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# Discrete-Continuous Jacobi-Sobolev Spaces and Fourier Series

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**Abstract**

Let $p \geq 1$, $\ell \in \mathbb{N}$, $\alpha, \beta > -1$ and $\varpi = (\omega_0, \omega_1, \ldots, \omega_{\ell-1}) \in \mathbb{R}^\ell$. Given a suitable function $f$, we define the discrete-continuous Jacobi-Sobolev norm of $f$ as:

$$\|f\|_{\mathsf{s},p} := \left( \sum_{k=0}^{\ell-1} \left| f^{(k)}(\omega_k) \right|^p + \int_{-1}^{1} \left| f^{(\ell)}(x) \right|^p d\mu^{\alpha,\beta}(x) \right)^{\frac{1}{p}},$$

where $d\mu^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta dx$. Obviously, $\|\cdot\|_{\mathsf{s},2} = \sqrt{\langle \cdot, \cdot \rangle_{\mathsf{s}}}$, where $\langle \cdot, \cdot \rangle_{\mathsf{s}}$ is the inner product.

$$\langle f, g \rangle_{\mathsf{s}} := \sum_{k=0}^{\ell-1} f^{(k)}(\omega_k)\, g^{(k)}(\omega_k) + \int_{-1}^{1} f^{(\ell)}(x)\, g^{(\ell)}(x) d\mu^{\alpha,\beta}(x).$$

In this paper, we summarize the main advances on the convergence of the Fourier-Sobolev series, in norms of type $L^p$, cases continuous and discrete. We study the completeness of the Sobolev space associated with the norm $\|\cdot\|_{\mathsf{s},p}$ and the density of the polynomials. Furthermore, we obtain the conditions for the convergence in $\|\cdot\|_{\mathsf{s},p}$ norm of the partial sum of the Fourier-Sobolev series of orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_{\mathsf{s}}$ .

## 1 Introduction

The convergence problem of the trigonometric Fourier series has been one of the main driving forces behind the mathematical analysis growth. For instance, the development of the Hilbert spaces theory

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is closely related to the problem of the convergence of the Fourier series for square integrable $2\pi$-periodic functions on $[-\pi,\pi]$,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left|f(x)-\sum_{|k|\leq n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iky}dy\right)e^{ikx}\right|^{2}dx=0.$$

In 1927, M. Riesz gave an analytic proof of the convergence of trigonometric Fourier series for $L^p$-integrable $2\pi$-periodic functions on $[-\pi,\pi]$ using complex methods (see [28]). This proof was one of the sources to develop the main harmonic analysis tools such as the Hardy-Littlewood maximal functions and the singular integrals. In the proof, Riesz strongly used the conjugated function, which is the first singular integral considered. Later A. P. Calderón and A. Zygmund studied singular integrals in higher dimensions using real analysis methods (see [3]).

Trough the Hilbert space theory the study of the $L^2$ convergence of the classical orthogonal polynomials Fourier series became a straightforward issue, since it only depends on the orthogonality. But we find a very different scenario in the case of the $L^p$ convergence when $p\neq 2$. The study of $L^p$-convergence of classical orthogonal polynomials Fourier series started in the 1940's with a serie of papers (see [24, 25, 26, 27]). H. Pollard considered the mean convergence of the Legendre, Gegenbauer and Jacobi Fourier expansions extending the harmonic analysis techniques used for Fourier series to this case (see Theorem A). Pollard used a special decomposition of the Christoffel-Darboux kernel such that the $L^p$ continuity of the Hilbert transform with respect to certain weights can be used. Later on B. Muckenhoupt revisited the convergence of Jacobi polynomials (see [19] and Theorem B).

Additionally, H. Pollard proved in [26] by means of the asymptotic relation between the Hermite polynomials and the Gegenbauer polynomials (see [34, (5.3.4)]) that the Hermite polynomials only converge in $L^p$-norm when $p=2$. Finally, due to the relation between the Hermite and the Laguerre polynomials, (see [34, (5.6.1)]), there exists also an anomalous behavior on the $L^p$-convergence of the Laguerre polynomials Fourier series.

Let $\mathbb{F}$ be a linear space of complex-valued functions with an inner product $\langle\cdot,\cdot\rangle$. The inner product $\langle\cdot,\cdot\rangle$ is said to be standard if $\langle zf(z),g(z)\rangle=\langle f(z),zg(z)\rangle$ for every $f,g\in\mathbb{F}$, i.e. standard means that the operator of multiplication by the independent variable $z$ is symmetric.

We denote by $\mathbb{P}$ the linear space of all polynomials. If $\mathbb{P}$ is a subspace of $\mathbb{F}$, well-known arguments allow to establish the existence of a unique (up to normalization) sequence of orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle$. If this inner product is standard, as an immediate consequence we get that the corresponding sequence of orthogonal polynomials satisfies a three-term recurrence relation. This is a very important property, connected with several fields such as: difference equations ([16]), numerical analysis ([10]), Fourier analysis ([12, 22, 35]), among others. When you do not have such a relation, these approaches are not generally available.

Otherwise, if an inner product is not symmetric with respect to the multiplication operator, we say that it is non-standard. The case of inner products modified by terms involving derivatives. Here we restrict ourselves to the Sobolev-type inner products, defined as:

$$\langle f,g\rangle_{\mathsf{s}}=\sum_{k=0}^{\ell}\int_{-\infty}^{\infty}f^{(k)}(x)g^{(k)}(x)d\mu_k(x),\qquad \text{where } \ell\in\mathbb{Z}_+ \text{ is fixed.} \tag{1}$$

and the associated Sobolev norms of type-$p$:

$$\|f\|_{\mathsf{s},p}=\Big(\sum_{k=0}^{\ell}\int_{-\infty}^{\infty}\left|f^{(k)}(x)\right|^{p}d\mu_k(x)\Big)^{1/p}\qquad \text{where } 1\leq p<\infty \text{ is fixed.} \tag{2}$$

Observe that the measures $\mu_k$ can be either a discrete measures or a continuous one. Thus there are essentially three types of such inner products:

I. *Discrete case.* The support $\mu_0$ contains infinitely many points and $\mu_1, \cdots, \mu_\ell$ are supported on finite subsets.

II. *Continuous case.* All the measures involved in the inner product (1) (i.e. the norm (2)) are supported on subsets with infinitely many points.

III. *Discrete-continuous case.* The support $\mu_\ell$ contains infinitely many points and $\mu_0, \cdots, \mu_{\ell-1}$ are supported on finite subsets.

The theory of Sobolev ortogonal polynomials has been studied intensively in the last 30 years, we recommend the survey [18] and the references therein to get a good overview. As it would expect, the orthogonal projection onto the usual $L^2$ space does not have the required properties, which leads naturally to consider inner products of the form (1). Nonetheless, there are only a few general results about the extremal polynomials with respect to the norm (2), with $p \neq 2$. Most of them, on the asymptotic behavior of the sequence of the corresponding extremal polynomials (see [9, 14, 15]).

From the anomalous behavior of the Hermite and Laguerre polynomials with respect to the $L^p$-convergence, it is fruitless to consider the convergence of the Fourier series of any of them in the Sobolev case. Therefore, due to the importance of the Jacobi polynomials in this topic, in the next section, we review some of their properties and also the main ideas of the proof of the $L^p$ convergence of their Fourier series. In the section 3, we review what is known about the convergence of the Fourier series of Jacobi-Sobolev polynomials.

Section 4 is dedicated to defining the discrete-continuous Jacobi-Sobolev spaces and to study necessary and sufficient conditions for its completeness. In the last section we prove the convergence of the Fourier series in the norm of the discrete-continuous Jacobi-Sobolev spaces.

# 2 Fourier series of Jacobi polynomials

The Jacobi polynomials with parameters $\alpha, \beta > -1$, $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ are orthogonal polynomials with respect to the inner product

$$\langle f, g\rangle_{\alpha,\beta} = \int_{-1}^{1} f(x)\, g(x)\, d\mu^{\alpha,\beta}(x), \tag{3}$$

where $d\mu^{\alpha,\beta}(x) = w_{\alpha,\beta}(x)dx = (1-x)^{\alpha}(1+x)^{\beta}dx$, is the *Jacobi measure* (or beta measure). We take the normalization of $P_n^{(\alpha,\beta)}$ such that

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \quad \text{where} \quad \binom{a}{b} = \Gamma(a+1)/\left(\Gamma(a-b+1)\Gamma(b+1)\right),$$

for $a, b \in \mathbb{R}$ and $\Gamma$ denotes the usual Gamma function (see [35, §10.1]). More precisely,

$$\int_{-\infty}^{\infty} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) d\mu^{\alpha,\beta}(x) = h_n^{(\alpha,\beta)} \delta_{n,m}, \quad \text{for } n, m = 0, 1, 2, \ldots,$$

and

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \qquad [34,(4.3.3)]. \tag{4}$$

Additionally,

$$\frac{dP_n^{(\alpha,\beta)}}{dx}(x) = \frac{(n+\alpha+\beta+1)}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x) \qquad [34,(4.21.7)]. \tag{5}$$

Also, $\left\{P_n^{(\alpha,\beta)}\right\}_{n\geq 0}$ satisfy the three term recurrence relation,

$$\begin{aligned} 2n(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) =& (2n+\alpha+\beta-1)\left((2n+\alpha+\beta)x+\alpha^2-\beta^2\right)P_{n-1}^{(\alpha,\beta)}(x) \\ &-2(n+\alpha-1)(n+\beta-1)\frac{2n+\alpha+\beta}{2n+\alpha+\beta-2}P_{n-2}^{(\alpha,\beta)}(x), \end{aligned}$$

for $n\geq 2$; with $P_0^{(\alpha,\beta)}(x)=1$, and $P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta)$.

For $1\leq p\leq\infty$ let consider the Banach space $L^p(\mu^{\alpha,\beta})$ of $p$-th power integrable functions with respect to $d\mu^{\alpha,\beta}$ on $[-1,1]$, embedded with the norm

$$\|f\|_{\alpha,\beta,p} = \begin{cases} \left(\int_{-1}^1 |f(x)|^p d\mu^{\alpha,\beta}(x)\right)^{\frac{1}{p}}, & \text{if } p\in[1,\infty), \\ \inf\{\alpha>0 : \nu(\{|f|>\alpha\})=0\}, & \text{if } p=\infty; \end{cases} \tag{6}$$

where $\nu$ denotes the Lebesgue measure on $[-1,1]$.

So, $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ is a sequence of orthogonal polynomials in $L^2(\mu^{\alpha,\beta})$. Let

$$p_n^{(\alpha,\beta)} = (h_n^{(\alpha,\beta)})^{-1/2} P_n^{(\alpha,\beta)} \tag{7}$$

be the $n$-th Jacobi orthonormal polynomial with respect to the inner product (3), then $\|p_n^{(\alpha,\beta)}\|_{\alpha,\beta,2}=1$, where $\|\cdot\|_{\alpha,\beta,2}=\sqrt{\langle\cdot,\cdot\rangle_{\alpha,\beta}}$. From (4)-(5), we have the following property for the $k$-th derivative of the orthonormal Jacobi polynomials:

$$\begin{aligned} \left(p_n^{(\alpha,\beta)}(x)\right)^{(k)} =& A_{n.k}\, p_{n-k}^{(\alpha+k,\beta+k)}(x), \quad \text{where} \\ A_{n,k}^{\alpha,\beta} =& \sqrt{\frac{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+k+1)}{\Gamma(n-k+1)\,\Gamma(n+\alpha+\beta+1)}} \neq 0, \end{aligned}$$

and $f^{(k)}$ denotes the $k$-th derivative of a function $f$.

Given $f\in L^1(\mu^{\alpha,\beta})$, set

$$a_n := \left\langle f, p_n^{(\alpha,\beta)}\right\rangle_{\alpha,\beta} = \int_{-1}^1 f(x)p_n^{(\alpha,\beta)}(x)d\mu^{(\alpha,\beta)}(x), \tag{8}$$

the $n$-th Fourier coefficient and define the partial sum operators of $f$ as

$$S_n^{\alpha,\beta}(f,x) = \sum_{k=0}^n a_k p_k^{(\alpha,\beta)}(x). \tag{9}$$

The analysis of the convergence of Fourier-Jacobi series in $L^p(\mu^{\alpha,\beta})$ has a long history. The first result was obtained by H. Pollard in [26] for Gegenbauer polynomials and then in [27] for Jacobi polynomials,

**Theorem A.** *[27, Th. A] Let* $\alpha,\beta \geq -1/2$,

$$M(\alpha,\beta) = 4\,\max\left\{\frac{\alpha+1}{2\alpha+3},\frac{\beta+1}{2\beta+3}\right\} \text{ and } m(\alpha,\beta) = 4\,\min\left\{\frac{\alpha+1}{2\alpha+1},\frac{\beta+1}{2\beta+1}\right\}.$$

*Then, for any* $f \in L^p(\mu^{\alpha,\beta})$ *with* $M(\alpha,\beta) < p < m(\alpha,\beta)$*, the Fourier-Jacobi expansion of* $f$ *converges to* $f$ *in* $L^p(\mu^{\alpha,\beta})$*, i.e.*

$$\lim_{n\to\infty} \|f - S_n^{\alpha,\beta}(f,\cdot)\|_{\alpha,\beta,p} = \lim_{n\to\infty} \left(\int_{-1}^{1} \left|f(x) - S_n^{\alpha,\beta}(f,x)\right|^p d\mu^{\alpha,\beta}(x)\right)^{\frac{1}{p}} = 0. \tag{10}$$

Furthermore, in [27, Th. B], Pollard also proved that the preceding result fails if $p < M(\alpha,\beta)$ or $p > m(\alpha,\beta)$. In [21], J. Newman and W. Rudin, study the case of the extreme points $p = M(\alpha,\beta)$ or $p = m(\alpha,\beta)$ proving that in that case the convergence also fails.

According to Banach-Steinhaus uniform boundedness principle (see [32, §5.8]), in order to prove (10), it is enough to prove the uniform $L^p(\mu^{\alpha,\beta})$-boundedness of the partial sums, i.e. there exists a constant $C > 0$ such that

$$\|S_n^{\alpha,\beta}(f,\cdot)\|_{\alpha,\beta,p} \leq C\,\|f\|_{\alpha,\beta,p},$$

for all $f \in L^p(\mu^{\alpha,\beta})$ and for all $n \geq 0$.

B. Muckenhoupt extended Pollard's result for all $\alpha,\beta > -1$ (instead of $\alpha,\beta > -1/2$) in the Corollary of Theorem 1 of [19]. Muckenhoupt's proof uses Pollard's decomposition but now for the weights $w_{ap,bp}(x) = (1-x)^{ap}(1+x)^{bp}$, where $a,b \in \mathbb{R}$, such that

$$\left|a+\frac{1}{p}-\frac{\alpha+1}{2}\right| \leq \min\left\{\frac{1}{4},\frac{\alpha+1}{2}\right\} \quad \text{and} \quad \left|b+\frac{1}{p}-\frac{\beta+1}{2}\right| \leq \min\left\{\frac{1}{4},\frac{\beta+1}{2}\right\}. \tag{11}$$

Note that, if $a = \alpha/p$ and $b = \beta/p$, we get the Jacobi weights. It is easy to see that these conditions, for the case $\alpha,\beta > -1/2$, give Pollard's conditions $M(\alpha,\beta) < p < m(\alpha,\beta)$. The Muckenhoupt's proof is based on the following inequality, which we will use later.

**Theorem B.** *[19, Th. 1] Let* $\alpha,\beta > -1$, $1 < p < \infty$ *and* $a,b \in \mathbb{R}$ *such that* (11) *holds. Then there exists a constant, C, independent of f and n, such that*

$$\int_{-1}^{1} \left|S_n^{\alpha,\beta}(f,x)(1-x)^a(1+x)^b\right|^p dx \leq C\int_{-1}^{1} \left|f(x)(1-x)^a(1+x)^b\right|^p dx,$$

*where* $S_n^{\alpha,\beta}(f,\cdot)$ *is given by* (8)-(9).

Additionally, in [19, Th. 2], Muckenhoupt also proved that condition (11) is also necessary.

# 3 Fourier series of Sobolev orthogonal polynomials

From now on, we will focus our attention on summarizing the main advances on the convergence of the Fourier-Sobolev series with respect to $L^p$-norms given by (2). The study of the convergence in norm (2) of the orthogonal polynomials Fourier series faces several problems. First of all, in general, the Sobolev polynomials do not satisfy a three-term recurrence relation and, therefore, there is not a Christoffel-Darboux formula for their Dirichlet-Szegő kernel.As a consequence, the argument outlined above can no be applied to this case. Additionally, since there are three different types of Sobolev inner products, we will need three different methods to study them separately. Let us consider the results that have been obtained so far in each case:

I. *Discrete case.* In [4], the authors give a complete characterization of the boundedness of the partial sum operators with respect to the inner product (1) where $\ell = 1$, $\mu_0 = \mu^\alpha + M(\delta_{-1} + \delta_1)$ and $\mu^\alpha$ is the Gegenbauer probability measure,

$$d\mu^\alpha(x) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma^2(\alpha)}(1-x^2)^\alpha dx, \quad \alpha > -1/2,$$

and $\mu_1 = N(\delta_{-1} + \delta_1)$. Due to the lack of a Christoffel-Darboux formula for Sobolev orthogonal polynomials, the main tools involve a special case of a general weighted transplantation theorem, see [20, Th. 1.6] and the other one is a particular case of the multiplier result [20, Th. 1.10]. Using these tools, they prove (see [4, Th. 1.1]) the boundedness of the partial sum operators in the corresponding space for

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1}.$$

Then, using the density of the polynomials $\{q_n\}_{n\geq 0}$ in the $L^p(\mu^\alpha)$ space (see [30]) and the Banach-Steinhaus uniform boundedness principle, they obtain that, for these values of $p$, the sequence of the partial sums $\{S_n(f,\cdot)\}_{n\geq 0}$ of $f$ converges to the function $f$, in $\|\cdot\|_{p,\alpha}$ (see [4, Cor. 1.3]).

II. *Continuous case.* The first article in this direction is due to F. Marcellán, Y. Quintana and A. Urieles [17]. They consider the inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)d\mu^{\alpha,\beta}(x) + \int_{-1}^{1} f'(x)g'(x)d\mu^{\alpha+1,\beta+1}(x).$$

In this case, it follows from (5) that the corresponding Jacobi-Sobolev polynomials are up to a constant factor the classical Jacobi polynomials and, therefore, they satisfy a three term-recurrence relation. In the paper, the authors actually only focus on the case of the Legendre-Sobolev polynomials, i.e. $\alpha = \beta = 0$. The existence of a recurrence relation for them (see [17, (3.35)]) implies that an explicit formula can be given for the Sobolev-Dirichlet-Szegő kernel corresponding to the $n$-th partial sum. Unfortunately the corresponding part for the derivatives generates a hypersingular transform which is not easy to handle without having to ask the function a lot of regularity. Consequently, the final result is not entirely satisfactory (see [17, Th. 3.1]).

The second article in this direction is due to Ó. Ciaurri and J. Mínguez [5]. They consider the general case of the Fourier series of Jacobi-Sobolev polynomials, $\alpha,\beta > -1$, with an arbitrary number order derivatives, i.e. orthogonal polynomials with respect to Sobolev-type inner product

$$\langle f,g\rangle = \sum_{k=0}^{\ell} \int_{-1}^{1} f^{(k)}(x)g^{(k)}(x)\,d\mu^{\alpha+k,\beta+k}(x), \quad \alpha,\beta > -1, \quad \ell \geq 1.$$

Again, by (5), the corresponding Jacobi-Sobolev polynomials $\{q_n^{(\alpha,\beta,\ell)}\}_{n\geq 0}$ are multiples of the classical Jacobi polynomials. They consider an alternative approach following basically what they had developed in [5], applying Muckenhoupt's multipliers and transplantation operators for Jacobi expansions (see [20, Th. 5.1, Th. 7.1 and Th. 1.10], [7]) and also Pollard's results [27].

They prove in [5, Th. 1], the boundedness of the partial sum operators $S_n^{\alpha,\beta,\ell}(f,\cdot)$ in the corresponding space when

$$\max\left\{\frac{4(\alpha+\ell+1)}{2(\alpha+\ell)+3},\frac{4(\beta+\ell+1)}{2(\beta+\ell)+3}\right\} < p < \min\left\{\frac{4(\alpha+\ell+1)}{2(\alpha+\ell)+1},\frac{4(\beta+\ell+1)}{2(\beta+\ell)+1}\right\}.$$

Then, using the density of the polynomials $\{q_n^{(\alpha,\beta,\ell)}\}_{n\geq 0}$ in the corresponding space, see [30] and the Banach-Steinhaus uniform boundedness principle, they obtain that this result is equivalent to the convergence of $S_n^{\alpha,\beta,\ell}(f,\cdot)$ to $f$ in the corresponding norm (see [5, Corollary 2]).

Let $(d\mu_0,d\mu_1)$ be a pair of positive Borel measures on the real line with finite moments of all orders and $P_{i,n}$ be the monic orthogonal polynomial of degree $n$ with respect to $d\mu_i$. The pair $(d\mu_0,d\mu_1)$ is called *coherent* if there exists a sequence of nonzero real numbers $\{a_n\}_{n\geq 1}$ such that

$$P_{1,n}(z) = \frac{P'_{0,n+1}(z)}{n+1} + a_n\frac{P'_{0,n}(z)}{n}, \quad n\geq 1.$$

The notion of coherent pairs of measures was introduced in [11] and it is closely related with the Sobolev inner product

$$\langle f,g\rangle = \int_a^b f(x)g(x)d\mu_0(x) + \int_a^b f'(x)g'(x)d\mu_1(x), \tag{12}$$

where $-\infty\leq a<b\leq\infty$. For a review of the study of the coherent pair of measures we recommended [18, §5]. The study of convergence of Fourier series of orthogonal polynomials with respect to the inner product (12) has been carried out in [6].

III: *Discrete-continuous case.* In this case, as far as we know, a general result about convergence in the $L^p$-norm of the Fourier series is still unknown. In [33], the author consider the case of the inner product

$$\langle f,g\rangle = \sum_{k=0}^{\ell-1} f^{(k)}(-1)g^{(k)}(-1) + \int_{-1}^1 f^{(\ell)}(t)g^{(\ell)}(t)(1-t)^\alpha dt.$$

He proved that the associated Fourier series converges uniformly in $[-1,1]$.

We will consider a more general case in this direction in the next section.

# 4 Discrete-Continuous Jacobi-Sobolev Spaces

First, we need to consider the natural linear spaces for the problem. Let $p\geq 1$, $\ell\in\mathbb{N}$, $\alpha,\beta>-1$ and $\varpi=(\omega_0,\omega_1,\dots,\omega_{\ell-1})\in\mathbb{R}^\ell$. Given a suitable function $f$, we define the Jacobi-Sobolev norm as:

$$\|f\|_{\mathsf{s},p} = \left(\sum_{k=0}^{\ell-1}\left|f^{(k)}(\omega_k)\right|^p + \int_{-1}^1\left|f^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x)\right)^{\frac{1}{p}}. \tag{13}$$

It is clear that $\|\cdot\|_{\mathsf{s},p}$ is a norm on the linear space of all polynomials $\mathbb{P}$ and we define the Jacobi-Sobolev discrete-continuous space $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$ as the completion of the normed space $(\mathbb{P},\|\cdot\|_{\mathsf{s},p})$. In other words, $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$ is the collection of all equivalence class of Cauchy sequences in $(\mathbb{P},\|\cdot\|_{\mathsf{s},p})$, under the following equivalence relation: $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ Cauchy sequences in $(\mathbb{P},\|\cdot\|_{\mathsf{s},p})$ are equivalent $\{P_n\}_{n\geq 0}\sim\{Q_n\}_{n\geq 0}$ if and only if $\lim_{n\to\infty}\|P_n-Q_n\|_{\mathsf{s},p}=0$.

On the other hand, let $U^{\alpha,\beta}_{\varpi,p}$ be the linear space of functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ satisfying:

1. $f$ has derivative at $\omega_k$ of order $k$, $k=1,2,\dots,\ell-1$.

2. $f$ is $\ell$ times differentiable almost everywhere on $[-1,1]$.

3. $f^{(\ell)}\in L^p(\mu^{\alpha,\beta})$.

Obviously, (13) is a seminorm on $U^{\alpha,\beta}_{\varpi,p}$ and $\mathbb{P}\subset U^{\alpha,\beta}_{\varpi,p}$. Let us write $f\sim g$ if and only if $\|f-g\|_{\mathsf{s},p}=0$. It is a straightforward result that this is an equivalence relation on $U^{\alpha,\beta}_{\varpi,p}$. Denote the linear space of the equivalence classes of this relation by $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$. Given $F\in\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ and if $f$ is any element of $F$, then we define the norm in $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ as $\|F\|_{\mathsf{s},p}=\|f\|_{\mathsf{s},p}$. Thus, $(\mathbf{U}^{\alpha,\beta}_{\varpi,p},\|\cdot\|_{\mathsf{s},p})$ is a normed space.

In what follows, we will introduce some notions which will be useful in the sequel. Given $\vec{x}=(x_0,x_1,\dots,x_{m-1})$, $\vec{y}=(y_0,y_1,\dots,y_{m-1})\in\mathbb{R}^m$, there exists a unique polynomial $\mathscr{A}_{\vec{x},\vec{y}}$ of degree at most $m-1$, such that

$$\mathscr{A}^{(k)}_{\vec{x},\vec{y}}(x_k)=y_k,\quad k=0,1,\dots,m-1.$$

The polynomial $\mathscr{A}_{\vec{x},\vec{y}}$ is known as the *Abel-Goncharov interpolation polynomial*, associated with $\vec{y}$ on $\vec{x}$ (see [1, Ch. 3], [8, §2.6], or [36, §8]). The Abel-Goncharov interpolation polynomial is a generalization of Taylor's polynomial, which corresponds to the case $x_{m-1}=x_{m-2}=\cdots=x_0$.

The polynomial $\mathscr{A}_{\vec{x},\vec{y}}$ is given by the explicit expression

$$\mathscr{A}_{\vec{x},\vec{y}}(z)=\sum_{k=0}^{m-1}y_k\mathscr{G}_{\vec{x},k}(z),$$

where $\mathscr{G}_{\vec{x},0}\equiv 1$ and $\mathscr{G}_{\vec{x},k}$, for each $k=1,\dots,m-1$, is the polynomial of degree $k$, generated by the $k$-th iterated integral

$$\mathscr{G}_{\vec{x},k}(s)=\int_{x_0}^{s}\int_{x_1}^{s_1}\cdots\int_{x_{k-2}}^{s_{k-2}}\int_{x_{k-1}}^{s_{k-1}}ds_k ds_{k-1}\cdots ds_2 ds_1.$$

The polynomial $\mathscr{G}_{\vec{x},k}$ is called the *k-th Goncharov's polynomial associated with* $\vec{x}$ and satisfies

$$\mathscr{G}^{(\nu)}_{\vec{x},k}(x_\nu)=\begin{cases}0, & \text{if } \nu\neq k,\\ 1, & \text{if } \nu=k.\end{cases}\tag{14}$$

**Proposition 4.1.** *Let $p\in[1,\infty)$ and $F\in\mathbf{U}^{\alpha,\beta}_{\varpi,p}$, then there exists a sequence of polynomials $\{P_n\}_{n\geq 0}$ such that*

$$\|P_n-f\|_{\mathsf{s},p}\longrightarrow 0,$$

*for every $f\in F$. As a consequence, $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ can be identified as a normed subspace of $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$.*

*Proof.* Let $f \in F \in \mathbf{U}^{\alpha,\beta}_{\varpi,p}$, then $f^{(\ell)} \in L^p(\mu^{\alpha,\beta})$. From the density of the linear space of continuous functions on $L^p(\mu^{\alpha,\beta})$ (see [32, Theorem 3.14]) and Weierstrass' approximation theorem [2, §4.1], for all $n \in \mathbb{N}$, there exists a polynomial $p_n$ such that

$$\|f^{(\ell)} - p_n\|_{\alpha,\beta,p} < \frac{1}{n}.$$

Now, consider the vector $f(\varpi) = (f(\omega_0), f'(\omega_1), \ldots, f^{(\ell-1)}(\omega_{\ell-1}))$ and the polynomial

$$P_n(s) = \mathscr{A}_{\varpi,f(\varpi)}(s) + \int_{\omega_0}^{s}\int_{\omega_1}^{s_1}\cdots\int_{\omega_{\ell-2}}^{s_{\ell-2}}\int_{\omega_{\ell-1}}^{s_{\ell-1}} p_n(s_\ell)ds_\ell ds_{\ell-1}\cdots ds_2 ds_1.$$

It is clear that this polynomial satisfies $P_n^{(k)}(\omega_k) = f^{(k)}(\omega_k)$ for $k = 0,1,\ldots,\ell-1$, and $P_n^{(\ell)} \equiv p_n$. Thus we have

$$\|f - P_n\|_{\mathsf{s},p} = \|f^{(\ell)} - P_n^{(\ell)}\|_{\alpha,\beta,p} = \|f^{(\ell)} - p_n\|_{\alpha,\beta,p} < \frac{1}{n},$$

and we get the first statement. Note that the sequence $\{P_n\}_{n\geq 0}$ does not depend on the function $f \in F$ chosen. Indeed, if $h \sim f$, then

$$\|P_n - h\|_{\mathsf{s},p} \leq \|P_n - f\|_{\mathsf{s},p} + \|f - h\|_{\mathsf{s},p} = \|P_n - f\|_{\mathsf{s},p} \longrightarrow 0.$$

The identification of $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ as a normed subspace of $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$ follows from the fact that $\{P_n\}_{n\geq 0}$ is a Cauchy sequence on $(\mathbb{P}, \|\cdot\|_{\mathsf{s},p})$

$$\|P_n - P_m\|_{\mathsf{s},p} \leq \|P_n - f\|_{\mathsf{s},p} + \|f - P_m\|_{\mathsf{s},p}.$$

Then, if $F$ is the equivalence class of $f$ it can be identified with the equivalence class of $\{P_n\}_{n\geq 0}$ in $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$. □

From now on, we will not distinguish between the equivalent class $F$ and any representative of it $f$, and we will write $\mathbf{U}^{\alpha,\beta}_{\varpi,p} \subset \mathbf{W}^{\alpha,\beta}_{\varpi,p}$. It is worth to remark that although $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ is the function space of minimum requirements over $f$ such that $\|f\|_{\mathsf{s},p}$ makes sense, this is not broad enough to describe every element of $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$, at least not for every combination of the parameters $p, \alpha, \beta, \ell$ and $\varpi$ (see Theorem 3).

In this section we obtain all the combinations of the values $p, \alpha, \beta$ and $\varpi$ for which $\mathbf{W}^{\alpha,\beta}_{\varpi,p} = \mathbf{U}^{\alpha,\beta}_{\varpi,p}$. The study of completeness of Sobolev spaces for the general vectorial measures $\vec{\mu}$, was given by J. M. Rodríguez et al. in [29, 30]. The Sobolev space defined in [29, 30], in the case of our specific vectorial measure $\vec{\mu} = (\delta_{\omega_0}, \delta_{\omega_1}, \ldots, \delta_{\omega_{\ell-1}}, \mu^{\alpha,\beta})$, agrees with our space $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ in the essential case when $\omega_i \in [-1,1], i = 0,1,\ldots,\ell-1$.

We are going to prove the completeness of $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ by means of [29, Th 5.1]. In this paper the authors introduce several definitions which we will follow in order to use the theorem mentioned above. We also prove that for the complementary combinations of the parameters $\alpha, \beta, \ell, \varpi$, the space $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ is not complete and, consequently, $\mathbf{U}^{\alpha,\beta}_{\varpi,p} \neq \mathbf{W}^{\alpha,\beta}_{\varpi,p}$.

**Definition 4.1.** *(see [13, Definition 1.4]) and [29, Definition 2]) Let $p \geq 1$ and $K$ be a closed interval on the real line. A weight function $w$ defined on $K$ satisfies condition $B_p(K)$ if $w^{-1} \in L^{\frac{1}{p-1}}(\nu_K)$, where $\nu_K$ denotes the Lebesgue measure on $K$, and we write $w \in B_p(K)$.*

*If $A \subset \mathbb{R}$ and $w$ is a weight function defined on $A$, then we say that $w \in B_p(A)$ when $w \in B_p(K)$ for every closed interval $K \subset A$.*

**Example 4.1.** *Consider the Jacobi weight function* $w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$*, with* $\alpha,\beta > -1$*.*

1. *If* $p = 1$*, then*

$$w_{\alpha,\beta} \in \begin{cases} B_1([-1,1]), & \text{if } \alpha,\beta \in (-1,0], \\ B_1((-1,1]), & \text{if } \alpha \in (-1,0], \beta \in (0,\infty), \\ B_1([-1,1)), & \text{if } \alpha \in (0,\infty), \beta \in (-1,0], \\ B_1((-1,1)), & \text{if } \alpha,\beta \in (0,\infty). \end{cases}$$

2. *If* $p > 1$*, then*

$$w_{\alpha,\beta} \in \begin{cases} B_p([-1,1]), & \text{if } \alpha,\beta \in (-1,p-1), \\ B_p((-1,1]), & \text{if } \alpha \in (-1,p-1), \beta \in [p-1,\infty), \\ B_p([-1,1)), & \text{if } \alpha \in [p-1,\infty), \beta \in (-1,p-1), \\ B_p((-1,1)), & \text{if } \alpha,\beta \in [p-1,\infty). \end{cases}$$

**Proposition 4.2.** *Let* $\alpha,\beta > -1, p \geq 1$*,* $k = 0,1,\dots,\ell$*; and consider the vectorial weight*

$$w(x) = (w_0(x), w_1(x), \dots, w_{\ell-1}(x), w_\ell(x)) = (0,0,\dots,0,w_{\alpha,\beta}(x)),\ x \in [-1,1], \tag{15}$$

$$\text{the interval} \quad I_k = \begin{cases} (kp-1,(k+1)p-1], & \text{if } k = 0,1,\dots,\ell-1, \\ (\ell p-1,\infty), & \text{if } k = \ell, \end{cases} \tag{16}$$

*and the vectorial weight* $\overline{w}^k = (\overline{w}^k_0, \overline{w}^k_1, \dots, \overline{w}^k_\ell)$*, where*

$$\overline{w}^k_j(x) = \begin{cases} \mathbb{1}_{[-1,0]}(x), & \text{if } j = 0,1,\dots,\ell-k-1, \\ (1+x)^{\beta-(\ell-j)p}\mathbb{1}_{[-1,0]}(x), & \text{if } j = \ell-k,\dots,\ell-1, \\ w_{\alpha,\beta}(x), & \text{if } j = \ell, \end{cases}$$

*and* $\mathbb{1}_A$ *denotes the characteristic function of the set* $A \subset \mathbb{R}$*.*

*If* $k$ *is such that* $\beta \in I_k$*, then* $\overline{w}^k$ *is a right completion of* $w$ *with respect to* $-1$ *(see [29, Definition 5]). Analogously, we can found a left completion of* $w$ *with respect to* $1$*. Namely*

$$\overline{w}^k_j(x) = \begin{cases} \mathbb{1}_{[0,1]}(x), & \text{if } j = 0,1,\dots,\ell-k-1, \\ (1-x)^{\alpha-(\ell-j)p}\mathbb{1}_{[0,1]}(x), & \text{if } j = \ell-k,\dots,\ell-1, \\ w_{\alpha,\beta}(x), & \text{if } j = \ell, \end{cases}$$

*where* $\alpha \in I_k$*.*

*Proof.* It is straightforward to check $\overline{w}^k_\ell = w_\ell$ and there exists $\varepsilon = 1$ such that $\overline{w}^k_j = w_j$ for $x \notin [-1,-1+\varepsilon] = [-1,0]$. Then it only remains to prove that $\widetilde{w}^k_j = \overline{w}^k_j - w_j = \overline{w}^k_j$, $0 \leq j \leq \ell-1$, satisfies

1. $\widetilde{w}^k_j \in L^1([-1,0])$.

2. $\Lambda_p\left(\widetilde{w}^k_j, \overline{w}^k_{j+1}\right) < \infty$ where $\Lambda_p(u,v) := \sup_{-1<r<0} \left(\int_{-1}^{r} u\right) \|v^{-1}\|_{L^{1/(p-1)}([r,-1+\varepsilon])}$.

The first statement comes from

$$\int_{-1}^{0}\widetilde{w}_j^k(x)dx = \int_{-1}^{0}\overline{w}_j^k(x)dx$$

$$= \begin{cases} 1, & \text{if } j = 0,1,\dots,\ell-k-1, \\ \displaystyle\int_{-1}^{0}(1+x)^{\beta-(\ell-j)p}dx, & \text{if } j = \ell-k,\dots,\ell-1, \end{cases} < \infty,$$

where the integral in the second case is finite because of $\beta-(\ell-j)p \geq \beta-kp > -1$. For the second statement we will split the proof in several cases.

• Case $p = 1$, $0 \leq j \leq \ell-k-2$,

$$\Lambda_p(\widetilde{w}_j^k,\overline{w}_{j+1}^k) = \Lambda_p(\overline{w}_j^k,\overline{w}_{j+1}^k) = \sup_{-1<r<0}\left(\int_{-1}^{r}dx\right)\left\|\mathbb{1}_{[r,0]}^{-1}\right\|_{L^\infty([r,0])}$$
$$= \sup_{-1<r<0}(r+1) = 1.$$

• Case $p = 1$, $j = \ell-k-1$,

$$\Lambda_p(\widetilde{w}_{\ell-k-1}^k,\overline{w}_{\ell-k}^k) \leq \max\{1,2^{-\alpha}\}\sup_{-1<r<0}\left\{\left(\int_{-1}^{r}dx\right)\sup_{r<x<0}(1+x)^{-(\beta-k)}\right\}$$
$$= \max\{1,2^{-\alpha}\}\sup_{-1<r<0}(r+1) = \max\{1,2^{-\alpha}\},$$

where we have used that $-\beta+k \geq -(k+1)p+1+k = 0$.

• Case $p = 1$, $\ell-k \leq j \leq \ell-1$,

$$\Lambda_p(\widetilde{w}_j^k,\overline{w}_{j+1}^k) = \Lambda_p(\overline{w}_j^k,\overline{w}_{j+1}^k) = \sup_{-1<r<0}\left(\int_{-1}^{r}\overline{w}_j^k(x)dx\right)\left\|\left(\overline{w}_{j+1}^k\right)^{-1}\right\|_{L^\infty([r,0])}$$
$$\leq \max\{1,2^{-\alpha}\}\sup_{-1<r<0}\left\{\int_{-1}^{r}\overline{w}_j^k(x)dx\sup_{r<x<0}(1+x)^{-(\beta-(\ell-j)+1)}\right\}$$
$$= \frac{\max\{1,2^{-\alpha}\}}{\beta-(\ell-j)+1}\sup_{-1<r<0}\left\{(1+r)^{\beta-(\ell-j)+1}(1+r)^{-(\beta-(\ell-j)+1)}\right\}$$
$$= \frac{\max\{1,2^{-\alpha}\}}{\beta-(\ell-j)+1},$$

where we have used that $\beta-(\ell-j)+1 > kp-(\ell-j) = k-(\ell-j) \geq 0$.

• Case $p > 1$, $0 \leq j \leq \ell-k-2$,

$$\Lambda_p(\widetilde{w}_j^k,\overline{w}_{j+1}^k) = \sup_{-1<r<0}\left(\int_{-1}^{r}dx\right)\left(\int_{r}^{0}dx\right)^{p-1} = \sup_{-1<r<0}(r+1)(-r)^{p-1} \leq 1.$$

• Case $p > 1$, $j = \ell-k-1$, $\beta < (k+1)p-1$,

$$\Lambda_p(\widetilde{w}_{\ell-k-1}^k,\overline{w}_{\ell-k}^k) \leq \max\{1,2^{-\alpha}\}\sup_{-1<r<0}\left(\int_{-1}^{r}dx\right)\left(\int_{r}^{0}(1+x)^{-\frac{\beta-kp}{p-1}}dx\right)^{p-1}$$
$$= \frac{\max\{1,2^{-\alpha}\}(p-1)^{p-1}}{(-\beta+(k+1)p-1)^{p-1}}\sup_{-1<r<0}(r+1)\left(1-(1+r)^{\frac{-\beta+(k+1)p-1}{p-1}}\right)^{p-1}$$
$$\leq \frac{\max\{1,2^{-\alpha}\}(p-1)^{p-1}}{(-\beta+(k+1)p-1)^{p-1}},$$

where we have used that $-\beta+(k+1)p-1>0$ in the last inequality and also to calculate the second integral.
• Case $p>1$, $j=\ell-k-1$, $\beta=(k+1)p-1$,

$$\begin{aligned}\Lambda_p(\widetilde{w}^k_{\ell-k-1},\overline{w}^k_{\ell-k}) &\le \max\{1,2^{-\alpha}\}\sup_{-1<r<0}\left(\int_{-1}^{r}dx\right)\left(\int_r^0(1+x)^{-\frac{\beta-kp}{p-1}}dx\right)^{p-1}\\ &=\max\{1,2^{-\alpha}\}\sup_{-1<r<0}(r+1)\left(\int_r^0(1+x)^{-1}dx\right)^{p-1}\\ &=\sup_{-1<r<0}(r+1)\,(-\ln(r+1))^{p-1}<\infty,\end{aligned}$$

where we have used that, by L'Hôpital rule, $\lim_{x\to0^+}x(\ln(1/x))^p=0$.
• Case $p>1$, $\ell-k\le j\le\ell-1$,

$$\begin{aligned}\Lambda_p(\widetilde{w}^k_j,\overline{w}^k_{j+1}) &\le \max\{1,2^{-\alpha}\}\sup_{-1<r<0}\int_{-1}^{r}\overline{w}^k_j(x)dx\left(\int_r^0(1-x)^{-\frac{\beta-(\ell-j-1)p}{p-1}}dx\right)^{p-1}\\ &=\frac{\max\{1,2^{-\alpha}\}(p-1)^{p-1}}{(\beta-(\ell-j)p+1)^p}\sup_{-1<r<0}\left(1-(1+r)^{\frac{\beta-(\ell-j)p+1}{p-1}}\right)^{p-1}\\ &=\frac{\max\{1,2^{-\alpha}\}(p-1)^{p-1}}{(\beta-(\ell-j)p+1)^p},\end{aligned}$$

where we have used that $\beta-(\ell-j)p+1>kp-(\ell-j)p\ge0$.

□

**Proposition 4.3.** *Denote by $\Omega^{(k)}$ the set of k-regular points of the vectorial weight* (15) *(see [29, Definition 6]).*

- *If $p>1$, then for $m=1,2,\ldots,\ell$;*

$$\Omega^{(\ell-m)}=\begin{cases}[-1,1], & \text{if } \alpha,\beta\in(-1,mp-1),\\ (-1,1], & \text{if } \alpha\in(-1,mp-1),\beta\in[mp-1,\infty),\\ [-1,1), & \text{if } \alpha\in[mp-1,\infty),\beta\in(-1,mp-1),\\ (-1,1), & \text{if } \alpha,\beta\in[mp-1,\infty).\end{cases}$$

- *If $p=1$, then for $m=1,2,\ldots,\ell$;*

$$\Omega^{(\ell-m)}=\begin{cases}[-1,1], & \text{if } \alpha,\beta\in(-1,m-1],\\ (-1,1], & \text{if } \alpha\in(-1,m-1],\beta\in(m-1,\infty),\\ [-1,1), & \text{if } \alpha\in(m-1,\infty),\beta\in(-1,m-1],\\ (-1,1), & \text{if } \alpha,\beta\in(m-1,\infty).\end{cases}$$

*Proof.* From $w_{\alpha,\beta}\in B_p((-1,1))$ it follows that $y$ is $(\ell-1)$-regular for every $y\in(-1,1)$ and the analysis of the regularity of $y=1$ and $y=-1$ are analogous. So we will only focus our attention on the point $y=-1$. First, we are going to prove that if $p>1$ and $\beta\in(-1,mp-1)\subset\cup_{i=0}^{m-1}I_i$, where $I_i$ is as in (16), then $y=-1$ is an $(\ell-m)$-right regular point.

Take $0\le k\le m-1$ such that $\beta\in I_k$ .

• Case $\beta\in(kp-1,(k+1)p-1)$,

$$\int_{-1}^{0}\left(\overline{w}_{\ell-k}^{k}\right)^{-\frac{1}{p-1}}dx=\begin{cases}\int_{-1}^{0}(x+1)^{-\frac{\beta-kp}{p-1}}dx, & \text{if } k\ge1,\\ \int_{-1}^{0}(1-x)^{-\frac{\alpha}{p-1}}(1+x)^{-\frac{\beta}{p-1}}dx, & \text{if } k=0,\end{cases}<\infty,$$

where the integral in the first case is finite due to $\frac{\beta-kp}{p-1}<\frac{p-1}{p-1}=1$. Note that $\ell-k\ge\ell-m+1$.

• Case $\beta=(k+1)p-1$, $\int_{-1}^{0}\left(\overline{w}_{\ell-(k+1)}^{k}\right)^{-\frac{1}{p-1}}dx=1$. Note that, as $\beta<mp-1$, we get $k\le m-2$, so $\ell-(k+1)\ge\ell-m+1$.

In both cases, we have proved that there exists $\ell-m<j\le\ell$ such that $\overline{w}_j^k\in B_p([-1,0])$, so $y=-1$ is $(\ell-m)$-right regular.

Now, consider $p=1$, and $\beta\in(-1,m-1]=\cup_{k=0}^{m-1}I_k$. Take $0\le k\le m-1$ such that $\beta\in I_k$.

$$\sup_{-1\le x\le0}\left\{\left(\overline{w}_{\ell-k}^{k}(x)\right)^{-1}\right\}=\begin{cases}\sup\limits_{-1\le x\le0}\left\{(x+1)^{-\beta+k}\right\}, & \text{if } k\ge1,\\ \sup\limits_{-1\le x\le0}\left\{(1-x)^{-\alpha}(1+x)^{-\beta}\right\}, & \text{if } k=0,\end{cases}<\infty.$$

Note that $\ell-k\ge\ell-m+1$, so we have proved that there exists $\ell-m<j\le\ell$ such that $\overline{w}_j^k\in B_p([-1,0])$, so $y=-1$ is $(\ell-m)$-right regular.

Now, we are going to prove that in the other case the point $y=-1$ is not $(\ell-m)$-right regular. But first, we are going to prove the following auxiliary result. If $w$ is a non-negative Lebesgue measurable function, then

$$\|w^{-1}\|_{L^{\frac{1}{p-1}}([a,b])}\ge\frac{(b-a)^p}{\int_a^b w(x)dx},\quad a<b. \tag{17}$$

This inequality comes from Holder inequality

$$\begin{aligned}(b-a)^p=\left(\int_a^b dx\right)^p=\left(\int_a^b w^{\frac{1}{p}}w^{-\frac{1}{p}}dx\right)^p\le\left(\|w^{\frac{1}{p}}\|_{L^p([a,b])}\|w^{-\frac{1}{p}}\|_{L^{\frac{p}{p-1}}([a,b])}\right)^p\\ =\|w\|_{L^1([a,b])}\|w^{-1}\|_{L^{\frac{1}{p-1}}([a,b])}.\end{aligned}$$

Let $\overline{w}$ be a right completion of $w$ with respect to $y=-1$. Then, we will prove by induction that for $0<s<\frac{\varepsilon}{\ell+1}$, (where $\varepsilon$ is the given one for $\overline{w}$ in the definition of right completion) we have

$$\|\overline{w}_{\ell-k}^{-1}\|_{L^{\frac{1}{p-1}}(J_{k,s})}\ge M_k s^{-\beta+(k+1)p-1},\quad k=0,1,\dots,\ell, \tag{18}$$

where $J_{k,s}=[-1+(\ell-k)s,-1+(\ell-k+1)s]$ and $M_k$ is some positive constant independent of $s$. For $k=0$ we have

• Case $p > 1$, $\beta \in [mp-1,\infty)$,

$$\begin{aligned}\|\overline{w}_\ell^{-1}\|_{L^{\frac{1}{p-1}}(J_{0,s})} =&\left(\int_{-1+\ell s}^{-1+(\ell+1)s}(1-x)^{-\frac{\alpha}{p-1}}(1+x)^{-\frac{\beta}{p-1}}dx\right)^{p-1}\\ \geq&\min\left\{\frac{1}{2^\alpha},\frac{1}{(2-\varepsilon)^\alpha}\right\}\left(\int_{-1+\ell s}^{-1+(\ell+1)s}(1+x)^{-\frac{\beta}{p-1}}dx\right)^{p-1}\\ =&M_0 s^{-\beta+p-1},\end{aligned}$$

where

$$M_0 = \begin{cases}\min\left\{\frac{1}{2^\alpha},\frac{1}{(2-\varepsilon)^\alpha}\right\}\frac{(p-1)^{p-1}}{(\beta-p+1)^{p-1}}\left(\ell^{-\frac{\beta}{p-1}}-(\ell+1)^{-\frac{\beta}{p-1}}\right)^{p-1}, & \text{if } \beta > p-1,\\ \min\left\{\frac{1}{2^\alpha},\frac{1}{(2-\varepsilon)^\alpha}\right\}\left[\ln\left(\frac{\ell+1}{\ell}\right)\right]^{p-1}, & \text{if } \beta = p-1,\end{cases} > 0.$$

• Case $p = 1$, $\beta \in (m-1,\infty)$,

$$\begin{aligned}\|\overline{w}_\ell^{-1}\|_{L^\infty(J_{0,s})} =& \sup_{-1+\ell s<x<-1+(\ell+1)s}\left\{(1-x)^{-\alpha}(1+x)^{-\beta}\right\}\\ \geq&\min\{2^{-\alpha},(2-\varepsilon)^{-\alpha}\}\sup_{-1+\ell s<x<-1+(\ell+1)s}\left\{(1+x)^{-\beta}\right\} = M_0 s^{-\beta},\end{aligned}$$

where $M_0 = \min\{2^{-\alpha},(2-\varepsilon)^{-\alpha}\}\ell^{-\beta}$.

Now, suppose that (18) is true for $k-1$. Then from (17) it follows

$$\begin{aligned}\|\overline{w}_{\ell-k}^{-1}\|_{L^{\frac{1}{p-1}}(J_{k,s})} \geq& \frac{s^p}{\int_{-1+(\ell-k)s}^{-1+(\ell-k+1)s}\overline{w}_{\ell-k}(x)dx}\\ \geq& \frac{s^p\|\overline{w}_{\ell-k+1}^{-1}\|_{L^{\frac{1}{p-1}}(J_{k-1,s})}}{\int_{-1}^{-1+(\ell-k+1)s}\widetilde{w}_{\ell-k}(x)dx\|\overline{w}_{\ell-k+1}^{-1}\|_{L^{\frac{1}{p-1}}([-1+(\ell-k+1)s,-1+\varepsilon])}}\\ \geq& \frac{s^p\|\overline{w}_{\ell-k+1}^{-1}\|_{L^{\frac{1}{p-1}}(J_{k-1,s})}}{\Lambda_p(\widetilde{w}_{\ell-k},\overline{w}_{\ell-k+1})} \geq \frac{M_{k-1}s^{-\beta+(k+1)p-1}}{\Lambda_p(\widetilde{w}_{\ell-k},\overline{w}_{\ell-k+1})}.\end{aligned}$$

Taking $M_k = \frac{M_{k-1}}{\Lambda_p(\widetilde{w}_{\ell-k},\overline{w}_{\ell-k+1})} > 0$ we conclude the proof of (18). Thus we have

$$\|\overline{w}_j^{-1}\|_{L^{\frac{1}{p-1}}(J_{\ell-j,s})} \geq M_{\ell-j}s^{-\beta+(\ell-j+1)p-1}, \quad j = 0,1,\ldots,\ell,$$

and from here it follows

• Case $p > 1$, $\beta \in [mp-1,\infty)$, $\lim_{s\to 0}\|\overline{w}_j^{-1}\|_{L^{\frac{1}{p-1}}([-1+js,-1+(j+1)s])} \neq 0$, for $j = \ell-m+1,\ldots,\ell$.

• Case $p = 1$, $\beta \in (m-1,\infty)$, $\lim_{s\to 0}\|\overline{w}_j^{-1}\|_{L^\infty([-1+(\ell-j)s,-1+(\ell-j+1)s])} = \infty$, for $j = \ell-m+1,\ldots,\ell$.

Thus $\overline{w}_j \notin B_p([-1,-1+\varepsilon])$ for $j = \ell-m+1,\ldots,\ell$, so $y = -1$ is not $(\ell-m)$-right regular. □

**Proposition 4.4.** *Let* $\mu_{[-1,1]} = (\mu_0,\dots,\mu_{\ell-1},\mu^{\alpha,\beta})$ *be a vectorial measure , where*

$$\mu_j = \begin{cases} \delta_{\omega_j}, & \text{if } \omega_j \in [-1,1], \\ 0, & \text{if } \omega_j \notin [-1,1] \end{cases}, \quad j = 0,1,\dots,\ell-1.$$

*Then* $\mu_{[-1,1]}$ *is a vectorial measure in* $[-1,1]$ *of type 2 (see [30, Definition 11]) if and only if* $\omega_j \in \Omega^{(j)}$, *for every* $j$ *such that* $\omega_j \in [-1,1]$.

*Recall that* $\Omega^{(k)}$ *denotes the set of* $k$*-regular points of the vectorial weight* (15) *given explicitly in Proposition 4.3.*

*Proof.* The condition $\omega_j \in \Omega^{(j)}$ is just the condition in the definition of strongly $p$-admissible vectorial measure (see [29, Definition 8]) and $\mu_{[-1,1]}$ is finite (each component measure is finite). Then, we only have to check that there exist real numbers $-1 \le a_1 < a_2 < a_3 < a_4 \le 1$ such that

1. $w_\ell \in B_p([a_1,a_4])$,
2. if $-1 < a_1$, then $w_j$ is comparable to a non-decreasing weight in $[-1,a_2]$, for $0 \le j \le \ell$,
3. if $a_4 < 1$, then $w_j$ is comparable to a non-decreasing weight in $[a_3,1]$, for $0 \le j \le \ell$,

where $w_j = d\mu_j/dx$, $j = 0,1,\cdots,\ell$, see [30, Definition 1]. Therefore, we obtain $w_j \equiv 0$, $j = 0,1,\dots,\ell-1$; and $w_\ell(x) = w_{\alpha,\beta}(x)$. Since $w'_\ell(x) = (1-x)^{\beta-1}(1+x)^{\alpha-1}(\beta-\alpha-(\beta+\alpha)x)$ it follows that there exists $0 < \varepsilon < 1$ (depending on $\alpha$ and $\beta$) such that

$$w'(x) \begin{cases} > 0 \text{ for } x \in (-1,-1+\varepsilon], & \text{if } \beta > 0, \\ < 0 \text{ for } x \in [1-\varepsilon,1), & \text{if } \alpha > 0. \end{cases}$$

Thus, from Example 4.1, the three conditions are easy verified taking $a_2 = -1+\varepsilon$, $a_3 = 1-\varepsilon$ and

$$\bullet \text{ Case } p > 1 \quad \begin{cases} a_1 = -1, a_4 = 1, & \text{if } \alpha,\beta \in (-1,p-1), \\ a_1 = -1+\frac{\varepsilon}{2}, a_4 = 1, & \text{if } \alpha \in (-1,p-1), \beta \in [p-1,\infty), \\ a_1 = -1, a_4 = 1-\frac{\varepsilon}{2}, & \text{if } \alpha \in [p-1,\infty), \beta \in (-1,p-1), \\ a_1 = -1+\frac{\varepsilon}{2}, a_4 = 1-\frac{\varepsilon}{2}, & \text{if } \alpha,\beta \in [p-1,\infty). \end{cases}$$

$$\bullet \text{ Case } p = 1 \quad \begin{cases} a_1 = -1, a_4 = 1, & \text{if } \alpha,\beta \in (-1,0], \\ a_1 = -1+\frac{\varepsilon}{2}, a_4 = 1, & \text{if } \alpha \in (-1,0], \beta \in (0,\infty), \\ a_1 = -1, a_4 = 1-\frac{\varepsilon}{2}, & \text{if } \alpha \in (0,\infty), \beta \in (-1,0], \\ a_1 = -1+\frac{\varepsilon}{2}, a_4 = 1-\frac{\varepsilon}{2}, & \text{if } \alpha,\beta \in (0,\infty). \end{cases}$$

□

**Proposition 4.5.** *If* $m \in \mathbb{Z}_+$, *then there exist* $m+1$ *positive real numbers* $\{\lambda_{m,k}\}_{k=0}^m$ *such that*

$$\left[\frac{1}{(1\pm x)\ln(1\pm x)}\right]^{(m)} = \frac{(\mp 1)^m}{(1\pm x)^{m+1}\ln(1\pm x)} \sum_{k=0}^{m} \frac{\lambda_{m,k}}{\ln^k(1\pm x)}. \tag{19}$$

*Proof.* We are going to prove it by induction on $m$. The case $m=0$ is trivial. Suppose the formula (19) holds for $m=n$ and we are going to prove it for $m=n+1$.

$$\begin{aligned}\left(\frac{1}{(1\pm x)\ln(1\pm x)}\right)^{(n+1)} &= \left(\frac{(\mp 1)^n}{(1\pm x)^{n+1}\ln(1\pm x)}\sum_{k=0}^{n}\frac{\lambda_{n,k}}{\ln^k(1\pm x)}\right)' \\ &= (\mp 1)^n\sum_{k=0}^{n}\left(\frac{\lambda_{n,k}}{(1\pm x)^{n+1}\ln^{k+1}(1\pm x)}\right)' \\ &= \frac{(\mp 1)^{n+1}}{(1\pm x)^{n+2}\ln(1\pm x)}\sum_{k=0}^{n+1}\frac{(n+1)\lambda_{n,k}+k\lambda_{n,k-1}}{\ln^k(1\pm x)},\end{aligned}$$

where $\lambda_{n,-1}=\lambda_{n,n+1}=0$. Now the proof is completed taking $\lambda_{n+1,k}=(n+1)\lambda_{n,k}+k\lambda_{n,k-1}$, for $k=0,1,\dots,n+1$. $\square$

**Proposition 4.6.** *Let $p\geq 1$, $m\in\mathbb{N}$, $\beta>-1$, $a\in(-1,1)$ and*

$$\alpha\begin{cases}\geq mp-1, & \text{if } p>1,\\ > mp-1, & \text{if } p=1,\end{cases}$$

*then the functions*

$$\phi_m(x)=\begin{cases}\left[\ln\left(\ln\left(\frac{1}{1-x}\right)\right)\right]^{(m)}, & \text{if } \alpha=mp-1,\\ \left[\ln\left(\frac{1}{1-x}\right)\right]^{(m)}, & \text{if } \alpha>mp-1,\end{cases}\qquad x\in[a,1),$$

$$\phi_{-m}(x)=\begin{cases}\left[\ln\left(\ln\left(\frac{1}{1+x}\right)\right)\right]^{(m)}, & \text{if } \alpha=mp-1,\\ \left[\ln\left(\frac{1}{1+x}\right)\right]^{(m)}, & \text{if } \alpha>mp-1,\end{cases}\qquad x\in(-1,a],$$

*are continuous, satisfy $\phi_m\in L^p(\mu^{\alpha,\beta}|_{[a,1]})$, $\phi_{-m}\in L^p(\mu^{\alpha,\beta}|_{[-1,a]})$ and*

$$\lim_{x\to\pm 1}\int_a^x\int_a^{x_1}\cdots\int_a^{x_{m-1}}\phi_{\pm m}(x_m)dx_m dx_{m-1}\cdots dx_1=\infty$$

*Proof.* The first and the last statements are straightforward and the proofs of $\phi_m\in L^p(\mu^{\alpha,\beta}|_{[a,1]})$ and $\phi_{-m}\in L^p(\mu^{\alpha,\beta}|_{[-1,a]})$ are analogous, so we only going to prove $\phi_m\in L^p(\mu^{\alpha,\beta}|_{[a,1]})$. Using (19) we obtain

• Case $\alpha=mp-1$, $p>1$,

$$\begin{aligned}\|\phi_m\|^p_{L^p(\mu^{\alpha,\beta}|_{[a,1]})} &= \int_a^1\left|\left[\ln\left(\ln\left(\frac{1}{1-x}\right)\right)\right]^{(m)}\right|^p(1-x)^{mp-1}(1+x)^\beta dx \\ &= \int_a^1\left|\frac{1}{(1-x)^m\ln(1-x)}\sum_{k=0}^{m-1}\frac{\lambda_{m-1,k}}{\ln^k(1-x)}\right|^p(1-x)^{mp-1}(1+x)^\beta dx \\ &\leq \max\{(1+a)^\beta,2^\beta\}\int_a^1\frac{1}{\ln^p\left(\frac{1}{1-x}\right)}\left|\sum_{k=0}^{m-1}\frac{(-1)^k\lambda_{m-1,k}}{\ln^k\left(\frac{1}{1-x}\right)}\right|^p d\ln\left(\frac{1}{1-x}\right) \\ &\leq \max\{(1+a)^\beta,2^\beta\}\int_{\ln\left(\frac{1}{1-a}\right)}^{\infty}\frac{1}{y^p}\left|\sum_{k=0}^{m-1}\frac{(-1)^k\lambda_{m-1,k}}{y^k}\right|^p dy<\infty,\end{aligned}$$

where the last integral is convergent by the limit comparison test

$$\lim_{y\to\infty} \frac{\frac{1}{y^p}\left|\sum_{k=0}^{m-1}\frac{(-1)^k\lambda_{m-1,k}}{y^k}\right|^p}{\frac{1}{y^p}} = \lambda_{m-1,0}^{p} > 0.$$

• Case $\alpha > mp-1$, $p \geq 1$,

$$\begin{aligned}\int_a^1 |\phi_m(x)|^p d\mu^{\alpha,\beta}(x) &= \int_a^1 \left|\left[\ln\left(\frac{1}{1-x}\right)\right]^{(m)}\right|^p (1-x)^{\alpha}(1+x)^{\beta}dx \\ &\leq \frac{\max\{(1+a)^{\beta},2^{\beta}\}}{\alpha-mp+1}(1-a)^{\alpha-mp+1}.\end{aligned}$$

□

We are ready to prove the completeness of the spaces $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$.

**Theorem 3.** *Let $p \geq 1$, $\alpha,\beta > -1$. Then $\mathbf{U}^{\alpha,\beta}_{\varpi,p}$ is a complete space if and only if one of the following statements holds*

- $\ell = 1$.
- $\ell \geq 2$, $p > 1$ *and for* $m = 1,\ldots,\ell-1$,

$$\omega_{\ell-m} \in \begin{cases} \mathbb{R}, & \text{if } \alpha,\beta \in (-1,mp-1), \\ \mathbb{R}\setminus\{-1\}, & \text{if } \alpha \in (-1,mp-1), \beta \in [mp-1,\infty), \\ \mathbb{R}\setminus\{1\}, & \text{if } \alpha \in [mp-1,\infty), \beta \in (-1,mp-1), \\ \mathbb{R}\setminus\{-1,1\}, & \text{if } \alpha,\beta \in [mp-1,\infty). \end{cases}$$

- $\ell \geq 2$, $p = 1$, *and for* $m = 1,\ldots,\ell-1$;

$$\omega_{\ell-m} \in \begin{cases} \mathbb{R}, & \text{if } \alpha,\beta \in (-1,m-1], \\ \mathbb{R}\setminus\{-1\}, & \text{if } \alpha \in (-1,m-1], \beta \in (m-1,\infty), \\ \mathbb{R}\setminus\{1\}, & \text{if } \alpha \in (m-1,\infty), \beta \in (-1,m-1], \\ \mathbb{R}\setminus\{-1,1\}, & \text{if } \alpha,\beta \in (m-1,\infty). \end{cases}$$

*In addition, in the affirmative case we obtain $\mathbf{U}^{\alpha,\beta}_{\varpi,p} = \mathbf{W}^{\alpha,\beta}_{\varpi,p}$.*

Note that there is not restrictions over $\omega_0$ in any case.

*Proof.* Let $\{f_n\}_{n\geq 0} \subset \mathbf{U}^{\alpha,\beta}_{\varpi,p}$ be a Cauchy sequence. From Proposition 4.1 there exists a sequence of polynomials $\{P_n\}_{n\geq 0}$ such that $\|P_n - f_n\|_{\mathrm{s},p} < \frac{1}{n}$.

Consider the space $V^{k,p}([-1,1],\mu_{[-1,1]})$ defined in [30, Definition 9], where the measure $\mu_{[-1,1]}$ is as in Proposition 4.4 and whose norm is given by

$$\|f\|_{\tilde{\mathrm{s}},p} = \left(\sum_{\substack{k=0\\ \omega_k\in[-1,1]}}^{\ell-1}\left|f^{(k)}(\omega_k)\right|^p + \int_{-1}^{1}\left|f^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x)\right)^{\frac{1}{p}}.$$

From Proposition 4.4, [29, Remark 4 in Definition 16] and [29, Theorem 5.1] this space is complete. So, as the restriction of $P_n$ to $[-1,1]$ belongs to $V^{k,p}([-1,1],\mu_{[-1,1]})$ and

$$\|P_{n_1}-P_{n_2}\|_{\tilde{\mathsf{s}},p} \le \|P_{n_1}-P_{n_2}\|_{\mathsf{s},p} \le \frac{1}{n_1} + \|f_{n_1}-f_{n_2}\|_{\mathsf{s},p} + \frac{1}{n_2},$$

there exists a function $\tilde{f} \in V^{k,p}([-1,1],\mu_{[-1,1]})$ such that

$$\|P_n - \tilde{f}\|_{\tilde{\mathsf{s}},p} \longrightarrow 0. \tag{20}$$

On the other hand, since

$$\begin{aligned}\left|P_{n_1}^{(j)}(\omega_j) - P_{n_2}^{(j)}(\omega_j)\right|^p \le & \sum_{k=0}^{\ell-1} \left|P_{n_1}^{(k)}(\omega_k) - P_{n_2}^{(k)}(\omega_k)\right|^p \\ & + \int_{-1}^{1} \left|P_{n_1}^{(\ell)}(x) - P_{n_2}^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x) = \|P_{n_1}-P_{n_2}\|_{\mathsf{s},p},\end{aligned}$$

it follows that $\{P_n^{(j)}(\omega_j)\}_{n\ge 0}$, $j=0,1,\ldots,\ell-1$ is a Cauchy sequence in $\mathbb{R}$, then there exist values $f_{\omega_k}^k$ such that $P_n^{(k)}(\omega_k) \to f_{\omega_k}^k$, $k=0,1,\ldots,\ell-1$.

Note that, from (20) we obtain $\tilde{f}^{(k)}(\omega_k) = f_{\omega_k}^k$, for $\omega_k \in [-1,1]$, where the derivative of $\tilde{f}$ at $\pm 1$, as is usual, means the corresponding usual side derivative.

Consider the vector $f(\varpi) = (f_{\omega_0}^0, f_{\omega_1}^1, \ldots, f_{\omega_{\ell-1}}^{\ell-1})$ and the function

$$f(x) = \begin{cases} \tilde{f}(x), & \text{if } x \in [-1,1], \\ \mathscr{A}_{\varpi,f(\varpi)}(x), & \text{if } x \in \mathbb{R}\setminus[-1,1], \end{cases}$$

which belongs to $\mathbf{U}_{\varpi,p}^{\alpha,\beta}$ and satisfies

$$\begin{aligned}\|P_n - f\|_{\mathsf{s},p}^p &= \sum_{\omega_k \notin [-1,1]} \left|P_n^{(k)}(\omega_k) - f^{(k)}(\omega_k)\right|^p + \|P_n - f\|_{\tilde{\mathsf{s}},p}^p \\ &= \sum_{\omega_k \notin [-1,1]} \left|P_n^{(k)}(\omega_k) - f_{\omega_k}^k\right|^p + \|P_n - \tilde{f}\|_{\tilde{\mathsf{s}},p}^p \longrightarrow 0.\end{aligned}$$

Finally, from $\|f_n - f\|_{\mathsf{s},p} \le \|f_n - P_n\|_{\mathsf{s},p} + \|P_n - f\|_{\mathsf{s},p} \longrightarrow 0$, we conclude that $\mathbf{U}_{\varpi,p}^{\alpha,\beta}$ is a complete space. Thus, as $\mathbb{P} \subset \mathbf{U}_{\varpi,p}^{\alpha,\beta} \subset \mathbf{W}_{\varpi,p}^{\alpha,\beta}$ and $\mathbf{W}_{\varpi,p}^{\alpha,\beta}$ is the completion of $\mathbb{P}$ we obtain $\mathbf{W}_{\varpi,p}^{\alpha,\beta} = \mathbf{U}_{\varpi,p}^{\alpha,\beta}$.

The second part of the proof is to show that the conditions over $\omega_k$ are also necessary. Then $\ell \ge 2$ and let us consider the case $\omega_{\ell-m} = 1$ ($\alpha \ge mp-1$ if $p>1$ or $\alpha > m-1$ if $p=1$), the other case, when $\omega_{\ell-m} = -1$, is analogous. Take the sequence of functions

$$f_n(x) = \mathbb{1}_{[a,b_n]}(x) \int_a^x \int_a^{x_1} \cdots \int_a^{x_{\ell-1}} \phi_m(x_\ell)\, dx_\ell dx_{\ell-1} \cdots dx_1,$$

where $\phi_m$ is the function defined in Proposition 4.6 and the values $a$ and $b_n$, $n \ge 0$, are taken as $-1 < a < b_0 < 1$ and $\{b_n\}_{n\ge 0}$ is a non-decreasing sequence with 1 as limit and such that $\omega_k \notin \{a\} \cup [b_0,1)$ for $k=0,1,\ldots,\ell-1$. Since $\phi_m \in L^p(\mu^{\alpha,\beta}|_{[a,1]})$ it follows that $f_n^{(\ell)} \in L^p(\mu^{\alpha,\beta})$, so $f_n \in \mathbf{U}_{\varpi,p}^{\alpha,\beta}$ for $n \in \mathbb{N}$.

Note also that, from the condition on $a$ and $b_n$, $f_n$ vanishes on neighborhoods of each $\omega_j \notin (-1,1)$, and $f_n$ does not depend on $n$ over neighborhoods of each $\omega_j \in (-1,1)$. Now we are going to prove that $f_n$ is a Cauchy sequence

$$\begin{aligned}\|f_{n_1}-f_{n_2}\|_{s,p}^p &= \sum_{k=0}^{\ell-1}\left|f_{n_1}^{(k)}(\omega_k)-f_{n_2}^{(k)}(\omega_k)\right|^p+\int_{-1}^{1}\left|f_{n_1}^{(\ell)}(x)-f_{n_2}^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x),\\ &= \left|\int_{b_{n_2}}^{b_{n_1}}|\phi_m(x)|^p d\mu^{\alpha,\beta}(x)\right|.\end{aligned}$$

Since $\phi_m \in L^p(\mu^{\alpha,\beta}|_{[a,1]})$, the sequence $\int_{b_n}^1 |\phi_m(x)|^p d\mu^{\alpha,\beta}(x)$ tends to 0 when $n\to\infty$, so it is a Cauchy sequence on $\mathbb{R}$. Consequently, $f_n$ is a Cauchy sequence on $\mathbf{U}_{\varpi,p}^{\alpha,\beta}$. On the other hand, if $f \in \mathbf{U}_{\varpi,p}^{\alpha,\beta}$ is a function such that $f_n \to f$ in $\mathbf{U}_{\varpi,p}^{\alpha,\beta}$, then

$$\sum_{k=0}^{\ell-1}\left|f_n^{(k)}(\omega_k)-f^{(k)}(\omega_k)\right|^p+\int_{-1}^{1}\left|f_n^{(\ell)}(x)-f^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x)=\|f_n-f\|_{s,p}\longrightarrow 0.$$

Hence, $f_n^{(k)}(\omega_k)\longrightarrow f^{(k)}(\omega_k)$ for $k=0,1,\cdots,\ell-1$ and

$$\int_a^{b_n}\left|\phi_m(x)-f^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x)\longrightarrow 0.$$

Since $f \in \mathbf{U}_{\varpi,p}^{\alpha,\beta}$ we get $\phi_m - f^{(\ell)} \in L^p(\mu^{\alpha,\beta}|_{[a,1]})$, so

$$\int_a^{1}\left|\phi_m(x)-f^{(\ell)}(x)\right|^p d\mu^{\alpha,\beta}(x)=0,$$

and, as a consequence $f^{(\ell)} \equiv \phi_m$ a.e. on $[a,1]$. Thus we can write for $x\in[a,1)$

$$f^{(\ell-m)}(x)=\int_a^x\int_a^{x_1}\cdots\int_a^{x_{m-1}}\phi_m(x_m)dx_m dx_{m-1}\cdots dx_1+Q_{m-1}(x),\quad x\in[a,1),$$

where $Q_{m-1}$ is a polynomial of degree at most $m-1$. Now using that $\omega_{\ell-m}=1$ we obtain that $f$ has derivative of order $\ell-m$ on $[a,1]$ and the function $f^{(\ell-m)}$ is the derivative of $f^{(\ell-m-1)}$ on the closed interval $[a,1]$ (note that $m\le \ell-1$). From Darboux's theorem [31, Th. 5.12] it follows that $f^{(\ell-m)}([1-\varepsilon,1])$ is an interval for every sufficiently small $\varepsilon>0$ and we get the contradiction $\lim_{x\to 1^-} f^{(\ell-m)}(x)=\infty$ (see Proposition 4.6). □

**Corollary 3.1.**

- *If* $\ell=1$ *and* $p\ge 1$, *then* $\mathbf{W}_{\omega_0,p}^{\alpha,\beta}=\mathbf{U}_{\omega_0,p}^{\alpha,\beta}$, $\forall\omega_0\in\mathbb{R}$.
- *If* $\ell\ge 2$ *and* $p>1$, *then*

$$\mathbf{W}_{\varpi,p}^{\alpha,\beta}=\mathbf{U}_{\varpi,p}^{\alpha,\beta},\ \forall\varpi\in\mathbb{R}^\ell,\ \textit{if and only if}\ \alpha,\beta\in(-1,p-1).$$

- *If* $\ell\ge 2$ *and* $p=1$, *then*

$$\mathbf{W}_{\varpi,p}^{\alpha,\beta}=\mathbf{U}_{\varpi,p}^{\alpha,\beta},\ \forall\varpi\in\mathbb{R}^\ell,\ \textit{if and only if}\ \alpha,\beta\in(-1,0].$$

## 5 Sobolev-type orthogonal polynomials and Fourier series

On $\mathbb{P}$, let us consider the Sobolev-type inner product, related to the norm (13) with $p = 2$

$$\langle f, g\rangle_{\mathsf{s}} := \sum_{k=0}^{\ell-1} f^{(k)}(\omega_k)\, g^{(k)}(\omega_k) + \int_{-1}^{1} f^{(\ell)}(x)\, g^{(\ell)}(x) d\mu^{\alpha,\beta}(x). \tag{21}$$

Let $P_{\varpi,n}^{(\alpha,\beta)}$ be the $\ell$-th iterated integral of $p_n^{(\alpha,\beta)}$ (the $n$-th Jacobi orthonormal polynomial, as in (7)), normalized by the conditions

$$\left(P_{\varpi,n}^{(\alpha,\beta)}\right)^{(k)}(\omega_k) = 0, \quad k = 0, 1, \ldots, \ell-1, \tag{22}$$

or equivalently,

$$P_{\varpi,n}^{(\alpha,\beta)}(s) = \int_{\omega_0}^{s}\int_{\omega_1}^{s_1} \cdots \int_{\omega_{\ell-2}}^{s_{\ell-2}}\int_{\omega_{\ell-1}}^{s_{\ell-1}} p_n^{(\alpha,\beta)}(s_\ell) ds_\ell ds_{\ell-1} \cdots ds_2 ds_1.$$

**Theorem 4.** *The polynomials*

$$q_n(x) = \begin{cases} \mathscr{G}_{\varpi,n}(x), & \textit{if } 0 \le n \le \ell-1, \\ P_{\varpi,n-\ell}^{(\alpha,\beta)}(x), & \textit{if } \ell \le n, \end{cases} \tag{23}$$

*constitute a family of orthonormal polynomials with respect to* (21) *on* $\mathbf{W}_{\varpi,2}^{\alpha,\beta}$*, which is also complete.*

The algebraic and analytical properties of the polynomials $\{q_n\}_{n\ge 0}$ have been studied in [23].

*Proof.* First, note that, for each $n \ge 0$, $q_n$ is a polynomial of degree $n$ and from (14) and (22)

$$q_n^{(k)}(\omega_k) = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{if } n \ne k, \end{cases} \qquad \text{for } 0 \le k \le \ell-1. \tag{24}$$

Now, we are going to prove that $\{q_n\}_{n\ge 0}$ is an orthonormal polynomial sequence with respect to (21).

If $m, n \ge \ell$, then from (23) and (24)

$$\langle q_m, q_n\rangle_{\mathsf{s}} = \int_{-1}^{1} p_{m-\ell}^{(\alpha,\beta)}(x)\, p_{n-\ell}^{(\alpha,\beta)}(x) d\mu^{\alpha,\beta}(x) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \ne m. \end{cases}$$

Otherwise, we get that $q_m^{(\ell)} q_n^{(\ell)} \equiv 0$ and from (24)

$$\langle q_m, q_n\rangle_{\mathsf{s}} = \sum_{k=0}^{\ell-1} q_m^{(k)}(\omega_k)\, q_n^{(k)}(\omega_k) = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \ne m. \end{cases}$$

The completeness is a direct consequence of the fact that $\mathbf{W}_{\varpi,p}^{\alpha,\beta} = \overline{\mathbb{P}}$. □

Our next step is to define rigorously what means the Fourier series of $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$, where $p \geq 1$, $\ell \in \mathbb{N}$, $\alpha, \beta > -1$ and $\varpi = (\omega_0, \omega_1, \ldots, \omega_{\ell-1}) \in \mathbb{R}^{\ell}$.

For $f \in \mathbf{U}^{\alpha,\beta}_{\varpi,p}$ we denote $S_n(f,x) := \sum_{k=0}^{n} \langle f, q_k \rangle_{\mathsf{s}} \, q_k(x)$. Under the above assumptions of Theorem 3, $\mathbf{W}^{\alpha,\beta}_{\varpi,p} = \mathbf{U}^{\alpha,\beta}_{\varpi,p}$ but, in general, we only know that $\mathbf{W}^{\alpha,\beta}_{\varpi,p} = \overline{\mathbf{U}^{\alpha,\beta}_{\varpi,p}}$. However, if $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p} \setminus \mathbf{U}^{\alpha,\beta}_{\varpi,p}$, then $f$ can be identified as a Cauchy sequence $\{f_n\}_{n\geq 0} \subset \mathbf{U}^{\alpha,\beta}_{\varpi,p}$ with norm $\|f\|_{\mathsf{s},p} = \lim_{n\to\infty} \|f_n\|_{\mathsf{s},p}$.

The existence of this limit is guaranteed since $\{\|f_n\|_{\mathsf{s},p}\}_{n\geq 0}$ is a Cauchy sequence in $\mathbb{R}$. On the other hand, if $\{f_n\}_{n\geq 0}$ is a Cauchy sequence in $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$ we get

$$\|f_n - f_m\|^p_{\mathsf{s},p} = \sum_{k=0}^{\ell-1} \left| f_n^{(k)}(\omega_k) - f_m^{(k)}(\omega_k) \right|^p + \|f_n^{(\ell)}(x) - f_m^{(\ell)}\|^p_{\alpha,\beta,p},$$

where the norm $\|\cdot\|_{\alpha,\beta,p}$ was defined in (6). Therefore, the sequences $\{f_n^{(k)}(\omega_k)\}_{n\geq 0}$, $k = 0,1,\ldots,\ell-1$ and $\{f_n^{(\ell)}\}_{n\geq 0}$ are also Cauchy sequences on the complete spaces $\mathbb{R}$ and $L^p(\mu^{\alpha,\beta})$, respectively. So there exist the limits

$$f^k_{\omega_k} = \lim_{n\to\infty} f_n^{(k)}(\omega_k), \; k = 0,1,\ldots,\ell-1; \; \text{ and } \; f^{\ell} = \lim_{n\to\infty} f_n^{(\ell)}.$$

Then, we get

$$\|f\|^p_{\mathsf{s},p} = \lim_{n\to\infty} \|f_n\|^p_{\mathsf{s},p} = \lim_{n\to\infty} \sum_{k=0}^{\ell-1} \left| f_n^{(k)}(\omega_k) \right|^p + \|f_n^{(\ell)}\|^p_{\alpha,\beta,p} = \sum_{k=0}^{\ell-1} \left| f^k_{\omega_k} \right|^p + \|f^{\ell}\|^p_{\alpha,\beta,p}.$$

From the proof of Theorem 3, we see that there is not always a function $f$ satisfying

$$f^{(k)}(\omega_k) = f^k_{\omega_k}, \text{ for } k = 0,1,\cdots,\ell-1; \quad \text{and} \quad f^{(\ell)} \equiv f^{\ell} \text{ a.e. on } [-1,1].$$

But even so, we can understand the value $f^k_{\omega_k}$ as the $k$-th derivative of the equivalent class $f$ at $\omega_k$ and the function $f^{\ell} \in L^p(\mu^{\alpha,\beta})$ as its $\ell$-th derivative a.e. on $[-1,1]$. Then, in a natural way, we can define for every $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$

$$\langle f, q_n \rangle_{\mathsf{s}} = \sum_{k=0}^{\ell-1} f^k_{\omega_k} q_n^{(k)}(\omega_k) + \int_{-1}^{1} f^{\ell}(x) \, q_n^{(\ell)}(x) d\mu^{\alpha,\beta}(x),$$

and now the Fourier series of $f$ makes sense for all $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$.

From (23)-(24), it is straightforward to prove that if $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$, then

$$\langle f, q_n \rangle_{\mathsf{s}} = \begin{cases} f^{(n)}(\omega_n), & \text{if } 0 \leq n \leq \ell-1, \\ \left\langle f^{(\ell)}, p_{n-\ell}^{(\alpha,\beta)} \right\rangle_{\alpha,\beta}, & \text{if } \ell \leq n. \end{cases} \tag{25}$$

**Theorem 5.** *Let* $\alpha, \beta > -1$,

$$M(\alpha,\beta) = 2\,\frac{\max\{\alpha,\beta,-\frac{1}{2}\}+1}{\max\{\alpha,\beta,-\frac{1}{2}\}+\frac{3}{2}} \quad \textit{and} \quad m(\alpha,\beta) = 2\,\frac{\max\{\alpha,\beta,-\frac{1}{2}\}+1}{\max\{\alpha,\beta,-\frac{1}{2}\}+\frac{1}{2}}. \tag{26}$$

*If $M(\alpha,\beta) < p < m(\alpha,\beta)$, then, for any $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$, its Fourier expansion with respect of the polynomials $\{q_n\}_{n\geq 0}$, converges to $f$ in $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$, i.e.*

$$\lim_{n\to\infty} \|f - S_n(f,\cdot)\|_{\mathrm{s},p} = 0.$$

It is not difficult to verify that the definition of $M(\alpha,\beta)$ and $m(\alpha,\beta)$ in (26) is the same as in Theorem A (when $\alpha,\beta \geq -1/2$) and in (11) (with $a = \alpha/p$ and $b = \beta/p$).

Let $\alpha,\beta > -1$, $\gamma = \max\{\alpha,\beta\}$,

$$M(\gamma) = 2\,\frac{\max\{\gamma,-\frac{1}{2}\}+1}{\max\{\gamma,-\frac{1}{2}\}+\frac{3}{2}} \quad \text{and} \quad m(\gamma) = 2\,\frac{\max\{\gamma,-\frac{1}{2}\}+1}{\max\{\gamma,-\frac{1}{2}\}+\frac{1}{2}}.$$

$(\gamma,p)$ is said to be a convergence pair if $M(\gamma) < p < m(\gamma)$. In Figure 1, the entire shaded region $\Delta$ (light and dark) is the set of all pairs $(\gamma,p)$ of convergence. The dark region $\Delta_0$, at the left, is the set of $(\gamma,p)$ such that $\mathbf{W}^{\alpha,\beta}_{\varpi,p}$ is a space of functions; i.e. $\mathbf{W}^{\alpha,\beta}_{\varpi,p} = \mathbf{U}^{\alpha,\beta}_{\varpi,p}$, for all $\varpi \in \mathbb{R}^\ell$ (see Corollary 3.1).

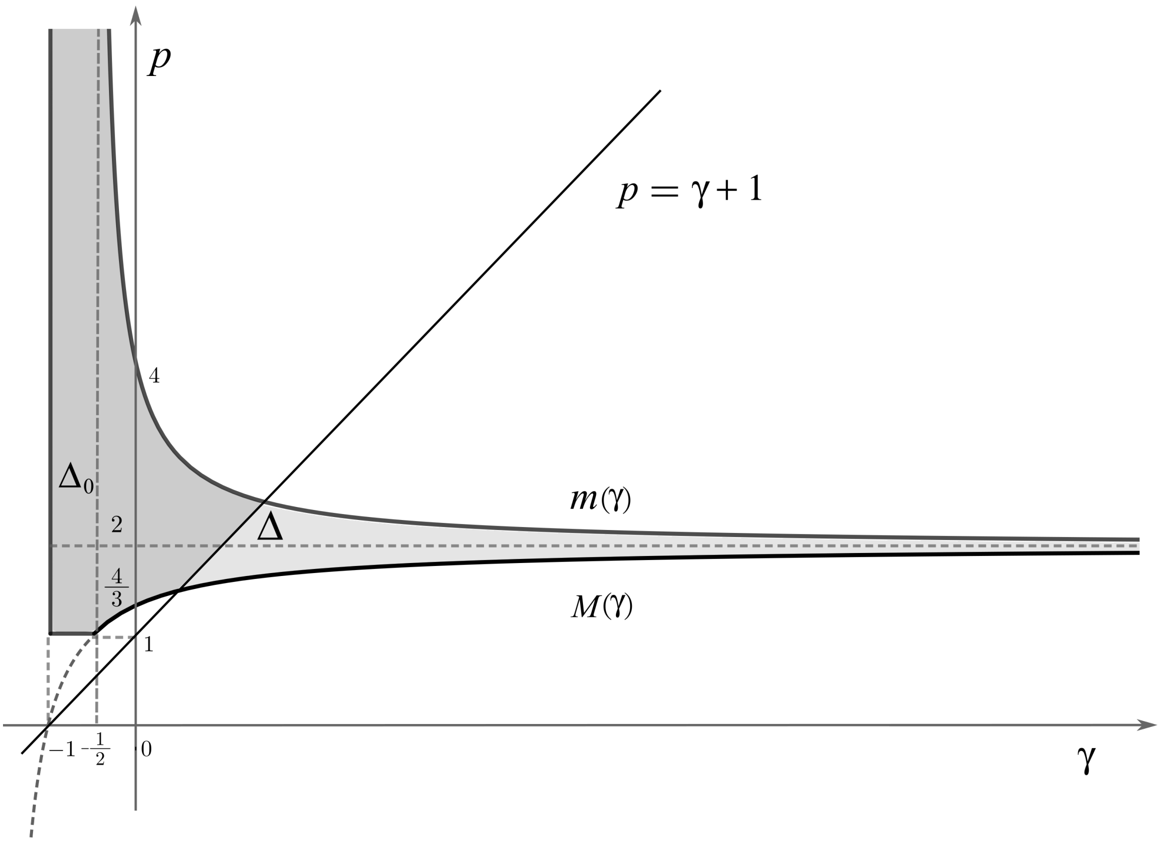


Figure 1: The open regions $\Delta = \{(\gamma,p) : M(\gamma) < p < m(\gamma)\}$ and $\Delta_0 = \{(\gamma,p) \in \Delta : p > \gamma+1\}$.

*Proof of Theorem 5.* Given $f \in L^p(\mu^{\alpha,\beta})$, let $S^{\alpha,\beta}_n(f,\cdot)$ be the $n$-th partial sum of the Fourier-Jacobi series of $f$

$$S^{\alpha,\beta}_n(f,x) = \sum_{k=0}^{n} \left\langle f, p^{(\alpha,\beta)}_k \right\rangle_{\alpha,\beta} p^{(\alpha,\beta)}_k(x).$$

Now, given $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$, from (25)

1.- If $0 \leq k \leq \ell-1$, then $S_n^{(k)}(f,\omega_k) = \begin{cases} 0, & \text{if } n<k, \\ f^{(k)}(\omega_k), & \text{if } n \geq k. \end{cases}$

2.- If $n \geq \ell$, then $S_n^{(\ell)}(f,x) = S_{n-\ell}^{\alpha,\beta}(f^{(\ell)},x)$.

From Theorem B, with $a = \alpha/p$ and $b = \beta/p$, there exists a constant $C$, independent on $n$ and $f$, such that $\|S_{n-\ell}^{\alpha,\beta}(f^{(\ell)},x)\|_{\alpha,\beta,p} \leq C\|f^{(\ell)}\|_{\alpha,\beta,p}$.

We first prove that there exists a constant $C_1$, independent on $n$ and $f$, such that

$$\|S_n(f,\cdot)\|_{\mathsf{s},p} \leq C_1\|f\|_{\mathsf{s},p}, \qquad \text{for all } f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}. \tag{27}$$

If $n \geq \ell$, then we have

$$\begin{aligned} \|S_n(f,\cdot)\|_{\mathsf{s},p}^p &= \sum_{k=0}^{\ell-1} \left|S_n^{(k)}(f,\omega_k)\right|^p + \int_{-1}^{1} \left|S_n^{(\ell)}(f,x)\right|^p d\mu^{\alpha,\beta}(x) \\ &= \sum_{k=0}^{\ell-1} \left|f^{(k)}(\omega_k)\right|^p + \|S_{n-\ell}^{\alpha,\beta}(f^{(\ell)},\cdot)\|_{\alpha,\beta,p}^p \\ &\leq \sum_{k=0}^{\ell-1} \left|f^{(k)}(\omega_k)\right|^p + C^p\|f^{(\ell)}\|_{\alpha,\beta,p}^p \leq \max\{1,C^p\}\|f\|_{\mathsf{s},p}^p. \end{aligned}$$

If $0 \leq n \leq \ell-1$, then we get

$$\begin{aligned} \|S_n(f,\cdot)\|_{\mathsf{s},p}^p &= \sum_{k=0}^{\ell-1} \left|S_n^{(k)}(f,\omega_k)\right|^p + \int_{-1}^{1} \left|S_n^{(\ell)}(f,x)\right|^p d\mu^{\alpha,\beta}(x) \\ &\leq \sum_{k=0}^{\ell-1} \left|f^{(k)}(\omega_k)\right|^p \leq \|f\|_{\mathsf{s},p}^p. \end{aligned}$$

This yields formula (27) with $C_1 = \sqrt[p]{\max\{1,C^p\}}$.

Given $f \in \mathbf{W}^{\alpha,\beta}_{\varpi,p}$ and $\varepsilon > 0$, then there exists $P \in \mathbb{P}$, such that $\|f-P\|_{\mathsf{s},p} < \frac{\varepsilon}{C_1+1}$, where $C_1$ is given by (27). Note that for all $n$ greater than or equal to the degree of $P$ we get $P \equiv S_n(P,\cdot)$ and, as a consequence,

$$\begin{aligned} \|S_n(f,\cdot)-f\|_{\mathsf{s},p} \leq& \|S_n(f,\cdot)-P\|_{\mathsf{s},p} + \|P-f\|_{\mathsf{s},p} = \|S_n(f,\cdot)-S_n(P,\cdot)\|_{\mathsf{s},p} + \|P-f\|_{\mathsf{s},p} \\ =& \|S_n(f-P,\cdot)\|_{\mathsf{s},p} + \|P-f\|_{\mathsf{s},p} \leq (C+1)\|P-f\|_{\mathsf{s},p} < \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \|f-S_n(f,\cdot)\|_{\mathsf{s},p} = 0$. □


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