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Quantile-Regression Inference With Adaptive Control of Size

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Abstract

Regression quantiles have asymptotic variances that depend on the conditional densities of the response variable given regressors. This paper develops a new estimate of the asymptotic variance of regression quantiles that leads any resulting Wald-type test or confidence region to behave as well in large samples as its infeasible counterpart in which the true conditional response densities are embedded. We give explicit guidance on implementing the new variance estimator to control adaptively the size of any resulting Wald-type test. Monte Carlo evidence indicates the potential of our approach to deliver powerful tests of heterogeneity of quantile treatment effects in covariates with good size performance over different quantile levels, data-generating processes and sample sizes. We also include an empirical example. Supplementary material is available online.

Keywords: Regression quantile, asymptotic variance, standard error, conditional density estimation.

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1 Introduction

Consider an independent and identically distributed (iid) sample $(\boldsymbol{X}_1, Y_1), \ldots, (\boldsymbol{X}_n, Y_n)$, where each Y_i is scalar-valued, and where, for some fixed d, each $\boldsymbol{X}_i$ is a d-dimensional regressor. We assume that the conditional distribution of the ith response variable Y_i given $\boldsymbol{X}_i$ satisfies

$$\Pr\left[Y_i \leq \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \middle| \boldsymbol{X}_i \right] = \alpha \tag{1}$$

almost surely (a.s.) for some fixed quantile $\alpha \in (0, 1)$, where $\boldsymbol{\beta}(\alpha) \in \mathbb{R}^d$ is unknown and $\boldsymbol{X}_i^\top$ denotes the transpose of $\boldsymbol{X}_i$. The relation (1) specifies a linear α-quantile regression model. Models of conditional quantiles, such as the model given above in (1), have taken on an important role in the statistical sciences. They generally offer researchers the possibility of being able to engage in a systematic analysis of the effects of a set of conditioning variables on all aspects of the conditional distribution of a response variable. A notable characteristic of this approach is the ability it gives researchers to model only the quantiles of interest to a given empirical study without the need to construct an explicit model for the other regions of the response density. For example, a researcher may by varying the quantile index α examine the specific effects of regressors on any point of the conditional distribution of the response variable. Thus the differential effects of some medical intervention (X) on survival time (Y) can be analyzed separately for low-risk and high-risk individuals by constructing estimates of the conditional quantile function of Y given $\boldsymbol{X}$ for various quantiles. The monograph of Koenker (2005) and the volume edited by Koenker et al. (2017) provide comprehensive reviews of quantile-regression methodology, along with illustrative examples of its application in various disciplines.

There are several proposals available for quantile regression inference. Some of these proposals, such as certain methods involving resampling (He, 2017, contains a comprehensive review), approaches based on the asymptotic behavior of regression rank scores (Gutenbrunner and Jurečková, 1992), direct methods (Zhou and Portnoy, 1996; Fan and Liu, 2016) or more recent

Bayesian approaches (Yang and He, 2012; Feng et al., 2015; Yang et al., 2016) differ from Wald-type methods by avoiding the need to estimate conditional density functions for the purpose of asymptotic variance estimation of conditional quantile estimators. Wald-type procedures, however, do generally retain the attractive feature of computational simplicity, and perhaps for this reason remain popular in empirical practice.

In this paper we develop a new estimator of the asymptotic covariance matrix of a given regression quantile. The new estimator is explicitly intended to induce the Wald-type tests or confidence regions in which it is embedded to behave as well in large samples as their empirically infeasible counterparts in which the true, as opposed to estimated, conditional densities appear. The asymptotic variance estimator proposed here induces the empirical size distortions of Wald-type tests to vanish at the same rate enjoyed by the corresponding tests incorporating the actual conditional density functions, i.e., the disparity between the actual and nominal sizes of these tests vanishes at the *adaptive* rate.

There is of course a long history on estimation of the asymptotic variance of quantile regression parameters and the corresponding Wald-type tests. Among existing procedures, two implementations that are particularly popular are those of Powell (1991) and Hendricks and Koenker (1992). We show that the proposals of Powell (1991) and Hendricks and Koenker (1992) both induce Wald-type tests whose empirical size distortions cannot vanish at the adaptive rates that become possible when these tests incorporate the asymptotic variance estimator that we develop below.

The proposed estimator for the conditional density evaluated at the conditional quantile has applications beyond the formulation of Wald-type tests with adaptive control of size. This estimator can be used for counterfactual wage decompositions in a quantile regression setting (Machado and Mata, 2005). It has been used for developing improved specification tests for linear quantile regression (Escanciano and Goh, 2014). Semiparametrically efficient inference in linear quantile regression requires, either explicitly or implicitly, an estimator of the so-called efficient score, which involves the conditional density evaluated at the quantile (Newey and Powell, 1990; Komunjer and Vuong, 2010). Finally, estimates of conditional densities are also needed in semiparametric extensions of the basic linear quantile regression model, e.g., Ma and He (2016) and Feng and Zhu (2016). Further applications of our estimator such as these are of independent interest.

Finally, we note that this paper is partly motivated by a recent contribution of Portnoy (2012) to the effect that the first-order asymptotic normal approximation for regression quantiles is associated with an error bound of order $O_p\left(n^{-1/2}(\log n)^{3/2}\right)$. This in turn implies, as we show below, the benchmark $O_p\left(n^{-1/2}(\log n)^{3/2}\right)$-rate at which size distortions for Wald-type tests regarding quantile-regression parameters converge when the conditional response densities are assumed to be known. An important point to note is that the error bound of nearly $n^{-1/2}$-order elucidated by Portnoy (2012) is smaller than the error bound of nearly $n^{-1/4}$-order associated with the classic Bahadur representation for regression quantiles. In particular, the larger error of nearly $n^{-1/4}$-order is in fact larger in magnitude than the estimation error associated with any set of reasonable estimates of the conditional response densities, including those proposed by Powell (1991) and Hendricks and Koenker (1992). This would apparently suggest that the rate-adaptive implementation of Wald-type tests proposed in this paper is at best of second-order importance. The smaller error bound shown by Portnoy (2012) effectively allows one to consider the question of optimally implementing Wald-type tests in this context as a methodological issue of first-order importance.

The remainder of this paper proceeds as follows. The next section develops the asymptotic properties of our proposed kernel estimator of the conditional response density evaluated at the conditional quantile of interest. Section 3 analyzes the size distortions of tests of linear restrictions of quantile coefficients based on the asymptotic distribution of regression α-quantiles. This section also discusses conditions for our Wald-type tests to exhibit size distortions that decay at the adaptive rate in large samples. Section 4 presents the results of a series of simulation experiments which illustrate the potential of our methods to deliver accurate and powerful tests, and which are motivated from our empirical application, which in turn is discussed in Section 5. An online supplement includes precise statements of the assumptions underlying our theoretical results, proofs of those results, additional simulation evidence, details on implementation and further discussion of the empirical example.

2 The New Estimator

Consider the α-quantile regression model given above in (1). For each quantile $\alpha \in (0,1)$, the *regression α-quantile* (Koenker and Bassett, 1978) is defined as

$$\hat{\boldsymbol{\beta}}_n(\alpha) \equiv \arg\min_{\boldsymbol{b}\in\mathbb{R}^d} \sum_{i=1}^{n} \rho_\alpha \left(Y_i - \boldsymbol{X}_i^\top \boldsymbol{b}\right),$$

where $\rho_\alpha(u) = u\left(\alpha - 1\{u \leq 0\}\right)$.

For each $i = 1, \ldots, n$, let $f_i(y)$ and $F_i(y)$ denote the conditional density and cumulative distribution function (cdf), respectively, of Y_i given $\boldsymbol{X}_i$, evaluated at y. If one assumes that for each i, $F_i(y)$ is absolutely continuous, and that $f_i(y)$ is finite and bounded away from zero at $y = \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)$, then under Assumption 1 as given in Appendix A of the supplementary material, the regression α-quantile is asymptotically normal with

$$\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \hat{\boldsymbol{\beta}}(\alpha)\right) \xrightarrow{d} N\left(0, \boldsymbol{V}(\alpha)\right), \tag{2}$$

where $\boldsymbol{V}(\alpha) = \alpha(1-\alpha)\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)$ (e.g., Koenker, 2005, Theorem 4.1), and where

$$\boldsymbol{G}_0(\alpha) = E\left[f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top\right]; \tag{3}$$

$$\boldsymbol{H} = E\left[\boldsymbol{X}_i\boldsymbol{X}_i^\top\right]. \tag{4}$$

Standard Wald-type inferential procedures based on (2) naturally require the estimation of the matrix $\boldsymbol{G}_0(\alpha)$, which in turn requires, at least implicitly, the estimation of the conditional density functions $f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$ $(i = 1, \ldots, n)$.

We propose an estimator of the conditional response densities $f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$, estimates of which in turn are used to specify a new estimator of the matrix $\boldsymbol{G}_0(\alpha)$ appearing in the asymptotic variance of the regression α-quantile. The new estimator of the conditional densities developed here explicitly exploits the behavior of the fitted conditional U_j-quantiles $\boldsymbol{X}_i^\top \hat{\boldsymbol{\beta}}_n\left(U_j\right)$ over a range of quantiles $U_1, \ldots, U_m$ that are iid realizations from a uniform distribution on $\mathcal{A} = [a_1, a_2]$. To motivate the new estimator, note the identity $F_i(y) = a_1 + \int_{a_1}^{a_2} 1\left\{y - F_i^{-1}(\alpha) \geq 0\right\} d\alpha$ for $a_1 \leq F_i(y) \leq a_2$. This suggests using a smooth approximation of the indicator function, which after differentiation leads one to the quantity $(a_2 - a_1) \cdot h^{-1} E\left[K\left(h^{-1}\left(y - F_i^{-1}(U)\right)\right) \middle| \boldsymbol{X}_i\right]$, where $K(\cdot)$ is a smoothing kernel satisfying the conditions of Assumption 2 in the supplementary material and where $U | \boldsymbol{X}_i \sim Unif[a_1, a_2]$, where $a_1 < \alpha < a_2$. This quantity should be a good approximation of $f_i(y)$ as $h \to 0$, where $h > 0$ is a scalar smoothing parameter. In order to avoid numerical integration, we approximate the integral by a finite sum with m terms. Note that we certainly could take $m = \infty$, but this would require numerical integration. In what follows, we let both m and the scalar smoothing parameter h depend on the sample size n, with $m \to \infty$ and $h \to 0$ as $n \to \infty$.

The discussion above leads to the estimator of $f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$ given by

$$\hat{f}_{ni}\left(\boldsymbol{X}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)\right) = \frac{a_2 - a_1}{m h_m} \sum_{j=1}^{m} K\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\hat{\boldsymbol{\beta}}_n\left(U_j\right) - \hat{\boldsymbol{\beta}}_n(\alpha)\right)\right) \tag{5}$$

for each $i = 1, \ldots, n$. The estimators $\hat{f}_{ni}\left(\boldsymbol{X}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)\right)$ given in (5) are in turn embedded in the following estimator of the matrix $\boldsymbol{G}_0(\alpha)$ as given above in (3):

$$\hat{\boldsymbol{G}}_n(\alpha) \equiv \frac{1}{n} \sum_{i=1}^{n} \hat{f}_{ni}\left(\boldsymbol{X}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)\right) \boldsymbol{X}_i \boldsymbol{X}_i^\top . \tag{6}$$

We are now in a position to state the main result of this section. Define for $\alpha \in \mathcal{A}$

$$\boldsymbol{D}_{nj}(\alpha) \equiv \sqrt{n}\left[\left(\hat{\boldsymbol{\beta}}_n\left(U_j\right) - \boldsymbol{\beta}\left(U_j\right)\right) - \left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)\right], \tag{7}$$

$\sigma_K^2 \equiv \int_{-1/2}^{1/2} w^2 K(w)dw$ and $\|K\|_2 \equiv \sqrt{\int_{-1/2}^{1/2} K^2(w)dw}$. In addition, we adopt henceforth the notation $g^{(k)}(\boldsymbol{X})$ to denote the kth-order derivative of any real-valued measurable function $g(\boldsymbol{X})$.

Theorem 1. *Under Assumptions 1–4 as given in Appendix A of the supplementary material, and for each $\alpha \in \mathcal{A}$,*

$$\hat{\boldsymbol{G}}_n(\alpha) = \boldsymbol{G}_0(\alpha) + \boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + \boldsymbol{T}_{3nm}(\alpha) + \boldsymbol{R}_{nm}(\alpha), \tag{8}$$

where

$$\boldsymbol{T}_{1nm}(\alpha) = \sigma_K^2 \cdot \frac{h_m^2}{2n} \sum_{i=1}^{n} f_i^{(2)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \boldsymbol{X}_i \boldsymbol{X}_i^\top,$$

$$\boldsymbol{T}_{2nm}(\alpha) = \sqrt{\frac{-\log h_m}{m h_m}} \cdot \|K\|_2 \cdot \frac{1}{n} \sum_{i=1}^{n} \sqrt{f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)} \boldsymbol{X}_i \boldsymbol{X}_i^\top,$$

$$\boldsymbol{T}_{3nm}(\alpha) = \frac{a_2 - a_1}{n m h_m^2} \sum_{i=1}^{n} \boldsymbol{X}_i^\top \left[\sum_{j=1}^{m} \frac{1}{\sqrt{n}} \boldsymbol{D}_{nj}(\alpha) K^{(1)}\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha)\right)\right)\right] \boldsymbol{X}_i \boldsymbol{X}_i^\top.$$

In addition, $\boldsymbol{T}_{1nm}(\alpha) = O_p\left(h_m^2\right)$, $\boldsymbol{T}_{2nm}(\alpha) = O_p\left(\sqrt{\log h_m^{-1}/(m h_m)}\right)$, $\boldsymbol{T}_{3nm}(\alpha) = O_p\left(n^{-1/2}\right)$ *and*

$$\begin{aligned}\boldsymbol{R}_{nm}(\alpha) &= O_p\left(\frac{1}{n} + \frac{1}{n^{3/2} h_m^4}\right) + o_p\left(h_m^2 + \sqrt{\frac{-\log h_m}{m h_m}}\right) \\ &= o_p\left(\boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + \boldsymbol{T}_{3nm}(\alpha)\right)\end{aligned}$$

as $n \longrightarrow \infty$.

The terms $\boldsymbol{T}_{1nm}(\alpha)$, $\boldsymbol{T}_{2nm}(\alpha)$ and $\boldsymbol{T}_{3nm}(\alpha)$ given in the statement of Theorem 1 are the leading second-order terms in an asymptotic expansion in probability, for a given $\alpha \in \mathcal{A}$, of $\hat{\boldsymbol{G}}_n(\alpha)$ about the estimand $\boldsymbol{G}_0(\alpha)$. Consider

$$\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \equiv \frac{a_2 - a_1}{m h_m} \sum_{j=1}^{m} K\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha)\right)\right), \tag{9}$$

which defines a natural, but empirically infeasible, kernel estimator of $f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$ that essentially relies on $\boldsymbol{\beta}(\alpha)$ and $\boldsymbol{\beta}\left(U_j\right)$, where $j \in \{1, \ldots, m\}$, being known. Then the term $\boldsymbol{T}_{1nm}(\alpha)$ appearing in the statement of Theorem 1 reflects the conditional asymptotic biases given $\boldsymbol{X}_i$ of the estimators $\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$, defined above in (9). The magnitude of the term $\boldsymbol{T}_{2nm}(\alpha)$, on the other hand, is driven by the conditional variance given $\boldsymbol{X}_i$ of $\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)$ about

$$(a_2 - a_1) \cdot E\left[h_m^{-1} K\left(h_m^{-1} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha)\right)\right) \middle| \boldsymbol{X}_i \right].$$

Lastly, the term $\boldsymbol{T}_{3nm}(\alpha)$ corresponds to the error involved in estimating $\boldsymbol{\beta}(\alpha)$ with $\hat{\boldsymbol{\beta}}_n(\alpha)$.

3 Wald-Type Tests With Adaptive Control of Size

We consider the empirical sizes of Wald-type tests of hypotheses of the form

$$H_0: \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = 0, \tag{10}$$

where $\boldsymbol{R}$ is a fully specified $(J \times d)$ matrix with rank J, $\boldsymbol{r} \in \mathbb{R}^J$ is fully specified and α is a fixed quantile in $\mathcal{A} = [a_1, a_2]$ with $0 < a_1 < a_2 < 1$. Define the following:

$$\hat{\boldsymbol{W}}_n \equiv \boldsymbol{W}_n(\hat{\boldsymbol{G}}_n(\alpha)), \tag{11}$$

$$\boldsymbol{W}_0 \equiv \boldsymbol{W}(\boldsymbol{G}_0(\alpha)), \tag{12}$$

where for a generic positive definite matrix $\boldsymbol{G}$ we define $\boldsymbol{W}_n(\boldsymbol{G}) \equiv (\boldsymbol{R}\boldsymbol{G}^{-1}\boldsymbol{H}_n\boldsymbol{G}^{-1}\boldsymbol{R}^\top)^{-1}$ and $\boldsymbol{W}(\boldsymbol{G}) \equiv (\boldsymbol{R}\boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{G}^{-1}\boldsymbol{R}^\top)^{-1}$ with $\boldsymbol{H}_n = n^{-1}\sum_{i=1}^n \boldsymbol{X}_i\boldsymbol{X}_i^\top$.

Wald-type tests in this context are based on the asymptotic normality of regression quantiles; as such, attention is naturally directed to the sampling behavior of asymptotically-χ_J^2 statistics of the form $\{n/[\alpha(1-\alpha)]\}(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{r})^\top \boldsymbol{W}_n(\boldsymbol{G}_n(\alpha))(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{r})$, where $\boldsymbol{G}_n(\alpha)$ is a consistent estimator of the matrix $\boldsymbol{G}_0(\alpha)$. The focus in this section is on the effect estimation of the matrix $\boldsymbol{G}_0(\alpha)$ exerts on the discrepancy between the empirical and nominal sizes of the associated Wald-type test.

We address the question of whether a Wald-type test of $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = 0$ admits the possibility of *adaptive size control* as $n \to \infty$. In particular, is it possible to implement the estimator $\hat{\boldsymbol{G}}_n(\alpha)$ given above in (6) in such a way as to make the discrepancy between the actual size and nominal level of a Wald-type test of H_0 vanish at the same rate as the infeasible test in which the matrix $\boldsymbol{G}_0(\alpha)$ is actually known? That the answer to this question is positive can be seen by considering the empirical size function of a nominal level-τ Wald test of H_0. Let $\chi_{J,\tau}^2$ denote the $(1-\tau)$-quantile of a χ_J^2-distribution, and let $\boldsymbol{Z}(\alpha) \sim N(0, \boldsymbol{V}(\alpha))$, where the covariance matrix $\boldsymbol{V}(\alpha)$ is as given above in (2). Then one can combine the asymptotic normality result in (2) with Theorem 1 to deduce the following representation of the size function:

$$
\begin{aligned}
&\Pr\left[\frac{n}{\alpha(1-\alpha)}\left(\hat{\boldsymbol{\beta}}_n(\alpha)^\top\boldsymbol{R}^\top - \boldsymbol{r}^\top\right)\hat{\boldsymbol{W}}_n\left(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{r}\right) > \chi_{J,\tau}^2 \middle| H_0\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}\boldsymbol{Z}(\alpha)^\top\boldsymbol{R}^\top\boldsymbol{W}_0\boldsymbol{R}\boldsymbol{Z}(\alpha)\right. \\
&> \chi_{J,\tau}^2 - \frac{1}{\alpha(1-\alpha)}\left(h_m^2\Lambda_{1n}(\alpha,0) + \sqrt{\frac{-\log h_m}{mh_m}}\Lambda_{2nm}(\alpha,0) + \frac{1}{\sqrt{n}}\Lambda_{3nm}(\alpha,0)\right) \\
&\left.-\Theta_n(0) - \Xi_{nm}(0)\right], \qquad (13)
\end{aligned}
$$

where $\Lambda_{1nm}(\alpha,0)$, $\Lambda_{2nm}(\alpha,0)$ and $\Lambda_{3nm}(\alpha,0)$ are $O_p(1)$, $\Theta_n(0)$ converges to zero at the same rate as the error committed by the first-order asymptotic approximation in (2), and where $\Xi_{nm}(0) = o_p\left(h_m^2 + \left[\log h_m^{-1}/(mh_m)\right]^{1/2} + n^{-1/2}\right)$. Precise expressions for $\Xi_{nm}(0)$, $\Lambda_{knm}(\alpha, 0)$ ($k = 1, 2, 3$) and $\Theta_n(0)$ are given in (31)–(35) of the supplementary material.

Inspection of (13) indicates that should the matrix $\boldsymbol{G}_0(\alpha)$ be assumed or in fact be known by the researcher, then the magnitude of the term $\Theta_n(0)$ indicates the rate of convergence of the size distortion of the infeasible Wald-type test in which $\boldsymbol{G}_0(\alpha)$ is known, i.e., the *adaptive rate of size control* as $n \to \infty$. It follows that the adaptive rate of size control is determined by the accuracy of the first-order asymptotic normal approximation for $\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)$.

An important question in this connection is whether the adaptive rate of size control is so large as to dominate the estimation error associated with any reasonable estimate of $\boldsymbol{G}_0(\alpha)$; in this case one might wonder if there is much point in concerning oneself with a size-optimal implementation of a given estimator of $\boldsymbol{G}_0(\alpha)$. This concern is particularly relevant if the first-order asymptotic normal approximation to $\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)$ is of nearly $n^{-1/4}$-order, as indicated by traditional analyses of the Bahadur representation for regression quantiles (e.g., Jurečková and Sen, 1996, Theorem 4.7.1). On the other hand, Portnoy (2012, Theorem 5) has recently established that in fact the error associated with the first-order normal approximation is of nearly $n^{-1/2}$-order, which is sufficiently small so as not to dominate strictly the estimation error committed by a typical estimate of $\boldsymbol{G}_0(\alpha)$ involving local smoothing. It follows that at least under the conditions imposed by Portnoy (2012, Theorem 5), the problem of constructing a size-optimal estimator of $\boldsymbol{G}_0(\alpha)$ by choice of a smoothing parameter should be of primary concern in empirical practice.

We consider an implementation of the estimator $\hat{\boldsymbol{G}}_n(\alpha)$ given above in (6) that causes the corresponding Wald-type test of $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = 0$ to exhibit adaptive size control as $n \to \infty$. The precise conditions on the bandwidth h_m and the grid size m are specified in Assumption 3 in Appendix A of the supplementary material. These conditions suffice to make the size distortion of the Wald-type test of H_0 vanish at the adaptive rate as $n \to \infty$:

Theorem 2. *Suppose the validity of Assumptions 1–4 as given in Appendix A of the supplementary material. Then the corresponding Wald-type test of H_0 based on $\hat{\boldsymbol{G}}_n(\alpha)$ exhibits adaptive size control as $n \to \infty$.*

The same conditions also cause the Wald-type confidence interval for a given linear combination of components of $\boldsymbol{\beta}(\alpha)$ to have a level error that vanishes at the rate enjoyed by the corresponding intervals in which $\boldsymbol{G}_0(\alpha)$ does not need to be estimated.

Practical recommendations on the implementation of bandwidth parameters and grid sizes that satisfy the conditions of Theorem 2 are given in Section 4 below and also in Appendix D of the supplementary material. In particular, Wald-type tests embedding our proposed estimator of $\hat{\boldsymbol{G}}_n(\alpha)$ implemented with a fixed (i.e., non-random) bandwidth are exhibited in Section 4 below and in Appendix E of the supplementary material. Appendix D of the supplementary material, on the other hand, derives an empirically feasible data-driven bandwidth that induces corresponding Wald-type tests to exhibit adaptive size control as $n \to \infty$.

Simulation evidence on the finite-sample performance of Wald-type tests implemented with the data-driven bandwidth are presented in Appendix E of the supplementary material.

The following corollary is immediate from Theorem 2 and Portnoy (2012, Theorem 5):

Corollary 1. *Suppose the validity of Assumptions 1–4 as given in Appendix A of the supplementary material. Then the following hold as $n \to \infty$:*

1. *The size distortion of the Wald-type test of $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = 0$ involving $\hat{\boldsymbol{G}}_n(\alpha)$ is $O_p\left(n^{-1/2}(\log n)^{3/2}\right)$; and*

2. *the level error of the Wald-type confidence interval involving $\hat{\boldsymbol{G}}_n(\alpha)$ for a linear combination of the elements of $\boldsymbol{\beta}(\alpha)$ is $O_p\left(n^{-1}(\log n)^3\right)$.*

Theorem 2 and Corollary 1 jointly establish that in this context the adaptive rate of size control of Wald-type tests is of nearly $n^{-1/2}$-order, and that a Wald-type test constructed using the proposed estimator $\hat{\boldsymbol{G}}_n(\alpha)$ given above in (6) can be implemented to exhibit this rate as $n \to \infty$.

Finally, Appendix C of the Supplementary material shows that the estimators of $\boldsymbol{G}_0(\alpha)$ proposed by Powell (1991) and Hendricks and Koenker (1992) cannot induce Wald-type tests that control size adaptively in large samples.

4 Numerical Evidence

We present in this section the results of a series of Monte Carlo simulations that are motivated by the empirical question examined in Section 5. These simulations evaluate the performance of Wald-type tests for testing the heterogeneity of quantile treatment effects (QTEs; see e.g., Doksum, 1974) in covariates. We naturally focus attention on the relative performance of Wald-type tests incorporating our proposed estimator of $\boldsymbol{G}_0(\alpha)$. We compare the empirical size and size-corrected power performance of our tests to those of ten alternative testing procedures available in version 5.35 of the `quantreg` package (Koenker, 2018) for the R statistical computing environment (R Core Team, 2016). The simulations presented here are all implemented in R; in particular, we make use of the `quantreg` package to generate simulations for each of the competing testing procedures that we considered. R code to implement the simulations presented here is included in the supplementary material.

We consider the data-generating process $Y = 1 + \sum_{j=1}^{4} X_j + D + \delta_a(U)DX_1 + F^{-1}(U)$, where $\{X_j\}_{j=1}^{4}$ are iid standard normal and independent of a treatment indicator D, which follows a Bernoulli distribution with probability $1/2$, where U is an independent U[0,1] and where $a \in \mathbb{R}$ denotes the parameter indexing the family of functions $\{\delta_a(\cdot) : a \in \mathbb{R}\}$. In this model the QTE for a given setting of a, expressed as a function of a quantile of interest α, is given by

$QTE(\alpha) = 1 + \delta_a(\alpha)X_1$.

It follows that for a given quantile α, a test of the hypothesis $H_0 : \delta_a(\alpha) = 0$ against $H_1 : \delta_a(\alpha) \neq 0$ corresponds to a test of the homogeneity of the α-QTE in X_1 against the alternative of heterogeneity.

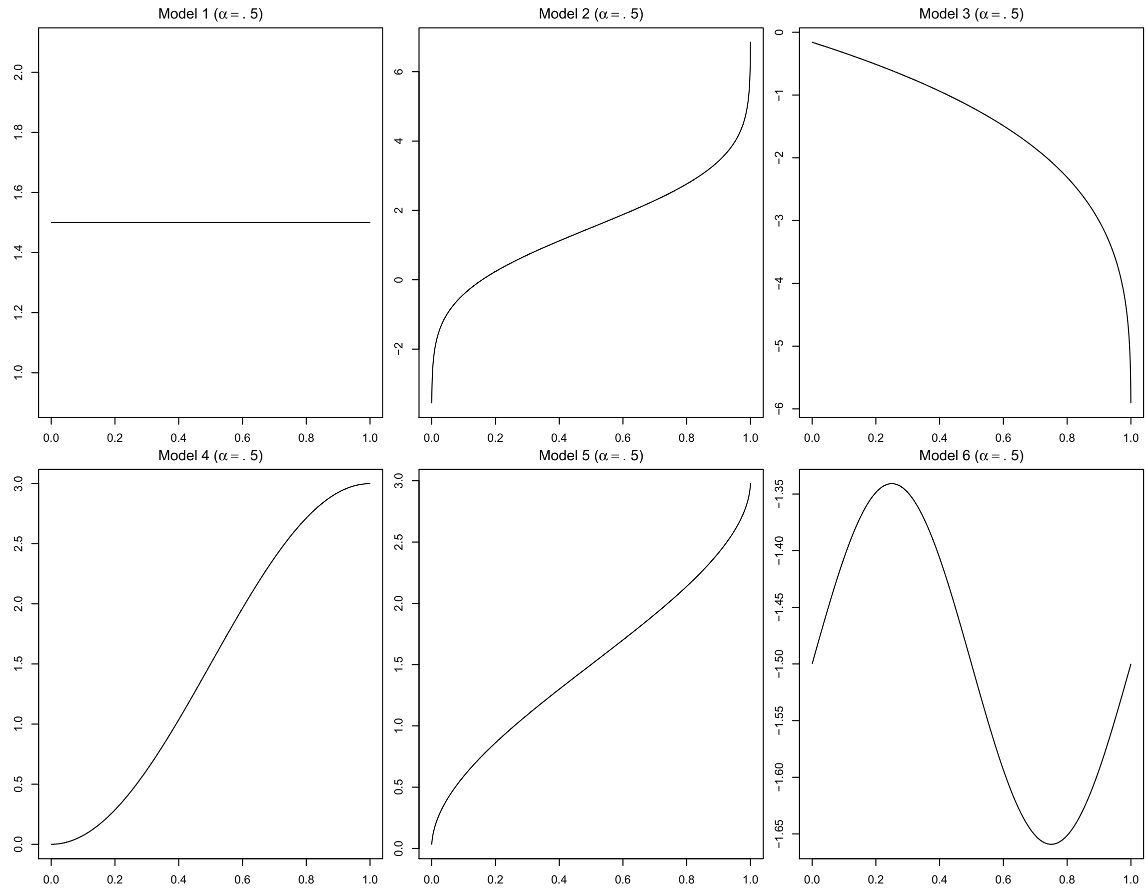
We set F in the simulations presented here to a standard normal distribution; results in which F denotes a Student-t distribution with three degrees of freedom are given in Appendix E.3 of the supplement. We consider the following specifications of the heterogeneity parameter $\delta_a(\alpha)$:

- Model 1: $\delta_a(U) = a$ (pure location).
- Model 2: $\delta_a(U) = a(1 + F^{-1}(U))$ (location-scale model).
- Model 3: $\delta_a(U) = (1 - 5a)G^{-1}(U) - G^{-1}(\alpha)$, with $G \sim Beta(1, 4)$.
- Model 4: $\delta_a(U) = 2aG^{-1}(U)$, with $G \sim Beta(0.5, 0.5)$.
- Model 5: $\delta_a(U) = 2aG^{-1}(U)$, with $G \sim Beta(2, 2)$.
- Model 6: $\delta_a(U) = (\sin(2\pi U) - \sin(2\pi\alpha) - 2\pi a)/2\pi$.

Each of these models satisfies the null hypothesis of treatment homogeneity when $a = 0$. Under the null, all models but Models 3 and 6 are pure location models. The alternative hypothesis corresponds to $a \neq 0$. Size-corrected power performance is considered against alternatives corresponding to the settings $a = 0.50$, 1.00 and 1.50. The corresponding heterogeneity parameters for Models 1–6 under $\alpha = .50$ are plotted in Figure 1 for the case where $a = 1.50$. It is clear that our specifications of Models 1–6 imply QTEs with very different functional forms.

The simulations presented below consider the size and power performance over 1000 Monte Carlo replications of nominal 5%-level tests for α-quantile regression parameters, where $\alpha \in \{.25, .50, .75\}$. Average CPU times over 1000 replications required to implement each of the tests

Figure 1: Heterogeneity parameters for Models 1–6 under α-QTE heterogeneity ($a = 1.50$), where $\alpha = 0.5$



examined here are also reported. We considered simulated samples of size $n \in \{100, 300\}$. The techniques used to compute the tests considered are as follows:

- `weg`: Wald-type tests incorporating our proposed estimator $\hat{\boldsymbol{G}}_n(\alpha)$, where α is the quantile of interest. The proposed estimator $\hat{\boldsymbol{G}}_n(\alpha)$ was implemented using the Epanechnikov kernel with m quantiles uniformly distributed over the range $[a_1, a_2] = [.01, , 99]$, with

$$m = \left\lfloor \left[\frac{k}{(\log n)^{\frac{11}{5}}} \right]^{\frac{5}{4}} \right\rfloor \tag{14}$$

  and $k = 5$. The bandwidth considered is given by

$$h_m = c \left(\frac{\log m}{m} \right)^{1/5} \tag{15}$$

  where $c = 1.5$. The choices of m and h_m are motivated from the theoretical results presented earlier in Section 3. The choice of m in (14) in particular coincides with the lower bound on the rate of divergence of m as a function of n in our asymptotic results. Appendix E.1 in the supplement contains extensive simulation results in which we vary the constants k and c. It is shown there that the choice of k is not as important in terms of finite-sample test performance as the choice of c. Our experience with several data-generating processes, including the ones above, suggest that the choice $c = 1.5$ performs very well. We nevertheless develop in Appendix D of the supplement a data-driven method for choosing the bandwidth constant c for a given value of m, which is similarly shown in Appendix E.2 to induce good test performance.

- `riid`: Rank tests assuming a location-shift model with iid errors (Koenker, 1994).

- `rnid`: Rank tests assuming a potentially heteroskedastic location-scale-shift model (Koenker and Machado, 1999).

- `wiid`: Wald-type tests assuming a location-shift model with iid errors, with scalar sparsity estimate computed as in Koenker and Bassett (1978).

- `wnid`: Wald-type tests assuming independent but not identically distributed errors incorporating the difference-quotient estimator denoted by $\hat{\boldsymbol{G}}_n^{HK}(\alpha)$ in (38) of the supplement and implemented using the Hall and Sheather (1988) rule-of-thumb bandwidth.

- `wker`: Wald-type tests assuming independent but not identically distributed errors incorporating the kernel estimator denoted by $\hat{\boldsymbol{G}}_n^{P}(\alpha)$ in (36) of the supplement, where $\hat{\boldsymbol{G}}_n^{P}(\alpha)$ was implemented using a uniform kernel supported on $[-1, 1]$ and the bandwidth $\delta_n^{P,HS} \equiv \Phi^{-1}\left(.50 + h_n^{HS}\right) - \Phi^{-1}\left(.50 - h_n^{HS}\right)$, where h_n^{HS} is the Hall and Sheather (1988) rule-of-thumb bandwidth.

- `bxy`: Bootstrap tests based on the (x, y)-pair method.

- `bpwy`: Bootstrap tests based on the Parzen et al. (1994) method of resampling the subgradient condition.

- `bmcmb`: Bootstrap tests based on the "MCMB-A" variant of the Markov chain marginal bootstrap method of He and Hu (2002), described in Kocherginsky et al. (2005). This variant of the method of He and Hu (2002), in common with the `riid` and `wiid` methods described above, assumes an underlying location-shift model with iid errors.

- `bwxy`: Bootstrap tests based on the generalized bootstrap of Bose and Chatterjee (2003) with unit exponential weights.

- `bwild`: Bootstrap tests based on the wild bootstrap method proposed by Feng et al. (2011).

The Wald-type tests computed using the `wiid`, `wnid` and `wker` methods were all implemented using the default bandwidth setting in the `quantreg` package (Koenker, 2018), namely the Hall and Sheather (1988) rule-of-thumb-bandwidth appropriate for inference regarding a population quantile. In addition, the bootstrap tests were all implemented with the default setting of 200 bootstrap resamples.

Each of `wiid`, `wnid`, `wker`, `bxy`, `bpwy`, `bmcmb`, `bwxy` and `bwild` was implemented by direct computation of the corresponding test statistic using the corresponding standard error returned by the `summary.rq` feature of `quantreg`. On the other hand, the rank-based procedures `riid` and `rnid` both involved direct inversion of the corresponding confidence interval obtained from the `summary.rq` feature.

The corresponding simulation results are displayed in Tables 1–6. These results include average CPU times in seconds over 1000 replications taken to compute each test statistic. These average timings correspond to simulations under the null (i.e., the setting $a = 0$) when the quantile of interest is given by $\alpha = 0.5$. Average timings for simulations in which $a \neq 0$ or $\alpha \neq 0.5$ are virtually identical.

We also examined in unreported work implementations of `wiid`, `wnid`, `wker` and `riid` available from the `anova.rq` feature of `quantreg`, but the resulting tests were found to exhibit empirical rejection probabilities that were virtually identical to those of the corresponding implementations of these tests using `summary.rq`. We also noticed that `anova.rq` has a noticeable tendency to run more slowly than `summary.rq` for `wiid`, `wnid` and `wker`, and more quickly than `summary.rq` for `riid`.

We see that the empirical size of the proposed method is accurate even with samples of sizes as small as $n = 100$, and is often more accurate than alternative methods, including resampling methods. We also see that the proposed Wald test has good size-corrected power across all six models, three quantiles and two sample sizes for relatively small deviations from the null, i.e. when the constant a is small. It seems clear that an analytical comparison of the asymptotic local relative efficiencies of the different tests considered here with that of the asymptotically uniformly most powerful test (Choi et al., 1996) would be interesting, although such an analysis seems beyond the scope of this paper. We note in passing that the conditional density estimator embedded in our method of inference can be instrumental in estimating the efficient score (Newey and Powell, 1990) and thus in developing asymptotically optimal inference for quantile regression.

Table 1: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 1. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$				CPU time ($\alpha = .50$)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
weg	5.6	16	31.6	52	4.5	24.5	56	81.3	5.1	21.8	43.7	68	0.0118
wiid	9.1	10	22.2	39.5	7.3	15.5	45.7	75.9	8.2	12.5	31.1	56.3	0.0025
wnid	8.1	8.3	18.7	37.9	6.8	17.5	51	80	7.4	12.2	33.1	59.9	0.0021
wker	1.3	13.2	31.5	53.7	0.3	17	51.2	80.8	1.9	17.7	41.8	69.5	0.0015
riid	7.9	8.6	21.4	39.4	8.6	17.7	46.5	76.9	7.5	15.3	35.5	61.5	0.0049
rnid	5.9	7.4	19	37.7	6.5	17.5	46.7	76.5	5.1	15.2	34.7	61.3	0.0156
bxy	3.1	9.6	23.6	44.7	2.9	16.7	49.8	80	3.2	14.8	37	65.7	0.0212
bpwy	1.2	9.7	23.7	44.3	2.4	17.1	49.4	80.4	1.6	17.5	41.1	69.6	0.0229
bmcmb	3.3	8.8	23.2	43.3	3.7	16	48.9	79.2	3.4	16.6	39.7	66.7	0.0137
bwxy	4.1	9.3	22.9	44.5	3	16	48.4	79.9	4.4	13.7	36	64.6	0.0218
bwild	6.9	10.9	24	46.2	7.2	14.1	42.7	76.1	6.2	16.2	37	65.4	0.0235
$n = 300$													
weg	5.4	32.1	79.8	97.7	3.2	40	84.5	98.1	4.1	36.7	85.4	98	0.0453
wiid	7.9	25.4	74.2	98.1	3.7	33.6	84.3	98.5	6	30.5	84.5	99.6	0.0026
wnid	8.2	26.2	76.1	98.6	3.9	34.9	86.4	98.6	5.9	32.5	84.7	99.3	0.0035
wker	3	28.4	79.5	99.3	1.3	34.5	85.9	98.7	2	34.3	87	99.7	0.0017
riid	7.7	27	75.8	97.6	5	31.4	80.5	98.1	5.6	31.7	81.6	98.8	0.0193
rnid	6.6	26.5	74.7	97.6	4.7	31.4	80.4	98	4.7	31	82.3	98.6	0.0311
bxy	4.4	29.4	79.2	98.3	2.5	34.1	84.4	98.4	3	32.7	85.5	99.4	0.0948
bpwy	3.4	28.9	78.8	98.7	2.2	34.4	84.9	98.4	2.3	34.5	85.9	99.3	0.0991
bmcmb	5.9	26.9	77.9	98.4	3.7	33.7	82.4	98.3	3.8	32.5	84.6	99.2	0.0369
bwxy	4.9	29.2	79.1	98.8	2.7	32	82.4	98.4	3.1	31.5	83.9	99.2	0.1002
bwild	7.1	29	79.1	98.7	4.8	32.3	82	98.3	4.9	31.9	85.7	99.6	0.1018

Table 2: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 2. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

n = 100		α = 0.25				α = 0.5				α = 0.75			CPU time (α = .50)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
weg	5.7	15.4	31.3	50.5	5.9	20.6	46.5	69.6	6.4	22.2	48.8	67.5	0.0108
wiid	8.4	10.2	20.5	38.9	8.9	12.7	34.5	62.7	9	15.7	39.9	63.1	0.0022
wnid	7.4	7.3	21.8	40.2	9.1	12.6	37.2	65.3	8.6	14.5	42.1	64.4	0.002
wker	1.5	8	21.9	39.9	1.1	12.5	36.5	63.2	1.7	11.3	37.7	61	0.0014
riid	7.7	7.9	20.2	36.7	8.7	11.1	31.7	55.1	8.2	14.7	37.6	60.1	0.0047
rnid	5.8	7.5	20.2	36	7.2	11.4	31.1	54.1	6	14	35.9	57.1	0.0142
bxy	3.4	7.9	20.3	37.6	3.4	12.6	36.2	60.3	4.1	14.6	39.3	62.1	0.021
bpwy	1.8	7.1	20.8	40.2	2.9	12.8	37.1	62.7	2.5	12.7	40	62.6	0.0225
bmcmb	3.4	8	20.5	36.7	4.1	12.7	36.2	60.1	4.6	15.3	39.2	61	0.0131
bwxy	4.5	8.3	20.6	37.9	4.2	13.2	37.1	60.2	5.2	13.5	38.7	61.5	0.0216
bwild	7.4	7.3	18.7	35.4	8.4	12.9	35.1	57	7.3	14.1	38.5	59.2	0.0229
n = 300													
weg	4.1	24	64.5	88.1	3.2	41	83.9	97.7	4.9	42.5	85.4	97.3	0.0445
wiid	5.5	20.7	58.8	88.4	5	32.2	81.3	98.5	8	34	83.4	98.3	0.0025
wnid	5.9	19.6	60.1	88.6	4.8	35.6	84.6	98.5	8.4	36	86.5	98.9	0.0034
wker	2.3	18.4	57	86	1	35.9	82.2	97.9	2.3	36	85.7	98.7	0.0016
riid	6	17.9	55	83.9	5.4	31.8	77.7	96.5	7.5	35.6	82	97.3	0.0193
rnid	4.6	17.3	53.3	83.1	5.1	30.7	76.9	96.2	6.8	33.8	80.9	96.8	0.0311
bxy	2.6	20.7	58.8	84.2	3.7	32.7	79.9	96.9	3.7	38.4	84.6	98	0.0945
bpwy	2.4	18.1	55.3	83.7	3	32.7	79.1	97	3	38.5	84.7	98.4	0.0997
bmcmb	4.3	18.3	53.1	82.7	4.4	31.2	78.4	97	5	37.9	84.2	97.4	0.0369
bwxy	2.6	17.8	53.8	81.8	3.6	31.5	78.6	96.7	4	36.1	82.8	97.3	0.1003
bwild	5.1	19.1	55.8	84.1	5	30.7	78.9	96.4	6.1	36.1	84.6	98.5	0.1024

Table 3: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 3. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			CPU time ($\alpha = .50$)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
weg	5.9	12.5	24.3	43.7	4.8	21.3	43.2	63.8	5	28.3	57.6	79.4	0.0109
wiid	9.7	6.1	14.5	26.4	7.5	11	28.5	53.3	7.7	16.1	44.9	71.8	0.0023
wnid	7.9	8.4	19	36.8	6.7	11	31.9	56.8	7.2	18.2	47.4	72.1	0.002
wker	1.4	8.1	19.7	39.2	0.7	12.5	33.4	58	1.4	18.9	52.6	78.1	0.0014
riid	7.5	6.5	15.7	32.6	7.3	9.4	26.7	47.4	8	16.9	43.9	68.2	0.0048
rnid	5.3	6.7	16.6	32.2	6.5	9.3	27.8	45.6	5.5	17.3	45.4	68.3	0.0145
bxy	2.4	8.3	19.1	37.9	2.8	12.3	32.3	55.7	3	19.3	49.2	75.2	0.021
bpwy	1.2	8.1	20.3	38.2	2.4	11.6	31.8	54.2	1.5	18.7	50.2	75.7	0.0228
bmcmb	2.6	7.5	18.5	34.5	3.6	11.6	31.8	54.7	3.1	18.1	47.1	73	0.0133
bwxy	3.1	8.5	20.2	37.6	3.5	10.7	30.9	54.2	3.9	18.9	49.5	74.3	0.0215
bwild	6.3	7.7	18.5	35.7	7.6	10	27.7	50.2	7	17.1	47.2	73.6	0.0235
$n = 300$													
weg	4.9	18.3	53	83.1	4.3	29.6	75.8	96.4	6.1	44	88.9	98.4	0.044
wiid	6.6	12.5	46.4	81.3	6.9	24.4	74	96.3	6.9	41.1	91.1	99.5	0.0025
wnid	6.8	14.7	52.7	84.1	5.8	28.7	78.2	97.3	7.7	41.4	92	99.7	0.0035
wker	3.3	15.4	52.7	84.5	1.6	28.2	76.7	96.2	3.2	40	90.4	99.7	0.0017
riid	5.8	15.6	49.7	82.2	6.4	26	72.1	95	7.3	38.3	87.3	98.9	0.0193
rnid	5	15	48.1	80.5	6	25.4	70.4	94.4	6.4	37.9	86.5	99	0.0308
bxy	3.7	16.1	50.3	83.3	3.5	27.3	74.7	95.6	3.8	41.1	89.9	99.6	0.0946
bpwy	3.1	15.6	52	83.7	3	28.2	75.4	95.9	2.8	38.5	89.8	99.2	0.0993
bmcmb	4.7	14.8	49.7	81	4.7	28.7	76.5	96	5.1	40.8	90.5	99.4	0.0367
bwxy	3.7	14.9	51	82.8	3.7	28.5	75.7	96	4.2	39.9	90	99.6	0.1001
bwild	6.3	13.9	48.7	81.9	5.9	25.3	73.3	95.7	6.8	37.8	88.9	99.5	0.1021

Table 4: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 4. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			CPU time ($\alpha = .50$)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
weg	6.5	14.2	27	45.3	4.7	23.1	52.6	73.2	6.2	22.4	49.3	74.6	0.0115
wiid	9.8	6	16	30.4	7.5	14.2	41.1	68.6	9.8	13	36.8	66.9	0.0025
wnid	8.5	6.3	15.6	32.2	7.8	14	42.8	69	8.2	15.9	43.8	71.8	0.0021
wker	1.4	11.7	24.5	43.2	1.1	12.7	43	66.9	1.7	16.2	45.1	73.2	0.0015
riid	7.7	7.4	17.8	31.6	7.4	15.3	40.3	63.1	7.9	14.4	41	67.2	0.0049
rnid	5.4	8.2	18.8	34.5	6.3	13.8	39.5	62.1	5.5	15.9	41.7	68.4	0.0154
bxy	3.2	9.1	19.6	37.8	3.6	14.6	42.5	65.3	3.1	17.6	46.3	72.6	0.021
bpwy	1.5	8.5	20.7	38	2.7	13.8	40.6	64.2	1.1	17.1	47.3	74.9	0.0234
bmcmb	4.4	6.7	17.2	33.3	4.1	14	41.2	64.2	3.2	17.3	45.9	71.4	0.0136
bwxy	4.4	8.9	20.2	37.7	3.9	15	42.9	66.2	4.3	17.7	47	72.5	0.0216
bwild	7.4	9.2	20.7	37.3	6.7	13.6	40	64.3	7.8	15.3	41.5	68.7	0.0233
$n = 300$													
weg	4.9	24.9	59	85	3.9	36.5	81.5	97.3	5	45.4	87	98.5	0.0438
wiid	6.5	14.5	48.1	81.5	6.9	28.7	79.6	97.7	5.9	40.1	88.3	99.2	0.0025
wnid	7.3	17.6	53.3	84.3	7.2	28.2	79.6	97.6	5.9	42.9	90.3	99.4	0.0034
wker	3.5	23.3	59.9	87.1	2.1	28.9	78.6	97.6	2.3	41.1	88.9	98.8	0.0016
riid	7.2	17.4	49.2	81.6	8	26.7	76.5	96	5.6	40.4	86.6	98.2	0.0191
rnid	6.1	17.5	50.8	81.9	6.8	25.6	76.1	95.5	4.8	41	86	98.1	0.0306
bxy	4.5	18.6	52.2	82.5	3.8	27.8	77.6	96.4	3.3	40.8	87.6	98.3	0.0937
bpwy	4	18	55	84.5	4.4	28.9	77.2	96.5	2.3	42.2	87.9	98.5	0.0992
bmcmb	5.6	17	50.8	81.5	5.7	28.8	78.2	96.5	4.6	41.1	87.4	98.1	0.0367
bwxy	4.5	18.2	52.5	82.5	4.8	28.1	76.7	96.2	3.3	43.5	88.6	98.4	0.0993
bwild	6.6	17.9	53.1	82.9	6.5	25.9	75.7	96	5	41.5	88.3	98.7	0.1017

Table 5: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 5. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			CPU time ($\alpha = .50$)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
weg	5.8	17.3	34.4	52.8	4.7	19.9	40	62.6	6.7	17.5	39.8	64.9	0.0109
wiid	8.5	10.9	22.7	42.1	7.2	11.5	27.7	55.6	9.6	11.3	28.9	54.6	0.0023
wnid	8.2	10.1	25.5	46.1	6.7	11	32.5	59.5	8.2	10.7	31	57.4	0.002
wker	1.1	13.1	30.4	54.3	0.7	12.3	34	60.5	1.5	12.8	35.9	65.2	0.0014
riid	7.3	11.1	25.7	45.9	8.1	9.4	27.2	51.8	8.4	11	29.5	58.8	0.0049
rnid	5.3	11.2	26.1	45.8	7	10.7	27.3	51.9	6.2	11.7	28.9	56.8	0.0145
bxy	2.7	11.4	27.1	49.4	2.5	11.9	32.7	58.7	3.3	12.4	33.3	62.7	0.021
bpwy	1.2	12.1	28.8	50.7	2.6	12.5	33.8	60.3	2	12.1	34	64	0.0231
bmcmb	2.9	10.8	27.6	47.5	3.7	11.3	31.8	59.5	3.5	11.7	32.8	59.4	0.0134
bwxy	4.2	11.3	27.7	48.8	3.6	11.3	32.4	58.6	4.4	11.8	32.7	61.8	0.0215
bwild	6.8	12.4	26.9	47.2	7	9.7	28.6	53.9	7.4	10.5	31.8	61.2	0.0231
$n = 300$													
weg	5.4	26.6	71.2	94.9	4.1	34.6	78.2	96.3	4.8	40	84.1	97.6	0.0456
wiid	7.2	24	66.6	94.6	6.6	25.9	73.1	96.5	6.6	33.6	83.5	99.3	0.0027
wnid	6.9	24.5	68.4	95.6	6.3	29	76.7	97.5	7	37.4	86.6	99.3	0.0036
wker	2.7	26.5	72.1	96.6	1.7	30.4	77.3	97.8	2.6	38.3	87.9	99.4	0.0017
riid	6.4	20.3	63.8	91.7	5.9	25.9	72.7	95.5	6.9	33.3	82.5	98.5	0.0193
rnid	5.4	22.2	66.5	92.9	5.5	26.7	73.3	95.5	5.7	34.1	83.8	98.6	0.0318
bxy	3.6	24.7	70.3	95.5	3.8	29.4	75.4	97.4	4	34.3	84.9	99	0.0944
bpwy	3.5	23	68.2	95	3.6	28.6	75.9	97.1	2.7	37.7	85.8	99.2	0.0997
bmcmb	4.9	24	68.6	95.3	5	28.1	75.5	96.9	4.6	36.2	85.4	99.3	0.0373
bwxy	4	24.4	69.7	95.7	4.1	29.2	75.7	97.1	4.1	35	85.2	99.1	0.1
bwild	6.4	23.1	69.1	95.7	6	28.1	74.7	97.1	5.7	35.3	85.1	99.1	0.1026

Table 6: Empirical rejection percentages (size and size-corrected powers) and average execution time, Model 6. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			CPU time ($\alpha = .50$)
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50	
`weg`	5.2	24.5	49.4	74	3.6	21.5	59.2	88.4	5.6	12.8	30.2	54.9	0.0109
`wiid`	9.7	12.2	31.5	57.4	7.2	13.6	44.1	80.4	10	6.7	18.7	40	0.0023
`wnid`	7.3	16.3	39	66.6	5.9	16.3	52.4	86.7	8.1	7.8	21.8	43.7	0.002
`wker`	1.3	20.2	47.4	75.8	0.8	16.3	53.5	89.2	2.2	8.9	26.2	52.1	0.0014
`riid`	8.4	15	36.8	62.3	7.5	14.7	46.5	80.5	7.8	5.7	19.8	41.2	0.0049
`rnid`	6.7	13.1	35	60.1	5.5	15	46.8	82.8	5.6	6.2	20.8	43.7	0.0145
`bxy`	2.7	17.6	41.4	70.9	2.4	16.7	52.5	86.9	3.1	8.7	25.1	50.4	0.0209
`bpwy`	1.5	17.7	42.7	71.8	1.9	16.9	51.1	87.2	1.7	8	22.5	48.7	0.0228
`bmcmb`	3.1	15.8	40.3	69.1	3.4	15.8	51.8	85.7	3.6	8.6	23.7	50.7	0.0133
`bwxy`	3.9	17.7	41.7	71.5	2.9	17.5	52.7	87.3	4.2	8	23.2	49.4	0.0214
`bwild`	6.9	16.2	40.2	70	6.7	14	46	83.3	7.3	7.3	21.8	46.2	0.0231
$n = 300$													
`weg`	5	46.8	86.4	98	5.2	32.2	79.9	98.4	4.2	21	61.8	93.2	0.044
`wiid`	6.4	39.7	87.4	99.2	8.3	25.9	76.7	97.9	6.5	13.9	54.3	91.3	0.0024
`wnid`	6.4	42.8	89.3	99.7	8.1	26.7	78.4	98.9	6.5	16.6	60.6	94.2	0.0034
`wker`	3.2	43.6	91	99.7	2.4	31.5	83.4	99	2.6	16	60.5	94.5	0.0016
`riid`	6.9	39.3	86.5	99	7.5	25.2	71.8	97	6.4	15.1	56.7	92	0.0194
`rnid`	6.1	39.7	86.1	99	6.9	27.8	76.3	97.6	5.3	15.2	55.7	92	0.031
`bxy`	3.2	43.8	89.3	99.4	4.4	30.3	80.1	98.4	3.1	16.1	59.3	93.5	0.0945
`bpwy`	3.2	42.5	88.1	99.4	4.2	29.4	80.9	98.5	3.1	16.3	58.1	93.2	0.0997
`bmcmb`	5.2	40.8	88.3	99.4	6.3	28.3	78.6	98.5	4.7	15.2	58.3	93.3	0.0368
`bwxy`	4.6	39.8	87.7	99.5	5	28.8	79.4	98.4	3.5	16.5	58.5	93.7	0.0999
`bwild`	5.7	39.7	87.7	99.5	6.7	29.2	80.4	98.7	6.1	14.2	56.8	93.1	0.1025

The simulations presented here, along with further simulations reported in the supplementary material, indicate the potential of Wald-type tests based on our proposed method to deliver good size accuracy and reasonable power across a range of quantiles and data-generating processes. These simulations also support the theoretical results presented earlier in Section 2 inasmuch as the size accuracy of the test tends to outperform those of the other Wald-type tests considered over the three different quantiles and six data-generating processes considered in our simulations.

5 Empirical Example

We consider the reemployment bonus experiments conducted in Pennsylvania by the United States Department of Labor between July 1988 and October 1989 (Corson et al., 1992). This experiment involved the randomized assignment of new claimants for unemployment insurance (UI) benefits into one of several treatment groups or a control group. Claimants assigned to the control group were handled according to the usual procedures of the unemployment insurance system, while claimants assigned to treatment were awarded cash bonuses if they were able to demonstrate full-time reemployment within a specified qualifying period.

The corresponding data were previously analyzed using quantile-regression methods by Koenker and Bilias (2001) and Koenker and Xiao (2002); Koenker and Bilias (2001) also discuss older literature evaluating similar experiments. We follow Koenker and Xiao (2002) by focusing solely on a single treatment group, which combined with the control group yields a sample of size $n = 6384$. The corresponding dataset is publicly available and can be downloaded from `http://www.econ.uiuc.edu/~roger/research/inference/Penn46.ascii`. Claimants for unemployment benefits that were assigned to this treatment were offered a bonus equal to six times the usual weekly benefit if they secured full-time employment within 12 weeks. Because approximately 20% of the subjects were reemployed within one week and another 20%

were not reemployed within a 26-week follow-up window, Koenker and Xiao (2002) assume a quantile-regression specification of the form $F^{-1}_{\log T|\boldsymbol{X}}(\alpha) = \boldsymbol{X}^\top \boldsymbol{\beta}(\alpha)$, where $\alpha \in [.20, .80]$, where T denotes the duration of unemployment in weeks and where the regressors contained in $\boldsymbol{X}$ include a constant term, an indicator for assignment to treatment and the fourteen demographic or socioeconomic control variables listed in Koenker and Xiao (2002, p. 1603).

We depart from the specification of Koenker and Xiao (2002) by including interactions of the treatment indicator with each of the control variables used by these authors. We also include interactions of the indicator for gender with indicators for race, Hispanic ethnicity and number of dependents. We consider, for a given quantile in the interval $[.20, .80]$, the hypothesis that the treatment interaction terms in $\boldsymbol{X}$ are jointly insignificant, i.e., that the effect of treatment at a given quantile in $[.20, .80]$ does not vary with any of the control variables included in $\boldsymbol{X}$. Appendix F of the supplementary material presents some additional evidence specific to the question of whether the effect of treatment in this context varies by age or by participants' stated expectation of being recalled to a previously held job.

Figure 2 reports p-values for the hypothesis of covariate homogeneity in treatment over each quantile in a grid of 300 points in $[.20, .80]$. Our test is implemented using our proposed method with the data-driven bandwidth with $k = 5$ discussed in detail in Appendix D of the supplement. We also compare the p-values from tests implemented using our method with the corresponding p-values from the alternative testing methods considered in the simulations reported above. In particular, the `wiid`, `wnid`, `wker`, `bxy`, `bpwy`, `bmcmb`, `bwxy` and `bwild` methods were implemented by direct computation of the corresponding Wald-type statistic using the estimated asymptotic covariance matrix generated by the `summary.rq` feature of version 5.35 of the `quantreg` package (Koenker, 2018) for the R statistical computing environment (R Core Team, 2016). The `riid` method, on the other hand, was implemented by direct invocation of the `anova.rq` feature of `quantreg`.

One can see from Figure 2 that our proposed procedure implies significant covariate-heterogeneity in quantile treatment effects at the .10-level over nearly all quantiles between .43 and .74. Unreported results indicate that the joint significance observed at these quantiles is driven largely by the significance of two covariates, namely the interaction between treatment and an indicator variable for being younger than 35 years of age, and the interaction between treatment and an indicator for whether a given participant expected to be recalled to previous employment. Additional results reported in Appendix F of the supplement reveal significant differences in quantile treatment effects between participants younger than 35 and those aged 35 and older for nearly all quantiles between .50 and .80. In particular, the corresponding participants aged 35 and older are shown to exit unemployment significantly more slowly than those younger than 35.

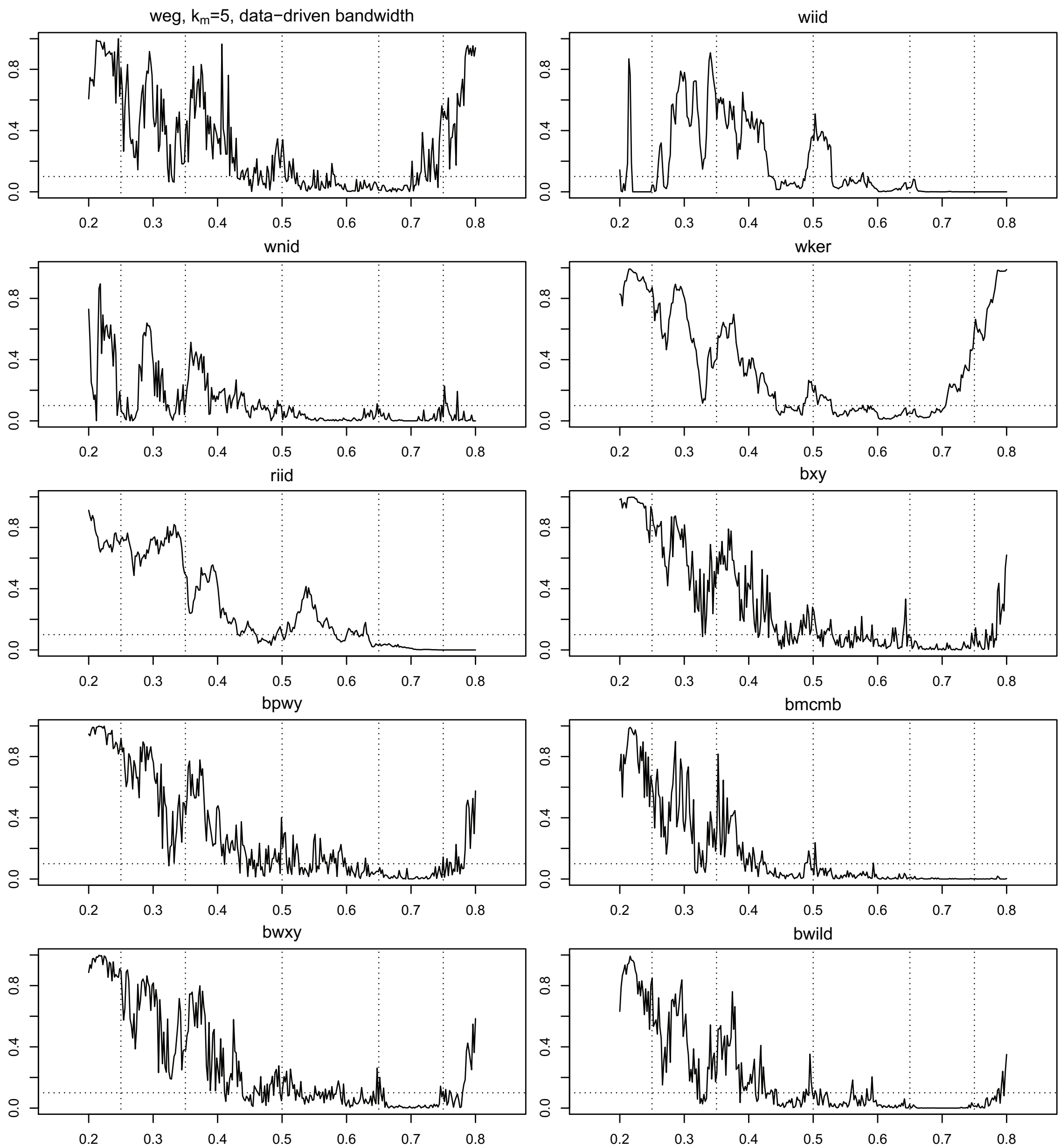
Significant differences in quantile treatment effects between participants expecting recall to a previous job and those not expecting recall are also shown in Appendix F to exist for nearly all quantiles between .43 and .74. This last result is potentially important in evaluating the cost-effectiveness of the program given the experiment's exclusion of all claimants for unemployment insurance for whom inclusion in the treatment group was deemed not to provide a sufficient encouragement "to search for work more diligently and to accept suitable employment more rapidly than would be the case otherwise" (Corson et al., 1992, p. 10). The experimenters specifically excluded from the study all claimants who indicated a definite expectation of being recalled to a previous employer on a specific date within 60 days of filing their applications for UI benefits. These claimants were deemed to be so secure in their expectation of future full-time employment that any bonus paid to them upon resuming full-time employment would be interpreted as a windfall. Included in the experiment, however, were those claimants who indicated some expectation of being recalled to a previous job, although with no definite date of recall. The experimenters deemed claimants in this category to be similar to claimants with no stated expectation of returning to a previous job in terms of their assumed response to a promised bonus payment upon resuming full-time employment within the qualifying period. The results presented in Appendix F of the supplement indicate that UI claimants who indicated some expectation of being recalled, although not to the extent of having a specific date of recall, in fact differ in their responses to treatment than those claimants who indicated no expectation of recall whatsoever.

Figure 2 also shows that the other testing methods considered varied in the extent to which the hypothesis of covariate-homogeneity in the treatment effect was rejected over quantiles in the interval $[.20, .80]$. In particular, none of the additional inference methods considered was seen to imply the same range of quantiles corresponding to covariate heterogeneity in the corresponding quantile treatment effects that was revealed by our method. For example, `wiid` yielded significance at all quantiles greater than .53. We note in addition that some p-values for tests implemented using `wker` in fact exceed .98 for most quantiles above .78, which suggests that the corresponding regression-quantile covariance matrices were not well estimated by `wker`.

In view of the rejection, reported by Koenker and Xiao (2002), of the null of a linear location-shift model for quantiles on the interval $[.25, .75]$, we interpret the `wiid` method's conclusion of significance at all quantiles greater than .53 as misleading, and likely driven by misspecification of the assumed location-shift model. As such, inferences resulting from other methods that assume a linear location-shift model (i.e., `riid` and `bmcmb`) are similarly likely to be misleading.

In summary, we have used our proposed method of inference to show that the effect of treatment on the duration of employment tends to vary with individual characteristics of the experimental subjects only over a relatively narrow range of quantiles between .43 and .74. These ranges of quantiles corresponding to covariate heterogeneity in the effect of treatment is not matched by any of the other testing methods considered. It follows that our proposed method permits an understanding of the effectiveness of a particular unemployment relief policy distinct from that produced by other methods of inference.

Figure 2: Pennsylvania reemployment bonus experiment: 6384 observations. p-values for pointwise tests of covariate-homogeneity in treatment effect, α-quantile regressions, $\alpha \in [.20, .80]$. The dotted horizontal line denotes significance at the 10% level.



SUPPLEMENTARY MATERIAL

Appendices: Appendix A contains precise statements of the assumptions used in Theorems 1 and 2; Appendix B contains proofs of Theorems 1 and 2; Appendix C shows that the estimators of $\boldsymbol{G}_0(\alpha)$ proposed by Powell (1991) and Hendricks and Koenker (1992) cannot induce Wald-type tests that control size adaptively in large samples; Appendix D describes a data-driven, as opposed to a fixed, bandwidth to implement our proposed estimate of $\boldsymbol{G}_0(\alpha)$; Appendix E reports further simulation evidence on the finite-sample performance of our proposed method relative to its competitors, while Appendix F contains further investigation of the empirical example presented in Section 5. (qdf60supp.pdf)

R programs: We also include R code that enables reproduction of the simulation results in Section 4 and Appendix E and of the empirical analyses reported in Section 5 and Appendix F. (qdf60code.zip)

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Quantile-Regression Inference With Adaptive Control of Size
SUPPLEMENTARY MATERIAL

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Abstract

This supplement contains precise statements of the assumptions underlying the theoretical results in the main body of our paper. This supplement also contains proofs of those results, analyses of the established estimators of $\boldsymbol{G}_0(\alpha)$ proposed by Powell (1991) and Hendricks and Koenker (1992), details regarding a data-driven bandwidth that can be used to implement the proposed Wald-type testing procedure, results of further simulations and further discussion of the empirical example presented in the paper.

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A Assumptions

Precise statements of the regularity conditions underlying the theorems presented in the main body of our paper are collected here. Henceforth, $(\mathbf{X}_1^\top, Y_1), \ldots, (\mathbf{X}_n^\top, Y_n)$ is am iid sample of size n. Let $\|\cdot\|$ denote the Euclidean norm, let $\mathcal{X}$ denote the common support of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, and define

$$\mathcal{B} \equiv \left\{ \boldsymbol{x}^\top \boldsymbol{\beta}(\alpha) : \boldsymbol{x} \in \mathcal{X}, \alpha \in \mathcal{A} \right\}. \tag{1}$$

Assumption 1. *The following conditions hold: (1) Uniformly in* $\alpha \in \mathcal{A}$, *where* $\mathcal{A} = [a_1, a_2]$ *for* $0 < a_1 < a_2 < 1$, $\boldsymbol{G}_n(\alpha) \equiv n^{-1} \sum_{i=1}^n f_i \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \boldsymbol{X}_i \boldsymbol{X}_i^\top = \boldsymbol{G}_0(\alpha) \left(1 + O_p \left(n^{-1/2} \right) \right)$ *as* $n \to \infty$*; (2)*

$$E \left[\|\boldsymbol{X}_i\|^4 \right] < \infty; \tag{2}$$

and (3) $\boldsymbol{G}_0(\alpha)$ *and* $\boldsymbol{H} = E \left[\boldsymbol{X}_i \boldsymbol{X}_i^\top \right]$ *are positive definite.*

Assumption 2. *The smoothing kernel* $K(\cdot)$ *satisfies the following conditions: (1)* $K(\cdot)$ *is nonnegative, symmetric and bounded with support* $[-1/2, 1/2]$, *with* $\|K\|_2 \equiv \sqrt{\int_{-1/2}^{1/2} K^2(w) dw} \in (0, \infty)$*; (2)* $\int_{-1/2}^{1/2} K(w) dw = 1$ *and* $\left| \int_{-1/2}^{1/2} w^k K(w) dw \right| < \infty$ *for* $k \leq 4$*; and (3)* $K(\cdot)$ *is three-times continuously differentiable on* $\mathbb{R}$, *where the derivatives* $K^{(k)}(w)$ *satisfy* $\int_{-1/2}^{1/2} \left| K^{(k)}(w) \right| dw < \infty$ *for* $k = 1, 2, 3$.

Assumption 3. *For* $m \to \infty$ *as* $n \to \infty$ *at a rate no slower than* $[n/(\log n)^{11/5}]^{5/4}$, *the bandwidth sequence* $\{h_m\}$ *satisfies (1)* $h_m \to 0$*; (2)* $m h_m^5$, $n h_m^4 \to \infty$*; and (3)* $m h_m / \left[(\log m)^2 \sqrt{\log h_m^{-1}} \right] \to \infty$.

Assumption 4. *For each each* $i = 1, \ldots, n$*: (1) The conditional moment* $\Pr \left[Y_i \leq \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \middle| \boldsymbol{X}_i \right] = \alpha$ *holds almost surely for* $\alpha \in [a_1, a_2]$. *(2) The conditional distribution function* F_i *is absolutely continuous, with corresponding density* f_i *such that* $f_i(\cdot)$ *is uniformly continuous on the closure of* $\mathcal{B}$. *In addition, there exists an interval* $[-b, b]$ *with* $b \in (0, \infty)$ *such that* $\mathcal{B} \subset [-b, b]$, *where* $\mathcal{B}$ *is as given above in (1). (3) The densities* $f_i(y)$ *are five-times differentiable for all* $y \in \mathcal{B}$ *with* $\max_{1 \leq i \leq n} \sup_{y \in \mathcal{B}} \left| f_i^{(k)}(y) \right| < \infty$ *a.s. for each* $k = 0, 1, \ldots, 5$. *(4) There exist constants* $0 < l_1 \leq l_2 < \infty$ *such that* $0 < l_1 \leq f_i(y) \leq l_2 < \infty$ *for all* $y \in \mathcal{B}$.

Assumption 5. *The first component of the design vector* $\boldsymbol{X}_i$ *is an intercept, i.e.,* $\boldsymbol{X}_i = [\ 1 \quad \tilde{\boldsymbol{X}}_i^\top\ ]^\top$ *for some* $(d-1)$*-variate* $\tilde{\boldsymbol{X}}_i$. *Let* $\boldsymbol{\beta}^{(1)}(\alpha) \equiv (d/d\alpha) \boldsymbol{\beta}(\alpha)$, *i.e., the gradient vector, and let* $\tilde{\phi}_i(t)$ *denote the conditional characteristic function given* $\tilde{\boldsymbol{X}}_i$ *of the random variable*

$$\tilde{\boldsymbol{X}}_i \cdot \left(1 \left\{ Y_i \leq \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(\alpha) + n^{-1/2} \boldsymbol{\beta}^{(1)}(\alpha) \right) \right\} - \alpha \right).$$

For any $\epsilon > 0$, *there exists* $\eta \in (0, 1)$ *such that* $\inf_{\|t\| > \epsilon} \prod_{i=1}^n \tilde{\phi}_i(t) \leq \eta^n$ *uniformly in* $\alpha \in [\epsilon, 1 - \epsilon]$.

Remark 1. *The requirement of part 1 of Assumption 1 that $\boldsymbol{G}_n(\alpha)$ converge uniformly in $\alpha \in \mathcal{A}$ to $\boldsymbol{G}_0(\alpha)$ is used only in the proof of Theorem 3 appearing below in Appendix C. Theorems 1 and 2, whose proofs appear below in Appendix B only involve a requirement of pointwise convergence, i.e., that $\boldsymbol{G}_n(\alpha)$ converge to $\boldsymbol{G}_0(\alpha)$ for any $\alpha \in \mathcal{A}$.*

Remark 2. *Although we do not make this explicit in the conditions of Assumption 3, our results do allow for stochastic bandwidths. This is certainly relevant to the discussion presented below in Appendix D in which a data-driven bandwidth is derived.*

Remark 3. *Part 1 of Assumption 4 requires the correct specification of the quantile regression model on $[a_1, a_2]$. This condition may restrict the choice of a_1 and a_2 in practice. Specification tests developed in e.g., Escanciano and Goh (2014) and related papers can be used to check this condition. On the other hand, unreported simulations do suggest that Wald tests incorporating the proposed estimator of $\boldsymbol{G}_0(\alpha)$ have a certain degree of robustness in finite samples to incorrect specification of the underlying quantile regression model on a given interval of quantiles $[a_1, a_2]$. An extension of the analysis presented in this paper to the case where our estimator of $\boldsymbol{G}_0(\alpha)$ is computed using a shrinking neighborhood $[a_{1n}, a_{2n}]$ of the quantile level α, on which the quantile-regression specification holds, may be desirable.*

Remark 4. *Assumption 5 is taken from Portnoy (2012) and can be shown to hold if the distribution of $\tilde{\boldsymbol{X}}_i$ is appropriately smooth and bounded. In addition, the condition of part 2 of Assumption 4 , which implies that $\|\boldsymbol{X}_i\|$ is uniformly bounded on its support, is required in the derivation of both the precise form of the quantity $\boldsymbol{T}_{2nm}(\alpha)$ appearing in the statement of Theorem 1 of our paper and of the error rate appearing in the statement of Portnoy (2012, Theorem 5). Relaxation of this condition will not affect the $O_p\left(\sqrt{\log h_m^{-1}/(mh_m)}\right)$ convergence rate of $\boldsymbol{T}_{2nm}(\alpha)$ stated in Theorem 1 of our paper, but will likely increase the power of $\log n$ that appears in the conclusion of Portnoy (2012, Theorem 5); see in this connection the discussion in Portnoy (2012, p. 1720).*

B Proofs of Theorems 1 and 2

B.1 A useful lemma

We begin by introducing a useful lemma on uniform-in-bandwidth rates of convergence of kernel-type estimators that is instrumental in proving our main results. Recall that $\{U_j\}_{j=1}^m$ denotes a random sample of size m, distributed as $U \sim Unif[a_1, a_2]$ for $[a_1, a_2] \subset (0, 1)$. For a given $\boldsymbol{x} \in \mathbb{R}^d$, let $F_{\boldsymbol{x}}(\cdot)$ and $f_{\boldsymbol{x}}(\cdot)$ denote the cdf and Lebesgue density of the random variable $\boldsymbol{x}^\top \boldsymbol{\beta}(U)$. Define

$$g_{m,h}(w, \boldsymbol{x}) \equiv \frac{1}{m} \sum_{j=1}^{m} \varphi \left(\frac{\boldsymbol{x}^\top \boldsymbol{\beta}(U_j) - w}{h} \right),$$

where $\varphi(\cdot)$ is either $(a_2 - a_1) K(\cdot)$, $(a_2 - a_1) K^{(1)}(\cdot)$ or $(a_2 - a_1) K^{(2)}(\cdot)$. Let $\mathcal{X} \subset \mathbb{R}^d$ denote the support of $\boldsymbol{x}$, and let wp. 1 stand for with probability one.

Lemma 1. *Under Assumptions 2–4, for $c > 0$, and $0 < h_0 < 1$,, the following holds wp. 1:*

$$\limsup_{m \to \infty} \sup_{c \log m/m \le h \le h_0} \sup_{\boldsymbol{x} \in \mathcal{X}} \sup_{w \in \mathbb{R}} \frac{\sqrt{m} \left| g_{m,h}(w, \boldsymbol{x}) - E\left[g_{m,h}(w, \boldsymbol{x}) \right] \right|}{\sqrt{h |\log h|}} \equiv A(c) < \infty, \tag{3}$$

for $\varphi(\cdot)$ equal to $(a_2 - a_1) K(\cdot)$, $(a_2 - a_1) K^{(1)}(\cdot)$ or $(a_2 - a_1) K^{(2)}(\cdot)$.

Proof. We provide the proof for $\varphi(\cdot) = (a_2 - a_1) K(\cdot)$; the proof for $\varphi(\cdot) = (a_2 - a_1) K^{(1)}(\cdot)$ or $\varphi(\cdot) = (a_2 - a_1) K^{(2)}(\cdot)$ is the same. In particular, Lemma 1 follows from an application of the main result of Mason and Swanepoel (2011, p. 73) applied to the class of functions

$$\mathcal{G} = \left\{ (u, h) \to K \left(\frac{\boldsymbol{x}^\top \boldsymbol{\beta}(u) - w}{h} \right) : \boldsymbol{x} \in \mathcal{X}, w \in \mathbb{R} \right\}.$$

We proceed to verify their conditions. Since K is bounded, Mason and Swanepoel (2011, Condition (G.i)) is trivially satisfied. To verify Mason and Swanepoel (2011, Condition (G.ii)) note that

$$\begin{aligned}
E\left[K^2 \left(\frac{\boldsymbol{x}^\top \boldsymbol{\beta}(U) - w}{h} \right) \right] &= \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} K^2 \left(\frac{\boldsymbol{x}^\top \boldsymbol{\beta}(u) - w}{h} \right) du \\
&= \frac{1}{a_2 - a_1} \int_{\boldsymbol{x}^\top \boldsymbol{\beta}(a_1)}^{\boldsymbol{x}^\top \boldsymbol{\beta}(a_2)} K^2 \left(\frac{q - w}{h} \right) f_{\boldsymbol{x}}(q) dq \\
&= \frac{h}{a_2 - a_1} \int_{(\boldsymbol{x}^\top \boldsymbol{\beta}(a_1) - w)/h}^{(\boldsymbol{x}^\top \boldsymbol{\beta}(a_2) - w)/h} K^2(t) f_{\boldsymbol{x}}(w + ht) dt \\
&\le \frac{h}{a_2 - a_1} \|K\|_2^2 l_2,
\end{aligned}$$

where $\|K\|_2 = \sqrt{\int_{-1/2}^{1/2} K^2(w) dw} < \infty$ by Assumption 2 and where l_2 is as given in Assumption 4 above. Hence, Mason and Swanepoel (2011, Condition (G.ii)) holds. The class

$$\mathcal{G}_0 = \left\{ u \to K \left(\frac{\boldsymbol{x}^\top \boldsymbol{\beta}(u) - w}{h} \right) : \boldsymbol{x} \in \mathcal{X}, w \in \mathbb{R}, h \in (0, 1] \right\}$$

is a VC class, which is also pointwise measurable, see e.g. Nolan and Pollard (1987). It follows that Mason and Swanepoel (2011, Conditions (F.i) and (F.ii)) hold. This completes the proof. □

B.2 Proof of Theorem 1

Begin by noting that

$$
\begin{aligned}
\hat{\boldsymbol{G}}_n(\alpha) &= \frac{1}{n}\sum_{i=1}^{n} \hat{f}_{ni}\left(\boldsymbol{X}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top \\
&= \frac{a_2-a_1}{nmh_m}\sum_{i=1}^{n}\sum_{j=1}^{m} K\left(\frac{1}{\sqrt{n}h_m}\boldsymbol{X}_i^\top\left[\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n\left(U_j\right)-\boldsymbol{\beta}\left(U_j\right)\right)-\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{\beta}(\alpha)\right)\right.\right. \\
&\left.\left.+\sqrt{n}\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right]\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top \\
&= \frac{a_2-a_1}{nmh_m}\sum_{i=1}^{n}\sum_{j=1}^{m} K\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)+\frac{1}{\sqrt{n}h_m}\boldsymbol{X}_i^\top\boldsymbol{D}_{nj}(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top,
\end{aligned}
$$

where

$$
\boldsymbol{D}_{nj}(\alpha) = \sqrt{n}\left[\left(\hat{\boldsymbol{\beta}}_n\left(U_j\right)-\boldsymbol{\beta}\left(U_j\right)\right)-\left(\hat{\boldsymbol{\beta}}_n(\alpha)-\boldsymbol{\beta}(\alpha)\right)\right].
$$

By a Taylor expansion we accordingly have

$$
\begin{aligned}
\hat{\boldsymbol{G}}_n(\alpha) = \frac{a_2-a_1}{nmh_m}\sum_{i=1}^{n}\sum_{j=1}^{m}&\left[K\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right)\right. \\
&+\frac{1}{\sqrt{n}h_m}\boldsymbol{X}_i^\top\boldsymbol{D}_{nj}(\alpha)K^{(1)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right) \\
&\left.+\frac{1}{2nh_m^2}\left(\boldsymbol{X}_i^\top\boldsymbol{D}_{nj}(\alpha)\right)^2K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\bar{\boldsymbol{\beta}}\left(U_j\right)-\bar{\boldsymbol{\beta}}(\alpha)\right)\right)\right]\boldsymbol{X}_i\boldsymbol{X}_i^\top,
\end{aligned}
$$

where

$$
\left\|\left(\bar{\boldsymbol{\beta}}\left(U_j\right)-\bar{\boldsymbol{\beta}}(\alpha)\right)-\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right\| < \left\|\frac{1}{\sqrt{n}}\boldsymbol{D}_{nj}(\alpha)\right\|
$$

for each $j = 1,\ldots,m$.

Then

$$
\hat{\boldsymbol{G}}_n(\alpha) = \frac{1}{n}\sum_{i=1}^{n}\tilde{f}_i\left(\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top
$$

$$
+ \frac{a_2 - a_1}{nmh_m^2} \sum_{i=1}^{n} \boldsymbol{X}_i^\top \left[\sum_{j=1}^{m} \frac{1}{\sqrt{n}} \boldsymbol{D}_{nj}(\alpha) K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right] \boldsymbol{X}_i \boldsymbol{X}_i^\top
$$

$$
+ \frac{a_2 - a_1}{nmh_m^3} \sum_{i=1}^{n} \boldsymbol{X}_i^\top \left[\sum_{j=1}^{m} \frac{1}{n} \boldsymbol{D}_{nj}(\alpha) \boldsymbol{D}_{nj}(\alpha)^\top K^{(2)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\bar{\boldsymbol{\beta}}\left(U_j\right) - \bar{\boldsymbol{\beta}}(\alpha) \right) \right) \right] \boldsymbol{X}_i \cdot \boldsymbol{X}_i \boldsymbol{X}_i^\top, \tag{4}
$$

where $\tilde{f}_i \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right)$ denotes the empirically infeasible estimator of $f_i \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right)$ defined in 9 of our paper.

In what follows, the three terms appearing in the representation of $\hat{\boldsymbol{G}}_n(\alpha)$ given in (4) are analyzed in sequence. For convenience, we suppress the dependence on n of the quantile grid size m. We show that the following holds as $n \to \infty$ for a fixed quantile $\alpha \in \mathcal{A}$:

1. $$
\frac{1}{n} \sum_{i=1}^{n} \tilde{f}_i \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \boldsymbol{X}_i \boldsymbol{X}_i^\top = \frac{1}{n} \sum_{i=1}^{n} f_i \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \boldsymbol{X}_i \boldsymbol{X}_i^\top + T_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + o_p \left(h_m^2 + \sqrt{\frac{-\log h_m}{mh_m}} \right), \tag{5}
$$

   where $\boldsymbol{T}_{1nm}(\alpha)$ and $\boldsymbol{T}_{2nm}(\alpha)$ are as given above in the statement of Theorem 1, and where $\boldsymbol{T}_{1nm}(\alpha) = O_p\left(h_m^2\right)$ and $\boldsymbol{T}_{2nm}(\alpha) = O_p\left(\sqrt{-\log h_m/(mh_m)}\right)$. It follows that the remainder term in (5) is of strictly smaller order than $\boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha)$.

2. We also show that

$$
\boldsymbol{T}_{2nm}(\alpha) \equiv \frac{a_2 - a_1}{nmh_m^2} \sum_{i=1}^{n} \boldsymbol{X}_i^\top \left[\sum_{j=1}^{m} \frac{1}{\sqrt{n}} D_{nj}(\alpha) K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right] \boldsymbol{X}_i \boldsymbol{X}_i^\top = O_p\left(\frac{1}{\sqrt{n}} \right); \tag{6}
$$

3. and finally that

$$
\frac{a_2 - a_1}{nmh_m^3} \sum_{i=1}^{n} \boldsymbol{X}_i^\top \left[\sum_{j=1}^{m} \frac{1}{n} \boldsymbol{D}_{nj}(\alpha) \boldsymbol{D}_{nj}(\alpha)^\top K^{(2)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\bar{\boldsymbol{\beta}}\left(U_j\right) - \bar{\boldsymbol{\beta}}(\alpha) \right) \right) \right] \boldsymbol{X}_i \cdot \boldsymbol{X}_i \boldsymbol{X}_i^\top
$$

$$= O_p\left(\frac{1}{n} + \frac{1}{n^{\frac{3}{2}} h_m^4}\right). \tag{7}$$

Combining (4) with (5)–(7) yields the desired conclusion; namely, that

$$\begin{aligned}
\hat{\boldsymbol{G}}_n(\alpha) &= \frac{1}{n}\sum_{i=1}^{n} \tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \boldsymbol{X}_i \boldsymbol{X}_i^\top + T_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + \boldsymbol{R}_{nm}(\alpha) \\
&= G_0(\alpha) + \boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + T_{3nm}(\alpha) + \boldsymbol{R}_{nm}(\alpha),
\end{aligned}$$

where $\boldsymbol{R}_{nm}(\alpha)$ denotes the sum of the remainder term in (5) and the expression in (7). In particular,

$$\boldsymbol{R}_{nm}(\alpha) = o_p\left(h_m^2 + \sqrt{\frac{-\log h_m}{m h_m}}\right) + O_p\left(\frac{1}{n} + \frac{1}{n^{\frac{3}{2}} h_m^4}\right) = o_p\left(\boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha)\right).$$

Claim 1. The following holds:

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left[\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) - f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right] \boldsymbol{X}_i \boldsymbol{X}_i^\top \\
&= \boldsymbol{T}_{1nm}(\alpha) + \boldsymbol{T}_{2nm}(\alpha) + o_p\left(h_m^2 + \sqrt{\frac{-\log h_m}{m h_m}}\right),
\end{aligned}$$

where

$$\boldsymbol{T}_{1nm}(\alpha) = O_p\left(h_m^2\right)$$

and

$$\boldsymbol{T}_{2nm}(\alpha) = O_p\left(\sqrt{\frac{-\log h_m}{m h_m}}\right).$$

We begin by establishing the rates of convergence of $\boldsymbol{T}_{1nm}(\alpha)$ and $\boldsymbol{T}_{2nm}(\alpha)$. In particular, we have

$$\begin{aligned}
\|\boldsymbol{T}_{1nm}(\alpha)\| &\le \frac{h_m^2}{2}\int_{-1/2}^{1/2} w^2 K(w) dw \cdot \max_{1\le i\le n}\left|f_i^{(2)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right| \cdot \frac{1}{n}\sum_{i=1}^{n}\|\boldsymbol{X}_i\|^2 \\
&= O_p\left(h_m^2\right),
\end{aligned} \tag{8}$$

where use has been made of the conditions of Assumptions 1, 2 and 4 that $E\left[\|\boldsymbol{X}_i\|^4\right] < \infty$, $\int_{-1/2}^{1/2} w^2 K(w)dw < \infty$ and $\max_{1\leq i\leq n}\sup_{y\in\mathcal{B}}\left|f_i^{(2)}(y)\right| < \infty$ a.s., where $\mathcal{B}$ is as given above in (1). In addition, we have

$$\begin{aligned}\|\boldsymbol{T}_{2nm}(\alpha)\| &\leq \sqrt{\frac{-\log h_m}{mh_m}\cdot\|K\|_2^2\cdot\max_{1\leq i\leq n}\sup_{y\in\mathcal{X}_i}|f_i(y)|}\cdot\frac{1}{n}\sum_{i=1}^n\|\boldsymbol{X}_i\|^2\\ &= O_p\left(\sqrt{\frac{-\log h_m}{mh_m}}\right)\end{aligned} \tag{9}$$

via a similar argument.

Next, define the quantities

$$V_{\tilde{f}_i}(\alpha) \equiv \tilde{f}_i\left(\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)\right) - E\left[\frac{a_2-a_1}{h_m}K\left(\frac{1}{h_m}\boldsymbol{X}_i^\top(\boldsymbol{\beta}(U)-\boldsymbol{\beta}(\alpha))\right)\middle|\boldsymbol{X}_i\right]$$

and

$$B_{\tilde{f}_i}(\alpha) \equiv E\left[\frac{a_2-a_1}{h_m}K\left(\frac{1}{h_m}\boldsymbol{X}_i^\top(\boldsymbol{\beta}(U)-\boldsymbol{\beta}(\alpha))\right)\middle|\boldsymbol{X}_i\right] - f_i\left(\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)\right)$$

for each $i\in\{1,\ldots,n\}$. We have

$$\frac{1}{n}\sum_{i=1}^n\tilde{f}_i\left(\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top = \frac{1}{n}\sum_{i=1}^n f_i\left(\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top + \frac{1}{n}\sum_{i=1}^n V_{\tilde{f}_i}(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top + \frac{1}{n}\sum_{i=1}^n B_{\tilde{f}_i}(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top. \tag{10}$$

Recall that $\mathcal{X}\subset\mathbb{R}^d$ denotes the common support of $\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3,\ldots$. In addition, let $f_{\boldsymbol{x}}(\cdot)$ denote the density of the conditional distribution of Y_i given $\boldsymbol{X}_i=\boldsymbol{x}$, and let

$$\tilde{f}_{\boldsymbol{x}}(\boldsymbol{x}^\top\boldsymbol{\beta}(\alpha)) \equiv (mh_m)^{-1}\sum_{j=1}^m K\left(h_m^{-1}\boldsymbol{x}^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right).$$

Recall the condition of Assumption 4 that for some finite $b>0$, $\mathcal{B}\subset[-b,b]$. The following holds almost surely:

$$\left\|\frac{1}{n}\sum_{i=1}^n V_{\tilde{f}_i}(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top - \boldsymbol{T}_{2nm}(\alpha)\right\|$$

$$
\begin{aligned}
&= \left\| \sqrt{\frac{\log h_m^{-1}}{m h_m}} \cdot \frac{1}{n} \sum_{i=1}^{n} \sqrt{f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)} \boldsymbol{X}_i \boldsymbol{X}_i^\top \right. \\
&\left. \cdot \left(\sqrt{\frac{m h_m}{\log h_m^{-1}}} \cdot \frac{\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) - E\left[\tilde{f}_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \middle| \boldsymbol{X}_i\right]}{\sqrt{f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)}} - \|K\|_2 \right) \right\| \\
&\le \sup_{\boldsymbol{x} \in \mathcal{X}} \sup_{y \in [-b,b]} \left| \sqrt{\frac{m h_m}{\log h_m^{-1}}} \cdot \frac{\tilde{f}_{\boldsymbol{x}}(y) - E\left[\tilde{f}_{\boldsymbol{x}}(y)\right]}{\sqrt{f_{\boldsymbol{x}}(y)}} - \|K\|_2 \right| \\
&\cdot \sqrt{\frac{\log h_m^{-1}}{m h_m}} \cdot \frac{1}{n} \sum_{i=1}^{n} \sqrt{f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)} \|\boldsymbol{X}_i\|^2 .
\end{aligned} \tag{11}
$$

By Giné et al. (2004, p. 185) we have for any $z \in \mathbb{R}$ that

$$
\lim_{n \to \infty} \Pr\left[(-\log h_m) \left(\frac{\sqrt{m h_m}}{\sqrt{-\log h_m}} \sup_{\boldsymbol{x} \in \mathcal{X}} \sup_{y \in [-b,b]} \left| \frac{\tilde{f}_{\boldsymbol{x}}(y) - E\left[\tilde{f}_{\boldsymbol{x}}(y)\right]}{\sqrt{f_{\boldsymbol{x}}(y)}} \right| - \|K\|_2 \right) \le z \right] = \exp\left(-e^{-z}\right).
$$

It follows that

$$
\left| \sqrt{\frac{m h_m}{\log h_m^{-1}}} \cdot \sup_{\boldsymbol{x} \in \mathcal{X}} \sup_{y \in [-b,b]} \frac{\tilde{f}_{\boldsymbol{x}}(y) - E\left[\tilde{f}_{\boldsymbol{x}}(y)\right]}{\sqrt{f_{\boldsymbol{x}}(y)}} - \|K\|_2 \right| = o_p(1). \tag{12}
$$

Combining (12) with (11), the condition of Assumption 4 that $f_i(\cdot)$ is uniformly bounded on $\mathcal{B}$ and the condition that $n^{-1} \sum_{i=1}^{n} \|\boldsymbol{X}_i\|^2 = O_p(1)$ (implied by (2) of Assumption 1) yields the result that

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} V_{\tilde{f}_i}(\alpha) \boldsymbol{X}_i \boldsymbol{X}_i^\top - \boldsymbol{T}_{2nm}(\alpha) \right\| = o_p\left(\sqrt{\frac{-\log h_m}{m h_m}} \right). \tag{13}
$$

Next, consider that for each $i \in \{1, \ldots, n\}$ and sufficiently small h_m that

$$
\begin{aligned}
&E\left[\frac{a_2 - a_1}{h_m} K\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha)\right) \right) \middle| \boldsymbol{X}_i \right] \\
&= \int_{a_1}^{a_2} \frac{1}{h_m} K\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(u) - \boldsymbol{\beta}(\alpha)\right) \right) du \\
&= \int_{\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_1)}^{\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_2)} \frac{1}{h_m} K\left(\frac{1}{h_m} \left(t_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \right) f_i\left(t_i\right) dt_i
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-1/2}^{1/2} K(w_i)\, f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) + h_m w_i\right) dw_i \\
&= f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \int_{-1/2}^{1/2} K(w_i)\, dw_i + \frac{h_m^2}{2} f_i^{(2)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \int_{-1/2}^{1/2} w_i^2 K(w_i)\, dw_i \\
&+ \frac{h_m^3}{6} f_i^{(3)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \int_{-1/2}^{1/2} w_i^3 K(w_i)\, dw_i + o_p\left(h_m^3\right)
\end{aligned}
$$

as $n \to \infty$, where we have exploited the conditions of Assumptions 2 and 4 that $\left|\int_{-1/2}^{1/2} w^4 K(w) dw\right| < \infty$ and $\max_{1 \le i \le n} \sup_{y \in \mathcal{B}} \left| f_i^{(4)}(y)\right| < \infty$ a.s.

It follows that

$$
\begin{aligned}
&\left\| \frac{1}{n} \sum_{i=1}^n B_{\tilde{f}_i}(\alpha) \boldsymbol{X}_i \boldsymbol{X}_i^\top - \boldsymbol{T}_{1nm}(\alpha) \right\| \\
&= \frac{h_m^3}{6} \int_{-1/2}^{1/2} w^3 K(w) dw \cdot \left\| \frac{1}{n} \sum_{i=1}^n f_i^{(3)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \boldsymbol{X}_i \boldsymbol{X}_i^\top \right\| + o_p\left(h_m^3\right) \\
&\le \frac{h_m^3}{6} \int_{-1/2}^{1/2} w^3 K(w) dw \cdot \max_{1 \le i \le n} \left| f_i^{(3)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) \right| \cdot \frac{1}{n} \sum_{i=1}^n \|\boldsymbol{X}_i\|^2 + o_p\left(h_m^3\right) \\
&= O_p\left(h_m^3\right) O_p(1) \\
&= o_p\left(h_m^2\right), \qquad (14)
\end{aligned}
$$

where we have additionally exploited the conditions of Assumptions 2 and 4 that $\int_{-1/2}^{1/2} w^3 K(w) dw < \infty$ and $\max_{1 \le i \le n} \sup_{y \in \mathcal{B}} \left| f_i^{(3)}(y)\right| < \infty$ a.s.

The desired conclusion follows from the combination of (8)–(10) and (13)–(14).

Claim 2. The following holds:

$$
\boldsymbol{T}_{2nm}(\alpha) = O_p\left(\frac{1}{\sqrt{n}}\right)
$$

By Lemma 1 applied with $\varphi(\cdot) = (a_2 - a_1) K^{(1)}(\cdot)$,

$$
(a_2 - a_1) \cdot \sqrt{\frac{m h_m^3}{\log h_m^{-1}}}
$$

$$
\cdot \left| \frac{1}{mh_m^2} \sum_{j=1}^{m} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta} \left(U_j \right) - \boldsymbol{\beta}(\alpha) \right) \right) \right| - E \left[\frac{1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \middle| \boldsymbol{X}_i \right] \right|
$$

$$
< \infty, \tag{15}
$$

a.s. as $n \to \infty$.

Let $\mathcal{R}_+$ and $\mathcal{R}_-$ denote the regions $\{t \in [\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_1), \boldsymbol{X}_i^\top \boldsymbol{\beta}(a_2)] : K^{(1)} \left(\left(t - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) / h_m \right) > 0\}$ and $\{t \in [\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_1), \boldsymbol{X}_i^\top \boldsymbol{\beta}(a_2)] : K^{(1)} \left(\left(t - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) / h_m \right) < 0\}$, respectively. Then using integration by parts and applying the assumption that $K(\cdot)$ has bounded support, we have for a sufficiently small h_m that

$$
\begin{aligned}
& E \left[\frac{a_2 - a_1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \middle| \boldsymbol{X}_i \right] \\
&= \int_{a_1}^{a_2} \frac{1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(u) - \boldsymbol{\beta}(\alpha) \right) \right) \right| du \\
&= \int_{\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_1)}^{\boldsymbol{X}_i^\top \boldsymbol{\beta}(a_2)} \frac{1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \left(t_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right) \right| f_i \left(t_i \right) dt_i \\
&= \frac{-1}{h_m} \left(\int_{\mathcal{R}_+} K \left(\frac{1}{h_m} \left(t_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right) f_i^{(1)} \left(t_i \right) dt_i - \int_{\mathcal{R}_-} K \left(\frac{1}{h_m} \left(t_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right) f_i^{(1)} \left(t_i \right) dt_i \right) \\
&= - \int_{\left\{ K^{(1)} > 0 \right\}} K \left(w_i \right) f_i^{(1)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) + h_m w_i \right) dw_i + \int_{\left\{ K^{(1)} < 0 \right\}} K \left(w_i \right) f_i^{(1)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) + h_m w_i \right) dw_i \\
&= f_i^{(1)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \left(\int_{\left\{ K^{(1)} < 0 \right\}} K(w) dw - \int_{\left\{ K^{(1)} > 0 \right\}} K(w) dw \right) \\
&+ h_m f_i^{(2)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \left(\int_{\left\{ K^{(1)} < 0 \right\}} w K(w) dw - \int_{\left\{ K^{(1)} > 0 \right\}} w K(w) dw \right) + O_p \left(h_m^2 \right) \\
&\leq \left| f_i^{(1)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right| \left| \int_{\left\{ K^{(1)} < 0 \right\}} K(w) dw - \int_{\left\{ K^{(1)} > 0 \right\}} K(w) dw \right| \\
&+ h_m \left| f_i^{(2)} \left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right| \cdot \left| \int_{\left\{ K^{(1)} < 0 \right\}} w K(w) dw - \int_{\left\{ K^{(1)} > 0 \right\}} w K(w) dw \right| + O_p \left(h_m^2 \right),
\end{aligned} \tag{16}
$$

where the assumption that $\max_i \sup_{y \in \mathcal{B}} \left| f_i^{(3)} \left(y \right) \right| < \infty$ a.s. (Assumption 4) has also been used.

Combine (15) and (16) to deduce that

$$
\begin{aligned}
&\max_i \frac{a_2 - a_1}{mh_m^2} \sum_{j=1}^{m} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \\
&\leq \max_i \left| \frac{a_2 - a_1}{mh_m^2} \sum_{j=1}^{m} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right| - E\left[\frac{a_2 - a_1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \Bigg| \boldsymbol{X}_i \right] \right| \\
&+ \max_i E\left[\frac{a_2 - a_1}{h_m^2} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \Bigg| \boldsymbol{X}_i \right] \\
&= O_p\left(\sqrt{\frac{-\log h_m}{mh_m^3}} \right) + \max_i \left| f_i^{(1)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right| \left| \int_{\left\{K^{(1)}<0\right\}} K(w)dw - \int_{\left\{K^{(1)}>0\right\}} K(w)dw \right| + O_p\left(h_m\right) \\
&= \max_i \sup_{\alpha \in \mathcal{A}} \left| f_i^{(1)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right| \left| \int_{\left\{K^{(1)}<0\right\}} K(w)dw - \int_{\left\{K^{(1)}>0\right\}} K(w)dw \right| + O_p\left(\sqrt{\frac{-\log h_m}{mh_m^3}} + h_m \right),
\end{aligned} \tag{17}
$$

where the assumptions that $\max_i \sup_{y \in \mathcal{B}} f_i(y) < \infty$ a.s. (Assumption 4), $\int_{-1/2}^{1/2} \left(K^{(1)}(w) \right)^2 dw < \infty$ (Assumption 2) and $\max_i \sup_{y \in \mathcal{B}} \left| f_i^{(3)}(y) \right| < \infty$ (Assumption 4) have been used.

Therefore

$$
\begin{aligned}
&\max_i \left| \frac{a_2 - a_1}{mh_m^2} \sum_{j=1}^{m} \frac{1}{\sqrt{n}} \boldsymbol{D}_{nj}(\alpha) K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \\
&\leq \max_i \frac{a_2 - a_1}{mh_m^2} \sum_{j=1}^{m} \left| K^{(1)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) \right| \cdot \left[\frac{1}{\sqrt{n}} \max_j \left\| \boldsymbol{D}_{nj}(\alpha) \right\| \right] \\
&\leq \left(\max_i \left| f_i^{(1)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha) \right) \right| \cdot \left| \int_{\left\{K^{(1)}<0\right\}} K(w)dw - \int_{\left\{K^{(1)}>0\right\}} K(w)dw \right| + O_p\left(\sqrt{\frac{-\log h_m}{mh_m^3}} + h_m \right) \right) \\
&\cdot O_p\left(\frac{1}{\sqrt{n}} \right) \\
&= O_p\left(\frac{1}{\sqrt{n}} \right),
\end{aligned} \tag{18}
$$

where we have used (17), the condition of Assumption 4 that $\max_i \sup_{y\in\mathcal{B}} \left|f_i^{(1)}(y)\right| < \infty$ and the result (e.g., Angrist et al., 2006, Theorem 3) that

$$\max_j \left\|\boldsymbol{D}_{nj}(\alpha)\right\| \leq 2\sup_{\alpha\in\mathcal{A}} \left\|\sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)\right\|$$
$$= O_p(1).$$

Applying (18) we have

$$\left\|\frac{a_2-a_1}{n^{\frac{3}{2}}mh_m^2}\sum_{i=1}^n\sum_{j=1}^m \boldsymbol{X}_i^\top\left(\frac{1}{\sqrt{n}}\boldsymbol{D}_{nj}(\alpha)\right)K^{(1)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top\right\|$$
$$\leq \max_i\left|\frac{a_2-a_1}{mh_m^2}\sum_{j=1}^m\frac{1}{\sqrt{n}}\boldsymbol{D}_{nj}(\alpha)K^{(1)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right)\right|\cdot\frac{1}{n}\sum_{i=1}^n\|\boldsymbol{X}_i\|^3$$
$$= O_p\left(\frac{1}{\sqrt{n}}\right)O_p(1)$$
$$= O_p\left(\frac{1}{\sqrt{n}}\right),$$

where we have also applied condition (2) of Assumption 1. The desired conclusion follows.

Claim 3: The following holds:

$$\frac{a_2-a_1}{nmh_m^3}\sum_{i=1}^n \boldsymbol{X}_i^\top\left[\sum_{j=1}^m\frac{1}{n}\boldsymbol{D}_{nj}(\alpha)\boldsymbol{D}_{nj}(\alpha)^\top K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\bar{\boldsymbol{\beta}}\left(U_j\right)-\bar{\boldsymbol{\beta}}(\alpha)\right)\right)\right]\boldsymbol{X}_i\cdot\boldsymbol{X}_i\boldsymbol{X}_i^\top$$
$$= O_p\left(\frac{1}{n}+\frac{1}{n^{\frac{3}{2}}h_m^4}\right).$$

We first show that

$$\frac{a_2-a_1}{nmh_m^3}\sum_{i=1}^n \boldsymbol{X}_i^\top\left[\sum_{j=1}^m\frac{1}{n}\boldsymbol{D}_{nj}(\alpha)\boldsymbol{D}_{nj}(\alpha)^\top K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top\left(\boldsymbol{\beta}\left(U_j\right)-\boldsymbol{\beta}(\alpha)\right)\right)\right]\boldsymbol{X}_i\cdot\boldsymbol{X}_i\boldsymbol{X}_i^\top$$

$$= O_p\left(\frac{1}{n}\right). \tag{19}$$

In this connection, Lemma 1 applied with $\varphi(\cdot) = (a_2 - a_1)\, K^{(2)}(\cdot)$ yields

$$\begin{aligned}
&(a_2 - a_1) \cdot \lim_{m\to\infty} \sqrt{\frac{m h_m^5}{\log h_m^{-1}}} \\
&\cdot \left| \frac{1}{m h_m^3} \sum_{j=1}^{m} K^{(2)}\left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha)\right)\right) - E\left[\frac{1}{h_m^3} K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top (\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha))\right) \middle| \boldsymbol{X}_i\right]\right| < \infty
\end{aligned} \tag{20}$$

almost surely, where we have exploited the assumption that $\int_{-1/2}^{1/2} \left(K^{(2)}(w)\right)^2 dw < \infty$. In addition, by a derivation similar to that leading to (16) above, we have for all $\alpha \in \mathcal{A}$ that

$$E\left[\frac{a_2 - a_1}{h_m^3} \left|K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top (\boldsymbol{\beta}(U) - \boldsymbol{\beta}(\alpha))\right)\right| \middle| \boldsymbol{X}_i\right] < \infty \tag{21}$$

wp. 1. Combine (20) and (21) to deduce (19), to wit:

$$\begin{aligned}
&\left\| \frac{a_2 - a_1}{n m h_m^3} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{n} \boldsymbol{X}_i^\top \boldsymbol{D}_{nj}(\alpha) \boldsymbol{D}_{nj}(\alpha)^\top \boldsymbol{X}_i K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha)\right)\right) \boldsymbol{X}_i \boldsymbol{X}_i^\top \right\| \\
&\leq \frac{a_2 - a_1}{n} \cdot \max_j \left\|\boldsymbol{D}_{nj}(\alpha)\right\|^2 \\
&\cdot \max_i \frac{1}{m h_m^3} \sum_{j=1}^{m} \left|K^{(2)}\left(\frac{1}{h_m}\boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha)\right)\right)\right| \cdot \frac{1}{n} \sum_{i=1}^{n} \|\boldsymbol{X}_i\|^4 \\
&= O_p\left(\frac{1}{n}\right) O_p(1) \\
&= O_p\left(\frac{1}{n}\right),
\end{aligned}$$

where we have applied the assumption that $\max_i \sup_{y\in\mathcal{B}} \left|f_i^{(2)}(y)\right| < \infty$ (Assumption 4) and the condition (2) of Assumption 1.

Next, we show that

$$\frac{a_2 - a_1}{n^2 m h_m^3} \sum_{i=1}^{n} \sum_{j=1}^{m} \left\| \boldsymbol{D}_{nj}(\alpha) \right\|^2 \left| K^{(2)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right) - K^{(2)} \left(\frac{1}{h_m} \boldsymbol{X}_i^\top \left(\bar{\boldsymbol{\beta}}\left(U_j\right) - \bar{\boldsymbol{\beta}}(\alpha) \right) \right) \right| \|\boldsymbol{X}_i\|^4$$

$$= O_p\left(\frac{1}{n^{\frac{3}{2}} h_m^4} \right). \tag{22}$$

In particular, (22) follows directly from the mean value theorem, $\left\|K^{(3)}\right\|_\infty < \infty$, the result

$$\max_j \left\| \left(\bar{\boldsymbol{\beta}}\left(U_j\right) - \bar{\boldsymbol{\beta}}(\alpha) \right) - \left(\boldsymbol{\beta}\left(U_j\right) - \boldsymbol{\beta}(\alpha) \right) \right\| = O_p\left(\frac{1}{\sqrt{n}} \right)$$

(e.g., Angrist et al., 2006, Theorem 3) and condition (2) of Assumption 1.

The desired conclusion follows from (19) and (22).

B.3 Proof of Theorem 2

Begin by considering the following expansion of $\hat{\boldsymbol{W}}_n$ about $\boldsymbol{W}_n(\boldsymbol{G}_0(\alpha))$, which is a consequence of Theorem 1 appearing in the main body of our paper:

$$\hat{\boldsymbol{W}}_n = \boldsymbol{W}_n(\boldsymbol{G}_0(\alpha)) + \boldsymbol{U}_{1nm}(\alpha) + \boldsymbol{U}_{2nm}(\alpha) + \boldsymbol{U}_{3nm}(\alpha), \tag{23}$$

where $\hat{\boldsymbol{W}}_n$ is as given in (11) of our paper, and where we have

$$\boldsymbol{W}_n(\boldsymbol{G}_0(\alpha)) = \boldsymbol{W}_0 + O_p\left(n^{-\frac{1}{2}}\right) \tag{24}$$

by Assumption 1, where $\boldsymbol{W}_0$ is as given above in (12) of our paper. In addition, we have by the binomial inverse theorem,

$$\begin{aligned} \boldsymbol{U}_{1nm}(\alpha) = \boldsymbol{W}_0 \cdot \Big[\boldsymbol{R} \Big(& \boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{T}_{1nm}(\alpha) \boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{H} \boldsymbol{G}_0^{-1}(\alpha) \\ & + \boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{H} \boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{T}_{1nm}(\alpha) \boldsymbol{G}_0^{-1}(\alpha) \Big) \boldsymbol{R}^\top \Big] \cdot \boldsymbol{W}_0 \\ & + \text{smaller-order terms} \\ & \equiv \bar{\boldsymbol{U}}_{1nm}(\alpha) + \text{smaller-order terms}; \end{aligned} \tag{25}$$

$$\boldsymbol{U}_{2nm}(\alpha) = \boldsymbol{W}_0 \cdot \Big[\boldsymbol{R} \Big(\boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{T}_{2nm}(\alpha) \boldsymbol{G}_0^{-1}(\alpha) \boldsymbol{H} \boldsymbol{G}_0^{-1}(\alpha)$$

$$
\begin{aligned}
&+\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{T}_{2nm}(\alpha)\boldsymbol{G}_0^{-1}(\alpha)\Big)\boldsymbol{R}^\top\Big]\cdot\boldsymbol{W}_0\\
&+\text{ smaller-order terms}\\
&\equiv \bar{\boldsymbol{U}}_{2nm}(\alpha)+\text{ smaller-order terms;}
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\boldsymbol{U}_{3nm}(\alpha) = \boldsymbol{W}_0\cdot &\Big[\boldsymbol{R}\Big(\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{T}_{2nm}(\alpha)\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\\
&+\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{T}_{3nm}(\alpha)\boldsymbol{G}_0^{-1}(\alpha)\Big)\boldsymbol{R}^\top\Big]\cdot\boldsymbol{W}_0\\
&+\text{ smaller-order terms}\\
&\equiv \bar{\boldsymbol{U}}_{3nm}(\alpha)+\text{ smaller-order terms.}
\end{aligned}
\tag{27}
$$

It follows that

$$
\begin{aligned}
\boldsymbol{U}_{1nm}(\alpha) &= O_p\left(\boldsymbol{T}_{1nm}(\alpha)\right) = O_p\left(h_m^2\right),\\
\boldsymbol{U}_{2nm}(\alpha) &= O_p\left(\boldsymbol{T}_{2nm}(\alpha)\right) = O_p\left(\sqrt{\frac{-\log h_m}{mh_m}}\right)
\end{aligned}
$$

and that

$$
\boldsymbol{U}_{3nm}(\alpha) = O_p\left(\boldsymbol{T}_{2nm}(\alpha)\right) = O_p\left(\frac{1}{\sqrt{n}}\right).
$$

Combining (23)–(27) we find that for each $\alpha \in \mathcal{A}$,

$$
\begin{aligned}
\hat{\boldsymbol{W}}_n &= \boldsymbol{W}_0 + \bar{\boldsymbol{U}}_{1nm}(\alpha) + \bar{\boldsymbol{U}}_{2nm}(\alpha) + \bar{\boldsymbol{U}}_{3nm}(\alpha)\\
&\quad + \text{ smaller-order terms.}
\end{aligned}
\tag{28}
$$

Next, consider that if $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha)-\boldsymbol{r} = \boldsymbol{0}$ is true, then the first-order asymptotic approximation for $\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha)$ has the form

$$
\sqrt{n}\left(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r}\right) = \boldsymbol{R}(\boldsymbol{Z}(\alpha) + \boldsymbol{S}_{1n}),
\tag{29}
$$

where $\boldsymbol{Z}(\alpha) \sim N(\boldsymbol{0}, \boldsymbol{V}(\alpha))$ where $\boldsymbol{V}(\alpha)$ is as given above in (2) of our paper, and $\boldsymbol{S}_{1n} = o_p(1)$ as $n \to \infty$.

We can now consider the empirical size of a nominal level-τ Wald test of $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = \boldsymbol{0}$ incorporating the proposed estimator $\hat{\boldsymbol{G}}_n(\alpha)$ of $\boldsymbol{G}_0(\alpha)$. In particular, the representation appearing in (13) of our paper follows from the representations in (28) and (29). Let $\boldsymbol{S}_{2nm}$ denote the $o_p\left(h_m^2 + \sqrt{\log h_m^{-1}/(mh_m)} + n^{-1/2}\right)$ remainder term in (28). Then

$$
\begin{aligned}
&\pi_0\left(h_m\right) \\
&\equiv \Pr\left[\frac{n}{\alpha(1-\alpha)}\left(\hat{\boldsymbol{\beta}}_n^\top(\alpha)\boldsymbol{R}^\top - \boldsymbol{r}^\top\right)\hat{\boldsymbol{W}}_n\left(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r}\right) > \chi^2_{J,\tau}\middle| H_0\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}\left[(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\left(\boldsymbol{W}_0 + \bar{\boldsymbol{U}}_{1nm}(\alpha) + \bar{\boldsymbol{U}}_{2nm}(\alpha) + \bar{\boldsymbol{U}}_{3nm}(\alpha)\right)\right.\right. \\
&\cdot\left.\left.\boldsymbol{R}\left(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}\right) + (\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\boldsymbol{S}_{2nm}\boldsymbol{R}\left(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}\right)\right] > \chi^2_{J,\tau}\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\left(\boldsymbol{W}_0 + \bar{\boldsymbol{U}}_{1nm}(\alpha) + \bar{\boldsymbol{U}}_{2nm}(\alpha) + \bar{\boldsymbol{U}}_{3nm}(\alpha)\right)\right. \\
&\cdot\left.\boldsymbol{R}\left(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}\right) > \chi^2_{J,\tau} - \Xi_{nm}(0)\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}\left(\boldsymbol{Z}(\alpha)^\top\boldsymbol{R}^\top\boldsymbol{W}_0\boldsymbol{R}\boldsymbol{Z}(\alpha) + \boldsymbol{Z}(\alpha)^\top\boldsymbol{R}^\top\boldsymbol{W}_0\boldsymbol{R}\boldsymbol{S}_{1n} + \boldsymbol{S}_{1n}^\top\boldsymbol{R}^\top\boldsymbol{W}_0\boldsymbol{R}\boldsymbol{Z}(\alpha) + \boldsymbol{S}_{1n}^\top\boldsymbol{R}^\top\boldsymbol{W}_0\boldsymbol{R}\boldsymbol{S}_{1n}\right.\right. \\
&\left.\left.+h_m^2\Lambda_{1nm}(\alpha,0) + \sqrt{\frac{-\log h_m}{mh_m}}\Lambda_{2nm}(\alpha,0) + \frac{1}{\sqrt{n}}\Lambda_{3nm}(\alpha,0)\right) > \chi^2_{J,\tau} - \Xi_{nm}(0)\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}\boldsymbol{Z}(\alpha)^\top\boldsymbol{R}^\top\boldsymbol{W}_0^{-1}\boldsymbol{R}\boldsymbol{Z}(\alpha) > \chi^2_{J,\tau}\right. \\
&\left.-\frac{1}{\alpha(1-\alpha)}\left(h_m^2\Lambda_{1nm}(\alpha,0) + \sqrt{\frac{-\log h_m}{mh_m}}\Lambda_{2nm}(\alpha,0) + \frac{1}{\sqrt{n}}\Lambda_{3nm}(\alpha,0)\right) - \Theta_n(0) - \Xi_{nm}(0)\right],
\end{aligned}
\tag{30}
$$

where

$$
\Xi_{nm}(0) = (\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\boldsymbol{S}_{2nm}\boldsymbol{R}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}). \tag{31}
$$

In addition, the quantities $\Lambda_{1nm}(\alpha,0)$, $\Lambda_{2nm}(\alpha,0)$ and $\Lambda_{3nm}(\alpha,0)$ appearing in (30) are given by

$$
\Lambda_{1nm}(\alpha,0) = \frac{1}{h_m^2}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\bar{\boldsymbol{U}}_{1nm}(\alpha)\boldsymbol{R}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}), \tag{32}
$$

$$
\Lambda_{2nm}(\alpha,0) = \sqrt{\frac{mh_m}{-\log h_m}}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n})^\top\boldsymbol{R}^\top\bar{\boldsymbol{U}}_{2nm}(\alpha)\boldsymbol{R}(\boldsymbol{Z}(\alpha)+\boldsymbol{S}_{1n}) \tag{33}
$$

and

$$\Lambda_{3nm}(\alpha, 0) = \sqrt{n}(\boldsymbol{Z}(\alpha) + \boldsymbol{S}_{1n})^\top \boldsymbol{R}^\top \bar{\boldsymbol{U}}_{3nm}(\alpha)\boldsymbol{R}(\boldsymbol{Z}(\alpha) + \boldsymbol{S}_{1n}), \tag{34}$$

while

$$\Theta_n(0) = \frac{1}{\alpha(1-\alpha)} \left(2\boldsymbol{Z}(\alpha)^\top \boldsymbol{R}^\top \boldsymbol{W}_0 \boldsymbol{R}\boldsymbol{S}_{1n} + \boldsymbol{S}_{1n}^\top \boldsymbol{R}^\top \boldsymbol{W}_0 \boldsymbol{R}\boldsymbol{S}_{1n}\right). \tag{35}$$

This establishes the representation of the size function given in (13) of our paper.

The remainder of Theorem 2 follows straightforwardly from the expression for the empirical size function given by $\pi_0(h_m)$ in (30) and the observation that if $G_0(\alpha)$ does not need to be estimated, the infeasible level-τ Wald test of $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) - \boldsymbol{r} = \boldsymbol{0}$ has an empirical size function given by

$$\begin{aligned}
&\Pr\left[\frac{n}{\alpha(1-\alpha)}\left(\hat{\boldsymbol{\beta}}_n(\alpha)^\top \boldsymbol{R}^\top - r^\top\right) \boldsymbol{W}_n(\boldsymbol{G}_0)\left(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r}\right) > \chi^2_{J,\tau} \middle| H_0\right] \\
&= \Pr\left[\frac{1}{\alpha(1-\alpha)}\boldsymbol{Z}(\alpha)^\top \boldsymbol{R}^\top \boldsymbol{W}_0 \boldsymbol{R}\boldsymbol{Z}(\alpha) > \chi^2_{J,\tau} - \Theta_n(0)\right].
\end{aligned}$$

C Analysis of Wald-type Tests Implemented Using the Estimators of Powell (1991) and Hendricks and Koenker (1992)

We show in this appendix that the well-known estimators of $\boldsymbol{G}_0(\alpha)$ proposed by Powell (1991) and Hendricks and Koenker (1992) for the express purpose of quantile-regression inference cannot actually be used to generate Wald-type tests that control size adaptively in large samples. For $i = 1, \ldots, n$ and α a fixed quantile in $\mathcal{A}$, let

$$\hat{\mathbf{G}}_n^P(\alpha) \equiv \frac{1}{nh_n}\sum_{i=1}^n K\left(\frac{Y_i - \mathbf{X}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)}{h_n}\right) \boldsymbol{X}_i \mathbf{X}_i^\top \tag{36}$$

denote the kernel-based estimator of $\mathbf{G}_0(\alpha)$ proposed by Powell (1991), where in this case $K(\cdot)$ is taken to denote a smoothing kernel satisfying the conditions of Assumption 2 above, while the bandwidth h_n is assumed to satisfy the constraints $h_n \longrightarrow 0$ and $nh_n^3 \longrightarrow \infty$ as $n \longrightarrow \infty$.

In addition, let

$$\hat{f}_{ni}^{HK}(\alpha) \equiv \frac{2h_n}{\mathbf{X}_i^\top \left(\hat{\boldsymbol{\beta}}_n(\alpha + h_n) - \hat{\boldsymbol{\beta}}_n(\alpha - h_n)\right)} \tag{37}$$

denote the difference-quotient estimator of $f_i\left(\boldsymbol{X}_i^\top\beta(\alpha)\right)$ suggested by Hendricks and Koenker (1992). Let

$$\hat{\boldsymbol{G}}_n^{HK}(\alpha) \equiv \frac{1}{n}\sum_{i=1}^{n}\hat{f}_{ni}^{HK}(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top \tag{38}$$

denote the corresponding estimator of $\mathbf{G}_0(\alpha)$.

We establish that optimal implementations of $\hat{\mathbf{G}}_n^P(\alpha)$ or $\hat{\mathbf{G}}_n^{HK}(\alpha)$ from the point of view of maximizing the rate of decay of the empirical size distortion of a Wald-type test of H_0 : $\mathbf{R}\beta(\alpha)-\mathbf{r}=\mathbf{0}$ are still sub-optimal in the sense that the resulting tests will exhibit size distortions that decay at rates that are strictly slower than the $O_p\left(n^{-1/2}(\log n)^{3/2}\right)$ adaptive rate. Similarly, a Wald-type confidence interval for a given linear combination of $\beta(\alpha)$ constructed using $\hat{\mathbf{G}}_n^P(\alpha)$ or $\hat{\mathbf{G}}_n^{HK}(\alpha)$ will not exhibit a level error that decays at the $O_p\left(n^{-1}(\log n)^3\right)$ adaptive rate.

Theorem 3. *Suppose the validity of Assumptions 1, 4 and 5 as given in Appendix A above. Let $\{h_n\}$ denote a bandwidth sequence in which $h_n \to 0$ and $nh_n^3 \to \infty$ as $n \to \infty$. We have the following as $n \to \infty$:*

1. *Suppose $K(\cdot)$ is a smoothing kernel satisfying the conditions of Assumption 2. Then the magnitude of the empirical size distortion of a Wald-type test of H_0 : $\mathbf{R}\beta(\alpha)-\mathbf{r}=\mathbf{0}$ in which the estimator $\hat{\mathbf{G}}_n^P(\alpha)$ is embedded can be no smaller than $n^{-2/5}$-order. This rate of convergence is attained when $h_n \propto n^{-1/5}$. In addition, the level error of a Wald-type confidence interval for a linear combination of the elements of $\beta(\alpha)$ that incorporates $\hat{\mathbf{G}}_n^P(\alpha)$ can be no smaller than $n^{-4/5}$-order, a magnitude attained when $h_n \propto n^{-1/5}$.*

2. *The magnitude of the empirical size distortion of a test of H_0 : $\mathbf{R}\beta(\alpha)-\mathbf{r}=\mathbf{0}$ based on (2) as given in the main body of the paper and in which the estimator $\hat{\mathbf{G}}_n^{HK}(\alpha)$ is embedded can be no smaller than $n^{-2/7}$-order. This rate of convergence is attained when $h_n \propto n^{-1/7}$. In addition, the level error of a Wald-type confidence interval for a linear combination of the elements of $\beta(\alpha)$ that incorporates $\hat{\mathbf{G}}_n^{HK}(\alpha)$ can be no smaller than $n^{-4/7}$-order, a magnitude attained when $h_n \propto n^{-1/7}$.*

Proof. The proof appears below in Appendix C.1. □

C.1 Proof of Theorem 3

We consider the two conclusions of Theorem 3 in sequence.

First conclusion of Theorem 3:

By Taylor expansion, the following holds for each $\alpha \in \mathcal{A}$:

$$\begin{aligned}&\hat{\boldsymbol{G}}_n^P(\alpha)\\&= \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{1}{h_n}\left(Y_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top - \frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^{n} K^{(1)}\left(\frac{1}{h_n}\bar{U}_i(\alpha)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top,\end{aligned} \tag{39}$$

where $\left|\bar{U}_i(\alpha) - \left(Y_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right| < \left|n^{-1/2}\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\right|$ for each $i = 1, \ldots, n$ and where

$$\boldsymbol{B}_n(\alpha) = \sqrt{n}\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right).$$

Consider the first term in (39). Standard calculations show that for each $\alpha \in \mathcal{A}$,

$$\frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{1}{h_n}\left(Y_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right)\boldsymbol{X}_i\boldsymbol{X}_i^\top = \boldsymbol{G}_0(\alpha) + O_p\left(h_n^2 + \frac{1}{\sqrt{nh_n}}\right). \tag{40}$$

Consider the second term in (39). In particular, for $U_i(\alpha) \equiv Y_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)$ we have

$$\begin{aligned}&\frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^{n} K^{(1)}\left(\frac{1}{h_n}\bar{U}_i(\alpha)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top\\&= \frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^{n} K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top\\&+ \frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^{n}\left(K^{(1)}\left(\frac{1}{h_n}\bar{U}_i(\alpha)\right) - K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top.\end{aligned} \tag{41}$$

Standard calculations show that

$$\frac{1}{nh_n^2}\sum_{i=1}^{n}\left|K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right| - E\left[\frac{1}{h_n^2}\left|K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right|\,\middle|\, \boldsymbol{X}_i\right] = O_p\left(\frac{1}{\sqrt{nh_n^3}}\right), \tag{42}$$

while

$$E\left[\frac{1}{h_n^2}\left|K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right|\,\middle|\, \boldsymbol{X}_i\right]$$

$$\leq \left| f_i^{(1)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right| \cdot \left| \int_{\{K^{(1)}<0\}} K(w)dw - \int_{\{K^{(1)}>0\}} K(w)dw \right|$$

$$+ h_n \left| f_i^{(2)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right| \cdot \left| \int_{\{K^{(1)}<0\}} wK(w)dw - \int_{\{K^{(1)}>0\}} wK(w)dw \right| + O_p\left(h_n^2\right) \tag{43}$$

as $n \to \infty$. Combining (42) with (43) we find that

$$\frac{1}{nh_n^2}\sum_{i=1}^n \left| K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right|$$

$$\leq \max_i \left| f_i^{(1)}\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)\right| \cdot \left| \int_{\{K^{(1)}<0\}} K(w)dw - \int_{\{K^{(1)}>0\}} K(w)dw \right|$$

$$+ O_p\left(\frac{1}{\sqrt{nh_n^3}} + h_n\right). \tag{44}$$

The asymptotic normality result given in (2) of our paper implies that $\boldsymbol{B}_n(\alpha) = O_p(1)$ for every $\alpha \in \mathcal{A}$. Combine this result with (44) above to deduce that the following holds for each $\alpha \in \mathcal{A}$:

$$\left\| \frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^n K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top \right\|$$

$$\leq \frac{1}{\sqrt{n}} \max_i \|\boldsymbol{X}_i\|^3 \cdot \|\boldsymbol{B}_n(\alpha)\| \cdot \frac{1}{nh_n^2}\sum_{i=1}^n \left| K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right|$$

$$= O_p\left(\frac{1}{\sqrt{n}}\right) \cdot O_p(1) \cdot O_p\left(1 + \frac{1}{\sqrt{nh_n^3}} + h_n\right)$$

$$= O_p\left(\frac{1}{\sqrt{n}}\right). \tag{45}$$

Next, note that

$$\frac{1}{n^{\frac{3}{2}}h_n^2}\sum_{i=1}^n \left(K^{(1)}\left(\frac{1}{h_n}\bar{U}_i(\alpha)\right) - K^{(1)}\left(\frac{1}{h_n}U_i(\alpha)\right)\right)\boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\boldsymbol{X}_i\boldsymbol{X}_i^\top.$$

$$= O_p\left(\frac{1}{n^{\frac{3}{2}}h_n^3}\right)$$

$$= o_p\left(\frac{1}{\sqrt{n}}\right), \tag{46}$$

a result that follows from an application of the mean value theorem, the assumption that $\left\|K^{(2)}\right\|_\infty < \infty$ and $\|\boldsymbol{X}_i\| < \infty$, (2) of our paper, the assumption that $nh_n^3 \to \infty$ and the result that

$$\begin{aligned}\max_i \left|\bar{U}_i(\alpha) - U_i(\alpha)\right| &< \max_i \left|n^{-\frac{1}{2}} \boldsymbol{X}_i^\top \boldsymbol{B}_n(\alpha)\right| \\ &\leq \frac{1}{n^{\frac{1}{2}}} \max_i \|\boldsymbol{X}_i\| \cdot O_p(1) \\ &= O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned}$$

Combining (39), (40), (41), (45) and (46) we find that

$$\begin{aligned}\hat{\boldsymbol{G}}_n^P(\alpha) &= \boldsymbol{G}_0(\alpha) + O_p\left(h_n^2 + \frac{1}{\sqrt{nh_n}} + \frac{1}{\sqrt{n}}\right) \\ &= \boldsymbol{G}_0(\alpha) + O_p\left(h_n^2 + \frac{1}{\sqrt{nh_n}}\right) \end{aligned} \tag{47}$$

The expansion of $\hat{\boldsymbol{G}}_n^P(\alpha)$ given in (47) can then be combined with the binomial inverse theorem to deduce an expansion of $\left[\boldsymbol{R}\left(\hat{\boldsymbol{G}}_n^P(\alpha)\right)^{-1} \boldsymbol{H}_n \left(\hat{\boldsymbol{G}}_n^P(\alpha)\right)^{-1} \boldsymbol{R}^\top\right]^{-1}$ about $\left(\boldsymbol{R}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{R}^\top\right)^{-1}$ of the form

$$\begin{aligned}&\left[\boldsymbol{R}\left(\hat{\boldsymbol{G}}_n^P(\alpha)\right)^{-1} \boldsymbol{H}_n \left(\hat{\boldsymbol{G}}_n^P(\alpha)\right)^{-1} \boldsymbol{R}^\top\right]^{-1} \\ &= \left(\boldsymbol{R}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{R}^\top\right)^{-1} + \boldsymbol{U}_{1n}^P(\alpha) + \boldsymbol{U}_{2n}^P(\alpha) + \boldsymbol{R}_{n2}^P(\alpha),\end{aligned} \tag{48}$$

where for each $\alpha \in \mathcal{A}$, we have $\boldsymbol{U}_{1n}^P(\alpha) = O_p\left(h_n^2\right)$, $\boldsymbol{U}_{2n}^P(\alpha) = O_p\left((nh_n)^{-1/2}\right)$ and $\boldsymbol{R}_{n2}^P(\alpha) = o_p\left(h_n^2 + (nh_n)^{-1/2}\right)$ as $n \to \infty$.

The remainder of the proof of the first part of Theorem 3 follows arguments similar to those used in the proof of Theorem 2.

Second conclusion of Theorem 3:

Let $\alpha \in \mathcal{A} \equiv [a_1, a_2] \subset (0, 1)$ be a fixed quantile. Define

$$\boldsymbol{D}_n(\alpha) \equiv \sqrt{2nh_n}\left[\left(\hat{\boldsymbol{\beta}}_n(\alpha + h_n) - \boldsymbol{\beta}(\alpha + h_n)\right) - \left(\hat{\boldsymbol{\beta}}_n(\alpha - h_n) - \boldsymbol{\beta}(\alpha - h_n)\right)\right].$$

In particular, $\boldsymbol{D}_n(\alpha)$ denotes an instance of an appropriately normalized regression-quantile spacing local to $\left(\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{\beta}(\alpha)\right)$ whose asymptotic behavior is analyzed in Portnoy (2012). We make use of the following result derived from Portnoy (2012, Theorem 1, Ingredients 1–7 of the proof of Theorem 2):

Proposition 1 ((Portnoy, 2012)). *Suppose Assumptions 1, 4 and 5 hold. Then there exists a constant W such that for $\|\boldsymbol{D}_n(\alpha)\| \leq W\sqrt{\log n}$, the density of $\boldsymbol{D}_n(\alpha)$ at t satisfies*

$$f_{\boldsymbol{D}_n(\alpha)}(t) = \phi_{\boldsymbol{D}(\alpha)}(t)\left(1 + O\left(\sqrt{\frac{(\log n)^3}{2nh_n}}\right)\right),$$

where $\phi_{\boldsymbol{D}(\alpha)}(\cdot)$ denotes the density of a mean-zero Gaussian random vector $\boldsymbol{D}(\alpha)$ with covariance structure given in Portnoy (2012, eq. (7.3)).

In particular, Proposition 1 implies the existence of a mean-zero Gaussian random variable $\boldsymbol{D}(\alpha)$ such that

$$\|\boldsymbol{D}_n(\alpha) - \boldsymbol{D}(\alpha)\| = O_p\left(\sqrt{\frac{(\log n)^3}{nh_n}}\right). \tag{49}$$

Now consider the conditional density estimator $\hat{f}_{ni}^{HK}(\alpha)$ given in (37) above. For each $\alpha \in \mathcal{A}$ we have

$$\begin{aligned}
\hat{f}_{ni}^{HK}(\alpha) &= \frac{2h_n}{\boldsymbol{X}_i^\top \left[\frac{1}{\sqrt{2nh_n}}\boldsymbol{D}_n(\alpha) + (\boldsymbol{\beta}(\alpha + h_n) - \boldsymbol{\beta}(\alpha - h_n))\right]} \\
&= \frac{1}{\boldsymbol{X}_i^\top \left[\frac{1}{2^{\frac{3}{2}}\sqrt{nh_n^3}}\boldsymbol{D}_n(\alpha) + \frac{1}{2h_n}(\boldsymbol{\beta}(\alpha + h_n) - \boldsymbol{\beta}(\alpha - h_n))\right]} \\
&= \frac{1}{\frac{1}{2^{\frac{3}{2}}\sqrt{nh_n^3}} \cdot \boldsymbol{X}_i^\top \boldsymbol{D}_n(\alpha) + \frac{1}{f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right)} + O_p\left(h_n^2\right)} \\
&= f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) + O_p\left(h_n^2 + \frac{(\log n)^{\frac{3}{2}}}{nh_n^2} + \frac{1}{\sqrt{nh_n^3}}\right) \\
&= f_i\left(\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)\right) + O_p\left(h_n^2 + \frac{1}{\sqrt{nh_n^3}}\right),
\end{aligned}$$

where (49) and Taylor expansions of $\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha + h_n)$ and $\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha - h_n)$ about $\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)$ have been applied. Arguments similar to those used in the proof of Theorem 1 can then be used to show that for each $\alpha \in \mathcal{A}$,

$$\hat{\boldsymbol{G}}_n^{HK}(\alpha) = \boldsymbol{G}_0(\alpha) + \boldsymbol{T}_{1n}^{HK}(\alpha) + \boldsymbol{T}_{2n}^{HK}(\alpha) + \boldsymbol{R}_{n1}^{HK}(\alpha), \tag{50}$$

where $\boldsymbol{T}_{1n}^{HK}(\alpha) = O_p\left(h_n^2\right)$, $\boldsymbol{T}_{2n}^{HK}(\alpha) = O_p\left(\left(nh_n^3\right)^{-1/2}\right)$ and $\boldsymbol{R}_{n1}^{HK}(\alpha) = o_p\left(h_n^2 + \left(nh_n^3\right)^{-1/2}\right)$. The expansion of $\hat{\boldsymbol{G}}_n^{HK}(\alpha)$ given in (50) can then be combined with the binomial inverse theorem to deduce an expansion of $\left[\boldsymbol{R}\left(\hat{\boldsymbol{G}}_n^{HK}(\alpha)\right)^{-1}\boldsymbol{H}_n\left(\hat{\boldsymbol{G}}_n^{HK}(\alpha)\right)^{-1}\boldsymbol{R}^\top\right]^{-1}$ about $\boldsymbol{W}_0$ of the form

$$\begin{aligned}&\left[\boldsymbol{R}\left(\hat{\boldsymbol{G}}_n^{HK}(\alpha)\right)^{-1}\boldsymbol{H}_n\left(\hat{\boldsymbol{G}}_n^{HK}(\alpha)\right)^{-1}\boldsymbol{R}^\top\right]^{-1}\\ &= \left(\boldsymbol{R}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{H}\boldsymbol{G}_0^{-1}(\alpha)\boldsymbol{R}^\top\right)^{-1} + \boldsymbol{U}_{1n}^{HK}(\alpha) + \boldsymbol{U}_{2n}^{HK}(\alpha) + \boldsymbol{R}_{n2}^{HK}(\alpha),\end{aligned} \tag{51}$$

where for each $\alpha \in \mathcal{A}$, we have $\boldsymbol{U}_{1n}^{HK}(\alpha) = O_p\left(h_n^2\right)$, $\boldsymbol{U}_{2n}^{HK}(\alpha) = O_p\left(\left(nh_n^3\right)^{-1/2}\right)$ and $\boldsymbol{R}_{n2}^{HK}(\alpha) = o_p\left(h_n^2 + \left(nh_n^3\right)^{-1/2}\right)$ as $n \to \infty$.

The remainder of the proof of the second conclusion of Theorem 3 follows arguments similar to those used in the proof of Theorem 2.

D A Data-Driven and Rate-Optimal Bandwidth

We present details regarding the derivation and estimation of a particular rate-optimal bandwidth usable in the implementation of our proposed estimate of $\boldsymbol{G}_0(\alpha)$. This bandwidth differs from the fixed bandwidth used in the simulations presented in Section 4 of the paper in that it is data-driven, i.e., it involves a leading constant that must be estimated. This bandwidth is nevertheless rate-optimal in that it decays at a rate such that any Wald-type test in which the corresponding estimate of $\boldsymbol{G}_0(\alpha)$ is embedded exhibits adaptive size control as $n \to \infty$. We present simulation evidence on the finite-sample performance of the resulting data-driven bandwidth in Appendix E.2 below. We also make use of our proposed data-driven bandwidth in the empirical analysis presented in Section 5 of our paper.

Assuming that the vector $\boldsymbol{\beta}(\alpha)$ of α-quantile regression coefficients is d-dimensional, let $\boldsymbol{R}$ denote a fully specified $(J \times d)$-matrix of rank J. In addition, let $\boldsymbol{r} \in \mathbb{R}^J$ be fully specified. Suppose that one wishes to test the hypothesis $H_0 : \boldsymbol{R}\boldsymbol{\beta}(\alpha) = \boldsymbol{r}$. Consider the size function given in (13) of our paper. Differentiating the size function with respect to h_m we find that the magnitude of the empirical size distortion is minimized by the solution to

$$h_m^5 = \frac{1}{16m}[\left(\log h_m^{-1}\right)^{\frac{1}{2}} - \left(\log h_m^{-1}\right)^{-\frac{1}{2}} h_m^2]^2 \left(\frac{\Lambda_{2nm}(\alpha,0)}{\Lambda_{1nm}(\alpha,0)}\right)^2, \tag{52}$$

where $\Lambda_{1nm}(\alpha,0)$ and $\Lambda_{2nm}(\alpha,0)$ are as given above in (32) and (33), respectively.

Note that

$$\frac{1}{16m}[\left(\log h_m^{-1}\right)^{\frac{1}{2}} - \left(\log h_m^{-1}\right)^{-\frac{1}{2}} h_m^2]^2 \left(\frac{\Lambda_{2nm}(\alpha,0)}{\Lambda_{1nm}(\alpha,0)}\right)^2 \approx \frac{1}{m}\log h_m^{-1}\left(\frac{\Lambda_{2nm}(\alpha,0)}{4\Lambda_{1nm}(\alpha,0)}\right)^2$$

when m is large. It follows that for large m an approximate solution to (52) is given implicitly by the relation

$$\frac{h_m^5}{\log h_m^{-1}} = \left(\frac{\Lambda_{2nm}(\alpha,0)}{4\Lambda_{1nm}(\alpha,0)}\right)^2 \frac{1}{m},$$

which implies that for large m, the optimal value of h_m has the form

$$h_m^* = \kappa\left[\left(\frac{\Lambda_{2nm}(\alpha,0)}{\Lambda_{1nm}(\alpha,0)}\right)^2 \cdot \frac{\log m}{m}\right]^{\frac{1}{5}}, \tag{53}$$

where $\kappa > 0$ is a proportionality constant. Experimentation with simulations involving various settings of κ suggest that the choice $\kappa = 1$ works well in practice.

We present in this connection a plug-in estimate of the optimal bandwidth h_m^* given in (53) with $\kappa = 1$. In particular, we show how one might estimate the unknown quantities in the leading constant appearing in the expression for h_m^*. Let $\hat{\boldsymbol{\beta}}_n(\alpha)$ denote the regression α-quantile based on a random sample of observations given by $(\boldsymbol{X}_1^\top, Y_1), \ldots, (\boldsymbol{X}_n^\top, Y_n)$. Let $\boldsymbol{X}$ denote the $(n \times d)$ matrix of regressors whose ith row is given by $\boldsymbol{X}_i^\top$ $(i = 1, \ldots, n)$. We propose to estimate the optimal bandwidth given in (53) under the setting $\kappa = 1$ by

$$\hat{h}_m^* \equiv \left[\left(\frac{\hat{\Lambda}_{2nm}(\alpha,0)}{\hat{\Lambda}_{1nm}(\alpha,0)}\right)^2 \cdot \frac{\log m}{m}\right]^{\frac{1}{5}}, \tag{54}$$

where

$$\hat{\Lambda}_{1nm}(\alpha,0) = (\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r})^\top \hat{\boldsymbol{U}}_{1nm}(\alpha)(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r});$$

$$\hat{\Lambda}_{2nm}(\alpha,0) = (\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r})^\top \hat{\boldsymbol{U}}_{2nm}(\alpha)(\boldsymbol{R}\hat{\boldsymbol{\beta}}_n(\alpha) - \boldsymbol{r}),$$

and where the quantities $\hat{\boldsymbol{U}}_{1nm}(\alpha)$ and $\hat{\boldsymbol{U}}_{2nm}(\alpha)$ are given by

$$\begin{aligned}\hat{\boldsymbol{U}}_{1nm}(\alpha) = (\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top)^{-1}\cdot\Big(&\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\hat{\boldsymbol{T}}_{1nm}(\alpha)\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\\ &+\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\hat{\boldsymbol{T}}_{1nm}(\alpha)\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top\Big)(\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top)^{-1}\end{aligned} \tag{55}$$

and

$$\begin{aligned}\hat{\boldsymbol{U}}_{2nm}(\alpha) = (\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top)^{-1}\Big(&\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\hat{\boldsymbol{T}}_{2nm}(\alpha)\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\\ &+\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\hat{\boldsymbol{T}}_{2nm}(\alpha)\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top\Big)(\boldsymbol{R}\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{X}^\top\boldsymbol{X})\tilde{\boldsymbol{G}}_n(\alpha)^{-1}\boldsymbol{R}^\top)^{-1},\end{aligned} \tag{56}$$

respectively, where $\tilde{\boldsymbol{G}}_n(\alpha)$ denotes our proposed estimate of the matrix $\boldsymbol{G}_0(\alpha)$ given by $\hat{\boldsymbol{G}}_n(\alpha)$ in (6) of the main body of our paper, but implemented with the preliminary bandwidth

$$\tilde{h}_{m1} \equiv m^{-\frac{1}{5}}. \tag{57}$$

The quantities $\hat{\boldsymbol{T}}_{1nm}(\alpha)$ and $\hat{\boldsymbol{T}}_{2nm}(\alpha)$ appearing in (55) and (56), respectively, are given by

$$\hat{\boldsymbol{T}}_{1nm}(\alpha) = \frac{1}{n}\boldsymbol{X}^\top\tilde{\boldsymbol{f}}''_{nm}(\alpha)\boldsymbol{X}$$

and

$$\hat{\boldsymbol{T}}_{2nm}(\alpha) = \frac{1}{n}\boldsymbol{X}^\top\tilde{\boldsymbol{S}}_{nm}(\alpha)\boldsymbol{X}, \tag{58}$$

where $\tilde{\boldsymbol{f}}''_{nm}(\alpha)$ denotes the diagonal $(n\times n)$-matrix whose ith diagonal element is a kernel estimate of the second derivative of the conditional response density at the point $\boldsymbol{X}_i^\top\boldsymbol{\beta}(\alpha)$ given by

$$\tilde{\boldsymbol{f}}^2_{ni}(\alpha) \equiv \frac{1}{m\tilde{h}^3_{m2}}\sum_{j=1}^{m}K^{(2)}\left(\frac{1}{\tilde{h}_{m2}}\boldsymbol{X}_i^\top(\hat{\boldsymbol{\beta}}_n(U_j) - \hat{\boldsymbol{\beta}}_n(\alpha))\right),$$

where $U_1,\ldots,U_m$ denote a random sample of $Unif[a_1,a_2]$-variates generated by Monte Carlo, and where $K(\cdot)$ denotes the standard Gaussian kernel, $K^{(2)}(\cdot)$ denotes its second derivative and

$\tilde{h}_{m2} = m^{-1/9}$. Similarly, the quantity $\tilde{\boldsymbol{S}}_{nm}(\alpha)$ appearing in (58) is the diagonal $(n \times n)$-matrix whose ith diagonal element is the square root of a kernel estimate of the conditional response density at the point $\boldsymbol{X}_i^\top \boldsymbol{\beta}(\alpha)$ given by

$$\boldsymbol{S}_{ni}(\alpha) \equiv \sqrt{\frac{1}{m\tilde{h}_{m1}} \sum_{j=1}^{m} K\left(\frac{1}{\tilde{h}_{m1}} \boldsymbol{X}_i^\top (\hat{\boldsymbol{\beta}}_n(U_j) - \hat{\boldsymbol{\beta}}_n(\alpha))\right)},$$

where again $U_1, \ldots, U_m$ is a simulated random sample of $Unif[a_1, a_2]$-variates and $K(\cdot)$ is the standard Gaussian kernel. The bandwidth $\tilde{h}_{m1}$ is as given above in (57).

E Further Numerical Evidence

This appendix contains additional simulation evidence in the context of the same family of data-generating processes described in Section 4 of the main text. These additional simulations also consider the size and power performance of tests of the same hypothesis of no treatment-effect heterogeneity described in the paper. Samples of sizes $n = 100$ and $n = 300$, generated by 1000 Monte Carlo replications, continue to be examined. The simulations reported in Appendices E.1, E.2 and E.4, like those reported in Section 4 of the paper, involve $N(0,1)$-errors. Appendix E.3 presents results for the same family of data-generating processes, but with t_3-errors. Appendix E.4 presents results for "F-tests" of the joint hypothesis of QTE-homogeneity in two different covariates , while Appendices E.1, E.2 and E.3 contain results for the same "t"-test of QTE-homogeneity in a single covariate considered in Section 4 of the main text.

E.1 Sensitivity analysis: Results induced by our method with a fixed bandwidth and varying choices of k and c, $N(0,1)$-errors

This appendix analyzes the sensitivity of Wald-type tests implemented according to our procedure to variation in the the pseudo-sample size m or to the smoothing parameter h_m, which are assumed to take the forms given in (14) and (15) in the main text, respectively. Wald-type tests implemented according to our procedure are seen to have size or power performance that is not much affected across quantiles or models by variation in the pseudo-sample size provided that the pseudo-sample is sufficiently large. In particular, variation in the tuning parameter k when $k \geq 5$ is seen not to exert strong effects on the size or power performance of Wald-type tests implemented with our method. The size performance of these same tests, on the other hand, is seen to be somewhat more sensitive to variation in the bandwidth leading constant, in particular, to variation in the parameter c. We also note the relative insensitivity of the size-corrected power of these tests to variation in c.

In what follows, Tables 1–6 display empirical sizes and size-corrected powers for Wald-type tests implemented according to our procedure with the bandwidth constant c appearing in (15) of the main text fixed at $c = 1.5$. Tables 7–12 display the same quantities, but for tests implemented according to the proposed procedure in which the pseudo-sample size constant k appearing in (14) is fixed at $k = 5$.

Table 1: Empirical rejection percentages (size and size-corrected powers), Model 1. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	7.2	15.7	31.3	46.5	4.7	18.9	38.3	57.6	5.9	18.9	40.2	57.0
3	5.6	19.4	36.6	58.6	3.7	23.4	42.4	65.7	3.9	19.6	37.9	59.7
5	6.0	17.0	37.8	61.9	4.2	23.5	42.8	66.7	4.6	19.4	38.4	61.7
7	6.1	18.7	38.7	60.9	4.7	19.0	46.7	75.4	5.4	19.9	45.8	71.9
9	6.5	19.4	47.8	77.7	5.5	20.4	43.8	70.0	6.0	19.5	44.0	69.9
11	6.5	20.4	45.6	67.6	3.9	24.0	54.2	81.0	7.2	16.5	38.2	66.6
13	6.0	20.8	40.8	69.2	6.1	19.8	47.9	75.9	5.5	20.6	45.7	73.2
15	6.4	20.1	40.5	66.2	5.1	21.7	43.0	67.9	6.2	22.9	47.7	75.4
$n = 300$												
1	4.8	31.6	69.5	86.4	3.8	32.4	70.1	88.7	4.9	29.1	65.7	82.1
3	6.0	26.6	66.0	90.6	5.1	33.0	74.6	95.1	4.7	29.5	72.4	94.2
5	5.0	30.2	74.7	95.6	5.2	37.3	85.1	99.3	5.2	28.2	73.4	95.2
7	3.8	38.5	81.8	98.4	4.3	39.7	87.6	99.3	5.0	38.9	83.3	98.6
9	5.1	32.4	80.8	98.3	3.7	41.8	89.6	99.9	6.3	30.7	76.9	97.1
11	5.8	32.1	78.7	98.3	5.5	34.9	82.2	98.8	6.6	32.1	81.1	98.7
13	4.6	35.4	81.0	98.7	4.8	37.4	86.1	99.4	6.0	32.0	80.2	98.6
15	6.7	28.8	77.6	97.9	5.2	40.2	88.2	99.8	7.0	27.2	73.5	97.9

Table 2: Empirical rejection percentages (size and size-corrected powers), Model 2. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	6.4	15.8	23.6	33.4	3.9	22.2	39.4	53.5	5.5	20.2	36.9	48.9
3	5.2	17.9	28.2	42.7	4.2	20.5	43.6	62.7	6.2	21.5	47.1	68.3
5	5.0	16.0	31.2	50.7	5.5	16.8	33.4	57.0	5.2	26.1	54.8	75.0
7	6.6	14.8	29.1	48.1	3.6	24.4	49.0	69.2	5.6	22.7	53.4	72.5
9	7.9	10.2	26.3	46.1	5.4	22.4	49.9	71.9	5.7	26.8	52.5	75.5
11	6.9	15.1	29.7	47.7	5.8	24.5	50.8	75.0	6.1	27.3	57.7	80.1
13	6.3	19.2	31.8	48.8	4.6	25.1	50.1	72.0	6.4	25.6	52.0	73.6
15	6.5	15.9	33.0	52.5	5.7	20.2	48.5	70.9	5.0	28.3	57.2	80.2
$n = 300$												
1	5.8	18.1	46.3	71.3	3.9	33.0	70.6	85.1	5.2	38.5	75.2	84.8
3	5.4	21.9	49.4	76.7	3.5	36.0	78.1	96.2	5.0	38.3	77.7	94.1
5	4.1	22.2	57.9	87.1	3.4	39.1	86.2	98.5	5.2	40.9	83.0	96.1
7	5.4	21.7	54.2	83.4	3.8	34.1	78.1	95.5	6.2	43.2	86.5	97.8
9	5.6	21.0	56.8	86.6	3.8	40.9	88.0	98.7	4.2	41.8	86.4	98.1
11	5.5	20.2	52.1	83.3	4.0	45.4	87.8	97.7	5.7	42.6	87.1	98.2
13	4.8	22.8	58.7	86.9	4.7	37.1	82.2	97.9	7.7	44.0	89.5	98.5
15	6.9	19.4	60.2	88.4	4.8	36.3	84.3	97.3	5.7	39.7	87.2	98.4

Table 3: Empirical rejection percentages (size and size-corrected powers), Model 3. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	5.8	15.7	25.5	36.2	4.2	17.8	30.4	43.9	5.7	20.8	38.8	53.8
3	5.6	14.0	24.9	43.5	3.8	18.9	38.6	60.6	6.5	22.6	47.1	69.8
5	6.2	12.6	28.8	50.8	4.9	18.8	43.9	67.7	4.9	26.4	53.3	74.4
7	6.6	14.1	29.0	49.5	4.2	22.9	49.2	73.6	6.5	22.4	47.9	70.6
9	5.9	13.8	23.7	39.7	5.5	15.8	39.0	60.0	7.2	24.6	52.9	73.1
11	6.9	12.9	26.0	42.9	4.5	19.0	44.9	67.2	6.8	24.6	52.2	75.8
13	6.9	15.8	27.8	45.3	5.0	19.0	42.0	64.0	6.0	26.4	55.5	76.4
15	6.8	13.6	28.9	49.1	4.4	22.4	44.2	66.9	6.9	22.7	53.2	73.5
$n = 300$												
1	5.2	16.9	39.3	63.6	3.6	31.5	67.4	82.7	7.2	37.5	71.2	86.0
3	5.3	16.8	54.2	84.4	3.5	35.1	78.1	95.7	5.3	39.6	80.6	95.0
5	5.9	17.1	49.8	81.7	4.9	31.4	83.4	97.3	5.8	48.9	90.1	98.8
7	4.9	18.7	56.2	87.4	5.1	33.0	81.4	96.6	5.1	44.4	86.8	98.9
9	7.7	13.0	46.1	78.3	4.0	33.9	77.5	96.2	4.9	44.0	88.1	98.1
11	6.5	15.2	45.5	81.6	2.9	41.0	86.4	99.0	5.6	43.8	88.4	98.8
13	6.4	14.5	51.1	82.5	4.9	30.1	78.6	97.1	5.3	51.2	93.2	99.4
15	6.1	15.2	49.4	82.6	5.1	33.4	83.7	98.4	5.9	51.3	91.6	99.6

Table 4: Empirical rejection percentages (size and size-corrected powers), Model 4. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	5.9	17.1	27.7	40.6	5.0	18.5	38.2	52.2	4.9	20.8	40.4	54.7
3	4.6	17.7	33.9	49.6	4.2	23.7	50.5	71.2	4.8	21.3	45.4	67.6
5	7.4	15.8	33.9	52.8	4.4	22.1	49.5	71.7	6.4	19.4	46.2	72.1
7	6.5	14.5	27.1	48.1	3.9	22.4	49.8	74.9	5.3	27.9	57.7	80.4
9	7.0	15.8	27.5	47.7	3.9	19.9	44.0	68.4	7.2	20.2	50.0	75.3
11	7.4	15.5	29.3	46.7	5.0	23.1	50.8	74.4	6.8	22.4	50.1	74.3
13	6.5	14.8	34.3	56.1	3.9	23.2	51.8	73.6	6.1	19.6	46.0	70.0
15	7.6	16.4	33.9	54.1	5.7	19.8	41.9	68.2	6.4	21.6	44.9	70.9
$n = 300$												
1	5.4	23.7	51.2	71.8	3.3	40.8	79.2	91.4	6.9	35.9	71.6	85.7
3	5.3	22.3	60.1	86.2	3.5	40.0	82.8	97.0	4.4	43.9	87.0	97.2
5	6.7	18.3	50.2	79.6	4.6	37.4	86.9	98.5	4.2	42.7	86.9	98.5
7	5.9	24.2	65.0	90.7	3.9	42.2	87.3	99.4	6.2	42.3	87.3	98.9
9	4.9	29.1	67.2	92.2	5.0	37.3	83.6	98.2	6.4	48.8	93.3	99.4
11	5.5	20.5	60.4	88.6	5.2	36.8	80.9	97.6	6.5	47.4	92.9	99.7
13	5.4	24.9	67.4	90.9	4.0	44.1	90.7	99.0	5.6	39.5	87.0	98.9
15	4.8	29.9	67.0	92.0	5.0	41.7	89.3	99.1	6.1	44.2	92.8	99.3

Table 5: Empirical rejection percentages (size and size-corrected powers), Model 5. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	5.7	14.3	28.5	45.6	5.1	18.2	34.7	49.6	5.9	18.5	35.7	50.6
3	5.1	18.4	34.6	58.9	3.9	19.6	44.2	69.8	5.5	20.7	44.4	66.7
5	5.1	17.2	35.1	56.9	5.6	18.9	41.4	63.8	6.1	21.2	50.7	75.9
7	5.8	17.0	39.0	63.0	4.4	22.8	51.0	78.6	4.7	24.9	52.7	76.2
9	5.8	18.1	33.7	54.7	5.6	19.8	45.9	70.3	4.6	25.7	54.2	76.5
11	6.1	17.9	41.6	64.8	5.0	20.6	48.9	75.9	6.3	21.1	47.1	73.4
13	5.9	20.1	38.2	62.4	5.7	21.6	47.8	72.7	5.3	23.1	52.1	80.0
15	7.0	13.8	31.5	55.2	5.6	17.4	42.9	72.4	6.0	23.6	49.7	76.3
$n = 300$												
1	4.3	24.1	57.4	81.3	3.7	35.4	74.7	89.6	5.7	30.4	64.2	81.4
3	4.8	29.3	71.3	93.4	4.5	34.4	81.1	97.6	6.0	36.5	78.8	96.5
5	5.5	24.7	61.6	88.2	5.3	31.2	79.8	97.6	5.6	35.0	81.8	98.4
7	4.3	32.5	73.8	94.9	5.2	37.3	83.5	99.1	5.0	37.3	84.1	98.9
9	5.0	27.2	67.0	92.5	4.1	40.5	84.7	99.2	6.1	38.0	83.7	98.3
11	5.9	26.7	70.2	94.4	4.6	37.5	87.6	99.6	5.5	35.1	82.5	98.5
13	5.3	25.8	71.0	94.8	3.3	42.1	90.6	99.2	5.9	45.5	90.9	99.7
15	5.7	29.1	73.1	95.0	4.0	41.3	85.7	98.7	5.7	42.5	90.2	99.5

Table 6: Empirical rejection percentages (size and size-corrected powers), Model 6. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
k/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1	7.7	21.0	42.3	61.9	3.7	23.0	43.3	61.5	6.3	13.7	28.6	45.1
3	4.9	25.9	54.5	78.9	4.1	22.2	46.2	71.3	6.3	13.9	28.0	49.5
5	6.3	20.3	43.3	68.1	5.6	18.2	38.4	67.3	6.2	15.3	34.5	62.8
7	7.3	25.4	51.3	73.7	4.3	21.7	45.5	72.1	5.6	16.0	39.7	65.5
9	5.7	25.9	53.7	77.5	5.3	20.0	47.3	73.2	8.3	11.0	27.4	53.7
11	6.0	24.0	47.5	71.4	4.9	22.0	48.0	77.1	6.2	14.0	30.2	53.3
13	7.7	26.8	55.7	82.4	5.4	22.1	42.3	65.3	7.8	15.6	34.1	61.2
15	5.8	26.3	51.0	77.0	4.6	22.3	50.6	75.7	8.0	12.9	33.0	64.3
$n = 300$												
1	5.3	41.3	75.7	88.6	3.7	35.3	74.0	90.4	5.3	22.0	53.7	79.0
3	4.4	42.1	80.5	95.9	4.4	36.1	82.3	97.7	5.5	19.4	55.6	85.6
5	4.2	43.2	82.0	97.7	3.5	38.8	85.4	98.8	5.9	19.0	66.8	94.4
7	5.3	47.8	88.8	98.9	3.2	43.0	88.7	99.6	6.7	22.2	69.9	96.2
9	7.1	43.4	89.0	99.6	4.5	39.7	88.0	99.7	5.5	18.2	64.6	93.4
11	6.4	40.1	83.4	98.6	4.3	34.8	87.9	99.3	6.3	19.7	65.6	95.6
13	5.5	50.5	93.4	99.9	5.2	36.8	88.0	99.1	5.8	23.0	68.9	96.4
15	4.3	48.5	87.8	99.1	4.5	36.8	86.8	99.7	6.1	23.8	77.1	98.3

Table 7: Empirical rejection percentages (size and size-corrected powers), Model 1. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $k = 5$

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	7.80	20.60	33.60	50.20	7.30	17.80	45.50	71.50	8.50	17.50	40.20	64.40
1.25	7.20	17.80	40.10	63.80	6.00	20.80	43.90	70.70	6.30	18.70	38.10	62.00
1.50	6.30	20.70	44.40	72.20	4.10	20.80	49.90	78.70	5.10	19.60	46.60	73.90
1.75	6.00	18.30	46.80	75.80	4.30	21.30	51.70	81.60	5.20	19.30	39.90	64.60
2.00	3.10	23.10	47.30	73.00	2.00	30.60	63.10	87.90	3.90	18.40	38.10	65.30
$n = 300$												
1.00	6.20	33.00	72.00	93.20	6.00	33.20	76.20	96.90	8.20	30.40	77.20	94.70
1.25	5.00	37.80	81.30	98.00	6.00	30.30	75.20	96.60	6.50	33.00	78.50	96.40
1.50	5.00	29.30	70.80	94.10	4.70	38.20	84.90	98.90	5.70	27.20	68.30	93.60
1.75	4.30	36.70	83.90	98.70	3.40	40.60	88.70	99.40	4.60	34.30	80.90	98.00
2.00	3.50	36.10	83.30	99.20	2.70	39.30	84.90	98.80	4.30	35.50	85.50	99.50

Table 8: Empirical rejection percentages (size and size-corrected powers), Model 2. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $k = 5$

$n = 100$			$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$	
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	9.50	16.30	29.50	44.70	8.00	21.90	49.20	72.00	7.50	23.40	44.40	64.10
1.25	7.90	14.80	27.70	44.50	4.20	20.90	45.70	67.40	7.30	21.20	43.10	64.60
1.50	7.20	13.40	25.50	44.20	4.90	20.90	44.20	66.20	6.10	23.80	48.10	66.70
1.75	4.90	14.40	30.00	49.00	3.00	21.30	47.00	69.60	4.40	22.00	42.20	60.00
2.00	3.90	14.90	28.90	44.40	2.90	23.40	48.50	70.00	4.30	22.20	46.90	68.80
$n = 300$												
1.00	6.00	24.80	56.40	82.20	6.00	36.20	79.20	95.20	6.90	40.40	78.10	93.60
1.25	5.60	23.10	56.90	83.00	5.20	33.70	78.90	95.40	5.50	43.20	85.60	96.30
1.50	5.80	18.30	52.70	80.50	4.10	44.10	85.70	98.00	6.50	47.20	89.60	98.50
1.75	3.40	22.70	59.00	86.30	3.40	37.20	84.80	97.50	4.80	50.40	91.90	98.60
2.00	4.40	24.80	63.10	90.10	2.90	44.90	87.60	98.70	3.60	52.80	93.80	99.10

Table 9: Empirical rejection percentages (size and size-corrected powers), Model 3. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $k = 5$

$n = 100$			$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$	
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	8.60	12.20	22.50	39.70	6.80	18.70	36.10	56.70	8.60	22.70	43.80	62.70
1.25	6.50	13.60	24.50	41.70	5.60	19.20	45.30	69.00	6.40	27.10	54.40	76.10
1.50	4.60	14.70	29.60	49.50	4.10	20.50	45.40	67.20	5.00	27.90	58.00	78.10
1.75	5.40	15.40	27.50	47.10	3.40	20.40	49.20	75.30	4.30	32.10	66.30	82.70
2.00	3.90	15.20	32.40	52.20	2.80	18.20	46.70	71.10	4.40	27.00	58.30	80.80
$n = 300$												
1.00	8.90	14.20	42.00	76.60	4.90	35.60	79.90	95.80	7.40	45.90	86.60	97.10
1.25	6.50	13.70	41.10	73.90	5.80	25.40	68.90	92.60	5.90	42.40	84.90	96.60
1.50	5.70	16.60	49.90	82.90	3.50	38.50	83.50	97.70	5.20	48.80	91.50	98.40
1.75	3.70	19.70	52.40	85.80	3.40	34.90	80.00	98.10	4.90	44.60	86.80	98.60
2.00	4.70	17.30	50.40	82.90	3.70	36.40	81.80	97.20	3.60	50.90	93.20	99.30

Table 10: Empirical rejection percentages (size and size-corrected powers), Model 4. 1000 Monte Carlo replications; procedure “weg” implemented with fixed bandwidth, $k = 5$

$n = 100$			$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$	
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	8.00	15.90	27.00	41.70	7.60	18.20	40.60	61.20	8.20	25.20	57.40	78.80
1.25	6.20	20.00	33.30	51.80	5.90	21.20	42.40	63.80	7.80	21.90	46.50	66.50
1.50	6.10	14.00	28.10	46.70	5.20	21.40	47.30	71.10	5.50	23.10	48.50	70.30
1.75	4.80	14.50	30.00	51.10	3.60	20.40	42.30	66.30	4.90	22.10	47.50	67.30
2.00	3.40	21.40	42.30	64.70	2.70	25.80	53.40	75.70	3.90	20.10	39.90	65.70
$n = 300$												
1.00	7.40	22.90	54.30	79.40	6.50	36.40	80.50	96.30	7.20	41.10	87.20	97.80
1.25	6.40	23.10	58.90	84.80	4.30	39.80	82.50	96.80	5.80	40.10	85.70	97.30
1.50	4.70	22.90	57.00	85.10	5.00	37.40	84.50	98.70	4.70	46.20	89.00	98.80
1.75	4.20	26.20	60.50	87.60	3.60	34.50	81.20	97.50	5.30	53.00	92.80	99.70
2.00	3.40	23.30	63.10	90.50	3.20	43.30	87.20	99.10	4.80	39.50	87.20	98.80

Table 11: Empirical rejection percentages (size and size-corrected powers), Model 5. 1000 Monte Carlo replications; procedure “weg” implemented with fixed bandwidth, $k = 5$

$n = 100$			$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$	
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	9.30	16.30	31.70	49.70	7.60	21.30	47.90	73.60	7.70	19.80	40.90	63.10
1.25	6.10	18.30	37.50	59.60	5.30	21.80	43.80	71.50	7.70	19.70	37.00	57.50
1.50	5.70	18.00	35.30	56.80	4.00	25.40	50.30	74.30	6.00	19.80	39.70	64.10
1.75	4.40	19.10	37.50	60.60	3.70	25.50	55.20	80.70	5.80	21.00	52.00	76.90
2.00	4.00	16.50	37.00	57.80	2.30	22.30	46.10	73.10	3.50	22.40	45.90	69.50
$n = 300$												
1.00	6.80	28.00	65.80	90.60	4.90	39.20	83.70	98.60	6.20	39.70	81.70	97.50
1.25	6.70	23.70	58.70	84.40	6.30	31.60	78.90	97.00	5.60	35.80	78.80	96.50
1.50	5.50	24.20	62.90	90.60	4.50	34.60	80.70	98.00	4.20	43.80	88.20	98.90
1.75	5.70	27.10	70.80	93.70	2.90	38.50	88.50	99.40	5.10	41.10	88.40	98.90
2.00	4.50	27.00	69.90	94.20	3.10	35.80	86.10	99.30	4.00	36.10	80.80	98.30

Table 12: Empirical rejection percentages (size and size-corrected powers), Model 6. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $k = 5$

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
c/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
1.00	8.70	20.80	41.60	64.20	6.90	20.40	46.40	73.60	9.50	14.90	29.50	50.60
1.25	5.90	23.20	41.50	58.20	6.00	21.40	42.90	68.30	7.50	12.00	32.80	55.60
1.50	5.60	23.70	53.80	77.30	5.40	19.00	47.00	73.80	6.10	14.70	31.80	58.10
1.75	5.20	23.60	43.50	66.70	3.10	20.50	45.70	72.70	4.40	15.40	34.80	57.90
2.00	4.00	25.80	50.50	75.30	2.60	24.20	52.40	80.20	3.90	16.60	38.30	66.20
$n = 300$												
1.00	7.10	45.50	86.90	97.90	5.20	34.70	77.60	96.60	6.30	19.50	61.10	87.60
1.25	5.40	47.80	88.40	98.30	4.40	41.40	88.90	99.20	5.80	23.20	71.50	95.80
1.50	4.60	47.90	88.60	98.70	4.10	40.10	87.60	98.90	6.30	18.90	58.20	89.50
1.75	5.30	43.90	88.40	99.10	2.50	45.70	88.70	99.60	4.40	21.00	68.50	94.50
2.00	4.00	47.50	88.40	98.70	2.80	45.70	93.00	99.70	4.10	20.60	65.40	94.90

E.2 Data-driven bandwidth: Results induced by our method with $k = 5$ and the data-driven bandwidth derived in Appendix D

This appendix presents the performance of Wald-type tests implemented according to our procedure but in which the bandwidth takes the form given by h_m^* above in (54). The rate-optimal data-driven bandwidth given above in (54) is seen to induce Wald-type tests with good size and power performance across quantiles and data-generating processes. These results are for the same "t-test" of QTE-homogeneity in a single covariate considered in Section 4 of the main text, and involve the same series of six data-generating processes with $N(0, 1)$-errors considered in the main text. Tables 13–18 below repeat the relevant entries in Tables 1–6 in the main text for ease of reference.

Table 13: Empirical rejection probabilities (size and size-corrected powers), Model 1. 1000 Monte Carlo replications; procedure "weg" implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7.2	14.2	36.8	59.6	4.7	21.3	45.9	72.2	4.8	19.8	45.3	71.1
wiid	9.1	10	22.2	39.5	7.3	15.5	45.7	75.9	8.2	12.5	31.1	56.3
wnid	8.1	8.3	18.7	37.9	6.8	17.5	51	80	7.4	12.2	33.1	59.9
wker	1.3	13.2	31.5	53.7	0.3	17	51.2	80.8	1.9	17.7	41.8	69.5
riid	7.9	8.6	21.4	39.4	8.6	17.7	46.5	76.9	7.5	15.3	35.5	61.5
rnid	5.9	7.4	19	37.7	6.5	17.5	46.7	76.5	5.1	15.2	34.7	61.3
bxy	3.1	9.6	23.6	44.7	2.9	16.7	49.8	80	3.2	14.8	37	65.7
bpwy	1.2	9.7	23.7	44.3	2.4	17.1	49.4	80.4	1.6	17.5	41.1	69.6
bmcmb	3.3	8.8	23.2	43.3	3.7	16	48.9	79.2	3.4	16.6	39.7	66.7
bwxy	4.1	9.3	22.9	44.5	3	16	48.4	79.9	4.4	13.7	36	64.6
bwild	6.9	10.9	24	46.2	7.2	14.1	42.7	76.1	6.2	16.2	37	65.4
$n = 300$												
weg	6.1	26.9	71.4	93.9	4.6	36.4	85.1	98.8	5.9	27.3	68.8	93.3
wiid	7.9	25.4	74.2	98.1	3.7	33.6	84.3	98.5	6	30.5	84.5	99.6
wnid	8.2	26.2	76.1	98.6	3.9	34.9	86.4	98.6	5.9	32.5	84.7	99.3
wker	3	28.4	79.5	99.3	1.3	34.5	85.9	98.7	2	34.3	87	99.7
riid	7.7	27	75.8	97.6	5	31.4	80.5	98.1	5.6	31.7	81.6	98.8
rnid	6.6	26.5	74.7	97.6	4.7	31.4	80.4	98	4.7	31	82.3	98.6
bxy	4.4	29.4	79.2	98.3	2.5	34.1	84.4	98.4	3	32.7	85.5	99.4
bpwy	3.4	28.9	78.8	98.7	2.2	34.4	84.9	98.4	2.3	34.5	85.9	99.3
bmcmb	5.9	26.9	77.9	98.4	3.7	33.7	82.4	98.3	3.8	32.5	84.6	99.2
bwxy	4.9	29.2	79.1	98.8	2.7	32	82.4	98.4	3.1	31.5	83.9	99.2
bwild	7.1	29	79.1	98.7	4.8	32.3	82	98.3	4.9	31.9	85.7	99.6

Table 14: Empirical rejection probabilities (size and size-corrected powers), Model 2. 1000 Monte Carlo replications; procedure “weg” implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$			$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$	
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7	12.4	21.3	39.2	6.2	17.4	38	58.7	6.1	20.5	44.1	65.4
wiid	8.4	10.2	20.5	38.9	8.9	12.7	34.5	62.7	9	15.7	39.9	63.1
wnid	7.4	7.3	21.8	40.2	9.1	12.6	37.2	65.3	8.6	14.5	42.1	64.4
wker	1.5	8	21.9	39.9	1.1	12.5	36.5	63.2	1.7	11.3	37.7	61
riid	7.7	7.9	20.2	36.7	8.7	11.1	31.7	55.1	8.2	14.7	37.6	60.1
rnid	5.8	7.5	20.2	36	7.2	11.4	31.1	54.1	6	14	35.9	57.1
bxy	3.4	7.9	20.3	37.6	3.4	12.6	36.2	60.3	4.1	14.6	39.3	62.1
bpwy	1.8	7.1	20.8	40.2	2.9	12.8	37.1	62.7	2.5	12.7	40	62.6
bmcmb	3.4	8	20.5	36.7	4.1	12.7	36.2	60.1	4.6	15.3	39.2	61
bwxy	4.5	8.3	20.6	37.9	4.2	13.2	37.1	60.2	5.2	13.5	38.7	61.5
bwild	7.4	7.3	18.7	35.4	8.4	12.9	35.1	57	7.3	14.1	38.5	59.2
$n = 300$												
weg	5.7	15	45.9	77.2	5	32.6	77.1	96.1	5.6	35.4	81.2	95.7
wiid	5.5	20.7	58.8	88.4	5	32.2	81.3	98.5	8	34	83.4	98.3
wnid	5.9	19.6	60.1	88.6	4.8	35.6	84.6	98.5	8.4	36	86.5	98.9
wker	2.3	18.4	57	86	1	35.9	82.2	97.9	2.3	36	85.7	98.7
riid	6	17.9	55	83.9	5.4	31.8	77.7	96.5	7.5	35.6	82	97.3
rnid	4.6	17.3	53.3	83.1	5.1	30.7	76.9	96.2	6.8	33.8	80.9	96.8
bxy	2.6	20.7	58.8	84.2	3.7	32.7	79.9	96.9	3.7	38.4	84.6	98
bpwy	2.4	18.1	55.3	83.7	3	32.7	79.1	97	3	38.5	84.7	98.4
bmcmb	4.3	18.3	53.1	82.7	4.4	31.2	78.4	97	5	37.9	84.2	97.4
bwxy	2.6	17.8	53.8	81.8	3.6	31.5	78.6	96.7	4	36.1	82.8	97.3
bwild	5.1	19.1	55.8	84.1	5	30.7	78.9	96.4	6.1	36.1	84.6	98.5

Table 15: Empirical rejection probabilities (size and size-corrected powers), Model 3. 1000 Monte Carlo replications; procedure "weg" implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7.4	10.9	21.5	39.8	6.2	17	36.9	57.5	6.3	23.3	53.1	76.8
wiid	9.7	6.1	14.5	26.4	7.5	11	28.5	53.3	7.7	16.1	44.9	71.8
wnid	7.9	8.4	19	36.8	6.7	11	31.9	56.8	7.2	18.2	47.4	72.1
wker	1.4	8.1	19.7	39.2	0.7	12.5	33.4	58	1.4	18.9	52.6	78.1
riid	7.5	6.5	15.7	32.6	7.3	9.4	26.7	47.4	8	16.9	43.9	68.2
rnid	5.3	6.7	16.6	32.2	6.5	9.3	27.8	45.6	5.5	17.3	45.4	68.3
bxy	2.4	8.3	19.1	37.9	2.8	12.3	32.3	55.7	3	19.3	49.2	75.2
bpwy	1.2	8.1	20.3	38.2	2.4	11.6	31.8	54.2	1.5	18.7	50.2	75.7
bmcmb	2.6	7.5	18.5	34.5	3.6	11.6	31.8	54.7	3.1	18.1	47.1	73
bwxy	3.1	8.5	20.2	37.6	3.5	10.7	30.9	54.2	3.9	18.9	49.5	74.3
bwild	6.3	7.7	18.5	35.7	7.6	10	27.7	50.2	7	17.1	47.2	73.6
$n = 300$												
weg	5.2	17.4	52.4	82.3	4.8	30.3	79.4	97.1	4.9	49.5	90.5	98.4
wiid	6.6	12.5	46.4	81.3	6.9	24.4	74	96.3	6.9	41.1	91.1	99.5
wnid	6.8	14.7	52.7	84.1	5.8	28.7	78.2	97.3	7.7	41.4	92	99.7
wker	3.3	15.4	52.7	84.5	1.6	28.2	76.7	96.2	3.2	40	90.4	99.7
riid	5.8	15.6	49.7	82.2	6.4	26	72.1	95	7.3	38.3	87.3	98.9
rnid	5	15	48.1	80.5	6	25.4	70.4	94.4	6.4	37.9	86.5	99
bxy	3.7	16.1	50.3	83.3	3.5	27.3	74.7	95.6	3.8	41.1	89.9	99.6
bpwy	3.1	15.6	52	83.7	3	28.2	75.4	95.9	2.8	38.5	89.8	99.2
bmcmb	4.7	14.8	49.7	81	4.7	28.7	76.5	96	5.1	40.8	90.5	99.4
bwxy	3.7	14.9	51	82.8	3.7	28.5	75.7	96	4.2	39.9	90	99.6
bwild	6.3	13.9	48.7	81.9	5.9	25.3	73.3	95.7	6.8	37.8	88.9	99.5

Table 16: Empirical rejection probabilities (size and size-corrected powers), Model 4. 1000 Monte Carlo replications; procedure "weg" implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	4.9	13.3	24.4	41.8	4.8	15.8	38.6	60.2	5	22.5	46.2	66.9
wiid	9.8	6	16	30.4	7.5	14.2	41.1	68.6	9.8	13	36.8	66.9
wnid	8.5	6.3	15.6	32.2	7.8	14	42.8	69	8.2	15.9	43.8	71.8
wker	1.4	11.7	24.5	43.2	1.1	12.7	43	66.9	1.7	16.2	45.1	73.2
riid	7.7	7.4	17.8	31.6	7.4	15.3	40.3	63.1	7.9	14.4	41	67.2
rnid	5.4	8.2	18.8	34.5	6.3	13.8	39.5	62.1	5.5	15.9	41.7	68.4
bxy	3.2	9.1	19.6	37.8	3.6	14.6	42.5	65.3	3.1	17.6	46.3	72.6
bpwy	1.5	8.5	20.7	38	2.7	13.8	40.6	64.2	1.1	17.1	47.3	74.9
bmcmb	4.4	6.7	17.2	33.3	4.1	14	41.2	64.2	3.2	17.3	45.9	71.4
bwxy	4.4	8.9	20.2	37.7	3.9	15	42.9	66.2	4.3	17.7	47	72.5
bwild	7.4	9.2	20.7	37.3	6.7	13.6	40	64.3	7.8	15.3	41.5	68.7
$n = 300$												
weg	4.6	26.6	64.8	90.2	5.4	30	73.5	94.2	5.7	42.3	89.6	98.4
wiid	6.5	14.5	48.1	81.5	6.9	28.7	79.6	97.7	5.9	40.1	88.3	99.2
wnid	7.3	17.6	53.3	84.3	7.2	28.2	79.6	97.6	5.9	42.9	90.3	99.4
wker	3.5	23.3	59.9	87.1	2.1	28.9	78.6	97.6	2.3	41.1	88.9	98.8
riid	7.2	17.4	49.2	81.6	8	26.7	76.5	96	5.6	40.4	86.6	98.2
rnid	6.1	17.5	50.8	81.9	6.8	25.6	76.1	95.5	4.8	41	86	98.1
bxy	4.5	18.6	52.2	82.5	3.8	27.8	77.6	96.4	3.3	40.8	87.6	98.3
bpwy	4	18	55	84.5	4.4	28.9	77.2	96.5	2.3	42.2	87.9	98.5
bmcmb	5.6	17	50.8	81.5	5.7	28.8	78.2	96.5	4.6	41.1	87.4	98.1
bwxy	4.5	18.2	52.5	82.5	4.8	28.1	76.7	96.2	3.3	43.5	88.6	98.4
bwild	6.6	17.9	53.1	82.9	6.5	25.9	75.7	96	5	41.5	88.3	98.7

Table 17: Empirical rejection probabilities (size and size-corrected powers), Model 5. 1000 Monte Carlo replications; procedure "weg" implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	5.6	13.7	32.4	59.2	5.4	15.3	37.7	63.3	6.3	17.4	41	64.4
wiid	8.5	10.9	22.7	42.1	7.2	11.5	27.7	55.6	9.6	11.3	28.9	54.6
wnid	8.2	10.1	25.5	46.1	6.7	11	32.5	59.5	8.2	10.7	31	57.4
wker	1.1	13.1	30.4	54.3	0.7	12.3	34	60.5	1.5	12.8	35.9	65.2
riid	7.3	11.1	25.7	45.9	8.1	9.4	27.2	51.8	8.4	11	29.5	58.8
rnid	5.3	11.2	26.1	45.8	7	10.7	27.3	51.9	6.2	11.7	28.9	56.8
bxy	2.7	11.4	27.1	49.4	2.5	11.9	32.7	58.7	3.3	12.4	33.3	62.7
bpwy	1.2	12.1	28.8	50.7	2.6	12.5	33.8	60.3	2	12.1	34	64
bmcmb	2.9	10.8	27.6	47.5	3.7	11.3	31.8	59.5	3.5	11.7	32.8	59.4
bwxy	4.2	11.3	27.7	48.8	3.6	11.3	32.4	58.6	4.4	11.8	32.7	61.8
bwild	6.8	12.4	26.9	47.2	7	9.7	28.6	53.9	7.4	10.5	31.8	61.2
$n = 300$												
weg	6.3	22	62.8	89.9	5.6	29	78.1	97	5.2	36.6	84.7	98.1
wiid	7.2	24	66.6	94.6	6.6	25.9	73.1	96.5	6.6	33.6	83.5	99.3
wnid	6.9	24.5	68.4	95.6	6.3	29	76.7	97.5	7	37.4	86.6	99.3
wker	2.7	26.5	72.1	96.6	1.7	30.4	77.3	97.8	2.6	38.3	87.9	99.4
riid	6.4	20.3	63.8	91.7	5.9	25.9	72.7	95.5	6.9	33.3	82.5	98.5
rnid	5.4	22.2	66.5	92.9	5.5	26.7	73.3	95.5	5.7	34.1	83.8	98.6
bxy	3.6	24.7	70.3	95.5	3.8	29.4	75.4	97.4	4	34.3	84.9	99
bpwy	3.5	23	68.2	95	3.6	28.6	75.9	97.1	2.7	37.7	85.8	99.2
bmcmb	4.9	24	68.6	95.3	5	28.1	75.5	96.9	4.6	36.2	85.4	99.3
bwxy	4	24.4	69.7	95.7	4.1	29.2	75.7	97.1	4.1	35	85.2	99.1
bwild	6.4	23.1	69.1	95.7	6	28.1	74.7	97.1	5.7	35.3	85.1	99.1

Table 18: Empirical rejection probabilities (size and size-corrected powers), Model 6. 1000 Monte Carlo replications; procedure “weg” implemented with data-driven bandwidth, $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	5.4	20.1	42.6	64.1	5.3	19	41	67.9	5.7	14.1	32.3	56.9
wiid	9.7	12.2	31.5	57.4	7.2	13.6	44.1	80.4	10	6.7	18.7	40
wnid	7.3	16.3	39	66.6	5.9	16.3	52.4	86.7	8.1	7.8	21.8	43.7
wker	1.3	20.2	47.4	75.8	0.8	16.3	53.5	89.2	2.2	8.9	26.2	52.1
riid	8.4	15	36.8	62.3	7.5	14.7	46.5	80.5	7.8	5.7	19.8	41.2
rnid	6.7	13.1	35	60.1	5.5	15	46.8	82.8	5.6	6.2	20.8	43.7
bxy	2.7	17.6	41.4	70.9	2.4	16.7	52.5	86.9	3.1	8.7	25.1	50.4
bpwy	1.5	17.7	42.7	71.8	1.9	16.9	51.1	87.2	1.7	8	22.5	48.7
bmcmb	3.1	15.8	40.3	69.1	3.4	15.8	51.8	85.7	3.6	8.6	23.7	50.7
bwxy	3.9	17.7	41.7	71.5	2.9	17.5	52.7	87.3	4.2	8	23.2	49.4
bwild	6.9	16.2	40.2	70	6.7	14	46	83.3	7.3	7.3	21.8	46.2
$n = 300$												
weg	5	43.4	86.1	98	6.2	28.5	76	97.4	6	18	60.2	91.5
wiid	6.4	39.7	87.4	99.2	8.3	25.9	76.7	97.9	6.5	13.9	54.3	91.3
wnid	6.4	42.8	89.3	99.7	8.1	26.7	78.4	98.9	6.5	16.6	60.6	94.2
wker	3.2	43.6	91	99.7	2.4	31.5	83.4	99	2.6	16	60.5	94.5
riid	6.9	39.3	86.5	99	7.5	25.2	71.8	97	6.4	15.1	56.7	92
rnid	6.1	39.7	86.1	99	6.9	27.8	76.3	97.6	5.3	15.2	55.7	92
bxy	3.2	43.8	89.3	99.4	4.4	30.3	80.1	98.4	3.1	16.1	59.3	93.5
bpwy	3.2	42.5	88.1	99.4	4.2	29.4	80.9	98.5	3.1	16.3	58.1	93.2
bmcmb	5.2	40.8	88.3	99.4	6.3	28.3	78.6	98.5	4.7	15.2	58.3	93.3
bwxy	4.6	39.8	87.7	99.5	5	28.8	79.4	98.4	3.5	16.5	58.5	93.7
bwild	5.7	39.7	87.7	99.5	6.7	29.2	80.4	98.7	6.1	14.2	56.8	93.1

E.3 Results for models with Student-t errors

This appendix repeats the simulations presented in Section 4 of the main text, but in which the $N(0,1)$-errors specified are replaced with t_3-errors. The corresponding simulation results are displayed in Tables 19–24.

We see that the empirical size accuracy and size-adjusted power of Wald-type tests induced by the proposed estimate of $\boldsymbol{G}_0(\alpha)$ are quite competitive with the other methods considered.

Table 19: Empirical rejection percentages (size and size-corrected powers), Model 1 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	6.1	17.7	37.4	62.1	4.6	20.9	44.4	70.9	8.5	14.9	28.9	52.6
wiid	8.6	12.9	27.6	51.5	7.9	10.7	32.2	58.6	10.4	9.1	21.7	41.4
wnid	5.3	14.2	35	60.9	6.6	12.1	38.3	67.4	7.1	10.1	27.1	51.8
wker	1.2	12	30.8	57.8	0.5	15.8	43	73.3	2.4	13.6	34	59
riid	6.9	13.4	33.7	58.1	5.9	14.4	37.4	65.5	8.3	10.1	26.5	49.2
rnid	4.3	13	33.1	57.6	4.6	13.9	38	65.9	6	8.7	24.4	47.7
bxy	1.9	14.5	36.8	62	2.3	15	41.3	70.9	2.8	11.1	29	55.4
bpwy	0.9	13.1	34.7	59.7	1.4	15.6	41.4	72.2	1.9	14.5	34.3	60.5
bmcmb	2.9	15	34	59.5	2.9	16.1	41.9	72.4	3.6	10.4	27.7	54.1
bwxy	2.8	14.4	37.5	62.9	2.4	16.4	40.7	71.9	3.9	12.5	31.3	57.9
bwild	6.3	13.1	32	57.7	4.7	14.1	38.2	67.3	7.5	9.9	25.7	51.2
$n = 300$												
weg	6.9	21.8	61.9	89.1	4.5	33.8	80.9	98.3	6.6	29.9	69.2	93.9
wiid	8.5	16.3	57.1	90.3	7	21.9	74.5	97.5	7.8	22.7	70.4	95.1
wnid	7.3	20.7	67.3	94	6.1	28	79.3	98.2	7.3	24.2	72.6	95.9
wker	2.7	20.4	66.7	93.2	1.2	33.5	82.9	98.8	3.6	26.9	73.7	96.6
riid	6.9	18.3	62.3	91	6.1	27.8	78.5	98	5.9	28.3	74.7	95.6
rnid	5.6	19.7	63.7	91.2	5.7	27.9	78	97.9	5.1	26.6	74.3	94.9
bxy	3.2	22.2	68.4	93.2	3.2	29.6	80.1	98.3	3.2	30.6	76.7	97
bpwy	2.9	22.7	68.1	93.9	3.2	29.3	80.7	98.3	2.4	28.5	75.1	97
bmcmb	5.5	21.4	65.6	92.9	5	28.6	79.7	98.2	4.8	29.5	75.9	96.3
bwxy	3.9	22.8	69.2	93.9	3.6	29.4	81.1	98.3	3.7	30.1	76.6	97.3
bwild	7.3	18.9	63.7	92.2	5.8	26.4	77.1	98	6.4	26.3	73.2	96.1

Table 20: Empirical rejection percentages (size and size-corrected powers), Model 2 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7.1	12	24.3	40.7	4	19.5	39.5	57.2	7.5	20.6	42.7	60.2
wiid	9.8	7.9	15.9	31.3	6	10.6	25	45.4	8.8	14.6	34.7	56.6
wnid	6.3	9.1	20.8	36	6.5	10.3	28.3	47.9	6.5	15.5	37.8	57.9
wker	1.9	7.4	16.5	32	0.7	11.1	31	49.6	2	17.3	40.3	61.1
riid	7.4	8.8	17.2	30.1	6.4	8.7	23.7	39.8	7.1	15.4	34.2	54.3
rnid	5.2	8.5	17.1	29.1	5.1	9.4	23.7	40.2	4.9	15.1	34.2	53.8
bxy	2.2	8.2	18	32.2	2	10.5	28.7	45.8	2.4	16.7	39.7	59.6
bpwy	0.9	9.3	19.3	33.6	1.6	11.1	30.3	47.9	1.5	19.1	43.3	62.5
bmcmb	2.2	8.2	18.2	31.7	2.8	11.3	29.2	45.7	2.6	16	36.2	55.6
bwxy	3.3	8.2	18.8	31.6	2.2	10.4	29.6	47.6	2.9	18.3	40.3	59.2
bwild	7.3	7.8	16	30.5	5.2	10.3	25.2	42.2	6.9	14.8	35.4	56.2
$n = 300$												
weg	6.2	14.9	40.5	71.4	4.1	36	76.4	95.8	6.4	39.4	81.5	95.8
wiid	6.9	12.5	38.4	70.1	5.9	27.1	72.8	95.2	6.8	34.6	80.4	96.1
wnid	6	15.1	44.2	74	5	30.8	77.1	96.3	5.4	40.7	86.3	97.3
wker	3.3	12.2	37.1	67.6	0.9	35.8	78.1	96.2	3.4	38.7	82.5	96.3
riid	8.7	10.8	31	59.9	5.8	27.1	68	91.5	7.5	31.4	74	92.1
rnid	7.7	10.9	30.7	58.6	5.1	27.5	67.2	91.1	6.1	33.3	75	93
bxy	3.8	13.9	36.7	64.9	2.7	30	72.5	93.9	3.8	36.7	80.3	95
bpwy	3.5	12.4	37.2	66.7	2.8	30	73	93.3	3.9	35.7	78.9	95.4
bmcmb	5.7	13.1	35.8	62.8	3.9	30.2	72.3	93.1	4.7	39.1	80.5	95
bwxy	4.1	13.7	36.6	63.8	3.1	30.1	72.5	93.6	4.3	35.6	79	94.7
bwild	6.8	13.2	34.1	63.9	4.8	29	71.5	93.3	6.2	34.3	78.4	94.6

Table 21: Empirical rejection percentages (size and size-corrected powers), Model 3 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7.1	10.9	18.8	33.4	5.6	18.7	39.3	61.5	7.8	19.4	41.4	62.8
wiid	10.5	4.8	10.2	20.7	6.6	13.8	34.2	57.5	10.5	10.7	28	48.2
wnid	6.2	6.3	13.5	29.1	6	13.9	35.1	59.7	5.6	19.1	41.3	66.2
wker	1.5	7.4	18	35.3	0.5	14.7	35.9	60.6	1.3	17.9	40.9	65.6
riid	6.3	7.3	16.2	30.6	7	13.6	32	54.6	6.7	15.6	35	58.1
rnid	4.4	6.5	15.5	30.2	4.8	14.7	34	56.4	4.6	17.7	36.7	60.1
bxy	2.6	6.2	16.5	32.2	2.4	14.7	34.1	60	2.3	20.3	42.9	66.9
bpwy	1.2	6.7	15.6	31.8	1.6	14.2	33.7	59	0.8	20.7	43.8	68.5
bmcmb	3.4	5.3	13.5	27.1	3.2	12.1	30.6	54.6	2.6	17.5	38.1	63.2
bwxy	3.1	6.9	17.2	33.7	3.3	14.4	34.5	59.8	2.5	20.6	43.5	67.5
bwild	5.9	6.3	15.2	30.7	5.8	12.7	31	55.6	6.9	16.6	37.1	60.2
$n = 300$												
weg	5.3	15.5	41	71.4	3.3	27.8	75.3	95.3	6.9	33.7	79.7	96.1
wiid	8.5	9.5	31.5	64.7	6.1	20.3	66.8	94.9	8.1	28.1	78.3	97.6
wnid	7	9.9	35.8	71	5.2	23.3	73.8	96.7	8.1	31.3	82.2	98.1
wker	3.2	11.4	36.5	71.5	1.2	27.3	77.4	97.1	3.7	32.5	83.5	98.1
riid	6.1	12.3	35.5	67.6	6.3	20.4	66.4	94.2	6.7	27.6	75.4	96.6
rnid	5.3	13.3	35.3	67.6	5.8	21.1	68.1	93.3	5.9	29.8	77.4	96.5
bxy	2.9	13	37.5	71.8	3.7	21.9	70.2	94.6	3.9	31.1	81.2	97.6
bpwy	3.1	11.8	37.2	72.2	2.7	23	70.3	94.7	3.2	32.9	82.8	98.1
bmcmb	5.4	11.4	34.8	68.7	4.4	23.4	71	95.1	5.8	32.1	82	97.8
bwxy	3.8	12.5	36.3	71.7	3.4	23.6	70.7	94.8	4.3	32	81.6	97.7
bwild	7	11.1	34.7	70.2	5	22.8	70.8	95.4	6.8	27.9	79.4	97.7

Table 22: Empirical rejection percentages (size and size-corrected powers), Model 4 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	6.9	13.4	21.3	37.5	6.5	17.2	38.2	60.5	6.3	21	45.2	66.4
wiid	10	7.3	14.5	26	8	8.9	27.7	52.5	9.4	12.1	31.6	57.9
wnid	6.8	8.4	16.3	31.9	7.2	12.8	36.3	60.3	5.6	14.8	40.8	65.9
wker	1.7	9.1	18	33.5	0.8	12.8	37.1	60.6	1.7	13.4	40.4	66.7
riid	7.4	7.5	17.2	31.2	8.5	11.1	30.8	53.4	5.8	14	36.3	60.5
rnid	4.3	7.7	17.3	30.2	7.1	10.7	30.4	51.3	4.7	14.4	34.9	60
bxy	1.6	9.1	19.4	34.8	3.7	11.4	35.3	57.6	2.5	15	41.1	67.1
bpwy	0.8	9	18.8	35.4	2.8	12.2	36.6	58.9	1.4	17.2	43.7	68.3
bmcmb	2.3	7.5	15.6	29.9	4.2	10.9	33.3	55.7	2.8	13.4	36.7	62.6
bwxy	2.8	9.2	19.4	34.7	4.1	11.4	35.3	56.8	3	16.5	42.9	66.8
bwild	7.6	8.9	17.8	31.4	6.7	11.1	32.5	55.3	6.5	12.9	37.2	62.5
$n = 300$												
weg	7.1	14.8	47.4	78.4	4.4	36.4	83	97.5	7.1	32.2	77.7	95.8
wiid	7.3	11.3	42	78	6.7	29.6	80.7	97.6	7.7	26.9	77.3	96.8
wnid	7.7	12.6	44.6	80.6	6.3	32	81.4	98.5	7.3	31.6	83.4	98.5
wker	3.1	12.9	44.3	77.9	0.9	34.5	84	98.5	4.1	30.1	82.3	98.4
riid	6.8	13.6	45.6	77.5	7.3	27.3	75.8	95.7	7	30.1	79.2	97.3
rnid	5.6	14.4	46.4	77.5	6.7	26.5	74	94.4	5.5	32	80.4	97.3
bxy	3.7	13.3	45.9	77.6	3.3	32.2	80	97.2	4.3	33	83.9	98.2
bpwy	3.2	14.5	47.3	79.4	3	32.4	79.8	97.2	3.4	33.2	83.4	98.3
bmcmb	5.1	13.5	46.7	78.3	4.7	32.2	79.4	97.9	5.8	29.6	81.1	97.8
bwxy	4	14.5	46.4	78.1	3.4	32.2	79.5	96.9	4.6	31.4	82	98.2
bwild	7.4	11	40.9	74.4	6.2	29.6	78	96.2	7.2	28.6	80.2	97.6

Table 23: Empirical rejection percentages (size and size-corrected powers), Model 5 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	8.7	11.4	24.6	42.5	4	22.9	45.1	67.3	6.8	19.1	44.9	72.4
wiid	10.1	7.7	15.3	30.7	6.3	10.9	29.6	55.2	8.6	11.7	33.2	58.7
wnid	7	7.8	19.2	38.9	6	13.3	33.3	60	6.1	14.1	41.2	69.7
wker	2.1	9.2	22	43.7	0.5	17.8	42.7	70.3	0.8	14.7	43	74.4
riid	7.5	8	17.6	35.1	5.7	15.5	35.3	59.6	6.5	14.4	37.8	65.7
rnid	5	8.9	19.5	37.9	4.2	14.6	35	60.2	4.4	15.2	38.2	66.3
bxy	2.7	9.8	22.5	43.3	2	16.6	38.6	65.2	1.4	16	44	75
bpwy	1.4	9.3	21.8	42.6	1.5	17	39.6	66.8	0.7	18.6	46	75.9
bmcmb	3.2	8.8	20.6	38.5	2.5	16	38.2	65	2.1	16.5	43.8	72.2
bwxy	3.6	9.4	21.7	41.7	2.4	16.5	40	65.7	2.2	17.9	45	74.7
bwild	7.2	9.2	19.5	37.4	5	14.9	35	60.8	5.9	15.6	40.3	70.1
$n = 300$												
weg	6.6	23.7	59.2	88.7	3.7	32.9	81.7	97.5	5.2	33.5	78.6	95.7
wiid	6.8	20.6	57.4	89.4	5.8	28.6	80	98.5	6.6	27.1	75.3	96.7
wnid	5.9	24.1	63.5	93.4	5.5	30.9	81.1	98.7	6.4	29.3	79.9	97.8
wker	3.3	20.1	59.7	91.6	1.1	31.3	81.9	98.3	2.9	29.5	81.1	97.8
riid	7.1	20.7	58.6	88.8	7.2	26.4	76.8	96.6	6.6	26.2	73.8	96.4
rnid	5.6	20.9	57	88.3	6.6	25.2	75.7	96.5	5.6	26.9	74.6	96.7
bxy	3.4	22.5	60.4	91.4	3.7	29.2	79.7	97.8	3.8	29.6	78.3	97.3
bpwy	2.6	22.2	59.2	91.1	3.3	29.7	79.5	97.9	3.1	29.3	79	97.5
bmcmb	4.9	21.7	60.7	91.4	4.9	30.5	80.2	98	5.2	26.1	77.7	96.8
bwxy	3.6	22.7	61.7	91.8	3.7	28.8	80	97.7	4.3	27.4	77.1	96.8
bwild	7.1	20.2	57	90.2	6.1	26.8	77.2	97.6	6	28.2	79	97.3

Table 24: Empirical rejection percentages (size and size-corrected powers), Model 6 with t_3-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	6	19.9	41.4	64.5	4.1	23.1	47.2	73.3	7.5	13.1	29.5	54.8
wiid	9.7	9.9	24.6	46.3	6.3	15.2	36.6	66.4	10	6.9	19	39.3
wnid	6.5	15.6	35.6	61.3	5.4	15.6	41.9	73.2	5.5	9.6	27.1	54.5
wker	1.9	12.3	32.6	59	0.1	20	47	76.6	1.6	9.3	27.1	56.4
riid	6.4	16.2	35.2	58.9	6.1	16.7	40.7	69.6	6.4	9.5	25.2	49.5
rnid	4.1	17	36.7	60.8	5	17.3	39.8	68.9	4.6	9.6	25	49.8
bxy	1.7	18.1	40	64.5	1.2	19.6	46.6	77.4	2.1	10.6	31.1	58.7
bpwy	0.8	17.1	40.1	64.3	0.8	18.6	44.4	75.5	1.1	10.5	29	57
bmcmb	2.4	14.9	34.9	61.3	1.5	19.2	45.1	75.3	2.5	9	27	55
bwxy	2.5	17	38.7	64.5	1.4	20	45.8	76.3	2.9	10.9	30	58.6
bwild	6.5	13.4	33.8	59	5	16.4	40.7	70.8	5.4	8.8	25.1	53.2
$n = 300$												
weg	6.4	35.4	74.8	94.9	3.8	42.1	86.4	98.3	5.3	19.8	57.7	87.9
wiid	7.5	30.2	72.4	96.7	6	31	80.2	98.7	5.8	14.6	55.2	88.7
wnid	6.3	37.8	81.1	98.7	7.2	29.9	81.8	98.8	4.9	18.4	58.8	92
wker	3.5	33.1	77.8	98.5	0.9	38.2	87.2	99.1	2	18	60.2	91.7
riid	6.6	31.8	74.9	96.8	6.1	32.6	81.2	98.5	5.6	16.7	55.7	88.5
rnid	5.2	33.9	76.3	97.4	5.2	33.2	81.4	98.5	4.6	17.1	57	89.3
bxy	3.4	35.9	78.8	98.3	2.8	34.9	84.8	98.9	2.4	19	61.5	91.6
bpwy	2.8	35.7	79.1	98.2	2.4	34.4	84	98.6	2	18.9	60.2	90.9
bmcmb	4.8	34.7	78	98	4.9	34.1	83.5	99	3.7	16.1	57.3	90.3
bwxy	3.9	37.3	80	98.1	2.9	36.1	85.1	99	2.3	18.1	59.8	90.9
bwild	6.8	34.2	77.1	98	5.4	32.5	83	98.9	5.1	16.3	57.5	90.9

E.4 Results for tests of a joint hypothesis

This appendix considers a joint hypothesis of significance for a bivariate subvector of the vector of coefficients in a linear quantile regression. Specifically, we consider the family of data-generating processes given by $Y = 1 + \sum_{j=1}^{4} X_j + D + \delta_a(U)DX_1 + \gamma(U)X_5 + F^{-1}(U)$, where $\{X_j\}_{j=1}^{4}$, D, U and $\{\delta_a(\cdot) : a \in \mathbb{R}\}$ are as described in Section 4 of the main text, and where $P[\gamma(U) \equiv 0] = 1$, $X_5 \sim N(0,1)$ and X_5 is independent of $[\ X_1 \quad X_2 \quad X_3 \quad X_4 \quad D \quad U\ ]^\top$. That is, X_5 is an irrelevant regressor. In what follows we consider, for quantiles $\alpha \in \{.25, .50, .75\}$, tests of the null hypothesis $H_0 : \delta_a(\alpha) = \gamma(\alpha) = 0$. We examine the empirical power of these tests against alternatives in which $\delta_a(\alpha) \neq 0$ with $a \in \{.50, 1.00, 1.50\}$ and $\gamma(\alpha) = 0$.

The corresponding simulation results are displayed in Tables 25–30 for samples of sizes $n = 100$ and $n = 300$. These tables present the results of "F-test" implementations of our proposed procedure with pseudo-sample size m given by expression (14) in the main text with

$k = 5$. The corresponding bandwidth h_m is as given by (15) in the main text with constant $c = 1.5$. We also present the results of "F-tests" implemented using most of the other testing methods considered in Section 4 of the main text. Each of these other testing methods, with the exception of `riid`, were implemented by direct computation of the corresponding test statistic using the corresponding estimated asymptotic covariance generated by the `summary.rq` feature of the `quantreg` package. We also examined implementations of `wiid`, `wnid` and `wker` using `anova.rq`, but found that these implementations generated tests having empirical performances that were virtually identical to those of their counterparts implemented using `summary.rq`.

We note that `riid` can only be applied to tests of joint hypotheses using `anova.rq`. We also note that at present there exists no possibility of applying the `rnid` method to tests of joint hypotheses within `quantreg`.

We see that the empirical sizes and size-corrected powers of Wald-type tests induced by the proposed estimate of $\boldsymbol{G}_0(\alpha)$ are competitive with the alternative methods available. These results, along with those reported above in this appendix, supply further evidence of the potential of our method to generate tests with good size and power performance.

Table 25: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 1 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	6.7	19.7	37.9	61.8	4.5	22.3	43.3	65.8	6.7	19.2	37.1	59.4
wiid	11.9	9.5	22.6	43	8.1	10.3	24.8	51.9	13	7.4	16.8	35.9
wnid	11.9	9.2	24.1	47.4	8.1	12.1	31.2	59.1	13	6.5	18.6	38.2
wker	1.3	9.1	28.8	56.3	0.5	12.8	34.9	64.8	1.2	10.2	28.2	54.3
riid (anova.rq)	3.7	12	31.5	57.7	3	11.9	34	63.5	4.8	8	22.7	49.2
bxy	3.3	10	26	54.1	2.1	13.2	32.9	62.2	3	10.8	26.2	51.3
bpwy	1.5	9.1	26.3	53.4	1.5	12.9	33.1	62.3	1.3	9.7	25.8	51.4
bmcmb	3.8	7.5	23.7	48.4	2.8	12.5	30.8	59.9	3.2	10.1	26	50.2
bwxy	4.7	9.3	26.3	53.2	2.8	13.7	33.1	62	4.3	9.3	25.7	51.2
bwild	8.7	6.9	24.3	47.7	8	11.5	27.9	55.5	9.6	9.7	24.8	48.7
$n = 300$												
weg	5.7	26.5	68.4	94.1	4.7	29	72.7	95.5	5.4	30.6	68	92
wiid	8.7	17	57.7	92.7	6.9	17.2	63.2	95.4	8.8	19.6	59.2	91.2
wnid	8.7	19	64.4	95.4	6.9	20.3	68.7	96.8	8.8	21.6	65.2	94.7
wker	3.7	19.4	67.6	96.3	1.5	20.2	70.9	97.7	3.1	24.5	68.1	95.5
riid (anova.rq)	4.7	21.3	69.8	96.5	4.4	25	73.3	97.8	5.4	22.5	67.4	95.6
bxy	4	20.9	67.3	96.7	4.4	19.1	66.1	96.1	4.3	24.2	66.7	94.3
bpwy	3.7	20.3	67.5	96	4.3	18.8	64.4	95.8	3.5	23.3	66.5	94.2
bmcmb	5.8	20.8	67.7	96	6.5	20	65.3	96	5.9	23.4	63.4	94.4
bwxy	4.8	19.5	66.2	96.1	4.4	19.8	65.8	95.8	4.4	24.5	65.8	94.6
bwild	8.1	21.4	68.1	96.2	7.2	19.5	65.6	95.2	7.6	22	63.4	93.4

Table 26: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 2 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
`weg`	8.1	18.3	27.4	42.7	5	20.7	41.2	63	7.5	21.8	42.5	63.2
`wiid`	11.1	8	14.2	26.2	7.9	9.4	26	46.7	11.7	10.4	26.4	50.6
`wnid`	11.1	7.6	12.9	26.4	7.9	10.8	29	50.1	11.7	11.5	30	51.6
`wker`	1.3	11.1	20.5	35.6	0.6	8.7	25.8	48.3	1.5	13	35.6	62.4
`riid` (`anova.rq`)	4.1	7.9	15.9	29.6	4.2	11.7	27.1	48.7	5.2	10.7	30.9	54.4
`bxy`	3.6	7.6	15.6	29.7	2	9	23.8	45.2	3.1	12.7	33	59.6
`bpwy`	1.5	8.1	15.8	31.2	1.6	8.7	23.9	44.7	1.5	11.8	32	58
`bmcmb`	3.7	9.5	14.9	29.5	3.5	10.1	25.6	45	3.5	13.1	32.1	56
`bwxy`	5.1	8.4	15.3	30.2	3.3	9.3	26.1	46.9	4.9	12.1	32.6	57.8
`bwild`	8.8	8.4	14.2	30	7.9	9.5	22.1	42.6	8.3	12.1	30.8	55.1
$n = 300$												
`weg`	6.2	21	55.4	81.5	3.4	34.9	75.7	94	5.4	33.8	71.6	92.1
`wiid`	8.4	14.8	46.5	81.4	7.3	21.9	67.4	93.2	8.1	24.3	62.2	91.4
`wnid`	8.4	18.7	56	84.3	7.3	24.2	73.3	95	8.1	25	69.6	94.3
`wker`	2.7	12.8	46.3	79.3	1.1	26.6	71.7	94.1	2.8	30.7	72.9	94.6
`riid` (`anova.rq`)	6.3	14.1	44.2	75.9	4.5	25.8	70.8	94.3	5.3	23.5	69.5	93.3
`bxy`	4.4	14.3	45.3	73.1	3.6	24.4	67.8	91.6	4	28.1	70	92.2
`bpwy`	3.7	14.2	46.8	75.9	2.9	25.7	67.1	91.6	3.8	27.3	68.8	92.3
`bmcmb`	6.3	12.9	43.8	73.8	5.1	23.7	66.3	90.6	5.7	26.5	65	90.9
`bwxy`	5.8	12.2	41.5	71.1	3.5	23.3	66.1	91.8	4.6	26.7	67.8	91.1
`bwild`	8.2	13.2	44.3	74.6	5.9	24.6	66.1	91.5	6.8	27.9	69	92.3

Table 27: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 3 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7	16.9	25.8	37.5	4.3	25.5	38.7	57.8	7.2	24.8	47.1	65
wiid	10.2	6.3	11.8	22.7	8	7.7	17.5	32.9	11.3	10.9	24.1	47.3
wnid	10.2	6.3	12	23.5	8	9	20	36.6	11.3	10.9	27.6	47
wker	1.3	6.6	15	29.1	0.4	11.7	24.8	42.6	1.7	11.9	37.4	61.7
riid (anova.rq)	4	6.7	15.1	29.7	4.3	8.6	19.8	38.5	4.8	13.5	31.4	55.1
bxy	2.9	6.8	16.3	28.2	2.7	11.1	21.7	37.8	3.3	12.8	34.9	57.6
bpwy	1.3	7.4	14.5	28.5	2.2	9.5	19.6	36	1.9	12.9	36.5	59.9
bmcmb	3.8	5.6	14.6	23.6	3.2	11	20.1	36.1	3.8	11.4	31.2	52
bwxy	4.8	6.2	13.8	26.7	3.4	11.2	21.8	40	5.5	11.5	32.9	55.5
bwild	7.7	6.9	15.3	27.4	8.5	10	18.4	33.6	8.3	12.9	35.2	56
$n = 300$												
weg	6.3	17	41.3	71.3	4.3	29.7	72.8	95.3	5.7	42.6	85.3	98
wiid	8.4	12.1	35.4	69.2	6.9	22.3	66.4	95.7	6.4	37.5	84	98.7
wnid	8.4	14.4	42	74	6.9	24.4	70.8	96.6	6.4	42	88.7	99.5
wker	2.7	8.4	32.5	68.6	1.3	24.5	71.2	96.9	2.7	45.4	91	99.6
riid (anova.rq)	4.7	10.3	36.8	72.3	5.2	22.9	71.4	96.6	4.4	38.2	87.1	98.7
bxy	4.6	9.5	31.9	64.7	3.7	23.6	67.6	94.3	4	38.4	86.4	98.6
bpwy	3.7	10	34.1	67.4	3.7	23.9	69.5	94.6	2.5	41.9	87.9	98.7
bmcmb	6	10.3	31.2	64.9	5.8	22.7	67.2	93.7	5.3	39.8	86.6	98.4
bwxy	4.9	10.2	34.1	66.3	3.9	24.8	69.6	94.8	4.2	38.7	85.6	98.7
bwild	8.1	9.6	31.2	66.9	7	20.9	64	91.8	7.3	38.7	86.5	98.7

Table 28: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 4 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
`weg`	7.7	19.8	28.9	40.5	4.3	22.4	43.3	66.4	7.8	22.9	40.8	60
`wiid`	12.9	6	10.7	18.5	7.2	13.7	30.6	56.3	12.1	8.1	18.5	36.5
`wnid`	12.9	5.8	10.4	22.7	7.2	14.7	33.4	57.5	12.1	8.8	21	41.2
`wker`	1.7	9.2	18.2	34.2	0.4	9.6	28	51	1.7	9.3	27.4	50.9
`riid` (`anova.rq`)	4.9	5.9	14.9	26.7	3.9	12	28.5	53.1	6.1	7.7	22.1	43.7
`bxy`	3.8	7.7	14.6	25.7	2.2	11.7	27.4	49.4	4.7	8.4	24.3	45.9
`bpwy`	1.6	7.2	14.6	27.4	2.1	11.1	26.5	50.5	2	10.4	27.9	50.8
`bmcmb`	4.2	7.4	13.9	26.7	3.4	10.8	27.5	49.6	4.3	10.9	26.6	44.7
`bwxy`	4.9	8.8	16.3	30.2	3	11.6	29.6	52.5	5.3	11	27.6	48.9
`bwild`	8.6	8.5	15.2	26.9	6.5	11.7	28.2	47.7	10.4	7.4	21.2	40
$n = 300$												
`weg`	6.8	19.3	49.3	76.4	4.9	32.1	73.7	93.5	5.5	38.6	81.2	97.8
`wiid`	8.4	12.7	39.4	73.4	6.3	23.6	67.4	93.4	8.7	24.9	76.4	97.9
`wnid`	8.4	16.5	47.4	79.9	6.3	26.2	71.5	95.9	8.7	31.8	82.8	99
`wker`	3.3	13.4	42.6	76.1	1.2	24.8	69.6	95.5	2.2	30.2	85.2	99.1
`riid` (`anova.rq`)	5.7	12.1	45.2	79.8	4	26.4	71.4	95	5	30.6	83.4	98.9
`bxy`	4.4	14.3	42.1	73	3.8	23.9	65	91.8	4.8	26.4	79	98
`bpwy`	4	14.6	43	74.3	3.6	23.5	64.7	91.7	3.8	28.3	80.5	98.4
`bmcmb`	7.2	13.8	40	70	5.7	24	66.7	92.5	6.5	26.5	77.7	97.8
`bwxy`	5.3	14.1	43	73	4.3	24.1	65.6	91.6	5.3	26.1	79	98.4
`bwild`	9.3	14	40.5	71.5	7.9	22.5	63	91.2	7.3	24.5	76.3	97.9

Table 29: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 5 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$		$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$		
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
weg	7.2	17.9	29.7	45.9	4.7	21.7	41	64.6	8.2	19.5	41.6	64.6
wiid	12.3	6.1	11.8	23.9	8.7	8.2	21.3	46.2	11.1	9.2	22.6	48
wnid	12.3	5.7	15.2	27.9	8.7	8.9	25.6	51.2	11.1	9.1	26.9	54.2
wker	1.1	7	17	36.3	0.3	9.7	29.2	56.3	1.6	10.6	32.6	63.2
riid (anova.rq)	5.1	6.5	16.9	34.7	4.8	8.9	28.6	54.8	3.9	12.1	33.9	62.3
bxy	3.7	5.6	15.3	31.8	3.2	8.9	27.1	51.4	3.6	10.8	30.8	58.6
bpwy	1.5	6.7	15.4	35.5	2.3	8.9	27	52.9	1.8	12.4	33.4	61.7
bmcmb	3.9	6.4	14.9	30.9	3.8	8	22.7	48.5	3.5	10.3	31.6	58.3
bwxy	5.3	6.2	16.1	34	4.3	7.7	26.1	52.8	4.7	11.9	32.6	60.2
bwild	10.1	7	15.6	32	8.6	10	25.4	49.9	10	9.2	26.6	52.5
$n = 300$												
weg	5	26.2	64	89.9	4.2	34.7	78.4	97	5.5	35.9	79.4	97.4
wiid	8.2	14.7	49.4	85.6	7.7	22.1	72.1	97.8	9.4	25.7	74.1	97.6
wnid	8.2	18.6	58.2	89.4	7.7	23.1	75.5	98.7	9.4	28.7	79	98.4
wker	2.9	18.2	61.4	91.4	0.9	23.6	75.7	98.2	2.9	33.1	83	99.1
riid (anova.rq)	4.1	21.6	60.3	92.8	4.3	27.6	77	97.7	5.2	29.2	81.1	98.6
bxy	3.9	16.7	55.6	87.9	4.3	23.3	71.6	96.6	4.3	29	79.2	98.4
bpwy	3	16.7	57.4	88.2	4.2	22.7	70.8	97	3.3	29.9	79.4	98.3
bmcmb	5.5	17.1	56.5	87.8	6.5	23.7	71.2	96.7	6.2	29.2	78.1	97.8
bwxy	4.7	16.8	57.5	87.5	4.8	24.2	72.6	97.3	5	29.6	79.2	98
bwild	7.3	17	56.5	88	7.8	20.8	68.2	96.2	8.3	29.9	78.5	97.9

Table 30: Empirical rejection percentages (size and size-corrected powers), "F-test". Model 6 with $N(0,1)$-errors. 1000 Monte Carlo replications; procedure "weg" implemented with fixed bandwidth, $c = 1.5$ and $k = 5$; other procedures implemented using summary.rq unless otherwise indicated.

$n = 100$	$\alpha = 0.25$				$\alpha = 0.5$				$\alpha = 0.75$			
Method/a	0	0.50	1.00	1.50	0	0.50	1.00	1.50	0	0.50	1.00	1.50
`weg`	9.1	18.5	38.5	61.5	4.5	21.4	44.9	72.2	7.8	14.9	29.6	50
`wiid`	14	7.8	20.8	42.4	10.5	9	24.1	51.4	12.2	6.8	15.1	35
`wnid`	14	9.4	22.6	44.6	10.5	10	27.4	57.7	12.2	5.8	13.8	29.5
`wker`	1.3	11.4	30.3	58.8	0.2	12.6	36.9	68.6	1.6	7.3	20.1	44.6
`riid` (`anova.rq`)	5.5	10.7	29.7	55.3	4.3	11.8	35.7	67.9	4.1	7.2	21	43.9
`bxy`	3.8	11	30.5	56.7	2.4	11.9	34.5	66	3.4	6.1	19.2	41.8
`bpwy`	2	11.2	28	53.7	1.8	12.9	38.8	68.3	1.5	6.6	17.3	38.7
`bmcmb`	4.2	10.1	26.7	51.4	3	12.7	35.1	66.5	3.6	6.9	17.2	37.6
`bwxy`	6	10	26.2	53.1	2.9	11.5	36.5	66.5	4.8	6.4	18.3	41.1
`bwild`	10.5	11.2	29.4	55.9	8.4	10.6	31.3	60.8	8.9	5.6	15.7	38
$n = 300$												
`weg`	6.3	35.6	75.2	95.7	3.2	36.6	80.7	98.6	5.5	18.3	49.3	83.1
`wiid`	9.2	24.1	67.3	94.7	6.3	24.8	73.9	97.2	7.8	11.1	40.9	80.7
`wnid`	9.2	29.1	73.3	97.4	6.3	29.8	78.8	98.5	7.8	12.8	45.3	86.2
`wker`	2.9	36.6	81.8	98.9	0.8	30.8	80.4	98.9	3	13.4	48.9	89.3
`riid` (`anova.rq`)	5.7	28.4	77.5	97.8	4.6	27.6	77.1	99	4.7	11.2	47.4	88.4
`bxy`	4.1	31.4	76.3	97.2	3.6	28.7	76.8	98	3.9	11.1	44.4	85.2
`bpwy`	3.5	31.8	76.4	97.4	3.3	27.9	76.3	97.6	3	12.7	48	86.3
`bmcmb`	5.8	30.8	75.5	97.4	4.8	30.5	77.7	98.5	5	12.2	46.9	87.1
`bwxy`	4.5	31.1	75.4	97.3	3.3	29	77.5	98.1	4.7	11.9	44.9	85.5
`bwild`	7.6	29.4	74.2	97.3	6.6	25.9	73.1	97.3	7.8	11.3	43.5	84.3

F Additional Material on the Empirical Example

We present in this appendix further details regarding the empirical application considered in Section 5 of our paper. Recall that we are concerned with estimating the distinct effects of treatment for experimental subjects at each quantile in a grid of 300 evenly spaced points in [.20, .80]. This is done in the context of the quantile-regression model

$$F^{-1}_{\log T|\boldsymbol{X}}(\alpha) = \boldsymbol{X}^\top \boldsymbol{\beta}(\alpha), \tag{59}$$

where $\alpha \in [.20, .80]$, where T denotes the duration of unemployment in weeks and where the regressors contained in $\boldsymbol{X}$ include a constant term, an indicator for assignment to treatment and various demographic or socioeconomic control variables listed in Koenker and Xiao (2002, p. 1603).

We depart in this appendix from the general question of treatment-effect heterogeneity considered in Section 5 of the main text by focusing on the questions of whether the effects of treatment by quantile differ significantly according to the age of the participants and also according to whether participants have some expectation to be recalled to a previously held job, although not to the extent of having a definite date of recall within 60 days of filing their applications for unemployment insurance (UI) benefits (Corson et al., 1992, p. 9). We note in this connection that of the 6384 participants in this experiment, 3460 (54%) were under the age of 35, while 753 (12%) indicated to the experimenters some expectation of being recalled to previous employment. Participants in the latter category were assumed by the experimenters to be similar to claimants with no stated expectation of returning to a previous job in terms of their assumed response to a promised bonus payment upon securing new employment within the qualifying period. On the other hand, UI claimants who indicated both an expectation of recall and a definite recall date were disqualified from participation in the experiment as their stated confidence in returning to work was assumed to make their hypothetical behavioral response to treatment systematically different from UI claimants expressing less confidence in returning to full-time employment.

Figure 1 displays, in the context of the model given above in (59), estimated differences in treatment effects between workers younger than 35 and those aged 35 and older at the time of the experiment. These estimated differences in treatment effects are plotted for each quantile in a grid of 300 points in $[.20, .80]$. The shaded area in Figure 1 indicates the union of 90% confidence intervals for the estimated difference in treatment effects at each quantile. These confidence intervals are computed using our proposed method with data-driven bandwidth given above in (54) and where the pseudo-sample size m is given by (14) in the main text with $k = 5$. These confidence intervals imply that workers younger than 35 tend to exit unemployment as a result of the treatment significantly more quickly than workers 35 and older for nearly all quantiles in the interval $[.50, .80]$.

Estimated differences in treatment effects between workers with some expectation of being recalled to previous employment and those with no such expectation are displayed in Figure 2 for each quantile in a grid of 300 points in $[.20, .80]$. The shaded area in Figure 2, like that in Figure 1, denotes the union of 90% confidence intervals for the estimated difference in treatment effects, pointwise by quantile. These confidence intervals are computed in the same way as was done when generating the shaded area appearing in Figure 1. The confidence intervals in Figure 2 imply that the treatment has the effect of actually increasing unemployment durations for workers expecting a recall to a previous job for nearly all quantiles in the interval $[.43, .74]$. All this suggests that the cash bonus may not be as relevant as originally hoped for those claimants who indicated some degree of confidence in the temporary nature of their current spell of unemployment. In other words, this result suggests that the inclusion of these claimants in the experiment is as potentially problematic as the hypothetical inclusion of those excluded claimants who indicated both an expectation of recall and a definite recall date.

In summary, we have used our proposed method of inference to show that the treatment tends to cause participants having some expectation of being recalled to a previous job to exit unemployment more slowly than those not expecting to be recalled. This result further illustrates the utility, in terms of understanding behavioral responses to changes in unemployment insurance rules, of accounting for heterogeneity in treatment effects via the introduction of simple interaction terms in quantile-regression models.

Figure 1: Pennsylvania reemployment bonus experiment: 6384 observations. Differences in estimated treatment effects by quantile for workers younger than 35 and workers aged 35 and older, α-quantile regressions, $\alpha \in [.20, .80]$. The shaded area denotes the union of pointwise 90% confidence intervals, computed according to our proposal with data-driven bandwidth and $k = 5$, for each of 300 quantiles in $[.20, .80]$. Dotted vertical lines denote the .25-, .35-, .50-, .65- and .75-quantiles.

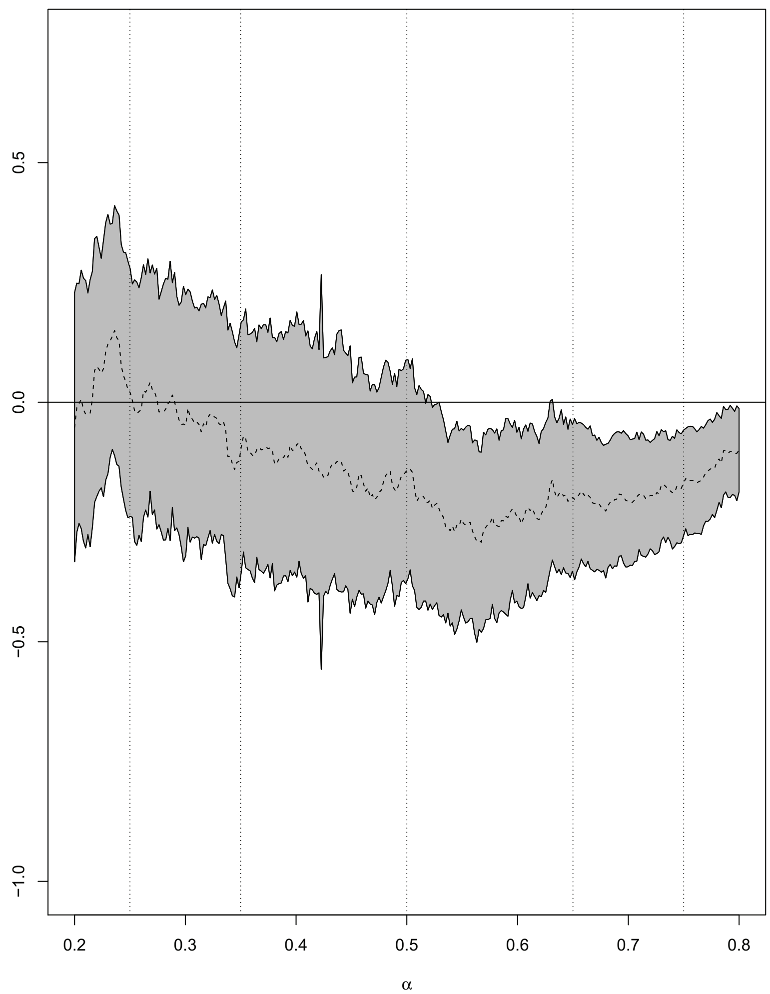
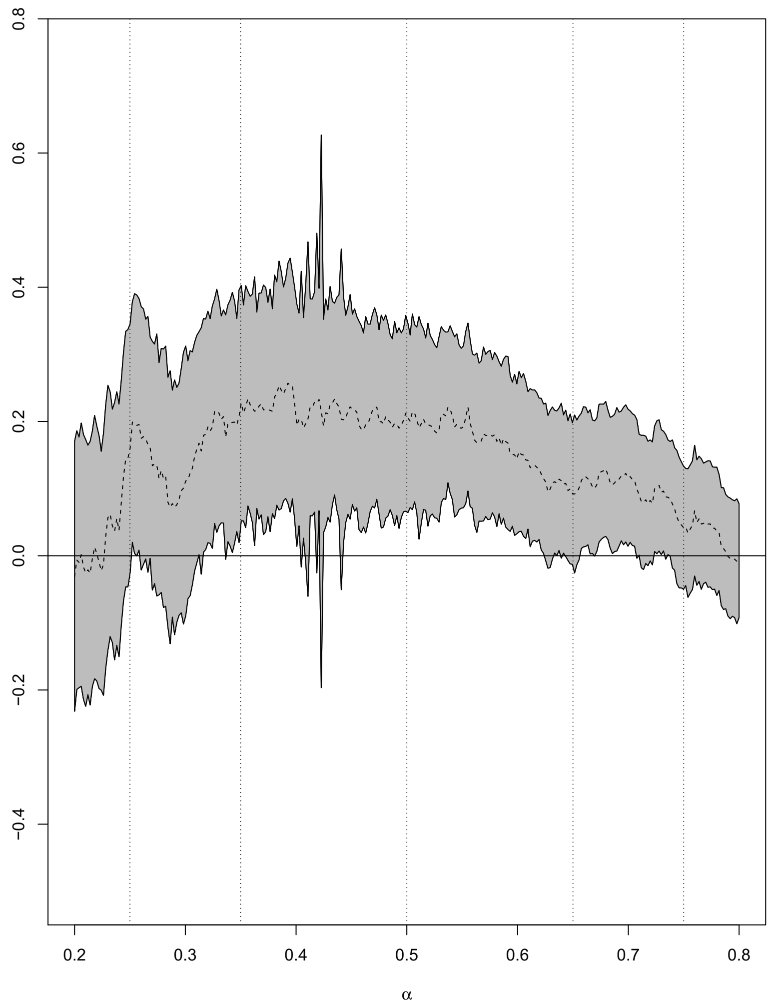


Figure 2: Pennsylvania reemployment bonus experiment: 6384 observations. Differences in estimated treatment effects by quantile for workers expecting and not expecting to be recalled to a previous job, α-quantile regressions, $\alpha \in [.20, .80]$. The shaded area denotes the union of pointwise 90% confidence intervals, computed according to our proposal with data-driven bandwidth and $k = 5$, for each of 300 quantiles in $[.20, .80]$. Dotted vertical lines denote the .25-, .35-, .50-, .65- and .75-quantiles.



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