

Article

On the Schrödinger Equation for Time-Dependent Hamiltonians with a Constant Form Domain

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Abstract: We study two seminal approaches, developed by B. Simon and J. Kisyński, to the well-posedness of the Schrödinger equation with a time-dependent Hamiltonian. In both cases, the Hamiltonian is assumed to be semibounded from below and to have a constant form domain, but a possibly non-constant operator domain. The problem is addressed in the abstract setting, without assuming any specific functional expression for the Hamiltonian. The connection between the two approaches is the relation between sesquilinear forms and the bounded linear operators representing them. We provide a characterisation of the continuity and differentiability properties of form-valued and operator-valued functions, which enables an extensive comparison between the two approaches and their technical assumptions.

Keywords: Schrödinger equation; time-dependent Hamiltonian; Hilbert scales; time-dependent domain

MSC: 35J10; 35Q41; 46N50; 47B25; 81Q10



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1. Introduction

A quantum dynamical system is a first-order linear evolution equation on a separable, complex Hilbert space \mathcal{H} , where the evolution is determined by a family of densely defined, self-adjoint operators $\{H(t)\}_{t \in I}$, where $I \subset \mathbb{R}$ is an interval. This family of operators is called the Hamiltonian of the system. To simplify the notation, it is denoted by $H(t)$. In the most general case, this represents the dynamics of a quantum system subjected to external time-varying forces. The dynamics is given by the time-dependent (or non-autonomous) Schrödinger equation:

$$\frac{d}{dt}\Psi(t) = -iH(t)\Psi(t),$$

with some initial datum $\Psi(s) = \Psi_0$.

Different from what happens in the autonomous case (i.e., $H(t) = H$), where the existence and uniqueness of the solution are guaranteed by the celebrated Stone theorem on one-parameter unitary groups [1–3], additional regularity conditions are required when $H(t)$ depends nontrivially on time. In particular, when dealing with unbounded Hamiltonians, the domain $\text{dom } H(t)$ can possibly depend nontrivially on t . In such cases, even by requiring the expression of the Hamiltonian to be reasonably “well-behaved” (e.g., continuously differentiable) as a function of time, the well-posedness of the problem above is not ensured; standard methods such as product integrals or Dyson expansions cannot be applied directly.

Singular potentials in the Schrödinger equation are a convenient way of analysing different types of nontrivial boundary conditions. One example is point-like interactions,

which model impurities. This has applications in physically relevant systems such as propagation in quantum waveguides [4–6] or quantum dots [7]. Furthermore, recent results allowing for the numerical computation of the spectrum for nontrivial boundary conditions [8,9] opened the possibility to control the state of a quantum system by modifying the boundary conditions. The latter has implications in the development of new quantum technologies.

The problem of time-dependent Hamiltonians for the class of models $H(t) = -\Delta + V(t)$, with Δ being the Laplace operator on $L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, and $V(t)$ being a time-dependent potential, has been studied extensively in the literature. However, these results cannot be applied directly to treat the cases with non-constant boundary conditions, e.g., singular potentials, and one needs to rely on more abstract settings or take a case-by-case approach. The situation where $V(t)$ is a time-dependent singular potential has also been considered [10–14], for example point (also called Dirac) interactions with a time-dependent position or strength [15–17], possibly satisfying specific scaling properties [18,19]. In these latter situations, the domains of the unbounded Hamiltonians $H(t)$ depend explicitly on the parameter t . Alternatively, one may consider the Laplace operator on a bounded region with time-dependent boundary conditions. In such a case, the time dependence of the operator is entirely encoded in its domain. Results in this direction have been found for a quantum particle in a one-dimensional cavity with moving walls, which can be transformed into a fixed-boundary problem [20–23]. More recently, time-dependent boundary conditions in a graph-like manifold, potentially relevant in quantum control theory, have been investigated [24,25].

From a mathematical point of view, the non-autonomous Schrödinger equation is a first-order linear evolution problem in a Hilbert space with an unbounded generator that depends on time, including the case in which its domain depends on time. The first one addressing the problem in this abstract setting was T. Kato [26,27]. In this first approach, he considered the operator domain $\text{dom } H(t)$ to be constant and provided sufficient conditions for the existence of solutions. Furthermore, J.L. Lions [28] addressed the existence of solutions for problems of this type for the case in which the operator defining the dynamics, i.e., the Hamiltonian, is an elliptic differential operator with smooth coefficients. K. Yosida [29,30] studied the case in which $H(t)$ is positive and introduced a family of approximations $H_n(t) = H(t)(n^{-1} - H(t))^{-1}$, currently commonly known as Yosida's approximations. Even if his intention was to address the problem on which $\text{dom } H(t)$ was not constant, T. Kato noticed that Yosida's original assumptions implied that $\text{dom } H(t)$ had to be constant; see the remark on page 429 in [30].

It was J. Kisyński [31] to be the first one to improve Yosida's techniques in a way allowing for time-varying domains. Independently, B. Simon [32] took an approach very similar to that of J. Kisyński, albeit only considering the case in which the form domain of the operator is constant; see Definition 5. In the case of strictly positive operators, the form domain is the domain of the operator $H(t)^{1/2}$ obtained by the functional calculus of self-adjoint operators. There is also another work by T. Kato [33] addressing the problem for time-dependent domains, and further refinements of this approach have been developed over the years. A more recent work on the abstract non-autonomous Schrödinger problem is [34]. The stability properties of time-dependent Hamiltonians have also been investigated via non-standard analysis techniques [35] and, more recently, in [25], where sharper bounds to the norms of the solutions obtained by B. Simon were found. Finally, let us point out that many practical models use the nonlinear Schrödinger equation, i.e., they involve nonlinear terms, and a certain energy law is preserved. For instance, there are some existing works on the existence and uniqueness of solutions for certain nonlinear hyperbolic PDEs (with no dissipation) [36], as well as the numerical approximation to these PDEs, such as [37–40].

The case of study of the research presented in this article is when $H(t)$ is a family of self-adjoint operators, uniformly bounded from below and with a constant form domain. No other assumptions on the particular form of the operator were made. The problem of the existence and uniqueness of the solutions was studied by B. Simon and J. Kisyński

with similar approaches. B. Simon pointed out the relation between both approaches in his book (see footnote 21 in [32], p. 56, and his remark after Theorem II.27). Referring to his approach to the existence theorem (Theorem 4), he wrote ([32], pp. 58–59):

“[T]his theorem (in a slightly different form) is contained in the work of Kisyński [31]. [...] Yosida’s techniques are also behind Kisyński’s approach. [...] I discovered Kisyński’s paper only after completing this proof myself and have not checked carefully the differences, if any, in the details of the two proofs.”

The connection between both approaches is not immediate, and even the relation between the necessary conditions taken is not completely obvious. To the best of our knowledge, the details of this connection are yet to be thoroughly understood.

The aim of this work is to revise, in a common language, both approaches to the well-posedness problem for the non-autonomous Schrödinger equation with a constant form domain and (possibly) non-constant operator domain and to clarify the connections between them. The connection between J. Kisyński’s and B. Simon’s is subtle, and the relationship between their approaches and the necessary conditions taken is not straightforward.

In showing the explicit relation between both approaches, we revise the state-of-the-art of the problem. Since we did not assume any specific expression for the Hamiltonian $H(t)$, our work may also provide a useful reference for mathematical and theoretical physicists interested in such problems. This may inspire, for example, new developments in the study of quantum Hamiltonians with time-dependent boundary conditions. We applied our results to a simple model with a point-like interaction whose strength depends on time. The example shows explicitly the conditions imposed by B. Simon and J. Kisyński and highlights the analogies and subtle differences between the approaches.

The paper is organised as follows. In Section 2, we study the regularity properties of form-valued and operator-valued functions (Section 2.3). To do this, we recall first some basic notions about operators and forms on a Hilbert space (Section 2.1) and we present the canonical construction of the scale of Hilbert spaces associated with a semibounded self-adjoint operator (Section 2.2). In Section 3, we introduce the non-autonomous Schrödinger equation, both in its strong and weak formulation, for a Hamiltonian with a constant form domain (Section 3.1). Then, we revise B. Simon’s (Section 3.2) and J. Kisyński’s (Section 3.3) approaches to the well-posedness problem for the Schrödinger equation. Finally, in Section 3.4, we analyse the relations between both approaches using the results obtained in Section 2.

2. Regularity of Operator-Valued and Form-Valued Functions

2.1. Operators and Forms on Hilbert Spaces

Let \mathcal{H} be a complex, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In this work, we only considered operators $T : \text{dom } T \subset \mathcal{H} \rightarrow \mathcal{H}$ densely defined on \mathcal{H} . Self-adjoint operators semibounded from below are closely related to closed sesquilinear forms; see for instance [41], Chapter VI, and [42], Section VIII.6. Throughout this work, we observe the convention establishing that a sesquilinear form is anti-linear on its first argument and linear on the second one. We only considered Hermitian sesquilinear forms $h : \text{dom } h \times \text{dom } h \rightarrow \mathbb{C}$ defined on dense subspaces of the Hilbert space \mathcal{H} : i.e., $\text{dom } h$ is a dense subspace of \mathcal{H} .

For convenience, we state the following consequence of the Uniform Boundedness Principle.

Theorem 1. *Let \mathcal{H} be a Hilbert space, and let \mathcal{F} be a family of bounded sesquilinear forms on \mathcal{H} . If for every $\Psi, \Phi \in \mathcal{H}$, the set $\{|h(\Psi, \Phi)| : h \in \mathcal{F}\}$ is bounded, then the set:*

$$\{|h(\Psi, \Phi)| : \Psi, \Phi \in \mathcal{H}, \|\Psi\| \leq 1, \|\Phi\| \leq 1, h \in \mathcal{F}\}$$

is bounded.

Proof. For a fixed $\Psi \in \mathcal{H}$, consider the family of bounded operators from \mathcal{H} to \mathbb{C} :

$$\mathcal{T}_\Psi = \{\Phi \mapsto h(\Psi, \Phi) : \Phi \in \mathcal{F}\}.$$

By assumption, for every $\Psi, \Phi \in \mathcal{H}$, there is a constant $K_{\Psi, \Phi} > 0$ such that $|h(\Psi, \Phi)| \leq K_{\Psi, \Phi}$. Therefore, by the Uniform Boundedness Principle, the set:

$$\{|h(\Psi, \Phi)| : \Phi \in \mathcal{H}, \|\Phi\| \leq 1, h \in \mathcal{F}\}$$

is bounded. Applying again the Uniform Boundedness Principle to the family of bounded operators $\{\Psi \mapsto h(\Psi, \Phi) : \Phi \in \mathcal{H}, \|\Phi\| \leq 1, h \in \mathcal{F}\}$, the result follows. \square

An important concept to relate sesquilinear forms with operators is the notion of closed and semibounded quadratic form. Recall that a Hermitian sesquilinear form is said to be *semibounded* if there exists $m > 0$ such that $h(\Phi, \Phi) > -m\|\Phi\|^2$ for all $\Phi \in \text{dom } h$; in such a case, m is its semibound.

Definition 1. Let h be a semibounded Hermitian sesquilinear form with dense domain $\text{dom } h$, and let m be the semibound of h . We define the graph norm of the sesquilinear form h by:

$$\|\Phi\|_h := \sqrt{(1 + m)\|\Phi\|^2 + h(\Phi, \Phi)}, \quad \Phi \in \text{dom } h.$$

We say that h is closed if $\text{dom } h$ is closed with respect to the graph norm $\|\cdot\|_h$.

We recall next an important result; cf. [41], Section VI.2.

Theorem 2 (Representation theorem). Let h be a Hermitian, closed, semibounded sesquilinear form with dense domain $\text{dom } h \subset \mathcal{H}$. Then, there exists a unique, self-adjoint, semibounded operator T with domain \mathcal{D} and the same lower bound, such that:

(i) $\Phi \in \mathcal{D}$ if and only if $\Phi \in \text{dom } h$ and there exists $\chi \in \mathcal{H}$ such that:

$$h(\Psi, \Phi) = \langle \Psi, \chi \rangle, \quad \forall \Psi \in \text{dom } h;$$

(ii) $h(\Psi, \Phi) = \langle \Psi, T\Phi \rangle$ for any $\Psi \in \text{dom } h, \Phi \in \mathcal{D}$;

(iii) \mathcal{D} is a core for h , that is $\overline{\mathcal{D}}^{\|\cdot\|_h} = \text{dom } h$.

Note that this theorem establishes a one-to-one correspondence between closed, semibounded Hermitian sesquilinear forms and semibounded self-adjoint operators and motivates the following definition.

Definition 2. Let h be a closed, semibounded, Hermitian sesquilinear form. The operator T given in Theorem 2 is said to be the operator representing h . Conversely, h is called the sesquilinear form represented by T .

2.2. Scales of Hilbert Spaces

The notion of scales of Hilbert spaces (also known as *Gelfand triples*) plays a central role in this article. We review the basic ideas of the construction in this subsection. Details and proofs of the following statements can be found for instance in [43], Ch. I.

Let $\mathcal{H}^+ \subset \mathcal{H}$ be a dense subspace of the Hilbert space \mathcal{H} , and let $\langle \cdot, \cdot \rangle_+$ be an inner product endowing \mathcal{H}^+ with the structure of a Hilbert space and such that the associated norm, $\|\cdot\|_+$, satisfies:

$$\|\Phi\| \leq \|\Phi\|_+, \quad \Phi \in \mathcal{H}^+.$$

By the Riesz representation theorem, the restriction of the inner product of \mathcal{H} on \mathcal{H}^+ can be represented using the inner product in \mathcal{H}^+ , i.e., there exists an operator $\hat{J} : \mathcal{H} \rightarrow \mathcal{H}^+$ such that:

$$\langle \Psi, \Phi \rangle = \langle \hat{J}\Psi, \Phi \rangle_+, \quad \Psi \in \mathcal{H}, \Phi \in \mathcal{H}^+.$$

This operator is injective and allows defining another inner product on \mathcal{H} ,

$$\langle \cdot, \cdot \rangle_- := \langle \hat{J}\cdot, \hat{J}\cdot \rangle_+.$$

Let \mathcal{H}^- be the completion of \mathcal{H} with respect to the norm $\|\cdot\|_-$ associated with $\langle \cdot, \cdot \rangle_-$. The operator \hat{J} can be extended by continuity to an isometric bijection $J : \mathcal{H}^- \rightarrow \mathcal{H}^+$. The spaces $\mathcal{H}, \mathcal{H}^\pm$ form the scale of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$.

Finally, since:

$$|\langle \Psi, \Phi \rangle| = |\langle J\Psi, \Phi \rangle_+| \leq \|J\Psi\|_+ \|\Phi\|_+ = \|\Psi\|_- \|\Phi\|_+, \quad \Psi \in \mathcal{H}, \Phi \in \mathcal{H}^+,$$

the inner product on \mathcal{H} can be continuously extended to a pairing:

$$(\cdot, \cdot) : \mathcal{H}^- \times \mathcal{H}^+ \cup \mathcal{H}^+ \times \mathcal{H}^- \rightarrow \mathbb{C}.$$

Note also that, by definition,

$$\langle \Psi, \Phi \rangle_\pm = (\Psi, \hat{J}^{\mp 1}\Phi), \quad \Psi, \Phi \in \mathcal{H}^\pm.$$

Let us introduce now the scale of Hilbert spaces associated with a sesquilinear form. Let $h : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a Hermitian, strictly positive sesquilinear form such that \mathcal{H}^+ is complete with respect to the norm induced by the inner product $\langle \cdot, \cdot \rangle_+ := h(\cdot, \cdot)$ and satisfying:

$$\|\Phi\|^2 \leq h(\Phi, \Phi), \quad \Phi \in \mathcal{H}^+.$$

The construction above can therefore be applied to define an associated scale of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$.

Let H be the positive, self-adjoint operator representing h , that is,

$$\langle \Psi, \Phi \rangle_+ = h(\Psi, \Phi) = \langle \Psi, H\Phi \rangle,$$

for all $\Psi \in \mathcal{H}^+$ and $\Phi \in \text{dom } H$. Note that, if H is strictly positive, $H^{-1} \in \mathcal{B}(\mathcal{H})$ is well defined and $\hat{J} = H^{-1}$. Therefore, the operators H, H^{-1} can be extended to $\hat{H} = J^{-1} \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ and $\hat{H}^{-1} = J \in \mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$. To simplify the notation, we denote these extensions and the original operators by the same symbols, H and H^{-1} . Note also that, since H is self-adjoint and positive, then $H^{\pm 1/2}$ are well-defined and the following identities hold:

$$H^{\pm 1/2}\mathcal{H}^\pm = \mathcal{H}, \quad H^{\pm 1/2}\mathcal{H} = \mathcal{H}^\mp$$

and

$$\langle \Psi, \Phi \rangle_\pm = \langle H^{\pm 1/2}\Psi, H^{\pm 1/2}\Phi \rangle,$$

for Φ, Ψ in the appropriate spaces.

This discussion leads to the following definition.

Definition 3. Let \mathcal{H} be a Hilbert space with associated inner product $\langle \cdot, \cdot \rangle$, $\mathcal{H}^+ \subset \mathcal{H}$ a dense subspace, and let $h : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a Hermitian, strictly positive, closed sesquilinear form. Denote by H the strictly positive self-adjoint operator representing h (cf. Definition 2), and define the inner products:

$$\langle \Psi, \Phi \rangle_\pm = (\Psi, H^{\pm 1}\Phi) \quad \Psi, \Phi \in \mathcal{H}^\pm,$$

where \mathcal{H}^- is the closure of \mathcal{H} with respect to the norm $\|\cdot\|_-$ induced by $\langle \cdot, \cdot \rangle_-$. The scale of Hilbert spaces associated with h is the scale:

$$(\mathcal{H}^+, \langle \cdot, \cdot \rangle_+) \subset (\mathcal{H}, \langle \cdot, \cdot \rangle) \subset (\mathcal{H}^-, \langle \cdot, \cdot \rangle_-).$$

The scale of Hilbert spaces associated with a semibounded sesquilinear form \tilde{h} with semibound m is the scale associated with the strictly positive sesquilinear form $h(\Psi, \Phi) = \tilde{h}(\Psi, \Phi) + m + 1$.

The scale of Hilbert spaces associated with a semibounded self-adjoint operator will be the scale of Hilbert spaces associated with the unique quadratic form representing it.

Let us end this subsection with a useful result on families of scales of Hilbert spaces sharing a common domain \mathcal{H}^+ .

Theorem 3. Let \mathcal{H}^+ be a dense subset of a Hilbert space \mathcal{H} , and let $\langle \cdot, \cdot \rangle_{+,1}$ and $\langle \cdot, \cdot \rangle_{+,2}$ be inner products that give rise to the scales of Hilbert spaces:

$$(\mathcal{H}^+, \langle \cdot, \cdot \rangle_{+,i}) \subset \mathcal{H} \subset (\mathcal{H}^-, \langle \cdot, \cdot \rangle_{-,i}), \quad i = 1, 2.$$

Denote by $\|\cdot\|_{\pm,i}$ the norm on \mathcal{H}^{\pm} , and let $c > 0$. The following statements are equivalent:

- (i) For all $\Phi \in \mathcal{H}^+$, $c^{-1}\|\Phi\|_{+,1} \leq \|\Phi\|_{+,2} \leq c\|\Phi\|_{+,1}$;
- (ii) For all $\Phi \in \mathcal{H}^-$, $c^{-1}\|\Phi\|_{-,1} \leq \|\Phi\|_{-,2} \leq c\|\Phi\|_{-,1}$.

Proof. Let A_i be the strictly positive self-adjoint operator with domain \mathcal{H}^+ such that:

$$\langle \Psi, \Phi \rangle_{+,i} = \langle A_i \Psi, A_i \Phi \rangle, \quad i = 1, 2.$$

By the closed graph theorem, the operator defined by $T := A_1 A_2^{-1}$ is a bounded operator on \mathcal{H} ; therefore, given $\Phi \in \mathcal{H}^+$,

$$\|\Phi\|_{+,1} = \|A_1 A_2^{-1} A_2 \Phi\| \leq \|T\| \|A_2 \Phi\| = \|T\| \|\Phi\|_{+,2}.$$

Analogously, one can obtain $\|\Psi\|_{+,2} \leq \|T^{-1}\| \|\Psi\|_{+,1}$ for $\Psi \in \mathcal{H}^+$.

Using the adjoint of T , $T^\dagger = A_2^{-1} A_1$, one can prove similar inequalities for the norms $\|\cdot\|_{-,i}$, $i = 1, 2$:

$$\|\Phi\|_{-,1} \leq \|T^{-\dagger}\| \|\Phi\|_{-,2}, \quad \|\Phi\|_{-,2} \leq \|T^\dagger\| \|\Phi\|_{-,1}. \tag{1}$$

If (i) holds, one has:

$$\|T\Phi\| = \|A_2^{-1} \Phi\|_{+,1} \leq c \|A_2^{-1} \Phi\|_{+,2} = c \|\Phi\|.$$

Similarly, it follows that $\|T^{-1}\| \leq c$, and therefore, $\|T^{\pm 1}\| = \|T^{\pm 1}\| \leq c$. By Equation (1), (ii) follows.

The other implication, i.e., (ii) implies (i), is proven analogously. \square

2.3. Regularity of Operator-Valued and Form-Valued Functions

In the statement of the following results, \mathcal{H}^+ is a Hilbert space with norm $\|\cdot\|_+$ and inner product $\langle \cdot, \cdot \rangle_+$, not necessarily part of a scale of Hilbert spaces. We use this notation to ease the reading, as later we use these results in the case that \mathcal{H}^+ is part of a scale. Let $I \subset \mathbb{R}$ be a real interval, and let $\mathcal{V} = \{v_t\}_{t \in I}$ be a family of bounded sesquilinear forms on \mathcal{H}^+ . The aim of this subsection is to investigate the relationship between the regularity of the functions $t \in I \mapsto v_t(\Psi, \Phi) \in \mathbb{C}$ for $\Psi, \Phi \in \mathcal{H}^+$ fixed and the regularity of the form-valued functions $t \in I \mapsto v_t \in \mathcal{V}$. Let us first introduce some technical lemmas.

Lemma 1. Let \mathcal{F} be an equicontinuous family of functions from a Hilbert space \mathcal{H}^+ to \mathbb{C} . Then, the function $f : \Phi \in \mathcal{H}^+ \mapsto \sup_{F \in \mathcal{F}} |F(\Phi)| \in \mathbb{C}$ is continuous.

Proof. Let $\Phi, \Phi_0 \in \mathcal{H}^+$. It holds that:

$$\begin{aligned} \sup_{F \in \mathcal{F}} |F(\Phi)| &= \sup_{F \in \mathcal{F}} [|F(\Phi)| - |F(\Phi_0)| + |F(\Phi_0)|] \\ &\leq \sup_{F \in \mathcal{F}} [|F(\Phi)| - |F(\Phi_0)|] + \sup_{F \in \mathcal{F}} |F(\Phi_0)|, \end{aligned}$$

Combining this inequality with the one obtained interchanging the roles of Φ and Φ_0 , it follows that:

$$\left| \sup_{F \in \mathcal{F}} |F(\Phi)| - \sup_{F \in \mathcal{F}} |F(\Phi_0)| \right| \leq \sup_{F \in \mathcal{F}} \left| |F(\Phi)| - |F(\Phi_0)| \right|.$$

Therefore, one has:

$$|f(\Phi) - f(\Phi_0)| = \left| \sup_{F \in \mathcal{F}} |F(\Phi)| - \sup_{F \in \mathcal{F}} |F(\Phi_0)| \right| \leq \sup_{F \in \mathcal{F}} \left| |F(\Phi)| - |F(\Phi_0)| \right|.$$

From this inequality and the equicontinuity of \mathcal{F} , the result follows. \square

Lemma 2. For every $t \in I \subset \mathbb{R}$, let $v_t : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a bounded sesquilinear form. Let $t_0 \in I$, and suppose that $\lim_{t \rightarrow t_0} v_t(\Psi, \Phi)$ exists for every $\Psi, \Phi \in \mathcal{H}^+$. Then, there is a neighbourhood B_{t_0} of t_0 and a constant K such that:

$$|v_t(\Psi, \Phi)| \leq K \|\Psi\|_+ \|\Phi\|_+, \quad \forall \Psi, \Phi \in \mathcal{H}^+, \quad \forall t \in B_{t_0}.$$

Moreover, $L(\Psi, \Phi) = \lim_{t \rightarrow t_0} v_t(\Psi, \Phi)$ defines a bounded sesquilinear form on \mathcal{H}^+ with:

$$|L(\Psi, \Phi)| \leq K \|\Psi\|_+ \|\Phi\|_+.$$

Proof. For every $\Psi, \Phi \in \mathcal{H}^+$, the existence of $\lim_{t \rightarrow t_0} v_t(\Psi, \Phi)$ implies that there is a neighbourhood B_{t_0} of t_0 and a constant $K_{\Psi, \Phi} > 0$ such that, for $t \in B_{t_0}$, we have $|v_t(\Psi, \Phi)| \leq K_{\Psi, \Phi}$. Therefore, by Theorem 1, there is $K > 0$ such that, for every $t \in B_{t_0}$ and every $\Psi, \Phi \in \mathcal{H}^+$,

$$|v_t(\Psi, \Phi)| \leq K \|\Psi\|_+ \|\Phi\|_+.$$

It is straightforward to check that $L(\Psi, \Phi)$ is a sesquilinear form, and since the previous bound holds independently of $t \in B_{t_0}$, it follows that $|L(\Psi, \Phi)| \leq K \|\Psi\|_+ \|\Phi\|_+$. \square

Lemma 3. For each $t \in I \subset \mathbb{R}$, let $v_t : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a bounded sesquilinear form such that $\lim_{t \rightarrow t_0} v_t(\Psi, \Phi)$ exists for every $\Psi, \Phi \in \mathcal{H}^+$, and denote by L the bounded sesquilinear form defined by $L(\Psi, \Phi) = \lim_{t \rightarrow t_0} v_t(\Psi, \Phi)$. Then:

$$\lim_{t \rightarrow t_0} \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|v_t(\Psi, \Phi) - L(\Psi, \Phi)|}{\|\Phi\|_+ \|\Psi\|_+} = 0.$$

Proof. Fix $\Psi_0, \Phi_0 \in \mathcal{H}^+$ such that $\|\Psi_0\|_+ \leq 1$ and $\|\Phi_0\|_+ \leq 1$, and denote by B_{t_0} the neighbourhood of Lemma 2. By the definition of the limit, for any $\varepsilon > 0$, there exists δ such that, defining $B_{t_0}(\delta) := \{t \in I : |t - t_0| < \delta\} \subset B_{t_0}$, for $t \in B_{t_0}(\delta)$:

$$|v_t(\Psi_0, \Phi_0) - L(\Psi_0, \Phi_0)| < \frac{\varepsilon}{2}.$$

Therefore, for $t \in B_{t_0}(\delta)$,

$$|v_t(\Psi_0, \Phi) - L(\Psi_0, \Phi)| \leq \sup_{t \in B_{t_0}(\delta)} |v_t(\Psi_0, \Phi - \Phi_0) - L(\Psi_0, \Phi - \Phi_0)| + \frac{\varepsilon}{2}. \tag{2}$$

Consider the function $f_{\Phi_0} : \mathcal{H}^+ \rightarrow [0, \infty)$ defined by:

$$f_{\Phi_0}(\Phi) := \sup_{t \in B_{t_0}(\delta)} |v_t(\Psi_0, \Phi - \Phi_0) - L(\Psi_0, \Phi - \Phi_0)|.$$

By Lemma 2, the family of functions $\{\Phi \mapsto v_t(\Psi_0, \Phi - \Phi_0) - L(\Psi_0, \Phi - \Phi_0) : t \in B_{t_0}(\delta)\}$ is equicontinuous, and therefore, Lemma 1 implies that f_{Φ_0} is a continuous map. This implies that f_{Φ_0} is also weakly continuous, and therefore, $U_{\Phi_0} := f_{\Phi_0}^{-1}(\{x \in \mathbb{R} : |x| < \varepsilon/2\})$ is an open neighbourhood of Φ_0 in the weak topology of \mathcal{H}^+ . By Equation (2) and the definition of U_{Φ_0} , for every $\Phi \in U_{\Phi_0}$ and every $t \in B_{t_0}(\delta)$,

$$|v_t(\Psi_0, \Phi) - L(\Psi_0, \Phi)| \leq \varepsilon.$$

That is, for every $\Psi_0, \Phi_0 \in \mathcal{H}^+$, $\|\Phi_0\|_+ \leq 1$, and $\varepsilon > 0$, there is δ and a neighbourhood of Φ_0 in the weak topology, U_{Φ_0} , such that, for every $\Phi \in U_{\Phi_0}$ and $|t - t_0| < \delta$,

$$|v_t(\Psi_0, \Phi) - L(\Psi_0, \Phi)| < \varepsilon.$$

The family $\{U_\Phi : \|\Phi\|_+ \leq 1, \Phi \in \mathcal{H}^+\}$ is an open covering of the closed unit ball on \mathcal{H}^+ , and by the weak compactness of the unit ball, there is a finite subcovering $\{U_n\}_{n=1}^N$. It follows that, for every U_n , $1 \leq n \leq N$, there is δ_n such that, for $\Phi \in U_n$ and $|t - t_0| < \delta_n$:

$$|v_t(\Psi_0, \Phi) - L(\Psi_0, \Phi)| \leq \varepsilon.$$

Therefore, for $|t - t_0| < \min_n \delta_n$ and every $\Phi \in \mathcal{H}^+$ with $\|\Phi\|_+ \leq 1$, one has:

$$|v_t(\Psi_0, \Phi) - L(\Psi_0, \Phi)| \leq \varepsilon.$$

Repeating the argument for Ψ yields that the limit is uniform on Ψ, Φ in the closed unit ball, from which the result follows. \square

We can now prove the main results of this subsection.

Proposition 1. For $t \in I \subset \mathbb{R}$, let $v_t : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a family of bounded sesquilinear forms. Then:

(i) If, for every $\Psi, \Phi \in \mathcal{H}^+$, the map $t \mapsto v_t(\Psi, \Phi)$ is continuous, then:

$$\lim_{t \rightarrow t_0} \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|v_t(\Psi, \Phi) - v_{t_0}(\Psi, \Phi)|}{\|\Phi\|_+ \|\Psi\|_+} = 0.$$

If, in addition, I is compact, there exists $M > 0$ such that $|v_t(\Psi, \Phi)| \leq M \|\Psi\|_+ \|\Phi\|_+$ for every $t \in I$ and every $\Psi, \Phi \in \mathcal{H}^+$;

(ii) If, for every $\Psi, \Phi \in \mathcal{H}^+$, the map $t \mapsto v_t(\Psi, \Phi)$ is differentiable, then:

$$\lim_{t \rightarrow t_0} \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{1}{\|\Phi\|_+ \|\Psi\|_+} \left| \frac{v_t(\Psi, \Phi) - v_{t_0}(\Psi, \Phi)}{t - t_0} - \dot{v}_{t_0}(\Psi, \Phi) \right| = 0,$$

where $\dot{v}_t(\Psi, \Phi)$ denotes the derivative of $t \mapsto v_t(\Psi, \Phi)$. If, in addition, I is compact and $t \mapsto \dot{v}_t(\Psi, \Phi)$ is continuous for every Ψ, Φ , there is $M > 0$ such that $|\dot{v}_t(\Psi, \Phi)| < M \|\Psi\|_+ \|\Phi\|_+$ for every $t \in I$.

Proof. By Lemma 3, the continuity of $t \mapsto v_t(\Psi, \Phi)$ for each $\Psi, \Phi \in \mathcal{H}^+$ implies:

$$\lim_{t \rightarrow t_0} \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|v_t(\Psi, \Phi) - v_{t_0}(\Psi, \Phi)|}{\|\Phi\|_+ \|\Psi\|_+} = 0,$$

and its differentiability implies:

$$\lim_{t \rightarrow t_0} \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{1}{\|\Phi\|_+ \|\Psi\|_+} \left| \frac{v_t(\Psi, \Phi) - v_{t_0}(\Psi, \Phi)}{t - t_0} - \dot{v}_{t_0}(\Psi, \Phi) \right| = 0.$$

The uniform bound in (i) follows from Lemma 2 and the compactity of I , while the uniform bound in (ii) follows applying Lemma 2 and the continuity of $t \mapsto \dot{v}_t(\Psi, \Phi)$. \square

Given a scale of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$, by the Riesz representation theorem, any bounded sesquilinear form $v : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ can be associated with a unique operator $V \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ defined by $v(\Psi, \Phi) = \langle \Psi, V\Phi \rangle$. Let us now transfer the previous results for form-valued functions to the operator-valued functions $t \in I \mapsto V_t \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$. For convenience, we denote by $\|\cdot\|_{\pm, \mp}$ the operator norm in $\mathcal{B}(\mathcal{H}^\pm, \mathcal{H}^\mp)$.

Proposition 2. *Let $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ be a scale of Hilbert spaces and $I \subset \mathbb{R}$ compact. For $t \in I$, let $v_t : \mathcal{H}^+ \times \mathcal{H}^+ \rightarrow \mathbb{C}$ be a family of Hermitian, bounded, sesquilinear forms and $V(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ be defined as the unique operators such that $\langle \Psi, V(t)\Phi \rangle = v_t(\Psi, \Phi)$ for all $\Psi, \Phi \in \mathcal{H}^+$. Then:*

(i) *For every $t \in I$, the operator norm of $V(t) \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ satisfies:*

$$\|V(t)\|_{+,-} = \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|v_t(\Psi, \Phi)|}{\|\Psi\|_+ \|\Phi\|_+}.$$

If, in addition, the map $t \mapsto v_t(\Psi, \Phi)$ is continuously differentiable for every $\Psi, \Phi \in \mathcal{H}^+$, then:

- (ii) *The map $t \mapsto V(t)$ is continuous with respect to the $\|\cdot\|_{+,-}$ norm, and the family $\{V(t)\}_{t \in I}$ is uniformly bounded;*
- (iii) *The derivative $\frac{d}{dt} V(t)$ exists in the $\|\cdot\|_{+,-}$ -norm sense, and it is uniformly bounded, that is there is a constant K such that for every $t \in I$,*

$$\left\| \frac{d}{dt} V(t) \right\|_{+,-} \leq K.$$

Proof. By the definition of the operator norm, one has that:

$$\|V(t)\|_{+,-} = \sup_{\Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{\|V(t)\Phi\|_-}{\|\Phi\|_+} = \sup_{\substack{\Phi \in \mathcal{H}^+ \setminus \{0\} \\ \Psi \in \mathcal{H}^- \setminus \{0\}}} \frac{|\langle \Psi, V(t)\Phi \rangle_-|}{\|\Psi\|_- \|\Phi\|_+},$$

where we used the equality $\|\zeta\|_- = \sup_{\Psi \neq 0} \frac{|\langle \Psi, \zeta \rangle_-|}{\|\Psi\|_-}$, which holds for any $\zeta \in \mathcal{H}^-$.

Using the isomorphism $J : \mathcal{H}^- \rightarrow \mathcal{H}^+$ and the pairing (\cdot, \cdot) associated with the scale of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ (cf. Section 2.2), the previous equation can be written as:

$$\|V(t)\|_{+,-} = \sup_{\substack{\Phi \in \mathcal{H}^+ \setminus \{0\} \\ \Psi \in \mathcal{H}^- \setminus \{0\}}} \frac{|(J\Psi, V(t)\Phi)|}{\|\Psi\|_- \|\Phi\|_+} = \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|\langle \Psi, V(t)\Phi \rangle|}{\|\Psi\|_+ \|\Phi\|_+},$$

where we used the fact that J is an isometric bijection from \mathcal{H}^- to \mathcal{H}^+ . By the definition of $V(t)$, we therefore have:

$$\|V(t)\|_{+,-} = \sup_{\Psi, \Phi \in \mathcal{H}^+ \setminus \{0\}} \frac{|v_t(\Psi, \Phi)|}{\|\Psi\|_+ \|\Phi\|_+},$$

which proves (i). Property (ii) follows immediately from Proposition 1 and (i).

To prove (iii), note that, by Lemma 2, $\dot{v}_t(\Psi, \Phi) := \frac{d}{dt} v_t(\Psi, \Phi)$ defines a bounded sesquilinear form in \mathcal{H}^+ . Let $\dot{V}(t)$ be the unique operator in $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ such that

$(\Psi, \dot{V}(t)\Phi) = \dot{v}_t(\Psi, \Phi)$ for every $\Psi, \Phi \in \mathcal{H}^+$. We claim that $\dot{V}(t) = \frac{d}{dt}V(t)$ in the norm sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$, that is,

$$\lim_{t \rightarrow t_0} \left\| \frac{V(t) - V(t_0)}{t - t_0} - \dot{V}(t_0) \right\|_{+,-} = 0.$$

By (i), this is equivalent to:

$$\lim_{t \rightarrow t_0} \sup_{\substack{\Psi, \Phi \in \mathcal{H}^+ \\ \|\Psi\|_+ = 1 = \|\Phi\|_+}} \left| \frac{v_t(\Psi, \Phi) - v_{t_0}(\Psi, \Phi)}{t - t_0} - \dot{v}_{t_0}(\Psi, \Phi) \right| = 0,$$

which holds by Proposition 1(ii), as does the uniform bound. \square

3. Existence of Dynamics

3.1. Quantum Dynamics and Sesquilinear Forms

We begin this section with some definitions.

Definition 4. Let $I \subset \mathbb{R}$ be a compact interval, and let $H(t), t \in I$, be a time-dependent Hamiltonian. The (strong) Schrödinger equation is the linear evolution equation:

$$\frac{d}{dt}\Psi(t) = -iH(t)\Psi(t).$$

Despite this equation being a linear ordinary differential equation, the existence and uniqueness of solutions is not guaranteed in the most general case of a time-dependent Hamiltonian with a time-varying domain. Notice that it is implicitly imposing to the solutions that $\Psi(t) \in \text{dom } H(t)$ for all values of t . This condition makes it hard to construct explicit sequences that approximate them.

As stated in the Introduction, we consider the case of a time-dependent Hamiltonian with a constant form domain.

Definition 5. Let $I \subset \mathbb{R}$ be a compact interval and \mathcal{H}^+ a dense subspace of \mathcal{H} , and let $H(t), t \in I$, be a family of self-adjoint operators on \mathcal{H} such that, for any $t \in I$, the operator $H(t)$ is densely defined on $\text{dom } H(t)$. We say that $H(t), t \in I$, is a time-dependent Hamiltonian with constant form domain \mathcal{H}^+ if:

- (i) There is $m > 0$ such that $\langle \Phi, H(t)\Phi \rangle \geq -m\|\Phi\|^2$, for every $\Phi \in \text{dom } H(t)$ and $t \in I$;
- (ii) For any $t \in I$, the domain of the Hermitian sesquilinear form h_t associated with $H(t)$ (cf. Theorem 2) is \mathcal{H}^+ .

For the case in which the closed sesquilinear forms associated with $H(t)$ via Theorem 2 have a constant domain, one can define a weak version of the Schrödinger equation.

Definition 6. Let $I \subset \mathbb{R}$ be a compact interval, and let $H(t), t \in I$, be a time-dependent Hamiltonian with constant form domain \mathcal{H}^+ . For $t \in I$, let h_t be the sesquilinear form uniquely associated with the self-adjoint operator $H(t)$. The weak Schrödinger equation is the linear evolution equation:

$$\frac{d}{dt}\langle \Phi, \Psi(t) \rangle = -ih_t(\Phi, \Psi(t)), \quad \text{for every } \Phi \in \mathcal{H}^+.$$

The strong Schrödinger equation is the usual Schrödinger equation. We use the adjective *strong* when we want to emphasise the difference with the weak Schrödinger equation. It is immediate to prove that solutions of the strong Schrödinger equation are also solutions of the weak Schrödinger equation when the latter is defined.

The solutions of the initial-value problems for the weak and strong Schrödinger equations are given in terms of what are known as unitary propagators.

Definition 7. A unitary propagator is a two-parameter family of unitary operators $U(t, s)$, $s, t \in \mathbb{R}$, satisfying:

- (i) $U(t, s)U(s, r) = U(t, r)$;
- (ii) $U(t, t) = \mathbb{I}$;
- (iii) $U(t, s)$ is jointly strongly continuous in t and s .

Definition 8. We say that the strong (weak) Schrödinger equation is solvable if there exists a unitary propagator $U(t, s)$, $t, s \in \mathbb{R}$, such that, for any $\Psi \in \text{dom } H(s)$ ($\Psi \in \mathcal{H}^+$), the equation with initial value Ψ at time $s \in \mathbb{R}$ admits the curve $\Psi(t) := U(t, s)\Psi$ as its unique solution.

Following the discussion in Section 2.2, given a time-dependent Hamiltonian $H(t)$, $t \in I$, with constant form domain \mathcal{H}^+ , for each fixed time t , we can associate $H(t)$ with a scale of Hilbert spaces:

$$(\mathcal{H}^+, \langle \cdot, \cdot \rangle_{+,t}) \subset (\mathcal{H}, \langle \cdot, \cdot \rangle) \subset (\mathcal{H}_t^-, \langle \cdot, \cdot \rangle_{-,t}),$$

where $\langle \Psi, \Phi \rangle_{\pm,t} := \langle (H(t) + m + 1)^{\pm 1/2} \Psi, (H(t) + m + 1)^{\pm 1/2} \Phi \rangle$, and \mathcal{H}_t^- denotes the closure of \mathcal{H} with respect to the norm defined by $\|\Phi\|_{-,t}^2 := \langle \Phi, \Phi \rangle_{-,t}$. We denote $\mathcal{H}_t^+ = (\mathcal{H}^+, \langle \cdot, \cdot \rangle_{+,t})$, and with a slight abuse of notation, we use the symbol \mathcal{H}_t^- to represent the Hilbert space $(\mathcal{H}_t^-, \langle \cdot, \cdot \rangle_{-,t})$. We denote by $(\cdot, \cdot)_t : \mathcal{H}_t^+ \times \mathcal{H}_t^- \cup \mathcal{H}_t^- \times \mathcal{H}_t^+ \rightarrow \mathbb{C}$ the canonical pairings. The unbounded operators $H(t)$ with a constant form domain can be continuously extended to bounded operators $\tilde{H}(t) : \mathcal{H}_t^+ \rightarrow \mathcal{H}_t^-$. When they are defined, the inverse operators, $H(t)^{-1} \in \mathcal{B}(\mathcal{H})$, can be extended to bounded operators from \mathcal{H}_t^- to \mathcal{H}_t^+ , which coincide with $\tilde{H}(t)^{-1}$. As already done in Section 2.2, in order to simplify the notation, we drop the tilde, denoting the extensions and the original operators with the same symbols. We also denote by $\|\cdot\|_{\pm, \mp, t}$ the norm in $\mathcal{B}(\mathcal{H}_t^\pm, \mathcal{H}_t^\mp)$.

Scales of Hilbert spaces are the key objects needed to prove the main results in both J. Kisyński’s and B. Simon’s seminal works, which we shall discuss in the next subsections.

3.2. B. Simon’s Approach

Let us now introduce the main ideas used by B. Simon in [32]. The following assumptions for proving the existence of dynamics were made.

Assumption 1. Let H_0 be a positive operator on a Hilbert space, and let $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ be its associated scale of Hilbert spaces. For $0 \leq t \leq T$, let $H(t)$ be a family of bounded operators from \mathcal{H}^+ to \mathcal{H}^- such that $(\Psi, H(t)\Phi) = (H(t)\Psi, \Phi)$ for all $\Psi, \Phi \in \mathcal{H}^+$. We assume that there is $C > 0$ independent of t such that:

- (S1) $C^{-1}(H_0 + 1) \leq H(t) \leq C(H_0 + 1)$;
- (S2) $B(t) = \frac{d}{dt}H(t)^{-1}$ exists in the sense of the norm of \mathcal{H} and satisfying:

$$\|H(t)^{1/2}B(t)H(t)^{1/2}\| \leq C.$$

The approach by B. Simon defines the Hamiltonian of the system through a family of operators from \mathcal{H}^+ to \mathcal{H}^- . This simplifies some usual problems for the existence of quantum dynamics. On the one hand, a self-adjoint extension of $H(t)$, as an unbounded operator on \mathcal{H} , is fixed (see [32], Lemma II.6). On the other hand, it simplifies the problem raised by the time dependence of $\text{dom } H(t)$. Having a common domain as operators from \mathcal{H}^+ to \mathcal{H}^- sidesteps some of the technical difficulties that appear in this case.

Under these hypothesis, B. Simon proved the existence of dynamics in [32], Appendix II.7.

Theorem 4 ([32], Thm. II.27). Let $H(t)$ satisfy Assumption 1. Then, for any $\Phi_0 \in \mathcal{H}^+$, there is a unique function $\Phi(t) \in \mathcal{H}^+$ such that:

- (i) $\Phi(t)$ is continuous in the \mathcal{H}^+ weak topology, i.e., for all $\Psi \in \mathcal{H}^-$, $t \mapsto \langle \Psi, \Phi(t) \rangle$ is continuous;

- (ii) For any $\Psi \in \mathcal{H}^+$, $\Phi(t)$ solves the weak Schrödinger equation (cf. Definition 6) with initial condition Φ_0 :

$$\frac{d}{dt} \langle \Psi, \Phi(t) \rangle = -i \langle \Psi, H(t) \Phi(t) \rangle, \quad \Phi(0) = \Phi_0.$$

Moreover:

- (iii) $\lim_{t \rightarrow t_0} \left\| \frac{\Phi(t) - \Phi(t_0)}{t - t_0} + iH(t_0)\Phi(t_0) \right\|_- = 0$;
- (iv) $\|\Phi(t)\| = \|\Phi_0\|$;
- (v) $\Phi(t)$ is $\|\cdot\|$ -continuous.

Thus, the map $U(t, s) : \Phi(s) \in \mathcal{H}^+ \mapsto \Phi(t) \in \mathcal{H}^+$ is unitary on \mathcal{H} and can be extended to a unitary propagator.

The proof of B. Simon is based on K. Yosida’s idea ([30], pp. 425–429), whose original theorem only applies to the case in which $\text{dom } H(t)$ does not depend on t . The proof is based on the introduction of an approximating family of Hamiltonians $H_n(t) = H(t)(1 + n^{-1}H(t))^{-1}$, which can be seen as a bounded operator in $\mathcal{B}(\mathcal{H}^-)$; the existence of unitary propagators $U_n(t, s)$ for $H_n(t)$ is provided by a Dyson expansion. Using these propagators, for any $\Phi_0 \in \mathcal{H}^+$, approximated solutions $\Phi_n(t) = U_n(t, s)\Phi_0$ for the dynamical equation for $H(t)$ are constructed, and it is shown that Assumption 1 ensures that $\{\Phi_n(t)\}_{n \in \mathbb{N}} \subset \mathcal{H}^+$ has a convergent subsequence, whose limit satisfies the properties of the theorem.

3.3. J. Kisyński’s Approach

In [31], J. Kisyński studied the existence of solutions for the equation:

$$\frac{d}{dt} \Phi(t) = A(t)\Phi(t), \quad \Phi(0) = \Phi_0, \tag{3}$$

on a Banach space, with $A(t)$ being an unbounded linear operator defined on a dense domain $\text{dom } A(t)$, which can depend on time. He first proved the existence of solutions for the case in which $\text{dom } A(t)$ is constant by using Yosida’s approximation; then, he provided sufficient conditions for the existence of solutions in the more general case in which $\text{dom } A(t)$ does depend on t . These general results are finally applied to the case in which $A(t) = -i\Lambda(t)$, with $\Lambda(t)$ being a positive, self-adjoint operator on a Hilbert space, such that $\text{dom } \Lambda(t)^{1/2}$ is independent of t ([31], Secs. 7 and 8).

In order to prove the existence of a unitary propagator for this case, J. Kisyński assumed the following.

Assumption 2 ([31], Hyp. 7.1). Let \mathcal{H} be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{H}^+ \subset \mathcal{H}$ be a dense subspace:

- (K1) For every $t \in [0, T]$, $\langle \cdot, \cdot \rangle_{+,t}$ is an inner product on \mathcal{H}^+ endowing it with the structure of a Hilbert space \mathcal{H}_t^+ , which is continuously contained on \mathcal{H} ;
- (K2) For every $t \in [0, T]$ and every $\Psi, \Phi \in \mathcal{H}^+$, the function $t \mapsto \langle \Psi, \Phi \rangle_{+,t}$ is in $C^n(0, T)$ for $n \geq 1$.

Under these hypotheses, for each $t \in [0, T]$, J. Kisyński constructed the scale of Hilbert spaces $\mathcal{H}_t^+ \subset \mathcal{H} \subset \mathcal{H}_t^-$ associated with $\langle \Psi, \Phi \rangle_{+,t}$ (see [31], Lemmas 7.2–7.6). Then, the following operators are defined (cf. [31], Lemmas 7.7–7.10).

Proposition 3. Under Assumption 2, for every $t \in [0, T]$, the following statements hold:

- (i) Define the domain:

$$\mathcal{D}(\Lambda_0) = \{ \Phi \in \mathcal{H}^+ : \sup\{|\langle \Psi, \Phi \rangle_{+,t}| : \Psi \in \mathcal{H}^+, \|\Psi\| = 1\} < \infty \}.$$

Then, the equality:

$$\langle \Psi, \Lambda_0(t)\Phi \rangle = \langle \Psi, \Phi \rangle_{+,t}, \quad \Phi \in \mathcal{D}(\Lambda_0), \quad \Psi \in \mathcal{H}^+,$$

defines a self-adjoint, positive unbounded operator on \mathcal{H} ;

- (ii) Let $\Lambda(t)$ be the closure of $\Lambda_0(t)$ on \mathcal{H}_t^- ; then, $\Lambda(t)$ is a self-adjoint, positive operator on \mathcal{H}_t^- with $\text{dom } \Lambda(t) = \mathcal{H}^+$. Moreover, $\Lambda(t)^{-1}$ is the canonical isomorphism $J : \mathcal{H}_t^- \rightarrow \mathcal{H}_t^+$ (cf. Section 2.2) for the scale of Hilbert spaces $\mathcal{H}_t^+ \subset \mathcal{H} \subset \mathcal{H}_t^-$;
- (iii) Let $\mathcal{D}(\Lambda_1) = \{\Phi \in \mathcal{D}(\Lambda_0) : \Lambda_0\Phi \in \mathcal{H}^+\}$ and $\Lambda_1\Phi = \Lambda_0\Phi$ for $\Phi \in \mathcal{D}(\Lambda_1)$. Then, Λ_1 is a self-adjoint, positive unbounded operator on \mathcal{H}_t^+ .

Finally, in [31], Sec. 8, J. Kisyński showed the following theorem on the existence of unitary propagators. Starting his construction from a family of inner products (which are sesquilinear forms), he constructed the associated scales of Hilbert spaces and then focused on the operators representing these inner products. This approach has the same advantages as B. Simon’s: the self-adjointness of the operators is guaranteed by the hermiticity of the inner products and the representation theorem (see Theorem 2). Additionally, it sets the problem on a constant (form) domain \mathcal{H}^+ .

Theorem 5 ([31], Thm. 8.1). *Under Assumption 2 with $n \geq 1$, there is a unique unitary propagator $U(t, s)$ for the problem (3) such that:*

- (i) *The continuous extension of $U(t, s)$ to $\mathcal{B}(\mathcal{H}^-)$ is strongly continuous for $s, t \in [0, T]$;*
- (ii) *$U(t, s)\mathcal{H}^+ = \mathcal{H}^+$ and the restriction $U(t, s) \in \mathcal{B}(\mathcal{H}^+)$ is strongly continuous for $s, t \in [0, T]$;*
- (iii) *$t, s \mapsto U(t, s)$ is a strongly continuously differentiable function with values on $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ for $t, s \in [0, T]$, and in this sense:*

$$\frac{d}{dt}U(t, s) = -i\Lambda(t)U(t, s) \quad \text{and} \quad \frac{d}{ds}U(t, s) = -iU(t, s)\Lambda(s).$$

Moreover, if Assumption 2 is satisfied with $n \geq 2$, it also holds that:

- (iv) *$U(t, s)\mathcal{D}(\Lambda_1(s)) = \mathcal{D}(\Lambda_1(t))$ for $t, s \in [0, T]$ and for $s \in [0, T]$ and $\Phi \in \mathcal{D}(\Lambda_1(s))$ fixed, the function $t \mapsto U(t, s)\Phi$ is continuously differentiable in $[0, T]$ in the sense of \mathcal{H}^+ , and there, it satisfies:*

$$\frac{d}{dt}U(t, s)\Phi = -i\Lambda_1(t)U(t, s)\Phi;$$

- (v) *$U(t, s)\mathcal{D}(\Lambda_0(s)) = \mathcal{D}(\Lambda_0(t))$ for $t, s \in [0, T]$ and for $s \in [0, T]$ and $\Phi \in \mathcal{D}(\Lambda_0(s))$ fixed, the function $t \mapsto U(t, s)\Phi$ is continuously differentiable in $[0, T]$ in the sense of \mathcal{H} , and there, it satisfies:*

$$\frac{d}{dt}U(t, s)\Phi = -i\Lambda_0(t)U(t, s)\Phi.$$

3.4. Relations between the Approaches of B. Simon and J. Kisyński

We start by examining the similarities between the two approaches. Even though the starting point for each of them is apparently different, the tools they use are analogous. The link allowing us to relate both approaches are the scales of Hilbert spaces. B. Simon started its construction with the scale of Hilbert spaces associated with an operator H_0 , $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$ and then considered the family of operators $H(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$. On the other hand, J. Kisyński started from a family of inner products $\langle \cdot, \cdot \rangle_{+,t}$ and built the scales of Hilbert spaces $\mathcal{H}_t^+ \subset \mathcal{H} \subset \mathcal{H}_t^-$ associated with the sesquilinear forms $\langle \cdot, \cdot \rangle_{+,t}$ and the family of operators $\Lambda(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ representing them. Comparing Theorems 4 and 5, it is clear that the role the family of operators $H(t)$ plays in the approach by B. Simon is the same that the family $\Lambda(t)$ plays in J. Kisyński’s. Therefore, the problem addressed by them is the same if $h_t(\cdot, \cdot) := \langle \cdot, \cdot \rangle_{+,t}$ is the sesquilinear form associated with $H(t)$. For the rest of this section, we assume that this equality holds.

Before further studying the relation between both approaches, let us briefly review the assumptions B. Simon and J. Kisyński made to prove the existence of a unitary propagator. There are two main ingredients playing a central role in their proofs: the regularity of the generators $H(t)$ and the uniform equivalence of the norms $\|\cdot\|_{+,t} := \sqrt{\langle \cdot, \cdot \rangle_{+,t}}$, i.e., $c^{-1}\|\cdot\|_{+,t'} \leq \|\cdot\|_{+,t} \leq c\|\cdot\|_{+,t'}$.

Although both of them use repeatedly the equivalence of the norms, neither B. Simon nor J. Kisyński imposed it as an explicit assumption; still, the equivalence *does* follow from their assumptions. Let us first examine how this uniform equivalence of the norms appears in each of the approaches.

Proposition 4. Let $H_0, H(t)$ be as in Assumption 1; for $\Phi \in \mathcal{H}^+$, define the norms:

$$\|\Phi\|_{+,t} := \sqrt{(\Phi, H(t)\Phi)}, \quad \|\Phi\|_0 := \sqrt{(\Phi, (H_0 + 1)\Phi)}.$$

Then, (S1) holds if and only if the norms $\|\cdot\|_{+,t}$ are equivalent to $\|\cdot\|_0$ uniformly on $t \in [0, T]$.

Proof. It is enough to note that $\|\Phi\|_0^2 = (\Phi, (H_0 + 1)\Phi)$ and $\|\Phi\|_{+,t}^2 = (\Phi, H(t)\Phi)$. Therefore, taking square roots in (S1), the uniform equivalence of the norms follows. Conversely, taking squares on the uniform equivalence of the norms yields (S1). \square

Remark 1. Note that, since we have $\|\cdot\|_0 \sim \|\cdot\|_{+,t}$ with constant uniform on t , for any $t_0 \in [0, T]$, there is K such that:

$$K^{-1}\|\cdot\|_{+,t_0} \leq \|\cdot\|_{+,t} \leq K\|\cdot\|_{+,t_0},$$

for every $t \in [0, T]$.

Proposition 5. Let $\langle \cdot, \cdot \rangle_{+,t}$ be inner products as in Assumption 2. Then, (K1) implies that, for every $t \in [0, T]$, the norms $\|\cdot\|_{+,t}$ are equivalent. If, moreover, $t \mapsto \langle \cdot, \cdot \rangle_{+,t}$ is in $C^1([0, T])$, then the equivalence of the norms is uniform on t .

Proof. By (K1), \mathcal{H}_t^+ is continuously embedded in \mathcal{H} ; that is, there is $K > 0$ such that:

$$\|\Phi\| \leq K\|\Phi\|_{+,t}, \quad \forall \Phi \in \mathcal{H}_t^+.$$

Therefore, $\langle \cdot, \cdot \rangle_{+,t}$, as a Hermitian sesquilinear form densely defined on \mathcal{H} , is strictly positive, and there exists a self-adjoint, strictly positive operator $A(t) : \mathcal{H}^+ \rightarrow \mathcal{H}$ such that:

$$\langle \Psi, \Phi \rangle_{+,t} = \langle A(t)\Psi, A(t)\Phi \rangle, \quad (\Psi, \Phi \in \mathcal{H}^+).$$

Since it is positive and self-adjoint, $A(t)^{-1} : \mathcal{H} \rightarrow \mathcal{H}^+$ is a bounded self-adjoint operator. By the closed graph theorem, $A(t')A(t)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator, and for $\Phi \in \mathcal{H}^+$:

$$\|\Phi\|_{+,t'} = \|A(t')A(t)^{-1}A(t)\Phi\| \leq C_{t,t'}\|A(t)\Phi\| = C_{t,t'}\|\Phi\|_{+,t},$$

where $C_{t,t'} := \|A(t')A(t)^{-1}\|$.

The rest of the proof follows [31], Lemma 7.3, where it is proven using the Uniform Boundedness Principle that the constant can be chosen independently of t and t' . \square

Corollary 1. Let $\langle \cdot, \cdot \rangle_{+,t}$, $t \in [0, T]$, be inner products satisfying Assumption 2, and let $H(t)$ be the operator in $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ such that $\langle \Psi, \Phi \rangle_{+,t} = (\Psi, H(t)\Phi)$ for every $\Psi, \Phi \in \mathcal{H}^+$. Then, $H(t)$ satisfies (S1) of Assumption 1 with $H_0 = H(t_0)$ for some t_0 fixed.

Proof. Let $\|\cdot\|_{+,t} = \sqrt{\langle \cdot, H(t)\cdot \rangle}$. By Proposition 5, for any t, t_0 , we have:

$$K^{-1}\|\Phi\|_{+,t_0} \leq \|\Phi\|_{+,t} \leq K\|\Phi\|_{+,t_0}.$$

For a fixed t_0 , define the norm $\|\Phi\|_0 := \sqrt{\|\Phi\|_{+,t_0}^2 + \|\Phi\|^2}$. Let us show that $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{+,t}$. Since $\|\Phi\|_{+,t_0} \leq \|\Phi\|_0$, it follows that $\|\Phi\|_{+,t} \leq K\|\Phi\|_0$. Furthermore, since $\|\Phi\|_{+,t} \geq \|\Phi\|$,

$$K^{-2}\|\Phi\|_0^2 \leq K^{-2}\|\Phi\|_{+,t_0}^2 + \|\Phi\|^2 \leq 2\|\Phi\|_{+,t}^2.$$

Hence, there is $\tilde{K} > 1$ such that:

$$\tilde{K}^{-1}\|\Phi\|_{+,t} \leq \|\Phi\|_0 \leq \tilde{K}\|\Phi\|_{+,t}.$$

By Proposition 4, the preceding inequalities imply (S1). □

Using the structure of the scales of Hilbert spaces, from the equivalence of the norms $\|\cdot\|_{+,t}$, the equivalence of the norms $\|\cdot\|_{-,t}$ follows as well (see Theorem 3). This motivates us to work with a reference norm $\|\cdot\|_{\pm}$ and use the uniform equivalence when a particular $\|\cdot\|_{\pm,t}$ is convenient. The most convenient choice is to take $\|\cdot\|_{\pm} = \|\cdot\|_{\pm,t_0}$ for some reference t_0 . We denote by $\langle \cdot, \cdot \rangle_+$ the inner product such that $\|\cdot\|_+^2 = \langle \cdot, \cdot \rangle_+$ and by \mathcal{H}^{\pm} the Hilbert space $(\mathcal{H}^{\pm}, \langle \cdot, \cdot \rangle_{\pm})$.

Let us study now the regularity of the generators $H(t)$.

Proposition 6. *Let $H(t)$ be as in Assumption 1. Then, (S2) holds if and only if $t \mapsto H(t)$ is differentiable in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ and, for every $t \in [0, T]$,*

$$\left\| \frac{d}{dt} H(t) \right\|_{+,-} \leq C.$$

Proof. Note that, if (S2) holds,

$$\|B(t)\|_{-,+} = \sup_{\Phi \in \mathcal{H}^-} \frac{\|B(t)\Phi\|_+}{\|\Phi\|_-} \leq K \sup_{\Psi \in \mathcal{H}} \frac{\|H(t)^{1/2}B(t)H(t)^{1/2}\Psi\|}{\|\Psi\|} \leq KC$$

where we used the equivalence of the norms $\|\cdot\|_{\pm,t} \sim \|\cdot\|_{\pm}$ (cf. Proposition 4) and the assumption $\|H(t)^{1/2}B(t)H(t)^{1/2}\| < C$. Therefore, $\|B(t)\|_{-,+} < KC$, so that $B(t)$ can be continuously extended to an operator in $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$.

Define the operator $T_{\delta}(t) = \delta^{-1}[H(t + \delta)^{-1} - H(t)^{-1}] - B(t)$ in $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$. It follows that:

$$\|T_{\delta}(t)\|_{-,+} = \sup_{\Phi \in \mathcal{H}^-} \frac{\|T_{\delta}(t)\Phi\|_+}{\|\Phi\|_-} = \|A_0^{1/2}T_{\delta}(t)A_0^{1/2}\|,$$

where A_0 is the strictly positive self-adjoint operator such that $\|\cdot\|_{\pm} = \|A_0^{\pm 1/2} \cdot\|$. Hence, $H(t)^{-1}$ is differentiable in the sense of $\|\cdot\|_{-,+}$ if $H(t)^{-1}$ is differentiable in the sense of $\|\cdot\|$. By the product rule, this is equivalent to $H(t)$ being differentiable in the sense of $\|\cdot\|_{+,-}$.

Conversely, assume $H(t)$ is differentiable in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$. By the product rule, $H(t)^{-1}$ is differentiable in the sense of $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$. Denote by $B(t)$ the derivative of $H(t)^{-1}$ in the sense of $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$, and consider again the operator $T_{\delta}(t) = \delta^{-1}[H(t + \delta)^{-1} - H(t)^{-1}] - B(t)$. For $\Phi \in \mathcal{H}$, we have:

$$\|T_{\delta}(t)\Phi\| \leq \|T_{\delta}(t)\Phi\|_+ \leq \|T_{\delta}(t)\|_{-,+}\|\Phi\|_- \leq \|T_{\delta}(t)\|_{-,+}\|\Phi\|.$$

Therefore, $\|T_{\delta}(t)|_{\mathcal{H}}\| \leq \|T_{\delta}(t)\|_{-,+}$, and the differentiability of $H(t)^{-1}$ in the sense of $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$ implies the differentiability in the sense of $\mathcal{B}(\mathcal{H})$.

Finally,

$$H(t)^{1/2} \frac{d}{dt} H(t)^{-1} H(t)^{1/2} = H(t)^{-1/2} \frac{d}{dt} H(t) H(t)^{-1/2},$$

and it follows that $\|\frac{d}{dt}H(t)\|_{+,-,t} = \|H(t)^{1/2}B(t)H(t)^{1/2}\|$. The equivalence of the norms $\|\cdot\|_{\pm,t} \sim \|\cdot\|_{\pm}$ implies that $\|H(t)^{1/2}B(t)H(t)^{1/2}\|$ is bounded uniformly on t if and only if $\|\frac{d}{dt}H(t)\|_{+,-}$ is bounded uniformly on t . \square

Remark 2. Note that, in proving Proposition 6, we also showed that:

- (i) $\frac{d}{dt}H(t)^{-1}$ exists in the sense of $\mathcal{B}(\mathcal{H})$ if and only if it exists in the sense of $\mathcal{B}(\mathcal{H}^-, \mathcal{H}^+)$, which holds if and only if $\frac{d}{dt}H(t)$ exists in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$;
- (ii) $\|H(t)^{1/2}\frac{d}{dt}H(t)^{-1}H(t)^{1/2}\| = \|H(t)^{-1/2}\frac{d}{dt}H(t)H(t)^{-1/2}\|$.

Proposition 7. Let $\langle \cdot, \cdot \rangle_{+,t}$ be inner products as in Assumption 2, and let $H(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ be the family of bounded operators defined by $(\Psi, H(t)\Phi) = \langle \Psi, \Phi \rangle_+$. Then, (K2) implies that $H(t)$ is differentiable in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ and that there is $C > 0$ such that:

$$\left\| \frac{d}{dt}H(t) \right\|_{+,-} \leq C.$$

Proof. This is Proposition 2(iii) applied to $v_t(\Psi, \Phi) = \langle \Psi, \Phi \rangle_+$, which is continuously differentiable by assumption (see (K2)). \square

Corollary 1 and Propositions 6 and 7 yield immediately the following results showing the connection between B. Simon’s and J. Kisyński’s approaches. First of all, J. Kisyński’s assumptions imply B. Simon’s assumptions:

Theorem 6. Let $\langle \cdot, \cdot \rangle_{+,t}$ be a family of inner products satisfying Assumption 2, and let $H(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ be the family of bounded operators defined by $(\Psi, H(t)\Phi) = \langle \Psi, \Phi \rangle_{+,t}$. Then, the family $H(t)$ satisfies Assumption 1 with $H_0 = H(t_0)$ for some $t_0 \in [0, T]$.

Proof. By (K1), $H(t)$ are strictly positive. By Corollary 1, (S1) holds. Proposition 7 implies that $H(t)$ is differentiable in the norm sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$, with $\|\frac{d}{dt}H(t)\|_{+,-} < C$, and by Proposition 6, this implies (S2). \square

Remark 3. Note that the regularity obtained for $H(t)$ is higher than required by (S2): not only is $t \mapsto H(t)$ differentiable, but it is also continuously differentiable. This is implied by Proposition 2 and the fact that $t \mapsto \langle \cdot, \cdot \rangle_{+,t}$ is continuously differentiable. However, this point is crucial since, without the derivative $\frac{d}{dt}\langle \cdot, \cdot \rangle_{+,t}$ being continuous, the uniform bound for $\frac{d}{dt}H(t)^{-1}$ would not be recovered.

The converse implication (i.e., B. Simon’s assumptions imply J. Kisyński’s) also holds under the additional requirement of the continuity of $\frac{d}{dt}H(t)^{-1}$.

Theorem 7. Let $H_0, H(t)$ be operators as in Assumption 1. If $H(t)$ satisfies Assumption 1 and $t \in [0, T] \mapsto B(t) = \frac{d}{dt}H(t)^{-1} \in \mathcal{B}(\mathcal{H})$ is continuous, then $\langle \cdot, \cdot \rangle_{+,t} := (\cdot, H(t)\cdot)$ defines a family of inner products on \mathcal{H}^+ satisfying Assumption 2 with $n = 1$.

Proof. From Assumption 1, \mathcal{H}^+ is a dense subspace of \mathcal{H} . Moreover, $\langle \cdot, \cdot \rangle_0 := (\cdot, (H_0 + 1)\cdot)$ endows \mathcal{H}^+ with the structure of a Hilbert space topologically embedded in \mathcal{H} . By assumption, $\langle \cdot, \cdot \rangle_{+,t} := (\cdot, H(t)\cdot)$ defines a family of inner products on \mathcal{H}^+ , and by (S1), they induce on \mathcal{H}^+ topologies that are equivalent to the one induced by $\langle \cdot, \cdot \rangle_0$. Therefore, $(\mathcal{H}^+, \langle \cdot, \cdot \rangle_{+,t})$ are Hilbert spaces topologically embedded in \mathcal{H} , and (K1) holds.

By Proposition 6, (S2) implies that $H(t)$ is differentiable in $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$. By Proposition 2, this implies that the limit:

$$\lim_{t \rightarrow t_0} \sup_{\substack{\Psi, \Phi \in \mathcal{H}^+ \\ \|\Psi\|_+ = 1 = \|\Phi\|_+}} \frac{\langle \Psi, \Phi \rangle_{+,t} - \langle \Psi, \Phi \rangle_{+,t_0}}{t - t_0}$$

exists, which clearly implies that for every $\Psi, \Phi \in \mathcal{H}^+$ fixed, $t \mapsto \langle \Psi, \Phi \rangle_{+,t}$ is differentiable. Moreover, if $H(t)$ is continuously differentiable in $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$, so is the map $t \mapsto \langle \Psi, \Phi \rangle_{+,t}$ for every $\Psi, \Phi \in \mathcal{H}^+$. \square

3.5. Example: Particle in a Circle with a Point-Like Interaction

Consider the Hamiltonian of a free particle moving in the circle with a point-like interaction. We consider the case in which the strength of the point-like interaction varies with time and use the methods developed in the previous section to give necessary conditions for the existence of the solution of the non-autonomous Schrödinger equation.

As Hilbert space \mathcal{H} , we consider the space of the square integrable function on the interval $[0, 2\pi]$. The Hamiltonian is given by the Laplace operator on the self-adjoint domain:

$$\mathcal{D}_\alpha = \{ \Phi \in \mathcal{H}^2([0, 2\pi]) : \Phi(0) = \Phi(2\pi); \quad \Phi'(0) - \Phi'(2\pi) = \alpha\Phi(0) \},$$

where $\mathcal{H}^k([0, 2\pi])$ stands for the Sobolev space of order k . Each value of α determines a different self-adjoint extension. We consider the non-autonomous problem in which α is a function of time. Notice that the time dependence of the operator does not appear in its functional form, but only at the boundary condition. The family of closed sesquilinear forms associated with this family of self-adjoint operators is:

$$h_\alpha(\Phi, \Psi) = \langle \Phi'(x), \Psi'(x) \rangle + \alpha \overline{\Phi(0)} \Psi(0)$$

with domain $\text{dom } h_\alpha = \{ \Phi \in \mathcal{H}^1([0, 2\pi]) : \Phi(0) = \Phi(2\pi) \}$; we refer to [44–46] for further details.

To find the family of operators $H(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$, we need the trace operator. The trace operator, γ , is the operator that maps a function to its boundary values. It is well-defined, continuous on $\mathcal{H}^1([0, 2\pi])$, and surjective; cf. [47]. Now, notice that the domains of the forms do not depend on the parameter α , and the family of self-adjoint operators defined above has constant form domain $\mathcal{H}^+ := \{ \Phi \in \mathcal{H}^1([0, 2\pi]) : \Phi(0) = \Phi(2\pi) \}$, which is a closed subspace of $\mathcal{H}^1([0, 2\pi])$ because the trace operator is continuous on $\mathcal{H}^1([0, 2\pi])$. The boundary term of the sesquilinear form above can be expressed using the trace operator and the pairing (\cdot, \cdot) in the scale of Hilbert spaces as:

$$\alpha \overline{\Phi(0)} \Psi(0) = (\Phi, \alpha \gamma^\dagger \gamma \Psi),$$

where γ^\dagger is the adjoint operator with respect to the pairing. Hence, the family of time-dependent operators $H(t) : \mathcal{H}^+ \rightarrow \mathcal{H}^-$ becomes:

$$H(t) = \bar{\Delta} + \alpha(t) \gamma^\dagger \gamma,$$

where $\bar{\Delta}$ is the continuous extension of the Laplacian, Δ , to \mathcal{H}^+ .

It is well known that h_a is semibounded from below with a bound that depends continuously on α ; cf. [44]. It can be proven that (S1) holds as long as $\sup_{t \in [0, T]} |\alpha(t)| < M$. Proposition 6 establishes that (S2) holds if the derivative of the operators $H(t)$ in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ exists for all $t \in [0, T]$ and is uniformly bounded, i.e., for any $\alpha : [0, T] \rightarrow \mathbb{R}$ continuous with the bounded derivative. Theorem 4 shows that for any such α , a unique solution of the weak Schrödinger equation exists. The same result is obtained under the Assumptions (K1) and (K2). Condition (K1) is satisfied by construction, and (K2) is satisfied for n if and only if $\alpha : [0, T] \rightarrow \mathbb{R}$ is in $C^n[0, T]$. Notice that for $n > 1$ and since $[0, T]$ is compact, both the function α and its derivative are bounded in the interval. Theorem 5 also establishes that a unique solution of the weak Schrödinger equation exists. One has to require, however, the stronger condition of the continuous differentiability of the function α . If $\alpha \in C^2[0, T]$, then Theorem 5 establishes that strong solutions of the Schrödinger equation exist, i.e., solutions of the Schrödinger equation in the usual sense.

4. Conclusions

In this article, after carefully revising the two seminal approaches of J. Kisyński and B. Simon to the well-posedness of the Schrödinger problem associated with a time-varying Hamiltonian, we compared the assumptions at the roots of their results by using Hilbert scales as our primary tool. We studied the relation between one-parameter families of sesquilinear forms and operators representing them. In particular, we studied the connection between their respective continuity and differentiability properties. These results, presented in Section 2, have their own interest and allow comparing the approaches presented in Section 3.

Non-continuous Hamiltonians, in the sense described in the article, cannot lead to physically meaningful examples in a context with time-dependent domains, i.e., time-dependent boundary conditions. In this context, it is important from the physical point of view to have good characterisations of the well posedness of the Schrödinger equation. The comparison between the approaches manifests that the conditions imposed by J. Kisyński are easier to check as it is easier to check the continuity/differentiability properties of functions (K2) than the existence of uniform relative bounds for the operators (S1), and we provided proof that the latter implies the former.

J. Kisyński gave necessary conditions for the existence of solutions of the non-autonomous Schrödinger equation for the case in which $\text{dom } H(t)^{1/2}$ has a nontrivial dependence in time. This situation cannot be tackled by the framework proposed by B. Simon. In the case of a constant form domain, the necessary conditions imposed by J. Kisyński are more restrictive, as he required the continuity of the first derivatives, as opposed to B. Simon, who just required uniformly bounded derivatives. However, B. Simon only gave necessary conditions to find solutions of the weak Schrödinger equation and was able to prove the continuity of the solutions, but not differentiability in the sense of \mathcal{H} . However, the weak Schrödinger equation was not as physically meaningful as the strong Schrödinger equation. J. Kisyński's extra regularity conditions allowed him to prove not only the differentiability of the solutions in the sense of \mathcal{H} , but also in the sense of \mathcal{H}^+ . These imply that these solutions are solutions of the strong Schrödinger equation.

The analysis presented in this article was devoted to the linear Schrödinger equation and the techniques used originated in the theory of linear operators in Banach and Hilbert spaces. Whether these techniques can be applied to treat also the nonlinear Schrödinger equation is an interesting perspective.

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References

1. Stone, M.H. Linear Transformations in Hilbert Space: III. Operational Methods and Group Theory. *Proc. Natl. Acad. Sci. USA* **1930**, *16*, 172. [[CrossRef](#)]
2. Stone, M.H. On One-Parameter Unitary Groups in Hilbert Space. *Ann. Math.* **1932**, *33*, 643–648. [[CrossRef](#)]
3. von Neumann, J. Über Einen Satz von Herrn M.H. Stone. *Ann. Math.* **1932**, *33*, 567–573. [[CrossRef](#)]
4. Facchi, P.; Ligabò, M.; Lonigro, D. Spectral properties of the singular Friedrichs–Lee Hamiltonian. *J. Math. Phys.* **2021**, *62*, 032102. [[CrossRef](#)]
5. Leonforte, L.; Valenti, D.; Spagnolo, B.; Carollo, A.; Ciccarello, F. Dressed emitters as impurities. *Nanophotonics* **2021**, *10*, 4251–4259. [[CrossRef](#)]
6. Lonigro, D.; Facchi, P.; Pascazio, S.; Pepe, F.V.; Pomarico, D. Stationary excitation waves and multimerization in arrays of quantum emitters. *New J. Phys.* **2021**, *23*, 103033. [[CrossRef](#)]
7. Dell’Antonio, G.; Exner, P.; Geyler, V. Singular interactions in quantum mechanics: Solvable models. *J. Phys. A Math. Gen.* **2005**, *38*, E01. [[CrossRef](#)]
8. Iborat, A.; Pérez-Pardo, J.M. Numerical Solutions of the Spectral Problem for Arbitrary Self-Adjoint Extensions of the One-Dimensional Schrödinger Equation. *SIAM J. Numer. Anal.* **2013**, *51*, 1254–1279. [[CrossRef](#)]
9. López-Yela, A.; Pérez-Pardo, J.M. Finite element method to solve the spectral problem for arbitrary self-adjoint extensions of the Laplace–Beltrami operator on manifolds with a boundary. *J. Comput. Phys.* **2017**, *347*, 235–260. [[CrossRef](#)]
10. Wüller, U. Existence of the Time Evolution for Schrödinger Operators with Time Dependent Singular Potentials. *Annales de l’IHP Physique Théorique* **1986**, *44*, 155–171.
11. Yajima, K. Existence of Solutions for Schrödinger Evolution Equations. *Commun. Math. Phys.* **1987**, *110*, 415–426. [[CrossRef](#)]
12. Yajima, K. Schrödinger Equations with Time-Dependent Unbounded Singular Potentials. *Rev. Math. Phys.* **2011**, *23*, 823–838. [[CrossRef](#)]
13. Hughes, R.J. Unitary Propagators for Time-Dependent Hamiltonians with Singular Potentials. *J. Math. Anal. Appl.* **1995**, *193*, 447–464. [[CrossRef](#)]
14. Carles, R. Nonlinear Schrödinger Equation with Time Dependent Potential. *Commun. Math. Sci.* **2011**, *9*, 937–964. [[CrossRef](#)]
15. Mantile, A. Stability and Control of a Confined 1D Quantum System with Time-Dependent Delta Potentials. *J. Phys. A Math. Theor.* **2011**, *44*, 145305. [[CrossRef](#)]
16. Posilicano, A. The Schrödinger Equation with a Moving Point Interaction in Three Dimensions. *Proc. Am. Math. Soc.* **2007**, *135*, 1785–1793. [[CrossRef](#)]
17. Dell’Antonio, G.; Figari, R.; Teta, A. The Schrödinger Equation with Moving Point Interactions in Three Dimensions. In *Stochastic Processes, Physics and Geometry: New Interplays, I (Leipzig, 1999)*; Canadian Mathematical Society: Ottawa, ON, Canada, 2000; Volume 28, CMS Conference Proceedings, pp. 99–113.
18. Šamaj, L. Evolution of Quantum Systems with a Scaling Type Time-Dependent Hamiltonians. *Int. J. Mod. Phys. B* **2002**, *16*, 3909–3914. [[CrossRef](#)]
19. Combesure, M. The Quantum Stability Problem for Some Class of Time-Dependent Hamiltonians. *Ann. Phys.* **1988**, *185*, 86–110. [[CrossRef](#)]
20. Munier, A.; Burgan, J.; Feix, M.; Fijalkow, E. Schrödinger Equation with Time-Dependent Boundary Conditions. *J. Math. Phys.* **1981**, *22*, 1219–1223. [[CrossRef](#)]
21. Makowski, A.; Dembiński, S. Exactly Solvable Models with Time-Dependent Boundary Conditions. *Phys. Lett. A* **1991**, *154*, 217–220. [[CrossRef](#)]
22. Di Martino, S.; Anza, F.; Facchi, P.; Kossakowski, A.; Marmo, G.; Messina, A.; Militello, B.; Pascazio, S. A Quantum Particle in a Box with Moving Walls. *J. Phys. A Math. Theor.* **2013**, *46*, 365301. [[CrossRef](#)]
23. Facchi, P.; Garnerò, G.; Marmo, G.; Samuel, J. Moving Walls and Geometric Phases. *Ann. Phys.* **2016**, *372*, 201–214. [[CrossRef](#)]
24. Balmaseda, A.; Pérez-Pardo, J. Quantum Control at the Boundary. In *Classical and Quantum Physics*; Marmo, G., Martín de Diego, D., Muñoz Lecanda, M., Eds.; Springer International Publishing: Berlin/Heidelberg, Germany, 2019; Springer Proceedings in Physics, pp. 57–84.
25. Balmaseda, A.; Lonigro, D.; Pérez-Pardo, J. Quantum Controllability on Graph-like Manifolds through Magnetic Potentials and Boundary Conditions. *arXiv* **2021**, arXiv:2108.00495.
26. Kato, T. Integration of the Equation of Evolution in a Banach Space. *J. Math. Soc. Jpn.* **1953**, *5*, 208–234. [[CrossRef](#)]
27. Kato, T. On Linear Differential Equations in Banach Spaces. *Commun. Pure Appl. Math.* **1956**, *9*, 479–486. [[CrossRef](#)]
28. Lions, J.L. *Equations Différentielles Operationnelles: Et Problèmes aux Limites*; Springer: Berlin/Heidelberg, Germany, 1961.
29. Yosida, K. On the Integration of the Equation of Evolution. *J. Fac. Sci. Univ. Tokyo Sec. I* **1963**, *9*, 397–402.
30. Yosida, K. *Functional Analysis*, 1st ed.; Academic Press: New York, NY, USA, 1965.
31. Kiszyński, J. Sur les opérateurs de Green des problèmes de Cauchy abstraits. *Stud. Math.* **1964**, *3*, 285–328. [[CrossRef](#)]
32. Simon, B. *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*; Princeton Series in Physics; Princeton University Press: Princeton, NJ, USA, 1971.
33. Kato, T. Linear Evolution Equations of “Hyperbolic” Type, II. *J. Math. Soc. Jpn.* **1973**, *25*, 648–666. [[CrossRef](#)]
34. Maspero, A.; Robert, D. On Time Dependent Schrödinger Equations: Global Well-Posedness and Growth of Sobolev Norms. *J. Funct. Anal.* **2017**, *273*, 721–781. [[CrossRef](#)]

35. Sloan, A. The Strong Convergence of Schrödinger Propagators. *Trans. Am. Math. Soc.* **1981**, *264*, 557–570.
36. Wang, C.; Wise, S.M. Global smooth solutions of the three-dimensional modified phase field crystal equation. *Methods Appl. Anal.* **2010**, *17*, 191–212. [[CrossRef](#)]
37. Wang, C.; Wise, S.M. An energy stable and convergent finite-difference scheme for the modified phase field crystal equation. *SIAM J. Numer. Anal.* **2011**, *49*, 945–969. [[CrossRef](#)]
38. Baskaran, A.; Hu, Z.; Lowengrub, J.S.; Wang, C.; Wise, S.M.; Zhou, P. Energy stable and efficient finite-difference nonlinear multigrid schemes for the modified phase field crystal equation. *J. Comput. Phys.* **2013**, *250*, 270–292. [[CrossRef](#)]
39. Baskaran, A.; Lowengrub, J.S.; Wang, C.; Wise, S.M. Convergence analysis of a second order convex splitting scheme for the modified phase field crystal equation. *SIAM J. Numer. Anal.* **2013**, *51*, 2851–2873. [[CrossRef](#)]
40. Wang, L.; Chen, W.; Wang, C. An energy-conserving second order numerical scheme for nonlinear hyperbolic equation with an exponential nonlinear term. *J. Comput. Appl. Math.* **2015**, *280*, 347–366. [[CrossRef](#)]
41. Kato, T. *Perturbation Theory for Linear Operators*; Classics in Mathematics; Springer: Berlin, Germany, 1995.
42. Reed, M.; Simon, B. *Methods of Modern Mathematical Physics I: Functional Analysis*; Academic Press: New York, NY, USA, 1980.
43. Berezanskii, Y. *Expansions in Eigenfunctions of Self-Adjoint Operators*; Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, USA, 1968; Volume 17.
44. Ibort, A.; Lledó, F.; Pérez-Pardo, J.M. Self-Adjoint Extensions of the Laplace–Beltrami Operator and Unitaries at the Boundary. *J. Funct. Anal.* **2015**, *268*, 634–670. [[CrossRef](#)]
45. Ibort, A.; Lledó, F.; Pérez-Pardo, J.M. On Self-Adjoint Extensions and Symmetries in Quantum Mechanics. *Ann. Henri Poincaré* **2015**, *16*, 2367–2397. [[CrossRef](#)]
46. Balmaseda, A.; Di Cosmo, F.; Pérez-Pardo, J.M. On Z -Invariant Self-Adjoint Extensions of the Laplacian on Quantum Circuits. *Symmetry* **2019**, *11*, 1047. [[CrossRef](#)]
47. Adams, R.; Fournier, J. *Sobolev Spaces*; Pure and Applied Mathematics; Elsevier Science: Amsterdam, The Netherlands, 2003.