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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR A FOURTH-ORDER THIN FILM EQUATION VIA REGULARIZATION APPROACHES

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Abstract. This paper is devoted to some aspects of well-posedness of the Cauchy problem (the CP, for short) for a quasilinear degenerate fourth-order parabolic *thin film equation* (the TFE–4)

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{0.1}$$

where $n > 0$ is a fixed exponent, with bounded smooth compactly supported initial data. Dealing with the CP (for, at least, $n \in (0, \frac{3}{2})$) requires introducing classes of infinitely changing sign solutions that are oscillatory close to finite interfaces. The main goal of the paper is to detect proper solutions of the CP for the degenerate TFE–4 by uniformly parabolic analytic ε-regularizations at least for values of the parameter n sufficiently close to 0.

Firstly, we study an analytic "homotopy" approach based on *a priori* estimates for solutions of uniformly parabolic analytic ε-regularization problems of the form

$$u_t = -\nabla \cdot (\phi_\varepsilon(u) \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+,$$

where $\phi_\varepsilon(u)$ for $\varepsilon \in (0, 1]$ is an analytic ε-regularization of the problem (0.1), such that $\phi_0(u) = |u|^n$ and $\phi_1(u) = 1$, using a more standard classic technique of passing to the limit in integral identities for weak solutions. However, this argument has been demonstrated to be non-conclusive, basically due to the lack of a complete optimal estimate-regularity theory for these types of problems.

Secondly, to resolve that issue more successfully, we study a more general similar analytic "homotopy transformation" in both the parameters, as $\varepsilon \to 0^+$ and $n \to 0^+$, and describe *branching* of solutions of the TFE–4 from the solutions of the notorious *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N,$$

which describes some qualitative oscillatory properties of CP-solutions of (0.1) for small $n > 0$ providing us with the uniqueness of solutions for the problem (0.1) when n is close to 0.

Finally, *Riemann-like problems* occurring in a boundary layer construction, that occur close to nodal sets of the solutions, as $\varepsilon \to 0^+$, are discussed in other to get uniqueness results for the TFE–4 (0.1).

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1. Introduction: the Cauchy and free boundary problems for the TFE–4

1.1. Main model and their applications. In this paper, we study some aspects of well-posedness of the Cauchy problem (the CP) for a nowadays well-known fourth-order quasilinear evolution equation of parabolic type, called the *thin film equation* (the TFE–4), with an exponent $n > 0$,

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \,, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

with bounded, sufficiently smooth, and compactly supported initial data u_0 (not necessarily positive) with an arbitrary dimension $N \geq 1$. Note that these initial conditions could be relaxed, (for example $u_0 \in L^1 \cap L^\infty$) however, it is not the purpose of this work to analyse the problem from the perspective of different possible initial conditions.

Equation (1.1) arises in numerous physical related areas with applications in thin film, lubrication theory, and in several other hydrodynamic-type problems. In particular, those equations model the dynamics of a thin film of viscous fluid, as the spreading of a liquid film along a surface, where u stands for the height of the film. Then clearly assuming $u \geq 0$ naturally leads to a *free boundary problem* (an FBP) setting; see below. Specifically, when $n = 3$, we are dealing with a problem in the context of lubrication theory for thin viscous films that are driven by surface tension and when $n = 1$ with Hele–Shaw flows. However, in this work, we are considering solutions of changing sign. Such solutions can have some biological motivations [26], to say nothing of general PDE theory, where the CP-settings were always key.

1.2. Main results. As a more successful approach, among others, to clarify the well-posedness of the CP, we perform an analytic "homotopic" approach from our original equation (1.1) to an equation from which we can extract information about the solutions of equation (1.1),

Namely, we develop a *homotopic deformation* from the TFE–4 (1.1) to the classic and well-known *bi-harmonic equation*

$$u_t = -\Delta^2 u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \,, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

It is well-known that, for any smooth compactly supported data u_0, satisfying the natural "growth condition at infinity"

$$u_0 \in L^2_{\rho^*}(\mathbb{R}^N), \quad \text{where} \quad \rho^*(y) = \mathrm{e}^{-a|y|^{4/3}}, \quad a = \text{const.} > 0 \text{ small},$$

the *bi-harmonic* equation (1.2) admits a unique classic solution given by the convolution Poisson-type integral,

$$\tilde{u}(x,t) = b(x,t) \, * \, u_0(x) \equiv t^{-\frac{N}{4}} \int\limits_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{4}}) u_0(z) \, \mathrm{d}z, \tag{1.3}$$

where $b(x,t)$ is the fundamental solution

$$b(x,t) = t^{-\frac{N}{4}} F(y), \quad y = \tfrac{x}{t^{1/4}}, \tag{1.4}$$

of the operator $\frac{\partial}{\partial t} + \Delta^2$. The oscillatory rescaled kernel $F(y)$ is the unique solution of the linear elliptic problem

$$\mathbf{B}F \equiv -\Delta_y^2 F + \tfrac{1}{4}\, y \cdot \nabla_y F + \tfrac{N}{4}\, F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \textstyle\int_{\mathbb{R}^N} F(y) \, \mathrm{d}y = 1. \tag{1.5}$$

<u>First regularization.</u> Thus, firstly, we perform a homotopic deformation assuming that $n > 0$ is a sufficiently small and *fixed* exponent. We are then actually talking about some "homotopic classes" (understood here not in the classic sense from degree operator theory) of degenerate parabolic PDEs.

More precisely, we say that the TFE (1.1) is "homotopic to the linear PDE (1.2)" if there exists a family of uniformly parabolic equations (a *homotopic deformation*) with a coefficient

$$\phi_\varepsilon(u) > 0 \quad \text{analytic in both variables } u \in \mathbb{R} \text{ and } \varepsilon \in (0,1],$$

with unique analytic solutions $u_\varepsilon(x,t)$ of the problem

$$u_t = -\nabla \cdot (\phi_\varepsilon(u) \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}$$

such that

$$\phi_1(u) = 1, \quad \text{and} \quad \phi_\varepsilon(u) \to |u|^n \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{uniformly on compact subsets}\,, \tag{1.7}$$

so that $u(x,t)$ can be approximated by $u_\varepsilon(x,t)$ as $\varepsilon \to 0^+$.

A possible and quite natural homotopic path (to be used in this work) is

$$\phi_\varepsilon(u) := \varepsilon^n + (1-\varepsilon)(\varepsilon^2 + u^2)^{\frac{n}{2}}, \quad \varepsilon \in (0,1]. \tag{1.8}$$

Then, indeed, the non-degenerate uniformly parabolic equation (1.6) admits a unique (at least, locally in time) classic solution $u = u_\varepsilon(x,t)$, which is an analytic function in all the three variables x, t, and ε, just using classic parabolic theory [11, 15], for any $\varepsilon \in (0,1]$.

Therefore, the homotopic deformation described above is basically a continuous deformation from the TFE–4 to the bi-harmonic equation through the non-degenerate equation (1.6), (1.7) for $\varepsilon \in (0,1]$, for which we also know key features, such as the uniqueness of solutions, oscillatory changing sign properties of them and the well-posedness of the Cauchy problem.

Hence, given the well-defined analytic functional family (a curve or a path),

$$\mathcal{P}_\phi = \{u_\varepsilon(x,t)\}_{\varepsilon \in (0,1]}, \tag{1.9}$$

as usual in extended semigroup theory, a solution $u(x,t)$ of the CP for the TFE–4 (1.1) is then a function, satisfying in a standard pointwise (or even uniformly, if better estimates are available), or in a weaker sense,

$$u_\varepsilon(x,t) \to u(x,t) \quad \text{as} \quad \varepsilon \to 0^+. \tag{1.10}$$

Consequently, as a first step we perform such a homotopic deformation from the TFE–4 (1.1) to the *bi-harmonic equation* (1.2), through the non-degenerate ε-regularization equation (1.6), (1.7), using energy methods as those used by Bernis–Friedman in their pioneering work [5].

However, as we will see, passing to the limit in some more or less standard integral identities for $\{u_\varepsilon\}$ is not sufficient to convincingly distinguish the CP and FBP solutions (both can admit the same weak setting).

In other words, the weak formulation of the CP does not allow one to clarify key features of the CP-solutions. So, in this process, open questions remain, and this standard analysis does not solve the problem of uniqueness of solutions of the CP.

Indeed, in general, under necessary natural estimates on $\{u_\varepsilon\}$, deformation (1.10) is able to detect *a solution* of the CP, and does not guarantee its *conventional uniqueness* via existence of a limit in (1.10). So that, (1.10) may contain several (or even infinitely many) *partial limits*, since the end point $\varepsilon = 0^+$ of the path (1.9) is singular; see Section 2.

Moreover, nothing will also guarantee that a proper CP-solution *is actually independent of the type of an analytic ε-regularization $\phi_\varepsilon(u)$ in the path* (1.9). This is the strongest type of uniqueness, which is called the *absolute uniqueness*. In fact, this is very difficult to achieve for higher-order PDEs[1].

Note that, even for second-order parabolic equations, such a type of uniqueness of *proper solutions* (extremal: e.g., *minimal* ones in blow-up problems), which is guaranteed by the Maximum Principle and comparison techniques, was proved under the assumption on the *monotonicity in ε of any ε-regularization applied* (then any of such regularization leads to the unique proper minimal solution); see [16, Ch. 7]. Hence, for the TFE–4, we do not stress upon such a strong uniqueness issue.

[1]It seems that, in a most general setting, such absolute uniqueness results are non-existent in principle (though these could be useless if they exist).

<u>SECOND REGULARIZATION: A DOUBLE LIMIT.</u> Therefore, secondly, our further improved definition of the proper CP-solutions assumes also a second limit as $n \to 0^+$ (with a simpler regularization; see below), i.e., a continuous connection with solutions of the *bi-harmonic equation* (1.2) which we know explicitly. Thus, $u(x,t)$ in (1.10) is a CP-solution of the TFE–4 (1.1) if

$$u(x,t) \to \tilde{u}(x,t) \quad \text{as} \quad n \to 0^+, \tag{1.11}$$

where $\tilde{u}(x,t)$ is denoted by (1.3).

Actually, instead of independent limits (1.10), (1.11), we will need to perform a kind of *double* limit as ε, $n \to 0^+$, where special restrictions on two parameters will be required.

However, as discussed at the end of Section 2 even by this double limit as ε, $n \to 0^+$, using standard integral identities, it is not still possible to identify the problem which the limit belongs to. Hence, something different must be done.

Additionally open questions remain regarding regularity of the solutions for the TFE–4 (1.1), especially when $N \geq 2$. Note that, for $N = 1$, a complete analysis with several important uniform estimates was already done in [5]. This issue is essentially the reason why the energy methods do not work in $\mathbb{R}^N$ with $N \geq 2$ since extensions of that analysis are not straightforward.

Consequently, as a different and simpler, but more general, regularizing version, a homotopic path as $n \to 0^+$ and $\varepsilon \to 0^+$ can be used separately for a proper definition of the solutions of the Cauchy problem for the TFE–4 (1.1). However, a certain relation between the parameters ε and n must be imposed.

Hence, using the following simpler approximation in (1.6):

$$\phi_\varepsilon(u) = (\varepsilon^2 + u^2)^{\frac{n}{2}}, \tag{1.12}$$

this assumes that a correct solution of the CP for the TFE–4 (1.1) is that one, which can be continuously deformed via this "double" limit to the unique solution of the *bi-harmonic equation* (1.2).

Now, we can state the main result of this paper, which provides us with a definition of the proper solution of the Cauchy problem (1.1) as well as the uniqueness of solutions and well-posedness of the CP, when the parameter n is sufficiently close to zero.

Theorem 1.1. *Assume the condition*

$$n\,|\ln \varepsilon(n)| \to 0 \quad \textit{as} \quad n \to 0^+, \tag{1.13}$$

holds and the regularization family $\{u_\varepsilon\}$ *is uniformly bounded. Then:*

(i) *The solution of the perturbed problem*

$$u_t = -\nabla \cdot ((\varepsilon^2 + u^2)^{\frac{n}{2}} \nabla \Delta u), \quad u(x,0) = u_0(x),$$

converges uniformly to the solution (1.3) *of the Cauchy problem of the bi-harmonic equation*

$$u_t = -\Delta^2 u, \quad u(x,0) = u_0(x),$$

in the limit $\varepsilon = \varepsilon(n) \to 0^+$ *and* $n \to 0^+$*, which is unique.*

(ii) *Provided that the convolution*

$$\varphi_1(t) = -\int_0^t \nabla b(t-s) * \ln |\tilde{u}(s)| \nabla \Delta \tilde{u}(s)\, \mathrm{d}s,$$

remains uniformly bounded for the solution (1.3)*, the rate of convergence as* $n \to 0$ *of the* (*formal*) *asymptotic expansion* $u = \tilde{u} + V$ *is given by*

$$V := n\varphi_1 + o(n), \quad \textit{with} \quad \varphi_1(t) = -\int_0^t \nabla b(t-s) * \ln |\tilde{u}(s)| \nabla \Delta \tilde{u}(s)\, \mathrm{d}s.$$

Of course, a similar result can be more easily established for the previous, single-limit path (1.8), but the present double-limit one is indeed more general, though assumes the necessary restriction (1.13) on parameters. Remember that this restriction is crucial to have such results.

Consequently, thanks to the previous result, we can assure that there exists a branch of solutions of the TFE-4 (1.1) emanating at $n = 0^+$ from the unique solution of the parabolic bi-harmonic equation (1.2), deforming the solutions of the TFE-4 (1.1) as $n \to 0^+$ and $\varepsilon \to 0^+$ via an analytic path (through the ε regularization (1.12)) and inheriting the oscillatory and changing sign properties of the linear flow.

Also, since the parabolic bi-harmonic equation (1.2) is well-posed, through the homotopy deformation performed by Theorem 1.1 we can achieve such a well-posedness for the CP (1.1), at least when the parameter n is sufficiently close to zero.

Note that we are going to develop a branching theory for difficult *nonlinear degenerate parabolic PDEs*, requiring special relationships between $n \to 0^+$ and the regularization parameter $\varepsilon = \varepsilon(n) \to 0$. Without such a relation, no proper limits can be traced out at all.

In the final section, a further and deeper study of these limits (particularly, for establishing a proper uniqueness of CP-solutions of the TFE–4 via regularization) leads to difficult *boundary layer*-type problems that remain open in a sufficient generality. To be specific, we concentrate here on the analysis of the problematic limit as $\varepsilon \to 0^+$ for (1.12) in order to prove the uniqueness of solutions for the TFE–4 (1.1).

Although, we perform our analysis in arbitrary dimensions, the general (say, the absolute one) uniqueness result is still an open problem and probably non-achievable in principle. To do so, we need to understand the zero structure sets of the solutions, especially close to the interfaces. This leads us to analyze the *boundary layers* that occur close to those nodal sets. In other words, to study an asymptotic problem near the interfaces, usually called *Riemann's problems* for the TFE–4. Indeed, the well-posedness of the limiting problem when $\varepsilon \to 0^+$ will strongly depend on the asymptotics of the solution for the TFE–4 (1.1) close to the interfaces, as discussed in the final section of this paper.

1.3. **FBP and CP settings: similarities and distinctive features.** First serious attempts to create a proper mathematical theory for TFEs–4 were made in the 1980s. Since then, hundreds of papers and a number of monographs have dealt seriously with such higher-order parabolic PDEs. We refer to e.g. [12, 13, 22, 24] for most recent surveys and for extended lists of references concerning physical derivations of various models, key mathematical results, and further applications. Nowadays (and at least since the 1980s), such equations play quite a special and, even a key role, in general nonlinear PDE theory.

For this particular work it is crucial to note that the TFE–4 (1.1) is written for solutions of changing sign, which, as we show, can occur in the CP and also in some free-boundary problems.

As mentioned above, since the 1980s, it has been customary to consider *non-negative solutions* of the TFE–4, and moreover the existing mathematical PDE theory was created mostly for solutions

$$u = u(x,t) \geq 0.$$

However, *solutions of changing sign* have been under scrutiny already for a few years; see [8, 13, 14] and references therein. To deal with such a "dichotomy":

non-negative FBP- or *infinitely sign-changing* CP-solutions,

let us recall our main convention (to be discussed)

$$\text{(1.14)} \qquad \begin{array}{c}\text{oscillatory sign changing solutions are related to the CP, while} \\ \text{non-negative solutions appear for the standard FBP (as in [5, 6]),}\end{array}$$

for not that large $n > 0$, actually,

$$\text{for any} \quad 0 < n < n_{\rm h} = 1.7587... \, .$$

See [13] for any further details, where that upper bound is obtained using numerical analysis, and [18] for proper estimates of this *heteroclinic critical exponent* $n_{\rm h}$.

For both problems, the CP and FBP, we assume that the solutions satisfy the following standard free boundary conditions:

$$
\text{(1.15)} \qquad \begin{cases} u=0, & \text{zero-height,} \\ \nabla u = 0, & \text{zero contact angle,} \\ -\mathbf{n}\cdot \nabla(|u|^n \Delta u)=0, & \text{zero-flux (conservation of mass)} \end{cases}
$$

at the singularity surface (interface) $\Gamma_0[u]$, which is the lateral boundary of the support

$$
\text{supp}\, u \subset \mathbb{R}^N \times \mathbb{R}_+,
$$

where $\mathbf{n}$ stands for the unit outward normal to $\Gamma_0[u]$, which is assumed to be sufficiently smooth, at least, a.e.[2] The condition of zero flux might be interpreted as

$$
\lim_{\text{dist}(x,\Gamma_0[u])\downarrow 0} -\mathbf{n}\cdot \nabla(|u|^n \Delta u)=0,
$$

again for sufficiently smooth interfaces. Note that there can be other and different free-boundary conditions (e.g., without the zero contact angle one), and a lot of work has been directed to the discussion of that matter, for instance [27].

The problem is completed with bounded, smooth, integrable, compactly supported initial data:

$$
\text{(1.16)} \qquad u(x,0)=u_0(x) \quad \text{in} \quad \Gamma_0[u] \cap \{t=0\}.
$$

Thus, regarding the initial data, unlike the FBP, in the CP for (1.1) in $\mathbb{R}^N \times \mathbb{R}_+$, one needs to pose bounded compactly supported initial data (1.16) prescribed in the whole $\mathbb{R}^N$.

Intuitively (and rather loosely speaking), the principal difference between the CP and the FBP is that, for the CP, *no free boundary conditions* should be prescribed *a priori*, while, for the FBP, these must be clearly given as done in (1.15).

In this connection, it is worth mentioning that it is known that solutions of both the CP and the FBP can be obtained by proper ε-regularizations, i.e., *without* specifying the free-boundary conditions.

What is a principal fact is that those ε-regularizations for the CP and the FBP must be *different*, unless both problems coincide. The latter can actually happen for the TFE-4 for larger $n \ge n_{\rm h}$, when solutions of the CP lose their infinitely oscillatory properties, or even for $n \ge 2$ only. In general, the question of n's, for which "the CP = the FBP", remains open.

Note also that there exist many other FBPs with slightly different conditions on the interface (cf. e.g., *Stefan* and *Florin* classic FBPs for the heat equation, [16, Ch. 8]). Of course, for the CP, it would be very interesting to derive the actual and sharp enough conditions on the free boundary (of course, (1.15) are valid, but these have nothing to do with the optimal regularity for the CP), but this is difficult even in 1D, to say nothing of any proof; see [13, § 6].

Obviously, due to the zero flux condition in (1.15), for the FBP, the total mass of solutions is preserved in time. For the CP, as we know, this must be also true, but not that straightforward. Here, it is natural to declare another additional convention (to be treated as well): at least for $n \in (0, \frac{3}{2})$,

(1.17) the solutions of the CP are *smoother* at the interface than for the FBP.

Actually, we have got some evidence about that convention in [13]. Indeed, close to the interface for $t>0$, after an expected *"interface waiting time"*, in the sense of the "multiple zeros at the

[2] A very difficult result to prove, still a fully open problem for any $N \ge 2$. It seems that possible types of "singular cusps" of interfaces occurring via their self-focusing (e.g., the simplest Graveleau–Aronson-type radial "filling a hole" in PME theory known for (1.19) [2]) were not studied at all for (1.1).

interfaces", it follows that

$$\boxed{\text{CP:} \quad u(x,t) \in C_x^{[\frac{3}{n}]-1},} \tag{1.18}$$

where $[\cdot]$ denotes the integer part[3]. This means that, for small $n > 0$, the CP-solutions can be arbitrarily smooth in x at the interfaces. This will be dealt with rather seriously later.

Thus, under the same zero flux condition at finite interfaces (which should be established separately), the mass is preserved in the CP as well. Let

$$M(t) := \int u(x,t)\,\mathrm{d}x$$

be the mass of the solution, where integration is performed over smooth support ($\mathbb{R}^N$ is allowed for the CP only). Then, differentiating $M(t)$ with respect to t and applying the divergence theorem (under natural regularity assumptions on solutions and free boundary), we have that

$$J(t) := \tfrac{\mathrm{d}M}{\mathrm{d}t} = - \int\limits_{\Gamma_0 \cap \{t\}} \mathbf{n} \cdot \nabla(|u|^n \Delta u)\,.$$

Then, the mass is conserved if $J(t) \equiv 0$, which is assured by the flux condition in (1.15).

1.4. **Solutions of changing sign and comparison with porous medium flows.** As mentioned above, most of the existing results for the TFE–4 deal with *non-negative solutions* with compact support of various FBPs, which are often more physically relevant. In this context, we should point out that such approximations for *non-negative* and non-changing sign solutions, with various non-analytic (and non-smooth) regularizations, for example, of the form

$$\phi_\varepsilon(u) = |u|^n + \varepsilon,$$

which is not only non-analytic for $n < 2$, but even $\notin C^2$, have been widely used before in TFE–FBP theory as a key foundation; cf. [4], [5], and [7].

Indeed, as pointed out in [6], for non-negative solutions, when n approximates 0^+, the limiting problem is always a free boundary problem. Owing to the oscillatory behaviour of the solutions of the CP for the linear bi-harmonic equation (1.2), the CP cannot be the limiting problem in this case.

In other words, in general, positivity of solutions of some FBPs for the TFE–4 and other related equations is achieved via some uniformly parabolic, but sufficiently "singular" (and surely non-analytic) ε-regularization of the PDE. In fact, this creates special kinds of "obstacle FBPs", where the obstacle appears to get positive solutions. This was first proved by specific singular regularizations introduced in the seminal TFE-paper [5], devoted mainly to FBP settings.

On the other hand, it turned out that the classes of the so-called "oscillatory solutions of changing sign" of (1.1) were rather difficult to tackle rigorously by standard and classical methods. Specifically, due to the fact that any kind of a detailed analysis for higher-order equations is much more difficult than for their second-order counterparts, such as the notorious classic *porous medium equation* (PME–2)

$$u_t = \Delta(|u|^{n-1}u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \tag{1.19}$$

in view of the lack of the Maximum Principle, comparison methods, order-preserving semigroups, and potential properties of the operators involved. Thus, practically all the existing methods for monotone or variational operators are not applicable for (1.1).

[3]Note that inner, "transversal" zeros of solutions can be less regular, but these are not that interesting, unlike the key interface ones. However, "transversal" zeros are accumulated near interfaces, so (1.18) should not be understood literally, and it just shows the smoothness of the *non-oscillatory envelope* of solutions near interfaces (see [13, § 7]), or derivatives are assumed to be calculated over neighbourhoods without touching enough those "transversal" zeros.

The PME–2 (1.19) can be interpreted as a nonlinear degenerate extension of the classic *heat equation* for $n = 0$,

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+. \tag{1.20}$$

Note that passing to the limit $n \to 0^+$ in the PME–2 (1.19) for non-negative solutions used to be a difficult mathematical problem in the 1970s-80s, which exhibited typical features (but clearly simpler than those ones in the TFE case) of a "homotopy" transformation of PDEs. This study was initiated by Kalashnikov in 1978 [25] for the one-dimensional case. Further detailed results in $\mathbb{R}^N$ were obtained in [3]; see also [10]. More recent involved estimates were obtained in [28, 29] for the 1D PME–2 establishing the rate of convergence of solutions as $n \to 0^{\pm}$, such as $O(n)$ as $n \to 0^-$ (i.e, from $n < 0$, the fast diffusion range, where solutions are smoother) in $L^1(\mathbb{R})$ [28], and $O(n^2)$ as $n \to 0^+$ in $L^2(\mathbb{R} \times (0,T))$ [29].

However, most of such convergence results are also obtained for *non-negative* solutions of the PME–2. For solutions of changing sign, even for this second-order PME–2 (1.19), there are some open problems; see [17] for references and further details.

Thus, in the twenty-first century, higher-order TFEs such as (1.1), though looking like a natural and not-that-involved counterpart/extension of the PME–2 (1.19), corresponding mathematical TFE theory gets essentially more complicated with several key problems remaining open still (such as regularity, uniqueness in general, etc).

The outline of the paper is: in Section 2 we perform an homotopic approach passing to the limit in integral identities for weak solutions. In Section 3, applying a *branching argument* we study the homotopy *double limit* as $n, \varepsilon \to 0^+$. Finally, in Section 4 the limit as $\varepsilon \to 0^+$ is analysed in order to prove the uniqueness of solutions for the TFE–4 (1.1).

2. The ε-regularization problem: passage to the limit in a weak sense

2.1. **Preliminary estimates.** For any $\varepsilon \in (0, 1]$, we denote by $u_\varepsilon(x,t)$ the unique solution of the CP for the regularized non-degenerate uniformly parabolic equation (1.6), (1.8) with the same data u_0.

By classic parabolic theory [11, 15], this family

$$\{u_\varepsilon(x,t)\} \quad \text{is continuous (and analytic) in } \varepsilon \in (0,1],$$

in any natural functional topology, at least, on a time interval $[0,T]$. Indeed, we also have that all the derivatives are Hölder continuous in $\overline{\Omega} \times [0,T]$. However, as we show below, the main problem is the behaviour as $\varepsilon \to 0^+$, where the regularized PDE loses its uniform parabolicity.

Here, by Ω we denote either $\mathbb{R}^N$, or, equivalently, the bounded domain $\operatorname{supp} u \cap \{t\}$, i.e., the section of the support.

Note again that, for second-order parabolic equations with the Maximum Principle, such regularization-continuity approaches are typical for constructing unique proper extremal (say, minimal) solutions with various singularities (e.g., finite time blow-up, extinction, finite interfaces, etc.); see [16, Ch. 4-7] as a source of key references and basic results.

However, for higher-order degenerate parabolic flows admitting strongly oscillatory solutions of changing sign, such a homotopy-continuity approach generates a number of difficult problems. In fact, despite the fact that the passage to the limit as $\varepsilon \to 0^+$ looks like a reasonable way to define a proper solution of the TFE, we expect that there are always special classes of compactly supported initial data, for which such a limit is non-existent and, moreover, there are many partial limits, thus defining a variety of different solutions (meaning non-uniqueness).

Thus, to study such a limit (1.10) for the problem (1.6), (1.8), when $\varepsilon \to 0^+$, we firstly need some estimates for its regularized solutions $\{u_\varepsilon(x,t)\}$.

Proposition 2.1. *Let $u_\varepsilon(x,t)$ be the unique global solution of the CP for the regularized non-degenerate equation* (1.6), (1.8). *Then, for $t \in [0,T]$, there exists a positive constant $K > 0$, independent of ε and T, such that the following is satisfied:*

(i) $\int_\Omega |\nabla u_\varepsilon(x,t)|^2 \le K$, $\int_\Omega |u_\varepsilon(x,t)|^2 \le K$;

(ii) $\int_\Omega u_\varepsilon(x,t) \le K$;

(iii) $\|h_\varepsilon\|_{L^2(\Omega\times[0,T])} \le K$, *with* $h_\varepsilon := \phi_\varepsilon(u_\varepsilon)\nabla\Delta u_\varepsilon$.

Proof. Firstly, multiplying (1.6) by Δu_ε, integrating in $\Omega\times[0,t]$ for any $t\in[0,T]$, and applying the formula of integration by parts yields

$$\tfrac{1}{2}\int_\Omega |\nabla u_\varepsilon(x,t)|^2 + \int_0^t\int_\Omega \phi_\varepsilon(u)|\nabla\Delta u_\varepsilon|^2 = \tfrac{1}{2}\int_\Omega |\nabla u_\varepsilon(x,0)|^2, \tag{2.1}$$

thanks to the boundary conditions (1.15). Note that

$$\begin{aligned}&\int_\Omega [|\nabla u_\varepsilon(x,t+h)|^2 - |\nabla u_\varepsilon(x,t)|^2]\\ &= -\int_\Omega [\Delta u_\varepsilon(x,t+h)+\Delta u_\varepsilon(x,t)][u_\varepsilon(x,t+h)-u_\varepsilon(x,t)].\end{aligned}$$

Then, dividing that equality by h, passing to the limit as $h\downarrow 0$, and integrating between 0 and any $t\in[0,T]$, we find that[4]

$$\int_0^t\int_\Omega \Delta u_\varepsilon u_{\varepsilon,t} = \tfrac{1}{2}\int_\Omega |\nabla u_\varepsilon(x,t)|^2 - \tfrac{1}{2}\int_\Omega |\nabla u_\varepsilon(x,0)|^2,$$

which provides us with the necessary expression to obtain (2.1). Thus, from (2.1), we have

$$\int_\Omega |\nabla u_\varepsilon(x,t)|^2 \le K \quad \text{and} \quad \int_0^t\int_\Omega \phi_\varepsilon(u)|\nabla\Delta u_\varepsilon|^2 \le K, \tag{2.2}$$

since both terms of the left-hand side in (2.1) are always positive and the right-hand side is bounded by (1.16), for a positive constant $K>0$ that is independent of ε. Then, by Poincaré's inequality (when Ω is a bounded domain),

$$\int_\Omega |u_\varepsilon(\cdot,t)|^2 \le K \quad \text{for all} \quad t\in(0,T).$$

When $\Omega=\mathbb{R}^N$, for compactly supported data, all the solutions $u_\varepsilon(x,t)$ have exponential decay at infinity, which makes the integrations properly justified. Then the last inequality remains valid in certain $L^2_\rho(\mathbb{R}^N)$ and $H^1_\rho(\mathbb{R}^N)$ weighted spaces for an appropriately exponentially decaying weight ρ (say $\rho(y)=\mathrm{e}^{-a|y|^{4/3}}$).

On the other hand, in view of the finite propagation of perturbations in the TFE–4, on a given interval $t\in[0,T]$, we actually always deal with uniformly bounded supports of $u(x,t)$, while $u_\varepsilon(x,t)$ exhibit just extra exponentially small tails at infinity that do not affect the estimate.

Furthermore, from the mass conservation, we can also assure that

$$\int_\Omega u_\varepsilon(\cdot,t) \le K \quad \text{for all} \quad t\in(0,T).$$

Now, using (2.1) for $u_\varepsilon(\cdot,t)\in L^1(\Omega)$, we prove the following estimate:

$$\|h_\varepsilon\|_{L^2(\Omega\times[0,T])} \le K, \quad \text{where} \quad h_\varepsilon = \phi_\varepsilon(u_\varepsilon)\nabla\Delta u_\varepsilon. \tag{2.3}$$

[4]In fact, this is true from the beginning for classic C^∞-smooth solutions of (1.6), but we will need those manipulations in what follows.

To this end, from (2.2), we find

$$\int_0^t \int_\Omega [\varepsilon^n + (1-\varepsilon)(\varepsilon^2+u^2)^{\frac{n}{2}}]|\nabla\Delta u_\varepsilon|^2 \le K, \quad \text{so that}$$

$$\varepsilon^n \int_0^t \int_\Omega |\nabla\Delta u_\varepsilon|^2 \le K \quad \text{and} \quad \int_0^t \int_\Omega (\varepsilon^2+u^2)^{\frac{n}{2}}|\nabla\Delta u_\varepsilon|^2 \le K, \tag{2.4}$$

since $\varepsilon \in (0,1)$, with a constant $K>0$ independent of ε. Now, using Hölder's inequality,

$$\int_0^t \int_\Omega |h_\varepsilon|^2 \le 2\varepsilon^{2n} \int_0^t \int_\Omega |\nabla\Delta u_\varepsilon|^2 + 2\int_0^t \int_\Omega (\varepsilon^2+u^2)^{\frac{n}{2}}(\varepsilon^2+u^2)^{\frac{n}{2}}|\nabla\Delta u_\varepsilon|^2,$$

and by (2.4) (note also that $u_\varepsilon(\cdot,t) \in L^1(\Omega)$), we obtain (2.3). $\square$

Additionally, we get uniformly bounded estimates in L^∞ for the solutions of the TFE–4 equation (1.1) as well as for the solutions of the non-degenerate equation (1.6).

Indeed, without loss of generality, we consider a solution $u(x,t)$ of (1.1), and, by scaling techniques (see [19, 21]), prove its uniform L^∞ *a priori* bound.

The function $u(x,t)$ is assumed to satisfy the equation in the classic sense in the positivity and negativity domain, where it is C^∞ in x (or may be even analytic in x) by classic parabolic theory [15], so the proof does not use particular homogeneous smooth and sufficiently regular free boundary conditions at the zero set.

Implicitly, however, we have to assume that such free boundary conditions cannot lead to some "strong singularities" on the free boundaries, which are not small in L^∞ (a kind of "blow-up at the interfaces") and, hence, that may affect standard interior parabolic regularity phenomena in the uniform positivity/negativity domains.

In other words, we assume that, while solutions remain bounded and sufficiently smooth, homogeneous free boundary conditions imposed (if any; e.g., no conditions are assumed for the Cauchy problem in $\mathbb{R}^N \times \mathbb{R}_+$) move positive and negative humps of solutions sufficiently smoothly and without strong "collisions" and "self-focusing-like" behaviour leading to L^∞ blow-up at interface points.

The same analysis also directly applies to the regularized uniformly parabolic problems like (1.6), where the classic solution is unique, so it can be used in Theorem 1.1 and in other related results.

Proposition 2.2. *Under the above hypothesis, any solution of* (1.1) *is uniformly bounded.*

Proof. We argue by contradiction. Let there exist a monotone sequence $\{t_k\} \to +\infty$ (or $\{t_k\} \to T < \infty$, a finite blow-up time that makes no difference) and $\{x_k\} \subset \mathbb{R}^N$ such that

$$\sup_{(x,t)\in\mathbb{R}^N\times(0,t_k)} |u(x,t)| = |u(x_k,t_k)| = C_k \to +\infty \quad \text{monotonically.} \tag{2.5}$$

Then we rescale the TFE–4 and introduce a sequence of solutions $\{v_k(y,s)\}$ of the TFE–4:

$$\begin{aligned} & u_k(x,t) \equiv u(x+x_k, t+t_k) = C_k v_k(y,s), \quad x = a_k y, \quad t = b_k s, \quad \tfrac{b_k C_k^n}{a_k^4} = 1 \\ \Longrightarrow \quad & v_s = -\nabla\cdot(|v|^n \nabla\Delta v), \quad s > -\tfrac{t_k}{b_k} \to -\infty, \quad v_{k0}(y) = \tfrac{1}{C_k}\, u_0(x_k + a_k y). \end{aligned} \tag{2.6}$$

With this rescaling we are just performing a zoom around the point (x_k,t_k), in the region

$$B_\delta(0) \times (-\tau_k, 0), \quad \text{with} \quad \delta > 0 \quad \text{sufficiently small,}$$

defining the problem in a ball of radius $\frac{\delta}{a_k}$ for the variable y and in the interval

$$s \in \left(-\frac{t_k}{b_k}, 0\right).$$

Note that we have a negative interval since we have chosen $t+t_k$ and we approach the time from the left. Hence, the limiting problem will be defined in $\mathbb{R}^N \times (-\infty, 0)$. The only assumption is that the positive sequences satisfy

$$\{a_k\} \to 0 \quad \text{and} \quad \{b_k\} \to 0,$$

for scaling reasons. The sequence of solutions $\{v_k(y,s)\}$ thanks to (2.5) then satisfies that

$$|v_k(y,s)| \le 1 \quad \text{and} \quad |v_k(0,0)| = 1, \quad \text{for all} \quad s \in \left[-\frac{t_k}{b_k}, 0\right). \tag{2.7}$$

Moreover, by the uniform estimate as in Proposition 2.1(i), we have

$$\textstyle\int |\nabla_x u|^2 \mathrm{d}x \equiv \frac{C_k^2}{a_k^{N+2}} \int |\nabla_y v_k|^2 \mathrm{d}y \le K \quad \Longrightarrow \quad \int |\nabla_y v_k|^2 \mathrm{d}y \le K \frac{a_k^{N+2}}{C_k^2}. \tag{2.8}$$

Thus, passing to limit, as $k \to +\infty$, along a subsequence, we are then supposed to have an *ancient* solution $v_k \to v(y,s)$ defined for all $s<0$. According to the last estimate in (2.8), we will get a zero solution in the limit provided that

$$\kappa_k \equiv \frac{a_k^{N+2}}{C_k^2} \to 0. \tag{2.9}$$

Thus, we have to satisfy the following two properties only:

$$\begin{cases} b_k = \kappa_k^4 C_k^{\frac{8}{N+2}-n} \to 0, \\ a_k = \kappa_k^{\frac{1}{N+2}} C_k^{\frac{2}{N+2}} \to 0. \end{cases} \tag{2.10}$$

Evidently, both can be done easily provided that κ_k decays sufficiently fast. Hence, since passing to the limit, as $k \to +\infty$, in (2.6) for bounded solutions in the uniform positivity (negativity) domains of a parabolic equation is straightforward, the ancient solution is $v \equiv 0$. This contradicts the last assumption in (2.7), implying, by the interior parabolic regularity, that $v(y,0)$ must be non-trivial in a neighbourhood of $y=0$. This simply means that the TFE–4 does not have an internal mechanism to support indefinite growth (or blow-up) of solutions. $\square$

Remark. This type of scaling proof is rather standard, based on the famous Gidas–Spruck blow-up Method [23] for elliptic problems. For parabolic problems see [19] where a Kuramoto–Sivashinsky equation is analysed and [21] for higher-order equations with absorption terms.

2.2. **Passing to the limit.** To conclude this section, we show the existence of a weak solution of the CP for the degenerate parabolic TFE–4 (1.1) by passing to the limit:

(i) as ε goes to zero,

and also,

(ii) as $n \to 0^+$.

<u>Passing to the limit $\varepsilon \to 0^+$</u>. By Proposition 2.1, for bounded supports Ω, and the Aubin–Lions Lemma, the embedding

$$H_0^1(\Omega \times (0,T)) \hookrightarrow L^2(\Omega \times (0,T)),$$

is compact. Then, we can extract a convergent subsequence in $L^2(\Omega \times (0,T))$ as $\varepsilon \downarrow 0^+$ of solutions of (1.6), (1.8), labelled again by $u_\varepsilon(x,t)$, so

$$\lim_{\varepsilon \to 0^+} \left\| u_\varepsilon(\cdot,t) - U(\cdot,t) \right\|_{L^2(\Omega \times (0,T))} = 0, \tag{2.11}$$

where, for convenience, unlike in (1.10), this CP-solution of the TFE–4 (1.1) is denoted by the capital $U(\cdot,t)$.

Thus, the convergence of the regularized solutions of the CP (1.6), (1.8) is strong in $L^2(\Omega \times (0,T))$. In $\mathbb{R}^N$, we will use the appropriate L^2_ρ and Sobolev spaces H^1_ρ when necessary.

Note again that one difficulty we face is whether this limit depends on the taken subsequence or not.

In other words, this analysis, just with the limit $\varepsilon \to 0^+$, does not include any uniqueness result, which is expected to be a more difficult open problem for such nonlinear degenerate parabolic TFEs in non-fully divergence form and with non-monotone operators.

However, the principal issue of the analytic regularization via (1.6), (1.8) is that it is expected to lead to smoother CP-solutions at the interface than those for the standard FBP, at least for $n < \frac{3}{2}$.

The difference is that using the analytic regularized family $\{u_\varepsilon\}$ and imposing (1.15) it is assumed to guarantee that on the interface (being sufficiently smooth for simplicity),

$$\frac{\partial^2 u}{\partial \mathbf{n}^2} = 0 \quad \text{a.e., at least, for } n \in (0,1]. \tag{2.12}$$

In fact, proper oscillatory solutions of the CP are assumed to exhibit even more regularity at smooth interfaces [13] (cf. (1.18)): again in the sense of non-oscillatory envelopes[5],

$$\frac{\partial^l u}{\partial \mathbf{n}^l} = 0, \quad \text{where} \quad l = \left[\tfrac{3}{n}\right] - 1. \tag{2.13}$$

Therefore, as $n \to 0^+$, the smoothness of such solutions at the interfaces increases without bounds. Obviously, this is not the case for the FBP (a "positive obstacle" one) with standard conditions (1.15) and a usual quadratic ("parabolic") decay at the interfaces for any $n < \frac{3}{2}$, [22].

Lemma 2.1. *Let $u_\varepsilon(x,t)$ be the unique global solution of the CP for the regularized non-degenerate equation* (1.6), (1.8). *Then,*

$$\lim_{\varepsilon \to 0^+} \left\| u_\varepsilon(\cdot,t) - U(\cdot,t) \right\|_{L^2(\Omega \times (0,T))} = 0,$$

with $U(\cdot,t)$ a weak solution of the TFE–4 (1.1) *satisfying*

$$\int_0^T \int_\Omega \varphi_t U + \int_0^T \int_\Omega \nabla \varphi \cdot (|U|^n \nabla \Delta U) = 0 \quad \textit{for any} \quad \varphi \in C_0^\infty(\bar{\Omega} \times [0,T]).$$

Proof. Thus, as above and customary, multiplying (1.6) by a test function $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T])$ and integrating by parts in $\Omega \times [0,T]$ gives

$$-\int_0^T \int_\Omega \varphi_t u_\varepsilon - \int_0^T \int_\Omega \nabla \varphi \cdot (\phi_\varepsilon(u) \nabla \Delta u_\varepsilon) = 0.$$

Next, dealing with this equality and assuming $\phi_\varepsilon(u)$ of the form (1.8), we find

$$\int_0^T \int_\Omega \varphi_t u_\varepsilon + \varepsilon^n \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \Delta u_\varepsilon + (1-\varepsilon) \int_0^T \int_\Omega \nabla \varphi \cdot ((\varepsilon^2 + u^2)^{\frac{n}{2}} \nabla \Delta u_\varepsilon) = 0. \tag{2.14}$$

Applying Hölder's inequality, it is clear from (2.1) that there exists a subsequence labelled by $\{\varepsilon_k\}$, so that the second term of (2.14) approximates zero as $\varepsilon_k \downarrow 0^+$ for small $n > 0$,

$$\left| \varepsilon_k^n \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \Delta u_{\varepsilon_k} \right| \leq \varepsilon_k \Big(\varepsilon_k^{2(n-1)} \int_0^T \int_\Omega |\nabla \Delta u_{\varepsilon_k}|^2 \Big)^{\frac{1}{2}} \Big(\int_0^T \int_\Omega |\nabla \varphi|^2 \Big)^{\frac{1}{2}} \leq K \varepsilon_k \downarrow 0, \tag{2.15}$$

as $\varepsilon_k \downarrow 0^+$, for a positive constant $K > 0$.

[5]Or in the sense of partial limits along subsequences staying "sufficiently away" from less smooth transversal zeros concentrating at the interfaces.

Moreover, on the "good subsets" of uniform non-degeneracy

$$\mathcal{G}_{\varepsilon,\delta} := \{(x,t) \in \Omega \times [0,T]\, ;\, |u_\varepsilon(x,t)| > \delta > 0\} \tag{2.16}$$

for any fixed arbitrarily small $\delta > 0$, it is clear that the limiting solution as $\varepsilon \to 0^+$ is a weak solution of the TFE–4 (1.1).

Moreover, this represents a classical interior regularity result for parabolic equations on the sets of their uniform parabolicity. Indeed, by the regularity for the uniformly parabolic equation (1.6), we obtain that $u_{\varepsilon,t}$, ∇u_ε, and $\phi_\varepsilon(u)\nabla\Delta u_{\varepsilon,x}$ converge on compact subsets of $\mathcal{G} = \mathcal{G}_{0,0}$ (see below), at least in a weak sense.

In general, it is not that difficult to see that, as $\varepsilon = \varepsilon_k \to 0^+$ (along the lines of classic results in [5] and a number of later related others), we obtain a weak solution of the TFE–4, i.e.,

$$\int_0^T \int_{\mathcal{G}} \varphi_t U + \int_0^T \int_{\mathcal{G}} \nabla\varphi \cdot (|U|^n \nabla\Delta U) = 0, \tag{2.17}$$

where $U(x,t)$ is the limit obtained through (2.11), $\mathcal{G} = \mathcal{G}_{0,0}$ is associated with $U(x,t)$ (recall that it is smooth away from the nodal set by the interior parabolic regularity), and naturally assuming that $\varphi \in C_0^\infty(\mathcal{G})$.

However, in the "bad subsets"

$$\mathcal{B}_{\varepsilon,\delta} := \{(x,t) :\, |u_\varepsilon(x,t)| \le \delta\} \quad \text{for sufficiently small} \quad \delta \ge 0, \tag{2.18}$$

we must take $\varepsilon > 0$ also small enough and depending on δ. Indeed, let us fix an ε such that $0 < \varepsilon \le \delta$. Thus, applying Hölder's inequality to the third term in (2.14) over the set (2.18), where $|u_\varepsilon| \le \delta$, we have that

$$\begin{aligned} &\Big| \int_0^T \int_{\{|u_\varepsilon| \le \delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon) \Big| \\ &\quad \le \Big(\int_0^T \int_{\{|u_\varepsilon| \le \delta\}} |\nabla\varphi|^2 \Big)^{\frac{1}{2}} \Big(\int_0^T \int_{\{|u_\varepsilon| \le \delta\}} (1-\varepsilon)^2 (\varepsilon^2 + u_\varepsilon^2)^n |\nabla\Delta u_\varepsilon|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Then, since $\varphi \in C_0^\infty(\bar\Omega \times (0,\infty))$ and $\varepsilon \in (0,1)$, we get

$$\Big| \int_0^T \int_{\{|u_\varepsilon| \le \delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon) \Big| \le C \Big(\int_0^T \int_{\{|u_\varepsilon| \le \delta\}} (1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^n |\nabla\Delta u_\varepsilon|^2 \Big)^{\frac{1}{2}}$$

for a positive constant $C > 0$. Using that $|u_\varepsilon| \le \delta$, by (2.2), we find that

$$\begin{aligned} &\Big| \int_0^T \int_{\{|u_\varepsilon| \le \delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon) \Big| \\ &\quad \le C \Big(\int_0^T \int_{\{|u_\varepsilon| \le \delta\}} (1-\varepsilon)(\varepsilon^2 + \delta^2)^{\frac{n}{2}} (\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} |\nabla\Delta u_\varepsilon|^2 \Big)^{\frac{1}{2}}, \end{aligned}$$

and, hence, using (2.4),

$$\Big| \int_0^T \int_{\{|u_\varepsilon| \le \delta\}} \nabla\varphi \cdot ((1-\varepsilon)(\varepsilon^2 + u_\varepsilon^2)^{\frac{n}{2}} \nabla\Delta u_\varepsilon) \Big| \le C_1 \delta^{\frac{n}{2}} \to 0, \tag{2.19}$$

for a constant $C_1 > 0$, provided that $\delta \to 0$ as $\varepsilon \to 0^+$. This means that integration over "bad subsets" (2.18) leaves no trace in the final limit. So, the integral identity (2.17) for such a CP-solution U holds for arbitrary domains $\mathcal{G}$, so that U is a true weak solution of the CP (1.1). □

Remark. Although, we are able to prove the existence of a limit in a weak sense and the continuous deformation from the parabolic bi-harmonic equation (1.2) to the TFE–4 (1.1) through the regularized non-degenerate equation (1.6), (1.8), this also proves that a standard integral identity is not enough to specify directly the solutions of the CP. Indeed, due to free boundary conditions (1.15), the weak formulation (2.17) holds for any FBP-solution.

PASSAGE TO THE LIMIT $n \to 0^+$. In addition, the estimate (2.19) shows the actual rate of convergence if we perform a second "homotopy" limit as $n \to 0$, together with $\varepsilon \to 0$, in the analytic approximating flow (1.6), (1.8) in order to obtain, in this limit over (2.14), weak solutions of the bi-harmonic equation (1.2) and, hence, classical solutions by standard parabolic theory. For instance, to get such a convergence, one needs

$$\text{for} \quad \delta \sim \varepsilon, \quad n = n(\varepsilon) \to 0^+ \ \text{such that} \ \ \varepsilon^{\frac{n(\varepsilon)}{2}} \to 0^+. \tag{2.20}$$

Indeed, setting

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\frac{n(\varepsilon)}{2}} = 0, \quad \text{then} \quad \lim_{\varepsilon \to 0^+} n(\varepsilon) \ln \varepsilon = -\infty.$$

Then, we need the following:

$$n(\varepsilon) \ln \varepsilon \ll -1 \quad \Longrightarrow \quad n(\varepsilon) \gg \tfrac{1}{|\ln \varepsilon|},$$

which will provide us with the limit $n(\varepsilon)|\ln \varepsilon| \to +\infty$, and convergence of solutions, at least, in a weak sense.

However, this is not the end of the problem: indeed, under the condition (2.20) on the parameters, we definitely arrive, in the limit as ε, $n(\varepsilon) \to 0^+$, at a weak solution of the bi-harmonic equation (1.2) written in the following "mild" form:

$$\int_0^T \int_\Omega \varphi_t U + \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \Delta U = 0. \tag{2.21}$$

This is not a full (strongest) definition of weak solutions, since it assumes just a single integration by parts, so formally allows us to obtain, in the limit, a positive solution of the "obstacle" FBP for (1.2) with the corresponding conditions (1.15) (with $n = 0$), which can be constructed by a "singular" regularization as in [5]. Other solutions of different FBPs cannot be ruled out either.

Thus, it turns out that such a more or less standard passage to the limits in the integral identities *in principal cannot* recognize the desired difference between oscillatory solutions of the CP and others (possibly, non-negative or even positive ones) of the FBP posed for the TFE–4 (1.1). This emphasizes in general how hard a proper definition of the CP-setting can be. Therefore, a stronger version of homotopic limits is crucially necessary towards a proper identification of the CP-solutions.

Nevertheless, this first preliminary step in homotopy analysis declares some useful estimates and bounds on the parameters of regularization such as (2.20), which are necessary for a correct passing to the limit to get sign changing solutions of the linear bi-harmonic flow.

Moreover, even now, the homotopy concept as a connection to the linear PDE (1.6) can describe the origin (at $n = 0$) of the oscillatory solutions of TFEs and, hence, establish a transition to the *maximal regularity solutions* of the CP (1.1). Inevitably, bearing in mind the oscillatory character of the kernel $F(|y|)$ of the fundamental solution (1.4), the proper solutions of the CP are going to be oscillatory near finite interfaces, at least, for all small $n > 0$.

3. Branching through homotopy (1.12) as $n \to 0^+$ and $\varepsilon \to 0^+$

Now, performing a double limit as $n \to 0^+$ and $\varepsilon \to 0^+$, we ascertain the well-posedness of the TFE–4 (1.1) via a homotopy deformation from the solutions of the *bi-harmonic equation* (1.2), which are known to be oscillatory, to the solutions of the TFE–4 (1.1). This will provide us with one of the main results of this paper.

As a key example, consider the regularized CP (1.6), (1.12), with the same data:

$$u_\varepsilon : \quad u_t = -\nabla \cdot ((\varepsilon^2 + u^2)^{\frac{n}{2}} \nabla \Delta u), \quad u(x,0) = u_0(x), \tag{3.1}$$

where smooth bounded data $u_0(x)$ are compactly supported, or, at least, has exponential decay at infinity. Then, quite naturally, due to local parabolic properties, we may assume that, on any bounded fixed interval $t \in [0,T]$ and for any small $\varepsilon > 0$,

$$u_\varepsilon(x,t) \quad \text{has exponential decay as } x \to \infty. \tag{3.2}$$

Moreover, we may also suppose that $u_\varepsilon(x,t)$ has a natural spatial form of a "hump" concentrated near the origin on the given fixed time-interval $t \in [0,T]$ (in other words, for such a uniformly parabolic problem, there is no fast propagation at $x = \infty$, since parabolic flows do not allow this by interior regularity theory).

Contrary to what we did above we now choose

$$n \to 0^+,$$

as the main deformation parameter, and, for simplicity, will properly choose

$$\varepsilon = \varepsilon(n) \to 0.$$

Thus, we perform a passing to a single limit, though similar estimates can be obtained within a double (or even triple if necessary) limit strategy.

To this end, we rewrite this PDE in a perturbed version,

$$u_t = -\Delta^2 u + \nabla \cdot \big([1 - (\varepsilon^2 + u^2)^{\frac{n}{2}}] \nabla \Delta u\big), \tag{3.3}$$

and next, we use its equivalent integral form:

$$\begin{gathered} u_\varepsilon : \quad u(t) = b(t) * u_0 + \int\limits_0^t \nabla b(t-s) * F_{n,\varepsilon}(u(s)) \nabla \Delta u(s)\, {\mathrm d}s, \\ \text{where} \quad F_{n,\varepsilon}(u) := 1 - (\varepsilon^2 + u^2)^{\frac{n}{2}}. \end{gathered} \tag{3.4}$$

The convergence to the well-posed bi-harmonic solutions will crucially depend on the weak limit in the second perturbation term in (3.4):

$$F_{n,\varepsilon}(u) \equiv 1 - (\varepsilon^2 + u^2)^{\frac{n}{2}} \rightharpoonup 0 \quad \text{as} \quad n,\, \varepsilon(n) \to 0^+. \tag{3.5}$$

In other words, we need to clarify conditions, under which there occurs branching of proper CP-solutions of the TFE–4 at $n = 0^+$ from regular bounded classical solutions of the linear problem (1.2). Recall that this requires branching theory for a quasilinear degenerate *partial* differential equation, where a good functional setting will be key.

<u>Key assumption</u>. To perform our branching analysis, one needs to verify the following expansion:

$$F_{n,\varepsilon}(u) \equiv 1 - (\varepsilon^2 + u^2)^{\frac{n}{2}} = -\tfrac{n}{2} \ln(\varepsilon^2 + u^2)(1 + o(1)) \quad \text{as} \quad n \to 0^+ \tag{3.6}$$

on a fixed family $\{u_\varepsilon(x,t)\}$ of uniformly bounded and smooth solutions. Checking (3.6) on bad sets (2.18), i.e., for $u \approx 0$, yields the demand (cf. (2.20) if $\varepsilon \ll \mathrm{e}^{-\frac{1}{n}}$)

$$\boxed{n\,|\ln \varepsilon(n)| \to 0 \quad \text{as} \quad n \to 0^+.} \tag{3.7}$$

This is the key assumption on the regularization parameter $\varepsilon = \varepsilon(n) > 0$ and its relation with the main parameter n.

3.1. **Proof of Theorem 1.1.** Under condition (3.7), we perform a rather standard branching analysis which mimics that in the elliptic case shown in [1]. Actually, the integral equation (3.4) provides a nice opportunity to get a direct n-expansion of the solution to guarantee branching at $n = 0^+$.

Hence, substituting (3.6) into (3.4), we have that

$$u(t) = b(t) * u_0 - \tfrac{n}{2} \int_0^t \nabla b(t-s) * \ln(\varepsilon^2 + u^2) \nabla \Delta u(s)\,\mathrm{d}s + o(n) \quad \text{as} \quad n \to 0^+. \tag{3.8}$$

Next, in order to derive the actual rate of convergence to ascertain such a branching behaviour for small $n > 0$, we assume the following splitting expression:

$$u = \tilde u + V, \tag{3.9}$$

where $\tilde u$ is the unique solution (1.3) of the parabolic bi-harmonic equation (1.2) with the same initial data and with V being a small perturbation. Thus, setting

$$V := n\varphi_1 + o(n), \tag{3.10}$$

and substituting (3.9) into (3.8), omitting the terms $o(n)$ when necessary, yields

$$\begin{aligned} \tilde u(t) + n\varphi_1(t) = b(t) * u_0 \\ &- \tfrac{n}{2} \int_0^t \nabla b(t-s) * \ln\left(\varepsilon^2 + \tilde u^2(s) + 2n\tilde u(s)\varphi_1(s) + n^2\varphi_1^2(s)\right) \nabla \Delta \tilde u(s)\,\mathrm{d}s \\ &- \tfrac{n^2}{2} \int_0^t \nabla b(t-s) * \ln\left(\varepsilon^2 + \tilde u^2(s) + 2n\tilde u(s)\varphi_1(s) + n^2\varphi_1^2(s)\right) \nabla \Delta \varphi_1(s)\,\mathrm{d}s. \end{aligned}$$

Furthermore, by the convolution operation properties, we find that

$$\begin{aligned} \tilde u(t) + n\varphi_1(t) =&\, b(t) * u_0 \\ &- \int_0^t \nabla b(t-s) * \tfrac{n}{2} \ln(\varepsilon^2 + \tilde u^2(s) + 2n\tilde u(s)\varphi_1(s) + n^2\varphi_1^2(s)) \nabla \Delta \tilde u(s)\,\mathrm{d}s. \end{aligned}$$

Overall, this gives the following corrections in the expansion (3.9), (3.10):

$$\varphi_1(t) = -\textstyle\int_0^t \nabla b(t-s) * \ln|\tilde u(s)| \nabla \Delta \tilde u(s)\,\mathrm{d}s. \tag{3.11}$$

Thus, the above asymptotic expansion technique must assume that the convolution in (3.11) is always finite, e.g., that

$$\ln|\tilde u(y,s)| \in L^1_{\mathrm{loc}}(\mathbb{R}^N),$$

for any $s > 0$, i.e., $\tilde u(y,s)$ does not have zeros with an exponential decay in a neighbourhood. In particular, this is true if the solutions (1.3) have *transversal* zeros a.e. (this always happens in the radial geometry, for instance). The latter is rather plausible, but difficult to prove for general solutions of the bi-harmonic equation (1.2), so we need to include such an assumption.

Subsequently:

(i) We need to prove that the perturbation in (3.4) is asymptotically small. Obviously, this is guaranteed by the uniform estimate in Proposition 2.1 (iii) in the domain $\{|u| \ge z_*\}$, where z_* is such that

$$|\ln z| \le z^{\frac{n}{2}} \quad \text{for} \quad z \ge z_*. \tag{3.12}$$

Consider now the double integral in (3.4) over the domain, where

$$\{\varepsilon^2 \le \varepsilon^2 + u^2 \le z_*\}.$$

Then, the maximal singularity of the term $\ln(\varepsilon^2 + u^2)$ therein is achieved at $u = 0$ and is $\sim \ln \varepsilon$. Concerning the third derivative therein, quite similarly, as in the above case, we obtain from the same estimate of the Proposition 2.1(iii) that

$$\textstyle\iint |\nabla \Delta u_\varepsilon|^2 \le K \varepsilon^{-n}, \tag{3.13}$$

i.e., overall, the second perturbation term in (3.4) has the order, at most,

$$O(n\varepsilon^{-n}) \to 0 \quad \text{as} \quad n \to 0^+. \tag{3.14}$$

Indeed, by (3.7), $\varepsilon^{-n}(n) \to 1$, so that (3.14) holds.

(ii) As we have shown, the improved expansion (3.10) requires convergence of (3.11), as $n \to 0^+$, which is difficult to verify for sufficiently arbitrary solutions of the bi-harmonic equation.

It is worth mentioning that, thanks to the x- and t-analyticity of these solutions for $t \in (0, T]$, divergence of such integrals (if any), i.e., violation of such a rate of divergence due to formation of "flat" multiple zeros of $\tilde u(x, t)$, can occur at a finite number of points $(x, t) \in \mathbb{R}^N \times (0, T]$, so that this expansion is expected to hold, at least, in the a.e. sense for sufficiently arbitrary initial data. $\square$

Conclusion. Simultaneously, the expansion (3.9), (3.10) presents a proper definition of the CP-solutions of the TFE–4 (1.1):

> *solutions of the Cauchy problem for* (1.1) *are those, which can be deformed as $n \to 0^+$ via the analytic path* (3.1), (3.7) *to the unique solutions* (1.3) *of the bi-harmonic equation* (1.2) *with the same data.*

Inevitably, for small $n > 0$, such correct and well defined CP-solutions must inherit oscillatory and other sign-changing properties of this linear flow.

In other words, according to a homotopy approach, a proper setting of the CP for the TFE–4 (1.1) requires the whole set of solutions

$$\{u(x, t),\, n > 0\},$$

or, hence, a full 2D set

$$\{u_\varepsilon(x, t),\, \varepsilon > 0,\, n > 0\},$$

of regularized solutions. Therefore, a successful definition of the CP for an *individual* TFE–4 for a *fixed* value $n > 0$ may not be efficient or available.

4. Towards uniqueness: boundary layer and Riemann's problems as $\varepsilon \to 0^+$

We continue to study the regularized problem (3.1), and now we concentrate on the crucial limit (1.10). As we have mentioned, a *conventional* (i.e., associated with the regularization (1.12) currently applied) uniqueness naturally requires that

$$\limsup u_\varepsilon(x,t) = \liminf u_\varepsilon(x,t) = u(x,t) \quad \text{as} \quad \varepsilon \to 0^+. \tag{4.1}$$

Actually, this demands knowing that no essential ε-oscillations of $u_\varepsilon(x,t)$ occur in the "bad sets" (2.18), which are close to the nodal set of the limit solution $u(x,t)$. Indeed, if the behaviour of u_ε as $\varepsilon \to 0^+$ is non-oscillatory enough in these singular sets, then such oscillations cannot occur in principle in any good set of uniform positivity of solutions, where a strong interior parabolic regularity is in charge.

Therefore, proving the uniqueness (4.1) assumes a deeper understanding, than before, of *boundary layers* that occur close to nodal sets of the solutions, and mainly close to interfaces, where the strongest *singularity via the equation degeneracy* occurs. This leads to very difficult asymptotic questions, which are naturally referred to as to *Riemann's problems* for the TFE–4, each of which is associated with a prescribed type of singularity at the interface. We now present a short discussion of one such a Riemann problem.

4.1. First (artificial) Riemann's problem.

The regularized equation (3.1) has the clear advantage that ε can be scaled out by a number of scaling transformations:

$$u_\varepsilon(x,t) = \varepsilon v_\varepsilon(y,\tau), \quad \text{where} \quad y = \tfrac{x}{\varepsilon^\alpha}, \quad \tau = \tfrac{t}{\varepsilon^\beta}, \;\; \beta = 4\alpha - n, \tag{4.2}$$

and $\alpha \in \mathbb{R}$ is an arbitrary exponent. Then $v = v_\varepsilon$ solves an ε-independent equation with data $v_{0\varepsilon}$ now depending on ε:

$$v_\varepsilon: \quad v_\tau = -\nabla_y \cdot ((1+v^2)^{\frac{n}{2}} \nabla_y \Delta_y v), \quad v_{0\varepsilon}(y) = \tfrac{1}{\varepsilon}\, u_0(\varepsilon^\alpha y). \tag{4.3}$$

Consider a particular simpler case $\alpha = \frac{n}{4}$, for which $\beta = 0$ in (4.2), i.e., the time variable is not under scaling. Then we have the following problem:

$$u_\varepsilon(x,t) = \varepsilon v_\varepsilon\big(\tfrac{x}{\varepsilon^{n/4}}, t\big), \;\; v_t = -\nabla \cdot ((1+v^2)^{\frac{n}{2}} \nabla \Delta v), \quad v_{0\varepsilon}(y) = \tfrac{1}{\varepsilon}\, u_0(\varepsilon^{\frac{n}{4}} y). \tag{4.4}$$

To formulate a particular Riemann problem, let us assume that $0 \in \partial\, \mathrm{supp}\, u_0$ and

$$u_0(x) = \chi\big(\tfrac{x}{|x|}\big) |x|^{\frac{4}{n}} (1 + o(1)) \quad \text{as} \quad x \to 0, \tag{4.5}$$

where $\chi(s) \not\equiv 0$ is a continuous function on the unit sphere. In other words, (4.5) fixes a necessary type of degeneracy (vanishing) of initial data close to the origin being an interface point at the initial moment $t = 0$.

Later on, such a Riemann problem must be attached to *every* point $x_0 \in \partial \mathrm{supp}\, u_0$ of the initial interface, by changing $x \mapsto x - x_0$ in (4.5), or with other data asymptotics, where other scaling transformation from (4.2) (with a different α) can be applied.

Thus, for data as in (4.5), after scaling as in (4.4), we obtain the following rescaled initial data: uniformly on compact subsets in y,

$$v_{0\varepsilon}(y) = \chi\big(\tfrac{y}{|y|}\big) |y|^{\frac{4}{n}} (1 + o(1)) \to \chi\big(\tfrac{y}{|y|}\big) |y|^{\frac{4}{n}} \equiv v_0(y) \quad \text{as} \quad \varepsilon \to 0, \tag{4.6}$$

i.e., there exists a locally finite limit initial data. This gives *Riemann's problem* as a first approximation of a proper solution:

$$v_t = -\nabla \cdot ((1+v^2)^{\frac{n}{2}} \nabla \Delta v), \quad v_0(y) = \chi\big(\tfrac{y}{|y|}\big) |y|^{\frac{4}{n}}. \tag{4.7}$$

As usual, the first question is to check whether Riemann's problem (4.7) is globally well-posed or admits a singularity. It turns out that the latter holds:

Proposition 4.1. *In general, Riemann's problem* (4.7) *admits blow-up in finite time.*

Proof. To see this, we perform some easy estimates. Namely, as is well known, such a kind of blow-up is due to a fast growth of data in (4.7) as $y \to \infty$, where v also gets arbitrarily large and hence the PDE can be approximated by the initial TFE–4,

$$v_t = -\nabla \cdot (|v|^n \nabla \Delta v). \tag{4.8}$$

Looking for separate variable solutions of (4.8) yields

$$v(y,t) = \psi(t)\rho(y) \quad \Longrightarrow \quad \psi' = \psi^{n+1}, \quad \rho = -\nabla \cdot (|\rho|^n \nabla \Delta \rho). \tag{4.9}$$

The first ODE for $\psi > 0$ yields a typical blow-up behaviour,

$$\psi(t) = [n(T-t)]^{-\frac{1}{n}} \to \infty \quad \text{as} \quad t \to T^-, \tag{4.10}$$

where $T > 0$ is a finite blow-up time. Dealing with the second nonlinear elliptic equation in (4.9), it is easy to see that such an equality assumes the following algebraic balance:

$$\rho(y) \sim |y|^{\frac{4}{n}} \quad \text{as} \quad y \to \infty, \tag{4.11}$$

i.e., the same growth as data $v_0(y)$ in (4.7). Moreover, the equation for ρ admits an explicit radial solution of this form,

$$\rho(y) = C_* |y|^{\frac{4}{n}} \quad \text{in} \quad \mathbb{R}^N, \quad \text{where} \quad C_* = C_*(n,N) = \text{const.} > 0. \tag{4.12}$$

Of course, some special changing sign behaviour of the angular function $\chi(s)$ in data v_0 can result in global solvability of Riemann's problem (4.7). However, since the data is assumed to be arbitrary, we cannot rely on this. $\square$

The blow-up result in Proposition 4.1 is precisely the sign of possible *strong oscillations* that may occur as $\varepsilon \to 0$ close to such a degeneracy point. Note that, unlike second-order heat equations, blow-up in higher-order parabolic equations naturally means occurring infinitely many sign changes as $t \to T^-$, i.e., solutions get infinitely oscillatory close to blow-up time; see [20] and [9] as two examples. In this case, (4.1) may become non-achievable.

On one hand, this discrepancy does not mean the actual blow-up of the corresponding regularizing family $\{u_\varepsilon\}$ as $\varepsilon \to 0$. This example for special (and somehow artificial) data (4.5) just shows that such an independent asymptotic analysis, such as via (4.7), of the boundary layer is not allowed, as well as a full study of the ε-dependent problem (4.4) (including a careful control of the behaviour at $y \to \infty$ as $\varepsilon \to 0$, to avoid blow-up and essential oscillations).

On the other hand, we cannot exclude a possibility of the actual existence of non-small ε-oscillations of $u_\varepsilon(x,t)$ in a neighbourhood of the interface point for data (4.5) *provided that such a behaviour remains for some, possibly, extremely small time intervals.* However, and fortunately, this is not the case, and the generic behaviour near interfaces has different scaling laws. Then, the above blow-up solutions just describe a fast transition from data (4.5) to a solution $u(x,t)$ for arbitrarily small $t > 0$, which already gains the generic oscillatory interface behaviour to be described next.

4.2. **ε-regularization is stable near generic oscillatory interfaces.** As we have seen, the well-posedness of the limit problem at $\varepsilon = 0^+$ crucially depends on the asymptotics of the solution $u(x,t)$ close to the interface point.

Let us show that, for typical oscillatory sign-changing behaviour of solutions of the TFE–4, their ε-regularizations do not lead to blow-up.

We restrict our analysis to $N = 1$ and assume that, at $t = 0$, a solution profile $u(x, 0)$ of (1.1) already has a generic form close to $x = 0^+$ being its interface point. Namely, according to [13, § 7], as $x \to 0^+$,

$$u(x,0) \sim x^{\mu}[\varphi_*(s) + o(1)], \quad \text{where} \quad \mu = \tfrac{3}{n}, \quad s = \ln x, \tag{4.13}$$

and the oscillatory component $\varphi_*(s)$ is a periodic solution of the following ODE:

$$\varphi''' + 3(\mu - 1)\varphi'' + (3\mu^2 - 6\mu + 2)\varphi' + \mu(\mu - 1)(\mu - 2)\varphi + |\varphi|^{-n}\varphi = 0. \tag{4.14}$$

Existence of such a periodic function $\varphi_*(s)$ is known for $n > 0$ for some open intervals (see [13, § 7] and more estimates in [18])

$$n \in (0, n_{\rm h}), \quad \text{where} \quad n_{\rm h} = 1.7587... < n_+ = \tfrac{9}{3+\sqrt{3}} = 1.902...\,, \tag{4.15}$$

where n_+ is given by the maximal root $\mu_+ = \frac{3}{n_+}$ of the quadratic equation (see the third coefficient in the differential operator in (4.14))

$$3\mu^2 - 6\mu + 2 = 0 \quad \Longrightarrow \quad \mu_\pm = \tfrac{3 \pm \sqrt{3}}{3}.$$

Uniqueness of the periodic $\varphi_*(s)$ remains an open problem. It is worth mentioning that a huge number of numerics of the ODE (4.14) always confirmed that such a periodic solution is unique for all $n \in (0, n_{\rm h})$ and it is globally stable as $s \to +\infty$, i.e., *away* from the interface fixed at $y = 0^+$. It is important that both of these "good" properties remain valid up to a heteroclinic bifurcation at $n = n_{\rm h}$, at which such a $\varphi_*(s)$ is deformed into a heteroclinic connection of two constant equilibria,

$$\varphi_\pm = \pm\big[\mu_{\rm h}(\mu_{\rm h} - 1)(2 - \mu_{\rm h})\big]^{\frac{1}{n_{\rm h}}}, \quad \text{where} \quad \mu_{\rm h} = \tfrac{3}{n_{\rm h}} \in (1, 2)\,.$$

We must admit that (4.13) has been obtained from a nonlinear ODE for source-type similarity solutions of the TFE–4 in 1D. However, as customary, we expect that key evolution properties of this "fundamental" solution remain valid for more general solutions of the PDE including the asymptotics (4.13) close to the interface. Proving such a generic behaviour for the TFE–4 represents a very difficult open problem.

It follows from rescaled data in (4.3) that, for data from (4.13), one requires

$$\alpha = \tfrac{n}{3} \quad \text{and} \quad \beta = \tfrac{n}{3} \tag{4.16}$$

in the group of scaling transformations (4.2). Overall, this gives the following rescaled data for v_ε:

$$v_{0\varepsilon}(y) \sim y^{\frac{3}{n}}\big[\varphi_*\big(\tfrac{n}{3}\,\ln\varepsilon + \ln y\big) + o(1)\big]. \tag{4.17}$$

Therefore, as $\varepsilon \to 0^+$, the data becomes strongly oscillatory in view of fast phase changes in $\varphi_*(\cdot)$. But what is more important, the data remains uniformly bounded above by a slower, than in (4.6), power law:

$$|v_{0\varepsilon}(y)| \le \text{const.}\, |y|^{\frac{3}{n}} \quad \text{in} \quad \mathbb{R}. \tag{4.18}$$

It is not difficult to show that the problem for the rescaled equation in (4.7) with data as in (4.18) does not admit any blow-up, and the solution remains bounded on any compact subset. It is expected that this allows one to match this regular boundary layer with the outer expansion for $y \gg 1$ to get key features of solutions $v_\varepsilon(y,\tau)$ and hence of $u_\varepsilon(x,t)$ to see whether (4.1) actually holds. This is a general strategy towards uniqueness, and, clearly, much deeper and harder asymptotic analysis is required here to conclude.

Overall, this somehow shows that generic oscillatory behaviour of source-type is *stable* under analytic ε-regularizations, which then cannot lead to any non-uniqueness. However, this is just a 1D illustration of the regularization procedure. We still expect that there may exist special complicated configurations of data in $\mathbb{R}^N$ leading to some kind of "self-focusing" on interfaces implying to non-vanishing ε-oscillations as $\varepsilon \to 0$ close to interfaces and hence yielding non-uniqueness. In general, we expect that, for such higher-order parabolic (and other) strongly degenerate equations, any global concept of uniqueness cannot be a key issue in general PDE theory. Moreover, for such non-fully-divergent quasilinear degenerate PDEs, an (absolute) uniqueness can be non-achievable in principle.

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