\right)\Delta\tau \\ &= -\sum_{\ell\in\mathbb{N}^d}\mathcal{M}^+\left(\mathcal{I}^+_{a+\ell}\otimes\frac{X^\ell}{\ell!}S^+\right)\Delta M_R\tau = S^+\big(\mathcal{I}_a^+M_R\tau\big) = S^+M_R^+(\mathcal{I}_a^+\tau). \end{aligned}$$

▷

Similar computations are involved in Remark 4.2.6 and Proposition 4.2.8 of [7]. Note that it follows from (3.3) that the multiplicative map M_R^+ satisfies in that case the relation

$$M_R^+\big(\mathcal{I}_a^+(\tau)\big) = \mathcal{I}_a^+(M_R\tau).$$

One then proves similarly as in the proof of Lemma 7 that M_R^+ satisfies the co-interaction identity

$$\big(M_R^+\otimes M_R^+\big)\Delta^+ = \Delta^+M_R^+. \tag{3.6}$$

Given an admissible model (g,Π) on $\mathscr{T}$ set

$$\mathsf{g}^R := \mathsf{g}\circ M_R^+,\quad \Pi^R := \Pi\circ M_R.$$

It follows from the fact that M_R^+ commutes with the antipode S^+ that

$$(\mathsf{g}^R)^{-1} := \mathsf{g}^{-1}\circ M_R^+.$$

Corollary 8 – *The pair* $\big(\mathsf{g}^R,\Pi^R\big)$ *defines an admissible model on* $\mathscr{T}$*.*

Proof – On the one hand, identities (2.6) and (3.3) ensure that

$$\begin{aligned} \Pi_x^R\tau &= \big(\Pi^R\otimes(\mathsf{g}_x^R)^{-1}\big)\Delta\tau = \Big\{\Pi^R\otimes\big(\mathsf{g}_x^{-1}\circ M_R^+\big)\Big\}\Delta\tau = \big(\Pi\otimes\mathsf{g}_x^{-1}\big)(M_R\otimes M_R^+)\Delta\tau \\ &\overset{(2.6)}{=} \big(\Pi_x\otimes\mathsf{g}_x^{-1}\big)\,\delta_R\tau \overset{(3.3)}{=} \Pi_x(M_R\tau). \end{aligned}\tag{3.7}$$

It follows from this identity and the fact that M_R is degree preserving that Π_x^R satisfies the analytic estimates required from a model on $\mathscr{T}$. On the other hand, the co-interaction identity (3.6) gives

$$\begin{aligned} \mathsf{g}_{yx}^R &= \Big(\mathsf{g}_y^R\otimes(\mathsf{g}_x^R)^{-1}\Big)\Delta^+ = \big(\mathsf{g}_y\otimes\mathsf{g}_x^{-1}\big)\Big(M_R^+\otimes M_R^+\Big)\Delta^+ \\ &\overset{(3.6)}{=} \big(\mathsf{g}_y\otimes\mathsf{g}_x^{-1}\big)\Delta^+M_R^+ = \mathsf{g}_{yx}\circ M_R^+. \end{aligned}$$

It follows from this identity and the fact that M_R^+ is degree preserving that g_{yx}^R satisfies the analytic estimates required from a model on $\mathscr{T}$. ▷

Theorem 9 – *Assume* $\mathscr{T} = \big((\mathcal{T},\Delta),(\mathcal{T}^+,\Delta^+),(\mathcal{T}^-,\Delta^-)\big)$ *is the BHZ regularity structure associated with a system of singular stochastic PDEs. Let* $\mathcal{B}$ *stands for a linear basis of* $\mathcal{T}$*, and let* (g,Π) *be an admissible model on* $\mathscr{T}$*, with associated bracket map* $[\,\cdot\,]$ *in its paracontrolled representation* (3.1)*. For any degree preserving preparation map* R *the admissible model* $\big(\mathsf{g}^R,\Pi^R\big)$ *on* $\mathscr{T}$ *is parametrized by the family* $\big([M_R\tau]\in\mathcal{C}^{\mathsf{deg}(\tau)}(\mathbb{R}^d)\big)_{\tau\in\mathcal{B},\,\mathsf{deg}(\tau)\le 0}$*.*

Proof – The action of M_R on the parametrization set of the space of admissible models is thus given by

$$\Pi^R\tau = \Pi(M_R\tau) \overset{(3.4)}{=} \sum_{\mathbf{1}<\sigma\leq\tau} \mathsf{P}_{\mathsf{g}(M_R^+(\tau/\sigma))}[M_R\sigma] = \sum_{\mathbf{1}<\sigma\leq\tau} \mathsf{P}_{\mathsf{g}^R(\tau/\sigma)}[M_R\sigma].$$

The second equality follows from the fact that M_R^+ commutes with the antipode S^+ and from the formula (4.2) for g^R. The term $\mathsf{P}_f 1$ is equal to zero for any $f \in \mathcal{S}'(\mathbb{R}^d)$. Therefore, one can remove the term $\sigma = \mathbf{1}$. The fact that $M_R\sigma$ is a sum of terms of the same degree as the degree of σ shows that the preceding identity gives a parametrization of the model associated with Π^R by the $[M_R\sigma]$, for all σ with negative degree. ▷

Example – *An action of a renormalization group was observed previously in Bruned's work* [9] *on the renormalization of branched rough paths. This kind of rough paths was introduced by Gubinelli in* [16]*. Hairer & Kelly showed in* [20] *that they can be seen as geometric rough paths over a larger space. See e.g. Cass & Weidner's work* [13] *or Bailleul's work* [1] *for a quick grasp on branched rough paths.*

The regularity of a branched rough path is quantified by an exponent $\gamma \in (0,1)$, and a γ-branched rough path is indexed by decorated trees τ; denote by $|\tau|$ the number of nodes in τ. Fix $\gamma \in (0,1)$, and for a continuous function h on $[0,1]$ write h_t for its value at time t. Tapia and Zambotti exhibited in [23] *a free transitive action of the product space of Hölder spaces*

$$\mathcal{H}^\gamma := \Big\{ g = \big(g(\tau)\big)_{\tau\in\mathcal{B}} \in \prod_{\tau\in\mathcal{B}, |\tau|\leq 1/\gamma} \mathcal{C}^{\gamma|\tau|}([0,1])\,;\, g_0(\tau) = 0 \Big\},$$

where $\mathcal{B}$ is a certain collection of γ-branched rough paths, on the space of all γ-branched rough paths. One of the main results of [9] *provides an explicit formula for the map $g^M \in \mathcal{H}^\gamma$ sending any γ-branched rough path X to $X \circ M$, for a renormalization map M associated in this particular setting to a preparation map of BHZ, hence degree preserving, type. The map g^M is given in Theorem 4.4 of* [9] *and takes the form*

$$g_t^M(\tau) - g_s^M(\tau) = \big\langle \overline{X_{ts} \circ M}, \tau \big\rangle - \langle \overline{X}_{ts}, \tau \rangle. \tag{3.8}$$

where $\overline{X}$ is the Lyons-Victoir extension of X. The latter is not so explicit, so Theorem 9 gives a better description of the action of a renormalization map even in that setting. The paracontrolled parametrization bypasses in particular the problem emphasized in Remark 4.6 of [9] *related to the nonlinear character of the Lyons-Victoir extension map.* □

4 – The general case

We prove the second part of our main result, Theorem 2, in this section. Lemma 7 giving a simple form for δ_R, the co-interaction identity (3.4) and the commutation of the M_R^+ with the antipode S^+ does not hold in the case of a general preparation map R so one cannot use the mechanics of the proof of Theorem 9 in the general case. One can however give an explicit description of the admissible model $\mathsf{M}^R = \big(\mathsf{g}^R, \Pi^R\big)$ associated with R and infer from it an inductive description of the bracket map $[\,\cdot\,]^R$ associated with Π^R and giving a parametrization of the admissible model M^R.

4.1 Renormalised model associated with a preparation map

We will use in the next statement a density argument in the space of models that requires the introduction of a regularity structure $\mathscr{T}^{(\varepsilon)}$, indexed by a positive regularity exponent ε. The only difference between $\mathscr{T}^{(\varepsilon)}$ and $\mathscr{T}$ is the notion of degree $\mathsf{deg}^{(\varepsilon)}$ on $\mathscr{T}^{(\varepsilon)}$ is such that $\mathsf{deg}^{(\varepsilon)}(\tau) = \mathsf{deg}(\tau) - \varepsilon$, for all $\tau \in \mathcal{T}$ or $\tau \in \mathcal{T}^+\backslash\mathcal{T}_0^+$. The exponent ε is chosen small enough for $\alpha - \varepsilon$ to be positive for all $\alpha > 0$. Given now an admissible model $\big(\mathsf{g}, \Pi\big)$ on $\mathscr{T}$, set for all $\tau \in \mathcal{T}$ and $\sigma \in \mathcal{T}^+$

$$\Pi^R\tau := \Pi\big(M_R\tau\big), \quad (\mathsf{g}^R)^{-1}(\sigma) := \mathsf{g}^{-1}\big(M_R^+\sigma\big). \tag{4.1}$$

The map Π^R satisfies the admissibility condition from the fact that M_R commutes with the operators $\mathcal{I}_a$ and the admissibility of the map Π.

Proposition 10 – *The pair* $\big(\mathsf{g}^R,\Pi^R\big)$ *defines an admissible model on* $\mathscr{T}^{(\varepsilon)}$.

We have in particular

$$\mathsf{g}^R(\sigma)=\mathsf{g}\big(S^+M_R^+S^+\sigma\big),\quad \forall\,\sigma\in\mathcal{T}^+. \tag{4.2}$$

We will see as a corollary of Theorem 12 that $\big(\mathsf{g}^R,\Pi^R\big)$ is actually a model on $\mathscr{T}$. Bruned has proved in Section 3 of [8] a version of Proposition for *continuous admissible models*. The use of a density argument allows to extend the result to *all admissible models*.

Proof – Smooth models are models for which all the $\Pi\tau$ and $\mathsf{g}(\sigma)$ are smooth functions. We know from Theorem 2 in [4] or Theorem 5 in [5], giving paracontrolled parametrization of the space of admissible models, that the set of smooth admissible models on $\mathscr{T}$ is dense in the topology associated with the canonical injection of the set of models on $\mathscr{T}$ in the set of models on $\mathscr{T}^{(\varepsilon)}$. See also Theorem 2.14 in Singh and Teichmann's work [22] for a similar statement. It suffices then to prove that for any smooth model (g,Π) the pair $\big(\mathsf{g}^R,\Pi^R\big)$ defines an admissible model on $\mathscr{T}$ – this is what we prove in the following.

Identity (2.6) ensures that

$$\begin{aligned}\Pi_x^R\tau&=\big(\Pi^R\otimes(\mathsf{g}_x^R)^{-1}\big)\Delta\tau=\Big\{\Pi^R\otimes\big(\mathsf{g}_x^{-1}\circ M_R^+\big)\Big\}\Delta\tau\\&=\big(\Pi\otimes\mathsf{g}_x^{-1}\big)(M_R\otimes M_R^+)\Delta\tau\\&\overset{(2.6)}{=}\big(\Pi_x\otimes\mathsf{g}_x^{-1}\big)\,\delta_R\tau.\end{aligned} \tag{4.3}$$

It follows from this identity and the fact that δ_R is upper triangular, Lemma 5, that Π_x^R satisfies the analytic estimates required from a model on $\mathscr{T}$. Note that this holds for *all* admissible models (g,Π), smooth or not. (This point will be used in the proof of Corollary 13.)

Note that it follows from (4.3) and the admissibility of the model (g,Π) that one has for all $\tau\in\mathcal{T}$ and all x

$$\begin{aligned}\big(\mathsf{g}_x^R\big)^{-1}\big(\mathcal{I}_a^+\tau\big)&=\mathsf{g}_x^{-1}\Big(M_R^+\big(\mathcal{I}_a^+\tau\big)\Big)\\&=\mathsf{g}_x^{-1}\Big(\mathcal{M}^+\big(\mathcal{I}_a^+\otimes\mathrm{Id}\big)\delta_R\tau\Big)\\&\overset{(2.1)}{=}\Big((-D^aK*\Pi_x)(x)\otimes\mathsf{g}_x^{-1}\Big)\delta_R\tau\\&\overset{(4.3)}{=}-\Big(D^aK*\Pi_x^R\tau\Big)(x).\end{aligned} \tag{4.4}$$

Write for all x,y

$$\mathsf{g}_{yx}^R:=\Big(\mathsf{g}_x^R\otimes\big(\mathsf{g}_y^R\big)^{-1}\Big)\Delta^+,$$

and define now a multiplicative map from $\mathbb{R}[\mathcal{T}]$ into itself setting for all $\tau\in\mathcal{T}$

$$\widehat{\mathsf{g}_{yx}^R}(\tau):=\big(\mathrm{Id}\otimes\mathsf{g}_{yx}^R\big)\Delta\tau.$$

Denoting by μ_β the component of any $\mu\in\mathcal{T}$ in $\mathcal{T}_\beta$ in the grading $\bigoplus_{\beta\in A}\mathcal{T}_\beta$ of $\mathcal{T}$, the analytic estimates required from g_{yx}^R for $\big(\mathsf{g}^R,\Pi^R\big)$ to be a model on $\mathscr{T}$ are equivalent to having

$$\Big|\big(\widehat{\mathsf{g}_{yx}^R}(\sigma)\big)_\beta\Big|\lesssim|y-x|^{\mathsf{deg}(\sigma)-\beta} \tag{4.5}$$

for all $\sigma\in\mathcal{T}$ and all β with $\beta<\mathsf{deg}(\tau)$, for all x,y. We have

$$\widehat{\mathsf{g}_{yx}^R}(X_i)=X_i+(x_i-y_i)\mathbf{1},\quad\widehat{\mathsf{g}_{yx}^R}(\Xi)=\Xi.$$

The following identity is where working with smooth models helps – continuous models would make the job as well.

Lemma 11 – *One has the identity*

$$\widehat{\mathsf{g}_{yx}^R}\big(\mathcal{I}_a\tau\big)=\mathcal{I}_a\Big(\widehat{\mathsf{g}_{yx}^R}\tau\Big)-\sum_{|\ell|<\mathsf{deg}(\mathcal{I}_a\tau)}\frac{(X+x-y)^\ell}{\ell!}\,\Pi_x^R\Big(\mathcal{I}_{a+\ell}\big(\widehat{\mathsf{g}_{yx}^R}\tau\big)\Big)(y). \tag{4.6}$$

Proof – Note the pointwise evaluation of Π_x^R at a given point y; we work with smooth models to make sense of it – having a continuous model would be sufficient. We briefly recall how one can obtain (4.6) as the settings in [4] and [8] are not striclty speaking the same. The inductive relation (2.7) on Δ gives

$$\begin{aligned}\widehat{\mathsf{g}^R_{yx}}(\mathcal{I}_a\tau) &= \mathcal{I}_a\Big(\widehat{\mathsf{g}^R_{yx}}\tau\Big) + \sum_{|\ell+m|<\deg(\mathcal{I}_a^+\tau)} \frac{X^\ell}{\ell!}\,\mathsf{g}^R_{yx}\Big(\frac{X^m}{m!}\,\mathcal{I}^+_{a+\ell+m}(\tau)\Big) \\ &= \mathcal{I}_a\Big(\widehat{\mathsf{g}^R_{yx}}\tau\Big) + \sum_{|k|<\deg(\mathcal{I}_a^+\tau)} \frac{(X+x-y)^k}{k!}\,\mathsf{g}^R_{yx}\big(\mathcal{I}^+_{a+k}(\tau)\big)\end{aligned} \tag{4.7}$$

Rewriting g^R_{yx} under the form

$$\mathsf{g}^R_{yx} = \Big(\big((\mathsf{g}^R_x)^{-1}\circ S^+\big)\otimes(\mathsf{g}^R_y)^{-1}\Big)\Delta^+,$$

in order to use relation (4.4) giving $(\mathsf{g}^R_x)^{-1}$ and $(\mathsf{g}^R_y)^{-1}$, one has

$$\begin{aligned}\mathsf{g}^R_{yx}\big(\mathcal{I}_b(\tau)\big) &\overset{(2.9)}{=} (\mathsf{g}^R_y)^{-1}\big(\mathcal{I}^+_b(\tau)\big) - \sum_{m,n\in\mathbb{N}^d}\Big(\Big\{(\mathsf{g}^R_x)^{-1}\mathcal{I}^+_{b+m+n}\otimes\frac{(-x)^m}{m!}(\mathsf{g}^R_x)^{-1}\Big\}S^+\Delta\otimes\frac{y^n}{n!}(\mathsf{g}^R_y)^{-1}\Big)\Delta\tau \\ &= (\mathsf{g}^R_y)^{-1}\big(\mathcal{I}^+_b(\tau)\big) - \sum_{k\in\mathbb{N}^d}\frac{(y-x)^k}{k!}\Big((\mathsf{g}^R_x)^{-1}\mathcal{I}^+_{b+k}\otimes\Big((\mathsf{g}^R_x)^{-1}S^+\otimes(\mathsf{g}^R_y)^{-1}\Big)\Delta^+\Big)\Delta\tau \\ &= (\mathsf{g}^R_y)^{-1}\big(\mathcal{I}^+_b(\tau)\big) - \sum_{k\in\mathbb{N}^d}\frac{(y-x)^k}{k!}\Big((\mathsf{g}^R_x)^{-1}\mathcal{I}^+_{b+k}\otimes\mathsf{g}^R_{yx}\Delta^+\Big)\Delta\tau \\ &= (\mathsf{g}^R_y)^{-1}\big(\mathcal{I}^+_b(\tau)\big) - \sum_{k\in\mathbb{N}^d}\frac{(y-x)^k}{k!}\,(\mathsf{g}^R_x)^{-1}\Big(\mathcal{I}^+_{b+k}\big(\widehat{\mathsf{g}^R_{yx}}(\tau)\big)\Big),\end{aligned}$$

for all b. One gets identity (4.6) as follows from the preceding equality using the explicit expression for $(\mathsf{g}^R_x)^{-1}$ given in (4.4) and the relation $\Pi^R_x\circ\widehat{\mathsf{g}^R_{yx}} = \Pi^R_y$. Write $\widehat{\mathsf{g}^R_{yx}}(\tau) = \sum_i\lambda_i\tau_i$, with $\deg(\tau_i)\le\deg(\tau)$ – note that $\Pi^R_y\tau = \Pi^R_x\big(\widehat{g^R_{yx}}\tau\big) = \sum_i\lambda_i\Pi^R_x\tau_i$, then

$$\begin{aligned}\mathsf{g}^R_{yx}\big(\mathcal{I}^+_{a+\ell}(\tau)\big) &= -\Big(\big(D^{a+\ell}K\big)*\big(\Pi^R_y\tau\big)\Big)(y) - \sum_i\lambda_i\sum_{m\in\mathbb{N}^d}\frac{(y-x)^m}{m!}\big(\mathsf{g}^R_x\big)^{-1}\big(\mathcal{I}^+_{a+\ell+m}(\tau_i)\big) \\ &= \sum_i\lambda_i\Big\{-\Big(\big(D^{a+\ell}K\big)*\big(\Pi^R_x\tau_i\big)\Big)(y) - \sum_{m\in\mathbb{N}^d}\frac{(y-x)^m}{m!}\big(\mathsf{g}^R_x\big)^{-1}\big(\mathcal{I}^+_{a+\ell+m}(\tau_i)\big)\Big\} \\ &= \sum_i\lambda_i\Pi^R_x\big(\mathcal{I}^+_{a+\ell}(\tau_i)\big)(y) = \Pi^R_x\Big(\mathcal{I}^+_{a+\ell}\big(\widehat{\mathsf{g}^R_{yx}}(\tau)\big)\Big)(y).\end{aligned}$$

Together with identity (4.7), this gives the recursive formula (4.6). ▶

An elementary induction as in the proof of Proposition 3.16 in [8] shows then that the size bound (4.5) on $\widehat{\mathsf{g}^R_{yx}}(\sigma)$ holds for all $\sigma = \mathcal{I}_a(\tau)$, with $\tau\in\mathcal{T}$. This works as follows. Let $\alpha<\deg(\sigma)$. If $\alpha\in\mathbb{R}\setminus\mathbb{N}$, let us write $\widehat{\mathsf{g}^R_{yx}}(\tau) = \tau + \sum_i\lambda^i_{yx}\tau_i$, with constants λ^i_{yx} and

$$\deg(\tau_i) = \alpha_i < \deg(\tau), \qquad \big|(\tau_i)_{\alpha_i}\big| \lesssim |x-y|^{\deg(\tau)-\alpha_i};$$

then

$$\big|\big(\widehat{\mathsf{g}^R_{yx}}(\sigma)\big)_\alpha\big| = \big|\big(\mathcal{I}_a\big(\widehat{\mathsf{g}^R_{yx}}(\tau)\big)\big)_\alpha\big| \lesssim \sum_i \mathbb{1}_{\{\alpha_i+\beta-|a|=\alpha\}}|x-y|^{\deg(\tau)+\beta-|a|-\alpha_i} \lesssim |x-y|^{\deg(\sigma)-\alpha}.$$

Now, if $\alpha\in\mathbb{N}$ and $\alpha<\deg(\sigma)$, then

$$\begin{aligned}\big|\big(\widehat{\mathsf{g}^R_{yx}}(\sigma)\big)_\alpha\big| &= \left|\left(\sum_{\alpha\le|\ell|<\deg(\sigma)}\frac{(X+x-y)^\ell}{\ell!}\Pi^R_x\Big(\mathcal{I}_{a+\ell}\big(\widehat{\mathsf{g}^R_{yx}}(\tau)\big)(y)\right)_\alpha\right| \\ &\lesssim \sum_{\alpha\le|\ell|<\deg(\sigma)}|x-y|^{|\ell|-\alpha}\sum_{\gamma\le\deg(\tau)}|x-y|^{\gamma-|\ell|+\beta-|a|}\,\big|\big(\widehat{\mathsf{g}^R_{yx}}(\tau)\big)_\gamma\big|\end{aligned}$$

$$\lesssim \sum_{\alpha\leq|\ell|<\mathsf{deg}(\sigma)} |x-y|^{|\ell|-\alpha} \sum_{\gamma\leq \mathsf{deg}(\tau)} |x-y|^{\gamma-|\ell|+\beta-|a|}|x-y|^{\mathsf{deg}(\tau)-\gamma} \lesssim |x-y|^{\mathsf{deg}(\sigma)-\alpha}.$$

The multiplicativity of $\widehat{\mathsf{g}^R_{yx}}$ on $\mathbb{R}[\mathcal{T}]$ ensures that the bound (4.5) holds for all $\sigma\in\mathcal{T}$. ▷

Remark – *Proposition 10 can be proved in a different way, defining first a map $\delta_R^+:\mathcal{T}^+\to\mathcal{T}^+$ via the identity*

$$(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta^+\otimes\mathrm{Id})\delta_R^+ = \big(S^+M_R^+S^+\otimes M_R^+\big)\Delta^+.$$

The map $(\mathrm{Id}\otimes\mathcal{M}^+)(\Delta^+\otimes\mathrm{Id}):\mathcal{T}^+\otimes\mathcal{T}^+\to\mathcal{T}^+\otimes\mathcal{T}^+$, is indeed invertible, like in Lemma 3. The defining relation for δ_R^+ ensures that

$$\begin{aligned}\mathsf{g}^R_{yx} &= \big(\mathsf{g}^R_y\otimes(\mathsf{g}^R_x)^{-1}\big)\Delta^+ = \big(\mathsf{g}_y\otimes\mathsf{g}_x^{-1}\big)\Big(S^+M_R^+S^+\otimes M_R^+\Big)\Delta^+\\ &= \big(\mathsf{g}_{yx}\otimes(\mathsf{g}_x)^{-1}\big)\delta_R^+.\end{aligned}$$

A deep and fairly non-trivial result of Hairer & Quastel ensures that the map δ_R^+ is upper triangular if the map δ_R is upper triangular – see Lemma B.1 in [21]. *The size estimates on $\mathsf{g}^R_{yx}(\sigma)$, for any $\sigma\in\mathcal{T}^+$, follows then from the preceding formula for g^R_{yx} and Hairer & Quastel's result. It shows directly that $\big(g^R,\Pi^R\big)$ is an admissible model on $\mathscr{T}$ at the price of using Lemma B.1 of* [21] *as a blackbox. Our proof is elementary and does not use Hairer & Quastel's result; it follows the proof of Theorem 3.19 in* [8]. *We recover in Corollary 13 below the fact that $\big(g^R,\Pi^R\big)$ is an admissible model on $\mathscr{T}$ rather than just a model on $\mathscr{T}^{(\varepsilon)}$.*

4.2 Parametrization of renormalized models

Assume now that we work with any preparation map R on $\mathcal{T}$. The co-interaction identity (3.4) and the fact that M_R^+ commutes with S^+ are not guaranteed to hold anymore so the proof of Theorem 9 breaks down. We use instead the map δ_R which provides a connection between the renormalized model and the original model. We use the shorthand notation

$$\delta_R\tau =: \sum_{\sigma\leq^R\tau}\sigma\otimes\tau/^R\sigma \tag{4.8}$$

to describe the map δ_R. The sum is implicitly indexed by elements $\sigma\in\mathcal{T}$ in the basis $\mathcal{B}$ of $\mathcal{T}$ fixed throughout this work.

We need to introduce some basic definitions/results on the paraproduct P used in the representation result, Theorem 1, before stating our main result. Recall from [4] the definition of the two-parameter extension $\overline{\mathsf{P}}$ of the paraproduct operator in terms of the kernels Q_i of the Littlewood-Paley projectors – see e.g. Section 3.1 of [4]. For $j\geq 1$, set

$$P_j := \sum_{-1\leq i\leq j-2} Q_i,$$

and for a two variables real-valued distribution Λ on $\mathbb{R}^d\times\mathbb{R}^d$, and $j\geq1$, set for all $x\in\mathbb{R}^d$

$$\big(\mathbf{Q}_j\Lambda\big)(x) := \big\langle\Lambda, P_j(x-\cdot)\otimes Q_j(x-\cdot)\big\rangle.$$

The action of $\overline{\mathsf{P}}$ on Λ is given by

$$\overline{\mathsf{P}}\Lambda := \sum_{j\geq1}\mathbf{Q}_j\Lambda.$$

It coincides with the paraproduct operator when applied to product distrutions $\Lambda_{y,z}=a(y)b(z)$, in the sense that

$$\overline{\mathsf{P}}\big(a(y)b(z)\big) = \mathsf{P}_a b.$$

We use here a formal notation to emphasize the dependence of a distribution on $\mathbb{R}^d\times\mathbb{R}^d$ on its arguments. Recall also from [2] the definition of the operator

$$\mathsf{R}(a,b,c) := \mathsf{P}_a(\mathsf{P}_b c) - \mathsf{P}_{ab}c, \tag{4.9}$$

and the fact that it maps continuously $\mathcal{C}^\alpha(\mathbb{R}^d)\times\mathcal{C}^\beta(\mathbb{R}^d)\times\mathcal{C}^\gamma(\mathbb{R}^d)$ into $\mathcal{C}^{\alpha+\beta+\gamma}(\mathbb{R}^d)$, for all $\alpha,\beta\in(0,1)$ and $\gamma\in(-3,3)$. (See Proposition 3 in [2] – the parameters in the definition of the operators can be arranged so as to get the continuity of R for γ in any a priori fixed interval of regularity exponents. The preceding interval $(-3,3)$ has thus no special meaning.) We need also a key recursive identity which has been used in [4] – identity (2.5) therein. Rewriting the identity

$$\mathsf{\Pi}^R\tau = \sum_{\sigma\leq\tau} \mathsf{g}_x^R(\tau/\sigma)\,\mathsf{\Pi}_x^R\sigma$$

under the form

$$\mathsf{\Pi}_x^R\tau = \mathsf{\Pi}^R\tau - \sum_{\sigma<\tau} \mathsf{g}_x^R(\tau/\sigma)\,\mathsf{\Pi}_x^R\sigma$$

and iterating, we get first

$$\mathsf{\Pi}_x^R\tau = \mathsf{\Pi}^R\tau - \sum_{\sigma<\tau} \mathsf{g}_x^R(\tau/\sigma)\,\mathsf{\Pi}^R\sigma + \sum_{\sigma_2<\sigma_1<\tau} \mathsf{g}_x^R(\tau/\sigma_1)\mathsf{g}_x^R(\sigma_1/\sigma_2)\,\mathsf{\Pi}_x^R\sigma_2$$

and after a finite number of iterations

$$\mathsf{\Pi}_x^R\tau = \mathsf{\Pi}^R\tau - \sum_{n\geq1}(-1)^n \sum_{\sigma_n<\cdots<\sigma_1<\tau} \mathsf{g}_x^R(\tau/\sigma_1)\cdots\mathsf{g}_x^R(\sigma_{n-1}/\sigma_n)\,\mathsf{\Pi}^R\sigma_n. \tag{4.10}$$

Similarly, one has

$$\mathsf{\Pi}_x\tau = \mathsf{\Pi}\tau - \sum_{n\geq1}(-1)^n \sum_{\sigma_n<\cdots<\sigma_1<\tau} \mathsf{g}_x(\tau/\sigma_1)\cdots\mathsf{g}_x(\sigma_{n-1}/\sigma_n)\,\mathsf{\Pi}\sigma_n. \tag{4.11}$$

If one uses relation (4.3) to write

$$\mathsf{\Pi}_x^R\tau = \sum_{\sigma\leq^R\tau} \mathsf{g}_x^{-1}(\tau/^R\sigma)\,\mathsf{\Pi}_x\sigma$$

we obtain from (4.11) the identity

$$\mathsf{\Pi}_x^R\tau = \sum_{\sigma\leq^R\tau} \mathsf{g}_x^{-1}(\tau/^R\sigma)\,\mathsf{\Pi}\sigma - \sum_{n\geq1}(-1)^n \sum_{\sigma_n<\cdots<\sigma_1<\sigma\leq^R\tau} \mathsf{g}_x^{-1}(\tau/^R\sigma)\mathsf{g}_x(\sigma/\sigma_1)\cdots\mathsf{g}_x(\sigma_{n-1}/\sigma_n)\,\mathsf{\Pi}\sigma_n. \tag{4.12}$$

Theorem 12 – *The formula*

$$\begin{aligned}[\tau]^R = &\sum_{n\geq1}(-1)^{n-1} \sum_{\mathbf{1}<\tau_{n+1}<\cdots<\tau_1<\tau} \mathsf{R}\Big(\mathsf{g}^R(\tau/\tau_1)\cdots\mathsf{g}^R(\tau_{n-1}/\tau_n)\,;\,\mathsf{g}^R(\tau_n/\tau_{n+1})\,;\,[\tau_{n+1}]^R\Big)\\ &+\overline{\mathsf{P}}\Big(\big(\mathsf{\Pi}_y^R\tau\big)(z)\Big) + \mathsf{S}(\mathsf{\Pi}^R\tau),\end{aligned} \tag{4.13}$$

where $\overline{\mathsf{P}}\Big(\big(\mathsf{\Pi}_y^R\tau\big)(z)\Big)$ *is given by*

$$\begin{aligned}\overline{\mathsf{P}}\Big(\big(\mathsf{\Pi}_y^R\tau\big)(z)\Big) = &\sum_{\sigma\leq^R\tau} \mathsf{P}_{\mathsf{g}^{-1}(\tau/^R\sigma)}[\sigma] + \sum_{n\geq1}\sum_{\mathbf{1}<\sigma_n<\cdots<\sigma_1<\sigma\leq^R\tau}(-1)^{n-1}\\ &\mathsf{R}\Big(\mathsf{g}^{-1}(\tau/^R\sigma)\,\mathsf{g}(\sigma/\sigma_1)\cdots\mathsf{g}(\sigma_{n-1}/\sigma_n)\,;\,\mathsf{g}(\sigma_m/\sigma_{n+1})\,;\,[\sigma_{m+1}]\Big),\end{aligned} \tag{4.14}$$

defines inductively the bracket map $[\,\cdot\,]^R$ *in terms of the bracket map* $[\,\cdot\,]$.

Proof – One can repeat safely part of the proof of Proposition 12 in [4]. We proceed by induction on the size of the trees. By applying P to the identity (4.10), one has

$$\mathsf{P}_1\left(\mathsf{\Pi}^R\tau\right) = \sum_{n\geq1}(-1)^n \sum_{\tau_n<\cdots<\tau_1<\tau} \mathsf{P}_{\mathsf{g}^R(\tau/\tau_1)\ldots\mathsf{g}^R(\tau_{n-1}/\tau_n)}\mathsf{\Pi}^R\tau_n + \overline{\mathsf{P}}\Big(\big(\mathsf{\Pi}_y^R\tau\big)(z)\Big),$$

One has moreover

$$\mathsf{P}_1\left(\mathsf{\Pi}^R\tau\right) =: \mathsf{\Pi}^R\tau - \mathsf{S}(\mathsf{\Pi}^R\tau),$$

where $\mathsf{S}(\mathsf{\Pi}^R\tau)$ is a smooth term depending continuously in any Hölder topology on the distribution $\mathsf{\Pi}^R\tau$. In the end, we have

$$\Pi^R\tau = \sum_{n\geq 1}(-1)^n \sum_{\mathbf{1}<\tau_n<\cdots<\tau_1<\tau} \mathsf{P}_{\mathsf{g}^R(\tau/\tau_1)\dots\mathsf{g}^R(\tau_{n-1}/\tau_n)}\Pi^R\tau_n + \overline{\mathsf{P}}\Big(\big(\Pi^R_y\tau\big)(z)\Big) + \mathsf{S}(\Pi^R\tau)$$

We replace τ_n by the following expression

$$\Pi^R\tau_n = \sum_{\mathbf{1}<\tau_{n+1}<\tau_n} \mathsf{P}_{\mathsf{g}^R(\tau_n/\tau_{n+1})}[\tau_{n+1}]^R + [\tau_n]^R,$$

and using the definition (4.9) of the operator R, we get

$$\begin{aligned}[][\tau]^R = \sum_{n\geq 1}(-1)^{n-1} \sum_{\mathbf{1}<\tau_{n+1}<\cdots<\tau_1<\tau} \mathsf{R}\Big(\mathsf{g}^R(\tau/\tau_1)\cdots\mathsf{g}^R(\tau_{n-1}/\tau_n)\,;\,\mathsf{g}^R(\tau_n/\tau_{n+1})\,;\,[\tau_{n+1}]^R\Big) \\ + \overline{\mathsf{P}}\Big(\big(\Pi^R_y\tau\big)(z)\Big) + \mathsf{S}(\Pi^R\tau),\end{aligned} \tag{4.15}$$

from the same 'fantastic' telescopic sum as in the proof of Proposition 12 in [4]. The same mechanics is at work in the proof of identity (4.14). Indeed, since one has from identity (4.12)

$$\begin{aligned}\overline{\mathsf{P}}\Big(\big(\Pi^R_y\tau\big)(z)\Big) &= \sum_{\sigma\leq^R\tau} \mathsf{P}_{\mathsf{g}^{-1}(\tau/^R\sigma)}[\sigma] - \\ &\sum_{n\geq 1}(-1)^m \sum_{\sigma_n<\cdots<\sigma_1<\sigma\leq^R\tau} \mathsf{P}_{\mathsf{g}^{-1}(\tau/^R\sigma)\,\mathsf{g}(\sigma/\sigma_1)\cdots\mathsf{g}(\sigma_{n-1}/\sigma_n)}[\sigma_n],\end{aligned}$$

and

$$\Pi\sigma_n = \sum_{\sigma_{n+1}\leq\sigma_n} \mathsf{P}_{\mathsf{g}(\sigma_n/\sigma_{n+1})}[\sigma_{n+1}],$$

a telescopic sum appears and leaves formula (4.14). Formulas (4.15) and (4.14) give jointly an inductive formula giving $[\tau]^R$ in terms of the $[\tau']$, with $\tau'\in\mathcal{T}$. ▷

Corollary 13 – *The model $\big(g^R,\Pi^R\big)$ on $\mathscr{T}^{(\varepsilon)}$ is actually a model on $\mathscr{T}$.*

Proof – Given $\tau\in\mathcal{T}$ with $\mathsf{deg}(\tau)\leq 0$, we know from Proposition 10 in [4] that the double sum in (4.13) defines an element of $\mathcal{C}^{\mathsf{deg}(\tau)}(\mathbb{R}^d)$. Since the distribution $\Lambda = \big(\Pi^R_y\tau\big)(z)$ on $\mathbb{R}^d_y\times\mathbb{R}^d_z$ satisfies from (4.3) the estimate

$$\big\|\mathbf{Q}_j\Lambda\big\| \lesssim 2^{-j\mathsf{deg}(\tau)},$$

uniformly in $j\geq 1$, Proposition 8 in [4] tells us that $\overline{\mathsf{P}}\Big(\big(\Pi^R_y\tau\big)(z)\Big)$ is also an element of $\mathcal{C}^{\mathsf{deg}(\tau)}(\mathbb{R}^d)$. All the brackets $[\tau]^R$ are thus elements of $\mathcal{C}^{\mathsf{deg}(\tau)}(\mathbb{R}^d)$, so $\big(g^R,\Pi^R\big)$ turns out to be a model on $\mathscr{T}$ from Theorem 1, as the unique model on $\mathscr{T}$ associated to the brackets $[\,\cdot\,]^R$ provides canonically a model on $\mathscr{T}^{(\varepsilon)}$ that needs to coincide with $\big(g^R,\Pi^R\big)$, by uniqueness. ▷

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