Nash multiplicities and isolated points of maximum multiplicity

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Abstract

Let X be an algebraic variety defined over a field of characteristic zero, and let $\xi \in \underline{\text{Max}} \, \text{mult}(X)$ be a point in the closed subset of maximum multiplicity of X. We provide a criterion, given in terms of arcs, to determine whether ξ is isolated in $\underline{\text{Max}} \, \text{mult}(X)$. More precisely, we use invariants of arcs derived from the Nash multiplicity sequence to characterize when ξ is an isolated point in $\underline{\text{Max}} \, \text{mult}(X)$.

Introduction

In many situations, arc spaces are useful when one is looking for some information about the singularities of varieties. Let X be an algebraic variety over a field k, and $\xi \in X$ a point. An arc φ in $X = \text{Spec}(B)$ through ξ is a morphism

$$\varphi^* : \text{Spec}(K[[t]]) \longrightarrow X$$

mapping the closed point to ξ or, equivalently, a homomorphism of rings $\varphi : B \longrightarrow K[[t]]$ such that $\varphi(\mathcal{P}_\xi) \subset (t)$, where $\mathcal{P}_\xi \subset B$ is the prime ideal defining ξ. The order of the arc φ, which we will denote by $\text{ord}(\varphi)$, is the greatest positive integer n such that $\varphi(\mathcal{P}_\xi) \subset (t^n)$. Here K may be any field extension of k. Some examples of the study of the connections between arcs and singularities can be found, for instance, in the works of Ein, Ishii, Mustaţă, Reguera and Yasuda among others.

The Nash multiplicity sequence

We will be interested in a sequence of positive integers attached to an arc at a given point, the so called Nash multiplicity sequence, which was defined by M. Lejeune-Jalabert in [13] for hypersurfaces, and generalized later by M. Hickel in [10]. This sequence can be constructed as follows. Let ξ be a point in X and let φ be an arc through ξ. Also let $\Gamma_0 = \varphi \times i : B \times K[t] \to K[[t]]$ be the graph of φ, which is additionally an arc in $X_0 = X \times \mathbb{A}^1$ through $\xi_0 = (\xi, 0)$. Consider the following sequence of blow ups π_i at points:

(0.1)

Here π_i is the blowup at ξ_{i-1}, where $\xi_i = \text{Im}(\Gamma_i^*) \cap \pi_i^{-1}(\xi_{i-1})$ for $i = 1, \ldots, l, \ldots$, and Γ_i is the arc in X_i through ξ_i obtained by lifting Γ_0. Thus, the sequence is constructed by blowing up at closed points selected by the arcs Γ_i for $i \geq 0$. Let us recall that the multiplicity of X at a point $\eta \in X$ is given by an upper semicontinuous function

$$\text{mult}(X) : X \longrightarrow \mathbb{N}$$
$$\eta \longmapsto \text{mult}(X)(\eta) = \text{mult}_\eta(X) = \text{mult}(\mathcal{O}_{X,\eta}),$$

where $\text{mult}(\mathcal{O}_{X,\eta})$ stands for the multiplicity of the local ring $\mathcal{O}_{X,\eta}$ at the maximal ideal. Let us recall that this multiplicity can be computed as $a_d \cdot d!$, where a_d is the coefficient of the highest order term in the Hilbert polynomial of $\mathcal{O}_{X,\eta}$ (see [12, Ch. 11]). From a geometrical point of view, if X is defined over $\mathbb{C}$, then the multiplicty of X at η corresponds to the smallest of the ranks over the generic fiber when considering all possible local morphisms $(X, \eta) \to (\mathbb{C}^d, 0)$. If we denote by m_i the multiplicity of X_i at ξ_i for $i = 0, \ldots, l, \ldots$, then the Nash multiplicity sequence of φ in X at ξ is the sequence of positive integers

$$m_0 \geq m_1 \geq \ldots \geq m_l = m_{l+1} = \ldots \geq 1$$

(see [3, Section 2.2] for the detailed construction).

When X is a hypersurface, the Nash multiplicity sequence of an arc $\varphi \in \mathcal{L}(X)$ can be regarded as a refinement of the multiplicity of X at $\xi := \varphi(\langle t \rangle)$ in the following sense: On the one hand, the multiplicity function defines a stratification of X into locally closed subsets, and the multiplicity of X at a point corresponds to that of the stratum containing it. If X is a hypersurface embedded in a smooth scheme V, this stratification is determined by the order of vanishing of the partial derivatives applied to a local equation f defining X. On the other hand, consider the spaces of i-jets of X, which we shall denote by $\mathcal{L}_i(X)$ for $i \geq 0$, and the natural truncations from the arc space: $\pi_{X,i} : \mathcal{L}(X) \longrightarrow \mathcal{L}_i(X)$. Hence, $\pi_{X,i}(\mathcal{L}(X)) \subset \mathcal{L}_i(X)$, for $i \geq 0$, is the subset of i-jets in X which are the truncation of some arc in X. In [13], M. Lejeune-Jalabert proved that for each $i \geq 0$ there is a stratification of $\pi_{X,i}(\mathcal{L}(X)) \subset \mathcal{L}_i(X)$ into disjoint locally closed subsets: $\pi_{X,i}(\mathcal{L}(X)) = \cup_{1 \leq \mu_i \leq \ldots \leq \mu_0} \mathcal{H}_{\mu_0, \ldots, \mu_i}$. The Nash multiplicity sequence $(m_0, \ldots, m_i, \ldots)$ attached to the arc φ is determined by these stratifications: for $j \geq 0$, $\pi_{X,j}(\varphi) \in \mathcal{H}_{m_0, \ldots, m_j}$. In particular, for $j = 0$, $\pi_{X,0}(\mathcal{L}(X)) = \cup_{1 \leq \mu_0} \mathcal{H}_{\mu_0} \subset X$, corresponds to the stratification of X given by the multiplicity, so m_0 is just the multiplicity of X at ξ.

Throughout this paper, we use Hickel's approach since we state and prove our results for general varieties, not just hypersurfaces.

Our results

We will be particularly interested in the study of Nash multiplicity sequences of arcs through points of maximum multiplicity of X. Let $m := \max \text{mult}(X)$ be the maximum value achieved by $\text{mult}(X)$. We write

$$\underline{\text{Max}}\,\text{mult}(X) = \{\eta \in X : \text{mult}_\eta(X) \geq m\} = \{\eta \in X : \text{mult}_\eta(X) = m\}$$

for the closed subset of the singular locus of X consisting of the points of highest multiplicity. If X is a reduced equidimensional scheme, then X is regular if and only if the multiplicity equals one at every point (see [7, Section 2.10]). This is why the closed subset $\underline{\text{Max}}\,\text{mult}(X)$ is an object of interest in resolution of singularities.

If X is defined over a field k of characteristic zero, one can define the order of contact of an arc φ (through ξ) with $\underline{\text{Max}}\,\text{mult}(X)$,

$$r_{X,\varphi} \in \mathbb{Q}_{\geq 1}.$$

This order of contact is an invariant of the arc φ at the point ξ in X. This invariant can be computed as the order of a particular Rees algebra (see [3, Section 3]). From $r_{X,\varphi}$, one can obtain, for instance, the

number of blow ups as in (0.1) that are needed before the Nash multiplicity sequence decreases for the first time:

$$\rho_{X,\varphi} := \min_{i \in \mathbb{Z}_{>0}} \{i : m_i < m_0\} = [r_{X,\varphi}].$$

We call $\rho_{X,\varphi}$ the persistence of φ in X. In fact, sometimes it can be even more interesting to consider the quotient $\bar{r}_{X,\varphi} = \frac{r_{X,\varphi}}{\mathrm{ord}(\varphi)}$, which we refer to as the normalized order of contact of φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, since in this way one gets rid of the influence of the order of the arc. For the same reason, it is convenient to define also the normalized persistence of φ in X as $\bar{\rho}_{X,\varphi} = \frac{\rho_{X,\varphi}}{\mathrm{ord}(\varphi)}$. Indeed, the set

$$\Phi_{X,\xi} = \{\bar{r}_{X,\varphi}\}_\varphi \subset \mathbb{Q}_{\geq 1},$$

where φ runs over all arcs in X through ξ, is an invariant of X at ξ. This invariant reflects information about ξ coming from arcs. In particular, it uses information provided by the Nash multiplicity sequences of those arcs. It was proven in [3, Theorem 4.2.5] that this set has a minimum, which turns out to be the invariant $\mathrm{ord}_\xi^{(d)}(X)$, Hironaka's order in dimension d, which plays a key role in constructive resolution (see for example [11], [8] and [7, Sections 13 and 25]).

In this work we try to understand better what the set $\Phi_{X,\xi}$ can tell us about the singularities of X. We prove that the supremum of $\Phi_{X,\xi}$ actually allows us to determine whether ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ or not:

Theorem A. *(Main Theorem) Let X be a variety over a field k of characteristic zero, and let ξ be a point in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Then ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if the set $\Phi_{X,\xi}$ is upper bounded.*

In terms of the Nash multiplicity sequence it will mean (Corollary 2.4) that, whenever ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, then no arc through ξ can be found so that its normalized persistence in X is higher than a given integer (depending on X and ξ). On the other hand, if ξ belongs to a component of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ of dimension 1 or more, then there is no bound for how big $\bar{\rho}_{X,\varphi}$ will be for some arcs.

In the last section of this paper, we present an additional condition over X and isolated points of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ under which the supremum of $\Phi_{X,\xi}$ can be computed (Proposition 3.1). As we will see, this condition is also related to an invariant of constructive resolution of singularities: the τ invariant (see [1]). We will also show some illustrative examples there.

1 The order of contact of φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$

In what follows, we will assume X to be an algebraic variety of dimension d over a field k of characteristic zero such that $\max \mathrm{mult}(X) = b$, and ξ to be a point in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, which for simplicity we will assume to be closed. The notation used in this section will be the standard one through the rest of the paper. Details about Rees algebras in resolution and basic results used here can be found, for instance, in [9].

Let R be a regular ring which is of finite type over k. For us, a Rees algebra over R (or over $V = \mathrm{Spec}(R)$) is a graded ring $\mathcal{G} = \oplus_{i \in \mathbb{Z}_{\geq 0}} I_i W^i \subset R[W]$ with $I_0 = R$, which is finitely generated as an R-algebra. Note that this definition is more general than the (usual) one considering only algebras of the form $R[IW]$ for some ideal $I \subset R$. The singular locus of $\mathcal{G}$, $\mathrm{Sing}(\mathcal{G})$, is the subset of V composed by the points η of $\mathrm{Spec}(R)$ for which $\nu_\eta(f) \geq i$ for all $fW^i \in \mathcal{G}$, where $\nu_\eta(f)$ denotes the order of f in the regular local ring R_η. It can be shown that $\mathrm{Sing}(\mathcal{G})$ is a closed subset of V. By the order of an element $fW^i \in \mathcal{G}$ at a point $\eta \in \mathrm{Sing}(\mathcal{G})$, we mean the quotient $\frac{\nu_\eta(f)}{i} =: \mathrm{ord}_\eta(fW^i)$. The order of $\mathcal{G}$ at $\eta \in \mathrm{Sing}(\mathcal{G})$ is defined as

$\operatorname{ord}_{\eta}(\mathcal{G}) = \inf_{fW^i \in \mathcal{G}} \left\{ \operatorname{ord}_{\eta}(fW^i) \right\}$, and can actually be computed as $\operatorname{ord}_{\eta}(\mathcal{G}) = \min \left\{ \operatorname{ord}_{\eta}(fW^i) \right\}$, where the minimum runs over a finite set of generators of $\mathcal{G}$.

We say that a Rees algebra $\mathcal{G}$ over R represents the multiplicity of $X \hookrightarrow V = \operatorname{Spec}(R)$ at $\xi \in \underline{\operatorname{Max}}\operatorname{mult}(X)$ if

$$\operatorname{Sing}(\mathcal{G}) = \underline{\operatorname{Max}}\operatorname{mult}(X)$$

locally in a neighborhood of ξ, and this condition is stable under sequences of permissible transformations for $\mathcal{G}$, that is, after any sequence of the form

$$\begin{aligned} V &= V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_l} V_l \\ \mathcal{G} &= \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \ldots \longleftarrow \mathcal{G}_l \end{aligned} \tag{1.1}$$

where each π_j is either a smooth morphism or a blow up with center a regular closed subset of $\operatorname{Sing}(\mathcal{G}_{j-1})$ for $j = 1, \ldots, l$, as long as $\max \operatorname{mult}(X_l) = \max \operatorname{mult}(X)$ if X_l is the strict transform of X_{l-1} in V_l whenever π_l is a blow up, and the pullback if it is a smooth morphism. Here, the transform $\mathcal{G}_j$ of $\mathcal{G}_{j-1}$, for $j = 1, \ldots, l$, is

$$\mathcal{G}_j = \oplus_{i \geq 0} I_{i,j} W^i \quad \text{where} \quad I_{i,j} = I_{i,j-1} \mathcal{O}_{V_j} \cdot I(E_j)^{-i} \tag{1.2}$$

for any $i \geq 0$, being E_j the exceptional divisor of π_j. By being stable, we mean that

$$\operatorname{Sing}(\mathcal{G}_j) = \underline{\operatorname{Max}}\operatorname{mult}(X_j)$$

if X_j is the strict transform (or pullback, as corresponds) of X_{j-1} in V_j.

This justifies a notion of resolution of a Rees algebra. A resolution of $\mathcal{G}$ is a sequence as in (1.1) where the π_i are blow ups at regular closed subsets of $\operatorname{Sing}(\mathcal{G}_l)$, and such that $\operatorname{Sing}(\mathcal{G}_l) = \underline{\operatorname{Max}}\operatorname{mult}(X_l) = \emptyset$ (see [3, Sections 1.1 to 1.3]).

Remark 1.1. Given X and ξ, there is not a unique $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ representing the multiplicity of X at ξ. However, it can be shown that all Rees algebras representing the maximum multiplicity of X at ξ are somehow equivalent: they all undergo the same resolution, and they share the same order at any point of $\underline{\operatorname{Max}}\operatorname{mult}(X)$. This is the case, for instance, of the differential closure[1] of any Rees algebra $\mathcal{G}_X$ which represents the maximum multiplicity of X at ξ. For details about these facts, see [2].

Let us suppose, for simplicity, that $X = \operatorname{Spec}(B)$ is affine. Otherwise, since we will work locally, it is enough to consider open affine subsets of X. It is possible to find a local étale immersion $X \hookrightarrow \operatorname{Spec}(R)$ into a regular scheme of dimension $n > d$ and a Rees algebra $\mathcal{G}$ over R, representing the multiplicity of X locally in a neighborhood of ξ (see [14]). Under these hypotheses, we have a regular k-algebra S of dimension d and a projection $\beta : \operatorname{Spec}(R) \longrightarrow V^{(d)} = \operatorname{Spec}(S)$ inducing a finite projection

$$\beta_X : X \longrightarrow V^{(d)} = \operatorname{Spec}(S)$$

of generic rank b which is also transversal for $\mathcal{G}$, that is, $\operatorname{Ker}(d\beta)$ intersects the tangent space of $\mathcal{G}$ at ξ (see [7, 16.1]) only at 0, $d\beta$ being the morphism induced by β between the tangent spaces. This projection induces a homeomorphism between $\underline{\operatorname{Max}}\operatorname{mult}(X)$ and its image (see [7, Apendix A]) and an injective finite morphism of the form

$$S \longrightarrow B \cong S[x_1, \ldots, x_{n-d}]/I(X) = S[\overline{x}_1, \ldots, \overline{x}_{n-d}].$$

We obtain in this manner a local immersion of X in a smooth n-dimensional space $V^{(n)} = \operatorname{Spec}(R)$ in a neighborhood of ξ, where $R = S[x_1, \ldots, x_{n-d}]$. There exist $f_1, \ldots, f_{n-d} \in I(X) \subset R$ such that for certain $b_1, \ldots, b_{n-d} \in \mathbb{Z}_{>0}$, the Rees algebra

$$\mathcal{G} = R[f_1 W^{b_1}, \ldots, f_{n-d} W^{b_{n-d}}] \tag{1.3}$$

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[1] A Rees algebra $\mathcal{G} = \oplus_{i \geq 0} I_i W^i$ is differentially closed if, for any differential opperator D of order l, $0 \leq l \leq i$, we have $D(I_i) \subset I_{i-l}$ (see [16]).

represents the multiplicity of X locally in a neighborhood of ξ, and moreover, f_i is the minimal polynomial of $\overline{x}_i$ over S, and hence it is a monic polynomial in x_i with coefficients in S, for $i = 1, \ldots, n - d$ (see [14] for the result on the existence and construction of such a presentation, and [3] for notation as used here). Each f_i defines a hypersurface X_i in $\mathrm{Spec}(S[x_i])$. Assume that we choose the differentially closed algebra

$$\mathcal{G}_X^{(n)} = \mathrm{Diff}(R[f_1 W^{b_1}, \ldots, f_{n-d} W^{b_{n-d}}]), \tag{1.4}$$

which also represents the multiplicity of X locally in a neighborhood of ξ. We can suppose that the maximal ideal $\mathcal{M}_\xi$ of ξ in R is given by $< x_1, \ldots, x_{n-d}, z_1, \ldots, z_d >$ for a regular system of parameters $\{z_1, \ldots, z_d\}$ in S (see [7, Section 4]). The image $\xi^{(d)}$ of ξ by β_X is then defined by the maximal ideal $\mathcal{M}_{\xi^{(d)}} = < z_1, \ldots, z_d >$. Note that $R \longrightarrow B$ is surjective, and for any $i = 1, \ldots, n-d$ the following diagram commutes

$$
\begin{array}{ccccccccc}
\mathcal{G}_X^{(n)} & & R = S[x_1, \ldots, x_{n-d}] & \longrightarrow & S[x_1, \ldots, x_{n-d}]/(f_1, \ldots, f_{n-d}) & \longrightarrow & B & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & & & \\
\mathcal{G}_{X_i}^{(d+1)} & & S[x_i] & \longrightarrow & S[x_i]/(f_i) & & & & \\
& & \uparrow & & & & & & \\
\mathcal{G}_X^{(d)} \supset \mathcal{G}_{X_i}^{(d)} & & S & & & & & &
\end{array}
\tag{1.5}
$$

with $\beta_{X_i}^*$ the map into B.

The homomorphism $S \longrightarrow R$ happens to induce an elimination

$$\beta : V^{(n)} = \mathrm{Spec}(R) \longrightarrow V^{(d)} = \mathrm{Spec}(S)$$

for $\mathcal{G}_X^{(n)}$ (see [14]), that is, a transversal admissible projection, defining a homeomorphism between $\mathrm{Sing}(\mathcal{G}_X^{(n)}) \subset V^{(n)}$ and $\beta(\mathrm{Sing}(\mathcal{G}_X^{(n)})) \subset V^{(d)}$, and such that $\mathcal{G}_X^{(d)} = \mathcal{G}_X^{(n)} \cap S$ represents the multiplicity of $\beta(X)$ (see [5], [6], [7, 16 and Appendix A] and [15, Theorem 4.11 and Theorem 4.13] for properties and results on elimination). Such an elimination is equivalent to the possibility of reducing a problem of Resolution of Rees algebras in dimension n to a problem of Resolution of Rees algebras in dimension $d < n$. Since $\mathcal{G}_X^{(n)}$ is differentially closed, by means of the elimination via β, we have the following description for $\mathcal{G}_X^{(n)}$ (see [3, Example 1.5.4 and Example 1.5.15]):

$$\mathcal{G}_X^{(n)} = S[x_1][x_1 W] \odot \ldots \odot S[x_{n-d}][x_{n-d} W] \odot \mathcal{G}_X^{(d)}, \tag{1.6}$$

where we have used the fact that f_i is a monic polynomial in x_i for $i = 1, \ldots, n - d$, and $\mathcal{G} \odot \mathcal{H}$ denotes the smallest Rees algebra containing both $\mathcal{G}$ and $\mathcal{H}$. Furthermore, one may consider, for each $i \in \{1, \ldots, n-d\}$, the hypersurface $X_i \subset \mathrm{Spec}(S[x_i])$ defined by f_i, as it was done in [3, Section 4.2.1]. For each i, $\mathcal{G}_{X_i}^{(d+1)} = S[x_i][f_i W^{b_i}]$ represents the maximum multiplicity of X_i locally at $\beta_{X_i}(\xi)$. Then, $\mathcal{G}_X^{(n)}$ can be written as

$$\mathcal{G}_X^{(n)} = \mathcal{G}_{X_1}^{(d+1)} \odot \ldots \odot \mathcal{G}_{X_{n-d}}^{(d+1)}, \tag{1.7}$$

and

$$\mathcal{G}_X^{(d)} = \mathcal{G}_{X_1}^{(d)} \odot \ldots \odot \mathcal{G}_{X_{n-d}}^{(d)}. \tag{1.8}$$

Let φ be an arc in X through ξ which is not contained in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. We may project φ to an arc $\varphi^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$ via β_X, that is: $\varphi^{(d)} = \varphi \circ \beta_X^*$. We obtain a commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_{X,\xi} & \xrightarrow{\;\varphi\;} & \\
\beta_X^* \uparrow & \searrow^{\varphi^{(d)}} & K[[t]] \\
\mathcal{O}_{V^{(d)}, \xi^{(d)}} & &
\end{array}
$$

In particular, note that $\varphi(\mathcal{M}_\xi) \supset \varphi^{(d)}(\mathcal{M}_{\xi^{(d)}})$, so

$$\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(\mathcal{M}_\xi)) \leq \mathrm{ord}_t(\varphi^{(d)}(\mathcal{M}_{\xi^{(d)}})) = \mathrm{ord}(\varphi^{(d)}). \tag{1.9}$$

Along this paper, we will repeatedly define arcs through regular systems of parameters. For instance, to define an arc in $V^{(d)}$ through $\xi^{(d)}$, we will do it by giving the images of a r.s.p. $\{y_1, \ldots, y_d\} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$ by the arc. The fact that this description determines the arc completely is a consequence of the continuity of the completion map $\mathcal{O}_{V^{(d)},\xi^{(d)}} \longrightarrow \widehat{\mathcal{O}_{V^{(d)},\xi^{(d)}}}$ of $\mathcal{O}_{V^{(d)},\xi^{(d)}}$ at $\xi^{(d)}$, which allows us to define φ via a map $\widehat{\mathcal{O}_{V^{(d)},\xi^{(d)}}} \cong K[[y_1, \ldots, y_d]] \longrightarrow K[[t]]$. This map induces an arc in $V^{(d)}$ through $\xi^{(d)}$, if K is the residue field of $\mathcal{O}_{V^{(d)},\xi^{(d)}}$.

We denote by $\varphi(\mathcal{G}_X^{(n)})$ the Rees algebra over $K[[t]]$ generated by the images of the f_i in (1.4) by φ with their respective weights. That is,

$$\varphi(\mathcal{G}_X^{(n)}) = K[[t]][\varphi(f_1)W^{b_1}, \ldots, \varphi(f_{n-d})W^{b_{n-d}}].$$

Given X, $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ and an arc φ in X through ξ, the *order of contact of φ with* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, denoted by $r_{X,\varphi}$, is defined as the order of the Rees algebra $\varphi(\mathcal{G}_X^{(n)})$. The quotient $\bar{r}_{X,\varphi} = \frac{r_{X,\varphi}}{\mathrm{ord}(\varphi)}$ gives a more interesting version of this invariant because it avoids the influence of the order of the arc (see [3, Section 3.2]).

In order to express $\varphi(\mathcal{G}_X^{(n)})$ by means of the decomposition in (1.7), we may consider the projections of φ over the X_i, that we shall denote by $\varphi_i^{(d+1)}$, and which are actually arcs in the corresponding X_i through $\beta_{X_i}(\xi)$, because $f_i \in I(X)$ for $i = 1, \ldots, n-d$:

$$\tag{1.10}$$

The following Lemma shows how $r_{X,\varphi}$ can be computed using the expressions in (1.6) and (1.7).

Lemma 1.2. *[3, cf. Section 4] Let X be as in the beginning of this section, and let ξ be a point in* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. *Let φ be an arc in X through ξ. Then*

1. $r_{X,\varphi} = \mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ *and*

2. $\mathrm{ord}_t(\varphi(x_i)) \geq \mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ *for $i = 1, \ldots, n-d$.*

Proof. It follows from (1.6) and (1.7) that

$$r_{X,\varphi} = \mathrm{ord}_t(\varphi(\mathcal{G}_X^{(n)})) = \min_{i=1,\ldots,n-d} \left\{ \mathrm{ord}_t(\varphi_i^{(d+1)}(\mathcal{G}_{X_i}^{(d+1)})) \right\} =$$

$$= \min \left\{ \mathrm{ord}_t(\varphi(x_1)), \ldots, \mathrm{ord}_t(\varphi(x_{n-d})), \mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})) \right\} \leq \mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})).$$

On the other hand, for each i, by [3, Lemma 4.1.2],

$$\mathrm{ord}_t(\varphi_i^{(d+1)}(\mathcal{G}_{X_i}^{(d+1)})) = \min \left\{ \mathrm{ord}_t(\varphi_i^{(d+1)}(x_i)), \mathrm{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{X_i}^{(d)})) \right\} = \mathrm{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{X_i}^{(d)})), \tag{1.11}$$

so

$$r_{X,\varphi} = \min_{i=1,\ldots,n-d} \left\{ \operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{X_i}^{(d)})) \right\}.$$

But note that $\mathcal{G}_{X_i}^{(d)} \subset \mathcal{G}_X^{(d)}$ and $\varphi_i^{(d)} = \varphi^{(d)}$ (see (1.8) and (1.10)). Thus,

$$\varphi_i^{(d)}(\mathcal{G}_{X_i}^{(d)}) = \varphi^{(d)}(\mathcal{G}_{X_i}^{(d)}) \subset \varphi^{(d)}(\mathcal{G}_X^{(d)})$$

and

$$\operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{X_i}^{(d)})) \geq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})). \tag{1.12}$$

Consequently,

$$\operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})) \geq r_{X,\varphi} \geq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})),$$

proving 1. Now 2 is a consequence of (1.11), together with (1.12) and the fact that, for all $i = 1, \ldots, n-d$,

$$\varphi_i^{(d)}(x_i) = \varphi_i^{(d+1)}(x_i) = \varphi(x_i).$$

$\square$

2 Proof of the main result

In order to prove Theorem A, let us divide it in two one side implications, reformulated in Propositions 2.1 and 2.3 respectively, in a way that will be more convenient for their respective proofs. We first give a simple version of the proof of the easier one:

Proposition 2.1. *Let ξ be an isolated point of $\underline{\operatorname{Max}}\operatorname{mult}(X)$. Then there exists a positive integer $Q \in \mathbb{Z}_{>0}$, depending on X and ξ, such that for any arc φ in X through ξ,*

$$\bar{r}_{X,\varphi} \leq Q.$$

Proof. Consider the graded structure of a Rees algebra $\mathcal{G}_X^{(n)}$ representing the multiplicity of X in a neighborhood of ξ as in (1.4),

$$\mathcal{G}_X^{(n)} = \oplus_{i \geq 0} I_i W^i.$$

Since $\mathcal{G}_X^{(n)}$ is differentially closed, the set $\underline{\operatorname{Max}}\operatorname{mult}(X)$ is determined by the zeros of the ideal I_1 (see [16, Proposition 4.4]). Therefore, $\underline{\operatorname{Max}}\operatorname{mult}(X)$ being of dimension 0 is equivalent to $\sqrt{I_1}$ being a maximal ideal, which, for a (any) regular system of parameters $\{x_1, \ldots, x_{n-d}, z_1, \ldots, z_d\}$ in $\mathcal{O}_{X,\xi}$, is also equivalent to I_1 containing some ideal of the form

$$(x_1^{a_1}, \ldots, x_{n-d}^{a_{n-d}}, z_1^{a_{n-d+1}}, \ldots, z_d^{a_n})$$

for some positive integers $a_1, \ldots, a_n$. Note that this implies that

$$\mathcal{G}_X^{(n)} \supset \mathcal{O}_{X,\xi}[x_1^{a_1}W, \ldots, x_{n-d}^{a_{n-d}}W, z_1^{a_{n-d+1}}W, \ldots, z_d^{a_n}W].$$

Therefore,

$$\varphi(\mathcal{G}_X^{(n)}) \supset \mathcal{O}_{X,\xi}[\varphi(x_1^{a_1})W, \ldots, \varphi(x_{n-d}^{a_{n-d}})W, \varphi(z_1^{a_{n-d+1}})W, \ldots, \varphi(z_d^{a_n})W],$$

and

$$\operatorname{ord}_t(\varphi(\mathcal{G}_X^{(n)})) \leq \min \left\{ a_1 \cdot \operatorname{ord}_t(\varphi(x_1)), \ldots, a_{n-d} \cdot \operatorname{ord}_t(\varphi(x_{n-d})), a_{n-d+1} \cdot \operatorname{ord}_t(\varphi(z_1)), \ldots, a_n \cdot \operatorname{ord}_t(\varphi(z_d)) \right\}.$$

Thus

$$\bar{r}_{X,\varphi} \leq a_j \in \mathbb{Z}_{>0}$$

for any $j \in \{1, \ldots, n-d\}$ such that $\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(x_j))$ or any $j \in \{n-d+1, \ldots, n\}$ such that $\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(z_{j-n+d}))$. $\square$

The bound given by this proof is not optimal. In general, a rational number which will be smaller than the integer given by the a_j's can be found, yielding an optimal bound. Note that this rational number is an invariant of X at ξ, but since it is not needed in the proof of Theorem A, we ignore it here.

Remark 2.2. For some arcs, we can say more about $\bar{r}_{X,\varphi}$: If φ is such that $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(x_j))$ for some $j \in \{1, \ldots, n-d\}$, then $\bar{r}_{X,\varphi} = 1$. Indeed, $a_1 = \ldots = a_{n-d} = 1$ in the proof of Proposition 2.1, because $x_1, \ldots, x_{n-d} \in I_1$ (see (1.6)).

In the next section, a precise upper bound will be given under some special condition over X at ξ, in terms of orders of elimination algebras. This condition is related with the τ invariant of $\mathcal{G}_X^{(n)}$ at ξ.

We prove now the most delicate implication. To make the proof easier to understand, we will deal separately with an easy case first, even though it of course follows from the general one, which we prove afterwards. The reader unfamiliar with the techniques of resolution used in this proof, as well as definitions of strict and total transform of an ideal, can consult them in [4, Section 7] or [9].

Proposition 2.3. *If ξ lies in a component of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ of dimension greater or equal to 1, then for any $q \in \mathbb{Q}$, one can find an arc φ in X through ξ such that*

$$\bar{r}_{X,\varphi} > q.$$

Proof. Since $r_{X,\varphi} = \mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ if $\varphi^{(d)} = \varphi \circ \beta_X^*$ (see Lemma 1.2), our strategy here will be choosing an arc $\bar{\varphi}^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$ which gives $\mathrm{ord}_t(\bar{\varphi}^{(d)}(\mathcal{G}_X^{(d)}))$ big enough first, and then lifting it via β_X to an arc φ in X through ξ, proving afterwards that it satisfies the statement in the Proposition.

Suppose first that there exists a smooth curve $\widetilde{C} \subset \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ containing ξ. Then $C = \beta_X(\widetilde{C}) \subset V^{(d)}$ is a smooth curve containing $\xi^{(d)}$(see [14, Theorem 6.3]). Assume that C is defined by a prime ideal $J \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. Consider the family of arcs $\bar{\varphi}_N^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$, for $N \in \mathbb{Z}_{>0}$, given by

$$\bar{\varphi}_N^{(d)} : \mathcal{O}_{V^{(d)},\xi^{(d)}} \longrightarrow K[[t]],$$
$$J \longmapsto t^N,$$
$$\mathcal{M}_{\xi^{(d)}} \longmapsto t.$$

This can be done because we may assume that, in this situation, $J = (y_2, \ldots, y_d)$ for some regular system of parameters $\{y_1, \ldots, y_d\}$ of $\mathcal{O}_{V^{(d)},\xi^{(d)}}$. Then, such a family of arcs could be constructed by just defining $\bar{\varphi}_N^{(d)}(y_1) = t$ and $\bar{\varphi}_N^{(d)}(y_j) = t^N$ for $j = 2, \ldots, d$. For any $N \in \mathbb{N}$, the arc $\bar{\varphi}_N^{(d)}$ can be lifted to an arc φ_N in X through ξ satisfying $\bar{r}_{X,\varphi_N} \geq N$ as follows:

Note that we are under the hypothesis $d \geq 2$. Consider the ideal $\mathcal{P} = \mathrm{Ker}(\bar{\varphi}_N^{(d)}) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. There exists a prime ideal $\mathcal{Q}$ in $\mathcal{O}_{X,\xi}$ dominating $\mathcal{P}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{Q} \subset \mathcal{O}_{X,\xi} & \xrightarrow{\ \ \mu\ \ } & \mathcal{O}_{X,\xi}/\mathcal{Q} \\
\beta_X^* \big\uparrow & & \bar{\beta_X}^* \big\uparrow \\
\mathcal{P} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}} & \xrightarrow{\ \mu^{(d)}\ } & \mathcal{O}_{V^{(d)},\xi^{(d)}}/\mathcal{P}
\end{array}
$$

where the vertical arrows are finite morphisms, and both rings on the right side are 1-dimensional, so $\mathcal{Q}$ defines a curve. One can find a nontrivial arc $\tilde{\varphi}_N : \mathcal{O}_{X,\xi}/\mathcal{Q} \longrightarrow K[[t]]$ through $\mu(\xi)$,[2] which induces also an arc

$$\varphi_N = \tilde{\varphi}_N \circ \mu : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$$

---

[2] Here K will be the residue field of $\mathcal{O}_{X,\xi}/\mathcal{Q}$ at $\mu(\xi)$.

through ξ, and

$$\varphi_N^{(d)} = \varphi_N \circ \beta_X^* = \tilde{\varphi}_N \circ \bar{\beta}_X^* \circ \mu^{(d)} : \mathcal{O}_{V^{(d)},\xi^{(d)}} \longrightarrow K[[t]]$$

with

$$\mathrm{Ker}(\varphi_N^{(d)}) = \mathcal{P} = \mathrm{Ker}(\bar{\varphi}_N^{(d)}) = (y_2 - y_1^N, y_2 - y_j : 2 < j \leq d) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}.$$

Since $C \subset \mathrm{Sing}(\mathcal{G}_X^{(d)})$,

$$\mathrm{ord}_C(I_i \mathcal{O}_{V^{(d)},\xi^{(d)}}) \geq i \ \ \forall i \geq 0,$$

so $I_i \mathcal{O}_{V^{(d)},C} \subset J^i \mathcal{O}_{V^{(d)},C}$. But note that J is a regular prime in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$ defining C, so $I_i \mathcal{O}_{V^{(d)},\xi^{(d)}} \subset J^i$ for all $i \geq 0$. Consequently,

$$\mathcal{G}_X^{(d)} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}[JW],$$

and

$$\varphi_N^{(d)}(\mathcal{G}_X^{(d)}) \subset \varphi_N^{(d)}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[JW]).$$

Hence, for $\varphi_N^{(d)}$ constructed as above,

$$\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \geq \mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[JW])) = \mathrm{ord}_t(\varphi_N^{(d)}(J)).$$

Using also Lemma 1.2 and the fact that $\mathrm{ord}(\varphi_N) \leq \mathrm{ord}(\varphi_N^{(d)}) = \mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))$ (see (1.9)), we arrive to

$$\bar{r}_{X,\varphi_N} = \frac{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)}))}{\mathrm{ord}(\varphi_N)} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}(\varphi_N)} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}(\varphi_N^{(d)})} = \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))}.$$

Assume that $\varphi_N^{(d)}(y_j) = u_j t^{\alpha_j}$ for $j = 1,\dots,d$ for some u_j units in $K[[t]]$ and some $\alpha_j \in \mathbb{Z}_{>0}$. Then

$$\varphi_N^{(d)}(y_2 - y_1^N) = 0 = \varphi_N^{(d)}(y_2) - \varphi_N^{(d)}(y_1)^N = u_2 t^{\alpha_2} - u_1^N t^{\alpha_1 \cdot N} \ \ \text{and}$$

$$\varphi_N^{(d)}(y_2 - y_j) = 0 = \varphi_N^{(d)}(y_2) - \varphi_N^{(d)}(y_j) = u_2 t^{\alpha_2} - u_j t^{\alpha_j} \ \ \text{for} \ \ 2 < j \leq d.$$

Necessarily

$$\alpha_2 = \alpha_1 \cdot N \ \ \text{and}$$

$$\alpha_2 = \alpha_j \ \ \text{for} \ \ 2 < j \leq d,$$

so

$$\bar{r}_{X,\varphi_N} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))} = \frac{\min_{i=2,\dots,d}\{\alpha_i\}}{\min_{j=1,\dots,d}\{\alpha_j\}} = \frac{\alpha_2}{\alpha_1} = N$$

which, for a fixed $q \in \mathbb{Q}$, can be greater than q by just choosing N big enough.

Suppose now that $\widetilde{C} \subset \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ is not smooth. As before, assume that $C = \beta(\widetilde{C}) = V(J) \subset V^{(d)}$ for some ideal $J \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. Consider the following sequence:

$$
\begin{array}{ccccccccc}
V^{(d)} & = & V_0^{(d)} & \xleftarrow{\ \pi_1\ } & V_1^{(d)} & \xleftarrow{\ \pi_2\ } & \dots & \xleftarrow{\ \pi_r\ } & V_r^{(d)} \\
 & & \cup & & \cup & & \dots & & \cup \\
C & = & C_0 & & C_1' & & \dots & & C_r' \\
\xi^{(d)} & = & \xi_0^{(d)} & & \xi_1^{(d)} & & \dots & & \xi_r^{(d)}
\end{array}
\tag{2.1}
$$

where π_i is the blow up at the point $\xi_{i-1}^{(d)}$, and $\xi_i^{(d)} \in \pi_i^{-1}(\xi_{i-1}^{(d)}) \cap C_i'$ for $i = 1,\dots,r$, and such that the strict transform C_r' of C_0 by $\pi = \pi_1 \circ \dots \circ \pi_r$ is a smooth curve having normal crossings with the exceptional

divisor at $\xi_r^{(d)}$. Such a sequence can always be found, being an embedded desingularization of C. Let us look now at the total transform $J_r = J\mathcal{O}_{V_r^{(d)}}$ of the ideal J by π, which will be, locally in a neighborhood of $\xi_r^{(d)}$, of the form

$$J_r = \mathcal{M} \cdot J_r',$$

where J_r' is contained in the the ideal $I(C_r')$ defining the strict transform C_r' of C in $V_r^{(d)}$, and $\mathcal{M}$ is a locally a monomial. Let us choose a family of arcs $\bar{\varphi}_{N,r}^{(d)}$ in $V_r^{(d)}$ through $\xi_r^{(d)}$ for $N \in \mathbb{Z}_{>0}$ such that $\bar{\varphi}_{N,r}^{(d)}(I(C_r')) = t^N$ and $\bar{\varphi}_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}})) = t^a$ for some $a \in \mathbb{Z}_{>0}$ constant, as we did for the case of C smooth. For this, note that locally in a neighborhood of $\xi_r^{(d)}$, one can consider a regular system of parameters in $\mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$ given by

$$\{\tilde{y}_1 = I(H_1), \tilde{y}_2, \ldots, \tilde{y}_d\},$$

so that $I(C_r') = (\tilde{y}_2, \ldots, \tilde{y}_d)$, and moreover

$$\pi^*(\mathcal{M}_{\xi^{(d)}}) = I(H_1)^a$$

for $a \in \mathbb{N}$, where $H_1 = \pi_r^{-1}(\xi_{r-1})$ is the exceptional divisor of π_r, because of the way in which the centers of the π_i are chosen. Consider $\bar{\varphi}_{N,r}^{(d)}$ given as

$$\bar{\varphi}_{N,r}^{(d)} : \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} \longrightarrow K[[t]],$$
$$\tilde{y}_1 \longmapsto t,$$
$$\tilde{y}_j \longmapsto t^N, \text{ for } j = 2, \ldots, d,$$

which satisfies the desired properties. Note that π induces a sequence of permissible transformations of X via β_X:

$$
\begin{array}{ccccc}
X & = & X_0 & \xleftarrow{\quad \pi_X \quad} & X_r \\
\downarrow{\beta_X} & & & & \downarrow{\beta_{X_r}} \\
V^{(d)} & = & V_0^{(d)} \xleftarrow{\pi_1} V_1^{(d)} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_r} V_r^{(d)}
\end{array}
$$

For each $N \in \mathbb{Z}_{>0}$, $\bar{\varphi}_{N,r}^{(d)}$ can be lifted to an arc in X_r through $\xi_r^{(d)}$ via a diagram as in the regular case:

$$
\begin{array}{ccc}
\mathcal{Q} \subset \mathcal{O}_{X_r,\xi_r} & \xrightarrow{\quad \mu \quad} & \mathcal{O}_{X_r,\xi_r}/\mathcal{Q} \\
\beta_{X_r}^* \uparrow & & \bar{\beta}_{X_r}^* \uparrow \\
\mathcal{P} \subset \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} & \xrightarrow{\mu^{(d)}} & \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}/\mathcal{P}
\end{array}
$$

where $\mathcal{P} = \mathrm{Ker}(\bar{\varphi}_{N,r}^{(d)}) = \mathcal{Q} \cap \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$. As we did in the case of C a regular curve, we pick an arc[3]

$$\tilde{\varphi}_{N,r} : \mathcal{O}_{X_r,\xi_r}/\mathcal{Q} \longrightarrow K[[t]]$$

and obtain

$$\varphi_{N,r} = \tilde{\varphi}_{N,r} \circ \mu : \mathcal{O}_{X_r,\xi_r} \longrightarrow K[[t]],$$

so that $\mathrm{Ker}(\bar{\varphi}_{N,r}^{(d)}) = \mathrm{Ker}(\varphi_{N,r}^{(d)})$, where

$$\varphi_{N,r}^{(d)} = \varphi_{N,r} \circ \beta_{X_r}^* : \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} \longrightarrow K[[t]].$$

---

[3]Now K is the residue field of $\mathcal{O}_{X_r,\xi_r}/\mathcal{Q}$ at $\mu(\xi_r)$.

Note that $\mathrm{Ker}(\varphi_{N,r}^{(d)}) = (\tilde{y}_2 - \tilde{y}_1^N, \tilde{y}_2 - \tilde{y}_j : 2 < j \le d)$, so

$$\mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_2)) = \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_j)) = N \cdot \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_1))$$

for $2 < j \le d$, and that

$$\mathrm{ord}(\varphi_{N,r}^{(d)}) = \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}}))) = \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_1^a)) = a \cdot \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_1)),$$

so necessarily

$$\frac{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(I(C_r')))}{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}})))} = \frac{\min_{j=2,\ldots,d}\left\{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_j))\right\}}{\min_{i=1,\ldots,d}\left\{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(\tilde{y}_i))\right\}} = \frac{N}{a}. \tag{2.2}$$

Finally, we obtain

$$\varphi_N : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$$

by composing $\varphi_{N,r} \circ \pi_X^*$, and we also obtain its projection to $V^{(d)}$ as $\varphi_N^{(d)} = \varphi_{N,r}^{(d)} \circ \pi^*$. Note that the sequence of transformations in 2.1) is such that the multiplicity of X_i along the curve does not decrease along the process, and hence $C_i' \subset \beta_{X_i}(\underline{\mathrm{Max}}\,\mathrm{mult}(X_i))$ for $i = 0, \ldots, r$. As a consequence, it induces a sequence of permissible transformations of Rees algebras for $\mathcal{G}_X^{(d)}$ as in [15, Definition 6.1], since for all $i = 1, \ldots, r$, π_i is a blow up at a regular closed subset of $\mathrm{Sing}(\mathcal{G}_{X,i-1}^{(d)})$:

$$\begin{array}{ccccccc} V^{(d)} & = & V_0^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} \xleftarrow{\pi_2} \cdots & \xleftarrow{\pi_r} & V_r^{(d)} \\ \mathcal{G}_X^{(d)} & = & \mathcal{G}_{X,0}^{(d)} = \oplus_{i\ge 0}I_iW^i & \longleftarrow & \mathcal{G}_{X,1}^{(d)} \longleftarrow \cdots & \longleftarrow & \mathcal{G}_{X,r}^{(d)} = \oplus_{i\ge 0}I_{i,r}W^i \end{array} \tag{2.3}$$

where

$$I_i\mathcal{O}_{V_r^{(d)}} \subset I_{i,r}$$

for $i \ge 0$ (see (1.2)). In particular,

$$\mathcal{G}_X^{(d)}\mathcal{O}_{V_r^{(d)}} = \oplus_{i\ge 0}(I_i\mathcal{O}_{V_r^{(d)}})W^i \subset \oplus_{i\ge 0}I_{i,r}W^i.$$

Moreover,

$$\varphi_N^{(d)}(\mathcal{G}_X^{(d)}) = \varphi_{N,r}^{(d)}(\oplus_{i\ge 0}(I_i\mathcal{O}_{V_r^{(d)}})W^i) \subset \varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)}),$$

so

$$\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \ge \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)})).$$

Since $I(C_r')$ is a regular prime in $\mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$ defining a curve contained in $\mathrm{Sing}(\mathcal{G}_{X,r}^{(d)})$,

$$\mathcal{G}_{X,r}^{(d)} \subset \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}[I(C_r')W],$$

and hence

$$\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \ge \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)})) \ge \mathrm{ord}_t(\varphi_{N,r}^{(d)}(I(C_r'))). \tag{2.4}$$

On the other hand,

$$\mathrm{ord}(\varphi_N) = \mathrm{ord}_t(\varphi_N(\mathcal{M}_\xi)) \le \mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}})) = \mathrm{ord}(\varphi_N^{(d)}) = \mathrm{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}}))).$$

This, together with Lemma 1.2, (2.2), and (2.4) implies, for each $N \in \mathbb{Z}_{>0}$,

$$\bar{r}_{X,\varphi_N} = \frac{\mathrm{ord}_t(\varphi_N(\mathcal{G}_X^{(n)}))}{\mathrm{ord}(\varphi_N)} \ge \frac{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)}))}{\mathrm{ord}(\varphi_N^{(d)})} \ge \frac{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(I(C_r')))}{\mathrm{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}})))} = \frac{N}{a}.$$

Again, it is clear that for a fixed $q \in \mathbb{Q}$, we may choose N such that $\bar{r}_{X,\varphi_N} > q$. $\qquad\square$

As was stated in the introduction, our main result means, in terms of the Nash multiplicity sequence, that ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if there exists an upper bound for the number of blowups as in (0.1) needed before the Nash multiplicity sequence decreases for the first time (normalized by the order of φ), for any arc φ in X through ξ:

Corollary 2.4. *Let X be a variety over a field k of characteristic zero. A point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if $\sup_\varphi \left\{ \frac{\rho_{X,\varphi}}{\mathrm{ord}(\varphi)} \right\} < \infty$, where the supremum is taken over all arcs φ in X through ξ.*

Proof. The direct implication follows from [3, Corollary 4.3.3-1]. For the reverse one, assume that $\sup_\varphi \left\{ \frac{\rho_{X,\varphi}}{\mathrm{ord}(\varphi)} \right\} = q \in \mathbb{Q}_{>0}$ and get to a contradiction: for $N = [q] + 1 > q$, choose φ_{aN} as in the proof of Proposition 2.3, so that it satisfies $\bar{r}_{X,\varphi_{aN}} \geq N$. This implies

$$\rho_{X,\varphi_{aN}} = [r_{X,\varphi_{aN}}] \geq [N \cdot \mathrm{ord}(\varphi_{aN})] = N \cdot \mathrm{ord}(\varphi_{aN}).$$

But this is equivalent to

$$\frac{\rho_{X,\varphi_{aN}}}{\mathrm{ord}(\varphi_{aN})} \geq \frac{N \cdot \mathrm{ord}(\varphi_{aN})}{\mathrm{ord}(\varphi_{aN})} = N > q,$$

yielding a contradiction. $\qquad\qquad\square$

3 Consequences and examples

Assume now that $\tau_{\mathcal{G}_X^{(n)},\xi} = n - 1$. Recall that $\tau_{\mathcal{G}_X^{(n)},\xi}$ is the codimension of the largest linear subspace such that the addition of this subspace with the tangent cone[4] of $\mathcal{G}_X^{(n)}$ at ξ lies in the tangent cone again (see [1, Section 4] for details). Then, for some regular system of parameters $\{x_1', \ldots, x_{n-1}', z\} \subset R = \mathcal{O}_{V^{(n)},\xi}$ for $\mathcal{G}_X^{(n)}$ differentially closed representing the multiplicity of X at ξ, we have

$$x_1' W, \ldots, x_{n-1}' W \subset \mathcal{G}_X^{(n)},$$

and one can find an elimination map $V^{(n)} \xrightarrow{\beta_X^{(1)}} V^{(1)}$ (see [7, Sections 13.3 and 16.1]). This means, that finding a resolution of the algebra $\mathcal{G}_X^{(n)}$ is equivalent to finding a resolution of an algebra $\mathcal{G}_X^{(1)}$ over a smooth scheme of dimension 1, namely $V^{(1)}$. We may assume that, up to an étale extension, $R = S'[x_1', \ldots, x_{n-1}']$, where S' is a regular ring of dimension 1. Then

$$\mathcal{G}_X^{(n)} = R[x_1' W] \odot \ldots \odot R[x_{n-1}' W] \odot \mathcal{G}_X^{(1)} \tag{3.1}$$

where $\mathcal{G}_X^{(1)} \subset S'[W]$. Note that, in this situation:

$$\mathrm{ord}_\xi(\mathcal{G}_X^{(n)}) = 1 = \mathrm{ord}_{\xi^{(n-1)}}(\mathcal{G}_X^{(n-1)}) = \ldots = \mathrm{ord}_{\xi^{(2)}}(\mathcal{G}_X^{(2)}) < \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}),$$

so $\mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)})$ is the first interesting resolution invariant in this case.

Under these hypotheses ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, and hence Proposition 2.1 guarantees that $\Phi_{X,\xi}$ is upper bounded. It turns out that the additional condition on $\tau_{\mathcal{G}_X^{(n)},\xi}$ yields an improvement of that result:

---

[4]The tangent cone of $\mathcal{G}_X^{(n)} = \oplus_{i \geq 0} I_i W^i$ at ξ is the subspace of the tangent space of $V^{(n)}$ at ξ defined by the homogeneous ideal $\oplus_{i \geq 0} I_i \cdot (\mathcal{M}_\xi^i / \mathcal{M}_\xi^{i+1})$, where $\mathcal{M}_\xi$ is the maximal ideal of $V^{(n)}$ at ξ.

Proposition 3.1. *If $\tau_{\mathcal{G}_X^{(n)},\xi} = n-1$, then for any arc φ in X through ξ:*

$$\bar{r}_{X,\varphi} \leq \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}),$$

and this bound is sharp.

Proof. We may assume first that $\min_{i=1,\dots,n-1}\{\operatorname{ord}_t(\varphi(x_i'))\} = \operatorname{ord}_t(\varphi(x_1'))$. By (3.1), we obtain

$$r_{X,\varphi} \leq \min\left\{\operatorname{ord}_t(\varphi(x_1')), \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}, \tag{3.2}$$

where $\varphi^{(1)}$ is the projection of φ via the elimination map $\beta_X^{(1)} : \operatorname{Spec}(R) \longrightarrow \operatorname{Spec}(S')$. Note that, either $\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(x_1'))$ or $\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(z))$. In the first case,

$$1 \leq \bar{r}_{X,\varphi} \leq \frac{\min\left\{\operatorname{ord}_t(\varphi(x_1')), \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}}{\operatorname{ord}_t(\varphi(x_1'))} \leq 1,$$

which implies that

$$\bar{r}_{X,\varphi} = 1 < \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)})$$

In the second case,

$$\bar{r}_{X,\varphi} \leq \frac{\operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))}{\operatorname{ord}_t(\varphi(z))}.$$

Note that $\operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)})) \geq \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \operatorname{ord}_t(\varphi(z))$ (see [3, Lemma 4.1.6]). But actually this inequality is an equality here. This follows from the fact that $\mathcal{G}_X^{(1)} \subset S'[W]$ so, for all $gW^l \in \mathcal{G}_X^{(1)}$, we have that $\operatorname{ord}_t(\varphi(g)) = \operatorname{ord}_z(g) \cdot \operatorname{ord}_t(\varphi(z))$. One only needs to observe now that $\varphi^{(1)}(\mathcal{G}_X^{(1)}) = K[[t]][\varphi(g)W^l : gW^l \in \mathcal{G}_X^{(1)}]$, and the equality is clear. Hence

$$\bar{r}_{X,\varphi} \leq \frac{\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \operatorname{ord}_t(\varphi(z))}{\operatorname{ord}_t(\varphi(z))} = \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}).$$

To see that $\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)})$ is a sharp bound, consider an r.s.p. $\{x_1,\dots,x_{n-d}, z_1,\dots,z_{d-1}, z_d\} \subset R$ as in Section 1. Since $\tau_{\mathcal{G}_X^{(n)},\xi} = n-1$, we may assume that $x_1 W,\dots,x_{n-d}W, z_1 W,\dots,z_{d-1}W \in \mathcal{G}_X^{(n)}$. We may choose an arc $\bar{\varphi}^{(d)}$ in $V^{(d)}$ through $\beta_X(\xi)$ such that $\bar{\varphi}^{(d)}(z_d) = t$ and $\bar{\varphi}^{(d)}(z_1) = \dots = \bar{\varphi}^{(d)}(z_{d-1}) = t^a$, for some $a \in \mathbb{Z}_{>0}$, $a > \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) > 1$. This arc can be lifted to an arc φ in X through ξ, for which

$$\bar{r}_{X,\varphi} = \frac{\operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))}{\operatorname{ord}(\varphi)} \geq \frac{\operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))}{\operatorname{ord}(\varphi^{(d)})} = \frac{\min\left\{\operatorname{ord}_t(\varphi^{(d)}(z_1)),\dots,\operatorname{ord}_t(\varphi^{(d)}(z_{d-1})), \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}}{\operatorname{ord}(\varphi^{(d)})}$$

by Lemma 1.2 and (1.9), where $\varphi^{(d)} = \varphi \circ \beta_X^*$ and $\varphi^{(1)} = \varphi \circ (\beta_X^{(1)})^*$. Also, $\operatorname{Ker}(\varphi^{(d)}) = \operatorname{Ker}(\bar{\varphi}^{(d)}) = (z_d^a - z_1,\dots,z_d^a - z_{d-1})$, so for $i=1,\dots,d-1$ it is clear that

$$\operatorname{ord}_t(\varphi^{(d)}(z_i)) = a \cdot \operatorname{ord}_t(\varphi^{(d)}(z_d)) > \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \operatorname{ord}_t(\varphi^{(d)}(z_d)) = \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)})) > \operatorname{ord}_t(\varphi^{(d)}(z_d)).$$

Thus,

$$\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \geq \bar{r}_{X,\varphi} \geq \frac{\operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))}{\operatorname{ord}_t(\varphi^{(d)}(z_d))} = \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}).$$

$\square$

However, under the hypothesis of Proposition 3.1, sometimes it is possible to find arcs such that $\mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) = 1 < \bar{r}_{X,\varphi} < \mathrm{ord}_\xi(\mathcal{G}_X^{(1)})$. Let us show an example for this:

Example 3.2. Consider $X \hookrightarrow \mathrm{Spec}(k[x,y,z])$ defined by the equation $f = xy - z^5$ and $\xi = (0,0,0) = \underline{\mathrm{Max}}\,\mathrm{mult}(X)$, and let φ be the arc defined by $\varphi(x) = t^3$, $\varphi(y) = t^2$, $\varphi(z) = t$. Here

$$\mathcal{G}_X^{(3)} = \mathrm{Diff}(k[x,y,z][fW^2]) = k[x][xW] \odot \mathcal{G}_X^{(2)} = k[x,y][xW,yW] \odot \mathcal{G}_X^{(1)},$$

where $\mathcal{G}_X^{(2)} = k[y,z][yW,z^5W^2,z^4W]$ and $\mathcal{G}_X^{(1)} = k[z][z^5W^2,z^4W]$, so $\mathrm{ord}_{\xi^{(d)}}(\mathcal{G}_X^{(2)}) = 1$ and $\mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) = 5/2$. Note that $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(z)) = 1$. On the other hand,

$$r_{X,\varphi} = \mathrm{ord}_t(\varphi(\mathcal{G}_X^{(3)})) = \mathrm{ord}_t(\varphi^{(2)}(\mathcal{G}_X^{(2)})) = \min\left\{\mathrm{ord}_t(\varphi^{(2)}(y)), \mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\} = \min\{2, 5/2\} = 2.$$

Hence, for this example $1 < \bar{r}_{X,\varphi} = 2 < 5/2$.

Let us end our discussion with a couple of illustrative examples for Propositions 2.1 and 2.3 respectively. The first one shows an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ for which $\Phi_{X,\xi}$ is upper bounded by 3:

Example 3.3. Let $X = \{x^2y^3 - z^3s^4 = 0\} \hookrightarrow \mathrm{Spec}(k[x,y,z,s])$ and let $\xi = (0,0,0,0) = \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. We have

$$\mathcal{G}_X^{(4)} = \mathrm{Diff}(k[x,y,z,s][(x^2y^3 - z^3s^4)W^5]) =$$
$$= k[x,y,z,s][xW,yW,zsW,z^3W,s^2W,z^3sW^2,zs^2W^2,zs^4W^3,z^2s^3W^3,z^3s^2W^3,z^3s^3W^4,z^3s^4W^5].$$

Observe now that $xW, yW, z^3W, s^2W \in \mathcal{G}_X^{(4)}$, and $\varphi(x)W, \varphi(y)W, \varphi(z)^3W, \varphi(s)^2W \in \varphi(\mathcal{G}_X^{(4)})$. Then

$$r_{X,\varphi} \leq \min\{\mathrm{ord}_t(\varphi(x)), \mathrm{ord}_t(\varphi(y)), 3\cdot\mathrm{ord}_t(\varphi(z)), 2\cdot\mathrm{ord}_t(\varphi(s))\}.$$

If $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(x))$ or $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(y))$, then $\bar{r}_{X,\varphi} = 1$. If $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(z))$, then $\bar{r}_{X,\varphi} \leq 3$, and if $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(s))$, then $\bar{r}_{X,\varphi} \leq 2$. In any case

$$\bar{r}_{X,\varphi} \leq 3.$$

In our next example we construct, for a non isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, a family of arcs φ_N, $N \in \mathbb{Z}_{>0}$, for which $\bar{r}_{X,\varphi_N}$ equals a polynomial in N, namely $q(N) = N+2$, showing that $\Phi_{X,\xi}$ is not upper bounded:

Example 3.4. Let now $X = \{x^2y^3 - z^4s^5 = 0\}$, and let $\xi = (0,0,0,0)$ again. Now $\xi \subsetneq \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. In this case,

$$\mathcal{G}_X^{(4)} = \mathrm{Diff}(k[x,y,z,s][(x^2y^3 - z^4s^5)W^5]) = k[x,y,z,s][xW,yW,zsW,s^5W,zs^5W^2,z^2s^5W^3,z^3s^5W^4,z^4s^5W^5].$$

Consider the following family of arcs through ξ parametrized by $N \in \mathbb{Z}_{>0}$:

$$\varphi_N : k[x,y,z,s]/(x^2y^3 - z^4s^5) \longrightarrow K[[t]]$$
$$x \longmapsto t^{2N+2},$$
$$y \longmapsto t^{2N+5},$$
$$z \longmapsto t,$$
$$s \longmapsto t^{2N+3}.$$

Now

$$\varphi(\mathcal{G}_X^{(4)}) = K[[t]][t^{2N+2}W]$$

and $\mathrm{ord}(\varphi_N) = 1$, so

$$\bar{r}_{X,\varphi} = 2N+2,$$

which grows with N.

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