

## The kissing polynomials and their Hankel determinants

ANDREW F. CELSUS

5905 Hempstead Dr., Plano, TX 75093, USA

ALFREDO DEAÑO\*

Department of Mathematics, Universidad Carlos III de Madrid, Avda. de la universidad 30, 28911  
Leganés, Madrid, Spain

\*Corresponding author: [alfredo.deanho@uc3m.es](mailto:alfredo.deanho@uc3m.es)

DAAN HUYBRECHS

Department of Computer Science, KU Leuven, Celestijnenlaan 200A, BE-3001 Leuven, Belgium

AND

ARIEH ISERLES

DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Rd, Cambridge CB3  
0WA, UK

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In this paper, we investigate algebraic, differential and asymptotic properties of polynomials  $p_n(x)$  that are orthogonal with respect to the complex oscillatory weight  $w(x) = e^{i\omega x}$  on the interval  $[-1, 1]$ , where  $\omega > 0$ . We also investigate related quantities such as Hankel determinants and recurrence coefficients. We prove existence of the polynomials  $p_{2n}(x)$  for all values of  $\omega \in \mathbb{R}$ , as well as degeneracy of  $p_{2n+1}(x)$  at certain values of  $\omega$  (called *kissing points*). We obtain detailed asymptotic information as  $\omega \rightarrow \infty$ , using recent theory of multivariate highly oscillatory integrals, and we complete the analysis with the study of complex zeros of Hankel determinants, using the large  $\omega$  asymptotics obtained before.

**Keywords:** orthogonal polynomials, asymptotic approximation in the complex domain, numerical analysis, Hankel determinants.

### 1. The kissing polynomials

#### 1.1 Introduction and motivation

In this paper, we are concerned with orthogonal polynomials (OPs) with respect to the complex weight function  $w(z) = e^{i\omega z}$  on the interval  $[-1, 1]$ . Such polynomials have been recently investigated in the context of quadrature of highly oscillatory integrals, see for instance [Deaño \*et al.\* \(2017, Chapter 6\)](#) and the references therein, but the relevance of this work is broader; in particular, once, we consider more general complex-valued weight functions. The extension of standard theory of polynomials orthogonal with respect to real-valued Borel measures to the complex realm is far from straightforward and many familiar features are lost. As a rough rule of thumb, we may divide features of OPs into algebraic and analytic ones. Algebraic features, e.g. the existence of a three-term recurrence relation, do not depend on the weight function being real, but not-so analytic features. Obviously, we can no longer expect zeros to be real and to interlace but, more worryingly, the very *existence* of OPs is an analytic feature, being equivalent to Hankel determinants (in general, transcendental expressions) being nonzero. Thus, OPs

with respect to complex weight functions present a formidable challenge and call for a new arsenal of tools and techniques. This paper represents initial inroads into this fascinating subject.

In the context of quadrature of complex-valued highly oscillatory integrals, the problem is how to approximate efficiently the value of integrals of the form

$$I[f] = \int_{-1}^1 f(x)e^{i\omega x} dx, \quad \omega \gg 1, \quad (1.1)$$

where for simplicity, we assume that  $f(x)$  is an analytic function. The standard approach of directly applying numerical quadrature (e.g. Gaussian) to (1.1) is known to be highly inefficient when the oscillatory parameter  $\omega$  is large because, in order to attain good accuracy, the number of Gaussian quadrature nodes needs to scale like  $\mathcal{O}(\omega)$ .

Gaussian quadrature on the real line is a cornerstone in the numerical analysis of integrals. If we have

$$I[f] = \int_a^b f(x)w(x) dx, \quad (1.2)$$

where  $(a, b) \subseteq \mathbb{R}$  and the weight function  $w(x)$  is positive, a quadrature rule takes the form

$$Q[f] = \sum_{k=1}^n w_k f(x_k), \quad (1.3)$$

with quadrature nodes  $\{x_k\}_{k=1}^n$  and weights  $\{w_k\}_{k=1}^n$ ; if we use  $n$  nodes and weights, the quadrature rule  $Q[f]$  is called *Gaussian* if it is exact for polynomials of degree up to (and including)  $2n - 1$ , which is actually the optimal case. In this case, the quadrature rule makes use of OPs  $\{p_n(x)\}_{n \geq 0}$  with respect to  $w(x)$ , namely the quadrature nodes are chosen as the zeros of the  $n$ -th OP  $p_n(x)$  and the weights can similarly be constructed in terms of the OPs; see, e.g. [Gautschi \(2004\)](#) for more details.

The family of OPs constitutes a basis of the Hilbert space  $L^2([a, b], w(x))$ , with the standard inner product given by

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x) dx. \quad (1.4)$$

These OPs satisfy many important properties, among which we highlight the following.

- The polynomial  $p_n(x)$  is of degree exactly equal to  $n$  for all  $n \geq 0$ .
- The following orthogonality property is satisfied:

$$\int_a^b p_n(x)x^k w(x) dx = \begin{cases} 0, & k = 0, 1, \dots, n-1, \\ \chi_n > 0, & k = n. \end{cases}$$

- The zeros of  $p_n(x)$  are simple and they are located in the interval  $(a, b)$ .

- The OPs satisfy a *three-term recurrence relation* of the form

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x), \quad (1.5)$$

with initial data  $p_{-1}(x) = 0$ ,  $p_0(x) = 1$  and coefficients  $\alpha_n$  and  $\beta_n$  that can be written in terms of the inner product (1.4):

$$\alpha_n = \frac{\langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}, \quad n \geq 0, \quad \beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle} = \frac{\chi_n}{\chi_{n-1}}, \quad n \geq 1. \quad (1.6)$$

We refer the reader to the monographs Chihara (1978), Ismail (2005) and Szegő (1939) for these and other properties of OPs.

If we apply these ideas directly to (1.1), then the resulting bilinear form is

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)e^{i\omega x} dx, \quad (1.7)$$

and the corresponding monic OPs  $p_n^\omega(x)$  are defined as

$$\int_{-1}^1 p_n^\omega(x)x^k e^{i\omega x} dx = \begin{cases} 0, & k = 0, 1, \dots, n-1, \\ \chi_n(\omega), & k = n, \end{cases} \quad (1.8)$$

where  $n \in \mathbb{Z}_+$  and  $\omega \geq 0$ . For simplicity of notation, in the sequel, we omit the parameter  $\omega$  and write  $p_n(x)$  directly.

An important observation is that the bilinear form (1.7) is not an inner product because the weight function is no longer positive, and therefore it may happen that  $\langle f, f \rangle = 0$  for a non-zero function  $f(x)$  and the orthogonalization procedure to compute the OPs can potentially break down. Nevertheless, if such family of OPs exists, the complex Gaussian quadrature rule

$$\int_{-1}^1 f(x)e^{i\omega x} dx \approx \sum_{k=1}^n w_k f(x_k), \quad (1.9)$$

with complex nodes  $\{x_k\}_{k=1}^n$  and weights  $\{w_k\}_{k=1}^n$  inherits the optimal polynomial order of Gaussian quadrature that we mentioned above, and the optimal polynomial order translates into optimal asymptotic order in terms of the oscillatory parameter  $\omega$ :

$$\int_{-1}^1 f(x)e^{i\omega x} dx - \sum_{k=1}^n w_k f(x_k) = \mathcal{O}(\omega^{-2n-1}), \quad \omega \rightarrow \infty. \quad (1.10)$$

This property makes this construction attractive for numerical purposes, in particular for large values of  $\omega$ , a regime where standard quadrature of the oscillatory integral is problematic using classical methods, as observed before. We refer the reader to Deaño & Huybrechs (2009) for further details on these ideas, as well as the papers Deaño *et al.* (2010), Huybrechs *et al.* (2014) and the PhD thesis of Lejon (2016) for

other examples involving oscillatory integrals with stationary points, as well as the general monograph Deaño *et al.* (2017).

Since (1.7) involves non-Hermitian orthogonality and therefore satisfies  $\langle xf, g \rangle = \langle f, xg \rangle$ , the monic OPs satisfy a three-term recurrence relation analogous to (1.5): provided that  $p_n(x)$  and  $p_{n\pm 1}(x)$  exist for given  $n$  and  $\omega$ , we have

$$p_{n+1}(x) = [x - \alpha_n(\omega)]p_n(x) - \beta_n(\omega)p_{n-1}(x). \quad (1.11)$$

The initial values are taken as  $p_{-1}(x) = 0$ ,  $p_0(x) = 1$  and the coefficients  $\alpha_n(\omega)$  and  $\beta_n(\omega)$ , which are in general complex valued, can be written as

$$\alpha_n(\omega) = \frac{\langle xp_n, p_n \rangle}{\langle p_n, p_n \rangle}, \quad \beta_n(\omega) = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle} = \frac{\chi_n(\omega)}{\chi_{n-1}(\omega)}, \quad (1.12)$$

in terms of the bilinear form (1.7).

The location of the zeros of kissing polynomials is a central topic of this paper, and it is a nontrivial issue, even in the case when  $p_n(x)$  exists, since the classical result that the zeros are contained in the interval where orthogonality is defined no longer holds. In this direction, the zero distribution of complex OPs, in particular in the limit  $n \rightarrow \infty$ , has already been investigated for a few decades, with applications to rational approximation of functions in the complex plane and solutions of Painlevé equations, for example. The theory was developed by Gonchar & Rakhmanov (1987) for OPs with varying weights on the real line, where  $w(x) = w_n(x) = e^{-nV_n(x)}$ , using the essential ingredient of the equilibrium measure in the presence of an external field, given by  $V(x) = \lim_{n \rightarrow \infty} V_n(x)$ . This framework was later expanded for complex-valued weight functions and complex OPs in Stahl (1986), Rakhmanov (2011), Martínez-Finkelshtein & Rakhmanov (2016) and Kuijlaars & Silva (2015), among others. In this scenario, an essential part of the problem is the location of the specific curve, among all possible smooth deformations of the original contour, that attracts the zeros of the OPs as the degree gets large; this leads to the crucial concept of *S curve*, which is distinguished by a precise symmetry property that involves both the logarithmic potential of equilibrium measures and the external field  $V(x)$ . Recent extensions of this methodology include multiple OPs as well, which are connected to Hermite–Padé approximation.

## 1.2 Existence and the kissing pattern

The fact that in the present situation the weight function  $w(x) = e^{i\omega x}$  is not positive on the interval where orthogonality is defined has two important consequences that we will investigate in the rest of the paper.

In the first place, there is the question of existence of the OPs, which can be analysed recalling the fact that  $p_n(x)$  can be written in terms of the associated *Hankel determinants*: we construct the following  $n$ th *Hankel matrix*:

$$H_n(\omega) = \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \dots & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \dots & \mu_{n+1}(\omega) \\ \vdots & \vdots & & \vdots \\ \mu_n(\omega) & \mu_{n+1}(\omega) & \dots & \mu_{2n}(\omega) \end{bmatrix}, \quad n \geq 0, \quad (1.13)$$

where

$$\mu_m(\omega) = \int_{-1}^1 z^m e^{i\omega z} dz, \quad m \in \mathbb{Z}_+ \quad (1.14)$$

are the *moments* of the weight function, and we define the *Hankel determinant*

$$h_n(\omega) = \det H_n(\omega). \quad (1.15)$$

It is clear from the previous equations that  $h_n(\omega)$  is an analytic function of  $\omega$  in the complex plane.

The polynomial  $p_n(x)$  can then be written as follows:

$$p_n(x) = \frac{1}{h_{n-1}(\omega)} \det \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \cdots & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \cdots & \mu_{n+1}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(\omega) & \mu_n(\omega) & \cdots & \mu_{2n-1}(\omega) \\ 1 & x & \cdots & x^n \end{bmatrix}; \quad (1.16)$$

see for instance [Ismail \(2005, Chapter 2\)](#).

It follows directly from (1.16) that if  $h_{n-1}(\omega) = 0$ , then  $p_n(x)$  is not defined. For example, by direct calculation from (1.14), we have

$$\mu_0(\omega) = \frac{2 \sin \omega}{\omega} = h_0(\omega),$$

and therefore if  $\omega = k\pi$ , with  $k = 1, 2, \dots$ , then  $h_0(\omega) = 0$  and therefore  $p_1(x)$  is not defined. A few more cases are worked out explicitly in [Asheim et al. \(2014\)](#).

Secondly, even when the existence of  $p_n(x)$  is assured for some values of  $n$  and  $\omega$ , its roots lie in the complex plane. When  $\omega = 0$ ,  $p_n(x)$  is a multiple of the classical Legendre polynomial and the roots are real and in the interval  $(-1, 1)$ . For  $\omega > 0$ , we plot the trajectories of the zeros of  $p_n(x)$  in Figs 1 and 2; these zeros are symmetric with respect to the imaginary axis because

$$p_n(z) = (-1)^n \overline{p_n(-\bar{z})}, \quad z \in \mathbb{C}; \quad (1.17)$$

see [Asheim et al. \(2014\)](#) (writing the variable as  $z$  instead of  $x$ ), and the map  $z \mapsto -\bar{z}$  represents a reflection with respect to the imaginary axis. As we can see, the trajectories of the roots (as functions of  $\omega$ ) corresponding to polynomials of consecutive even and odd degree touch at a discrete set of frequencies  $\omega$ : the zeros of the polynomials *kiss* and this phenomenon motivates their name.

This kissing pattern, which consists of coincidence of roots of two consecutive polynomials  $p_{2N}(z)$  and  $p_{2N+1}(z)$ , results from degeneracy in the degree of the polynomials at certain critical values of  $\omega$ . Bearing in mind formula (1.16), it comes as no surprise that these critical values of  $\omega$  correspond exactly to the zeros of the Hankel determinant  $h_{n-1}(\omega)$ . This motivates our detailed study of these Hankel determinants in the sequel.

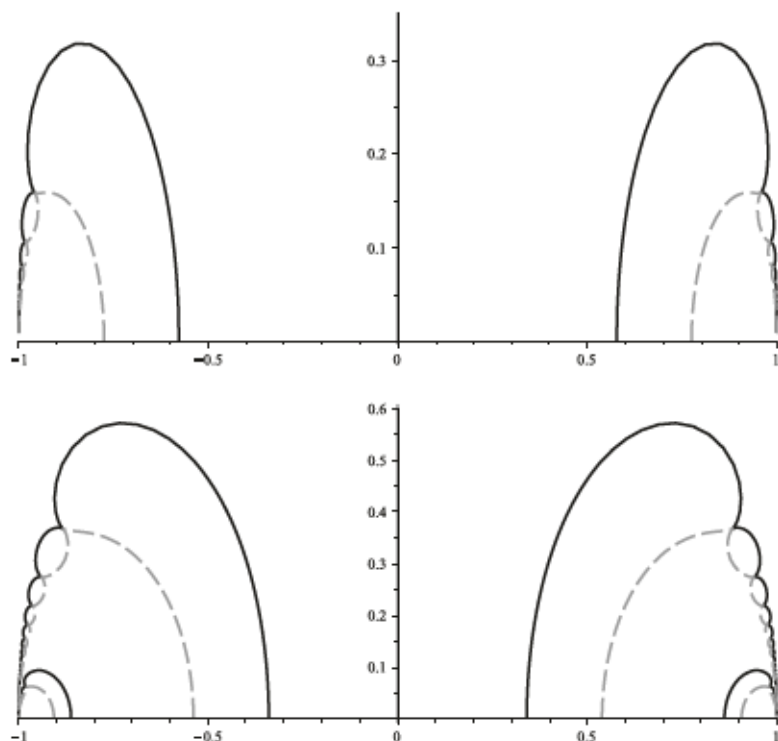


FIG. 1. Trajectories of the zeros of  $p_2(z)$  (dark, solid) and  $p_3(z)$  (grey, dashed), at the top, and  $p_4(z)$  (dark, solid) and  $p_5(z)$  (grey, dashed), at the bottom.

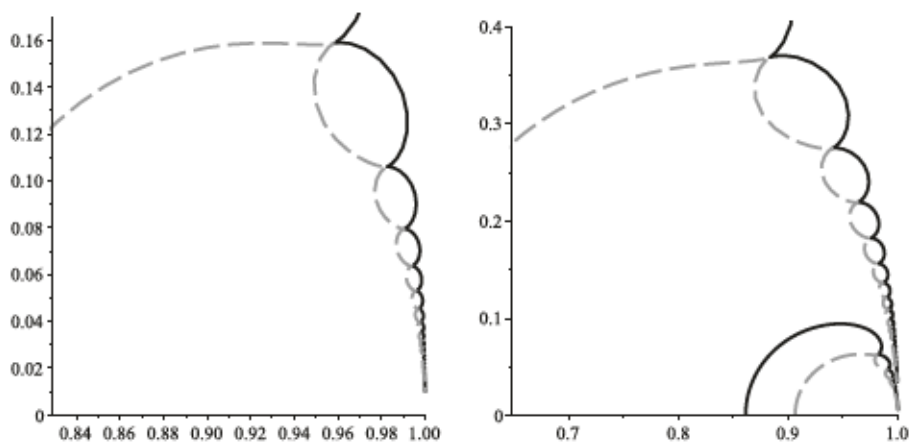


FIG. 2. Close-ups of the kissing patterns near the right endpoint  $+1$ , for  $p_2$  (dark, solid) and  $p_3$  (grey, dashed), on the left, and for  $p_4$  (dark, solid) and  $p_5$  (grey, dashed), on the right.

Using a different normalization, we can define a polynomial that always exists, regardless of the zeros of the Hankel determinant:

$$\tilde{p}_n(x) = \det \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \cdots & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \cdots & \mu_{n+1}(\omega) \\ \vdots & \vdots & & \vdots \\ \mu_{n-1}(\omega) & \mu_n(\omega) & \cdots & \mu_{2n-1}(\omega) \\ 1 & x & \cdots & x^n \end{bmatrix}.$$

It is clear that if  $h_{n-1}(\omega) \neq 0$ , then

$$\tilde{p}_n(z) = h_{n-1}(\omega)p_n(z). \quad (1.18)$$

Observe that, unlike  $p_n(z)$ , this new polynomial always exists, if  $h_{n-1}(\omega) = 0$  for some value of  $\omega$  then it has degree less than  $n$ . From the theory of quasi-OPs, or formal OPs, it is known that the degree of  $\tilde{p}_n(z)$  equals the dimension of the largest leading non-singular principal submatrix of the Hankel matrix  $H_{n-1}$  (Brezinski, 1980). This is the same as saying that the degree of  $\tilde{p}_n(z)$  is equal to the degree of the first existing polynomial  $p_k(z)$  of lower degree  $k \leq n$ . Plots indicate, and we will prove it later, that the degree of  $\tilde{p}_n(z)$  at a *kissing point* is actually  $n - 1$ .

Consider next the situation as  $\omega$  tends to a critical value  $\omega^* > 0$  such that  $h_n(\omega^*) = 0$ , for  $n \geq 1$ ; this can be understood by writing the three-term recurrence relation (1.11) in terms of the new polynomials, and then using (2.1), in terms of Hankel determinants:

$$h_{n-1}^2 \tilde{p}_{n+1}(z) = [h_n h_{n-1} z + i(\dot{h}_n h_{n-1} - h_{n-1} \dot{h}_n)] \tilde{p}_n(z) - h_n^2 \tilde{p}_{n-1}(z). \quad (1.19)$$

For critical values of  $\omega$ , where  $h_n$  vanishes, this expression simplifies. Indeed, if  $h_n(\omega^*) = 0$  and  $\dot{h}_n(\omega^*), h_{n-1}(\omega^*) \neq 0$ , equation (1.19) becomes

$$\tilde{p}_{n+1}(z) = i \frac{\dot{h}_n}{h_{n-1}} \tilde{p}_n(z), \quad (1.20)$$

i.e.  $\tilde{p}_{n+1}(z)$  is a scalar multiple of  $\tilde{p}_n(z)$ .

This means that at zeros of  $h_n$  the polynomial  $\tilde{p}_{n+1}(z)$  reduces to a constant multiple of  $p_n(z)$  of lower degree. Hence, their zeros coincide and the trajectories of both polynomials *kiss*. Also, in this situation, the monic polynomial  $p_{n+1}(z)$  blows up, and at least one of its zeros necessarily becomes infinite.

As a matter of fact, under the previous conditions, at this critical value  $\omega^*$ , the polynomial  $p_n(z)$  satisfies more orthogonality conditions than usual. The condition

$$\int_{-1}^1 p_n(z) p_n(z) e^{i\omega^* z} dz = \frac{h_n(\omega^*)}{h_{n-1}(\omega^*)} = 0$$

implies that

$$\int_{-1}^1 p_n(z) z^k e^{i\omega^* z} dz = 0, \quad k = 0, 1, \dots, n.$$

Thus, the polynomial  $z p_n(z)$  still satisfies  $n$  orthogonality conditions. In the theory of quasi-OPs, it is customary to replace in this case  $\tilde{p}_{n+1}(z)$  by  $z \tilde{p}_n(z)$ , in order to obtain a complete basis for the space of polynomials.

Such completeness is not necessary for the sake of quadrature rules and we forego a more complete description. Instead, we focus on an aspect of the kissing pattern that is unique to the polynomials at hand: the kisses in Figs 1 and 2 seemingly occur closer and closer to the endpoints  $\pm 1$  as  $\omega$  increases.

The paper is organized as follows. In Section 2, we present identities for the Hankel determinants, the recurrence coefficients and the kissing polynomials that hold for finite  $n$  and  $\omega$ ; such results belong to the integrable systems approach to OPs, which is relevant since the weight function for the kissing polynomials is an exponential deformation of the classical Legendre weight. More precisely, we present (i) a complex version of the Toda lattice equation and discrete string equations for the recurrence coefficients; (ii) a Toda equation and a differential equation in  $\omega$  related to Painlevé V for the Hankel determinants; (iii) raising and lowering operators, differential equations in  $z$  and in  $\omega$  for the kissing polynomials; and (iv) a differential equation for the motion of their zeros in terms of  $\omega$ .

In Section 3, we study properties of the Hankel determinants, recurrence coefficients and kissing polynomials as the parameter  $\omega \rightarrow \infty$ . Our results follow from the method of stationary phase for multivariate highly oscillatory integrals defined on the  $n$ -dimensional cube  $\mathcal{Y}_n = [-1, 1]^n$ , a methodology that requires elaborate combinatorial arguments in order to collect the dominant contributions from the vertices of  $\mathcal{Y}_n$ . Furthermore, our main result, Theorem 3.7, can be applied to more general cases where the function in the integrand is a smooth symmetric function of  $n$  variables. Additionally, in Theorem 3.10, we present an asymptotic result that relates the zeros of the kissing polynomials as  $\omega \rightarrow \infty$  with those of Laguerre polynomials. We observe that this asymptotic analysis is essentially different from the case when  $n$  is large, which is more standard in the theory of OPs and has been investigated in Suetin (2009), in Deaño (2014) and in Celsus & Silva (2020) (when  $\omega$  is fixed and when it is allowed to grow linearly with  $n$ ) and in Barhoumi *et al.* (2021) with  $\omega$  a general complex number. In these contributions, the asymptotic expansions are deduced by applying the Deift–Zhou steepest descent method to the corresponding Riemann–Hilbert problem, following ideas developed in Kuijlaars *et al.* (2004) for Jacobi polynomials modified by an analytic factor, see also the general monograph Deift (2000). In this context, the Riemann–Hilbert formulation for kissing polynomials also provides existence of  $p_n(z)$  for large enough  $n$ , together with asymptotic behaviour for  $z$  in different regions of the complex plane.

Section 4 is concerned with the main result of this paper for finite  $\omega$ , namely the proof that even-degree kissing polynomials always exist, for all  $n \geq 0$  and real  $\omega$ . While their existence for sufficiently large  $\omega$  is assured by the asymptotic analysis of Section 3, we need the material of Section 2 to leverage these results to all  $\omega \in \mathbb{R}$ .

Finally, the existence of *real* roots of Hankel determinants is critical to the existence of kissing polynomials. As it turns out, their *complex* roots describe an interesting ‘onion peel’ pattern in  $\mathbb{C}$ , which is described asymptotically as  $\omega \rightarrow \infty$  in Section 5, using the ideas from the previous section and in terms of the Lambert W function. The key idea in proving these results is a balance between algebraic and exponential terms in  $\omega$  in the asymptotic expansions calculated previously.



Finally, in Appendix A, we prove in detail several results at the interface of kissing polynomials, Riemann–Hilbert analysis and the theory of integrable systems, some of which are of an independent interest, which are germane to the analysis of Section 2.

## 2. Differential and difference equations

We have introduced the kissing polynomial  $p_n(z)$  as a polynomial of degree  $n$  in the variable  $z$  with parameter  $\omega$ . In this section, we explore a number of identities that can be obtained for the Hankel determinants  $h_n(\omega)$  and the OPs  $p_n(z)$  by considering different operations: more precisely, we can deduce differential identities with respect to  $z$  and  $\omega$ , as well as difference identities with respect to  $n$ .

We observe that our weight function  $w(x) = e^{i\omega x}$  is in fact a deformation of a classical weight function, namely the one for Legendre polynomials. Such modifications, with an exponential factor involving a parameter, have been widely studied in the literature for many families of OPs and using a variety of techniques: ladder operators (Ismail, 2005), integration by parts or Riemann–Hilbert methods. We refer the reader to Van Assche (2018) for more examples, including OPs on the unit circle and discrete OPs.

Time-dependent Jacobi polynomials studied in Basor *et al.* (2010), which are orthogonal with respect to the weight  $w(x, t) = (1-x)^\alpha(1+x)^\beta e^{-tx}$ , with  $\alpha, \beta > -1$  and  $t \in \mathbb{R}$ , are closely related to kissing polynomials, with the difference that the parameter  $t = -i\omega$  is purely imaginary in our case. A general analysis for complex parameter  $t$  has been recently described in Barhoumi *et al.* (2021).

Firstly, we present some properties of recurrence coefficients, more precisely a differential equation in  $\omega$ , which is a complex version of the classical Toda lattice, and two nonlinear difference equations in  $n$ , often known in the integrable systems community as *string equations*, using ideas from Magnus (1993, 1995). Next, we consider the Hankel determinants themselves, and in particular, the deformation with respect to  $\omega$  again, which leads to identification with solutions of the  $\sigma$ -Painlevé V equation. For kissing polynomials, following the technique of Ismail (2005, Chapter 22), we use the Riemann–Hilbert formulation to obtain a differential equation in the variable  $z$ . This differential equation is crucial in the proof of existence for the kissing polynomials of an even degree for all  $n \geq 0$  and  $\omega$ . Finally, by using an identity in Asheim *et al.* (2014) and the complex Toda lattice equations (2.5), along with ideas from Calogero (2003), we are able to study the behaviour of the polynomials as we deform the parameter  $\omega$ . All of the material in this section highlights the various connections between OPs and integrable systems, which can certainly be explored further.

### 2.1 Results for recurrence coefficients

The formulas (1.12) express the recurrence coefficients in terms of the bilinear form, but they are not very convenient for calculations. From the perspective of integrable systems, it is more interesting to write them in terms of Hankel determinants or subleading coefficients of the OPs.

**PROPOSITION 2.1** The recurrence coefficients in (1.11) admit the following expressions in terms of Hankel determinants (1.15):

$$\alpha_n(\omega) = -i \left[ \frac{\dot{h}_n(\omega)}{h_n(\omega)} - \frac{\dot{h}_{n-1}(\omega)}{h_{n-1}(\omega)} \right], \quad \beta_n(\omega) = \frac{h_n(\omega)h_{n-2}(\omega)}{h_{n-1}^2(\omega)}, \quad (2.1)$$

where  $\dot{h}_n$  indicates differentiation with respect to  $\omega$ . Furthermore,  $\alpha_n(\omega)$  can be written as follows:

$$\alpha_n(\omega) = \delta_{n,n-1}(\omega) - \delta_{n+1,n}(\omega), \quad (2.2)$$

where  $\delta_{n,n-1}(\omega)$  is the subleading coefficient of  $p_n(x)$ , that is,

$$p_n(x) = x^n + \delta_{n,n-1}(\omega)x^{n-1} + \dots, \quad (2.3)$$

*Proof.* Equation (2.2) follows directly from expanding the recurrence relation (1.11) in powers of  $x$  and then equating the terms multiplying  $x^n$ .

The proof of the second identity in (2.1) follows the standard one given for example in Ismail (2005, p.17); for the first one, we expand (1.16):

$$p_n(x) = \frac{1}{\tilde{h}_{n-1}} \det \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \cdots & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \cdots & \mu_{n+1}(\omega) \\ \vdots & \vdots & & \vdots \\ \mu_{n-1}(\omega) & \mu_n(\omega) & \cdots & \mu_{2n-1}(\omega) \\ 1 & x & \cdots & x^n \end{bmatrix} = x^n - \frac{\tilde{h}_{n-1}}{h_{n-1}} x^{n-1} + \dots,$$

where

$$\tilde{h}_{n-1} = \det \begin{bmatrix} \mu_0(\omega) & \mu_1(\omega) & \cdots & \mu_{n-2}(\omega) & \mu_n(\omega) \\ \mu_1(\omega) & \mu_2(\omega) & \cdots & \mu_n(\omega) & \mu_{n+1}(\omega) \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1}(\omega) & \mu_n(\omega) & \cdots & \mu_{2n-2}(\omega) & \mu_{2n-1}(\omega) \end{bmatrix}.$$

But as a consequence of the fact that  $\dot{\mu}_m = i\mu_{m+1}$  for  $m \geq 0$ , which follows from the exponential form of the weight function, see also (2.10) below, we have  $\dot{\tilde{h}}_{n-1} = -i\dot{h}_{n-1}$ , in terms of the derivative with respect to  $\omega$ . Therefore,

$$\delta_{n,n-1}(\omega) = i \frac{h_{n-1}}{\tilde{h}_{n-1}}. \quad (2.4)$$

This, together with (2.2), proves the result.  $\square$

**2.1.1 Differential equation in  $\omega$ .** As a function of the parameter  $\omega$ , the recurrence coefficients themselves satisfy a complex version of the classical Toda lattice equations, written in Flaschka variables; these equations are known to govern the deformation of the recurrence coefficients whenever the measure of orthogonality is a perturbation of a classical one with an exponential factor linear in the parameter, which in our case is  $\omega$ .

PROPOSITION 2.2 The recurrence coefficients satisfy the following differential–difference equations:

$$\begin{aligned}\dot{\alpha}_n(\omega) &= i[\beta_{n+1}(\omega) - \beta_n(\omega)] \\ \dot{\beta}_n(\omega) &= i\beta_n(\omega)[\alpha_n(\omega) - \alpha_{n-1}(\omega)],\end{aligned}\tag{2.5}$$

where  $\dot{\phantom{x}}$  indicates differentiation with respect to  $\omega$ .

*Proof.* The proof of this result mimics the one given in Ismail (2005, Theorem 2.8.1). We provide an alternative proof based on the Riemann–Hilbert formulation in the appendix.  $\square$

2.1.2 *Difference equation in  $n$ .* It has been well documented in the literature that recurrence coefficients of OPs with respect to exponential weights often satisfy certain nonlinear difference equations that are discrete integrable systems and sometimes correspond to discrete Painlevé equations (see for instance, Fokas *et al.*, 1991; Van Assche, 2018). Typically, these equations are derived *via* integration by parts, and in many cases, the computations are facilitated by the vanishing of boundary terms. However, by (1.8), directly integrating by parts in this case results in boundary terms, as the weight function does not vanish at  $x = \pm 1$ . To overcome this issue, we use ideas of Magnus (1993, 1995) and multiply the weight function by  $1 - x^2$  to kill off these boundary terms. This comes at a price, however, and we will see that the string equation for the kissing polynomials involves not only the recurrence coefficients but also the sub-leading coefficient of the polynomial. More precisely, we have the following result.

PROPOSITION 2.3 Assume  $\omega$  is such that  $\chi_n(\omega) \neq 0$  and  $\chi_{n-1}(\omega) \neq 0$ . Then, we have the following identities:

$$\begin{aligned}-2n\alpha_n + 2(n+2)\alpha_{n+1} + i\omega(\beta_{n+2} - \beta_n + \alpha_{n+1}^2 - \alpha_n^2) &= 0, \\ -\alpha_n^2 + 1 - (2n+3)\beta_{n+1} + (2n-1)\beta_n + i\omega[\beta_n(\alpha_n + \alpha_{n-1}) - \beta_{n+1}(\alpha_{n+1} + \alpha_n)] &= 0.\end{aligned}\tag{2.6}$$

*Proof.* Using the technique of Magnus, we write

$$\begin{aligned}0 &= \int_{-1}^1 \left[ (1-x^2) p_n^2(x) e^{i\omega x} \right]' dx \\ &= -2 \int_{-1}^1 x p_n^2(x) e^{i\omega x} dx + 2 \int_{-1}^1 (1-x^2) p_n(x) p_n'(x) e^{i\omega x} dx \\ &\quad + i\omega \int_{-1}^1 (1-x^2) p_n^2(x) e^{i\omega x} dx \\ &= -2(n+1)\alpha_n \chi_n + 2\delta_{n,n-1} \beta_n \chi_{n-1} + i\omega \left( \chi_n - \chi_{n+1} - \alpha_n^2 \chi_n - \beta_n^2 \chi_{n-1} \right),\end{aligned}$$

where we have used

$$x p_n'(x) = n p_n(x) - \delta_{n,n-1} p_{n-1}(x) + \dots$$

As  $\chi_{n-1} \neq 0$ , we may write

$$\beta_n = \frac{\chi_n}{\chi_{n-1}}$$

to simplify the previous equation and obtain

$$-2(n+1)\alpha_n + 2\delta_{n,n-1} + i\omega(1 - \beta_{n+1} - \beta_n - \alpha_n^2) = 0. \quad (2.7)$$

Using (2.2), we can write the previous string equation using only the recurrence coefficients, and we obtain the first equation in (2.6). Similarly, with more elaborate calculations, we have

$$\begin{aligned} 0 &= \int_{-1}^1 \left[ (1-x^2)p_n(x)p_{n-1}(x)e^{i\omega x} \right]' dx \\ &= -2 \int_{-1}^1 xp_n(x)p_{n-1}(x)e^{i\omega x} dx + \int_{-1}^1 (1-x^2)p_n'(x)p_{n-1}(x)e^{i\omega x} dx \\ &\quad + \int_{-1}^1 (1-x^2)p_n(x)p_{n-1}'(x)e^{i\omega x} dx + i\omega \int_{-1}^1 (1-x^2)p_n(x)p_{n-1}(x)e^{i\omega x} dx. \end{aligned}$$

If we write

$$p_n(x) = x^n + \delta_{n,n-1}x^{n-1} + \delta_{n,n-2}x^{n-2} + \dots,$$

then a direct calculation gives

$$xp_n'(x) = np_n(x) - \delta_{n,n-1}p_{n-1}(x) + (\delta_{n,n-1}\delta_{n-1,n-2} - 2\delta_{n,n-2})p_{n-2}(x) + \dots,$$

and from the recurrence relation, we deduce the identity

$$\delta_{n+1,n-1} = \delta_{n,n-2} - \alpha_n\delta_{n,n-1} - \beta_n, \quad (2.8)$$

in addition to (2.2). Then, we have

$$0 = -[2n+1 + i\omega(\alpha_n + \alpha_{n-1})]\chi_n + (n - \delta_{n,n-1}^2 + 2\delta_{n,n-2})\chi_{n-1}.$$

If  $\chi_{n-1} \neq 0$ , then we obtain

$$0 = -[2n+1 + i\omega(\alpha_n + \alpha_{n-1})]\beta_n + n - \delta_{n,n-1}^2 + 2\delta_{n,n-2}.$$

If we shift this equation  $n \mapsto n+1$  and subtract the two, using (2.2) and (2.8), we arrive at the second equation in (2.6).  $\square$

REMARK 2.1 Observe that if  $\omega = 0$ , then the first equation in (2.6) gives

$$\alpha_{n+1} = \frac{n}{n+2} \alpha_n,$$

with initial value  $\alpha_0 = 0$ , so  $\alpha_n = 0$  for  $n \geq 0$ , which corresponds to the case of Legendre polynomials. Then the second equation in (2.6) leads to

$$1 - (2n+3)\beta_{n+1} + (2n-1)\beta_n = 0 \Rightarrow \beta_{n+1} = \frac{2n-1}{2n+3}\beta_n + \frac{1}{2n+3}, \quad n \geq 0,$$

with initial value  $\beta_0 = 0$ . It can be shown by induction that the solution to this recursion is  $\beta_n = \frac{n^2}{4n^2-1}$  for  $n \geq 0$ , which agrees with the recurrence coefficient for monic Legendre polynomials.

Although we shall not make use of this identity in the sequel, it is worth noting it because of its potential connections to the theory of discrete integrable systems. Similar identities can be obtained with the same technique applied to the following integrals:  $\int_{-1}^1 [(1-x^2) p_n(x) p_{n-k}(x) e^{i\omega x}]' dx$ , with  $k \geq 1$ . However, we note that since  $p_{n-k}(x)$  can be written in terms of  $p_{n-k+1}(x)$  and  $p_{n-k+2}(x)$ , using the recurrence relation, then no essentially new identities should be expected once we have worked out the cases  $k = 0$  and  $k = 1$  in Proposition 2.3.

From a numerical point of view, it is known that direct computation of the recurrence coefficients using these nonlinear relations often suffers from stability issues (several examples can be found in Van Assche, 2018); however, string relations similar to Proposition 2.3 were used in Brower *et al.* (1993), Bleher (2011) and Bleher & Its (2003) as a foundation for stable numerical computation of recurrence coefficients, and these implications of the string relation could be pursued further.

## 2.2 Results for moments and Hankel determinants

In this section, we study several properties of the Hankel determinants that will be needed in the sequel. Since these determinants are constructed using the moments of the weight function  $\mu_n$  given in (1.14), we first present an auxiliary result about them.

LEMMA 2.4 The moments  $\mu_m(\omega)$  satisfy the following identities.

1. The linear recursion

$$\mu_{m+1}(\omega) = \frac{e^{i\omega} + (-1)^m e^{-i\omega}}{i\omega} - \frac{m+1}{i\omega} \mu_m(\omega), \quad m \geq 0, \quad (2.9)$$

starting with

$$\mu_0(\omega) = \int_{-1}^1 e^{i\omega x} dx = \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = \frac{2 \sin \omega}{\omega}.$$

2. The differential identity with respect to  $\omega$ :

$$\dot{\mu}_m = \frac{d}{d\omega} \mu_m = i\mu_{m+1}, \quad m \geq 0. \quad (2.10)$$

3. The following representations as confluent hypergeometric functions:

$$\mu_m(\omega) = \frac{M(m+1, m+2, i\omega) + (-1)^m e^{-i\omega} M(1, m+2, i\omega)}{m+1}, \quad m \geq 0; \quad (2.11)$$

see Olver *et al.* (2021, Chapter 13).

*Proof.* Formula (2.9) follows using integration by parts, and (2.10) is a straightforward consequence of the exponential form of the weight function. Formula (2.11) can be obtained from a standard integral representation of the confluent hypergeometric function:

$$M(a, b, x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt, \quad \operatorname{Re} b > \operatorname{Re} a > 0; \quad (2.12)$$

see Olver *et al.* (2021, (13.4.1)), identifying  $a = m+1$ ,  $b = m+2$  and  $x = i\omega$ .  $\square$

LEMMA 2.5 For any  $\omega \in \mathbb{R}$  and any  $n \geq 0$ , the Hankel determinant  $h_n(\omega)$  given by (1.15) is real valued. Furthermore,  $h_n(\omega)$  is an even function of  $\omega$ .

*Proof.* It follows directly from (1.14) that for  $n \geq 0$ ,

$$\overline{\mu_{2n}(\omega)} = \mu_{2n}(-\omega) = \mu_{2n}(\omega), \quad \overline{\mu_{2n+1}(\omega)} = \mu_{2n+1}(-\omega) = -\mu_{2n+1}(\omega), \quad (2.13)$$

and therefore even moments are real and odd ones are purely imaginary. Then, extracting a factor  $i^j$  from the  $j$ -th row and a factor  $i^k$  from each column of the matrix  $H_n(\omega)$  given by (1.13), we obtain

$$H_n(\omega) = i^{n(n+1)} \tilde{H}_n(\omega) = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \tilde{H}_n(\omega),$$

where the entries of the matrix on the right-hand side are  $(\tilde{H}_n(\omega))_{j,k} = i^{j+k} \mu_{j+k}(\omega)$ , for  $j, k \geq 0$ . Therefore, it follows from (2.13) that all these entries are real valued, and consequently, the determinant  $h_n(\omega)$  is real valued too. Finally, extracting  $(-1)^j$  from each row and  $(-1)^k$  from each column, we have

$$H_n(-\omega) = (-1)^{n(n+1)} H_n(\omega) = H_n(\omega),$$

and therefore,  $h_n(\omega) = h_n(-\omega)$ , so it is an even function.  $\square$

COROLLARY 2.6 As a consequence of Lemma 2.5 and definition (1.16), for  $\omega \in \mathbb{R}$

$$p_n^{-\omega}(z) = \overline{p_n^{\omega}(\bar{z})},$$

and in particular  $p_n^{-\omega}(x) = \overline{p_n^{\omega}(x)}$  for  $x \in \mathbb{R}$ .

Next, we present two useful auxiliary results; the first one is a complex version of the Toda evolution equation, which is well known in the theory of integrable systems and in random matrix theory (cf. for instance Bleher & Its, 2005, Section 2; Bleher, 2011, Proposition 18.1; or Bleher & Liechty, 2013, Theorem 1.4.2) but is presented with a proof for the sake of completeness.

LEMMA 2.7 It is true that

$$\ddot{h}_n(\omega)h_n(\omega) - \dot{h}_n^2(\omega) = -h_{n-1}(\omega)h_{n+1}(\omega), \quad n \geq 1, \quad (2.14)$$

where dot indicates differentiation with respect to  $\omega$ . Alternatively, we may write

$$\frac{d^2}{d\omega^2} \log h_{n-1}(\omega) = -\beta_n(\omega), \quad (2.15)$$

in terms of the recurrence coefficient  $\beta_n(\omega)$  in (1.11).

*Proof.* We recall a well-known alternative formula for the Hankel determinant; see for instance Ismail (2005, page 18). We start from

$$\chi_j(\omega) = \int_{-1}^1 p_j^2(z) e^{i\omega z} dz = \int_{-1}^1 z^j p_j(z) e^{i\omega z} dz = \frac{h_j(\omega)}{h_{j-1}(\omega)}, \quad j \geq 0,$$

using (1.16) and taking  $h_{-1}(\omega) := 1$ . From this formula, it follows that

$$h_{n-1}(\omega) = \prod_{j=0}^{n-1} \chi_j(\omega), \quad n \geq 1. \quad (2.16)$$

Note that in our context  $\chi_j(\omega)$  may have poles, since  $h_{j-1}(\omega)$  can have zeros, which is not the case in the standard theory. Still, by the above reasoning, the product formula for  $h_{n-1}(\omega)$  remains valid.

Next, we observe that

$$\dot{\chi}_j(\omega) = i \int_{-1}^1 z p_j(z)^2 e^{i\omega z} dz = i\alpha_j(\omega)\chi_j(\omega), \quad (2.17)$$

using the recurrence relation (1.11) and orthogonality. As a consequence,

$$h_{n-1}(\omega) = \sum_{\ell=0}^{n-1} \left[ \prod_{j \neq \ell} \chi_j(\omega) \right] \dot{\chi}_\ell(\omega) = \sum_{\ell=0}^{n-1} \left[ \prod_{j \neq \ell} \chi_j(\omega) \right] i\alpha_\ell(\omega)\chi_\ell(\omega) = ih_{n-1}(\omega) \sum_{\ell=0}^{n-1} \alpha_\ell(\omega). \quad (2.18)$$

We differentiate again, bearing in mind that  $\dot{\alpha}_\ell(\omega) = i[\beta_{\ell+1}(\omega) - \beta_\ell(\omega)]$ ; see (2.5):

$$\begin{aligned} \ddot{h}_{n-1}(\omega) &= i\dot{h}_{n-1}(\omega) \sum_{\ell=0}^{n-1} \alpha_\ell(\omega) - h_{n-1}(\omega) \sum_{\ell=0}^{n-1} [\beta_{\ell+1}(\omega) - \beta_\ell(\omega)] \\ &= \frac{[\dot{h}_{n-1}(\omega)]^2}{h_{n-1}(\omega)} - h_{n-1}(\omega)\beta_n(\omega) \\ &= \frac{[\dot{h}_{n-1}(\omega)]^2}{h_{n-1}(\omega)} - \frac{h_n(\omega)h_{n-2}(\omega)}{h_{n-1}(\omega)}, \end{aligned} \quad (2.19)$$

where we have used (2.1) and telescoped the second sum, taking  $\beta_0 = 0$ . Multiplying throughout by  $h_{n-1}(\omega)$  and shifting the index  $n - 1 \mapsto n$ , we obtain the result.

Finally, (2.15) follows from direct differentiation and formula (2.1), connecting recurrence coefficients with Hankel determinants.  $\square$

The previous identity is simple but has the disadvantage that it combines Hankel determinants with different values of  $n$ . It is possible to derive a differential identity without a shift in  $n$ , at the price of a more complicated structure.

**PROPOSITION 2.8** For  $n \geq 1$ , in the variable  $t = 2i\omega$ , the function

$$\sigma_n(t) = t \frac{d}{dt} \log h_{n-1}(t) - \frac{nt}{2} + n^2 \quad (2.20)$$

satisfies the Jimbo–Miwa–Okamoto  $\sigma$ -Painlevé V equation:

$$(t\sigma_n'')^2 = [\sigma_n - t\sigma_n' + 2(\sigma_n')^2 + 2n\sigma_n']^2 - 4(\sigma_n')^2(\sigma_n' + n)^2. \quad (2.21)$$

*Proof.* The proof is included in the appendix, Section A.5.  $\square$

**REMARK 2.2** This result follows as well from the work of Basor *et al.* (2010), who study the time-dependent Jacobi weight

$$w(x; t) = (1 - x)^\alpha (1 + x)^\beta e^{-tx}, \quad \alpha, \beta > -1, \quad t \in \mathbb{R}.$$

In order to arrive at (2.21), one identifies  $\alpha = \beta = 0$  and  $t = -i\omega$  and translates the results in Basor *et al.* (2010, Section 5), in particular (5.9) and (5.10) therein, to our parameter  $\omega$ .

**REMARK 2.3** In the study of Painlevé equations, locating poles of the solutions, or estimating regions in the complex plane that are free of them, is an active area of research. In this sense, Proposition 2.8 is relevant for the  $\sigma$ -Painlevé V equation, since the zeros of the Hankel determinant  $h_{n-1}(\omega)$ , which are a central topic of this paper, are poles of the function  $\sigma_n$  in (2.20), and therefore our results can be translated to that area.

In Fig. 3, we display  $\log |h_n(\omega)|$  as a function of  $\omega$  for different values of  $n$ , and we see a clear difference in behaviour depending on the parity of  $n$ . Based on this figure, and similar ones that can be obtained by direct computation in Maple, we formulate two results about the properties of the Hankel determinants:

1. two consecutive Hankel determinants  $h_n$  and  $h_{n+1}$  cannot have any common positive real zeros for  $n \geq 0$  and  $\omega > 0$ ;
2. for  $n \geq 0$  and  $\omega > 0$ , the determinants  $h_n$  and  $h_{n+2}$  do not have any common positive real zeros.

We first show that no two consecutive Hankel determinants can vanish simultaneously.

**LEMMA 2.9** There is no  $n \geq 1$  and  $\omega^* > 0$  such that

$$h_{n-1}(\omega^*) = h_n(\omega^*) = 0.$$



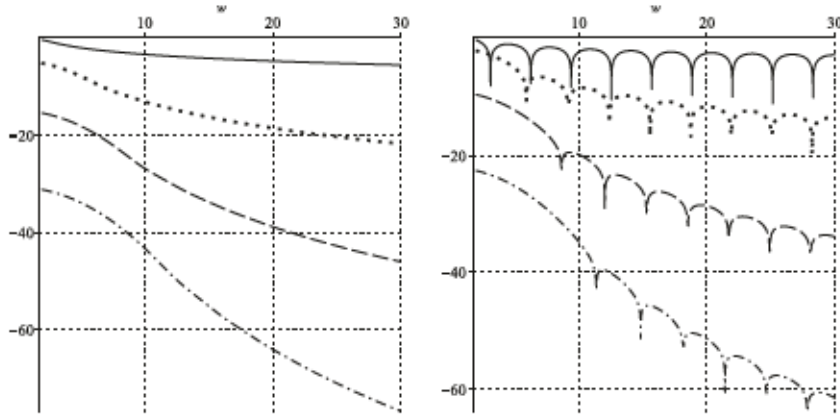


FIG. 3. On the left, plot in log-scale of  $\log |h_1(\omega)|$  (solid line),  $\log |h_3(\omega)|$  (dotted),  $\log |h_5(\omega)|$  (dashed) and  $\log |h_7(\omega)|$  (dashed-dotted). On the right, plot in log-scale of  $\log |h_2(\omega)|$  (solid line),  $\log |h_4(\omega)|$  (dotted),  $\log |h_6(\omega)|$  (dashed) and  $\log |h_8(\omega)|$  (dashed-dotted).

*Proof.* Assume that  $h_{n-1}(\omega^*) = h_n(\omega^*) = 0$  for some  $\omega = \omega^*$ . Then, by (2.14), we have  $\dot{h}_n = 0$  and so  $h_n$  has a root of order at least two at  $\omega = \omega^*$ . Since both terms on the left-hand side of (2.14) have a root of order at least two, so must the right-hand side, and this implies that either  $h_{n+1}$  or  $h_{n-1}$  have a root of order at least two.

In the latter case, two consecutive Hankel determinants have (at least) a double root. In the former case this happens too. Indeed, in this case, we have  $h_n(\omega^*) = \dot{h}_n(\omega^*) = h_{n-1}(\omega^*) = h_{n+1}(\omega^*) = 0$ . We can reformulate (2.14) as

$$\ddot{h}_{n+1}(\omega)h_{n+1}(\omega) - [\dot{h}_{n+1}(\omega)]^2 = -h_n(\omega)h_{n+2}(\omega), \quad n \geq 0.$$

It follows that  $\dot{h}_{n+1}(\omega) = 0$ , i.e. both  $h_n(\omega)$  and  $h_{n+1}(\omega)$  have a double root at  $\omega = \omega^*$ .

It remains to rule out two consecutive double roots. Let us assume that  $h_n$  and  $h_{n+1}$  have a common root of order  $\geq 2$  at  $\omega = \omega^*$ . In that case, the right-hand side of (1.19) vanishes at  $\omega = \omega^*$  but the left-hand side does not, since  $\tilde{p}_n$  does not vanish identically, unless also  $h_{n-1} = 0$ . Subsequently, if we shift the index down  $n \mapsto n - 1$  in (2.14), we obtain

$$\ddot{h}_{n-1}(\omega)h_{n-1}(\omega) - \dot{h}_{n-1}^2(\omega) = -h_{n-2}(\omega)h_n(\omega),$$

and if  $h_n(\omega^*) = h_{n-1}(\omega^*) = 0$ , we deduce that  $\dot{h}_{n-1}(\omega^*) = 0$  too, and therefore,  $h_{n-1}$  has a zero of order at least two at  $\omega = \omega^*$ . Continuing this reasoning leads to a chain of roots of order at least two and all Hankel determinants vanishing down to  $n = 0$ , which is a contradiction.  $\square$

It follows immediately that one cannot have  $h_n = \dot{h}_n = 0$  either, since then by (2.14) at least one of  $h_{n-1}$  or  $h_{n+1}$  has to vanish too. We can also exclude  $h_n = h_{n+2} = 0$ , which is our second result.

LEMMA 2.10 There is no  $n \geq 0$  and  $\omega^* > 0$  such that

$$h_n(\omega^*) = h_{n+2}(\omega^*) = 0.$$

*Proof.* The result is true by direct computation for  $n = 0$  and  $n = 1$ . Let us assume it is true up to  $n - 1$ , and assume that  $h_n(\omega^*) = 0$  for some  $\omega^* > 0$ . We intend to show that  $h_{n+2}(\omega^*) \neq 0$ .

We know that  $h_{n-1}(\omega^*) \neq 0$  by Lemma 2.9 and that  $h_{n-2}(\omega^*) \neq 0$  by our inductive assumption. It follows from (2.1) that  $\alpha_{n-1}$  is analytic at  $\omega^*$ . It also follows from (2.1) that  $\beta_n(\omega^*) = 0$ . Since  $h_n(\omega^*) \neq 0$  and  $h_{n-2}(\omega^*) \neq 0$ , this root of  $\beta_n$  is simple.

We reformulate the differential-difference equation (2.5) as

$$\begin{aligned}\beta_{n+1} &= -i\dot{\alpha}_n + \beta_n, \\ \alpha_{n+1} &= -i\frac{\dot{\beta}_{n+1}}{\beta_{n+1}} + \alpha_n.\end{aligned}$$

Plugging in a Taylor series of  $\alpha_{n-1}$  and  $\beta_n$  around  $\omega = \omega^*$  and using the above recursions shows, after straightforward computation, that  $\alpha_n$  and  $\alpha_{n+1}$  have a simple pole,  $\beta_{n+1}$  has a double pole,  $\beta_{n+2}$  has a simple root and  $\alpha_{n+2}$  is analytic at  $\omega^*$ . Using the expressions

$$\alpha_n = \frac{\langle zp_n, p_n \rangle}{\langle p_n, p_n \rangle} \quad \text{and} \quad \beta_n = \frac{\langle p_n, p_n \rangle}{\langle p_{n-1}, p_{n-1} \rangle},$$

this implies that  $\langle p_{n+1}, p_{n+1} \rangle$  has a simple zero at  $\omega = \omega^*$  and  $\langle p_{n+2}, p_{n+2} \rangle \neq 0$ . The latter in turn implies that  $h_{n+2}(\omega^*) \neq 0$ .  $\square$

We observe that this type of idea was used in a similar problem (but with a complex cubic potential on a union of infinite contours in  $\mathbb{C}$ ) in the thesis of Lejon (2016); in that setting, the combination of analogous properties for the Hankel determinants and the string (or Freud) equations lead to the proof of existence of the OPs of even degree. In the present case, the main issue is that the string equations have a more complicated structure because of the presence of boundary terms. This issue is addressed in Section 2.1.2.

### 2.3 Results for kissing polynomials

**2.3.1 Differential equation in  $z$ .** One standard result in the theory of classical OPs is the existence of a linear second-order differential equation; this result typically follows from combining two ladder operators (raising and lowering), which express  $p'_n(z)$  and  $p'_{n-1}(z)$  in terms of  $p_n(z)$  and  $p_{n-1}(z)$ :

$$\begin{aligned}\left[\frac{d}{dz} + B_n(z)\right]p_n(z) &= \beta_n A_n(z)p_{n-1}(z), \\ \left[\frac{d}{dz} - B_n(z) + v'(z)\right]p_{n-1}(z) &= -A_{n-1}(z)p_n(z),\end{aligned}$$

where  $w(x) = e^{-v(x)}$ . The coefficients  $A_n(x)$  and  $B_n(x)$  can be expressed in terms of recurrence coefficients and, if the support of the orthogonality measure is finite, of boundary values of the OPs and the weight function. The proof of the ladder relations relies on integration by parts and the Christoffel–Darboux identity, as shown by Chen & Ismail (1997) and also in Ismail (2005, Section 3.2); it is also possible to prove them using the Riemann–Hilbert problem for OPs, we refer the reader to Van Assche (2018, Chapter 4), and this is the methodology that we use for kissing polynomials in the appendix, in

particular equation (A.13). More precisely, we obtain

$$A_n(\omega) = -\frac{2n+1+i\omega(\alpha_n+z)}{z^2-1}, \quad B_n(\omega) = -\frac{nz-\delta_{n,n-1}+i\omega\beta_n}{z^2-1}.$$

Combination of these two ladder operators gives a second-order differential equation (depending on  $n$ ) for the OPs; see Chen & Ismail (1997, Theorem 2.2) or Ismail (2005, Theorem 3.2.3) for a general formulation. This is true for the kissing polynomials, too, and this identity will be a key element in showing existence of kissing polynomials of even degree later on.

We first present the following result, whose proof uses the Riemann–Hilbert problem for OPs and is given in the Appendix A.

**LEMMA 2.11** (Differential equation for kissing polynomials). Let  $\omega$  be such that  $h_{n-1}(\omega) \neq 0$ . Then the kissing polynomials satisfy the following second-order ODE:

$$p_n''(z) + \frac{R(z;\omega)}{Q(z;\omega)}p_n'(z) + \frac{S(z;\omega)}{Q(z;\omega)}p_n(z) = 0, \quad (2.22)$$

where  $Q, R, S$  are polynomials in  $z$ . Moreover, if  $h_n(\omega) = 0$ , then the only singular points of the differential equation are at  $z = \pm 1$ . If  $h_n(\omega) \neq 0$ , the differential equation also has a regular singular point at

$$z_*(\omega; n) = -\alpha_n - \frac{2n+1}{i\omega} \in i\mathbb{R}, \quad (2.23)$$

along with  $z = \pm 1$ .

This lemma has two immediate corollaries that will be used in the proof of existence of the even-degree kissing polynomials.

**COROLLARY 2.12** If  $h_n(\omega) = 0$ , then  $p_n(z)$  cannot have a zero of multiplicity greater than one on the imaginary axis.

*Proof.* If  $h_n(\omega) = 0$ , the imaginary axis consists solely of regular points of the second-order differential equation (2.22). If  $p_n(z)$  has a zero of multiplicity greater than one at  $z^* \in i\mathbb{R}$ , then  $p_n(z^*) = p_n'(z^*) = 0$ , and then  $p_n(z)$  is identically zero, by a standard argument about existence and uniqueness of solutions of second-order linear ODEs; see for instance Ince (1956, Chapter III).  $\square$

**COROLLARY 2.13** Assume that  $h_n \neq 0$  and  $h_{n-1} \neq 0$ . If  $p_n(z)$  has a zero at  $z_*(\omega)$ , then it is a double zero.

*Proof.* We may write

$$p_n(z) = \sum_{k=0}^{n-j} a_k (z - z_*)^{k+j}, \quad (2.24)$$

where  $j$  is yet to be determined and  $a_0 \neq 0$ . Using (A.18), we can expand  $R/Q$  and  $S/Q$  in a Laurent series about  $z_*$  as

$$\frac{R(z)}{Q(z)} = \sum_{k=-1}^{\infty} r_k(z - z_*)^k, \quad \frac{S(z)}{Q(z)} = \sum_{k=-1}^{\infty} s_k(z - z_*)^k. \quad (2.25)$$

Above, we compute  $r_{-1} = -1$ , which follows from (A.17) in the appendix. Plugging (2.24) and (2.25) into the differential equation

$$p_n''(z) + \frac{R(z)}{Q(z)}p_n'(z) + \frac{S(z)}{Q(z)}p_n(z) = 0,$$

and an examination of the coefficient of  $(z - z_*)^{j-1}$  give

$$a_0 j(j - 2) = 0. \quad (2.26)$$

As  $a_0 \neq 0$ , (2.26) implies that either  $j = 0$  or  $j = 2$ , completing the proof.  $\square$

**2.3.2 Differential equations in  $\omega$ .** Finally, we turn our attention to the behaviour of kissing polynomials as we deform the parameter  $\omega$ . The starting point of our analysis is the following relation, derived in Asheim *et al.* (2014, Theorem 3.2):

$$\dot{p}_n(z) = -i\beta_n p_{n-1}(z), \quad (2.27)$$

where we recall that  $\dot{p}_n(z) = \partial p_n(z)/\partial \omega$ . Using similar techniques to those used to derive the differential equation in the variable  $z$ , we are able to conclude that the kissing polynomials also satisfy a second-order differential equation in the parameter  $\omega$ .

**LEMMA 2.14** Assume that  $\check{\omega}$  is such that  $h_{n-1}(\check{\omega}) \neq 0$ , so that  $p_n(z)$  exists as a monic polynomial of degree  $n$  in a neighbourhood of  $\check{\omega}$ . Then, in this neighbourhood, the kissing polynomials satisfy

$$\ddot{p}_n + i(z - \alpha_n)\dot{p}_n - \beta_n p_n = 0, \quad (2.28)$$

where  $\dot{\phantom{x}}$  indicates differentiation with respect to the parameter  $\omega$ .

*Proof.* Using the recurrence relation (1.11), we may transform (2.27) to

$$\dot{p}_{n-1} = ip_n - i(z - \alpha_{n-1})p_{n-1}. \quad (2.29)$$

We may combine the two differential–difference equations, (2.27) and (2.29), as in the proof of Lemma 2.11 to obtain

$$\frac{i}{\beta_n}\ddot{p}_n + \left[ \frac{\partial}{\partial \omega} \left( \frac{i}{\beta_n} \right) - \frac{z - \alpha_{n-1}}{\beta_n} \right] \dot{p}_n - ip_n = 0.$$

Using the Toda equations (2.5), we can simplify this to

$$\ddot{p}_n + i(z - \alpha_n)\dot{p}_n - \beta_n p_n = 0,$$

completing the proof. □

Next, we study the behaviour of the zeros of  $p_n$  as functions of  $\omega$ , using techniques from Calogero (2003) and the differential equation (2.28).

LEMMA 2.15 Assume that  $\check{\omega}$  is such that  $h_{n-1}(\check{\omega}) \neq 0$ , so that  $p_n(z)$  exists as a monic polynomial of degree  $n$  in a neighbourhood of  $\check{\omega}$ . Denote by  $\{z_i(\omega)\}_{i=1}^n$  the  $n$  zeros of the polynomial  $p_n(z)$ . In this neighbourhood of  $\check{\omega}$ , the zeros evolve according to the dynamical system

$$\dot{z}_i = 2\dot{z}_i \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\dot{z}_j}{z_i - z_j} - i\dot{z}_i(z_i - \alpha_n), \quad i = 1, 2, \dots, n. \tag{2.30}$$

*Proof.* As  $p_n(z)$  is a monic polynomial of degree  $n$ , we write

$$p_n(z) = \prod_{i=1}^n [z - z_i(\omega)].$$

Differentiating with respect to  $\omega$  yields

$$\frac{\partial}{\partial \omega} p_n(z) \Big|_{z=z_i(\omega)} = -\dot{z}_i(\omega) \prod_{\substack{k=1 \\ k \neq i}}^n [z_i(\omega) - z_k(\omega)]$$

and

$$\frac{\partial^2}{\partial \omega^2} p_n(z) \Big|_{z=z_i(\omega)} = -\ddot{z}_i(\omega) \prod_{\substack{k=1 \\ k \neq i}}^n [z_i(\omega) - z_k(\omega)] + 2\dot{z}_i(\omega) \sum_{\substack{j=1 \\ j \neq i}}^n \dot{z}_j(\omega) \prod_{\substack{k=1 \\ k \neq i, j}}^n [z_i(\omega) - z_k(\omega)].$$

Evaluating (2.28) along any zero trajectory yields the following complex  $n$ -body problem,

$$\begin{aligned} & -\dot{z}_i(\omega) \prod_{\substack{k=1 \\ k \neq i}}^n [z_i(\omega) - z_k(\omega)] + 2\dot{z}_i(\omega) \sum_{\substack{j=1 \\ j \neq i}}^n \dot{z}_j(\omega) \prod_{\substack{k=1 \\ k \neq i, j}}^n [z_i(\omega) - z_k(\omega)] \\ & - i\dot{z}_i(\omega) [z_i(\omega) - \alpha_n(\omega)] \prod_{\substack{k=1 \\ k \neq i}}^n [z_i(\omega) - z_k(\omega)] = 0, \end{aligned}$$

for  $i = 1, 2, \dots, n$ , which upon simplification, results in (2.30). □

The above proof also lends some insight into the behaviour of the zeros for  $\omega > 0$ . It is clear that the initial positions of these zeros for  $\omega = 0$  are the zeros of the underlying Legendre polynomial. Let  $x_i$ ,  $i = 1, \dots, n$  denote the ordered zeros of the Legendre polynomial  $P_n$ , thus

$$z_i(0) = x_i, \quad i = 1, \dots, n.$$

Next, we observe that

$$-i\beta_n p_{n-1}(z_i(\omega)) = \frac{\partial}{\partial \omega} p_n(z) \Big|_{z=z_i(\omega)} = -\dot{z}_i(\omega) \prod_{\substack{k=1 \\ k \neq i}}^n [z_i(\omega) - z_k(\omega)]. \quad (2.31)$$

This equation may be compared with the more general formulas given by Ismail & Ma (2011) for the motion of zeros of OPs under Toda deformation, in particular equation (3.10) therein, with  $l = 1$  in their notation.

Evaluating equation (2.31) at  $\omega = 0$ , we deduce that

$$\dot{z}_i(0) = \frac{i\beta_{n,L} \tilde{P}_{n-1}(x_i)}{\prod_{\substack{k=1 \\ k \neq i}}^n (x_i - x_k)}, \quad i = 1, \dots, n,$$

where  $\beta_{n,L}$  are the recurrence coefficients for the Legendre polynomials and  $\tilde{P}_{n-1}(x)$  is the monic Legendre polynomial of degree  $n - 1$ . It is well known that

$$\beta_{n,L} = \frac{n^2}{(2n-1)(2n+1)}.$$

Therefore, we have that the zeros of the kissing polynomials move into the upper complex half-plane as soon as  $\omega > 0$ .

Another consequence of Lemma 2.15 is that the zeros are analytic functions of  $\omega$ , provided they are all simple and  $\alpha_n$  is not infinite. By (2.1), we see that  $\alpha_n$  is infinite when either  $h_n$  or  $h_{n-1}$  vanishes—that is,  $\alpha_n$  is infinite at kissing points. This should be read in light of the discussion in Section 1.2, where we have shown that if  $h_{n-1}$  vanishes,  $p_n(z)$  becomes a multiple of  $p_{n-1}(z)$ , and Figs 1 and 2 show that at these points the zero trajectories form cusp singularities.

### 3. Asymptotic analysis of multivariate oscillatory integrals

#### 3.1 General setting

In this section, we investigate the behaviour of Hankel determinants and the kissing polynomials as  $\omega \rightarrow \infty$ . This analysis is carried out using results from the asymptotic theory of highly oscillatory multidimensional integrals, actually a multivariate extension of the classical method of stationary phase. We note that analogous extensions of Laplace's and steepest descent methods, for integrals where the phase function is not purely imaginary, are presented in Desrosiers & Liu (2014).

Consider the  $n$ -fold integral

$$I_n[F] = \frac{1}{n!} \int_{-1}^1 \cdots \int_{-1}^1 F(x_0, \dots, x_{n-1}) e^{i\omega|x|} dx_0 \cdots dx_{n-1}, \tag{3.1}$$

where  $x = (x_0, \dots, x_{n-1})$ ,  $|x| := x_0 + x_1 + \dots + x_{n-1}$ , and we assume that

$$F(x) = f(x)g(x)^2, \tag{3.2}$$

where the function  $g$  is the *Vandermonde determinant*,

$$g(x) = g(x_0, x_1, \dots, x_{n-1}) = \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k) = \sum_{\pi} (-1)^{\sigma(\pi)} \prod_{k=0}^{n-1} x_k^{\pi(k)}, \tag{3.3}$$

$\pi \in \Pi_n$  being the set of all permutations of length  $n$ , acting for example on the  $n$ -tuple  $(x_0, \dots, x_{n-1})$ . In the sequel, we will also assume that the smooth function  $f$  is *symmetric in its arguments*.

The oscillatory integral (3.1) can be expanded asymptotically in inverse powers of  $\omega$ . Indeed, the integrand has the canonical form  $F(x)e^{i\omega\varrho(x)}$  of a non-oscillatory function  $F(x)$  multiplying an oscillatory exponential, with the so-called oscillator  $\varrho(x)$ —in this case, simply the linear function  $\varrho(x) = |x| = x_0 + x_1 + \dots + x_{n-1}$ . It is well known how to derive such expansion, for example using repeated integration by parts when  $F(x)$  is smooth (see, e.g. Wong, 2001). This is straightforward in principle but hampered by lengthy algebraic manipulations in our high-dimensional setting, since (3.1) is an  $n$ -fold integral. In the following, we will use the multi-index notation of Iserles & Nørsett (2006) to control the complexity.

**DEFINITION 3.1** For  $n \geq 1$ , we write  $\mathcal{Y}_n$  to denote the set of the  $2^n$  vertices of the  $n$ -dimensional cube  $[-1, 1]^n$ . For any  $v \in \mathcal{Y}_n$ , the index function  $s(v)$  of  $v \in \mathcal{Y}_n$  is the number of  $-1$  therein. We denote the set of vertices in  $\mathcal{Y}_n$  with  $r$  coordinates equal to  $-1$  as  $\mathcal{Y}_{n,r}$ , for  $r = 0, 1, \dots, n$ .

**PROPOSITION 3.2** As  $\omega \rightarrow \infty$ , the integral (3.1) admits the following asymptotic expansion:

$$\int_{[-1,1]^n} F(x)e^{i\omega|x|} dx \sim (-1)^n \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+n}} \sum_{|k|=m} \sum_{v \in \mathcal{Y}_n} (-1)^{s(v)} e^{i\omega|v|} \partial_x^k F(v). \tag{3.4}$$

Here,  $k = [k_0, k_1, \dots, k_{\ell}]$ , with  $k_j \in \mathbb{N}$ , is a multi-index and  $|k| = k_0 + k_1 + \dots + k_m$ , so  $\partial_x^k = \partial_{x_0}^{k_0} \partial_{x_1}^{k_1} \dots \partial_{x_m}^{k_m}$ .

Note that each term in the expansion, corresponding to some negative power of  $\omega$ , consists of summing over all partial derivatives of a certain total order  $m$  over all possible vertices of the cube  $[-1, 1]^n$ . One may think of these derivatives as originating from the integration by parts technique, and they are evaluated at the vertices because the endpoints of all univariate integrals involved are either  $+1$  or  $-1$ , and in this case, the integrand has no singularities or stationary points. Once one wants to study the large- $\omega$  expansion of these multiple oscillatory integrals, the main task at hand is to determine which vertices of  $\mathcal{Y}_n$  provide the leading term in the asymptotic expansion, and to calculate their contributions. Note that all derivatives of  $F(v)$  appear, for any multi-index  $k$ , in (3.4). In the main examples of interest

in this paper, because of the particular structure of  $F(v)$  in (3.2), many of these multi-indices feature together with a zero derivative and can be discarded, so determining precisely the order of the leading term, as will transpire in the sequel, is a delicate combinatorial task.

We note that the quantities of interest in this paper (Hankel determinants and kissing polynomials) can be written, using classical Heine's formula, as multiple integrals of the form (3.1). This is the starting point for their large- $\omega$  asymptotic analysis in this section.

### 3.2 Asymptotic analysis of Hankel determinants

The first step, which is a well-known identity in the theory of OPs and random matrices, is Heine's formula, see Ismail (2005) and Szegő (1939), that expresses the Hankel determinant as a multiple integral. While the proof of this result is well known, it is useful to provide it since it illuminates much of the work of this chapter.

LEMMA 3.3 For every  $n \in \mathbb{Z}_+$  it is true that

$$h_{n-1}(\omega) = \frac{1}{n!} \int_{-1}^1 \int_{-1}^1 \cdots \int_{-1}^1 \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k)^2 e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0. \quad (3.5)$$

*Proof.* We write the determinant in the following form:

$$\begin{aligned} h_{n-1}(\omega) &= \det \left( \int_{-1}^1 x_k^{j+k} e^{i\omega x_k} dx_k \right)_{j,k=0}^{n-1} \\ &= \int_{-1}^1 \int_{-1}^1 \cdots \int_{-1}^1 \det \left( x_k^{j+k} \right)_{j,k=0}^{n-1} e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0 \\ &= \int_{-1}^1 \int_{-1}^1 \cdots \int_{-1}^1 \prod_{k=0}^{n-1} x_k \det \left( x_k^j \right)_{j,k=0}^{n-1} e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0 \\ &= \int_{-1}^1 \int_{-1}^1 \cdots \int_{-1}^1 \prod_{k=0}^{n-1} x_k^k \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k) e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0, \end{aligned}$$

using the well-known formula for the determinant of a Vandermonde matrix, and the notation  $|x| = x_0 + x_1 + \cdots + x_{n-1}$  for the linear phase function. Let  $\pi$  be a permutation of  $(0, 1, \dots, n-1)$ . Then, changing the order of integration,

$$h_{n-1}(\omega) = (-1)^{\sigma(\pi)} \int_{-1}^1 \cdots \int_{-1}^1 \prod_{k=0}^{n-1} x_{\pi(k)}^k \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k) e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0,$$

where  $\sigma(\pi)$  is the sign of the permutation. Averaging over all  $n!$  permutations,

$$h_{n-1}(\omega) = \frac{1}{n!} \int_{-1}^1 \cdots \int_{-1}^1 g(x_0, \dots, x_{n-1}) \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k) e^{i\omega|x|} dx_{n-1} \cdots dx_1 dx_0,$$



where

$$g(x_0, \dots, x_{n-1}) = \sum_{\pi \in \Pi_n} (-1)^{\sigma(\pi)} \prod_{k=0}^{n-1} x_{\pi(k)}^k$$

and  $\Pi_n$  is the set of all the permutations of  $(0, 1, \dots, n-1)$ . We observe that  $g$  is itself the determinant of an  $n \times n$  Vandermonde matrix. Therefore,

$$g(x_0, \dots, x_{n-1}) = \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k) \quad (3.6)$$

and the proof of (3.5) is complete.  $\square$

It is clear from this result that the Hankel determinant  $h_{n-1}(\omega)$  corresponds to the choice  $f \equiv 1$  in (3.2). Using (3.6), we note that the non-oscillatory function

$$F(x) = \frac{1}{n!} g(x)^2 = \frac{1}{n!} \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k)^2, \quad n \in \mathbb{Z}_+ \quad (3.7)$$

is a polynomial of total degree  $(n-1)n$ . This implies that if  $f \equiv 1$ , then expansion (3.4) terminates, as all derivatives vanish once  $m \geq (n-1)n + 1$ . Since the expansion of  $h_{n-1}$  starts with  $\omega^{-n}$ , because of Proposition 3.2, we expect it to have the form

$$h_{n-1}(\omega) = \sum_{\ell=n}^{n^2} \frac{h_{n-1,\ell}}{\omega^\ell}. \quad (3.8)$$

The reason for the upper bound is that for the last significant value of  $\ell = (n-1)n$ , we have  $\ell + n = n^2$ .

By direct calculation, we find the first few expansions

$$\begin{aligned} h_0(\omega) &= \frac{2 \sin \omega}{\omega}, \\ h_1(\omega) &= \frac{4}{\omega^2} + \frac{2(\cos 2\omega - 1)}{\omega^4}, \\ h_2(\omega) &= -\frac{32 \sin \omega}{\omega^5} - \frac{64 \cos \omega}{\omega^6} + \frac{96 \sin \omega}{\omega^7} - \frac{32 \sin^3 \omega}{\omega^9}, \\ h_3(\omega) &= \frac{256}{\omega^8} + \frac{512(\cos 2\omega - 4)}{\omega^{10}} - \frac{3072 \sin 2\omega}{\omega^{11}} - \frac{768(11 \cos 2\omega - 2)}{\omega^{12}} + \frac{9216 \sin 2\omega}{\omega^{13}} \\ &\quad + \frac{6912(\cos 2\omega - 1)}{\omega^{14}} + \frac{576(\cos 2\omega - 1)^2}{\omega^{16}}. \end{aligned}$$

**REMARK 3.1** It might be deduced from (3.8) that  $h_{n-1}(\omega)$  has a high-order pole at  $\omega = 0$ , but the singularity is in fact removable, since from (3.5) it follows that  $h_{n-1}(\omega)$  is an analytic function of  $\omega$ ; the

case  $\omega = 0$  recovers the Legendre weight function, and the corresponding Hankel determinants are all positive. Actually, as  $\omega \rightarrow 0^+$ , we have

$$\begin{aligned} h_0(\omega) &= 2 - \frac{1}{3}\omega^2 + \frac{1}{60}\omega^4 + \mathcal{O}(\omega^6), \\ h_1(\omega) &= \frac{4}{3} - \frac{8}{45}\omega^2 + \frac{4}{315}\omega^4 + \mathcal{O}(\omega^6), \\ h_2(\omega) &= \frac{32}{135} - \frac{16}{525}\omega^2 + \frac{4}{2025}\omega^4 + \mathcal{O}(\omega^6), \\ h_3(\omega) &= \frac{256}{23625} - \frac{2048}{1488375}\omega^2 + \frac{1024}{11694375}\omega^4 + \mathcal{O}(\omega^6). \end{aligned}$$

The upper bound in (3.8) is sharp. However, the leading powers in these expressions are substantially higher than predicted by (3.8) and the discrepancy becomes more pronounced as  $n$  increases. Instead of (3.8), we have

$$h_{n-1}(\omega) = \sum_{\ell=\varepsilon_n}^{n^2} \frac{h_{n-1,\ell}}{\omega^\ell}, \quad (3.9)$$

where the leading powers are

$$\varepsilon_1 = 1, \quad \varepsilon_2 = 2, \quad \varepsilon_3 = 5, \quad \varepsilon_4 = 8, \quad \varepsilon_5 = 13, \quad \varepsilon_6 = 18, \quad \varepsilon_7 = 25, \dots$$

Note that for even  $n$  the values  $h_{n-1,\varepsilon_n}$  in the examples above are positive constants: this indicates the existence of even-degree polynomials for sufficiently large  $\omega$ . The factor  $\sin \omega$ , however, appearing for odd  $n$ , indicates asymptotic non-existence of odd-degree polynomials, approximately at integer multiples of  $\pi$  as  $\omega \rightarrow \infty$ .

The quest for the leading-order term in expansion (3.9) revolves around the study of the derivatives of the integrand at the vertices of the hypercube  $[-1, 1]^n$ . In particular, it is clear from the explicit expression (3.7) that the integrand vanishes to some order whenever two coordinates  $x_\ell$  and  $x_k$  coincide. This is the case at the vertices and, loosely speaking, the order is determined mostly by the difference between the number of  $+1$ s and  $-1$ s at a vertex.

Because of symmetry, without loss of generality, it is sufficient to consider the case of a vertex that contains  $r$  components equal to  $-1$ , i.e.  $s(v) = r$ , whereby we have

$$v = (\overbrace{-1, \dots, -1}^{r \text{ times}}, \overbrace{+1, \dots, +1}^{n-r \text{ times}}). \quad (3.10)$$

**DEFINITION 3.4** We consider the following factorization:

$$F(x) = f(x)\alpha_r^2(x_0, \dots, x_{r-1})\alpha_{n-r}^2(x_r, \dots, x_{n-1})\beta_{n,r}^2(x_0, \dots, x_{n-1}), \quad (3.11)$$

where

$$\alpha_r(x) = \prod_{0 \leq k < l \leq r-1} (x_l - x_k) \quad \text{and} \quad \beta_{n,r}(x) = \prod_{k=0}^{r-1} \prod_{l=r}^{n-1} (x_l - x_k). \tag{3.12}$$

Next, we need to consider derivatives of the function  $F(x)$ , evaluated at the set of vertices  $\mathcal{V}_n$ , as indicated by formula (3.4). A first important simplification is the following: by construction,  $\beta_{n,r}$  does not vanish at any vertex in  $\mathcal{V}_n$ , in fact  $\beta_{n,r}(v) = 2^{r(n-r)}$  for any  $v \in \mathcal{V}_n$ , whereas  $\alpha_r$  and  $\alpha_{n-r}$  may vanish at any  $v \in \mathcal{V}_n$ ; as a consequence, we can concentrate on derivatives of the function

$$F_\alpha(x) = \alpha_r(x)^2 \alpha_{n-r}(x)^2, \tag{3.13}$$

since any term involving derivatives of  $\beta_{n,r}$  will necessarily contain either  $\alpha_r$  or  $\alpha_{n-r}$  terms, whose value at a vertex  $v \in \mathcal{V}_n$  is 0. More precisely, we have the following result.

**PROPOSITION 3.5** Assume that  $f(v)$  does not vanish at any vertex in  $\mathcal{V}_n$ . The derivative of  $F_\alpha(x)$ , given by (3.13), evaluated at a vertex  $v \in \mathcal{V}_n$ , is nonzero only if the multi-index  $k = [k_0, \dots, k_{n-1}]$ , is split into  $k^{[1]} = [k_0, \dots, k_{r-1}]$  and  $k^{[2]} = [k_r, \dots, k_{n-1}]$ , where

$$k^{[1]} = \pi_1^{[1]} + \pi_2^{[1]}, \quad k^{[2]} = \pi_1^{[2]} + \pi_2^{[2]}, \quad \pi_i^{[1]} \in \Pi_r, \quad \pi_i^{[2]} \in \Pi_{n-r}.$$

Moreover, in that case,

$$\begin{aligned} \partial_x^k F_\alpha(v) &= G^2(r+1)G^2(n-r+1) \\ &\times \sum_{\pi_1^{[1]} + \pi_2^{[1]} = k^{[1]}} (-1)^{\sigma(\pi_1^{[1]}) + \sigma(\pi_2^{[1]})} \prod_{i=0}^{r-1} \binom{\pi_{1,i}^{[1]} + \pi_{2,i}^{[1]}}{\pi_{1,i}^{[1]}} \\ &\times \sum_{\pi_1^{[2]} + \pi_2^{[2]} = k^{[2]}} (-1)^{\sigma(\pi_1^{[2]}) + \sigma(\pi_2^{[2]})} \prod_{i=0}^{n-r-1} \binom{\pi_{1,i}^{[2]} + \pi_{2,i}^{[2]}}{\pi_{1,i}^{[2]}}, \end{aligned} \tag{3.14}$$

where  $G$  is the Barnes function,  $G(n) = \prod_{\ell=1}^{n-2} \ell!$ ; see Olver *et al.* (2021, Section 5.17).

*Proof.* Because  $f$  does not vanish at any vertex of  $\mathcal{V}_n$ , we need be concerned just with  $\alpha_r$  at  $-1$  and  $\alpha_{n-r}$  at  $+1$ . Since  $\alpha_r(x+c1) = \alpha_r(x)$  for all  $x \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$  because  $\alpha_r$  only depends on the differences between elements of  $x$ , it is sufficient to examine these expansions at  $x = 0$ .

It follows from (3.12) that  $\alpha_r(x)$  is a Vandermonde determinant, and then

$$\alpha_r(x) = \sum_{\pi \in \Pi_r} (-1)^{\sigma(\pi)} x_0^{\pi_0} x_1^{\pi_1} \dots x_{r-1}^{\pi_{r-1}},$$

where  $\Pi_r$  is the set of permutations of length  $r$  and  $\sigma(\pi)$  is the *sign* of  $\pi$ . We deduce that  $\partial_x^k \alpha_r(0) = 0$  unless  $k = \pi \in \Pi_r$ . In the latter case,

$$\partial_x^k \alpha_r(0) = (-1)^{\sigma(\pi)} \prod_{j=0}^{r-1} \pi_j! = (-1)^{\sigma(\pi)} G(r+1).$$

Consequently, by Leibniz's formula,

$$\begin{aligned} \partial_x^k \alpha_r^2(0) &= \sum_{k_1+k_2=k} \prod_{i=0}^{r-1} \binom{k_{1,i}+k_{2,i}}{k_{1,i}} \partial_x^{k_1} \alpha_r(0) \partial_x^{k_2} \alpha_r(0) \\ &= \sum_{\pi_1+\pi_2=\pi} \prod_{i=0}^{r-1} \binom{\pi_{1,i}+\pi_{2,i}}{\pi_{1,i}} (-1)^{\sigma(\pi_1)+\sigma(\pi_2)} G^2(r+1), \end{aligned}$$

with  $\pi_1, \pi_2 \in \Pi_r$  both permutations of length  $r$ , i.e. the only terms surviving in Leibniz's formula are those for which  $k_1$  and  $k_2$  are permutations.

Using the multi-index  $k = [k_0, \dots, k_{n-1}]$ , along with the definitions  $k^{[1]} = [k_0, \dots, k_{r-1}]$  and  $k^{[2]} = [k_r, \dots, k_{n-1}]$ , we have

$$\partial_x^k F_\alpha(v) = \partial_x^k [\alpha_r^2(-1)\alpha_{n-r}^2(+1)] = \partial_x^{k^{[1]}} [\alpha_r^2(0)] \partial_x^{k^{[2]}} [\alpha_{n-r}^2(0)].$$

This derivative is nonzero *only* for  $k^{[1]} = \pi_1^{[1]} + \pi_2^{[1]}$  and  $k^{[2]} = \pi_1^{[2]} + \pi_2^{[2]}$ , where  $\pi_i^{[1]} \in \Pi_r$  and  $\pi_i^{[2]} \in \Pi_{n-r}$ . Combined with the above, we arrive at (3.14).  $\square$

The expression (3.14) is only semi-explicit and it is fairly difficult to proceed analytically with conditions of the form  $k = \pi_1 + \pi_2$ . However, the expression is valid for any index  $k$ , and we are only interested in the derivative that corresponds to the leading order term in expansion (3.1). That is, we aim for the leading order term in (3.1), which corresponds to the smallest  $m = |k|$  such that  $\partial_x^k F(v)$  does not vanish.

It is clear from the preceding analysis that the number of  $(-1)$ s and  $(+1)$ s in  $v$  plays a crucial role in determining the leading term. For this reason, we give the following definition.

**DEFINITION 3.6** Given a vertex  $v \in \mathcal{V}_n$ , we define the *weight* of  $v$  as the difference (in absolute value) between the number of  $(+1)$ s and the number of  $(-1)$ s in  $v$ .

So far, we have been considering vertices with  $r$   $(-1)$ s and  $n-r$   $(+1)$ s, so the weight is  $|n-2r|$ .

It is straightforward to verify that for a derivative of order  $k$ , if we split the index into  $k^{[1]} = \pi_1^{[1]} + \pi_2^{[1]}$  and  $k^{[2]} = \pi_1^{[2]} + \pi_2^{[2]}$ , where  $\pi_i^{[1]} \in \Pi_r$  and  $\pi_i^{[2]} \in \Pi_{n-r}$ , then we have

$$|k| = (r-1)r + (n-r-1)(n-r), \quad (3.15)$$

since for  $\pi \in \Pi_m$  it is true that  $|\pi| = (m-1)m/2$ .

Since  $d|k|/dr = -2(n - 2r)$ , it follows that  $|k|$  is minimal for vertices with minimal weight, according to the previous definition. If  $n = 2N$  is even, this leads to  $r = n/2 = N$  and  $|k| = 2N(N - 1)$ , and if to  $r = (n + 1)/2$  or  $r = (n - 1)/2$  for odd  $n$ .

In the next result, we calculate the contributions given by the vertices in  $\mathcal{V}_{2N,N}$  (even case) and  $\mathcal{V}_{2N+1,N}$  and  $\mathcal{V}_{2N+1,N+1}$  (odd case).

**THEOREM 3.7** Let  $f(x) = f(x_0, x_1, \dots, x_{n-1})$  be a symmetric function of its  $n$  arguments.

- (i) If  $n = 2N$  is even, and  $v \in \mathcal{V}_{2N,N}$  is a vertex with weight 0 and such that  $f(v) \neq 0$ , then as  $\omega \rightarrow \infty$

$$I_{2N}[f] = \frac{4^{N^2}}{\omega^{2N^2}} f(v) G^4(N + 1) + \mathcal{O}(\omega^{-2N^2-1}). \quad (3.16)$$

- (ii) If  $n = 2N + 1$  is odd, and  $v_1 \in \mathcal{V}_{2N+1,N}$  and  $v_2 \in \mathcal{V}_{2N+1,N+1}$  are two vertices with weight 1, if  $f(v_1), f(v_2) \neq 0$ , then as  $\omega \rightarrow \infty$

$$I_{2N+1}[f] = -\frac{(-1)^{N+1}}{\omega^{2N(N+1)+1}} 4^{N(N+1)} G^2(N + 1) G^2(N + 2) [e^{i\omega} f(v_1) - e^{-i\omega} f(v_2)] + \mathcal{O}(\omega^{-2N(N+1)-2}). \quad (3.17)$$

If in addition  $f$  is an even function, then as  $\omega \rightarrow \infty$ , we have

$$I_{2N+1}[f] = \frac{2(-1)^N}{\omega^{2N(N+1)+1}} 4^{N(N+1)} G^2(N + 1) G^2(N + 2) f(v_1) \sin \omega + \mathcal{O}(\omega^{-2N(N+1)-2}).$$

*Proof.* Let us examine the factors in the expansion (3.1). There are  $\binom{n}{r}$  vertices with  $r$   $(-1)$ 's, permutations of (3.10): this is precisely the set  $\mathcal{V}_{n,r}$ . In the first case,  $r = n/2 = N$ . For each vertex with minimal weight, i.e. for each  $v \in \mathcal{V}_{2N,N}$ , we have  $(-1)^{s(v)} = (-1)^r = (-1)^N$  and  $e^{i\omega v^T 1} = 1$ . Furthermore,  $\beta_{n,r}(v)^2 = 4^{r(n-r)} = 4^{N^2}$  and  $f$  was assumed to be a symmetric function in its variables; hence,  $f$  is constant on  $\mathcal{V}_{2N,N}$ .

We have to sum over all derivatives of total order  $|k| = 2N(N - 1)$ . Since  $|k|$  is minimal, it follows from Leibniz's formula that

$$\partial_x^k F(v) = 4^{N^2} f(v) \partial_x^k F_\alpha(v),$$

where  $4^{N^2}$  is the contribution of  $\beta_{2N,N}^2$ . Each possible  $k$  is reached by a combination of permutations of length  $N$ . From (3.14), we find that

$$\begin{aligned} \sum_k \partial_x^k F_\alpha(v) &= G^4(N+1) \times \sum_{\pi_1^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_1^{[1]})} \sum_{\pi_2^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_2^{[1]})} \prod_{i=0}^{N-1} \binom{\pi_{1,i}^{[1]} + \pi_{2,i}^{[1]}}{\pi_{1,i}^{[1]}} \\ &\times \sum_{\pi_1^{[2]} \in \Pi_N} (-1)^{\sigma(\pi_1^{[2]})} \sum_{\pi_2^{[2]} \in \Pi_N} (-1)^{\sigma(\pi_2^{[2]})} \prod_{i=0}^{N-1} \binom{\pi_{1,i}^{[2]} + \pi_{2,i}^{[2]}}{\pi_{1,i}^{[2]}}. \end{aligned}$$

Identifying a sum with a determinant and permuting rows,

$$\sum_{\pi_2 \in \Pi_s} (-1)^{\sigma(\pi_2)} \prod_{i=0}^{s-1} \binom{\pi_{1,i} + \pi_{2,i}}{\pi_{1,i}} = \det(A_{\pi_{1,i,j}^{[s]}})_{i,j=0,\dots,s-1} = (-1)^{\sigma(\pi_1)} \det(A_{i,j}^{[s]})_{i,j=0,\dots,s-1},$$

where  $A_{i,j}^{[s]} = \binom{i+j}{j}$ ,  $i, j = 0, \dots, s-1$ .

It is easy to see that  $\det A^{[s]} = 1$ . Indeed, it follows from the definition of  $A_{i,j}^{[s]}$  that

$$A_{i,j}^{[s]} - A_{i,j-1}^{[s]} = A_{i-1,j}^{[s]}, \quad i, j = 1, \dots, s-1,$$

so subtracting the  $(j-1)$ st from the  $j$ th column, we have

$$\det A^{[s]} = \det \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & A^{[s-1]} \end{bmatrix}.$$

Consequently,  $\det A^{[s]} = \det A^{[s-1]}$ , and by induction, we have  $\det A^{[s]} = 1$  for  $s \geq 1$ . Alternatively, this result follows from identifying  $A^{[s]}$  as a classical Pascal matrix; see for instance [Edelman & Strang \(2004\)](#).

Therefore,

$$\sum_{\pi_1^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_1^{[1]})} \sum_{\pi_2^{[1]} \in \Pi_N} (-1)^{\sigma(\pi_2^{[1]})} \prod_{i=0}^{N-1} \binom{\pi_{1,i}^{[1]} + \pi_{2,i}^{[1]}}{\pi_{1,i}^{[1]}} = N! \quad (3.18)$$

and the same holds for  $\pi_1^{[2]}$  and  $\pi_2^{[2]}$  instead of  $\pi_1^{[1]}$  and  $\pi_2^{[1]}$ . Consequently,

$$\sum_k \partial_x^k F_\alpha(v) = (N!)^2 G^4(N+1).$$

Assembling everything in formula (3.4), the term corresponding to  $m = |k| = 2N(N - 1)$  in the asymptotic expansion becomes

$$\frac{(-1)^N 4^{N^2}}{(2N)!} \frac{1}{(-i\omega)^{2N^2}} \binom{2N}{N} (N!)^2 G(N+1)^4 = \frac{4^{N^2}}{\omega^{2N^2}} G(N+1)^4,$$

and so we arrive at (3.16).

Similar considerations hold in the odd case. In this case, either  $r = N$  and  $n - r = N + 1$  or vice versa, but these two cases are symmetric. They correspond to the sets  $\mathcal{Y}_{2N+1,N}$  and  $\mathcal{Y}_{2N+1,N+1}$  respectively.

For  $v \in \mathcal{Y}_{2N+1,N}$ , we have  $(-1)^{s(v)} = (-1)^N$  and  $e^{i\omega|v|} = e^{i\omega}$ . For  $v \in \mathcal{Y}_{2N+1,N+1}$ , we have  $(-1)^{s(v)} = (-1)^{N+1}$  and  $e^{i\omega|v|} = e^{-i\omega}$ . Furthermore, in both cases,  $\beta_{2N+1,N}(v) = 4^{N(N+1)}$  and  $m = |k| = 2N^2$ .  $\square$

In the particular case  $f(x) \equiv 1$ , we have the following asymptotic result for Hankel determinants.

**COROLLARY 3.8** As  $\omega \rightarrow \infty$ , the Hankel determinants satisfy

$$\begin{aligned} h_{2N-1}(\omega) &= \frac{4^{N^2}}{\omega^{2N^2}} G^4(N+1) + \mathcal{O}(\omega^{-2N^2-1}), \\ h_{2N}(\omega) &= \frac{2(-1)^N}{\omega^{2N(N+1)+1}} 4^{N(N+1)} G^2(N+1) G^2(N+2) \sin \omega + \mathcal{O}(\omega^{-2N(N+1)-2}). \end{aligned} \quad (3.19)$$

This asymptotic result is consistent with the existence of the kissing polynomial of even degree  $p_{2N}(z)$  for all  $\omega \geq 0$  (therefore,  $h_{2N-1}(\omega) \neq 0$ ). In the case of odd degree kissing polynomials, the corollary gives an estimate of kissing points for large  $\omega$ , asymptotically equispaced in line with zeros of the sine function.

### 3.3 Asymptotic behaviour of recurrence coefficients

We can use the previous results, in tandem with formulas (2.1) and (2.2), to derive large  $\omega$  asymptotics for the subleading coefficient  $\delta_{n,n-1}(\omega)$  and the recurrence coefficients  $\alpha_n(\omega)$  and  $\beta_n(\omega)$ .

Note that  $h_{n-1}(\omega)$  is an analytic function of  $\omega$ , so the derivative with respect to  $\omega$  has a similar asymptotic expansion:

$$\begin{aligned} \dot{h}_{2N-1}(\omega) &= -\frac{4^{N^2} 2N^2}{\omega^{2N^2+1}} G^4(N+1) + \mathcal{O}(\omega^{-2N^2-2}), \\ \dot{h}_{2N}(\omega) &= \frac{2(-1)^N}{\omega^{2N(N+1)+1}} 4^{N(N+1)} G^2(N+1) G^2(N+2) \cos \omega + \mathcal{O}(\omega^{-2N(N+1)-2}). \end{aligned} \quad (3.20)$$

Therefore, we have the following asymptotic results:

**PROPOSITION 3.9** For  $N \geq 0$ , the subleading coefficient and the recurrence coefficients of the kissing polynomial have the following asymptotic expansion as  $\omega \rightarrow \infty$ , excluding arbitrarily small but fixed

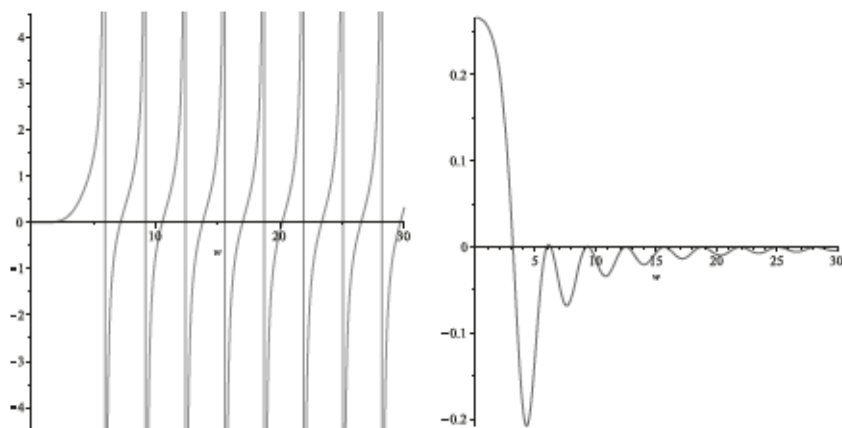


FIG. 4. Recurrence coefficients for increasing  $\omega$ ,  $\text{Im } \alpha_2(\omega)$  on the left and  $\text{Re } \beta_2(\omega)$  on the right.

neighbourhoods of the points  $\omega = k\pi$ ,  $k \in \mathbb{Z}$ :

$$\begin{aligned} \delta_{2N,2N-1}(\omega) &= i \frac{h_{2N-1}(\omega)}{h_{2N-1}(\omega)} = -i \frac{2N^2}{\omega} \left[ 1 + \mathcal{O}(\omega^{-1}) \right], \\ \delta_{2N+1,2N}(\omega) &= i \frac{\dot{h}_{2N}(\omega)}{h_{2N}(\omega)} = i \cot \omega \left[ 1 + \mathcal{O}(\omega^{-1}) \right], \\ \alpha_N(\omega) &= -i \left[ \frac{\dot{h}_N(\omega)}{h_N(\omega)} - \frac{\dot{h}_{N-1}(\omega)}{h_{N-1}(\omega)} \right] = (-1)^{N+1} i \cot \omega \left[ 1 + \mathcal{O}(\omega^{-1}) \right], \\ \beta_{2N}(\omega) &= \frac{h_{2N}(\omega)h_{2N-2}(\omega)}{h_{2N-1}^2(\omega)} = -\frac{4\Gamma(N+1)^2 \sin^2 \omega}{\Gamma(N)^2 \omega^2} \left[ 1 + \mathcal{O}(\omega^{-1}) \right], \\ \beta_{2N+1}(\omega) &= \frac{h_{2N+1}(\omega)h_{2N-1}(\omega)}{h_{2N}^2(\omega)} = \frac{1}{\sin^2 \omega} \left[ 1 + \mathcal{O}(\omega^{-1}) \right]. \end{aligned}$$

In Figs 4 and 5, we exhibit the recurrence coefficients  $\alpha_2(\omega)$ ,  $\beta_2(\omega)$ ,  $\alpha_3(\omega)$  and  $\beta_3(\omega)$ , as functions of increasing  $\omega$ . The recurrence coefficients were computed explicitly in Maple using the Hankel determinants and formula (2.1). It is particularly interesting to observe the different behaviour of  $\beta_n(\omega)$  depending on the parity of  $n$ . Recall from the proof of Lemma 2.10 that a simple root of  $h_n$  is associated with a double pole of  $\beta_{n+1}$  and simple roots of  $\beta_n$  and  $\beta_{n+2}$ ; that behaviour is consistent with this figure.

### 3.4 Asymptotic behaviour of the kissing polynomials near the endpoints

It is conjectured and motivated in [Asheim et al. \(2014\)](#) that the polynomials are approximately a multiple of Laguerre polynomials near the endpoints  $\pm 1$ , namely

$$p_{2n}(x) \sim \left( \frac{i}{\omega} \right)^{2n} L_n(-i\omega(x+1))L_n(-i\omega(x-1)), \quad \omega \rightarrow \infty, \quad (3.21)$$



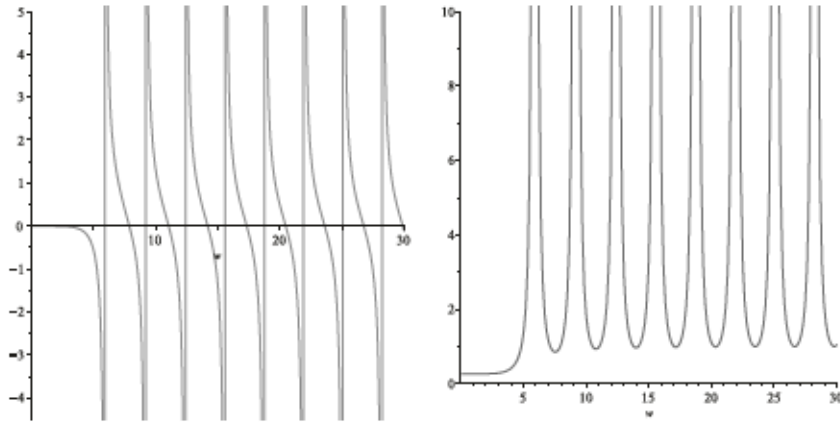


FIG. 5. Recurrence coefficients for increasing  $\omega$ ,  $\text{Im } \alpha_3(\omega)$  on the left and  $\text{Re } \beta_3(\omega)$  on the right.

where  $L_n$  is the  $n$ th Laguerre polynomial with parameter  $\alpha = 0$ ; see Szegő (1939). Thus, for large  $\omega$ , it seems that the OPs of even degree become approximately a product of lower degree OPs. This conjecture also implies that the roots shown in Fig. 1 behave like  $\pm 1 + \frac{ic}{\omega}$  for  $\omega \gg 1$ , where  $c$  is a root of the Laguerre polynomial  $L_n$ .

The connection with Laguerre polynomials in Asheim *et al.* (2014) arose from the role of the kissing polynomials in Gaussian quadrature for oscillatory integrals, in particular when applying the method of steepest descent. The lines emanating from the endpoints  $\pm 1$  upwards into the complex plane, parallel to the imaginary axis, are the steepest descent paths for the oscillatory weight function  $e^{i\omega x}$ . Along these contours, and up to a  $1/\omega$  scaling, the weight behaves like that of the Laguerre OPs:  $e^{-x}$ . The corresponding Gauss–Laguerre quadrature rules are known to be optimal for the evaluation of these two steepest descent integrals separately. The conjecture implies that the Gaussian quadrature rule for  $e^{i\omega x}$  on  $[-1, 1]$  locally behaves like a Gauss–Laguerre rule near both endpoints, for large  $\omega$ . An appealing benefit of the kissing polynomials is that for small  $\omega$  their roots converge to those of Legendre polynomials on  $[-1, 1]$ , whereas the steepest descent quadrature points  $\pm 1 + \frac{ic}{\omega}$  grow unbounded as  $\omega \rightarrow 0$ . For that reason a quadrature scheme based on the latter will always be asymptotic, but a quadrature scheme based on kissing polynomials can be numerically convergent even for small  $\omega$ .

The leading-order term in the asymptotic expansion of Hankel determinants  $h_{n-1}$  has already proved to be very revealing insofar as the existence of the polynomials is concerned. Yet, the expansion carries more information from which we can deduce the asymptotics of the roots. Using Heine’s formula for the OPs themselves,

$$P_n^\omega(x) = \frac{1}{n! h_{n-1}} \int_{-1}^1 \cdots \int_{-1}^1 \prod_{m=0}^{n-1} (x - x_m) \prod_{0 \leq k < \ell \leq n-1} (x_\ell - x_k)^2 e^{i\omega|x|} dx_0 \cdots dx_{n-1}, \quad (3.22)$$

see, e.g. Ismail (2005, Theorem 2.1.2), corresponds to the choice  $f(x) = \prod_{m=0}^{n-1} (x - x_m)$  in (3.1). Thus,

$$\tilde{P}_n \left( 1 - \frac{c}{i\omega} \right) = \frac{1}{n!} I_n \left[ \prod_{m=0}^{n-1} \left( 1 - \frac{c}{i\omega} - x_m \right) \right],$$

and we invoke the theory of Section 3.2 again, this time using

$$f(x_0, \dots, x_{n-1}) = \prod_{m=0}^{n-1} \left(1 - \frac{c}{i\omega} - x_m\right), \quad (3.23)$$

which is again a symmetric function, but now depending on  $\omega$ ; this fact introduces extra technicalities in the asymptotic calculation.

In this section, we prove the following result.

**THEOREM 3.10** As  $\omega \rightarrow \infty$ , the zeros  $\{z_k\}_{k=1}^{2N}$  of the polynomial  $p_{2N}(z)$  satisfy

$$z_k = \pm 1 + \frac{ic_k^{[N]}}{\omega} + \mathcal{O}(\omega^{-2}),$$

where  $c_1^{[N]}, \dots, c_N^{[N]} > 0$  are zeros of the  $N$ -th Laguerre polynomial. The same is true for  $p_{2N+1}(z)$ , except for a single zero on the pure imaginary axis.

**3.4.1 The case  $n = 2N$ .** The analysis in section Section 3.2 needs minor modifications in view of the fact that the function  $f$  itself now depends on  $\omega$ . The derivatives of  $F = fg^2$  have an expansion in inverse powers of  $\omega$  and we have to take this into account. Recall that contributions to the asymptotic expansion (3.4) can be thought of as originating from vertices  $v \in \mathcal{V}_{n,r}$  with  $r$   $(-1)$ s and  $n - r$   $(+1)$ s.

Our first question is, which vertices contribute to the leading order term in the expansion of  $I_n[f]$  in this case? Before, it was  $v \in \mathcal{V}_{2N,N}$ , i.e. vertices with minimal weight. In the current case, in spite of the dependence of  $f$  on  $\omega$ , little changes and the leading order term still originates in vertices with minimal weight.

Consider a general vertex  $v \in \mathcal{V}_{2N,N+t}$ , with  $-N \leq t \leq N$ . Without loss of generality, and using the same multi-index notation as in the previous section, see (3.10), we can take  $v = (-1^{[1]}, 1^{[2]})$ . Since  $f$  in (3.23) is linear in all its components, it is clear that  $\partial_x^k f(v) = 0$  unless  $k_0, k_1, \dots, k_{n-1} \in \{0, 1\}$ . Thus, suppose that  $k^{[1]} \in \mathbb{Z}_+^{N+t}$ ,  $k^{[2]} \in \mathbb{Z}_+^{N-t}$  such that the components satisfy  $k_i^{[1]}, k_i^{[2]} \in \{0, 1\}$ ,  $|k^{[1]}| = \kappa_1$  and  $|k^{[2]}| = \kappa_2$ . Then,

$$\begin{aligned} \partial_x^{(k^{[1]}, k^{[2]})} f(v) &= (-1)^{\kappa_1 + \kappa_2} \left(2 - \frac{c}{i\omega}\right)^{N+t-\kappa_1} \left(-\frac{c}{i\omega}\right)^{N-t-\kappa_2} \\ &= \frac{(-1)^{\kappa_1 + \kappa_2} 2^{N+t-\kappa_1} (-c)^{N-t-\kappa_2}}{(i\omega)^{N-t-\kappa_2}} \left[1 + \mathcal{O}(\omega^{-1})\right]. \end{aligned} \quad (3.24)$$

Note the lack of symmetry here: the roles of  $\kappa_1$  and  $\kappa_2$  are not interchangeable because  $f$  focuses on the right endpoint  $x = +1$ .

There are three contributions to the leading order exponent of  $\omega^{-1}$  in (3.4):

1. the dimension contributes  $n = 2N$ ;
2. the least order non-vanishing derivative of  $F_\alpha$ , recall (3.13), consists of permutations of length  $N+t$  and  $N-t$ , respectively. From (3.15), this contributes  $|k| = 2(N-1)N + 2t^2$  to our exponent;

3. a derivative of degree  $\kappa_1 + \kappa_2$  contributes  $\mathcal{O}(\omega^{-\kappa_1 - \kappa_2})$  to the  $\omega^{-m-n}$  term in (3.4) and, from (3.24), additionally contributes  $\mathcal{O}(\omega^{-N+t+\kappa_2})$  – altogether  $\mathcal{O}(\omega^{-N-\kappa_1+t})$ . We can choose  $\kappa_1 = 0$  (i.e. all derivatives can be only with respect to the trailing  $N - t$  components) and the contribution is  $N - t$ .

The total exponent is therefore  $2N^2 + N + 2t^2 - t$  and this, clearly, is minimized for  $t = 0$ . Though derivatives of  $f$  may contribute positive powers of  $\omega$ , vertices with larger weight (i.e. larger  $t$ ) contribute smaller powers of  $\omega$  and the latter effect is stronger. The resulting leading order behaviour,  $\omega^{-2N^2 - N}$ , is a factor  $\omega^N$  smaller than that in Theorem 3.7 simply because that is the size of  $f$  at an endpoint with weight 0.

Let us examine the derivatives of  $F = fg^2$  further. Corresponding to a vertex  $v \in \mathcal{V}_{2N,N}$ , the leading order term must differentiate  $g^2$  with permutations of length  $N$ . In addition, we may have derivatives of  $f$  with respect to the second set of variables. Denote by  $p_s \in \{0, 1\}^N$  any vector such that  $|p_s| = s$ , for  $s \in \{0, \dots, N\}$ , then we compute the derivatives from the above formula:

$$\partial_x^{(0,p_s)} f(v) = \frac{(-1)^s 2^N (-c)^{N-s}}{(i\omega)^{N-s}} \left[ 1 + \mathcal{O}(\omega^{-1}) \right], \tag{3.25}$$

which corresponds to the choice  $t = 0$ ,  $\kappa_1 = 0$  and  $\kappa_2 = s$  in (3.24). Derivatives of the former have been already analysed in Section 3.2. From (3.14), and recalling that  $\beta_{n,N+t}^2(v) = 2^{2(N+t)(N-t)}$ , so  $\beta_{n,N}^2(v) = 2^{2N^2}$ , we find that

$$\begin{aligned} \partial_x^{(k^{[1]}, k^{[2]})} g^2(v) &= 2^{2N^2} G^4(N+1) \sum_{\pi_1 + \pi_2 = k^{[1]}} (-1)^{\sigma(\pi_1) + \sigma(\pi_2)} \prod_{i=0}^{N-1} \binom{\pi_{1,i} + \pi_{2,i}}{\pi_{1,i}} \\ &\quad \times \sum_{\pi_3 + \pi_4 = k^{[2]}} (-1)^{\sigma(\pi_3) + \sigma(\pi_4)} \prod_{i=0}^{N-1} \binom{\pi_{3,i} + \pi_{4,i}}{\pi_{3,i}}. \end{aligned}$$

Note that the first sum in the previous formula is equal to  $N!$  because of (3.18).

The increase of the order of the derivative of  $F$  in expansion (3.4) comes at a cost of a factor  $(-i\omega)^{-s}$ . On the other hand, a higher-order derivative of  $f$  yields a factor of  $(i\omega)^{-N+s}$ , from (3.25). The product of these factors is  $(-1)^s (i\omega)^{-N}$  and has the same asymptotic size in  $\omega$  for all  $s$ . Hence, we need to consider all  $0 \leq s \leq N$ .

Since derivatives of  $g^2$  vanish unless the order of the derivative is a combination of permutations of length  $N$ , we need to consider all such combinations of permutations and all values of  $s$ . This leads to a sum of terms of the form

$$\partial_x^{(k^{[1]}, k^{[2]} + p_s)} F(v) = \prod_{p_{s,i}=1} (k_i^{[2]} + 1) \partial_x^{(k^{[1]}, k^{[2]})} g^2(v) \partial_x^{(0,p_s)} f(v),$$

for  $s \in \{0, 1, \dots, N\}$  and with  $k^{[1]}$  and  $k^{[2]}$  sums of two permutations in  $\Pi_N$ .

The  $s$ th term in this large sum is

$$\begin{aligned}
 & \sum_{|p_s|=s} \sum_{\pi_1, \pi_2, \pi_3, \pi_4 \in \Pi_N} \prod_{p_{s,j}=1} (k_i^{[2]} + 1) \partial_x^{(k^{[1]}, k^{[2]})} g^2(v) \partial_x^{(0, p_s)} f(v) \\
 = & 2^{2N^2+N} G^4(N+1) (-1)^s (i\omega)^{s-N} (-c)^{N-s} \sum_{|p_s|=s} \sum_{\substack{\pi_j \in \Pi_N \\ j=1,2,3,4}} \prod_{p_{s,j}=1} (\pi_{3,i} + \pi_{4,i} + 1) \\
 & \times (-1)^{\sigma(\pi_1)+\sigma(\pi_2)} \prod_{i=0}^{N-1} \binom{\pi_{1,i} + \pi_{2,i}}{\pi_{1,i}} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \binom{\pi_{3,i} + \pi_{4,i}}{\pi_{3,i}} \\
 = & 2^{2N^2+N} G^4(N+1) (-1)^s (i\omega)^{s-N} (-c)^{N-s} N! \sum_{|p_s|=s} \sum_{\pi_3, \pi_4 \in \Pi_N} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i} + \pi_{4,i} + p_{s,i})!}{\pi_{3,i}! \pi_{4,i}!}.
 \end{aligned}$$

In the last computation, we have used (3.18) and the fact that

$$\begin{aligned}
 & \sum_{\pi_3, \pi_4 \in \Pi_N} \prod_{p_{s,j}=1} (\pi_{3,i} + \pi_{4,i} + 1) (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \binom{\pi_{3,i} + \pi_{4,i}}{\pi_{3,i}} \\
 = & \sum_{\pi_3, \pi_4 \in \Pi_N} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i} + \pi_{4,i} + p_{s,i})!}{\pi_{3,i}! \pi_{4,i}!}.
 \end{aligned}$$

In this last sum, every  $p_s$  consists of  $s$  ones and  $N-s$  zeros; hence, there are  $\binom{N}{s}$  such vectors. Note that each gives exactly the same result because everything else in the relevant expression is constructed from two permutations. Therefore, we might just consider  $\binom{N}{s}$  times the vector

$$p_s^* = (\overbrace{0, \dots, 0}^{N-s \text{ times}}, \overbrace{1, \dots, 1}^{s \text{ times}}),$$

whereby

$$\begin{aligned}
 & \sum_{|p_s|=s} \sum_{\pi_3, \pi_4 \in \Pi_N} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i} + \pi_{4,i} + p_{s,i})!}{\pi_{3,i}! \pi_{4,i}!} \\
 = & \binom{N}{s} \sum_{\pi_3, \pi_4 \in \Pi_N} (-1)^{\sigma(\pi_3)+\sigma(\pi_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i} + \pi_{4,i} + p_{s,i}^*)!}{\pi_{3,i}! \pi_{4,i}!} = \binom{N}{s} \sum_{\pi \in \Pi_N} (-1)^{\sigma(\pi)} \det \mathcal{E}^{[N,s]}(\pi),
 \end{aligned}$$

where

$$\mathcal{E}_{ij}^{[N,s]}(\pi) = \begin{cases} \frac{(\pi_i+j)!}{\pi_i!j!}, & i = 0, \dots, N-s-1, \\ \frac{(\pi_i+j+1)!}{\pi_i!j!}, & i = N-s, \dots, N-1, \end{cases} \quad j = 0, \dots, N-1. \quad (3.26)$$

LEMMA 3.11 Let  $\pi$  be split into  $\pi^{[1]} = [\pi_0, \dots, \pi_{N-s-1}]$  and  $\pi^{[2]} = [\pi_{N-s}, \dots, \pi_{N-1}]$ , where  $\pi^{[1]} \in \Pi_{N-s}$  and  $\pi^{[2]} \in N-s + \Pi_s$ , then

$$(-1)^{\sigma(\pi)} \det \mathcal{E}^{[N,s]}(\pi) = \frac{N!^2}{s!(N-s)!^2}; \quad (3.27)$$

otherwise,  $\det \mathcal{E}^{[N,s]}(\pi) = 0$ .

*Proof.* Suppose first that the hypothesis on  $\pi^{[1]}$  and  $\pi^{[2]}$  holds.

We deduce, rearranging rows separately in the first  $N-s$  and the last  $s$  rows of  $\mathcal{E}^{[N,s]}(\pi)$ , that

$$(-1)^{\sigma(\pi)} \det \mathcal{E}^{[N,s]}(\pi) = \det \mathcal{E}^{[N,s]}(0, 1, \dots, N-1).$$

Since

$$\mathcal{E}_{ij}^{[N,s]}(0, 1, \dots, N-1) = \begin{cases} \binom{i+j}{i}, & i = 0, \dots, N-s-1, \\ (i+1)\binom{i+j+1}{i+1}, & i = N-s, \dots, N-1, \end{cases}$$

and  $j = 0, \dots, N-1$ , once we extract a factor of  $i+1$  from rows  $i = N-s, \dots, N-1$ , the outcome is

$$\mathcal{E}^{[N,s]}(0, 1, \dots, N-1) = \frac{N!}{(N-s)!} \mathcal{C}^{[N,s]}(0, 1, \dots, N-1),$$

where

$$\mathcal{C}_{ij}^{[N,s]}(0, 1, \dots, N-1) = \begin{cases} \binom{i+j}{i}, & i = 0, \dots, N-s-1, \\ \binom{i+j+1}{i+1}, & i = N-s, \dots, N-1, \end{cases} \quad j = 0, \dots, N-1.$$

The matrix  $\mathcal{C}$  is somewhat easier to manipulate. First, note that  $\mathcal{C}^{[N,0]}$  equals the Pascal matrix  $A^{[N]}$  used in the analysis before; hence,  $\det \mathcal{C}^{[N,0]} = 1$ .

In the case  $s = 1$ , easy calculation with binomial numbers confirms that

$$\mathcal{C}_{ij}^{[N,1]} - \mathcal{C}_{ij-1}^{[N,1]} = \mathcal{C}_{i-1j}^{[N,1]}, \quad j = 0, \dots, N-2$$

and

$$\mathcal{C}_{i,N-1}^{[N,1]} - \mathcal{C}_{i,N-2}^{[N,1]} = \binom{N+i-1}{i-1} + \binom{N+i-2}{i-1}.$$

Therefore,

$$\det \mathcal{C}^{[N,1]} = \det \begin{bmatrix} 1 & \mathbf{0}^\top \\ 1 & \mathcal{C}^{[N-1,1]} + \mathcal{C}^{[N-1,0]} \end{bmatrix},$$

where both matrices in the lower right block differ in only one column. This leads to

$$\det \mathcal{C}^{[N,1]} = \det \mathcal{C}^{[N-1,1]} + \det \mathcal{C}^{[N-1,0]} = \det \mathcal{C}^{[N-1,1]} + 1$$

and we deduce that  $\det \mathcal{C}^{[N,1]} = N$ .

Let us generalize the above to larger  $s$ . Note that

$$\mathcal{C}_{ij}^{[N,s]} - \mathcal{C}_{i,j-1}^{[N,s]} = \begin{cases} \mathcal{C}_{i-1,j}^{[N-1,s]} = \mathcal{C}_{i-1,j}^{[N,s-1]}, & j = 0, \dots, N-s-1, \\ \mathcal{C}_{i-1,N-s}^{[N-1,s]} + \mathcal{C}_{i-1,N-s-1}^{[N-1,s-1]}, & j = N-s, \\ \mathcal{C}_{i-1,j}^{[N-1,s]} = \mathcal{C}_{i-1,j}^{[N,s-1]}, & j = N-s+1, \dots, N-1, \end{cases}$$

An argument identical to the one we have used before shows that

$$\det \mathcal{C}^{[N,s]} = \det \mathcal{C}^{[N-1,s]} + \det \mathcal{C}^{[N-1,s-1]}, \quad s = 0, 1, \dots, N.$$

In case  $N = s$ , the same reasoning leads to  $\det \mathcal{C}^{[N,N]} = \det \mathcal{C}^{[N,0]} = 1$ . Induction then shows that  $\det \mathcal{C}^{[N,s]} = \binom{N}{s}$ , and (3.27) must be true.

Suppose now that  $\pi$  does not belong to  $\Pi_{N-s} \oplus (N-s + \Pi_s)$ , which makes sense only when  $s \in \{1, \dots, N-1\}$ . Then there exists (at least) one integer  $r \geq N-s$  such that  $\pi_r \in \pi^{[2]}$  and  $\pi_r = N-s-i$  for some  $i \geq 1$ . Among those, we choose  $r$  so that  $\pi_r$  is minimum, and we take  $t$  to be the index such that  $\pi_t = \pi_r + 1$ . Then either  $\pi_t \in \pi^{[1]}$  or  $\pi_t \in \pi^{[2]}$ . In the first case, using (3.26), the  $r$ -th row of  $\mathcal{C}^{[N,s]}(\pi)$  is

$$\frac{(N-s-i+j+1)!}{(N-s-i)!j!}, \quad j = 0, \dots, N-1,$$

and the  $t$ -th row of is

$$\frac{(N-s-i+j+1)!}{(N-s-i+1)!j!}, \quad j = 0, \dots, N-1.$$

Since the  $t$ -th row is a scalar multiple of the  $r$ -th row, the determinant vanishes and the proposition is true. If  $\pi_r \in \pi^{[2]}$ , then we repeat the previous reasoning with the index  $t$  and the index  $u$  such that  $\pi_u = \pi_t + 1$ . We continue this process and at some point we must find two indices with the property above, since it cannot happen that all permutations in  $\pi^{[1]}$  take smaller values than those in  $\pi^{[2]}$ .  $\square$

Let us assemble everything. Since there are precisely  $s!(N-s)!$  permutations belonging to  $\Pi_{N-s} \oplus (N-s + \Pi_s)$ , which produce nonzero determinants in the proposition, we obtain

$$\sum_{|p_s|=s} \sum_{\pi_3, \pi_4 \in \Pi_N} (-1)^{\sigma(\pi_3) + \sigma(\pi_4)} \prod_{i=0}^{N-1} \frac{(\pi_{3,i} + \pi_{4,i} + p_{s,i})!}{\pi_{3,i}! \pi_{4,i}!} = \frac{N!^3}{s!(N-s)!^2}$$

for  $s = 0, \dots, N$ .

The contribution of  $v$  includes the derivatives derived above, as well as an additional factor  $(-i\omega)^{-s}$  for each  $s$  arising from the  $\mathcal{O}(\omega^{-m-n})$  term in (3.4), and the factor  $(-1)^{\sigma(v)} = (-1)^N$ . This totals, after some manipulations,

$$\begin{aligned} & \sum_{s=0}^N (-1)^N (-i\omega)^{-s} 2^{2N^2+N} G(N+1)^4 (-1)^s (i\omega)^{s-N} (-c)^{N-s} \frac{N!^4}{s!(N-s)!^2} \\ &= (-i\omega)^{-N} 2^{2N^2+N} G(N+1)^4 N!^3 \sum_{s=0}^N \frac{1}{s!} \binom{N}{s} (-c)^s \\ &= (-i\omega)^{-N} 2^{2N^2+N} \frac{G(N+2)^4}{N!} L_N(c), \end{aligned}$$

where the latter simplification follows from the known explicit expression

$$L_N(c) = \sum_{s=0}^N \frac{1}{s!} \binom{N}{s} (-c)^s. \quad (3.28)$$

See for instance Chihara (1978, Chapter V) or Szegő (1939, Chapter 5).

There are  $\binom{2N}{N}$  vertices in  $\mathcal{V}_{2N,N}$ ; hence, the leading term in the expansion of the polynomial  $\check{p}_{2N}(1 - c/(i\omega))$  is

$$\frac{2^{2N^2+N} G(N+2)^4}{(-i\omega)^{2N^2} (-i\omega)^N N!^3} L_N(c) = (-i)^N \frac{2^{2N^2+N} G(N+2)^4}{\omega^{2N^2+N} N!^3} L_N(c).$$

Finally, we can divide by the leading term of  $h_{2N-1}$  (recall Theorem 3.7), which is  $4^{N^2} G(N+1)^4 \omega^{-2N^2}$ , to obtain the leading term of the monic polynomial:

$$p_{2N}\left(1 - \frac{c}{i\omega}\right) = \frac{(-2i)^N N!}{\omega^N} L_N(c) \left(1 + \mathcal{O}(\omega^{-1})\right), \quad \omega \rightarrow \infty.$$

This is precisely the result we wanted to show, in the case of polynomials of even degree.

**3.4.2 The case  $n = 2N + 1$ .** The calculation is very similar to the case of an even  $n$ ; hence, we review it briefly, emphasizing salient points. If  $n = 2N + 1$ , we consider a general vertex  $v \in \mathcal{V}_{2N+1, N+t}$ .

As before,  $\partial_x^k f(v) = 0$  unless the multi-indices are such that  $k^{[1]} \in \{0, 1\}^{N+t}$ ,  $k^{[2]} \in \{0, 1\}^{N+1-t}$ , with  $|k^{[1]}| = \kappa_1$  and  $|k^{[2]}| = \kappa_2$ . In this case, we have

$$\partial_x^{(k^{[1]}, k^{[2]})} f(v) = \frac{(-1)^{\kappa_1 + \kappa_2} 2^{N+t-\kappa_1} (-c)^{N+1-t-\kappa_2}}{(i\omega)^{N+1-t-\kappa_2}} \left[ 1 + \mathcal{O}(\omega^{-1}) \right]. \quad (3.29)$$

The contributions to the leading term are as follows.

1. The dimension contributes  $2N + 1$ .
2. The least order non-vanishing derivative of  $F_\alpha$ , that consists of permutations of length  $N + t$  and  $N + 1 - t$ . This amounts to  $|k| = 2N^2 + 2t(t - 1)$ .
3. By similar reason as for  $n = 2N$ , from (3.29), the leading power is  $\kappa_1 + \kappa_2$  (the degree of the derivative) plus  $N + 1 - t - \kappa_2$  (the above contribution)—altogether  $N + 1 - t + \kappa_1$ . Since we wish to minimize this, we need to take  $\kappa_1 = 0$  and the contribution is  $N - t + 1$ .

The total exponent is therefore  $2N^2 + 3N + 2 + t(2t - 3)$ , which is minimized for  $t \in \{-N, \dots, N + 1\}$  by  $t = 1$ . The exponent is then equal to  $(N + 1)(2N + 1)$ . Therefore, we need consider just vertices in  $\mathcal{V}_{2N+1, N+1}$ .

As before, derivatives of  $f$  in the term  $F = fg^2$  may have order  $s \geq 0$ . It is important to observe that the range of  $s$  is unchanged, i.e.  $0 \leq s \leq N$ , since it is constrained by the number of  $(+1)$ s in the vertex  $v \in \mathcal{V}_{2N+1, N+1}$ , which is  $N$  as before. Going through similar computations, the identity (3.28) quickly resurfaces in the leading order term.

#### 4. Existence of even-degree kissing polynomials

The goal of this section is to show that even-degree kissing polynomials exist for all  $n \geq 0$  and  $\omega > 0$ . In this proof of existence, we will make use of both the symmetry of the polynomials over the imaginary axis (see (1.17)) and the differential equation in  $z$  as stated in Lemma 2.11. We recall that Lemma 2.11 states that the polynomial  $p_n(z)$  satisfies a second-order, linear differential equation whose only singular points are at  $\pm 1$  and at the point

$$z_*(\omega) = -\alpha_n - \frac{2n + 1}{i\omega}. \quad (4.1)$$

We say that  $p_n(z)$  exists if there exists a monic polynomial of degree *exactly*  $n$  that satisfies the orthogonality conditions given in (1.8). Equivalently,  $p_n(z)$  exists for a given value of  $\omega$  if the Hankel determinant  $h_{n-1}(\omega)$  does not vanish. We have seen in Section 1.2 that as  $\omega \rightarrow \hat{\omega}$ , where  $\hat{\omega}$  satisfies  $h_{n-1}(\hat{\omega}) = 0$ , one or more of the zeros of  $p_n$  becomes infinite. Therefore, we will prove the existence of the even degree kissing polynomials for  $\omega > 0$  by showing that their zeros do not become infinite.

We first recall from (1.17) that for each  $\omega > 0$ , kissing polynomials obey the symmetry relation

$$p_n(z) = (-1)^n \overline{p_n(-\bar{z})}.$$

This immediately implies that zeros can become infinite for varying  $\omega$  in just one of two ways:

1. zeros tend to infinity in one or more pairs, symmetric about the imaginary axis or



2. an even number of zeros meets on the imaginary axis, forming there a single zero of multiplicity  $\geq 2$ . Once  $\omega$  increases, these zeros split and one (or more) of them travels to infinity along the imaginary axis.

We quickly rule out the first case above. We recall the polynomials  $\tilde{p}_n$ , defined in (1.18) as  $\tilde{p}_n(z) = h_{n-1}p_n(z)$ , which always exist as polynomials of degree  $\leq n$  (their degree just degenerates if the Hankel determinant vanishes).

LEMMA 4.1 If  $\hat{\omega}$  is such that  $h_{n-1}(\hat{\omega}) = 0$ , then  $\deg(\tilde{p}_n) = n - 1$  for  $\omega = \hat{\omega}$ .

*Proof.* We recall (1.20) (shifted from  $n \mapsto n - 1$ ), which states that if  $h_{n-1}(\hat{\omega}) = 0$ , then

$$\tilde{p}_n(z) = i \frac{\dot{h}_{n-1}(\hat{\omega})}{h_{n-2}(\hat{\omega})} \tilde{p}_{n-1}(z) = i \dot{h}_{n-1}(\hat{\omega}) p_{n-1}(z)$$

for  $\omega = \hat{\omega}$ . By Lemma 2.9 and the remarks immediately following it, we see that  $h_{n-2}(\hat{\omega}) \neq 0$ , so that  $p_{n-1}(z)$  exists as a monic polynomial of degree  $n - 1$ , and that  $\dot{h}_{n-1}(\hat{\omega}) \neq 0$ , as well. Therefore,  $\deg(\tilde{p}_n) = n - 1$ , completing the proof.  $\square$

It immediately follows that as  $\omega \rightarrow \hat{\omega}$ , precisely one zero escapes to infinity, which rules out the scenario of zeros tending to infinity in one or more symmetric pairs. We therefore turn our attention to the second case and rule out a zero of multiplicity greater than one forming on the imaginary axis. In order to accomplish this, we will need the following proposition.

PROPOSITION 4.2 Let  $\omega \in (\omega_1, \omega_2)$ , where  $\omega_1 < \omega_2$ , are such that  $h_{2n-1}(\omega) \neq 0$  and  $h_{2n}(\omega) \neq 0$  for all  $\omega \in (\omega_1, \omega_2)$ . Assume further that  $p_{2n-2}(z)$  exists as a monic polynomial of degree  $2n - 2$  and satisfies  $p_{2n-2}(z_*(\omega)) \neq 0$  for all  $\omega \in (\omega_1, \omega_2)$ . If  $p_{2n}(z_*(\omega)) = 0$ , then

$$\frac{d}{d\omega} p_{2n}(z_*(\omega)) \neq 0.$$

*Proof.* Using (2.27), we see that

$$\frac{d}{d\omega} p_{2n}(z_*(\omega)) = \frac{\partial}{\partial z} p_{2n}(z_*(\omega)) \dot{z}_*(\omega) - i\beta_{2n} p_{2n-1}(z_*(\omega)).$$

As both  $h_{2n}$  and  $h_{2n-1}$  are nonzero by assumption, we can use Corollary 2.13 to conclude that if  $p_{2n}(z)$  vanishes at  $z_*$ , then its first partial derivative in  $z$  must also vanish at  $z_*$ . Therefore, if  $dp_{2n}(z_*(\omega))/d\omega = 0$ , we would have

$$-i\beta_{2n} p_{2n-1}(z_*(\omega)) = 0.$$

If  $h_{2n-2}(\omega) \neq 0$ , the three-term recurrence would imply that  $p_{2n-2}(z_*(\omega)) = 0$ . If  $h_{2n-2}(\omega) = 0$ , we could use the fact that

$$\tilde{p}_{2n-1} = i \frac{\dot{h}_{2n-2}}{h_{2n-3}} \tilde{p}_{2n-2},$$

where  $\tilde{p}_{2n-1}(z) = h_{2n-2}p_{2n-1}(z)$ , to conclude again that  $p_{2n-2}(z_*(\omega)) = 0$ . In either case, we have a contradiction, completing the proof of the proposition.  $\square$

**THEOREM 4.3** For  $n = 0, 1, 2, \dots$  and  $\omega \in \mathbb{R}$ , the monic polynomial  $p_{2n}(z)$  exists and does not vanish on the imaginary axis.

*Proof.* Using Corollary 2.6, it is sufficient to consider  $\omega \geq 0$ . The statement is clearly true for  $k = 0$  and we proceed by induction. Therefore, we assume the theorem is true for  $k = 0, 1, \dots, n-1$ , and we show that  $p_{2n}(z)$  exists for all  $\omega > 0$  and does not vanish on the imaginary axis.

Assume for sake of contradiction that there exists an  $\omega > 0$  for which  $p_{2n}(z)$  fails to exist, and let  $\hat{\omega}$  be the smallest positive solution to  $h_{2n-1}(\omega) = 0$ . By the remarks preceding Lemma 4.1, we know there exists some  $0 < \omega_d < \hat{\omega}$  for which  $p_{2n}^{\omega_d}(z)$  has a purely imaginary zero of multiplicity greater than one. By Lemma 2.11 and standard analytic existence theorems for ODEs, we know that any purely imaginary zero of multiplicity greater than one must be located precisely at

$$z_*(\omega_d) = -\alpha_{2n} - \frac{4n+1}{i\omega_d}.$$

We next show that  $p_{2n}(z_*(\omega)) \neq 0$  for all  $\omega \in (0, \hat{\omega})$ , reaching a contradiction and thereby concluding that  $p_{2n}(z)$  exists for all  $\omega$ . Moreover, this in turn implies that  $p_{2n}$  does not vanish on the imaginary axis. To see this, note that when  $\omega = 0$ ,  $p_{2n}$  is the monic Legendre polynomial of degree  $2n$ , and as such is real valued and does not vanish on the imaginary axis. Had there existed an  $\omega_*$  for which  $p_{2n}^{\omega_*}(z)$  vanished somewhere on the imaginary axis, the symmetry of the polynomials across the imaginary axis would imply that there exists some  $\omega_d < \omega_*$  for which  $p_{2n}(z)$  had a zero of even multiplicity on the imaginary axis. Therefore, showing that  $p_{2n}(z_*(\omega)) \neq 0$  for all  $\omega > 0$  implies that  $p_{2n}(z)$  does not vanish on the imaginary axis.  $\square$

We want to show that  $p_{2n}(z_*(\omega)) \neq 0$  for all  $\omega \in (0, \hat{\omega})$ . Assume first that  $n$  is odd. As  $n$  is odd, and by assumption,  $p_{2n-2}(z)$  exists for all  $\omega$  and has no zeros on the imaginary axis, we have

$$p_{2n-2}(ix) > 0, \quad x \in \mathbb{R}, \quad \omega > 0. \quad (4.2)$$

Next, define  $\hat{\Omega} := \{0, \hat{\omega}\} \cup \{\omega : h_{2n}(\omega) = 0, \omega < \hat{\omega}\}$ , so that  $|z_*(\omega)| \rightarrow \infty$  as  $\omega \rightarrow \omega_* \in \hat{\Omega}$ . As  $p_{2n}$  exists on the interval  $(0, \hat{\omega})$ , we deduce that  $p_{2n}(z_*(\omega))$  is analytic on  $(0, \hat{\omega}) \setminus \hat{\Omega}$ . Observe that,  $n$  being odd,

$$p_{2n}(z_*(\omega)) \rightarrow -\infty, \quad \omega \rightarrow \omega_* \in \hat{\Omega}.$$

Recall that the goal is to show that  $p_{2n}(z_*(\omega))$  does not vanish in  $(0, \hat{\omega})$ . For the sake of contradiction, assume there exists some  $\omega_d < \hat{\omega}$  for which  $p_{2n}(z_*(\omega_d)) = 0$ , and define

$$\omega_0 := \sup_{\omega \in \hat{\Omega}} \{\omega : \omega < \omega_d\}, \quad \omega_3 := \inf_{\omega \in \hat{\Omega}} \{\omega : \omega > \omega_d\}.$$

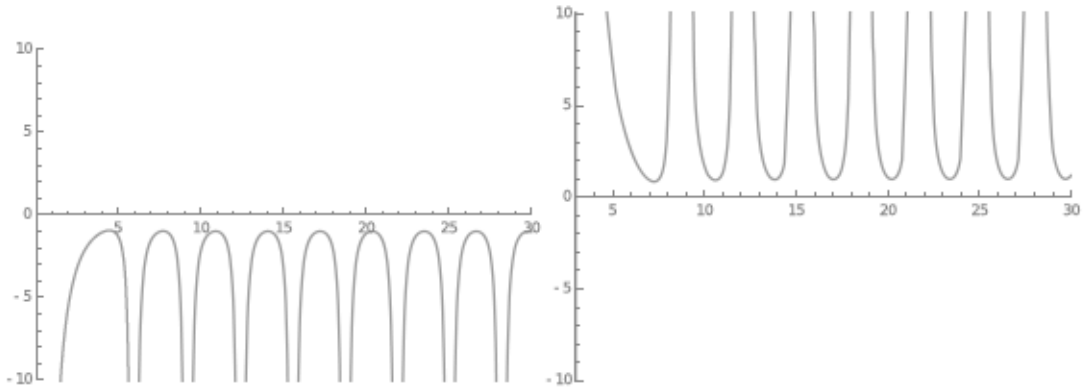


FIG. 6. Trajectory of  $p_2(z_*(\omega))$  (left) and of  $p_4(z_*(\omega))$  (right).

We then know that  $p_{2n}(z_*(\omega))$  is analytic in  $(\omega_0, \omega_3)$  vanishes somewhere in this interval and tends to  $-\infty$  as we approach the endpoints of this interval. By Proposition 4.2, we may conclude that

$$\frac{d}{d\omega} p_{2n}(z_*(\omega)) \neq 0, \quad \omega \in (\omega_0, \omega_3).$$

Therefore, there exist  $\omega_1, \omega_2$ , with  $\omega_0 < \omega_1 < \omega_2 < \omega_3$ , such that  $p_{2n}(z_*(\omega_1)) = p_{2n}(z_*(\omega_2)) = 0$  and

$$\frac{d}{d\omega} p_{2n}(z_*(\omega_1)) > 0, \quad \frac{d}{d\omega} p_{2n}(z_*(\omega_2)) < 0, \quad (4.3)$$

as well as

$$p_{2n}(z_*(\omega)) > 0, \quad \omega \in (\omega_1, \omega_2). \quad (4.4)$$

Next,

$$\frac{d}{d\omega} p_{2n}(z_*(\omega)) = \frac{\partial}{\partial z} p_{2n}(z_*(\omega)) z'_*(\omega) - i\beta_{2n} p_{2n-1}(z_*(\omega)). \quad (4.5)$$

As  $p_{2n}$  vanishes at  $z_*(\omega_1)$  and  $z_*(\omega_2)$ , we have by Corollary 2.13 that

$$\frac{\partial}{\partial z} p_{2n}(z_*(\omega_1)) = \frac{\partial}{\partial z} p_{2n}(z_*(\omega_2)) = 0,$$

which, using (4.3) and (4.5), results in

$$-i\beta_{2n}(\omega_1) p_{2n-1}(z_*(\omega_1)) > 0,$$

$$-i\beta_{2n}(\omega_2)p_{2n-1}(z_*(\omega_2)) < 0.$$

Using the three-term recurrence relation, and the fact that  $p_{2n}(z_*(\omega))$  vanishes at  $\omega_1$  and  $\omega_2$ , we may write these equations as

$$f(\omega_1)p_{2n-2}(z_*(\omega_1)) > 0, \quad f(\omega_2)p_{2n-2}(z_*(\omega_2)) < 0, \quad (4.7)$$

where

$$f(\omega) = -\frac{i\beta_{2n}(\omega)\beta_{2n-1}(\omega)}{z_*(\omega) - \alpha_{2n-1}(\omega)}.$$

We claim that  $f(\omega)$  is a well-defined, continuous function on  $[\omega_1, \omega_2]$  and is nonzero throughout this interval.

We prove this result using (2.1) and (4.1), we can write

$$f(\omega) = -\frac{h_{2n}h_{2n-3}}{h_{2n-1}} \left[ \frac{1}{h_{2n-2} \left( \frac{h_{2n}}{h_{2n}} - \frac{h_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega} \right)} \right].$$

The functions  $h_{2n}$ ,  $h_{2n-1}$ , and  $h_{2n-3}$  are analytic and do not vanish in  $[\omega_1, \omega_2]$  and we need to focus our attention just on the term in brackets. It is clear that if we can show that this term is never zero or infinite on  $[\omega_1, \omega_2]$ , the proof will be complete. The term in brackets is zero only at the poles of

$$g(\omega) := h_{2n-2} \left( \frac{h_{2n}}{h_{2n}} - \frac{h_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega} \right).$$

As  $g$  has poles only at the zeros of  $h_{2n}$  and at 0, we can conclude that  $f$  is never zero in  $[\omega_1, \omega_2]$ . We just need to show  $g(\omega)$  does not vanish on  $[\omega_1, \omega_2]$ , so that  $f$  is continuous on this interval and as such does not change sign. Note that  $g(\omega)$  is well defined and nonzero when  $h_{2n-2} = 0$ , so we must show that

$$\frac{h_{2n}}{h_{2n}} - \frac{h_{2n-2}}{h_{2n-2}} + \frac{4n+1}{\omega} = -i(z_* - \alpha_{2n-1})$$

does not vanish on  $[\omega_1, \omega_2]$  when  $h_{2n-2} \neq 0$ . As  $h_{2n-2} \neq 0$ , we may use the recurrence relation to show that, had  $z_* - \alpha_{2n-1}$  vanished, then

$$p_{2n}(z_*) = -\frac{h_{2n-1}h_{2n-3}}{h_{2n-2}^2} p_{2n-2}(z_*). \quad (4.8)$$

Were  $z_* - \alpha_{2n-1}$  to vanish at either  $\omega_1$  or  $\omega_2$ , where  $p_{2n}(z)$  also vanishes, then we would immediately deduce that  $p_{2n-2}$  vanishes there as well. Therefore, we can conclude that  $z_* - \alpha_{2n-1}$  does not vanish

at the endpoints of  $[\omega_1, \omega_2]$ . We have by (4.4) that  $p_{2n}(z_*) > 0$  for  $\omega \in (\omega_1, \omega_2)$ . On the other hand, as  $\omega < \omega_*$ , we have that  $h_{2n-1}$  is positive and by assumption  $h_{2n-3}$  is always positive as  $p_{2n-2}$  exists for all  $\omega$ . We also have from (4.2) that  $p_{2n-2}(z_*) > 0$  for all  $\omega$ , which, once combined with (4.8), yields a contradiction to  $p_{2n}(z_*) > 0$ . Therefore,  $f$  is continuous and cannot change sign on  $[\omega_1, \omega_2]$ .

As  $f(\omega)$  does not change sign on  $[\omega_1, \omega_2]$ , we know from (4.7) that  $p_{2n-2}(z_*(\omega))$  must change sign in  $(\omega_1, \omega_2)$ . However, this immediately implies the existence of  $\omega \in (\omega_1, \omega_2)$  for which  $p_{2n-2}$  vanishes on the imaginary axis, contradicting the inductive hypothesis. Therefore, we can conclude  $p_{2n}(z_*(\omega))$  does not vanish on  $(0, \hat{\omega})$ , as desired, concluding the proof.

We have seen above that the key to proving the existence of even-degree kissing polynomials was demonstrating that these polynomials can never form higher order zeros on the imaginary axis. Having proven that the even degree kissing polynomials do not have zeros of multiplicity greater than one on the imaginary axis, we may now take this a step further and show they do not have higher-order zeros anywhere in the complex plane.

**LEMMA 4.4** For any  $\omega \in \mathbb{R}$  such that  $h_{n-1}(\omega) \neq 0$ , and therefore such that  $p_n(z)$  exists as a monic polynomial of degree  $n$ , we have  $p_n(1) \neq 0$  and  $p_n(-1) \neq 0$ .

*Proof.* First, assume further that  $\omega$  is such that  $h_{n-2}(\omega) \neq 0$ , so that both  $p_n(z)$  and  $p_{n-1}(z)$  exist as monic polynomials of degree  $n$  and  $n-1$ , respectively. Using (A.13) in the appendix, we have

$$(z^2 - 1)p'_n(z) = N_1(z)p_n(z) + N_2(z)p_{n-1}(z),$$

where

$$N_1(z) = nz - i \left[ \frac{h_{n-1}}{h_{n-1}} - \omega \frac{h_n h_{n-2}}{h_{n-1}^2} \right], \quad N_2(z) = -\frac{i\omega h_{n-2} h_n}{h_{n-1}^2} [z - z_*(\omega)]$$

and we recall that

$$z_*(\omega) = -\alpha_n - \frac{2n+1}{i\omega}. \quad (4.9)$$

As  $h_{n-1}(\omega) \neq 0$ , we may write this in terms of the polynomials  $\tilde{p}_n$ , which exist for all  $\omega$ , as

$$\frac{z^2 - 1}{h_{n-1}} \tilde{p}'_n(z) = \frac{N_1(z)}{h_{n-1}} \tilde{p}_n(z) - \frac{i\omega h_n}{h_{n-1}^2} [z - z_*(\omega)] \tilde{p}_{n-1}(z). \quad (4.10)$$

As both  $N_1$  and  $N_2$  are well defined when  $h_{n-2}$  vanishes, (4.10) holds for any  $\omega$  provided  $h_{n-1}(\omega) \neq 0$ , by continuity.

Now, fix  $\omega$  so that  $h_{n-1}(\omega) \neq 0$ , and assume that  $\tilde{p}_n(1) = 0$ . Evaluating (4.10) at  $z = 1$ , we see that

$$\frac{i\omega h_n}{h_{n-1}^2} (1 - z_*(\omega)) \tilde{p}_{n-1}(1) = 0.$$

Note that  $1 - z_*(\omega) \neq 0$  as  $z_*(\omega) \in i\mathbb{R}$  for all  $\omega$ . First, consider the case  $h_n(\omega) \neq 0$ . We then immediately have that  $\tilde{p}_{n-1}(1) = 0$ . On the other hand, assume  $\omega = \hat{\omega}$  was such that  $h_n(\hat{\omega}) = 0$ . Then taking the

limit as  $\omega \rightarrow \hat{\omega}$  in (4.10), and using (2.1) and (4.9), we see that

$$\frac{\hat{\omega} \dot{h}_n(\hat{\omega})}{h_{n-1}^2(\hat{\omega})} \tilde{p}_{n-1}^{\hat{\omega}}(1) = 0, \quad (4.11)$$

where the superscript in  $\tilde{p}_n^{\hat{\omega}}(1)$  reminds us of the specific value of the parameter  $\omega$  therein. In light of Lemma 2.9 and the remarks immediately following the lemma, we see that  $\dot{h}_n(\hat{\omega}) \neq 0$ , so that in the case  $h_n(\omega) = 0$ , we still have  $\tilde{p}_{n-1}(1) = 0$ . Therefore,  $\tilde{p}_n(1) = 0$  implies that  $\tilde{p}_{n-1}(1) = 0$ .

We now show that  $\tilde{p}_n(1) = 0$  implies that  $\tilde{p}_{n-k}(1) = 0$  for  $k \in \{0, 1, \dots, n\}$ . As we have just shown, this statement is true for  $k = 0, 1$ , so we proceed by induction and assume it holds true for  $k = 0, 1, \dots, m-1 < n$  and demonstrate that it holds true for  $k = m$ .

We may use the three-term recurrence relation (1.19), where we shift the index  $n \mapsto n-m+1$  and use  $\tilde{p}_{n-m+1}(1) = \tilde{p}_{n-m+2}(1) = 0$  to conclude that

$$h_{n-m+1}^2(\omega) \tilde{p}_{n-m}(1) = 0.$$

Now, if  $h_{n-m+1}(\omega) \neq 0$ , we immediately deduce that  $\tilde{p}_{n-m}(1) = 0$ , completing the inductive step. On the other hand, assume that  $\omega = \hat{\omega}$  and  $h_{n-m+1}(\hat{\omega}) = 0$ . Shifting  $n \mapsto n-m+1$  in (4.10) and taking limits as  $\omega \rightarrow \hat{\omega}$ , we arrive (in a similar fashion to (4.11)) at

$$\frac{\hat{\omega} \dot{h}_{n-m+1}(\hat{\omega})}{h_{n-m}^2(\hat{\omega})} \tilde{p}_{n-m}^{\hat{\omega}}(1) = 0.$$

By Lemma 2.9, we have  $h_{n-m}(\hat{\omega}) \neq 0$  and  $\dot{h}_{n-m+1}(\hat{\omega}) \neq 0$ , so that  $\tilde{p}_{n-m}(1) = 0$ , completing the inductive step.

In particular, this chain of reasoning implies that  $\tilde{p}_0(1) = 0$ . However,  $\tilde{p}_0(z) \equiv 1$  and we have reached a contradiction. Consequently,  $\tilde{p}_n(1) \neq 0$ , which implies that  $p_n(1) \neq 0$  when  $h_{n-1}(\omega) \neq 0$ .

Finally, we may use the symmetry across the imaginary axis in (1.17) to conclude that  $p_n(1) \neq 0$  implies that  $p_n(-1) \neq 0$ , completing the proof.  $\square$

**COROLLARY 4.5** For  $n = 0, 1, 2, \dots$  and  $\omega \in \mathbb{R}$ , the monic polynomial  $p_{2n}(z)$  has  $2n$  simple zeros in the complex plane.

*Proof.* For sake of contradiction, assume the existence of some  $\hat{z}$  so that  $p_{2n}(\hat{z}) = 0$  and

$$\frac{\partial}{\partial z} p_{2n}(\hat{z}) = 0. \quad (4.12)$$

By Lemma 2.11, we know that  $\hat{z} \in \{-1, 1, z_*(\omega)\}$ . However, in the proof of Theorem 4.3, we showed that  $p_{2n}(z_*(\omega)) \neq 0$  for all  $\omega \in \mathbb{R}$ . Furthermore, Lemma 4.4 shows that  $p_{2n}(1) \neq 0$  and  $p_{2n}(-1) \neq 0$ , which contradicts the fact that  $p_{2n}(\hat{z}) = 0$ , proving that even-degree kissing polynomials have no zeros of nontrivial multiplicity.  $\square$

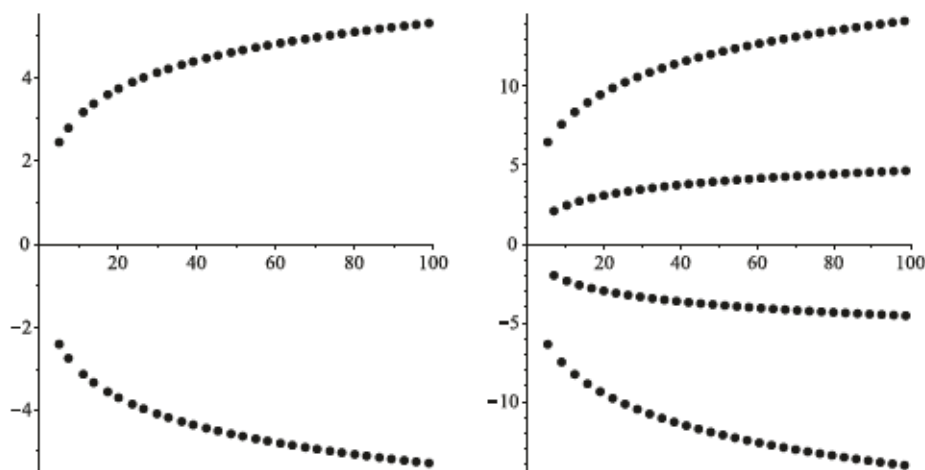


FIG. 7. Complex zeros of  $h_1(\omega)$  and  $h_3(\omega)$  (left and right, respectively) in the right complex half-plane.

## 5. Roots of $h_n(\omega)$ in the complex plane

In this section, we focus on the zeros of Hankel determinants in the complex plane, so unlike the rest of the paper, we will consider here  $\omega \in \mathbb{C}$ . First of all, we recall that  $h_n(\omega)$  is real for  $\omega \in \mathbb{R}$ , and since  $h_n(\omega)$  is an analytic function of  $\omega$ , by the Schwarz reflection principle, we have  $h_n(\bar{\omega}) = \overline{h_n(\omega)}$ , and all complex zeros must come in conjugate pairs. Also, since  $h_n(-\omega) = h_n(\omega)$ , we can restrict ourselves to the first quadrant of the complex plane.

Figures 7–10 display these zeros for different values of  $n$ . They follow very regular and symmetric patterns reminiscent of onion peels, which we intend to explain in this section, at least for large values of  $\omega$ . These patterns result from a delicate balancing act, in which algebraic powers of  $\omega^{-1}$  become comparable in size to decaying complex exponentials of the form  $e^{i\omega cz}$ ,  $c > 0$ , in the upper half of the complex plane. We revisit the asymptotic analysis of Chapter 3, this time taking complex exponential factors into account.

As before, we need to distinguish between two cases, corresponding to even and odd values of  $n$ .

### 5.1 The odd case: the roots of $h_{2N-1}(\omega)$

We commence from  $h_{2N-1}(\omega) = I_{2N}[1]$ . The contribution to  $h_{2N-1}(\omega)$  in the asymptotic expansion (3.4) corresponds, using the notation therein, to the layers  $\mathcal{V}_{2N,N\pm t}$ ,  $t = 0, \dots, N$ . We quantify the contribution of each layer with two numbers:

- a complex exponential factor: each vertex  $v \in \mathcal{V}_{2N,N\pm t}$  contributes  $e^{i\omega|v|} = e^{\mp 2i\omega t}$ ;
- the *leading* power of  $\omega$  ‘originating’ in vertices in the layer. As computed before, recall (3.15), that contributes a total power  $2N^2 - 2N + 2t^2$ . Therefore, in the end, we obtain  $\omega^{-2N-2(N^2-N+t^2)} = \omega^{-2N^2-2t^2}$ .

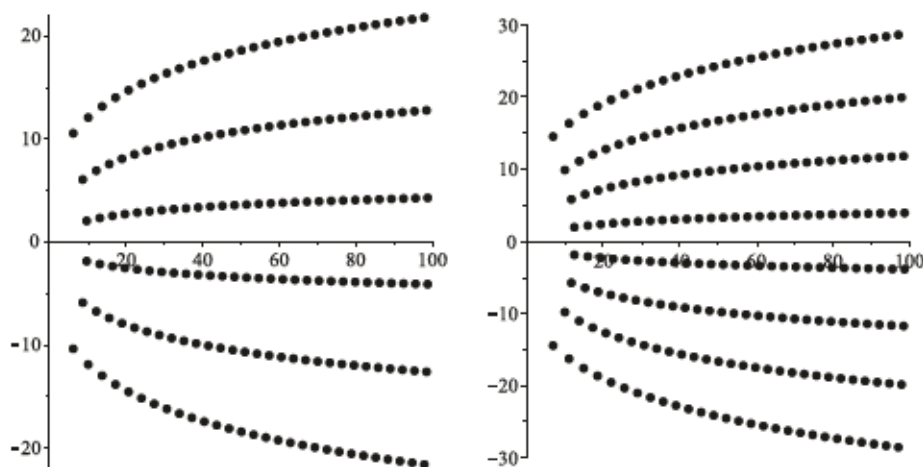


FIG. 8. Complex zeros of  $h_5(\omega)$  and  $h_7(\omega)$  (left and right, respectively) in the right complex half-plane.

In other words, as  $\omega \rightarrow \infty$ , each layer  $\mathcal{V}_{2N,N\pm t}$  contributes

$$Q_{2N,\pm t}(\omega) := \frac{e^{\mp 2i\omega t}}{\omega^{2N^2+2t^2}} \left[ c_{2N,t} + \mathcal{O}(\omega^{-1}) \right]. \quad (5.1)$$

If we concentrate on the case  $\text{Im } \omega > 0$ , then the  $-$  sign in (5.1) gives an increasing exponential, which is clearly dominant as  $|\omega| \rightarrow \infty$  in the upper right quadrant of the complex plane. The case  $\text{Im } \omega < 0$  follows by taking complex conjugates.

If  $h_n(\omega)$  vanishes for large  $\omega$ , then at least two terms in the asymptotic expansion must balance. Let  $t_1$  and  $t_2$  be two different indices in  $0, 1, \dots, N$ , with  $t_1 < t_2$  without loss of generality. Then, for  $\omega \rightarrow \infty$  on the real line, we have

$$\begin{aligned} & Q_{2N,t_1}(\omega) + Q_{2N,t_2}(\omega) \\ &= \frac{e^{-2i\omega t_1}}{\omega^{2N^2+2t_1^2}} \left[ c_{2N,t_1} + \mathcal{O}(\omega^{-1}) \right] + \frac{e^{-2i\omega t_2}}{\omega^{2N^2+2t_2^2}} \left[ c_{2N,t_2} + \mathcal{O}(\omega^{-1}) \right] \\ &= \frac{e^{-2i\omega t_1}}{\omega^{2N^2+2t_1^2}} \left[ c_{2N,t_1} + \mathcal{O}(\omega^{-1}) + c_{2N,t_2} \frac{e^{-2i\omega(t_2-t_1)}}{\omega^{2t_2^2-2t_1^2}} + \mathcal{O}(\omega^{2t_1^2-2t_2^2-1}) \right]. \end{aligned}$$

These terms do not have the same asymptotic order for  $\omega \in \mathbb{R}$  because the exponential term is oscillatory and bounded on the real line. However, a balance may take place as  $|\omega| \rightarrow \infty$  in the complex plane, in particular along a trajectory such that

$$c_{2N,t_2} \frac{e^{-2i\omega(t_2-t_1)}}{\omega^{2t_2^2-2t_1^2}} + c_{2N,t_1} = \mathcal{O}(|\omega|^{-1}), \quad |\omega| \rightarrow \infty. \quad (5.2)$$



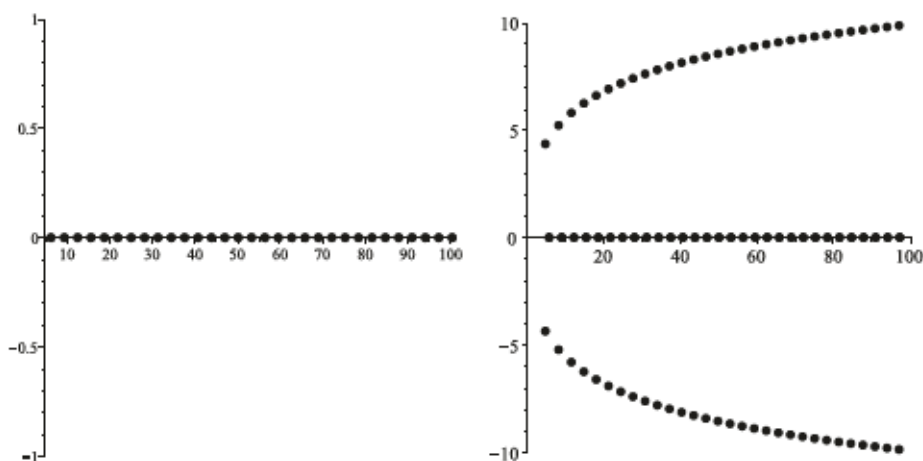


FIG. 9. Complex zeros of  $h_0(\omega)$  and  $h_2(\omega)$  (left and right, respectively) in the right complex half-plane.

More generally, this can happen if a trajectory exists in the complex plane such that, as  $|\omega| \rightarrow \infty$  along this trajectory,

$$e^{-i\omega} = \mathcal{O}(|\omega|^p), \quad \text{for some } p > 0. \quad (5.3)$$

We describe such trajectories later on in terms of the Lambert W function.

Assuming a trajectory for which (5.3) holds, we first show that there exists a couple of indices  $t_1$  and  $t_2$  such that the contributions  $Q_{2N,t_1}(\omega)$  and  $Q_{2N,t_2}(\omega)$  are of the same order of magnitude and, in addition, all the remaining  $Q_{2N,t}(\omega)$  for  $t \neq t_1, t_2$  are of smaller order. As shown in the next lemma, this happens precisely when  $t_1$  and  $t_2$  correspond to consecutive integers. Also, in this situation, we have  $p = t_1 + t_2$  in (5.3), as we show below.

**LEMMA 5.1** As  $|\omega| \rightarrow \infty$  such that (5.3) holds with  $p = t_1 + t_2$ ,  $Q_{2N,t_1}(\omega)$  and  $Q_{2N,t_2}(\omega)$  are of the same order of magnitude, and  $Q_{2N,t}(\omega)$  is of lower order for  $t \neq t_1, t_2$ , if and only if  $t_1 = k$  and  $t_2 = k + 1$ . In that case, we have

$$\begin{aligned} Q_{2N,k}(\omega), Q_{2N,k+1}(\omega) &= \mathcal{O}(|\omega|^{-2N^2+2k^2+2k}), \\ Q_{2N,t}(\omega) &= \mathcal{O}(|\omega|^{-2N^2-2t^2+4tk+2t}), \quad t \neq k, k+1. \end{aligned} \quad (5.4)$$

Note that  $-2N^2 - 2t^2 + 4tk + 2t = 2N^2 + 2k^2 + 2k - 2(k-t)(k-t+1) > 2N^2 + 2k^2 + 2k$  for  $t \neq k, k+1$ .

*Proof.* The first requirement, namely that  $Q_{2N,t_1}(\omega)$  and  $Q_{2N,t_2}(\omega)$  are of the same order of magnitude, means that

$$-2N^2 - 2t_1^2 + 2t_1p = -2N^2 - 2t_2^2 + 2t_2p,$$

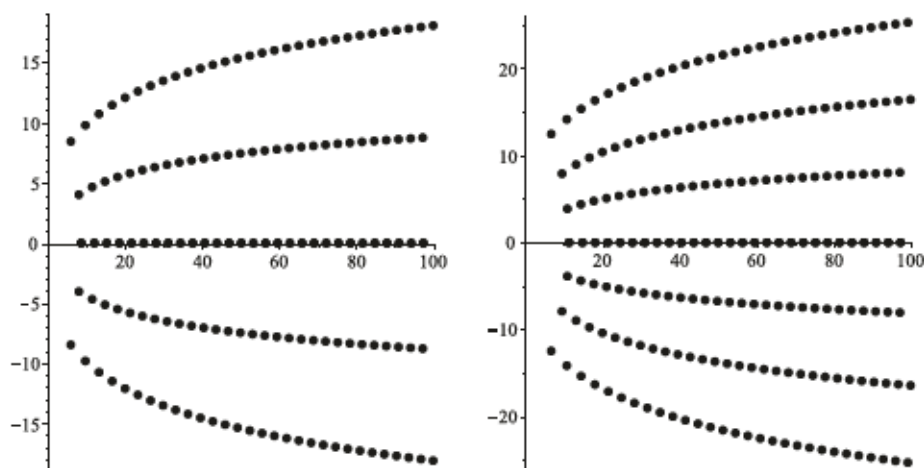


FIG. 10. Complex zeros of  $h_4(\omega)$  and  $h_6(\omega)$  (left and right, respectively) in the right complex half-plane.

where we have used (5.3). Therefore,  $p = t_1 + t_2$ , whereby

$$Q_{2N,t_1}(\omega), Q_{2N,t_2}(\omega) = \mathcal{O}(|\omega|^{-2N^2+2t_1t_2}).$$

The second requirement, i.e. that all other terms are of smaller order in  $\omega$ , becomes

$$-2N^2 - 2t^2 + 2t(t_1 + t_2) < -2N^2 + 2t_1t_2.$$

Therefore,  $(t - t_1)(t - t_2) > 0$  for all  $t \neq t_1, t_2$ . It is easy to verify that if  $t_1 = k, t_2 = k + 1$ , where  $k = 0, 1, \dots, N - 1$ , then indeed  $(t - k)(t - k - 1) > 0$  for all  $t \neq k, k + 1$ . Moreover, these are all possible such choices, for suppose that there exist  $t_1, t_2$  such that  $t_1 < t_2 - 1$ , then  $(t - t_1)(t - t_2) < 0$  for  $t = t_1 + 1$  and we reach a contradiction.

Finally, in this case, we check directly that (5.4) is satisfied with this choice.  $\square$

The lemma implies that we have exactly  $N$  different choices for the index  $k$ , each one corresponding to a separate ‘onion peel’ in the first quadrant in Figs 7 and 8. Furthermore, the coefficients  $c_{2N,t}$  can be computed explicitly.

LEMMA 5.2 For  $N \geq 1$  and  $0 \leq t \leq N$ , the coefficients  $c_{2N,t}$  are given by the following formulas:

$$\begin{aligned} c_{2N,t} &= 4^{N^2-t^2} G^2(N-t+1)G^2(N+t+1), \\ c_{2N+1,t} &= i(-1)^{N+t} 4^{(N-t)(N+t+1)} G^2(N+t+2)G^2(N-t+1). \end{aligned} \tag{5.5}$$

*Proof.* The proof follows along the same lines that were presented in Section 3.2. We first recall Proposition 3.2, which states that as  $\omega \rightarrow \infty$ ,

$$\int_{[-1,1]^n} F(x)e^{i\omega|x|} dx \sim (-1)^n \sum_{m=0}^{\infty} \frac{1}{(-i\omega)^{m+n}} \sum_{|k|=m} \sum_{v \in \mathcal{Y}_a} (-1)^{s(v)} e^{i\omega|v|} \partial_x^k F(v).$$

Above,  $k = [k_0, k_1, \dots, k_m]$ , with  $k_j \in \mathbb{N}$ , is a multi-index, and  $|k| = k_0 + k_1 + \dots + k_m$ , so  $\partial_x^k = \partial_{x_0}^{k_0} \partial_{x_1}^{k_1} \dots \partial_{x_m}^{k_m}$ . As in Definition 3.11, the function  $F$  that corresponds to the Hankel determinant can be split as

$$(2N)! F(x) = \alpha_{N\pm t}^2(x_0, \dots, x_{N\pm t-1}) \alpha_{N\mp t}^2(x_{N\pm t}, \dots, x_{2N-1}) \beta_{2N, N\pm t}^2(x_0, \dots, x_{2N-1}),$$

where

$$\alpha_r(x) = \prod_{0 \leq k < l \leq r-1} (x_l - x_k) \quad \text{and} \quad \beta_{2N, N\pm t}(x) = 2^{N^2-t^2}.$$

By (3.15), we see the leading order term corresponds to

$$|k| = m = 2N^2 + 2t^2 - 2N,$$

and therefore,  $c_{2N,t}$  is given by

$$c_{2N,t} = \frac{i^{2N^2+2t^2}}{(2N)!} \sum_{|k|=2N^2+2t^2-2N} \sum_{v \in \mathcal{Y}_{2N, N-t}} (-1)^{N\pm t} \partial_x^k F(v).$$

Using the same method of proof as in Theorem 3.7, we may simplify this to

$$c_{2N,t} = \frac{1}{(2N)!} \binom{2N}{N-t} 4^{N^2-t^2} \sum_{|k|=2N^2+2t^2-2N} \partial_x^k F_\alpha(v), \tag{5.6}$$

where

$$F_\alpha(x) = \alpha_{N\pm t}(x)^2 \alpha_{N\mp t}(x)^2.$$

Using Proposition 3.13 and the proof of Theorem 3.7 (see, in particular, (3.18)), we find that

$$\sum_{|k|=2N^2+2t^2-2N} \partial_x^k F_\alpha(v) = G^2(N-t+1)G^2(N+t+1)(N-t)!(N+t)!. \tag{5.7}$$

Combining (5.6) and (5.7), we arrive at

$$c_{2N,t} = 4^{N^2-t^2} G^2(N-t+1)G^2(N+t+1), \tag{5.8}$$

as desired. Similar considerations can be employed to compute the odd case,  $c_{2N+1,t}$ . □

As examples, let us consider the cases  $N = 1$  and  $N = 2$ .

- If  $N = 1$ , then we have  $t = 0, 1$ , so  $k = 0$ , and we obtain

$$Q_{2,0}(\omega) = \frac{1}{\omega^2} [c_{2,0} + \mathcal{O}(\omega^{-1})],$$

$$Q_{2,\pm 1}(\omega) = \frac{e^{\mp 2i\omega}}{\omega^4} [c_{2,1} + \mathcal{O}(\omega^{-1})],$$

with coefficients

$$c_{2,0} = 4G^2(2)G^2(2) = 4, \quad c_{2,1} = G^2(1)G^2(1) = 1,$$

from (5.5). Also,  $p = t_1 + t_2 = 2k + 1 = 1$ . So, along a trajectory for which  $e^{-i\omega} = \mathcal{O}(|\omega|)$  as  $|\omega| \rightarrow \infty$ , the previous two terms are balanced and they are both of order  $\mathcal{O}(|\omega|^{-2})$ .

- If  $N = 2$ , then we have  $t = 0, 1, 2$ , and

$$Q_{4,0}(\omega) = \frac{1}{\omega^8} [c_{4,0} + \mathcal{O}(\omega^{-1})],$$

$$Q_{4,\pm 1}(\omega) = \frac{e^{\mp 2i\omega}}{\omega^{10}} [c_{4,1} + \mathcal{O}(\omega^{-1})],$$

$$Q_{4,\pm 2}(\omega) = \frac{e^{\mp 4i\omega}}{\omega^{16}} [c_{4,2} + \mathcal{O}(\omega^{-1})],$$

with coefficients

$$c_{4,0} = 4^4 G^2(3)G^2(3) = 256,$$

$$c_{4,1} = 4^3 G^2(2)G^2(4) = 256,$$

$$c_{4,2} = 4^0 G^2(1)G^2(5) = 144,$$

from (5.5).

- If we choose  $k = 0$ , then  $t_1 = 0$  and  $t_2 = 1$ , so  $p = t_1 + t_2 = 1$ . Therefore, if  $e^{-i\omega} = \mathcal{O}(|\omega|)$  as  $|\omega| \rightarrow \infty$ , then there is a balance between  $Q_{4,0}(\omega)$  and  $Q_{4,1}(\omega)$  of order  $\mathcal{O}(|\omega|^{-8})$ . In this case, the remaining term is  $Q_{4,2}(\omega) = \mathcal{O}(|\omega|^{-12})$ , which is indeed of lower order.
- If we choose  $k = 1$ , then  $t_1 = 1$  and  $t_2 = 2$ , so  $p = t_1 + t_2 = 3$ . With  $e^{-i\omega} = \mathcal{O}(|\omega|^3)$  as  $|\omega| \rightarrow \infty$ , it follows that there is a balance between  $Q_{4,1}(\omega)$  and  $Q_{4,2}(\omega)$  of order  $\mathcal{O}(|\omega|^{-4})$ . Now, the remaining term is  $Q_{4,0}(\omega) = \mathcal{O}(|\omega|^{-8})$ , which is indeed of lower order.

Asymptotic approximations for the roots of  $h_{2N-1}(\omega)$  can be described in terms of the Lambert W function, which is a multivalued function that gives the solutions of the equation

$$we^w = z, \quad (5.9)$$

solving for  $w$  as a function of  $z$ . We refer the reader to [Olver et al. \(2021\)](#), Section 4.13) and also to [Corless et al. \(1996\)](#) for its definition and properties.

We choose  $k \in \{0, 1, \dots, N-1\}$  and we extract just the  $k$ th and  $(k+1)$ st terms from (5.1), then we obtain

$$\begin{aligned} h_{2N-1,k}(\omega) &= c_{2N,k} \frac{e^{-2i\omega k}}{\omega^{2N^2+2k^2}} + c_{2N,k+1} [1 + \mathcal{O}(|\omega|^{-1})] \\ &= \frac{e^{-2i\omega k}}{\omega^{2N^2+2k^2}} \left[ c_{2N,k} + c_{2N,k+1} \frac{e^{-2i\omega}}{\omega^{2(2k+1)}} + \mathcal{O}(|\omega|^{-1}) \right]. \end{aligned} \quad (5.10)$$

Here, we assume that there exists a trajectory such that  $e^{-i\omega} = \mathcal{O}(|\omega|^p) = \mathcal{O}(|\omega|^{2k+1})$  as  $|\omega| \rightarrow \infty$ , since  $p = t_1 + t_2 = 2k + 1$ .

**PROPOSITION 5.3** Let  $\omega_{k,\ell,m}$  be a root of  $h_{2N-1}(\omega)$ , where  $k = 0, 1, \dots, N-1$  identifies the layer,  $m \in \mathbb{Z}$  indexes groups of  $4k+2$  roots within each layer, and  $\ell = 0, 1, \dots, 4k+1$  labels these consecutive roots in such  $m$ -th group. Then, as  $m \rightarrow \infty$ , we have the approximation

$$\omega_{k,\ell,m} = -(2k+1)iW_m \left( \frac{i}{2k+1} \left| \frac{c_{2N,k+1}}{c_{2N,k}} \right| \frac{1}{4k+2} e^{\frac{\pi i \ell}{2k+1}} \right) + \mathcal{O}(m^{-1}). \quad (5.11)$$

Here,  $W_m$  is the  $m$ -th branch of the Lambert W function, and the coefficients  $c_{2N,j}$  are those given in Lemma 5.2.

*Proof.* Let us consider first the term in brackets in (5.10) but unperturbed:

$$c_{2N,k} + c_{2N,k+1} \frac{e^{-2iu}}{u^{2(2k+1)}} = 0 \Rightarrow u^{2(2k+1)} e^{2iu} = -\frac{c_{2N,k+1}}{c_{2N,k}}.$$

If we take roots and multiply by  $\frac{i}{2k+1}$ , we obtain  $4k+2$  solutions that we label  $u_{k,\ell}$ :

$$\frac{i u_{k,\ell}}{2k+1} e^{\frac{i u_{k,\ell}}{2k+1}} = \frac{i}{2k+1} \left| \frac{c_{2N,k+1}}{c_{2N,k}} \right| \frac{1}{4k+2} e^{\frac{\pi i \ell}{2k+1}}, \quad \ell = 0, 1, \dots, 4k+1.$$

This equation can be solved in terms of the Lambert W function, which is multivalued. Namely,

$$u_{k,\ell,m} = -(2k+1)iv_{k,\ell,m}, \quad v_{k,\ell,m} = W_m \left( \frac{i}{2k+1} \left| \frac{c_{2N,k+1}}{c_{2N,k}} \right| \frac{1}{4k+2} e^{\frac{\pi i \ell}{2k+1}} \right), \quad (5.12)$$

where  $m \in \mathbb{Z}$  indicates the branch. We observe that in the previous formula, we evaluate  $W_m$  at  $4k + 2$  equispaced points distributed on a circle of radius

$$r_k = \frac{1}{2k+1} \left| \frac{c_{2N,k+1}}{c_{2N,k}} \right|^{\frac{1}{4k+2}} = \frac{1}{2(2k+1)} \left[ \frac{(N+k)!}{(N-k-1)!} \right]^{\frac{1}{2k+1}}, \quad (5.13)$$

using (5.5). In the original equation (5.10), we have a remainder term  $\mathcal{O}(|\omega|^{-1})$ , so the solution is not exactly given by the values  $u_{k,\ell,m}$ . We observe that  $|u_{k,\ell,m}|$  gets large as  $m \rightarrow \infty$ , that is, as we pick different branches of the Lambert W function. Since the image of the  $m$ -th branch of W is contained in the strip  $(2m-2)\pi < \eta < (2m+1)\pi$ , see Corless *et al.* (1996, Section 4), we can estimate roughly  $W_m(z) = \mathcal{O}(m)$  as  $m \rightarrow \infty$ , therefore  $|u_{k,\ell,m}| = \mathcal{O}(m)$ , and we have an estimate for the remainder term

Following the calculations in Corless *et al.* (1996, Section 4), we write  $z = x + iy$  and  $w = \xi + i\eta$  in (5.9), and then separating real and imaginary parts, we have

$$x = e^\xi (\xi \cos \eta - \eta \sin \eta), \quad y = e^\xi (\eta \cos \eta + \xi \sin \eta).$$

As a consequence,

$$x^2 + y^2 = e^{2\xi} (\xi^2 + \eta^2),$$

which gives the image of a circle of radius  $r_k$ , given by (5.13), in the  $(\xi, \eta)$  plane:

$$\eta = \pm \sqrt{r_k^2 e^{-2\xi} - \xi^2}. \quad (5.14)$$

Each value of the parameter  $k = 0, 1, \dots, N-1$  gives a different circle of radius  $r_k$  in the  $z$  plane, which gets mapped to a different curve in the  $w$  plane by the Lambert W function in (5.12). Then, each group of  $4k + 2$  equispaced points on that circle gets mapped to the curve, each branch of W giving one cluster of points in the  $(\xi, \eta)$  plane. In Figs 11 and 12, we have plotted the curves separating branches of the Lambert W function (cf. Corless *et al.*, 1996, Figure 4), in dashed lines, together with the curve (5.14) in blue, in the case  $N = 0$  and  $N = 1$ . We indicate with red dots the images of the  $4k + 2$  points equispaced on the circle, given by each branch. Finally, multiplication by the factor  $-(2k+1)i$ , cf. (5.12), scales and rotates the points, in good agreement with the plots shown at the beginning of this chapter.

## 5.2 The even case: the roots of $h_{2N}$

We revisit the methodology of the last subsection, except that in the present case, we have the layers  $\mathcal{Y}_{2N+1,N-t}$  and  $\mathcal{Y}_{2N+1,N+1+t}$ ,  $t = 0, \dots, N$ . The exponential factor is  $e^{(2t+1)i\omega}$  in the first case, and  $e^{-(2t+1)i\omega}$  in the second.

For reasons of symmetry it is enough to look at one of these cases. Since we are interested in the upper-right quadrant, we choose the second case, where we have the dominant exponential factors again. Computing as before, the leading power of  $\omega$  is  $2N^2 + 2t^2 + 2t$ . Adding the dimension  $n = 2N + 1$ , we

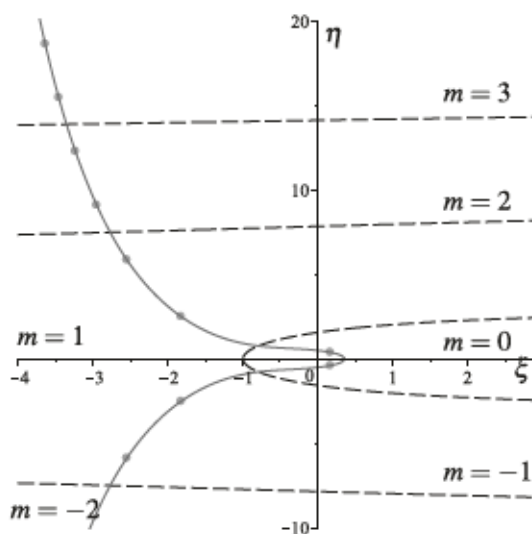


FIG. 11. Plot of the branches of the Lambert  $W_m$  function (dashed curves) and the curve (5.14), in solid line for  $N = 0$  and  $k = 0$ . Dots indicate the images of the points  $v_{k, \ell, m}$  in (5.12) by the different branches of the Lambert  $W$  function.

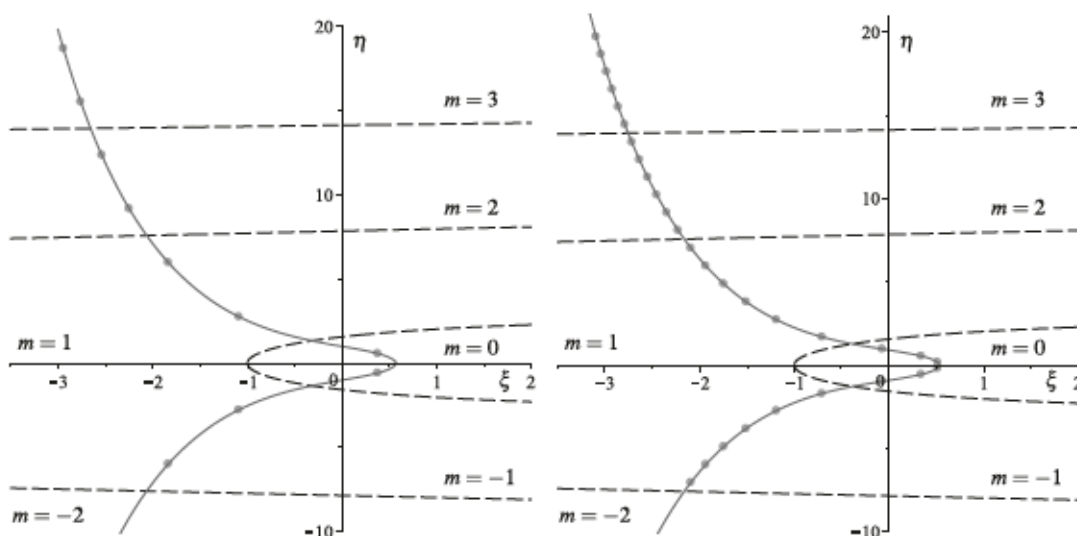


FIG. 12. Plot of the branches of the Lambert  $W_m$  function (dashed curves) and the curve (5.14), in solid line, for  $N = 1$  and  $k = 0$  (left) and for  $N = 1$  and  $k = 1$  (right). Dots indicate the images of the points  $v_{k, \ell, m}$  in (5.12) by the different branches of the Lambert  $W$  function.

deduce that the contribution of  $\mathcal{Y}_{2N+1, N+1+t}$  is for  $\omega \rightarrow \infty$  on the real line

$$Q_{2N+1, N+1+t}(\omega) := \frac{e^{-(2t+1)i\omega}}{\omega^{2N^2+2N+1+2t^2+2t}} \left[ c_{2N+1, t} + \mathcal{O}(\omega^{-1}) \right], \quad t = 0, \dots, N.$$

LEMMA 5.4 As  $|\omega| \rightarrow \infty$  such that (5.3) holds with  $p = t_1 + t_2 + 1$ , it is true that  $Q_{2N+1, N+1+t_1}(\omega)$  and  $Q_{2N+1, N+1+t_2}(\omega)$  are of the same order of magnitude, and  $Q_{2N, N+1+t}(\omega)$  is of lower order for  $t \neq t_1, t_2$ , if and only if  $t_1 = k$  and  $t_2 = k + 1$ , and in that case, we have

$$\begin{aligned} Q_{2N+1, N+1+k}(\omega), Q_{2N+1, N+2+k}(\omega) &= \mathcal{O}(\omega^{-2N^2-2N-2k^2+(2k+1)^2}), \\ Q_{2N+1, N+1+t}(\omega) &= \mathcal{O}(\omega^{-2N^2-2N-2t^2+(2t+1)(t_1+t_2)}). \end{aligned}$$

*Proof.* Assuming again that  $e^{-i\omega} = \mathcal{O}(|\omega|^p)$  for some  $p > 0$ , we have

$$Q_{2N+1, N+1+t}(\omega) = \mathcal{O}(\omega^{-2N^2-2N-1-2t^2-2t+2pt+p}).$$

We again choose  $t_1 < t_2$  according to the two rules: first, we impose

$$-2N^2 - 2N - 1 - 2t_1^2 - 2t_1 + 2p(2t_1 + 1) = -2N^2 - 2N - 1 - 2t_2^2 - 2t_2 + 2p(2t_2 + 1),$$

which gives  $p = t_1 + t_2 + 1$ . Then the total exponent is  $-2N^2 - 2N - 1$ . It follows that if  $t_1 = k$  and  $t_2 = k + 1$ , then the first part of (5.15) is satisfied. The second requirement reduces again to  $(t - t_1)(t - t_2 + 1) > 0$  for all  $t \neq t_1, t_2$ . It follows at once that  $t_1 = k, t_2 = k + 1$  for some  $k \in \{0, 1, \dots, N - 1\}$ ; otherwise, the inequality fails for  $t_1 + 1$  as in the even case.

All that remains is to check if the above choice of consecutive  $t_1, t_2$  works and indeed, trivially, it does: either  $t < t_1, t_2$  or  $t > t_1, t_2$  and in either case the inequality works.  $\square$

As an example, if  $N = 1$  we have  $t = 0, 1$ , so  $k = 0$  and therefore

$$\begin{aligned} Q_{3,2}(\omega) &= \frac{e^{-i\omega}}{\omega^5} [c_{3,0} + \mathcal{O}(\omega^{-1})], \\ Q_{3,3}(\omega) &= \frac{e^{-3i\omega}}{\omega^9} [c_{3,1} + \mathcal{O}(\omega^{-1})], \end{aligned} \tag{5.15}$$

with coefficients

$$c_{3,2} = -i4^2G^2(3)G^2(2) = -16i, \quad c_{3,3} = iG^2(4)G^2(1) = 4i,$$

from (5.5). Also,  $p = t_1 + t_2 + 1 = 2k + 2 = 2$ , so  $e^{-i\omega} = \mathcal{O}(|\omega|^2)$  as  $|\omega| \rightarrow \infty$ . It follows that there is a balance between  $Q_{3,2}(\omega)$  and  $Q_{3,3}(\omega)$  of order  $\mathcal{O}(|\omega|^{-3})$ .

PROPOSITION 5.5 Let  $\omega_{k,\ell,m}$  be a root of  $h_{2N}(\omega)$ , where  $k = 0, 1, \dots, N - 1$  identifies the layer,  $m \in \mathbb{Z}$  indexes groups of roots within each layer, and  $\ell = 0, 1, \dots, 4k + 3$  labels the  $4k + 4$  consecutive roots in such  $m$ -th group. Then, as  $m \rightarrow \infty$ , we have the approximation

$$\omega_{k,\ell,m} = -2i(k+1)W_m\left(\frac{i}{2(k+1)} \left| \frac{c_{2N+1,k+1}}{c_{2N+1,k}} \right|^{\frac{1}{4k+4}} e^{\frac{\pi i \ell}{4k+2}}\right) + \mathcal{O}(m^{-1}), \tag{5.16}$$

again in terms of the  $m$ -th branch of the Lambert W function and the coefficients  $c_{2N+1,j}$  in Lemma 5.2.



*Proof.* We choose  $t_1 = k$ ,  $t_2 = k + 1$ , therefore  $p = t_1 + t_2 + 1 = 2k + 2$ , and investigate the zeros of

$$h_{2N,k}(\omega) = \frac{e^{-(2k+1)i\omega}}{\omega^{2N^2+2N+1+2k^2+2k}} \left[ c_{2N+1,k} + c_{2N+1,k+1} \frac{e^{-2i\omega}}{\omega^{4k+4}} + \mathcal{O}(|\omega|^{-1}) \right].$$

Again the solution of the unperturbed problem can be constructed in terms of the Lambert W function, and we deduce (5.16) with a similar argument as in the previous case. As before, we can compute the coefficients explicitly:

$$\left| \frac{c_{2N+1,k+1}}{c_{2N+1,k}} \right|^{\frac{1}{4k+4}} = \frac{1}{2} \left[ \frac{(N+k+1)!}{(N-k-1)!} \right]^{\frac{1}{2k+2}}.$$

Note that this calculation does not include the real roots of  $h_{2N}$ , since in that case there is no balance between two different terms in the large  $\omega$  expansion.  $\square$

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## A. The Riemann–Hilbert problem and a proof of Lemma 2.8

### A.1 Riemann–Hilbert problem for kissing polynomials

Similarly to other families of orthogonal polynomials (OPs), we can formulate the kissing polynomials in terms of the solution of a Riemann–Hilbert problem. This formulation is well known in the literature, since the work of Fokas *et al.* (1992); we also refer the reader to the monograph by Deift (2000) and Chapter 22 in the book of Ismail (2005). In the present situation, since we are working with a complex weight function, we always must take care that the underlying quantities, in particular the OPs themselves, exist.

Define the following  $2 \times 2$  matrix

$$Y(z) = \begin{bmatrix} p_n(z) & \frac{1}{2\pi i} \int_{-1}^1 \frac{p_n(s)e^{i\omega s}}{s-z} ds \\ -2\pi i \kappa_{n-1}^2 p_{n-1}(z) & -\kappa_{n-1}^2 \int_{-1}^1 \frac{p_{n-1}(s)e^{i\omega s}}{s-z} ds \end{bmatrix}, \quad (\text{A.1})$$

where  $\kappa_n$  is the leading coefficient of the *orthonormal* kissing polynomial, obtained via

$$\frac{1}{\kappa_n^2(\omega)} = \chi_n(\omega). \quad (\text{A.2})$$

Note that the second column of (A.1) is formed of Cauchy transforms of analytic functions, and as such is analytic in  $\mathbb{C} \setminus [-1, 1]$ .

Provided that  $p_n(z)$  and  $p_{n-1}(z)$  both exist and are monic polynomials of degree  $n$  and  $n - 1$  respectively, that is, if  $h_{n-1}$  and  $h_{n-2}$  are not 0, then the matrix  $Y = Y_n(z)$  solves the following Riemann–Hilbert problem.

1.  $Y(z)$  is analytic for  $z \in \mathbb{C} \setminus [-1, 1]$ .

2. For  $x \in (-1, 1)$ ,  $Y(z)$  admits boundary values  $Y_{\pm}(x) = \lim_{\varepsilon \rightarrow 0} Y(x \pm i\varepsilon)$ , which are related by the jump

$$Y_+(x) = Y_-(x) \begin{bmatrix} 1 & e^{i\omega x} \\ 0 & 1 \end{bmatrix}.$$

3. As  $z \rightarrow \infty$ , we have the asymptotic behaviour

$$Y(z) = \left[ I + \mathcal{O}\left(\frac{1}{z}\right) \right] z^{n\sigma_3}, \quad (\text{A.3})$$

4. As  $z \rightarrow \pm 1$ , we have

$$Y(z) = \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z \mp 1|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z \mp 1|) \end{bmatrix}.$$

Above,  $\sigma_3$  is the Pauli matrix given by  $\sigma_3 = \text{diag}(1, -1)$ . We use the standard notation

$$f(z)^{\sigma_3} = \begin{bmatrix} f(z) & 0 \\ 0 & f(z)^{-1} \end{bmatrix}, \quad f(z) \neq 0.$$

We note that the assumption on  $h_{n-1}$  and  $h_{n-2}$  is needed in this case, given the fact that the weight function is not positive. Moreover, if  $h_{n-2}$  is zero, then the  $(2, 1)$  entry is not a polynomial of degree  $n-1$ , but lower.

This Riemann–Hilbert problem is the usual starting point for the Deift–Zhou method of nonlinear steepest descent, which is a powerful tool to derive large  $n$  asymptotics of OPs throughout the complex plane, yet we will follow the approach of Ismail (2005, Chapter 22) and use this formulation to derive algebraic and differential identities for the kissing polynomials, for finite  $n$ . To achieve this, we observe that it is possible to rewrite recurrence coefficients and norms of the OPs in terms of the correction matrices  $A_n(\omega)$  and  $B_n(\omega)$  at infinity: we write the asymptotic expansion (A.3) as

$$Y(z) = \left[ I + \frac{A_n(\omega)}{z} + \frac{B_n(\omega)}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right] z^{n\sigma_3}, \quad z \rightarrow \infty, \quad (\text{A.4})$$

where

$$A_n(\omega) = \begin{bmatrix} a_{11,n} & a_{12,n} \\ a_{21,n} & a_{22,n} \end{bmatrix}, \quad B_n(\omega) = \begin{bmatrix} b_{11,n} & b_{12,n} \\ b_{21,n} & b_{22,n} \end{bmatrix}, \quad (\text{A.5})$$

and the  $a_{ij,n}, b_{ij,n}$  are functions of  $n$  and  $\omega$ . We can calculate the relevant entries of (A.5) in terms of Hankel determinants by looking at the expansion of (A.1) at infinity, as follows:

$$a_{11,n} = i \frac{\dot{h}_{n-1}}{h_{n-1}}, \quad a_{21,n} = -\frac{2\pi i h_{n-2}}{h_{n-1}} = -\frac{2\pi i}{\chi_{n-1}}, \quad a_{12,n} = -\frac{h_n}{2\pi i h_{n-1}} = -\frac{\chi_n}{2\pi i} \quad (\text{A.6a})$$

and

$$a_{22,n} = -i \frac{\dot{h}_{n-1}}{h_{n-1}}, \quad b_{21,n} = 2\pi \frac{\dot{h}_{n-2}}{h_{n-1}}, \quad b_{12,n} = \frac{1}{2\pi} \frac{\dot{h}_n}{h_{n-1}}. \quad (\text{A.6b})$$

As a consequence of these formulas and of (2.1), we can write the recurrence coefficients in terms of the entries of the matrix  $A_n(\omega)$ :

$$\alpha_n(\omega) = a_{11,n} - a_{11,n+1}, \quad \beta_n(\omega) = a_{12,n} a_{21,n}.$$

### A.2 Transformation to constant jumps

In this appendix, we outline how to use Riemann–Hilbert methodology to obtain some of the results presented in the paper: first, we make a transformation of  $Y(z)$  so that the resulting matrix has constant jumps over the interval  $(-1, 1)$ . As such, we define

$$Z(z) = Y(z) \begin{bmatrix} e^{i\omega z/2} & 0 \\ 0 & e^{-i\omega z/2} \end{bmatrix}, \quad (\text{A.7})$$

so that  $Z$  solves the following Riemann–Hilbert problem.

1.  $Z(z)$  is analytic for  $z \in \mathbb{C} \setminus [-1, 1]$ .
2. For  $x \in (-1, 1)$ ,  $Z(z)$  admits boundary values  $Z_{\pm}(x) = \lim_{\varepsilon \rightarrow 0} Z(x \pm i\varepsilon)$ , which are related by the jump

$$Z_+(x) = Z_-(x) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

3. As  $z \rightarrow \infty$ , we have the asymptotic behaviour

$$Z(z) = \left[ I + \frac{A_n(\omega)}{z} + \frac{B_n(\omega)}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right] z^{n\sigma_3} e^{i\omega z\sigma_3/2}, \quad (\text{A.8})$$

with coefficients  $A_n(\omega)$  and  $B_n(\omega)$  given by (A.5), (A.6a) and (A.6b).

4. As  $z \rightarrow \pm 1$ , we have

$$Z(z) = \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(\log |z \mp 1|) \\ \mathcal{O}(1) & \mathcal{O}(\log |z \mp 1|) \end{bmatrix}.$$

By the standard technique of showing uniqueness of solutions to Riemann–Hilbert problems (Deift, 2000), we have  $\det Z(z) = 1$ , so that  $Z(z)$  is invertible for all  $z \in \mathbb{C}$ . Also, by taking derivative with respect to  $z$ , we are also able to conclude that  $Z'(z)$  solves the following Riemann–Hilbert problem.

1.  $Z'(z)$  is analytic for  $z \in \mathbb{C} \setminus [-1, 1]$ .
2. For  $x \in (-1, 1)$ ,  $Z'(z)$  admits boundary values  $Z'_{\pm}(x) = \lim_{\varepsilon \rightarrow 0} Z'(x \pm i\varepsilon)$ , which are related by the same jump as those for  $Z(z)$ .
3. As  $z \rightarrow \infty$ , we have the asymptotic behaviour

$$Z'(z) = \left[ \Gamma_0(\omega) + \frac{\Gamma_1(\omega)}{z} + \frac{\Gamma_2(\omega)}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right] z^{n\sigma_3} e^{i\omega z\sigma_3/2},$$

where the coefficients are

$$\begin{aligned} \Gamma_0(\omega) &= \frac{i\omega\sigma_3}{2}, & \Gamma_1(\omega) &= n\sigma_3 + \frac{i\omega A_n \sigma_3}{2}, \\ \Gamma_2(\omega) &= -A_n + nA_n \sigma_3 + \frac{i\omega}{2} B_n \sigma_3, \end{aligned} \quad (\text{A.9})$$

in terms of  $A_n(\omega)$  and  $B_n(\omega)$  given before.

4. As  $z \rightarrow \pm 1$ , we have

$$Z'(z) = \begin{bmatrix} \mathcal{O}(1) & \mathcal{O}(|z \mp 1|^{-1}) \\ \mathcal{O}(1) & \mathcal{O}(|z \mp 1|^{-1}) \end{bmatrix}.$$

### A.3 Differential equation and proof of Lemma 2.11

Note that, because of the fact that the jump matrix for  $Z(z)$  is independent of  $z$ , then both  $Z(z)$  and  $Z'(z)$  have the same jumps over the interval  $(-1, 1)$ ; then, we conclude that  $Z'(z)Z^{-1}(z)$  is analytic in  $\mathbb{C} \setminus \{\pm 1\}$ , while the singularities at the endpoints are at most simple poles. Therefore,  $(z^2 - 1)Z'(z)Z^{-1}(z)$  is an entire function. Using (A.8), we compute

$$Z^{-1}(z) = e^{-i\omega z \sigma_3 / 2} z^{-n \sigma_3} \left[ I + \frac{\Delta_1(\omega)}{z} + \frac{\Delta_2(\omega)}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right], \quad z \rightarrow \infty,$$

where

$$\Delta_1(\omega) = -A_n, \quad \Delta_2(\omega) = A_n^2 - B_n$$

and define

$$M(z) = \Gamma_0 z^2 + (\Gamma_0 \Delta_1 + \Gamma_1) z + (\Gamma_1 \Delta_1 + \Gamma_2 + \Gamma_0 \Delta_2 - \Gamma_0). \quad (\text{A.10})$$

Examining the asymptotics of  $Z^{-1}(z)$  and  $Z'(z)$  at infinity, we conclude that

$$(z^2 - 1)Z'(z)Z^{-1}(z) = M(z) + \mathcal{O}\left(\frac{1}{z}\right), \quad z \rightarrow \infty, \quad (\text{A.11})$$

and since this is an entire function, Liouville's theorem yields

$$(z^2 - 1)Z'(z) = M(z)Z(z). \quad (\text{A.12})$$

Looking at the leading column of (A.12) and taking advantage of (A.1) and (A.7), we have the following equations, which can be seen as ladder relations of kissing polynomials:

$$(z^2 - 1) \begin{bmatrix} p'_n(z) \\ p'_{n-1}(z) \end{bmatrix} = \begin{bmatrix} N_1(z) & N_2(z) \\ N_3(z) & N_4(z) \end{bmatrix} \begin{bmatrix} p_n(z) \\ p_{n-1}(z) \end{bmatrix}, \quad (\text{A.13})$$

where

$$N_1(z) = M_{11}(z) - \frac{i\omega}{2}(z^2 - 1), \quad N_2(z) = -2\pi i \kappa_{n-1}^2 M_{12}(z)$$

$$N_3(z) = -\frac{M_{21}(z)}{2\pi i \kappa_{n-1}^2}, \quad N_4(z) = M_{22}(z) - \frac{i\omega}{2}(z^2 - 1).$$

Using (A.5) and (A.10), we can simplify these to

$$N_1(z) = nz - i \left( \frac{h_{n-1}}{h_{n-1}} - \omega \frac{h_n h_{n-2}}{h_{n-1}^2} \right) = nz - i \left[ \frac{h_{n-1}}{h_{n-1}} - \omega \beta_n(\omega) \right], \quad (\text{A.14a})$$

$$N_2(z) = -\frac{i\omega h_{n-2} h_n}{h_{n-1}^2} (z - z_*(\omega; n)) = -i\omega \beta_n(\omega) (z - z_*(\omega; n)), \quad (\text{A.14b})$$

$$N_3(z) = i\omega [z - z_*(\omega; n - 1)], \quad (\text{A.14c})$$

and

$$\begin{aligned} N_4(z) &= -i\omega(z^2 - 1) - nz + i \left( \frac{\dot{h}_{n-1}}{h_{n-1}} - \omega \frac{h_n h_{n-2}}{h_{n-1}^2} \right) \\ &= -i\omega(z^2 - 1) - nz + i \left[ \frac{\dot{h}_{n-1}}{h_{n-1}} - \omega \beta_n(\omega) \right]. \end{aligned} \quad (\text{A.14d})$$

In (A.14b) and (A.14c), we have used the notation

$$z_*(\omega; n) := -\alpha_n - \frac{2n+1}{i\omega}. \quad (\text{A.15})$$

All of the  $N_i(z)$  are well defined provided  $h_{n-1} \neq 0$ , with  $N_3(z)$  needing the additional assumption of  $h_{n-2} \neq 0$  in order to be well defined. Combining the equations in (A.13) gives

$$\left[ (z^2 - 1) \frac{d}{dz} - N_4(z) \right] \left[ \frac{(z^2 - 1)p'_n(z)}{N_2(z)} - \frac{N_1(z)p_n(z)}{N_2(z)} \right] = N_3(z)p_n(z).$$

Equivalently, we can write

$$p''_n(z) + \frac{R(z)}{Q(z)}p'_n(z) + \frac{S(z)}{Q(z)}p_n(z) = 0, \quad (\text{A.16})$$

where

$$\begin{aligned} Q(z) &= (z^2 - 1)^2 N_2(z), \\ R(z) &= (z^2 - 1) N_2(z) (2z - N_1(z) - N_4(z)) - (z^2 - 1)^2 N'_2(z), \\ S(z) &= N_2(z) [N_1(z)N_4(z) - (z^2 - 1)N'_1(z)] - N_2^2(z)N_3(z) \\ &\quad + (z^2 - 1)N_1(z)N'_2(z). \end{aligned} \quad (\text{A.17})$$

Now, using (A.14b), we observe that if  $h_n = 0$ , then

$$N_2(z) = -\frac{\omega h_{n-2} \dot{h}_n}{h_{n-1}^2} \neq 0,$$

so that the only singular points of the ODE (A.16) are at  $\pm 1$ . On the other hand, if  $h_n \neq 0$ , then  $N_2$  has a simple, purely imaginary zero at  $z_*(\omega; n)$ , recall (A.15), which is purely imaginary and necessarily a singular point of the ODE (A.16). Simplifying yields

$$\begin{aligned} \frac{R(z)}{Q(z)} &= \frac{2z - N_1(z) - N_4(z)}{z^2 - 1} - \frac{1}{z - z_*(\omega; n)}, \\ \frac{S(z)}{Q(z)} &= \frac{N_1(z)N_4(z) - (z^2 - 1)N'_1(z)}{(z^2 - 1)^2} - \frac{N_2(z)N_3(z)}{(z^2 - 1)^2} + \frac{N_1(z)}{(z^2 - 1)(z - z_*(\omega; n))}. \end{aligned} \quad (\text{A.18})$$

Using (A.14a) and (A.14d), we see that both  $N_1(z)$  and  $N_4(z)$  are well defined when  $h_{n-2}$  vanishes, and by (A.15), we deduce that  $z_*$  does not depend on  $h_{n-2}$ . Finally, if  $\omega'$  is such that  $h_{n-2}(\omega') = 0$ , then

$$\lim_{\omega \rightarrow \omega'} N_2(z)N_3(z) = \frac{\omega' 2h_n(\omega') \dot{h}_{n-2}(\omega') [z - z_*(\omega'; n)]}{h_{n-1}^2(\omega')},$$

so that (A.16) holds even when  $h_{n-2} = 0$ , completing the proof of Lemma 2.11.

#### A.4 Recurrence relation

We observe that the Riemann–Hilbert formulation can be used in a systematic way to give alternative proofs of several identities that we use in the paper.

For example, if we consider the product  $Y_{n+1}(z)Y_n(z)^{-1}$ , it turns out that this is an entire function in  $z$ , since the jump matrix of  $Y_n(z)$  is actually independent of  $n$ . Using asymptotics at infinity (A.4), (A.5) and Liouville's theorem, we conclude that

$$Y_{n+1}(z)Y_n(z)^{-1} = C_n(z), \quad C_n(z) = \begin{bmatrix} z + a_{11,n+1} - a_{11,n} & -a_{12,n} \\ a_{21,n+1} & 0 \end{bmatrix}, \quad (\text{A.19})$$

from which, using (A.1), we identify the coefficients in the three-term recurrence relation (1.11), as well as the leading coefficient of the orthonormal polynomials:

$$\begin{aligned} a_{12,n} &= -\frac{\chi_n}{2\pi i}, & a_{21,n} &= -\frac{2\pi i}{\chi_{n-1}^2}, \\ a_{11,n} - a_{11,n+1} &= \alpha_n, & a_{12,n}a_{21,n} &= \beta_n. \end{aligned} \quad (\text{A.20})$$

There is an additional identity for the recurrence coefficient  $\alpha_n$ , which can be found in Deift (2000, Section 3.2) and Deift *et al.* (1999, Theorem 3.1),

$$\alpha_n = a_{11,n} + \frac{b_{12,n}}{a_{12,n}} = -a_{11,n+1} + \frac{b_{21,n+1}}{a_{21,n+1}}.$$

We can now combine the differential identity (A.12) (adding the subscript  $n$  for clarity) with (A.19):

$$Z'_n(z) = D_n(z)Z_n(z), \quad Z_{n+1}(z) = C_n(z)Z_n(z).$$

Once we compute  $Z'_{n+1}(z)$  in two different ways, we arrive at the compatibility relation

$$C'_n(z) = D_{n+1}(z)C_n(z) - C_n(z)D_n(z).$$

This equation gives four identities for recurrence coefficients: the (2, 2) entry is trivial, using (A.6a) as well as

$$b_{12,n} = \frac{1}{2\pi i} \delta_{n+1,n} \chi_n, \quad b_{21,n} = -2\pi i \frac{\delta_{n-1,n-2}}{\chi_{n-1}}.$$

The (1, 2) and (2, 1) entries recover (2.7), the identity obtained using Magnus's ideas, and the (1, 1) entry gives the second identity in (2.6).

#### A.5 Differential equation in $\omega$ and Painlevé V

Similarly, if we consider differentiation with respect to  $\omega$ , denoted by dot, we conclude that  $\dot{Z}_n(z)Z_n(z)^{-1}$  is an entire function in  $z$ , since the jump matrix of  $Z$  is independent of  $\omega$ . Using asymptotics at infinity



(A.4), (A.5) and Liouville's theorem,

$$\dot{Z}(z)Z(z)^{-1} = E_n(z), \quad E_n(z) = \frac{i}{2} \begin{bmatrix} z & -2a_{12,n} \\ 2a_{21,n} & -z \end{bmatrix}. \quad (\text{A.21})$$

Compatibility of  $Z_{n+1}(z) = C_n(z)Z_n(z)$  and (A.21) gives

$$\begin{aligned} \dot{C}_n = E_{n+1}C_n - C_nE_n &= \frac{i}{2} \begin{bmatrix} z & -2a_{12,n+1} \\ 2a_{21,n+1} & -z \end{bmatrix} \begin{bmatrix} z - \alpha_n & -a_{12,n} \\ a_{21,n+1} & 0 \end{bmatrix} \\ &\quad - \frac{i}{2} \begin{bmatrix} z - \alpha_n & -a_{12,n} \\ a_{21,n+1} & 0 \end{bmatrix} \begin{bmatrix} z & -2a_{12,n} \\ 2a_{21,n} & -z \end{bmatrix}. \end{aligned}$$

The (1, 1) entry of this equality gives  $\dot{\alpha}_n = i(\beta_{n+1} - \beta_n)$ , which is the first equation in (2.5), and the (1, 2) and (2, 1) entries give

$$-\dot{a}_{12,n} = i\alpha_n a_{12,n}, \quad \dot{a}_{21,n+1} = i\alpha_n a_{21,n+1},$$

so multiplying the first equation by  $a_{21,n}$  and the second one by  $a_{12,n}$  (and shifting the index), together with (A.20), we arrive at

$$\dot{\beta}_n = \frac{d}{d\omega}(a_{12,n}a_{21,n}) = ia_{12,n}a_{21,n}\alpha_n - ia_{21,n}a_{12,n}\alpha_{n-1} = i\beta_n(\alpha_n - \alpha_{n-1}),$$

which is the second equation in (2.5). These two identities can be derived by considering the  $\mathcal{O}(z^{-1})$  terms in (A.21) as well.

We can rewrite (A.11) as

$$Z'(z)Z(z)^{-1} = M_\infty + \frac{M_1}{z-1} + \frac{M_0}{z+1}, \quad (\text{A.22})$$

with coefficients

$$M_\infty = \Gamma_0, \quad M_1 = \frac{1}{2}(\Gamma_1\Delta_1 + \Gamma_2 + \Gamma_0\Delta_2) + \Gamma_0\Delta_1 + \Gamma_1, \quad M_0 = -\frac{1}{2}(\Gamma_1\Delta_1 + \Gamma_2 + \Gamma_0\Delta_2),$$

using (A.10). We change variable  $z = 2u - 1$ , to formulate the problem in the interval  $[0, 1]$  instead of  $[-1, 1]$  and consider the matrix

$$V(u) = 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} Z(2u-1), \quad (\text{A.23})$$

which has the following asymptotics at infinity:

$$\begin{aligned} V(u) &= 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} \left[ I + \frac{A_n(\omega)}{2u-1} + \frac{B_n(\omega)}{(2u-1)^2} + \mathcal{O}\left(\frac{1}{u^3}\right) \right] (2u-1)^{n\sigma_3} e^{\frac{i\omega(2u-1)\sigma_3}{2}} \\ &= \left[ I + \frac{V_{n,1}(\omega)}{u} + \mathcal{O}\left(\frac{1}{u^2}\right) \right] u^{n\sigma_3} e^{i\omega u\sigma_3}. \end{aligned} \quad (\text{A.24})$$

Here, the factor

$$V_{n,1}(\omega) = 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} \frac{A_n(\omega) - n\sigma_3}{2} e^{-\frac{i\omega}{2}\sigma_3} 2^{n\sigma_3} \quad (\text{A.25})$$

follows from expanding  $(2u-1)^{n\sigma_3}$ , and then (A.22) in the variable  $u$  becomes

$$V'(u)V(u)^{-1} = 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} \left( 2M_\infty + \frac{M_1}{u-1} + \frac{M_0}{u} \right) e^{-\frac{i\omega}{2}\sigma_3} 2^{n\sigma_3}. \quad (\text{A.26})$$

Similarly, (A.21) becomes

$$\dot{V}(u)V(u)^{-1} = \frac{i}{2}\sigma_3 + 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} E_n e^{-\frac{i\omega}{2}\sigma_3} 2^{n\sigma_3} = iu\sigma_3 + \begin{bmatrix} 0 & -i4^{-n}e^{i\omega}a_{12,n} \\ i4^n e^{-i\omega}a_{21,n} & 0 \end{bmatrix}. \quad (\text{A.27})$$

Since

$$2 \cdot 2^{-n\sigma_3} e^{\frac{i\omega}{2}\sigma_3} M_\infty e^{-\frac{i\omega}{2}\sigma_3} 2^{n\sigma_3} = i\omega\sigma_3,$$

if we identify the parameter  $2i\omega = t$ , then equations (A.26) and (A.27) match precisely (Jimbo & Miwa, 1981, (C.38) and (C.39)), that is, the linear system of differential equations associated with Painlevé V.

Using (2.4), the subleading coefficient is

$$\delta_{n,n-1}(\omega) = i \frac{d}{d\omega} \log h_{n-1}(\omega) = a_{11,n}(\omega),$$

written in terms of the first correction matrix at infinity of  $Y(z)$ , recall (A.4) and (A.5). Because of (A.7), (A.23) and (A.25), we have

$$(V_{n,1})_{11}(\omega) = \frac{a_{11,n}(\omega) - n}{2}$$

On the other hand, from Jimbo & Miwa (1981, pp. 443), this entry can be identified as  $(V_{n,1})_{11}(t) = -H_V(t)$ , where  $H_V(t)$  is the Hamiltonian for Painlevé V, with parameters  $\theta_0 = \theta_1 = 0$  and  $\theta_\infty = -2n$ . The function

$$\begin{aligned} \sigma_n(t) &= tH_V(t) + \frac{\theta_0 + \theta_\infty}{2}t + \frac{1}{4} \left( (\theta_0 + \theta_\infty)^2 - \theta_1^2 \right) \\ &= tH_V(t) - nt + n^2 \\ &= t \frac{d}{dt} \log h_{n-1}(t) - \frac{nt}{2} + n^2 \end{aligned}$$

satisfies the Jimbo–Miwa–Okamoto  $\sigma$ -Painlevé V equation:

$$(t\sigma_n'')^2 = [\sigma_n - t\sigma_n' + 2(\sigma_n')^2 + (v_0 + v_1 + v_2 + v_3)\sigma_n']^2 - 4 \prod_{j=0}^3 (\sigma_n' + v_j) \quad (\text{A.28})$$

with parameters

$$v_0 = 0, \quad v_1 = -\frac{\theta_0 - \theta_1 + \theta_\infty}{2}, \quad v_2 = -\theta_0, \quad v_3 = -\frac{\theta_0 + \theta_1 + \theta_\infty}{2};$$

see Jimbo & Miwa (1981, (C.45)), which becomes (2.21), identifying  $\theta_0$ ,  $\theta_1$  and  $\theta_\infty$  as above.