# Supply constrained location-distribution in not-for-profit settings

**Abstract**

There is a need to design location-distribution problems in not-for-profit settings, where very often limited resources have to be allocated to different demand regions and, thus, a combination of efficiency and equity goals can rightfully be considered. In this paper, we propose the use of a fractional efficiency measure and a new inequity measure related to the Gini coefficient. Both measures include the utility obtained by each demand region given a certain supply allocation. Considering these utility functions allows the decision maker to establish priorities by representing the heterogeneous effects of distributing supply to different demand locations (location effect) and in different volumes (diminishing returns effect). Based on the proposed efficiency and inequity measures, we design a supply constrained location-distribution model and suggest a novel bi-objective integer linear fractional programming approach to solve the model. Our resolution technique allows us to use multiple fractional objective measures that have not been commonly utilized before due to tractability and interpretation issues. Novel analytical results for the worst case performance of the proposed optimization tool are provided. Our numerical experiments assess computational efficiency and provide concrete managerial prescriptions. Finally, an illustrative application of our approach in the context of a food crisis in Angola is presented.

**Keywords.** location-distribution; utility; Gini index; efficiency; fractional programming

## 1 Introduction

Location-distribution optimization problems concern facility location decisions and supply distribution decisions from facilities to demand points. The usual objective of this type of problem is to minimize costs or maximize profits while fulfilling the recipients' demands given a certain service level. The setting for this study has three major characteristics that deviate from this standard problem. The first is that the operation is run by a public or nonprofit organization, so it has a not-for-profit goal whereby the costs is one of the factors to be considered but not the only one. The second characteristic is that the available supply is below the total demand level, thereby raising fairness issues because some portion of the demand will not be served. The third characteristic of our problem is the importance of considering the disparity between different demand locations (location effect) and the differences when distributing different volumes of supply in the same demand location (diminishing returns effect). These two effects can be represented in the demand location's utility functions, that constitute part of our proposed objective measures. Traditionally, when studying equity in location-distribution problems, the uniqueness of each demand location is solely represented by its population size and its specific geographical location with respect to the facilities that provide the goods or services (Marsh and Schilling 1994). Nonetheless, we claim that under the suggested setting not every individual should be treated equally because this is not what is observed in practice. We suggest the use of demand location's utility functions with respect to the supply received as a tool to represent heterogeneity. These utility functions can be designed in different ways such as based on experts' opinions (Gralla et al. 2014) or using regional population surveys that can identify the most vulnerable individuals (as shown in our case study in §6.)

There are several applied fields in which the suggested supply constrained location-distribution problem is applicable. Many of these settings belong to the public and nonprofit sectors such as the problems related to food distribution, vaccine distribution, blood bank distribution, and humanitarian aid distribution. We note that these applications naturally share the need to study the balance between efficiency and fairness, which are the two objectives that compose our problem. These applications can translate into interventions on a large scale (national or international), where distinct demand regions with different needs and effects of the intervention can be observed. In these settings, different utility functions per demand region can be considered to reflect these disparities.

The present paper proposes a supply constrained location-distribution model that has the form of an integer linear fractional programming (ILFP) problem. We suggest a general approach for solving ILFP problems via fractional programming. The efficiency of a process is defined as the ratio of outputs to inputs; hence, it constitutes a fractional form. Our second objective, the utility-based Gini coefficient, also has a fractional form as defined later. Owing to the not-for-profit nature of the problem, the model comprises two different objectives (efficiency and fairness) that form a bicriteria model. Thus, our resolution method effectively addresses bi-objective and fractional structures. Specifically, an $\epsilon$-constraint approach is used to address the bi-objective part of the problem, and a parametric approach is selected to address the single objective integer linear fractional programming part.

We outline the main contributions of our paper below:

• We suggest a fractional measure of efficiency and a novel and broader measure of fairness based on the Gini coefficient. Both measures use the utility functions of each demand location to represent heterogeneity between different demand locations (location effect) and heterogeneity between different levels of supply received at each location (diminishing returns effect). We believe that these effects are critical in location-distribution problems when there is a shortage and can be considered with our suggested measures and resolution approach.

• For tractability and ease of interpretation, most previous studies on optimization problems with efficiency and fairness objectives strongly prefer to employ non-fractional metrics. To address the tractability challenge, we develop a fractional optimization tool that can efficiently solve location-distribution problems with multiple fractional objectives. We also show that alternative non-fractional metrics that might be easier to interpret might ignore key trade-offs of the location-distribution problem.

• We theoretically and computationally analyze the speed of convergence of the Newton's and the sub-approximation approaches that are embedded in our resolution algorithm. In this analysis, we are the first to theoretically develop a complexity bound for the sub-approximation algorithm applied to ILFP problems. We also develop a new complexity bound for the Newton's method that generalizes a well-known result for ILFP problems with binary variables. Furthermore, we compare the computational efficiency of both methods and observe that the sub-approximation method is more efficient for a set of steep fixed facility location cost functions.

• The applicability of the model and solution approach is illustrated in a case study of food aid distribution in Angola. In particular, we present a concrete quantitative analysis of the problem related to the location of warehouses and distribution of food aid by the World Food Programme (WFP).

The paper is structured as follows. In the next section, we present a review of the literature. In Section 3, we introduce and characterize the suggested efficiency and fairness measures and present the model formulation. Section 4 presents the suggested resolution approach and provides theoretical results of the speed of convergence of the Newton's and sub-approximation methods for ILFP problems. In Section 5 all computational work is presented with a discussion of computational efficiency between the Newton's and sub-approximation methods and a second analysis that provides managerial insights. Section 6 shows a concrete design problem that uses actual data in the context of food aid distribution. Section 7 concludes the paper.

## 2 Literature review

**Not-for-profit location-distribution problems**

The location-distribution problem is a combination of location and distribution decisions in the same model that is usually defined as an integer programming (IP) problem or mixed-integer programming (MIP) problem (Nemhauser and Wolsey 1988). Consequently, considerable literature is centered on developing efficient optimization methods to solve this problem, that we also contribute to in this paper. When the purpose of the location-distribution problem is to serve not-for-profit objectives, the discussion is also extended to the choice of the objective metrics to use. We develop this discussion in the next few paragraphs.

There is a consensus in the literature that equity is a metric that is as important as efficiency and effectiveness when analyzing operations in the public and nonprofit sectors (Balcik et al. 2010, Karsu and Morton 2015, Savas 1978). Several publications study equity in facility location analysis (Marsh and Schilling 1994, Mulligan 1991, Ogryczak 2000). We highlight Marsh and Schilling (1994) who review twenty different equity measures for facility location, with the Gini coefficient being described as one of the measures. Its definition is based on the total distance traveled from each demand node to the nearest facility. Leclerc et al. (2012) review literature that models equity for allocating resources in public services systems with a special focus on Emergency Medical Services (EMS) resources. Another example is Balcik et al. (2010) that focus on incorporating equity in different vehicle routing applications such as disaster relief, mobility services, food distribution, and hazardous material transportation.

The Gini coefficient is an equity measure that has been previously used in facility and distribution problems. For example, Lejeune and Prasad (2013) study effectiveness-equity trade-offs in facility location models. In particular, they formulate bicriteria problems, where an average weighted distance is the measure of effectiveness and the Gini index is their measure of (in)equity. Procedures are developed to identify the optimal solution to both the effective and Gini measures on tree networks. Mandell (1991) studies an allocation problem in the form of bicriteria mathematical programming models to identify the trade-offs between overall output (effectiveness) and equity. The effectiveness measure used is the sum of output in each delivery area, while the equity measure is the Gini coefficient applied to units of supply distributed. The paper presents a case study of the allocation of new books among the branches of a public library to illustrate a real application of the model. In contrast with Mandell's model, we incorporate a novel and broader version of the

Gini coefficient based on the utility function of each demand location. This discussion is further elaborated in §3.1.2 and §3.1.3.

**Applications**

Below we outline a limited sample of applications where our suggested model and approach are evidently useful:

*Food delivery.* Food insecurity due to periodic droughts is common in Sub-Saharan Africa, where food aid is provided owing to seasonal food shortages. Rancourt et al. (2015) present a covering model to address the problem of locating distribution centers to deliver food aid in Kenya. They minimize the total welfare cost of all the stakeholders, including the provider's supply costs, the delivery organization's hand-out costs, and the beneficiary access cost to travel and collect the food. Food insufficiency and hunger also occur in developed countries, where the role of food banks is key to mitigate the problem. Demand is usually linked with population in need, and it typically exceeds the donated food supply. Under this setting, Orgut et al. (2016) claim that an optimal distribution should be equitable and effective and propose to minimize the amount of undistributed food while maintaining a user-specified upper bound on the absolute deviation of each service area from a perfectly equitable distribution. Moreover, in the context of a food bank network, Eisenhandler and Tzur (2018) develop a pickup and distribution model by suggesting a novel objective that integrates a measure of effectiveness (units supplied) and equity (the Gini coefficient). The food delivery problem is revisited in §6 as a case study.

*Disaster relief management.* There is an extensive body of literature on pre-disaster planning in humanitarian logistics that involves strategic planning of location of facilities and allocation of supplies. For example, Balcik and Beamon (2008) present a location-distribution problem, wherein a variant of the maximal covering location model with pre-positioned inventory decisions is developed to improve response time and the proportion of demand satisfied. The discussion of best objective functions has also attracted significant interest in disaster management. For example, Huang et al. (2012) explore efficiency, efficacy, and equity measures and study how these measures influence the structure of vehicle routing and distribution of resources. They suggest an interesting measure of equity that represents a convex disutility function that depends on the percentage of unsatisfied demand. Holguín-Veras et al. (2013) suggest social costs, defined as the summation of logistic and deprivation costs, as the preferred objective function for post-disaster humanitarian logistic models. Gralla et al. (2014) develop a piece-wise linear utility function to use as the objective of optimization models for humanitarian resource allocation problems across beneficiaries. This function is based on a conjoint survey involving experts' opinions.

*Healthcare.* An example of location-distribution problems in healthcare is the distribution of indoor residual spraying (IRS) activities to prevent malaria. In this context, Griffin et al. (2013) maximize the number of prevented malaria cases, where the number of people covered is less than the total population and the total costs are restricted to a specific budget. McCoy and Lee (2014) discuss the notions of inefficiency and inequity in healthcare delivery fleet management in developing countries. They consider utilitarian, proportionally fair, and egalitarian equity objectives. The mobile healthcare facility location routing problem is also of interest. Doerner et al. (2007) suggest a multicriteria combinatorial optimization problem to solve this problem, where the objectives are effectiveness of workforce employment, average accessibility in terms of distance for patients, and coverage. The supply chain management of blood banks is another location-distribution problem that can fit our model. For a review on this topic refer to Pierskalla (2005).

**Multiobjective integer linear fractional programming**

We formulate our supply constrained location-distribution model as a multiobjective ILFP. In general, a multiobjective optimization programming (MOP) problem with discrete variables can be solved by methods such as weighted sum scalarization (Geoffrion 1968), the compromise solution method (Yu 2013), and goal programming (Ignizio 1976). Particularly, the existing solution techniques for finding a set of all efficient solutions of a multiobjective ILFP are based on the cutting plane method (Abbas and Moulaı 2002) and the branch-and-cut method (Ait Mehdi et al. 2014). The major drawback of these methods, however, is that generating a whole set of efficient solutions is computationally challenging. Therefore, heuristic methods have been developed to approximate the set of efficient solutions. In particular, in our paper, we apply the $\epsilon$-constraint method (Haimes et al. 1971), one of the most widely used heuristics for MOP problems.

The resulting formulation of our model after applying the $\epsilon$-constraint method is an ILFP that is an NP-hard problem. Many researchers have developed techniques for solving this type of problem, including methods such as linearization (Williams 1974), branch-and-bound (Agrawal 1977, Robillard 1971), cutting plane (Granot and Granot 1977, Grunspan and Thomas 1973), enumerative (Arora et al. 1977, Granot and Granot 1976), approximation (Hashizume et al. 1987), and parametric (Radzik 1998, Wang et al. 2006). To solve the ILFP problems, we adopt the parametric method as it has been reported to be faster and more reliable than other existing algorithms (You et al. 2009). Stancu-Minasian (2012) offers an overview to the ILFP problem.

Parametric algorithms transform a problem with a fractional objective function into one with a non-fractional objective function. A limited number of general results about parametric algorithms have been proposed for the linear fractional combinatorial optimization (LFCO) problem, which is a special case of the fractional programming problem wherein all functions in the objective function and constraints are linear with binary variables defined on some combinatorial structure (Radzik 2013). For instance, Radzik (1993) and Wang et al. (2006) show that LFCO problems have polynomial bounds on the number of iterations of the Newton's method. In this paper, we extend the Newton's method's result in Radzik (1993) to ILFP problems with bounded variables and present a novel bounding result for the performance of the sub-approximation search algorithm. In particular, and to the best our knowledge, we are the first to demonstrate that the ILFP problem has polynomial bounds on the number of iterations of the sub-approximation algorithm.

# 3 Problem formulation

The goal of our problem is to distribute a total amount of supply $s$ to a set of $m$ demand locations, where each demand location has a known demand $b_j$ and the sum of all demands is larger than the total supply $s$. Thus, this is a supply shortage problem, where we suggest an objective that maximizes efficiency and minimizes inequity. We claim that the specific utility of receiving a certain amount of supply per demand location $j$ is a key factor in this type of problem; as a result, it plays a key role in both objective functions. The problem also needs to identify the facility locations to open, from which supply will be delivered to the demand locations.

The goal of this section is to introduce the two performance metrics (efficiency §3.1.1 and inequity §3.1.2) and the facility location-distribution problem (§3.2). The problem presented in this paper has integer variables and the objective function is expressed as a ratio of linear functions. This optimization problem is called integer linear fractional program and the standard form of this program is described as follows:

**Definition 1.** *An* ***Integer Linear Fractional Programming (ILFP) problem*** *is formulated as,*

$$\begin{aligned} \textit{maximize} \quad & \frac{f(x)}{g(x)}, \\ \textit{subject to} \quad & cx \leq b, \\ & x \in (\mathrm{Z}^+)^n \end{aligned}$$

*where $f(x)$ and $g(x)$ are linear functions and $g(x) > 0$ for all feasible solutions.*

### 3.1 Objective functions

For ease of exposition, we assume in this subsection that there is only one facility open; thus, we only use the subscript $j$ related to demand locations. Before we define our two objective functions, we formally characterize the utility functions that are part of both objectives. Each demand location $j$ has an associated non-negative utility $u_j$ that depends on how much supply is received. That is,

**Definition 2.** *We define the **utility function at location** $j$ as the function $u_j : [0, b_j] \rightarrow \mathbb{R}^+$ characterized as $u_j(x_j) = \sum_{t=1}^{l_j} c_j^t x_j^t$, where each slope $c_j^t$ is assigned to a range $[a_j^{t-1}, a_j^t]$ for each $t = 1, \ldots, l_j$ and is ordered in monotonically decreasing order $c_j^1 \geq c_j^2 \geq \ldots \geq c_j^{l_j}$. Moreover, we define the set of integer variables $x_j^t$ that can have a value between 0 and their upper limit $a_j^t - a_j^{t-1}$, so $x_j = \sum_{t=1}^{l_j} x_j^t$.*

According to this definition, we assume that our utility function $u_j$ is piece-wise linear and concave on its domain $[0, b_j]$. Assuming piece-wise linearity is convenient because it represents a good balance between accuracy and implementability (Gralla et al. 2014). The concavity assumption in utility functions is standard in microeconomics (Mas-Colell et al. 1995) and humanitarian settings (Huang et al. 2012, McCoy and Lee 2014, Toyasaki and Wakolbinger 2014). For example, similar to our concavity assumption, Huang et al. (2012) assume convex disutility functions because the rate of change of disutility is greater when the demand served is low. In other words, the last units delivered should have fewer weight than the first. The utility function is characterized by the points $(a_j^t, u_j(a_j^t))$ for $t = 1, \ldots, l_j$ as shown in Figure 1, where $a_j^t \in \mathbb{Z}^+$ and $l_j$ is the number of breakpoints. We denote $\bar{u}_j$ as the maximum utility yield at location $j$ when $b_j$ is delivered ($\bar{u}_j = u_j(b_j) = u_j(a_j^{l_j})$).

#### 3.1.1 Efficiency measure

In operational settings, efficiency is directly associated with productivity and is defined as the ratio of output per input. This measure is commonly used in settings such as manufacturing (e.g., the Toyota production system) and health care (Özcan 2007), and, if appropriately defined, it is also a pertinent measure in nonprofit settings despite often being overlooked (Beamon and Balcik 2008, Berenguer 2016). Regardless of its popularity as performance metric in for-profit settings and its broad use in empirical studies (e.g., Data Envelopment Analysis), this measure has

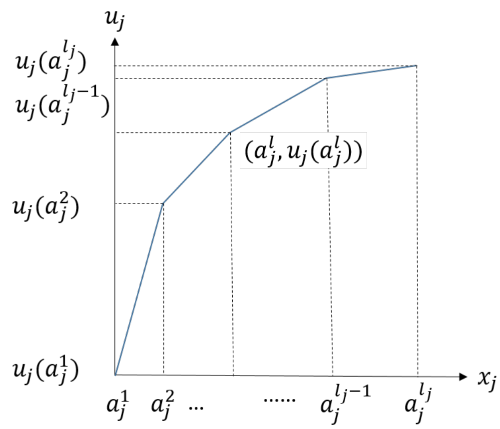


Figure 1: The slope of $u_j$ is monotonic decreasing.

not been commonly considered in analytical studies. Usually, non-fractional efficiency measures (e.g., output-related) are used in analytical models because they are more tractable and easier to interpret.

In the context of our problem, we use "utility per cost" as our efficiency measure. This measure has been used in welfare economics for assessing alternative uses of scarce health care resources in terms of output provided (Birch and Gafni 1992). Thus, in the numerator (i.e., output) we suggest the sum of demand location utilities obtained from receiving certain supply. In the denominator (i.e., input) we propose the total cost incurred in this operation. Indeed, minimizing costs is critical in our model because our problem assumes that all available supply should be distributed but does not have a specific budget limit. In terms of costs, there is a transportation cost per unit $d_j$ of serving demand location $j$ from the sole facility.

**Definition 3.** *Our efficiency measure ($EFF(x)$) for a specific supply allocation $x$ is defined as the ratio of the sum of demand locations' utilities to transportation costs* $EFF(x) = \dfrac{\sum_{j=1}^{m} u_j(x_j)}{\sum_{j=1}^{m} d_j x_j}$.

Note that the total cost in the denominator of our efficiency measure is not determined by the amount of supply (because this is fixed) but by the allocation of the supply.

**Example 1.** *As a concrete example, consider that a total amount of supply $s = 4$ needs to be distributed between two demand locations as described in Figure 2, where each demand location has a different utility function, $u_1 = \{(0,0),(1,2),(2,4),(3,6),(4,8)\}$ and $u_2 = \{(0,0),(1,6),(2,7.5),(3,7.75),(4,8)\}$. Additionally, the unit transportation cost to each location is defined as: $d_1(x) = 1, x \in [0,4]$, $d_2(x) = 0.8, x \in [0,2]$ and $d_2(x) = 0.6, x \in [3,4]$. In this setting, we observe that the optimal solutions to three different objectives (max. of utility, min. of cost and max. of utility per cost) are all* *different (see Table 1). In particular, the optimal solution to our suggested measure* $(1,3)$ *offers a balanced combination of overall utility (9.75) and cost (2.8).*

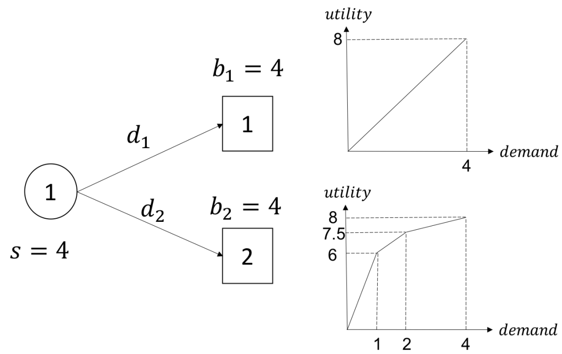


Figure 2: Example of resource allocation problem.

| Solution | Utility | Cost | Utility/Cost | Comment |
|---|---|---|---|---|
| (4,0) | 8 | 4 | 2 | - |
| (3,1) | 12 | 3.8 | 3.15 | Optimal solution of Max. utility |
| (2,2) | 11.5 | 3.6 | 3.19 | - |
| (1,3) | 9.75 | 2.8 | 3.48 | Optimal solution of Max. efficiency |
| (0,4) | 8 | 2.4 | 3.33 | Optimal solution of Min. cost |

Table 1: Objective values with different solutions.

#### 3.1.2 Inequity measure: utility-based Gini coefficient

Under the shortage setting presented in this paper, which is typical in public and nonprofit operations, an efficiency measure should be balanced with an equity measure (Balcik et al. 2010, Karsu and Morton 2015, Savas 1978). We next present our suggested inequity measure and discuss its appropriateness.

The income-based Gini coefficient is the most popular and the original definition of the Gini coefficient. It is based on the Lorenz curve, which is represented on a graph with the $x$-axis as the percentage of bottom household income and the $y$-axis as the percentage of national income. This curve is considered to be a measure of social inequality because the extent to which it sags below the 45° indicates the degree of inequality. In the context of our setting, different supply quantities are assigned to each demand location, where each location might have a different demand size. Mandell (1991) studied this setting and defined a version of the Gini coefficient directly based on the supply assigned per group ($x_j$), that we refer to as $G_{Mandell}$. We propose to consider the utility obtained for delivering a certain amount of supply to a specific group ($u_j(x_j)$) instead of directly considering ($x_j$), and we denote our measure as the *utility-based Gini coefficient* ($G_u$). This measure is developed in the following paragraphs, starting with the associated utility-based Lorenz curve.

Given $m$ groups, with the size of each group as $b_j$, for $j = 1, \ldots, m$, each group has a different utility function, as defined in Definition 2. For a given allocation $x = (x_1, x_2, \cdots, x_m)$, where $x_j$ is the amount of supply allocated to group $j$, we suppose the groups are labeled in non-decreasing order of relative utility such as $\frac{u_1(x_1)}{\bar{u}_1} \leq \frac{u_2(x_2)}{\bar{u}_2} \leq \ldots \leq \frac{u_m(x_m)}{\bar{u}_m}$.

To define the utility-based Lorenz curve, let $F_j$ and $\Phi_j$ be the cumulative maximum utility share and the cumulative utility received up to group $j$, for $j = 1, \ldots, m$ following the non-decreasing relative utility order, respectively. We assume that $F_0 = \Phi_0 = 0$ and define $F_j = \frac{\sum_{k=1}^{j} \bar{u}_k}{\sum_{k=1}^{m} \bar{u}_k}$ and $\Phi_j = \frac{\sum_{k=1}^{j} u_k(x_k)}{\sum_{k=1}^{m} u_k(x_k)}$ for $j = 1, \ldots, m$.

**Definition 4.** *Given a supply allocation $x$, the* ***utility-based Lorenz curve*** *is the red line represented in Figure 3, where each point on the curve, represents the cumulative proportion of utility received (Y-axis) ranked in non-decreasing size according to the cumulative proportion of maximum utility per demand location (X-axis).*

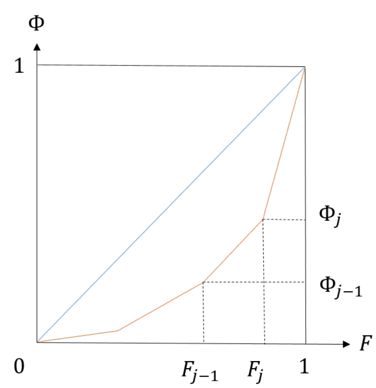


Figure 3: Utility-based Lorenz curve.

Note that the blue line that connects $(0, 0)$ and $(1, 1)$ is the line of equality. From these two lines, we can obtain a graphical depiction of the utility-based Gini coefficient in the same manner it is defined for other variables such as income.

**Definition 5.** *The* ***utility-based Gini coefficient*** *($G_u(x)$) is the area between the utility-based Lorenz curve (red curve) and the line of equality (blue line) divided by the area from the line of equality to the X-axis.*

Next, we present a novel exact mathematically tractable expression for the utility-based Gini coefficient based on the above graphical depiction.

**Proposition 1.** *The utility-based Gini coefficient is equivalent to the following mathematical expression:*

$$G_u(x) = \frac{\sum_{j=1}^{m-1} \sum_{k>j}^{m} |\bar{u}_k u_j(x_j) - \bar{u}_j u_k(x_k)|}{\sum_{j=1}^{m} \bar{u}_j \sum_{j=1}^{m} u_j(x_j)}.$$

*Proof.* The proof is in the Appendix. □

Note that $G_u(x)$ carries a constant coefficient in the denominator that has been conveniently removed in our suggested measure of inequity:

**Definition 6.** *The* ***inequity measure (Ineq)*** *based on $G_u$ is defined as*

$$Ineq(x) = G_u(x) \cdot (\sum_{j=1}^{m} \bar{u}_j) = \frac{\sum_{j=1}^{m-1} \sum_{k>j}^{m} |\bar{u}_k u_j(x_j) - \bar{u}_j u_k(x_k)|}{\sum_{j=1}^{m} u_j(x_j)} \qquad (1)$$

#### 3.1.3 Appropriateness of the utility-based Gini coefficient

Next, we discuss the difference between our suggested inequity measure ($Ineq$) and measure $G_{Mandell}$. The latter measure represents the case of all utility functions being defined as the identity function ($u(x_j) = x_j$) or, equivalently, all population in all demand regions being considered of equal importance. Thus, we have the following result:

**Lemma 1.** *When $u_j(x_j) = x_j$ for $j = 1, \ldots, m$ then $G_{Mandell}(x) = G_u(x)$.*

One of the main contributions of this paper is to attest the importance of accounting for the heterogeneity effects of distributing supply in different locations and at different volumes within the same location. When these differences are not considered we have the case of the $G_{Mandell}$. We claim that in the case of our not-for-profit supply constrained setting, in which we have different demand locations, accounting for heterogeneity is critical when making location and distribution decisions. Next, we review Marsh and Schilling (1994) seven characteristics of equity measures in the area of facility location and we identify the characteristics that we believe are not appropriate for supply constrained location-distribution settings. The general $Ineq$ and $G_{Mandell}$ measures satisfy the properties related to *tractability*, *appropriateness*, and *impartiality* but the general form $Ineq$ does not satisfy the most "desirable" analytical properties, which are the *principle of transfers* and the *scale invariance*. Nonetheless, there are particular utility function structures that prompt $Ineq$ to satisfy the *principle of transfers* and/or *scale invariance* properties. Before presenting these structures in Proposition 2, we define both properties.

**Definition 7.** *The principle of transfers is satisfied for equity measure $E(x)$ when a small transfer $(\epsilon)$ from one demand location to another demand location with smaller relative utility drives the inequity measure down, i.e. when $E(\ldots, x_j + \epsilon, \ldots, x_{\tilde{j}} - \epsilon, \ldots) \leq E(\ldots, x_j, \ldots, x_{\tilde{j}}, \ldots)$ for $\frac{u_j(x_j)}{\bar{u}_j} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}}$.*

**Definition 8.** *The scale invariance property is satisfied for equity measure $E(x)$ when the supply assigned to each demand location is multiplied by a positive constant and the inequity measure is unaffected, i.e. when $E(x) = E(Kx)$ where $K \geq 0$.*

**Proposition 2.** *Given the characterization of $u_j(x_j)$ in Definition 2, if*

$$u_j(x_j) = \begin{cases} ax_j \quad \text{for } j = 1, \ldots, m & Ineq(x) \text{ satisfies the principle of transfers and scale invariance} \\ a_j x_j \quad \text{for } j = 1, \ldots, m & Ineq(x) \text{ only satisfies scale invariance} \\ u(x_j) \quad \text{for } j = 1, \ldots, m & Ineq(x) \text{ only satisfies the principle of transfers} \\ \text{otherwise} & Ineq(x) \text{ does not satisfy any of the two properties} \end{cases}$$

*Proof.* The proof is in the Appendix. □

In Proposition 2, we observe that if the utility function at each location is linear (i.e., there is no concavity), the *scale invariance* property is satisfied (cases 1 and 2). In other words, as long as the effect of delivering one supply unit is constant throughout all levels of supply received at any demand location, the scale invariance property is satisfied. Additionally, we observe that if there is no location effect and if the utility function for each demand location is the same (cases 1 and 3), the *principle of transfers* is satisfied.

Precisely, because the general form of the $Ineq$ measure does not satisfy the *principle of transfers* and *scale invariance* criteria, $Ineq$ is a more appropriate measure than any inequity measure that satisfies these two criteria (e.g., $G_{Mandell}$) to represent the following two situations:

- different utility distributions are observed in different locations (i.e., location effect)
- diminishing marginal utility values are obtained for increased supply levels in the same location (i.e., diminishing returns effect)

The suitability of $Ineq$ in certain environments is illustrated in the managerial insights section (§5.2.3) and in the case study (§6), wherein each demand location has a different split of the proportion of population assigned to different food vulnerability levels that translates into different utility functions. The following Example 2 illustrates the difference between using $G_{Mandell}$ and $Ineq$ in a simple setting. For a detailed discussion about each property in Marsh and Schilling (1994), we direct you to the electronic companion (EC1).

**Example 2.** *We continue with the setting presented in Example 1 (Figure 2), where a total supply $s = 4$ needs to be served between two demand locations with a maximum demand of 4 units each ($b_1 = b_2 = 4$) and two different utility functions. For demand location 2, the marginal utility is greater than that of location 1 when the supply received is at 1 unit and, after this point, the slope becomes shallower than that of location 1. In this situation, the allocation with minimum $G_{Mandell}$ is $x_1 = x_2 = 2$ because this measure does not take utility functions into account. This distribution yields relative utilities of $\frac{u_1(x_1)}{\bar{u}_1} = \frac{4}{8}$ and $\frac{u_2(x_2)}{\bar{u}_2} = \frac{7.5}{8}$. In contrast, the optimal distribution of $Ineq$ is $x_1 = 3$ and $x_2 = 1$, resulting in the same relative utilities of $\frac{u_1(x_1)}{\bar{u}_1} = \frac{u_2(x_2)}{\bar{u}_2} = \frac{6}{8}$ in the two locations. Furthermore, using $Ineq$, a larger total relative utility of $\frac{12}{8}$ is obtained compared to $G_{Mandell}$ that is $\frac{11.5}{8}$.*

## 3.2 Model formulation

In this section, we present the multiple facility location-distribution model that combines the measures of efficiency ($EFF$) and inequity ($Ineq$). As a middle step, we note that to obtain an ILFP problem, the absolute values of the inequity objective (1) need to be linearized. In effect, we use the difference of two non-negative variables to represent the inner part of the absolute value of the inequity objective, $\tilde{d}^+_{jk} - \tilde{d}^-_{jk} = \bar{u}_k(\sum_{t=1}^{l_j} c^t_j x^t_j) - \bar{u}_j(\sum_{t=1}^{l_k} c^t_k x^t_k)$, $\forall j = 1, \ldots, m-1$, $\forall k > j$, where $\tilde{d}^+_{jk}$ and $\tilde{d}^-_{jk}$ are deviational non-negative integer variables. As a result, we use $\frac{\sum_{j=1}^{m-1}\sum_{k>j}^{m}(\tilde{d}^+_{jk}+\tilde{d}^-_{jk})}{\sum_{j=1}^{m}\sum_{t=1}^{l_j} c^t_j x^t_j}$ to represent $Ineq$.

At a strategic level, the decision maker may be willing to include into this bicriteria problem the optimal selection of facility locations among a list of $n$ candidate locations. According to the existence of multiple supplier facilities, we substitute $x_j$ with $x^t_{ij}$ that will represent the supply shipped from facility location $i$ to demand location $j$ respective to the range $[a^{t-1}_j, a^t_j]$. Note that each potential location $i$ has a maximum capacity to be distributed $s_i$, a fixed location cost $f_i$, a transportation cost per unit from $i$ to demand location $j$ $d_{ij}$, and a binary variable $y_i$ that determines whether a facility is operating in location $i$ or not. We note that the fixed location costs are added in the denominator of the efficiency measure (2).

Thus, the facility location-distribution problem $P$ is formulated as an ILFP problem as follows:

$$\text{maximize} \quad \frac{\sum_{j=1}^{m}\sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t}{\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}\left(\sum_{t=1}^{l_j} x_{ij}^t\right)} \tag{2}$$

$$\text{minimize} \quad \frac{\sum_{j=1}^{m-1}\sum_{k>j}^{m} \tilde{d}_{jk}^{+} + \tilde{d}_{jk}^{-}}{\sum_{j=1}^{m}\sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t} \tag{3}$$

$$\text{subject to} \quad \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{l_j} x_{ij}^t = s, \tag{4}$$

$$\sum_{t=1}^{l_j}\sum_{i=1}^{n} x_{ij}^t \leq b_j, \quad \forall j = 1,\ldots,m \tag{5}$$

$$\sum_{j=1}^{m}\sum_{t=1}^{l_j} x_{ij}^t \leq s_i y_i, \quad \forall i = 1\ldots,n \tag{6}$$

$$\sum_{i=1}^{n} x_{ij}^t \leq a_j^t - a_j^{t-1}, \quad \forall j = 1,\ldots,m, \forall t = 1,\ldots,l_j \tag{7}$$

$$\tilde{d}_{jk}^{+} - \tilde{d}_{jk}^{-} = \bar{u}_k(\sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t) - \bar{u}_j(\sum_{t=1}^{l_k} c_k^t \sum_{i=1}^{n} x_{ik}^t), \quad \forall j = 1,\ldots,m-1,\ \forall k > j \tag{8}$$

$$x_{ij}^t, \tilde{d}_{jk}^{+}, \tilde{d}_{jk}^{-} \in \mathbb{Z}^{+}, \quad \forall t = 1,\ldots,l_j, \forall j = 1,\ldots,m,\ \forall k > j \tag{9}$$

$$y_i \in \{0,1\}, \quad \forall i = 1,\ldots,n. \tag{10}$$

Constraint (4) forces all supply to be distributed between the different demand locations. Constraints (5) enforce that the supply distributed to each demand location $j$ does not exceed the maximum demand ($b_j$). Constraints (6) enforce that supply capacity at facility location $i$ is not exceed. Constraints (7) define the limits of integer variables $x_j^t$ that are confined to a specific portion $t$ of the utility function $u_j$. Constraints (8) define the auxiliary variables used to linearize the absolute value of the *Ineq* objective function. Constraints (9) and (10) are the integrality and binary constraints.

# 4 Resolution method: fractional programming

In this section, we present a two-phase algorithm for solving general bicriteria ILFP problems and show how this method solves our problem $P$ (§4.1). In the first phase of our algorithm, we solve an ILFP model to find the best achievable efficiency. The second phase then chooses an efficient solution that ensures the best equity level. After presenting the method, we propose two different parametric approaches (Newton's and sub-approximation methods) for solving the problem in the first phase (§4.2). Subsequently, we perform a worst-case analysis of the Newton's and sub-approximation methods based on the number of non-fractional versions of problems solved to obtain the solution (§4.3). The analytical results in this analysis demonstrate that the number of iterations to solve the ILFP problem is polynomially bounded.

### 4.1 Suggested algorithm

**First-phase algorithm ($P_1(\epsilon)$):** Firstly, we apply the $\epsilon$-constraint approach to our bi-objective problem $P$, where $\epsilon$ parametrically varies from 0 to $Ineq^*$. $Ineq^*$ is the value of the utility-based Gini coefficient obtained by applying the solution to the problem with the efficiency objective only (i.e., problem $(3.1.1), (4) \sim (10)$).

Thus, we define problem $P_1(\epsilon)$ as:

$$\begin{aligned} \text{maximize} \quad & \frac{\sum_{j=1}^{m}\sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t}{\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}\left(\sum_{t=1}^{l_j} x_{ij}^t\right)} && (2) \\ \text{subject to} \quad & \sum_{j=1}^{m-1}\sum_{k>j}^{m} \tilde{d}_{jk}^{+} + \tilde{d}_{jk}^{-} \leq \epsilon \sum_{j=1}^{m}\sum_{t=1}^{b_j} c_j^t \sum_{i=1}^{n} x_{ij}^t && (11) \\ & (4) \backsim (10). \end{aligned}$$

$P_1(\epsilon)$ is an ILFP that we solve using the parametric approach described in §4.2. By solving the problem for various values of $\epsilon$, we can obtain the trade-off curve between the efficiency ratio and the inequity measure (see an example in §6).

**Second-phase algorithm ($P_2(\epsilon)$):** To describe the solutions of our original problem $P$, we define the notion of noninferior solution.

**Definition 9.** *$x^*$=($\boldsymbol{y}$, $\boldsymbol{X}$, $\tilde{\boldsymbol{d}}^+$,$\tilde{\boldsymbol{d}}^-$) is a noninferior solution of $P$ if there exists no other feasible $x$ =($\boldsymbol{y}$, $\boldsymbol{X}$, $\tilde{\boldsymbol{d}}^+$,$\tilde{\boldsymbol{d}}^-$) such that $EFF(x) \geq EFF(x^*)$ and $Ineq(x) \leq Ineq(x^*)$.*

In this second stage, we solve an additional problem to obtain the noninferior solution of $P$.

The integer programming problem $P_2(\epsilon)$ is formulated as follows:

$$\text{minimize} \quad \sum_{j=1}^{m-1} \sum_{k>j}^{m} \tilde{d}_{jk}^{+} + \tilde{d}_{jk}^{-} \tag{12}$$

$$\text{subject to} \quad \frac{\sum_{j=1}^{m} \sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t}{\sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} \left( \sum_{t=1}^{l_j} x_{ij}^t \right)} = Z_1(\epsilon) \tag{13}$$

$$(4) \backsim (11),$$

where $Z_1(\epsilon)$ is an optimal objective function value of $P_1(\epsilon)$ from the first-phase of the algorithm. Constraint (13) ensures that we achieve the minimum inequity value by eliminating some feasible solutions in $P_1(\epsilon)$. The inefficient solutions can also be avoided by augmenting small derivative variables in the objective function (12) (Görmez et al. 2011).

Now, we can characterize the noninferior solutions of $P$ in terms of solutions of $P_2(\epsilon)$, using the following result from Chankong and Haimes (1983):

**Theorem 1** (Chankong and Haimes (1983))**.** *Given $\epsilon \in [0, Ineq^*]$, let ($\boldsymbol{y}$, $\boldsymbol{X}$, $\tilde{\boldsymbol{d}}^+$,$\tilde{\boldsymbol{d}}^-$) be an optimal solution of $P_2(\epsilon)$. Then ($\boldsymbol{y}$, $\boldsymbol{X}$, $\tilde{\boldsymbol{d}}^+$,$\tilde{\boldsymbol{d}}^-$) is a noninferior solution of $P$ if:*
*i) it is a unique solution; or*
*ii) the optimal value of $P_2(\epsilon^0)$ is strictly greater than $EFF$($\boldsymbol{y}$, $\boldsymbol{X}$, $\tilde{\boldsymbol{d}}^+$,$\tilde{\boldsymbol{d}}^-$) for any $\epsilon^0 \leq \epsilon$.*

## 4.2 Parametric algorithms

Note that $P_1(\epsilon)$ is an ILFP problem, and it can be solved by applying the parametric approach, which reduces the original fractional problem to a linear version of the problem (Radzik 1993). The parameterized problem with a parameter $\lambda \in \mathbb{R}$ ($P_\lambda(\epsilon)$) is defined as follows:

$$\text{maximize} \quad f(x) - \lambda g(x), \quad \text{subject to } x \in S, \tag{P$_\lambda$}$$

where $f(x) = \sum_{j=1}^{m} \sum_{t=1}^{l_j} c_j^t \sum_{i=1}^{n} x_{ij}^t$, $g(x) = \sum_{i=1}^{n} f_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} \left( \sum_{t=1}^{l_j} x_{ij}^t \right)$, and $S$ is a feasible set satisfying constraints (4) $\backsim$ (11). A series of the problem $P_\lambda(\epsilon)$ with $\lambda \in \mathbb{R}$ is then solved to generate a sequence of values $\{\lambda_i\}$ that converges to the optimal function value of the original problem $P_1(\epsilon)$. If we denote $h(\lambda)$ as the optimal objective function value of the problem $P_\lambda(\epsilon)$ with a given parameter value $\lambda$, then we know that $h(\lambda)$ is piece-wise linear, convex and decreasing in $\lambda$ (Dinkelbach 1967). Furthermore, the root of $h(\lambda) = 0$ is the optimal objective function value of $P_1(\epsilon)$. The following result gives properties of function $h(\lambda)$ to help find its root.

**Theorem 2** (Stancu-Minasian (2012))**.** *Let $x^*$ be an optimal solution of the integer linear fractional program and let $\lambda^* = \frac{f(x^*)}{g(x^*)}$. Then*

**(a)** $h(\lambda) > 0$ *if and only if* $\lambda < \lambda^*$*,*

**(b)** $h(\lambda) = 0$ *if and only if* $\lambda = \lambda^*$*,*

**(c)** $h(\lambda) < 0$ *if and only if* $\lambda > \lambda^*$*.*

Based on these properties of function $h$, some parametric approaches have been proposed for finding the root of $h(\lambda) = 0$ efficiently. Since the performance of a parametric algorithm mainly depends on the number of iterations and the time required to solve each linear problem ($P_\lambda$), knowing how many iterations are required to run a specific parametric algorithm is critical. In the following subsections, we introduce two parametric approaches: the Newton's and the sub-approximation methods.

**Newton's method:** The Newton's algorithm proposed by Dinkelbach (1967), as described in Table 2, is one of the most popular algorithms for solving ILFP problems. It generates a sequence of lower bounds approaching the optimal function value, $\lambda^*$. If the ILFP consists of binary variables (i.e., LFCO), it has been shown to have a polynomial bound on the number of iterations of $O(n^2 log(n))$, where $n$ is the number of variables of the original problem (Wang et al. 2006). In §4.3, we show a novel bounding result for the Newton's method applied to general ILFP problems.

| | |
|---|---|
| **Input:** | A piece-wise linear, decreasing and convex function $h(\lambda)$ with $\lambda_0$ such that $h(\lambda_0) > 0$, a feasible set of the fractional program $S$, and set $i = 0$. |
| **Step 1.** | Set $\lambda_{i+1} = \dfrac{f(x_i)}{g(x_i)}$, where $x_i = \arg\max_{x \in S}\{f(x) - \lambda_i g(x)\}$ |
| **Step 2.** | If $h(\lambda_{i+1}) = 0$, then set $x^* = x_{i+1}$ and terminate the algorithm. |
| **Step 3.** | If $h(\lambda_{i+1}) \neq 0$, then set $i = i + 1$, and go to Step 1. |
| **Output:** | $\lambda^*$ as the optimal objective value and $x^*$ as an optimal solution to the original problem. |

Table 2: Newton's algorithm (Dinkelbach 1967).

**Sub-approximation method:** A formal description of the sub-approximation algorithm is given in Table 3. In each iteration, an intersection point of the two subgradient lines at the lower and upper bounds is used to improve either the lower or upper bound for $\lambda^*$. The lower and upper bound approximations of the optimal objective value $\lambda^*$ are updated by considering the positions of the intersection point calculated in Step 1 of the algorithm. In the algorithm below, we observe that the lower bound is improved in Steps 2 and 3 and the upper bound is revised in Step 4.

| | |
|---|---|
| **Input:** | A piece-wise linear, decreasing and convex function $h(\lambda)$ defined in $[\rho_0, \gamma_0]$ with $h(\rho_0) > 0$ and $h(\gamma_0) < 0$, and set $i = 0$. |
| **Step 1.** | Compute the intersection point $(u_i, v_i)$ of the two subgradient lines of $h(\lambda)$ at $\rho_i$ and $\gamma_i$. If $\lvert h(u_i)\rvert < \delta$, then set $\lambda^* = u_i$ and $x^* = x_i$, where $x_i = \arg\max_{x \in \mathbb{X}}\{f(x) - u_i g(x)\}$, and terminate the algorithm. |
| **Step 2.** | If $v_i > 0$, then let $\rho_{i+1} = \frac{f(y_i)}{g(y_i)}$, where $y_i = \arg\max_{x \in \mathbb{X}}\{f(x) - \gamma_i g(x)\}$, $\gamma_{i+1} = \gamma_i$, $i = i + 1$, and go to **Step 1.** |
| **Step 3.** | If $v_i < 0$ and $h(u_i) > 0$, then let $\rho_{i+1} = u_i$, $\gamma_{i+1} = \gamma_i$, $i = i + 1$, and go to **Step 1.** |
| **Step 4.** | Otherwise ($v_i < 0$ and $h(u_i) < 0$), then let $\rho_{i+1} = \rho_i$, $\gamma_{i+1} = u_i$, $i = i + 1$, and go to **Step 1.** |
| **Output:** | $\lambda^*$ as the optimal objective value and $x^*$ as an optimal solution to the original problem. |

Table 3: sub-approximation algorithm.

### 4.3 Worst-case analysis

The goal of this section is to find polynomial bounds in the number of iteration for both the Newton's and sub-approximation algorithms to assess the efficiency of both algorithms in solving our problem and, by extension, any ILFP problem. Hereafter, we assume that all variables of ILFP problem are bounded by some number M. We use the following Lemma as a main tool for bounding the number of iterations of both algorithms.

**Lemma 2.** *Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_{\geq 0}$ and let $y_1, \ldots, y_q$ be vectors from $\{-M, -M+1, \ldots, M-1, M\}^n$, where $M$ is some positive integer. If for $\beta \in (0, 1)$ and $i = 1, \ldots, q-1$, $0 < y_{i+1}c \leq \beta y_i c$, then $q = O\left(n\left(\log\left(1 + \frac{1}{\beta}\right) + \log M + \log n\right)\right)$.*

*Proof.* The proof is in the Appendix. □

The following result is straightforward from Lemma 2.

**Corollary 1.** *Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ and let $y_1, \ldots, y_q$ be vectors from $\{0, 1, \ldots, M-1, M\}^n$, where $M$ is some positive integer. If for $\beta \in (0, 1)$ and $i = 1, \ldots, q-1$, $0 < y_{i+1}c \leq \beta y_i c$, then $q = O\left(n\left(1 + \frac{\log Mn}{\log\left(1 + \frac{1}{\beta}\right)}\right)\right)$.*

Based on this result, we can prove a polynomial bound on the number of iterations of the Newton's method to find an optimal solution of ILFP problems. Radzik (1993) proves that the Newton's method finds an optimal solution for an LFCO problem (i.e., an ILFP with binary variables) in $O(n^2 \log^2 n)$ iterations. He first defines a sequence $g_i = g(x_i)$, where $x_i =$

$\arg\max_{x\in S}\{f(x)-\lambda_i g(x)\}$, and analyzes the subsequence of $\{g_i\}$ such that $g_{i+1} \le \frac{1}{2}g_i$. It is then proved that the length of such subsequence is no more than $O(n\log n)$ (Lemma 3.7 of (Radzik 1993)). Moreover, the length of the consecutive subsequence $\{g_i\}$ that satisfies $g_{i+1} \ge \frac{1}{2}g_i$ is also at most $O(n\log n)$ (Lemma 3.8 of (Radzik 1993)). Therefore, his result on the LFCO problem is immediately obtained by combining these two results. We observe that the main idea of the above complexity bound can also be adopted for solving ILFP problems when variables are bounded by some number M. That is, if we set $c = b$ and $y_i = x_i$ from the proof of Lemma 3.7 of Radzik (1993) and use Corollary 1 with $\beta = \frac{1}{2}$, then we can obtain that a length of subsequence $\{g_i\}$ such that $g_{i+1} \le \frac{1}{2}g_i$ is no longer than $O(n\log(Mn))$. Similarly, if we set $c = b$ and $y_i = x_i$ from the proof of Lemma 3.8 of Radzik (1993) and use Corollary 1 with $\beta = \frac{1}{2}$, then we obtain that the consecutive subsequence $\{g_i\}$ such that $g_{i+1} \ge \frac{1}{2}g_i$ is at most $O(n\log(Mn))$. Combining these results immediately implies the following result.

**Proposition 3.** *The Newton's algorithm solves integer linear fractional problems in $O(n^2\log^2(Mn))$ iterations.*

Based on Lemma 2, Corollary 1 and Proposition 3, we can analyze the bound of the number of iterations of the sub-approximation algorithm. As described in Table 3, the lower bound $\{\rho_i\}$ is improved in Steps 2 and 3 and the upper bound $\{\gamma_i\}$ is revised in Step 4. In the following Lemma, we first show that the length of the lower bounds generated by the sub-approximation algorithm is at most the length of the sequence generated by the Newton's algorithm. Next, we prove that the length of upper bounds is limited by the number of intersection points generated in Step 1 of the sub-approximation algorithm.

**Lemma 3.** 1. *The number of lower bounds generated by the sub-approximation algorithm is at most $O(n^2\log^2(Mn))$.*

2. *The number of upper bounds generated by the sub-approximation algorithm is of the order of $O\left(n\left(1+\dfrac{\log Mn}{\log\left(1+\frac{1}{\beta}\right)}\right)\right)$ iterations.*

*Proof.* The proof is in the Appendix. □

Lemma 3 has the following implication on the total length of the sequence generated by the sub-approximation algorithm.

**Theorem 3.** *The sub-approximation algorithm solves integer linear fractional problems in* $O\left(\max\left(n^2\log^2(Mn), n\left(1+\frac{\log Mn}{\log\left(1+\frac{1}{\beta}\right)}\right)\right)\right)$ *iterations.*

This result shows that ILFP problems have a polynomial bound on the number of iterations of the sub-approximation method. To the best of our knowledge, we are the first to show a polynomial bound for this method. We note that a direct theoretical comparison of the bounds of the Newton's and sub-approximation algorithms (i.e., Proposition 3 and Theorem 3) is not applicable because the value of $\beta$ is determined by coefficients of the problem. In the next section, we provide a computational comparison.

# 5 Computational studies

In this section, we present our computational studies of problem $P$, with the proposed two-phase algorithm described in §4. As mentioned, the Newton's method is well known for solving ILFP problems (Schaible and Ibaraki 1983). Thus, part of our computational experiments is directed at studying how the sub-approximation method performs relative to Newton's method. In particular, the first experiment is designed to show a comparative performance of the two-phase algorithm when the Newton's and sub-approximation algorithms are used to solve the ILFP problem in the first-phase. In the second set of experiments, we discuss managerial insights by observing the impact of some system features to the system's solutions. In particular, we study the impact of supply levels, inequity levels, and different objective measures.

All computational experiments were run on a Linux-based workstation with a 2.5 GHz quad-core processor and 32 GB RAM. For both the Newton's and sub-approximation methods, the initial lower bound $\rho_0 = 0$ is used since $h(\rho_0) > 0$ follows from the assumptions of $P$. For the sub-approximation method, the initial upper bound $\gamma_0$ is given by $\gamma_0 = \beta \cdot \lambda_\epsilon^*$, where $\lambda_\epsilon^*$ is an optimal efficiency ratio of $P_2(\epsilon)$ and $\beta > 1$ so that $h(\gamma_0) < 0$ follows from Theorem 2. In our simulations, we specify $\beta = 2.0$ for setting the initial upper bounds and a tolerance level of $\delta = 10^{-6}$ as the termination condition for the algorithms.

## 5.1 Computational efficiency

To compare the performance of the two parametric algorithms (Newton's and sub-approximation methods), we solve a set of randomly generated problems, as described in Table 8 in the Appendix.

The key aspect in these experiments is that the fixed facility costs per each facility location $i = 1, \ldots, n$, are defined as $f(n,i) = e^{1.5 \times i \times ln10/n}$. The value of the fixed facility cost appears in the denominator of the efficiency ratio as described in Equation (2), which affects the steepness of the $h(\lambda)$. We observe that the facility costs generated by $f(n,i)$ make the $h(\lambda)$ function steeper at the left portion of the curve so that the problem becomes more difficult to solve. We expect that this experimental setting can help to decide which algorithm is more efficient for solving the ill-conditioned problems.

The test results are summarized for different pairs of $(\alpha, \epsilon)$ in Table 4 and in the electronic companion, in Tables 10, 11 and 12. In addition to CPU time (in seconds), the number of parameterized problems ($\mathrm{P}_\lambda$) solved is used as the measure of an algorithm's performance and is referred to as the number of function calls (func. calls). The optimal objective function values at the initial lower and upper bounds, $h(\rho_0)$ and $h(\gamma_0)$, are assumed to be given in the beginning of an algorithm and do not need function calls for computation. We observe that for both algorithms, the number of function calls and CPU time increase as the problem size becomes larger (i.e., as $n$ and $m$ increase). This is not surprising, because these problems have a steeper function $h$ curve as $n$ and $m$ increase. This causes difficulty in evaluating function values at approximated points. Additionally, in this setting and for the same $(n,m)$ values, we observe that the sub-approximation algorithm tends to require fewer function calls and less CPU time on average than that of the Newton's algorithm. This indicates that the sub-approximation algorithm seems to perform better as the problem becomes difficult to solve. For example, for particularly steep functions that appear in the denominator of the fractional objective. This is consistent with the results of an empirical study that compares both parametric methods for IFLP problems (Park 2016).

## 5.2 Managerial insights

Given the proven efficiency of our method, all remaining experiments in this paper use the two-phase algorithm with the sub-approximation method for solving problem $P$. Next, to infer useful managerial insights, we conduct a series of simulation experiments by observing the effects of applying changes to the supply levels, the allowed inequity levels, and the objectives functions on the network. Table 8 of the Appendix details the values of all parameters of this experiment.

| | Newton | | Sub-approximation | |
|---|---|---|---|---|
| (n,m) | func. calls | CPU time | func. calls | CPU time |
| (2,10) | 3.0 | 1.6 | 3.0 | 2.3 |
| (2,50) | 3.8 | 331.6 | 3.5 | 317.0 |
| (2,100) | 4.0 | 4,294.2 | 4.0 | 4,248.6 |
| (5,10) | 3.5 | 2.9 | 2.5 | 2.9 |
| (5,50) | 6.2 | 2,085.6 | 5.8 | 1,338.6 |
| (5,100) | 6.0 | 11,567.8 | 5.0 | 4,359.7 |
| (10,30) | 6.5 | 1,427.0 | 4.5 | 1,135.0 |
| (10,50) | 6.3 | 982.4 | 4.3 | 619.1 |
| (20,30) | 5.3 | 966.6 | 3.0 | 910.6 |
| (20,50) | 6.0 | 6,842.4 | 3.0 | 6,430.1 |
| (30,30) | 6.4 | 1,178.1 | 4.2 | 950.0 |
| (30,50) | 7.0 | 6,095.2 | 5.0 | 5,192.4 |

Table 4: Comparison of performances of two-phase algorithm with Newton's and sub-approximation for $\alpha = 0.3$ and $\epsilon = 0.3$. *Note*: Tables 10, 11 and 12 in the electronic companion show the results with other combinations of $(\alpha, \epsilon)$.

### 5.2.1 Impact of supply levels

Table 5 shows different solutions to problem $P$ with a different combination of maximum allowable inequity levels ($\epsilon$) and supply factors ($\alpha$). A first observation of Table 5 shows that as supply increases, the optimal solution tends to have more open facilities. For example, when $\alpha$ is changed from 0.6 (60%) to 0.8 (80%), an additional facility is open. This result is explained because higher facility location costs, due to the opening of more facilities, can be incurred and offset by higher levels of utility (due to higher supply levels being delivered) and lower levels of transportation costs (additional warehouses imply less distance between warehouses and demand locations).

In general, from Table 5 we observe that if the available amount of supplies increases, the efficiency can increase or decrease but inequity always decreases. Due to the nature of the efficiency ratio (where the utility is in the numerator and the fixed and transportation costs are in the denominator), the whole ratio value will be evaluated by marginal changes in these three values. Generally, a larger supply generates less inequity because it enlarges the feasible set, thereby providing more opportunities to evenly split the overall supply according to different utility levels.

| $\epsilon$ | $\alpha$ | EFF | Ineq | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $\alpha$ | EFF | Ineq | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.2 | 0.3714 | 0.0982 | 0 | 1 | 1 | 1 | 1 | 0.6 | 0.4808 | 0.1 | 1 | 1 | 1 | 1 | 1 |
| | 0.4 | 0.4789 | 0.1 | 1 | 1 | 1 | 1 | 1 | 0.8 | 0.4188 | 0.0995 | 1 | 1 | 1 | 1 | 1 |
| 0.3 | 0.2 | 0.6056 | 0.2999 | 0 | 1 | 1 | 0 | 1 | 0.6 | 0.5237 | 0.2347 | 1 | 1 | 1 | 0 | 1 |
| | 0.4 | 0.6400 | 0.2998 | 0 | 1 | 1 | 1 | 1 | 0.8 | 0.4217 | 0.1390 | 1 | 1 | 1 | 1 | 1 |
| 0.5 | 0.2 | 0.7961 | 0.498 | 0 | 1 | 1 | 0 | 1 | 0.6 | 0.5237 | 0.2355 | 1 | 1 | 1 | 1 | 1 |
| | 0.4 | 0.6562 | 0.382 | 0 | 1 | 1 | 0 | 1 | 0.8 | 0.4217 | 0.1390 | 1 | 1 | 1 | 1 | 1 |
| 0.7 | 0.2 | 0.8551 | 0.6388 | 0 | 1 | 1 | 0 | 0 | 0.6 | 0.5237 | 0.2347 | 1 | 1 | 1 | 0 | 1 |
| | 0.4 | 0.6562 | 0.382 | 0 | 1 | 1 | 0 | 1 | 0.8 | 0.4217 | 0.1390 | 1 | 1 | 1 | 1 | 1 |

Table 5: Results of varying maximum allowable inequity level ($\epsilon$) and supplies ($\alpha$) ($f = 100$).

### 5.2.2 Impact of inequity levels

We are also interested in studying how open facility locations are affected by different levels of allowable inequity. In addition to Table 5, Figure 4 depicts location details (demand locations (dots) and DCs (stars, circled if open)) of the results in Table 5 with $\alpha = 0.4$. The most interesting observation is that for the same experiment with decreasing maximum allowable inequity, the number of open facilities of the optimal solution tends to increase. This implies that considering a network with an increased number of open facilities can facilitate reduced levels of inequity. This is due to the fact that with more facilities open, more supply can reach different demand points at lower transportation costs despite the larger fixed location costs. Our case study results also provide evidence of this observation (§6.1).

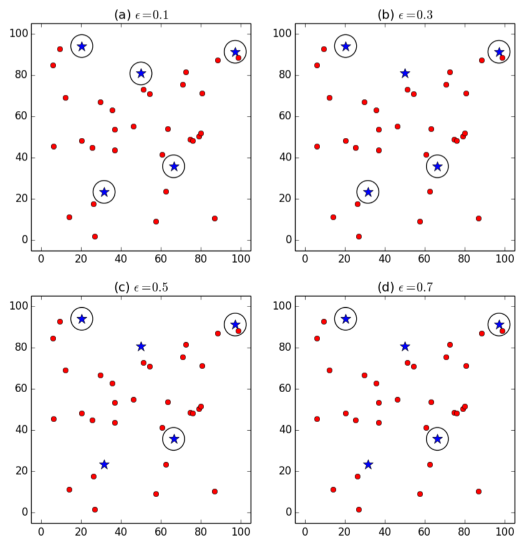


Figure 4: Facility location decisions with varying maximum allowable inequity level, $\epsilon$.

### 5.2.3 Impact of varying objective functions

In this section, we explore the impact of different objectives on the optimal solution. We start by solving problem $P$ with three different efficiency measures: maximizing total utility (*util.*), minimizing total cost (*cost*), and maximizing the efficiency ratio ($EFF$). Each row of Table 6 shows an optimal solution of problem $P$ with a different combination of maximum allowable inequity levels ($\epsilon$) and the last five columns include the facility location decisions. First, we observe that all facilities need to open for every case in the utility maximization problems. This implies that if cost is not considered in the decision-making process, unnecessary facilities might be open substantially increasing supply chain costs. Second, we observe that when minimizing costs the optimal inequity is the highest compared to the optimal inequity of the other two problems (utility and efficiency maximization). Thus, this experiment (Table 6) shows that using the efficiency ratio as the measure of efficiency provides well-balanced optimal costs, total utilities, and inequity values compared to the other two options.

| $\epsilon$ | objective | Ineq | Utility | Cost | EFF | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | util. | 0.10 | 1369 | 5128 | 0.2670 | 1 | 1 | 1 | 1 | 1 |
| | cost | 0.10 | 698 | 2193 | 0.3183 | 1 | 1 | 0 | 1 | 0 |
| | eff. | 0.10 | 1344 | 2444 | 0.5499 | 1 | 1 | 1 | 1 | 0 |
| 0.3 | util. | 0.29 | 1472 | 5530 | 0.2662 | 1 | 1 | 1 | 1 | 1 |
| | cost | 0.30 | 945 | 1847 | 0.5116 | 1 | 1 | 1 | 1 | 0 |
| | eff. | 0.30 | 1303 | 2020 | 0.6450 | 0 | 1 | 1 | 1 | 0 |
| 0.5 | util. | 0.29 | 1472 | 4964 | 0.2965 | 1 | 1 | 1 | 1 | 1 |
| | cost | 0.50 | 659 | 1602 | 0.4114 | 0 | 1 | 1 | 1 | 0 |
| | eff. | 0.46 | 1174 | 1733 | 0.6774 | 0 | 1 | 1 | 1 | 0 |
| 0.7 | util. | 0.29 | 1472 | 5571 | 0.2642 | 1 | 1 | 1 | 1 | 1 |
| | cost | 0.64 | 677 | 1578 | 0.4290 | 0 | 1 | 1 | 1 | 0 |
| | eff. | 0.46 | 1174 | 1733 | 0.6774 | 0 | 1 | 1 | 1 | 0 |

Table 6: Comparison of three different efficiency objectives when $\alpha = 0.4$ and $f = 100$.

In addition, we discuss the impact of considering different utility functions for each demand location compared with the case of considering all population equally. This translates into comparing the ”no utility” problem (with objectives $G_{Mandell}$ and $\frac{total\ supply}{cost}$) with problem $P$ (with objectives $Ineq$ and $EFF$). In Table 7 we solve both problems, where the most interesting observations are related to the number of opened DCs, the total utility covered, and the number of demand locations partially supplied. In particular, when solving the ”no utility” problem more DCs are utilized and less total utility is generated compared with solving $P$. Also, when observing the number of demand locations partially supplied, the solutions of $P$ have a smaller range of locations supplied between small $\epsilon$ ($\epsilon = 0.1$) and large $\epsilon$ ($\epsilon = 0.7$). For example, for varying allowable levels of inequity ($\epsilon$), for the ”no utility” problem the number of partially supplied locations varies from 30 to 12 which also translates into larger costs, but when considering $P$ the number of partially supplied locations only varies from 23 to 12. This is due to the concavity of the utility functions that compels problem $P$ to concentrate on supplying to those demand locations with steeper utility functions (i.e. the population that more critically needs the supply).

| $\epsilon$ | Problem | $G_{Mandell}$ | $Ineq$ | Actual Utility | Cost | $EFF$ | num. open DCs | num. partially supplied locations |
|---|---|---|---|---|---|---|---|---|
| 0.1 | no utility | 0.0899 | 0.2178 | 917 | 2096 | 0.4375 | 4 | 30 |
| 0.2 | no utility | 0.1966 | 0.2203 | 972 | 1934 | 0.5026 | 4 | 27 |
| 0.3 | no utility | 0.2982 | 0.2145 | 995 | 1805 | 0.5512 | 4 | 22 |
| 0.4 | no utility | 0.3999 | 0.2325 | 999 | 1670 | 0.5982 | 3 | 20 |
| 0.5 | no utility | 0.4998 | 0.2463 | 1019 | 1578 | 0.6458 | 3 | 18 |
| 0.6 | no utility | 0.5998 | 0.2699 | 990 | 1432 | 0.6913 | 3 | 14 |
| 0.7 | no utility | 0.6617 | 0.2761 | 1027 | 1436 | 0.7152 | 3 | 12 |
| 0.1 | $P$ | 0.367 | 0.0996 | 1022 | 1949 | 0.5244 | 4 | 23 |
| 0.2 | $P$ | 0.4388 | 0.1981 | 1029 | 1759 | 0.5850 | 4 | 20 |
| 0.3 | $P$ | 0.528 | 0.3 | 1036 | 1614 | 0.6419 | 3 | 17 |
| 0.4 | $P$ | 0.6102 | 0.3987 | 1009 | 1479 | 0.6822 | 3 | 15 |
| 0.5 | $P$ | 0.6648 | 0.4987 | 1033 | 1446 | 0.7144 | 3 | 12 |
| 0.6 | $P$ | 0.6616 | 0.5045 | 1027 | 1436 | 0.7152 | 3 | 12 |
| 0.7 | $P$ | 0.6616 | 0.5045 | 1027 | 1436 | 0.7152 | 3 | 12 |

Table 7: Comparison between the "no utility" problem and $P$ when $\alpha = 0.3$ and $f = 100$. *Note:* Table 14 in the electronic companion shows the results with $\alpha = 0.5$

# 6 Case study in food aid distribution

In this section, we illustrate the value of the model and tools suggested in this paper in the context of a food aid distribution problem. In particular, we consider the problem faced by the World Food Programme (WFP) in Angola during its post-civil war period (from 2002 to 2005) within the Protracted Relief and Recovery Operations (PRRO) project (Report 2005).

Two food security surveys were carried out in 2005 in the most food vulnerable regions of Angola: central highlands (WFP/VAM 2005) and south-eastern area of Angola (WFP/CFSVA 2005). Among other things, these surveys assessed humanitarian needs in the food sector and evaluated these needs by region. Each household (or person) is cataloged in one of four different vulnerability profiles that are named from most insecure to most secure: *food insecure*, *high vulnerable*, *moderate vulnerable*, and *low vulnerable* (WFP/VAM 2005). Some of the major indicators used to create these homogeneous groups are: assets, displacement, exposure to risks, food intake, education parameters, and sex of the head of the household. Between both surveys, there is a total of 13 different regions studied, where all regions of the central highlands have a portion of their population cataloged as *food insecure*, whereas for the south-eastern area no region has population cataloged at the highest criticality level, and all regions start in the *high vulnerable* range. Based on this classification, we propose the following utility function per region:

$$u_j(x_j) = u_j(x_j^1, x_j^2, x_j^3, x_j^4) = c^1 x_j^1 + c^2 x_j^2 + c^3 x_j^3 + c^4 x_j^4,$$

where $c^1 = 4 > c^2 = 3 > c^3 = 2 > c^4 = 1 > 0$, $x_j^1 \in [0, a_j^1]$ food insecure population range, $x_j^2 \in [0, a_j^2 - a_j^1]$ high vulnerable population range, $x_j^3 \in [0, a_j^3 - a_j^2]$ moderate vulnerable population range, and $x_j^4 \in [0, b_j - a_j^3]$ low vulnerable population range. Note that each demand region has a different utility function, as described in Figure 5, based on the different breakpoints of each demand location's piece-wise linear utility function. These breakpoints are determined based on the population assigned to each of the four vulnerability profile groups per demand location. Note that each coefficient (slope) of the utility function corresponds to a vulnerability group. These coefficient values represent a positive arbitrary decreasing sequence required to preserve the concavity of the utility function and we assume that these values do not change depending on the location because the characterization of each vulnerability group in the WFP reports does not change depending on the region of the country. We should note that, according to this definition, the utility functions of the regions in the south-eastern area start with a lower coefficient than the ones in the central area due to not having population of the highest vulnerability level. Further regional differences are considered in our analysis. In particular, we consider the proportion of population in each vulnerability group per region when using the proposed $Ineq$ measure. This inequity measure depends on the relative utility functions of the demand locations (introduced in §3.1.2), whose coefficients change depending on each location $j$ because $\bar{u}_j$ are different for each $j$:

$$\frac{u_j(x_j)}{\bar{u}_j} = \frac{c^1}{\bar{u}_j}x_j^1 + \frac{c^2}{\bar{u}_j}x_j^2 + \frac{c^3}{\bar{u}_j}x_j^3 + \frac{c^4}{\bar{u}_j}x_j^4 = \tilde{c}_j^1 x_j^1 + \tilde{c}_j^2 x_j^2 + \tilde{c}_j^3 x_j^3 + \tilde{c}_j^4 x_j^4,$$

where $\bar{u}_j = u_j(b_j) = c^1 a_j^1 + c^2 a_j^2 + c^3 a_j^3 + c^4 b_j$.

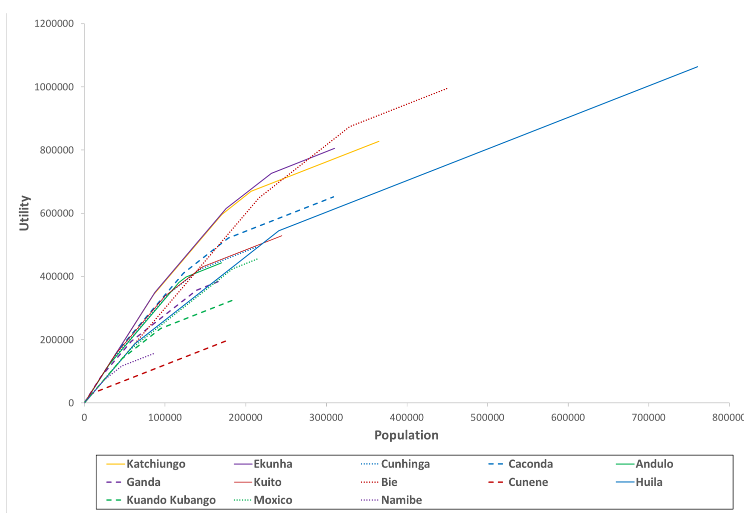


Figure 5: Utility function $(u_j(x_j))$ of each demand region.

Further location differences could be highlighted and considered when building the utility functions. For example, by considering factors that determine the capacity of recovery for each region such as each region's wealth, health, education levels, climate or external aid received for other purposes, etc. This regional information can be incorporated into the utility functions by normalizing their coefficients by these factors (Iancu and Trichakis 2014).

All demand locations (i.e., 13 regions) can be supplied by three WFP warehouses located in Luanda, Lobito, and Lubango, which are the major point of distribution of the country for WFP (See map in Figure 8 of the electronic companion). The capacity of Luanda, Lobito, and Lubango was of 19,000, 24,000, and 6,000 metric tonnes (MT), respectively. Our approach can help WFP solve their location-distribution problem, by determining which of these three warehouses to use for food distribution and what are the optimal supply allocations between warehouses and demand locations when food baskets are not enough to cover all total population. The PRRO project amount available in 2004 was of 121,000 MT. We assume that WFP distributes food every month and the average weight of the food aid per person is 25 kg. The capacity $s_i$ in MT is converted to a maximum number of people that each warehouse can supply and set to $s_1 = 633,3333$, $s_2 = 800,000$, and $s_3 = 200,000$, respectively. The fixed facility location cost $f_i$ is assumed to be increased in capacity and sets proportionally to the capacity as $f_1 = 5.67 \times 10^7$, $f_2 = 7.2 \times 10^7$, and $f_3 = 1.8 \times 10^7$. The transportation cost is set to equal the distance in kilometers currently provided by Google Maps. The parameter values for constructing the utility functions and demands can be directly extracted from reports WFP/VAM (2005) and WFP/CFSVA (2005) and are reported in Table 15 of the electronic companion alongside with the distances.

## 6.1 Results

If we solve the problem for WFP with different levels of maximum allowable inequity ($\epsilon$) from 0.05 to 0.75, we can observe some important key results (Figures 9 and 10 of the electronic companion present the detailed solutions). Related to facility location decisions, our analysis recommends that the warehouse in Luanda should never be used. The main reasons are that this warehouse is the one located the farthest away from the demand locations which are in the central and south-eastern areas of the country, and it is the second most expensive in terms of facility location costs. The other two warehouses are used up until $\epsilon = 0.7$ when only the warehouse in Lobito is used. Going from more facilities to less when lowering equity requirements has also been observed in §5.2.2.

It is also interesting to observe which locations are getting reduced supply as $\epsilon$ increases. As expected, at the smallest $\epsilon$ value, all regions are being partially supplied. Then, the first four demand locations that are completely cut from receiving supply as larger inequity is allowed are locations in the south-eastern area, where no population is cataloged as food insecure. These regions listed in discarding order are Moxico, Bie, Cunene, and Kuando Kubango. Nonetheless, transportation costs also play a key role in our model because the next regions to be discarded as $\epsilon$ increases are in the central area, which have population catalogued as food insecure, while the recommendation is to still keep supplying two regions in the south-eastern region. This is because of the close proximity of these two regions to the Lubango's warehouse. For very high $\epsilon$ value (i.e. when prioritizing maximum $EFF$), the recommendation is to focus on four demand regions with high volume of food insecure people and close proximity to the sole warehouse in Lobito.

### 6.2 Trade-off Curve

We construct a trade-off curve for this case study problem in Figure 6. The horizontal axis represents the inequity objective ($Ineq$) while the vertical axis represents the efficiency ratio ($EFF$). Each point on the curve is attained by the two-phase algorithm with varying the parameter $\epsilon$ from 0.05 to 0.75. We observe that the efficiency ratio increases for solutions with increased inequity value. In addition, the steepness of the curve indicates the marginal impact of inequity on efficiency. This implies that some decisions to improve inequity may be recommended if the marginal impact on efficiency is small. For instance, we may decide to change from Solution 14 to 13 (or 12) since inequity would improve 17.5% by worsening of efficiency by only 1.05%. Similarly, decision makers may sacrifice inequity if substantial improvements in efficiency can be attained. The electronic companion contains Table 16, which gives a detailed description of each solution (point).

### 6.3 Comparison with the no utility case

In this section, we use the case study setting to compare our proposed problem ($P$) with the problem that does not consider utility functions. Note that this alternative problem has $G_{Mandell}$ as the inequity measure and $\frac{total\ supply}{cost}$ as the efficiency measure. In Figure 7, we compare the solutions of both problems in terms of the relative utility at each demand region (Figures 11 and 12 in the electronic companion provide the detailed solutions of the no utility problem). The graph with $\epsilon$=0.05 (top) provides the solutions when both inequity and efficiency are relevant. We note that the solutions for both problems distribute supplies to all 13 demand regions but problem $P$ has more balanced relative utility values across the different demand regions as it is

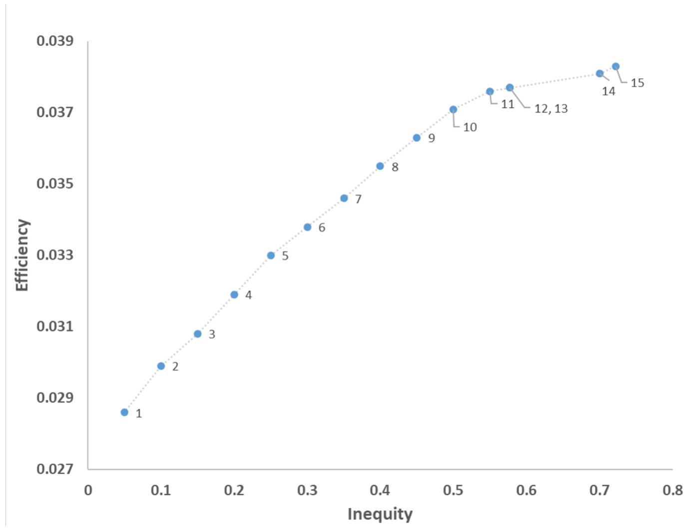


Figure 6: Efficiency vs. inequity trade-off curve.

featured in $Ineq$. We show a second solution that relaxes inequity a bit more ($\epsilon = 0.4$) and gives efficiency more protagonism. This case provides solutions that are quite different between the two problems. The solution from $P$ gives priority to certain more critical demand regions (central area) while also taking into account costs, as defined in $EFF = \frac{total\ utility}{costs}$. These two factors explain why four out of six demand regions in the south-eastern area are not being supplied. The no utility problem solution is still supplying 12 out of the 13 demand regions because it weights all individuals within and across locations equally. The third graph (bottom) represents the case of prioritizing efficiency while keeping a low level of equity ($\epsilon$=0.75). In this case, the solutions are also quite different because our suggested problem solution delivers all supply to regions with food insecure population, while the no utility case solution does not consider this factor and prioritizes cost minimization. There are also differences in terms of keeping warehouses open, the no utility problem keeps both facilities in Lobito and Lubango open for all inequity level cases because it does not consider ceasing to deliver in the south-eastern area due to not considering heterogeneity of vulnerability levels at different demand regions.

# 7 Concluding remarks

This paper proposes a model and solution approach to the location-distribution problem in not-for-profit settings where a limited volume of supply has to be allocated to different demand regions. In this context, we claim that to reflect this real setting not every individual (diminishing returns

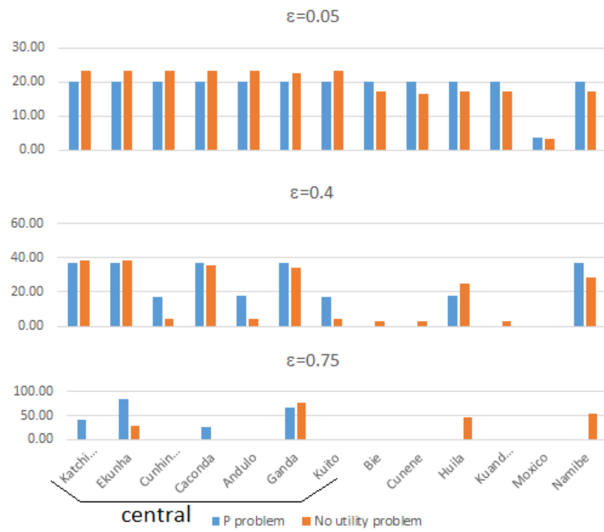


Figure 7: Relative utilities per demand region for both problems.

effect) or demand region (location effect) should be treated equally. For this reason, we suggest to include the utility function of each demand region depending on supply received as the tool to represent heterogeneity. A combination of efficiency and equity goals is suggested, where each demand location's utility function is part of both objectives. In particular, an efficiency ratio and a utility-based Gini coefficient measure are proposed, where the second one is a novel and broader version of the popular inequity measure, the Gini coefficient. Due to the fact that our efficiency and inequity objectives have a fractional form, the problem presented is an integer linear fractional programming (ILFP) problem.

To solve ILFP problems, we recommend a two-stage resolution approach where we theoretically analyze and computationally compare the speed of convergence of two different parametric approaches, the Newton's and the sub-approximation methods. We develop a new complexity bound for the sub-approximation algorithm applied to ILFP problems that states that a polynomial sequence of iterations is sufficient for solving ILFP problems. We also extend a well-known bounding result for the Newton's method applied to ILFP with binary variables to the general case with bounded integer variables. Furthermore, we explore the differences in computational efficiency between both methods for a set of difficult-to-solve facility location cost functions and confirm that the sub-approximation method performs well. The applicability of the model and solution approach is illustrated in a case study of food aid distribution in Angola based on the World Food Programme (WFP) food aid programs.

Going forward, there are interesting extensions and new directions for this work. One is to study the computational and practical implications of varying the number of breakpoints of the utility functions. Another interesting extension is to explore a multi-commodity case in which different types of supplies might have to be characterized differently when defining the utility functions.

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# Appendix

## A1. Proofs

*Proof of Proposition 1.* From equation (3) of Mandell (1991) the Gini coefficient is defined as:

$$G_{Mandell}(x) = \frac{\sum_{j=1}^{m}\sum_{k=1}^{m} b_j b_k |\frac{x_j}{b_j} - \frac{x_k}{b_k}|}{2(\sum_{j=1}^{m} b_j)^2 (\frac{\sum_{j=1}^{m} x_j}{\sum_{j=1}^{m} b_j})} \tag{14}$$

Now, we modify the above expression to the case of a utility-based measure by substituting all assigned relative allocations ($\frac{x_j}{b_j}$) by their relative utilities ($\frac{u(x_j)}{\bar{u}_j}$), where sub-indices are assigned in non-decreasing order of each demand location's relative utility. The goal of the following derivation is to connect this initial expression of $G$ with the graphical definition of the utility-based Gini coefficient presented in Definition 5.

$$\begin{aligned} G_u(x) &= \frac{\sum_{j=1}^{m}\sum_{k=1}^{m} \bar{u}_j \bar{u}_k |\frac{u_j(x_j)}{\bar{u}_j} - \frac{u_k(x_k)}{\bar{u}_k}|}{2(\sum_{j=1}^{m} \bar{u}_j)^2 \bar{u}} = \frac{\sum_{j=1}^{m}\sum_{k\leq j} \bar{u}_j \bar{u}_k \left(\frac{u_j(x_j)}{\bar{u}_j} - \frac{u_k(x_k)}{\bar{u}_k}\right)}{(\sum_{j=1}^{m} \bar{u}_j)^2 \bar{u}} \\ &= \frac{\sum_{j=1}^{m}\sum_{k\leq j}(\bar{u}_k u_j(x_j) - \bar{u}_j u_k(x_k))}{(\sum_{j=1}^{m} \bar{u}_j)^2 \bar{u}} = \frac{\sum_{j=1}^{m}(u_j(x_j)\sum_{k=1}^{j} \bar{u}_k - \bar{u}_j \sum_{k=1}^{j} u_k(x_k))}{(\sum_{j=1}^{m} \bar{u}_j)^2 \bar{u}} \\ &= \frac{\sum_{j=1}^{m}(u_j(x_j)\sum_{k=1}^{j} \bar{u}_k - \bar{u}_j \cdot \left(\sum_{j=1}^{m} \bar{u}_j\right) \cdot \bar{u} \cdot \Phi_j)}{(\sum_{j=1}^{m} \bar{u}_j)^2 \bar{u}} \end{aligned} \tag{15}$$

Equation (15) follows from the definition: $\Phi_j = \frac{\sum_{k=1}^{j} u_k(x_k)}{\sum_{k=1}^{m} u_k(x_k)} = \frac{\sum_{k=1}^{j} u_k(x_k)}{\left(\sum_{j=1}^{m} \bar{u}_j\right)\cdot \bar{u}}$ where $\bar{u} = \frac{\sum_{j=1}^{m} u_j(x_j)}{\sum_{j=1}^{m} \bar{u}_j}$. Now, since we have $\Phi_j - \Phi_{j-1} = \frac{u_j(x_j)}{\sum_{k=1}^{m} u_k(x_k)} = \frac{u_j(x_j)}{(\sum_{k=1}^{m} \bar{u}_k)\cdot \bar{u}}$, it yields,

$$\begin{aligned} G_u(x) &= \sum_{j=1}^{m}\left(F_j(\Phi_j - \Phi_{j-1}) - \frac{\bar{u}_j}{\sum_{j=1}^{m} \bar{u}_j}\Phi_j\right) = \sum_{j=1}^{m}\left(\left(F_j - \frac{\bar{u}_j}{\sum_{j=1}^{m} \bar{u}_j}\right)\Phi_j - F_j \Phi_{j-1}\right) \\ &= \sum_{j=1}^{m}(F_{j-1}\Phi_j - F_j \Phi_{j-1}) \end{aligned} \tag{16}$$

Equation (16) is proven to be the ratio of the area between the Lorenz curve and the line of equality to the area of the triangle below this line (Rao 1969). At last, we note that the utility-based Gini coefficient $G$ can be rewritten as $\frac{\sum_{j=1}^{m-1}\sum_{k>j} |\bar{u}_k u_j(x_j) - \bar{u}_j u_k(x_k)|}{\sum_{j=1}^{m} \bar{u}_j \sum_{j=1}^{m} u_j(x_j)}$. □

*Proof of Proposition 2.* Related to the principle of transfers property, we can show that this property is satisfied only when the utility function is the same across the different locations. So if we have $u_j(x_j) = u(x_j)$ for all $j$, we can easily prove that the principle of transfers is satisfied, i.e. $Ineq(\dots, x_j + \epsilon, \dots, x_{\tilde{j}} - \epsilon, \dots) \leq Ineq(\dots, x_j, \dots, x_{\tilde{j}}, \dots)$ for $\frac{u_j(x_j)}{\bar{u}_j} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}}$ with a small transfer $\epsilon > 0$. We use this equivalent form of $Ineq$, $Ineq = \frac{\sum_{\tilde{j}=1}^{m}\sum_{j\leq \tilde{j}} \bar{u}_{\tilde{j}} \bar{u}_j \left(\frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}} - \frac{u_j(x_j)}{\bar{u}_j}\right)}{(\sum_{\tilde{j}=1}^{m} \bar{u}_{\tilde{j}})\bar{u}}$ to prove this result. Due to using the same utility function and a sufficiently small positive $\epsilon$, we can order the relative utilities this way $\frac{u_j(x_j)}{\bar{u}_j} \leq \frac{u_j(x_j+\epsilon)}{\bar{u}_j} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}}-\epsilon)}{\bar{u}_{\tilde{j}}} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}}$. This implies that $(\frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}} - \frac{u_j(x_j)}{\bar{u}_j}) \geq (\frac{u_{\tilde{j}}(x_{\tilde{j}}-\epsilon)}{\bar{u}_{\tilde{j}}} - \frac{u_j(x_j+\epsilon)}{\bar{u}_j})$ which implies that $Ineq(\ldots, x_j+\epsilon, \ldots, x_{\tilde{j}}-\epsilon, \ldots) \leq Ineq(\ldots, x_j, \ldots, x_{\tilde{j}}, \ldots)$. However, note that when demand locations have different utility functions this ordering $\frac{u_j(x_j)}{\bar{u}_j} \leq \frac{u_j(x_j+\epsilon)}{\bar{u}_j} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}}-\epsilon)}{\bar{u}_{\tilde{j}}} \leq \frac{u_{\tilde{j}}(x_{\tilde{j}})}{\bar{u}_{\tilde{j}}}$ cannot be guaranteed.

Now, we are focusing on the scale invariance property to prove that this is only satisfied for linear functions (i.e., $u_j(x_j) = a_j x_j$). From the family of utility functions characterized in Definition 2, linear functions are the only ones that satisfy the scale invariant property function $u_j(Kx_j) = Ku_j(x_j)$. This property guaranties that $Ineq$ satisfies the scale invariance principle because $Ineq$ has a linear fractional form with respect to $u_j(x_j)$.

$$Ineq(Kx) = \frac{\sum_{j=1}^{m-1}\sum_{k>j}^{m} |\bar{u}_k u_j(Kx_j) - \bar{u}_j u_k(Kx_k)|}{\sum_{j=1}^{m} u_j(Kx_j)} = \frac{K}{K} Ineq(x) = Ineq(x).$$

□

*Proof of Lemma 2.* Consider the following polyhedron: $P = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : (\bar{y}_i - \frac{1}{\beta}\bar{y}_{i+1})z \geq 0 \quad \text{for } i = 1, \ldots, q-1, \bar{y}_q z = 1, Iz \geq 0\}$, where $I$ is a $n$-by-$n$ identity matrix. Let $A$ and $b$ denote the coefficient matrix and the right-hand-side vector of the system defining polyhedron $P$, respectively. This $P$ is not empty, since $z = \dfrac{\bar{c}}{\bar{y}_q \bar{c}}$ satisfies all constraints. Moreover, due to rank($A$)=$n$, $P$ includes the vertices. Therefore, there is a vertex $\tilde{c} \in P$ such that $\tilde{A}\tilde{c} = \tilde{b}$ for some subsystem $\tilde{A}z \leq \tilde{b}$ of $Az \leq b$ with invertible $n$-by-$n$ matrix $\tilde{A}$. By Cramer's rule, we have $\tilde{c}_i = \dfrac{\det \tilde{A}_i}{\det \tilde{A}} \quad \forall i = 1, \ldots, p$, where $\tilde{A}_i$ is the matrix formed by replacing the $i$-th column with $\tilde{b}$. Recall that the determinant of an $n$-by-$n$ matrix L is computed as $\det(L) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} l_{i,\sigma_i}$, where $S_n$ is a set of all permutations $\sigma$ of the set $\{1, 2, \ldots, n\}$. Thus, $|\det(L)| \leq l^n n!$, where $l = \max_{i,j=1,\ldots,n} |l_{ij}|$. Since the entries of $A$ and $b$ are from interval $\left[-\left(1+\frac{1}{\beta}\right)M, \left(1+\frac{1}{\beta}\right)M\right]$, $|\det(\tilde{A}_i)| \leq \left(\left(1+\frac{1}{\beta}\right)M\right)^n n!$ for $i = 1, \ldots, n$. Also, $|\det(\tilde{A})| \geq 1$. Thus, we have $\tilde{c}_i \leq \left(\left(1+\frac{1}{\beta}\right)M\right)^n n!$ for $i = 1, \ldots, n$. This yields,

$$\bar{y}_j \tilde{c} \leq M \sum_{i=1}^{n} \tilde{c}_i \leq M \cdot n \cdot \left(\left(1+\frac{1}{\beta}\right)M\right)^n \cdot n! \quad \text{for } j = 1, \ldots, m.$$

Finally, we obtain,

$$1 = \bar{y}_q \tilde{c} \leq \beta^{q-1} \bar{y}_1 \tilde{c} \leq \beta^{q-1} \cdot \left(1+\frac{1}{\beta}\right)^n \cdot M^{n+1} \cdot (n+1)!$$

$$\leftrightarrow \left(\frac{1}{\beta}\right)^{q-1} \leq \left(1+\frac{1}{\beta}\right)^n \cdot M^{n+1} \cdot (n+1)!$$

$$\leftrightarrow (q-1)\log\left(\frac{1}{\beta}\right) \leq (q-1)\log\left(1+\frac{1}{\beta}\right) \leq \log\left(\left(1+\frac{1}{\beta}\right)^n \cdot M^{n+1} \cdot (n+1)!\right)$$

So, $q = O\left(n\left(1+\frac{\log Mn}{\log\left(1+\frac{1}{\beta}\right)}\right)\right)$. □

*Proof of Lemma 3.* The lower bound is updated in Steps 2 or 3 of the algorithm. For both cases, the updated lower bound is closer to $\lambda^*$ than one that can be obtained by the Newton's algorithm at the current lower bound. Thus, we conclude that a number of lower bounds generated by sub-approximation algorithm is less than number of sequences solely generated by Newton's algorithm.

Related to the upper bound, let $i_1 < i_2 < \ldots < i_k$ be the iteration in which the upper bound $\{\gamma_i\}$ is generated. Let $\hat{x}_{i_k}$ and $\hat{x}_{i_{k+1}}$ be the values of the x-coordinate of the intersection points in iteration $i_k$ and $i_{k+1}$, respectively. Note that the x-coordinate value of the intersection point in iteration $i$ is computed as, $\hat{x}_i = \frac{f(x_i)-f(y_i)}{g(x_i)-g(y_i)}$, where $x_i = \arg\max_{x\in\mathbb{X}}\{f(x)-\rho_i g(x)\}$ and $y_i = \arg\max_{x\in\mathbb{X}}\{f(x)-\gamma_i g(x)\}$, i.e., $x_i$ and $y_i$ are optimal solutions of linearized problem at the lower bound and upper bound, respectively. Because $0 < \lambda^* \leq \hat{x}_{i_{k+1}} < \hat{x}_{i_k}$, this can be written as,

$$0 < \frac{f(x_{i_{k+1}})-f(y_{i_{k+1}})}{g(x_{i_{k+1}})-g(y_{i_{k+1}})} < \frac{f(x_{i_k})-f(y_{i_k})}{g(x_{i_k})-g(y_{i_k})}. \tag{17}$$

Since the sequence of $\{g(x_i)\}$ is decreasing and the sequence of $\{g(y_i)\}$ is increasing, we have $0 < g(x_{i_{k+1}}) - g(y_{i_{k+1}}) < g(x_{i_k}) - g(y_{i_k})$. Then, by multiplying $g(x_{i_{k+1}}) - g(y_{i_{k+1}})$ with the above equation (17), we obtain $0 < f(x_{i_{k+1}}) - f(y_{i_{k+1}}) < \frac{g(x_{i_{k+1}})-g(y_{i_{k+1}})}{g(x_{i_k})-g(y_{i_k})} f(x_{i_k}) - f(y_{i_k})$. Let us denote $\beta = \max_{k=0,\ldots,\tau} \frac{g(x_{i_{k+1}})-g(y_{i_{k+1}})}{g(x_{i_k})-g(y_{i_k})}$, where $\tau$ is the number of intersection points generated in Step 3. Finally, we have for $k = 0,\ldots,\tau$, $0 < f(x_{i_{k+1}}) - f(y_{i_{k+1}}) \leq \beta(f(x_{i_k}) - f(y_{i_k}))$, where $0 < \beta < 1$. Putting $s_k = f(x_{i_k}) - f(y_{i_k})$, we have that for $k = 0,\ldots,\tau$, $0 < s_{k+1} \leq \beta s_k$. We choose $c$ as a $2n$-dimensional vector as, $c = (a_1,\ldots,a_n,-b_1,\ldots,-b_n)$, and define another $2n$-dimensional vector $z_k$ for each $k = 0,\ldots,\tau$, such that the first $n$ components of $z_i$ cover $f(x_{i_k})$ and the last $n$ components of $z_i$ are $-f(y_{i_k})$. Note that $z_i$ consists of 0 and 1. Then we have $s_k = z_k c$, for $k = 0,\ldots,\tau$. Applying Corollary 1, we conclude that the number of the upper bounds generated by the sub-approximation algorithm is $\tau = O\left(n\left(1+\frac{\log Mn}{\log\left(1+\frac{1}{\beta}\right)}\right)\right)$.

□

## A2. Experimental designs

For more details about the experiments go to Tables 9 and 13 of the electronic companion.

| **Parameter definitions** | **Values §5.1** | **Values §5.2** |
|---|---|---|
| $n$: number of candidate facility locations | variable | 5 |
| $m$: number of demand locations | variable | 30 |
| $k$: number of random problems | 5 | - |
| $b_j$: total demand of demand location $j$ | draw from $U[1,10]^*$ | draw from $U[1,15]^*$ |
| $l_j$: number of breakpoints | 3 | 3 |
| $\alpha$: a factor for deciding amount of supply | 0.3 | 0.3 |
| $s$: total supply | $\lceil \alpha \cdot \sum_{j=1}^{m} b_j \rceil$ | $\lceil \alpha \cdot \sum_{j=1}^{m} b_j \rceil$ |
| $s_i$: maximum capacity of facility $i$ | $\sum_{j=1}^{m} b_j$ | $\sum_{j=1}^{m} b_j$ |
| $f_i$: fixed location costs at facility $i$ | $e^{1.5 \times i \times ln 10/n}$ | 100 |
| $(x,y)$: coordinates of facility and demand locations | drawn from $U[0,100]$ | drawn from $U[0,100]$ |
| $d_{ij}$: distance from facility $i$ to demand location $j$ | Euclidean distance* | Euclidean distance* |
| $c_j^t$: positive integer such that $c_j^1 \geq \ldots \geq c_j^{l_j}$ | partition $[0,b_j]$ into $l_j$ intervals, $[0,a_j^1],[a_j^1,a_j^2],\ldots,[a_j^{l_j-1},b_j]$, $c_j^t$ drawn from $U[a_j^t,a_j^{t+1}]^*$ | $c_j^{l_j}$ drawn from $U[1,5]^*$ $c_j^t$ drawn from $U[c_j^{t+1},c_j^{t+1}+t]^*$ where $t=1,\ldots l_j-1$ |

Table 8: Design of experiments in §5.1 and §5.2. *Note:* (*): rounded to the nearest integer.

# Electronic companion for:

# "Supply constrained location-distribution in not-for-profit settings"

## EC1. Characteristics of equity measures (Marsh and Schilling 1991)

- *Analytic tractability:* our paper contributes to show that $Ineq$ is a tractable measure when appropriately using fractional programming techniques. In contrast, note that the denominator of $G_{Mandell}$ is conveniently transformed into a constant value ($\sum_{j=1}^{m} x_j = s$) and this eliminates the fractional form of this measure.

- *Appropriateness:* this is a criterion related to managerial applicability. In general, the Gini coefficient is a popular measure that has been employed in a large number of studies. We believe that $Ineq$ is specially appropriate for the setting described in our paper because we assume that each demand region might have its own characteristics in terms of how the assigned supply is being satisfied. One particular example of different utility functions assigned to different demand locations is illustrated in the case study (§6).

- *Impartiality:* the 'equal treatment among equals' criterion is applicable to our measure. Even though we differentiate between each demand group by assuming each group has its unique utility function if two equal individuals (i.e., individuals with the same associated slope value) traded places the overall $Ineq$ would not change.

- *Principle of transfers:* in the context of our inequity measure, this principle would be satisfied if a small transfer of supply from a demand location with larger relative utility to one with lower relative utility drives the inequity measure down. Our our general $Ineq$ measure does not satisfy this criterion as the particular case of $G_{Mandell}$ does. This is due to the fact that each demand location has a different size and utility function.

- *Scale invariance:* A measure of inequity meets the scale invariance criterion if this measure is unaffected when the distribution to every demand location is multiplied by a positive constant. Our inequity measure does not satisfy this criterion in general due to the different shapes of each demand location's utility function. However, this is satisfied by the particular case of $G_{Mandell}$.

- *Pareto optimality:* it is satisfied if as the inequity measure improves (gets smaller) none of the demand locations is worse off. This criterion does not hold for general $Ineq$ or $G_{Mandell}$.

In fact, most of the inequity measures do not satisfy this principle.

- *Normalization:* it is satisfied when the inequity measure is defined in a way that it ranges from 0 to 1. This criterion is not satisfied by general $Ineq$ or $G_{Mandell}$. Note that this is a more restrictive criterion than the scale invariance property.

## EC2. Additional Tables of Section 5

| | | |
|---|---|---|
| **Input** | | $n$: a set of numbers for candidate facility locations, |
| | | $m$: a set of numbers for demand locations, |
| | | $k$: number of random problems, |
| | | $\alpha$: a factor for deciding amount of supply, |
| | | $D$: a value for the maximum demand |
| **For** | | each $l = 1, \dots, k$ **do** |
| | **For** | each $(i, j)$ for $i = 1, \dots, n$, $j = 1, \dots, m$ **do** |
| | | generate input parameters: |
| | | $b_j$ = random integer values from [1,D] |
| | | **For** each $t$ for $t = 1, \dots, l_j$ **do** |
| | | $c_j^t$ = random positive integer values such that |
| | | $c_j^1 \geq c_j^2 \geq \dots \geq c_j^{l_j}$ **endfor** |
| | | $s = \lceil \alpha \cdot \sum_{j=1}^m b_j \rceil$. |
| | | $s_i = \sum_{j=1}^m b_j$. |
| | | $f_i$ = positive integer values from function $f(i, n)$ |
| | | $d_{ij}$ = positive integer distance from facility and demand locations |
| | | Solve the problem by Newton and sub-approximation algorithms **endfor** |
| | Compute | average function calls and average CPU time for each algorithm **endfor** |
| **Result** | | average function calls and CPU time for each set of $(n, m)$ |

Table 9: Overall design of the experiments in §5.1.

| | Newton | | Sub-approximation | |
|---|---|---|---|---|
| (n,m) | func. calls | CPU time | func. calls | CPU time |
| (2,10) | 2.6 | 0.3 | 1.5 | 0.3 |
| (2,50) | 2.0 | 7.0 | 1.0 | 6.8 |
| (2,100) | 3.5 | 4697.1 | 3.0 | 2322.6 |
| (5,10) | 3.3 | 0.5 | 1.0 | 0.5 |
| (5,50) | 3.0 | 4.5 | 1.0 | 3.9 |
| (5,100) | 3.0 | 97.2 | 1.0 | 93.8 |
| (10,30) | 4.5 | 15.3 | 2.0 | 14.2 |
| (10,50) | 3.7 | 80.9 | 1.7 | 80.2 |
| (20,30) | 2.5 | 9.1 | 1.0 | 7.8 |
| (20,50) | 3.0 | 50.6 | 1.0 | 42.7 |
| (30,30) | 3.5 | 24.6 | 2.0 | 22.5 |
| (30,50) | 3.5 | 57.3 | 3.0 | 52.0 |

Table 10: Performance comparison of two-phase algorithm with Newton and sub-approximation for $\alpha = 0.3$ and $\epsilon = 0.6$

| (n,m) | Newton func. calls | Newton CPU time | Sub-approximation func. calls | Sub-approximation CPU time |
|---|---|---|---|---|
| (2,10) | 2.2 | 0.2 | 1.4 | 0.2 |
| (2,50) | 3.3 | 628.3 | 2.3 | 562.3 |
| (2,100) | 4.0 | 5851.9 | 4.0 | 4152.9 |
| (5,10) | 3.0 | 0.3 | 2.3 | 0.4 |
| (5,50) | 4.0 | 546.0 | 4.0 | 449.6 |
| (5,100) | 4.0 | 8815.8 | 3.0 | 6900.3 |
| (10,30) | 3.0 | 5.5 | 3.0 | 9.9 |
| (10,50) | 2.8 | 55.5 | 2.0 | 85.5 |
| (20,30) | 3.7 | 58.4 | 2.3 | 43.7 |
| (20,50) | 3.7 | 925.6 | 2.3 | 931.1 |
| (30,30) | 2.5 | 8.6 | 2.0 | 10.7 |
| (30,50) | 2.0 | 14.7 | 1.0 | 14.2 |

Table 11: Comparison of performances of two-phase algorithm with Newton and sub-approximation for $\alpha = 0.6$ and $\epsilon = 0.3$.

| (n,m) | Newton func. calls | Newton CPU time | Sub-approximation func. calls | Sub-approximation CPU time |
|---|---|---|---|---|
| (2,10) | 3.0 | 0.3 | 2.0 | 0.3 |
| (2,50) | 2.2 | 6.1 | 1.8 | 7.4 |
| (2,100) | 4.0 | 48.1 | 3.0 | 47.7 |
| (5,10) | 3.2 | 0.5 | 1.7 | 0.5 |
| (5,50) | 3.0 | 12.9 | 1.3 | 11.6 |
| (5,100) | 3.1 | 117.0 | 1.5 | 110.7 |
| (10,30) | 3.5 | 17.3 | 2.0 | 16.7 |
| (10,50) | 3.3 | 22.9 | 1.5 | 20.6 |
| (20,30) | 2.8 | 12.7 | 1.0 | 10.6 |
| (20,50) | 3.3 | 26.6 | 1.7 | 23.1 |
| (30,30) | 3.2 | 17.7 | 1.8 | 16.7 |
| (30,50) | 3.0 | 50.3 | 1.5 | 48.7 |

Table 12: Comparison of performances of two-phase algorithm with Newton and sub-approximation for $\alpha = 0.6$ and $\epsilon = 0.6$.

| | |
|---|---|
| **Input** | $n$: a number of candidate facility locations, |
| | $m$: a number of demand locations, |
| **do** | Generate input parameters for fixed $(n, m)$: |
| | $b_j$ = random integer values from [1,15] for $j = 1, \ldots, m$. |
| | $c_j^{l_j}$ = random positive integer values such that $c_j^1 \geq c_j^2 \geq \ldots \geq c_j^{l_j}$. |
| | $s_i = \sum_{j=1}^{m} b_j$ for $i = 1, \ldots, n$. |
| | $d_{ij} = \lceil d_{ij} \rceil$ = random distance from $x$ and $y$ coordinates of locations drawn from U[1,100], for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. |
| **For** | each $(\alpha, \epsilon, f)$, where $\alpha \in \{0, 1\}$, $\epsilon \in \{0, 1\}$ and $f = 100$ **do** |
| | Set $s = \lceil \alpha \cdot \sum_{j=1}^{m} b_j \rceil$ |
| | Solve the problem by the two-phase sub-approximation algorithm **endfor** |
| **Result** | Efficiency and Ineq for each set of $(\alpha, \epsilon, f)$ |

Table 13: Overall design of the experiments in §5.2.

| $\epsilon$ | Problem | $G_{Mandell}$ | $Ineq$ | Actual Utility | Cost | $EFF$ | num. open DCs | num. partially supplied locations |
|---|---|---|---|---|---|---|---|---|
| 0.1 | no utility | 0.0996 | 0.2223 | 1433 | 3121 | 0.4591 | 4 | 30 |
| 0.2 | no utility | 0.2 | 0.23 | 1510 | 2964 | 0.5094 | 4 | 27 |
| 0.3 | no utility | 0.2994 | 0.2324 | 1486 | 2648 | 0.5612 | 4 | 24 |
| 0.4 | no utility | 0.3994 | 0.2407 | 1540 | 2560 | 0.6016 | 4 | 21 |
| 0.5 | no utility | 0.465 | 0.2512 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.6 | no utility | 0.465 | 0.2512 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.7 | no utility | 0.465 | 0.2512 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.1 | $P$ | 0.371 | 0.0996 | 1599 | 2918 | 0.5480 | 4 | 23 |
| 0.2 | $P$ | 0.407 | 0.1997 | 1564 | 2645 | 0.5913 | 4 | 22 |
| 0.3 | $P$ | 0.4466 | 0.2997 | 1467 | 2395 | 0.6125 | 3 | 20 |
| 0.4 | $P$ | 0.465 | 0.3272 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.5 | $P$ | 0.465 | 0.3272 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.6 | $P$ | 0.465 | 0.3272 | 1444 | 2331 | 0.6195 | 3 | 18 |
| 0.7 | $P$ | 0.465 | 0.3272 | 1444 | 2331 | 0.6195 | 3 | 18 |

Table 14: Comparison between the “no utility” problem and $P$ when $\alpha = 0.5$ and $f = 100$.

## EC3. Additional information about Section 6 (case study)

| Regions | Demand (number of people) | | | | | Distance (km) | | |
|---|---|---|---|---|---|---|---|---|
| | food insecure | high | moderate | low | total | 1 | 2 | 3 |
| | $a_j^1$ | $a_j^2 - a_j^1$ | $a_j^3 - a_j^2$ | $a_j^4 - a_j^3$ | | Luanda | Lobito | Lubango |
| Katchiungo | 85,775 (23.5%) | 84,315 (23.1%) | 36,500 (10.0%) | 158,410 (43.4%) | 194,963 | 622 | 338 | 474 |
| Ekunha | 87,420 (28.2%) | 88,350 (28.5%) | 55,800 (18.0%) | 78,430 (25.3%) | 134,273 | 613 | 274 | 414 |
| Cunhinga | 32,250 (15.0%) | 72,025 (33.5%) | 38,700 (18.0%) | 72,025 (33.5%) | 110,777 | 653 | 552 | 758 |
| Caconda | 42,333 (13.7%) | 80,340 (26.0%) | 55,620 (18.0%) | 130,707 (42.3%) | 186,387 | 763 | 384 | 239 |
| Andulo | 29,580 (17.4%) | 87,720 (51.6%) | 8,500 (5.0%) | 44,200 (26.0%) | 52,731 | 562 | 461 | 666 |
| Ganda | 20,496 (12.2%) | 37,632 (22.4%) | 80,640 (48.0%) | 29,232 (17.4%) | 109,937 | 740 | 226 | 384 |
| Kuito | 37,975 (15.5%) | 61,985 (25.3%) | 46,550 (19.0%) | 98,490 (40.2%) | 145,099 | 684 | 583 | 821 |
| Bie | | 216,300 (48.0%) | 112,700 (25.0%) | 121,700 (27.0%) | 234,452 | 684 | 583 | 821 |
| Cunene | | 9,600 (5.4%) | 1,900 (1.1%) | 164,700 (93.5%) | 166,695 | 994 | 711 | 473 |
| Huila | | 62,300 (8.2%) | 178,700 (23.5%) | 519,300 (68.3%) | 698,092 | 992 | 478 | 175 |
| Kuando Kubango | | 46,800 (25.0%) | 47,500 (25.4%) | 92,900 (49.6%) | 140,475 | 1,006 | 756 | 505 |
| Moxico | | 56,500 (26.1%) | 127,500 (58.9%) | 32,500 (15.0%) | 160,074 | 1,071 | 970 | 1,176 |
| Namibe | | 23,900 (27.4%) | 22,500 (25.8%) | 40,800 (46.8%) | 63,373 | 948 | 434 | 185 |

Table 15: Regional demand and distance from warehouses data.

| Sol. | $\epsilon$ | Ineq | EFF | Open WH | Sol. | $\epsilon$ | Ineq | EFF | Open WH |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.05 | 0.05 | 0.0286 | $(0, 1, 1)$ | 8 | 0.40 | 0.40 | 0.0355 | $(0, 1, 1)$ |
| 2 | 0.10 | 0.10 | 0.0299 | $(0, 1, 1))$ | 9 | 0.45 | 0.45 | 0.0363 | $(0, 1, 1)$ |
| 3 | 0.15 | 0.15 | 0.0308 | $(0, 1, 1)$ | 10 | 0.50 | 0.0371 | 0.0672 | $(0, 1, 1)$ |
| 4 | 0.20 | 0.20 | 0.0319 | $(0, 1, 1)$ | 11 | 0.55 | 0.55 | 0.0376 | $(0, 1, 1)$ |
| 5 | 0.25 | 0.25 | 0.033 | $(0, 1, 1)$ | 12,13 | 0.60, 0.65 | 0.5774 | 0.0377 | $(0, 1, 1)$ |
| 6 | 0.30 | 0.30 | 0.0338 | $(0, 1, 1)$ | 14 | 0.70 | 0.70 | 0.0381 | $(0, 1, 0)$ |
| 7 | 0.35 | 0.35 | 0.0346 | $(0, 1, 1)$ | 15 | 0.75 | 0.7218 | 0.0383 | $(0, 1, 0)$ |

Table 16: Trade-off curve solutions. *Note*: All of 14 points on the trade-off curve are described in this table. The second column contains $\epsilon$ values used in Constraint (11) and the fifth column describes with 1 the open warehouses. For example, if we use $\epsilon = 0.05$ in the model, we need to open the second (Lubito) and third (Lubango) warehouses (i.e., $(y_1, y_2, y_3) = (0, 1, 1)$) to achieve 0.05 and 0.0286 as the inequity and efficiency measures, respectively.

Figures 9 and 10 contain the detailed solutions of our case study. The first column of each table is the maximum allowable value of $Ineq$ ($\epsilon$) in our model. For each $\epsilon$, the Table shows the number of people served by amounts of food shipped from warehouses to each region. For example, if we set $\epsilon = 0.05$, region 1 (Katchiungo) will receive the food from warehouse 2, which can serve 41,551 people. The last column represents a percentage of capacity being used for each warehouse. For instance, if we use $\epsilon = 0.05$, then no capacity will be used in warehouse 1 (i.e., warehouse 1 is not open), while warehouse 2 uses 25.77% of capacity. The supply rate is computed as $\frac{x_j}{b_j} \times 100$ and it is the percentage of supplies received to demand for each region $j$. The relative utility is computed as $\frac{u_j(x_j)}{\bar{u}_j} \times 100$, which is the percentage of relative utility as defined in our paper.

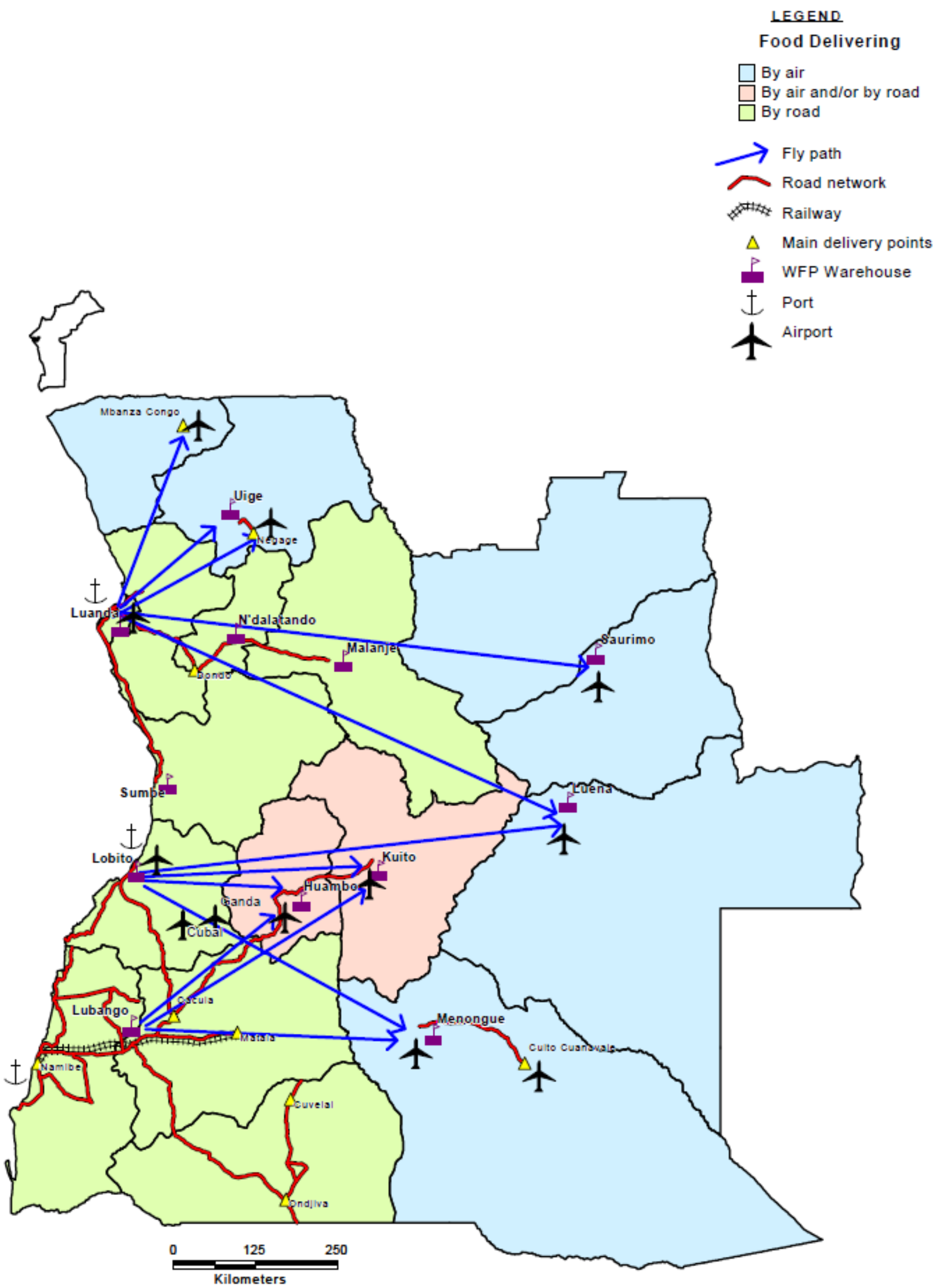


Figure 8: Map of WFP warehouses in early 2000s, where Luanda, Lobito and Lubango are the major ones (Source: WFP: A Report from the Office of Evaluation, Full Report of the Evaluation of the WFP ANGOLA Portfolio, 2002.)

| epsilon | | Katchiungo | Ekunha | Cunhinga | Caconda | Andulo | Ganda | Kuito | Bie | Cunene | Huila | ındo Kuba | Moxico | Namibe | capacity(%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 41551 | 40411 | 24831 | 0 | 22229 | 19352 | 26586 | 66685 | 0 | 0 | 0 | 5735 | 0 | 25.77 |
| | 3 | 0 | 0 | 0 | 32755 | 0 | 0 | 0 | 0 | 15014 | 75666 | 21980 | 0 | 10545 | 64.98 |
| | supply rate(%) | *11.38* | *13.04* | *11.55* | *10.60* | *13.08* | *11.52* | *10.85* | *14.80* | *8.52* | *9.95* | *11.74* | *2.65* | *12.09* | |
| | relative utility(%) | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *20.09* | *3.76* | *20.09* | |
| 0.1 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 44548 | 43326 | 26622 | 0 | 23833 | 20833 | 28503 | 52714 | 0 | 0 | 0 | 0 | 0 | 25.04 |
| | 3 | 0 | 0 | 0 | 35118 | 0 | 0 | 0 | 0 | 9600 | 83372 | 23565 | 0 | 11306 | 67.90 |
| | supply rate(%) | *12.20* | *13.98* | *12.38* | *11.37* | *14.02* | *12.40* | *11.63* | *11.70* | *5.45* | *10.97* | *12.59* | *0.00* | *12.97* | |
| | relative utility(%) | *21.53* | *21.53* | *21.53* | *21.53* | *21.54* | *21.54* | *21.53* | *15.88* | *14.60* | *21.53* | *21.53* | *0.00* | *21.54* | |
| 0.15 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 47368 | 46069 | 28307 | 0 | 25341 | 22584 | 30308 | 28722 | 0 | 0 | 0 | 0 | 0 | 23.82 |
| | 3 | 0 | 0 | 0 | 37341 | 0 | 0 | 0 | 0 | 9600 | 90622 | 25056 | 0 | 12022 | 72.77 |
| | supply rate(%) | *12.98* | *14.86* | *13.17* | *12.08* | *14.91* | *13.44* | *12.37* | *6.37* | *5.45* | *11.92* | *13.38* | *0.00* | *13.79* | |
| | relative utility(%) | *22.90* | *22.90* | *22.90* | *22.90* | *22.90* | *22.90* | *22.90* | *8.65* | *14.60* | *22.90* | *22.90* | *0.00* | *22.90* | |
| 0.20 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 50241 | 48863 | 30024 | 0 | 26878 | 24368 | 32146 | 4312 | 0 | 0 | 0 | 0 | 0 | 22.59 |
| | 3 | 0 | 0 | 0 | 39606 | 0 | 0 | 0 | 0 | 9600 | 98006 | 26545 | 0 | 12751 | 77.71 |
| | supply rate(%) | *13.76* | *15.76* | *13.96* | *12.82* | *15.81* | *14.50* | *13.12* | *0.96* | *5.45* | *12.89* | *14.18* | *0.00* | *14.62* | |
| | relative utility(%) | *24.29* | *24.29* | *24.29* | *24.29* | *24.29* | *24.29* | *24.29* | *1.30* | *14.60* | *24.29* | *24.26* | *0.00* | *24.29* | |
| 0.25 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 53785 | 52310 | 32142 | 0 | 28774 | 26569 | 34413 | 0 | 0 | 0 | 0 | 0 | 0 | 23.75 |
| | 3 | 0 | 0 | 0 | 42422 | 0 | 0 | 0 | 0 | 9600 | 107115 | 2560 | 0 | 13650 | 73.06 |
| | supply rate(%) | *14.74* | *16.87* | *14.95* | *13.73* | *16.93* | *15.81* | *14.05* | *0.00* | *5.45* | *14.09* | *1.37* | *0.00* | *15.65* | |
| | relative utility(%) | *26.00* | *26.00* | *26.00* | *26.00* | *26.00* | *26.00* | *26.00* | *0.00* | *14.60* | *26.00* | *2.34* | *0.00* | *26.00* | |
| 0.3 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 60152 | 58502 | 30400 | 0 | 27215 | 30522 | 32548 | 0 | 0 | 0 | 0 | 0 | 0 | 24.93 |
| | 3 | 0 | 0 | 0 | 49112 | 0 | 0 | 0 | 0 | 0 | 99623 | 0 | 0 | 15266 | 68.33 |
| | supply rate(%) | *16.48* | *18.87* | *14.14* | *15.89* | *16.01* | *18.17* | *13.28* | *0.00* | *0.00* | *13.10* | *0.00* | *0.00* | *17.51* | |
| | relative utility(%) | *29.08* | *29.08* | *24.59* | *29.08* | *24.59* | *29.08* | *24.59* | *0.00* | *0.00* | *24.59* | *0.00* | *0.00* | *29.08* | |
| 0.35 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 68424 | 66548 | 25962 | 0 | 23242 | 35660 | 27797 | 0 | 0 | 0 | 0 | 0 | 0 | 25.80 |
| | 3 | 0 | 0 | 0 | 57808 | 0 | 0 | 0 | 0 | 0 | 80533 | 0 | 0 | 17366 | 64.88 |
| | supply rate(%) | *18.75* | *21.47* | *12.08* | *18.71* | *13.67* | *21.23* | *11.35* | *0.00* | *0.00* | *10.59* | *0.00* | *0.00* | *19.92* | |
| | relative utility(%) | *33.08* | *33.08* | *21.00* | *33.08* | *21.00* | *33.08* | *21.00* | *0.00* | *0.00* | *21.00* | *0.00* | *0.00* | *33.08* | |
| 0.4 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 76708 | 74604 | 21005 | 0 | 19447 | 40804 | 22488 | 0 | 0 | 0 | 0 | 0 | 0 | 26.57 |
| | 3 | 0 | 0 | 0 | 66516 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 19468 | 61.79 |
| | supply rate(%) | *21.02* | *24.07* | *9.77* | *21.53* | *11.44* | *24.29* | *9.18* | *0.00* | *0.00* | *8.19* | *0.00* | *0.00* | *22.33* | |
| | relative utility(%) | *37.08* | *37.08* | *16.99* | *37.08* | *17.57* | *37.08* | *16.99* | *0.00* | *0.00* | *17.57* | *0.00* | *0.00* | *37.08* | |

Figure 9: Case study solutions.

| epsilon | | Katchiungo | Ekunha | Cunhinga | Caconda | Andulo | Ganda | Kuito | Bie | Cunene | Huila | ındo Kuba | Moxico | Namibe | capacity(%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.45 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 81917 | 79671 | 11201 | 0 | 19448 | 44039 | 11983 | 0 | 0 | 0 | 0 | 0 | 0 | 25.86 |
| | 3 | 0 | 0 | 0 | 71991 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 20790 | 64.62 |
| | supply rate(%) | 22.44 | 25.70 | 5.21 | 23.30 | 11.44 | 26.21 | 4.89 | 0.00 | 0.00 | 8.19 | 0.00 | 0.00 | 23.84 | |
| | relative utility(%) | 39.60 | 39.60 | 9.06 | 39.60 | 17.57 | 39.60 | 9.05 | 0.00 | 0.00 | 17.57 | 0.00 | 0.00 | 39.60 | |
| 0.5 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 86293 | 2186 | 0 | 19448 | 46435 | 2338 | 0 | 0 | 0 | 0 | 0 | 0 | 25.26 |
| | 3 | 0 | 0 | 0 | 76047 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 22518 | 67.03 |
| | supply rate(%) | 23.50 | 27.84 | 1.02 | 24.61 | 11.44 | 27.64 | 0.95 | 0.00 | 0.00 | 8.19 | 0.00 | 0.00 | 25.82 | |
| | relative utility(%) | 41.46 | 42.89 | 1.77 | 41.46 | 17.57 | 41.46 | 1.77 | 0.00 | 0.00 | 17.57 | 0.00 | 0.00 | 42.89 | |
| 0.55 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 87420 | 0 | 0 | 3015 | 56060 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 24.19 |
| | 3 | 0 | 0 | 0 | 84870 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 23900 | 71.28 |
| | supply rate(%) | 23.50 | 28.20 | 0.00 | 27.47 | 1.77 | 33.37 | 0.00 | 0.00 | 0.00 | 8.19 | 0.00 | 0.00 | 27.41 | |
| | relative utility(%) | 41.46 | 43.45 | 0.00 | 45.52 | 2.72 | 48.96 | 0.00 | 0.00 | 0.00 | 17.57 | 0.00 | 0.00 | 45.52 | |
| 0.6 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 87420 | 0 | 0 | 0 | 78624 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 26.23 |
| | 3 | 0 | 0 | 0 | 65321 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 23900 | 63.13 |
| | supply rate(%) | 23.50 | 28.20 | 0.00 | 21.14 | 0.00 | 46.80 | 0.00 | 0.00 | 0.00 | 8.19 | 0.00 | 0.00 | 27.41 | |
| | relative utility(%) | 41.46 | 43.45 | 0.00 | 36.53 | 0.00 | 66.52 | 0.00 | 0.00 | 0.00 | 17.57 | 0.00 | 0.00 | 45.52 | |
| 0.65 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 87420 | 0 | 0 | 0 | 78624 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 26.23 |
| | 3 | 0 | 0 | 0 | 65321 | 0 | 0 | 0 | 0 | 0 | 62300 | 0 | 0 | 23900 | 63.13 |
| | supply rate(%) | 23.50 | 28.20 | 0.00 | 21.14 | 0.00 | 46.80 | 0.00 | 0.00 | 0.00 | 8.19 | 0.00 | 0.00 | 27.41 | |
| | relative utility(%) | 41.46 | 43.45 | 0.00 | 36.53 | 0.00 | 66.52 | 0.00 | 0.00 | 0.00 | 17.57 | 0.00 | 0.00 | 45.52 | |
| 0.7 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 184089 | 0 | 42333 | 12519 | 78624 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 42.01 |
| | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | supply rate(%) | 23.50 | 59.38 | 0.00 | 13.70 | 7.36 | 46.80 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | |
| | relative utility(%) | 41.46 | 79.49 | 0.00 | 25.96 | 11.31 | 66.52 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | |
| 0.75 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 85775 | 196608 | 0 | 42333 | 0 | 78624 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 42.01 |
| | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | supply rate(%) | 23.50 | 63.42 | 0.00 | 13.70 | 0.00 | 46.80 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | |
| | relative utility(%) | 41.46 | 84.15 | 0.00 | 25.96 | 0.00 | 66.52 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | |

Figure 10: Case study solutions (continued).

| epsilon | | Katchiungo | Ekunha | Cunhinga | Caconda | Andulo | Ganda | Kuito | Bie | Cunene | Huila | ındo Kuba | Moxico | Namibe | capacity(%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 47918 | 46604 | 28637 | 0 | 25636 | 22318 | 30660 | 57678 | 0 | 0 | 0 | 4961 | 0 | 27.54 |
| | 3 | 0 | 0 | 0 | 37775 | 0 | 0 | 0 | 0 | 11426 | 61594 | 19012 | 0 | 9121 | 57.89 |
| | supply rate(%) | 13.13 | 15.03 | 13.32 | 12.22 | 15.08 | 13.28 | 12.51 | 12.80 | 6.48 | 8.10 | 10.16 | 2.29 | 10.46 | |
| | relative utility(%) | 23.16 | 23.16 | 23.16 | 23.16 | 23.16 | 22.69 | 23.16 | 17.37 | 16.45 | 17.37 | 17.37 | 3.26 | 17.37 | |
| 0.1 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 52055 | 50627 | 26239 | 0 | 23490 | 24245 | 28092 | 52847 | 0 | 0 | 0 | 0 | 0 | 26.83 |
| | 3 | 0 | 0 | 0 | 41036 | 0 | 0 | 0 | 0 | 10469 | 66911 | 17420 | 0 | 9909 | 60.73 |
| | supply rate(%) | 14.26 | 16.33 | 12.20 | 13.28 | 13.82 | 14.43 | 11.47 | 11.73 | 5.94 | 8.80 | 9.31 | 0.00 | 11.36 | |
| | relative utility(%) | 25.16 | 25.16 | 21.22 | 25.16 | 21.23 | 24.19 | 21.22 | 15.92 | 15.48 | 18.44 | 15.92 | 0.00 | 18.87 | |
| 0.15 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 56628 | 55075 | 22679 | 0 | 20303 | 26375 | 24282 | 45680 | 0 | 0 | 0 | 0 | 0 | 26.15 |
| | 3 | 0 | 0 | 0 | 44642 | 0 | 0 | 0 | 0 | 9049 | 72791 | 15057 | 0 | 10779 | 63.47 |
| | supply rate(%) | 15.51 | 17.77 | 10.55 | 14.45 | 11.94 | 15.70 | 9.91 | 10.14 | 5.14 | 9.57 | 8.04 | 0.00 | 12.36 | |
| | relative utility(%) | 27.37 | 27.37 | 18.34 | 27.02 | 18.35 | 25.85 | 18.35 | 13.76 | 13.76 | 19.55 | 13.76 | 0.00 | 20.53 | |
| 0.20 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 61110 | 59435 | 19191 | 0 | 17182 | 28463 | 20547 | 38654 | 0 | 0 | 0 | 0 | 0 | 25.48 |
| | 3 | 0 | 0 | 0 | 48175 | 0 | 0 | 0 | 0 | 7658 | 78551 | 12742 | 0 | 11632 | 66.15 |
| | supply rate(%) | 16.74 | 19.17 | 8.93 | 15.59 | 10.11 | 16.94 | 8.39 | 8.58 | 4.35 | 10.33 | 6.81 | 0.00 | 13.34 | |
| | relative utility(%) | 29.54 | 29.54 | 15.52 | 28.65 | 15.53 | 27.47 | 15.52 | 11.64 | 11.64 | 20.63 | 11.64 | 0.00 | 22.16 | |
| 0.25 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 65684 | 63883 | 15632 | 0 | 13996 | 30593 | 16737 | 31486 | 0 | 0 | 0 | 0 | 0 | 24.79 |
| | 3 | 0 | 0 | 0 | 51780 | 0 | 0 | 0 | 0 | 6238 | 84430 | 10378 | 0 | 12503 | 68.89 |
| | supply rate(%) | 18.00 | 20.61 | 7.27 | 16.76 | 8.23 | 18.21 | 6.83 | 6.99 | 3.54 | 11.10 | 5.54 | 0.00 | 14.34 | |
| | relative utility(%) | 31.75 | 31.75 | 12.64 | 30.30 | 12.65 | 29.13 | 12.64 | 9.48 | 9.49 | 21.73 | 9.48 | 0.00 | 23.82 | |
| 0.3 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 70258 | 68331 | 12074 | 0 | 10810 | 32723 | 12926 | 24317 | 0 | 0 | 0 | 0 | 0 | 24.11 |
| | 3 | 0 | 0 | 0 | 55386 | 0 | 0 | 0 | 0 | 4817 | 90309 | 8016 | 0 | 13373 | 71.63 |
| | supply rate(%) | 19.25 | 22.04 | 5.62 | 17.92 | 6.36 | 19.48 | 5.28 | 5.40 | 2.73 | 11.88 | 4.28 | 0.00 | 15.34 | |
| | relative utility(%) | 33.96 | 33.96 | 9.77 | 31.96 | 9.77 | 30.79 | 9.77 | 7.32 | 7.32 | 22.84 | 7.33 | 0.00 | 25.47 | |
| 0.35 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 74831 | 72779 | 8515 | 0 | 7624 | 34853 | 9116 | 17148 | 0 | 0 | 0 | 0 | 0 | 23.42 |
| | 3 | 0 | 0 | 0 | 58991 | 0 | 0 | 0 | 0 | 3398 | 96188 | 5653 | 0 | 14244 | 74.36 |
| | supply rate(%) | 20.50 | 23.48 | 3.96 | 19.09 | 4.48 | 20.75 | 3.72 | 3.80 | 1.93 | 12.65 | 3.02 | 0.00 | 16.33 | |
| | relative utility(%) | 36.17 | 36.17 | 6.89 | 33.62 | 6.89 | 32.45 | 6.89 | 5.17 | 5.17 | 23.94 | 5.17 | 0.00 | 27.13 | |
| 0.4 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 79405 | 77227 | 4955 | 0 | 4438 | 36983 | 5305 | 9981 | 0 | 0 | 0 | 0 | 0 | 22.74 |
| | 3 | 0 | 0 | 0 | 62597 | 0 | 0 | 0 | 0 | 1978 | 102067 | 3290 | 0 | 15114 | 77.10 |
| | supply rate(%) | 21.75 | 24.91 | 2.30 | 20.26 | 2.61 | 22.01 | 2.17 | 2.21 | 1.12 | 13.42 | 1.76 | 0.00 | 17.33 | |
| | relative utility(%) | 38.39 | 38.39 | 4.01 | 35.28 | 4.01 | 34.11 | 4.01 | 3.01 | 3.01 | 25.05 | 3.01 | 0.00 | 28.79 | |

Figure 11: Case study solutions of no utility problem.

| epsilon | | Katchiungo | Ekunha | Cunhinga | Caconda | Andulo | Ganda | Kuito | Bie | Cunene | Huila | ındo Kuba | Moxico | Namibe | capacity(%) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.45 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 83978 | 81675 | 1397 | 0 | 1251 | 39114 | 1495 | 2813 | 0 | 0 | 0 | 0 | 0 | 22.05 |
| | 3 | 0 | 0 | 0 | 66202 | 0 | 0 | 0 | 0 | 557 | 107946 | 927 | 0 | 15985 | 79.84 |
| | supply rate(%) | *23.01* | *26.35* | *0.65* | *21.42* | *0.74* | *23.28* | *0.61* | *0.62* | *0.32* | *14.20* | *0.50* | *0.00* | *18.33* | |
| | relative utility(%) | *40.60* | *40.60* | *1.13* | *36.94* | *1.13* | *35.77* | *1.13* | *0.85* | *0.85* | *26.16* | *0.85* | *0.00* | *30.45* | |
| 0.5 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 63431 | 89289 | 0 | 0 | 0 | 42761 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 20.36 |
| | 3 | 0 | 0 | 0 | 72374 | 0 | 0 | 0 | 0 | 0 | 118010 | 0 | 0 | 17475 | 86.61 |
| | supply rate(%) | *17.38* | *28.80* | *0.00* | *23.42* | *0.00* | *25.45* | *0.00* | *0.00* | *0.00* | *15.52* | *0.00* | *0.00* | *20.04* | |
| | relative utility(%) | *30.66* | *44.15* | *0.00* | *39.78* | *0.00* | *38.60* | *0.00* | *0.00* | *0.00* | *28.05* | *0.00* | *0.00* | *33.29* | |
| 0.55 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 24647 | 99477 | 0 | 0 | 0 | 47640 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.89 |
| | 3 | 0 | 0 | 0 | 80632 | 0 | 0 | 0 | 0 | 0 | 131475 | 0 | 0 | 19469 | 96.49 |
| | supply rate(%) | *6.75* | *32.09* | *0.00* | *26.09* | *0.00* | *28.36* | *0.00* | *0.00* | *0.00* | *17.29* | *0.00* | *0.00* | *22.33* | |
| | relative utility(%) | *11.91* | *47.95* | *0.00* | *43.57* | *0.00* | *42.40* | *0.00* | *0.00* | *0.00* | *30.58* | *0.00* | *0.00* | *37.08* | |
| 0.6 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 0 | 109107 | 0 | 0 | 0 | 54233 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.01 |
| | 3 | 0 | 0 | 0 | 68163 | 0 | 0 | 0 | 0 | 0 | 149673 | 0 | 0 | 22164 | 100.00 |
| | supply rate(%) | *0.00* | *35.20* | *0.00* | *22.06* | *0.00* | *32.28* | *0.00* | *0.00* | *0.00* | *19.69* | *0.00* | *0.00* | *25.42* | |
| | relative utility(%) | *0.00* | *51.54* | *0.00* | *37.84* | *0.00* | *47.53* | *0.00* | *0.00* | *0.00* | *34.00* | *0.00* | *0.00* | *42.22* | |
| 0.65 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 0 | 99827 | 0 | 0 | 0 | 63513 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.01 |
| | 3 | 0 | 0 | 0 | 38761 | 0 | 0 | 0 | 0 | 0 | 175284 | 0 | 0 | 25955 | 100.00 |
| | supply rate(%) | *0.00* | *32.20* | *0.00* | *12.54* | *0.00* | *37.81* | *0.00* | *0.00* | *0.00* | *23.05* | *0.00* | *0.00* | *29.76* | |
| | relative utility(%) | *0.00* | *48.08* | *0.00* | *23.77* | *0.00* | *53.36* | *0.00* | *0.00* | *0.00* | *38.82* | *0.00* | *0.00* | *48.13* | |
| 0.7 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 0 | 89539 | 0 | 0 | 0 | 73801 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.01 |
| | 3 | 0 | 0 | 0 | 6162 | 0 | 0 | 0 | 0 | 0 | 203677 | 0 | 0 | 30161 | 100.00 |
| | supply rate(%) | *0.00* | *28.88* | *0.00* | *1.99* | *0.00* | *43.93* | *0.00* | *0.00* | *0.00* | *26.79* | *0.00* | *0.00* | *34.59* | |
| | relative utility(%) | *0.00* | *44.24* | *0.00* | *3.78* | *0.00* | *58.70* | *0.00* | *0.00* | *0.00* | *44.16* | *0.00* | *0.00* | *53.47* | |
| 0.75 | warehouse | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.00 |
| | 2 | 0 | 54716 | 0 | 0 | 0 | 108624 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 17.01 |
| | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 209045 | 0 | 0 | 30955 | 100.00 |
| | supply rate(%) | *0.00* | *17.65* | *0.00* | *0.00* | *0.00* | *64.66* | *0.00* | *0.00* | *0.00* | *27.50* | *0.00* | *0.00* | *35.50* | |
| | relative utility(%) | *0.00* | *27.20* | *0.00* | *0.00* | *0.00* | *76.77* | *0.00* | *0.00* | *0.00* | *45.17* | *0.00* | *0.00* | *54.48* | |

Figure 12: Case study solutions of no utility problem (continued).