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Asymptotic zero distribution for a class of extremal polynomials

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We consider extremal polynomials with respect to a Sobolev-type p-norm, with $1 < p < \infty$ and measures supported on compact subsets of the real line. For a wide class of such extremal polynomials with respect to mutually singular measures (i.e. supported on disjoint subsets of the real line), it is proved that their critical points are simple and contained in the interior of the convex hull of the support of the measures involved and the asymptotic critical point distribution is studied. We also find the nth root asymptotic behavior of the corresponding sequence of Sobolev extremal polynomials and their derivatives.

Keywords: Extremal polynomials; Sobolev orthogonality; location of zeros; asymptotic behavior.

Mathematics Subject Classification: 42A05, 30C15, 26C05, 26C10, 33C47

1. Introduction

Let μ_0 be a positive Borel measure supported on an interval of the real line Δ_0 (which does not reduce to a point). For $1 < p < \infty$, we denote by $\mathbf{L}^p(\mu_0)$ the Banach space of all p-integrable functions on Δ_0 with respect to the measure μ_0, endowed with the norm

$$\|f\|_{0,p} = \left(\int_{\Delta_0} |f|^p d\mu_0\right)^{1/p}. \tag{1}$$

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We denote by $\mathbb{P}_n$ the space of polynomials (with complex coefficients) of degree $\leq n$, and by $\mathbb{P}_n^* \subset \mathbb{P}_n$ the subset of monic polynomials of exact degree n. It is well known that $\|\cdot\|_{0,p}$ is a *strictly convex norm*, i.e. the unit ball is a strictly convex set. Then, there exists a unique monic polynomial $P_n \in \mathbb{P}_n^*$ such that

$$\|P_n\|_{0,p} = \min_{Q \in \mathbb{P}_n^*} \|Q\|_{0,p}, \tag{2}$$

(cf. [4, Definition 7.5.1 and Theorem 7.5.3]). P_n is called the *nth monic extremal polynomial* relative to $\|\cdot\|_{0,p}$.

The study of zeros and critical points of extremal polynomials is of great interest because they can be interpreted in various ways from the standpoint of physics, function theory and numerical analysis. It is known that (1) defines a *Fejér or monotonic norm* when is restricted to the space $\mathbb{P}$ of polynomials (i.e. for distinct $f, g \in \mathbb{P}$ the condition $|f(z)| \leq |g(z)|$ for all $z \in \Delta_0$, with equality only if $g(z) = 0$, implies $\|f\|_{0,p} < \|g\|_{0,p}$). Hence, from *Fejér's convex hull theorem* [4, Theorem 10.2.2] we get that the zeros of P_n are simple and lie in Δ_0.

Let us mention a characterization of the solution of the extremal problem (2) (cf. [3, §2.2, Ex. 7-h]). A polynomial $P_n \in \mathbb{P}_n^*$ is the nth monic extremal polynomial in $\mathbf{L}^p(\mu_0)$ if and only if for all $Q \in \mathbb{P}_{n-1}$

$$\int_{\Delta_0} Q \operatorname{sgn}(P_n) |P_n|^{p-1} d\mu_0 = 0, \quad \text{where } \operatorname{sgn}(y) = \begin{cases} y/|y| & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases} \tag{3}$$

Hence, if P_n has a zero of multiplicity at least two at x^* then, $\frac{P_n(x)}{(x-x^*)^2}$ is a polynomial of degree $(n-2)$ and we have the contradiction

$$0 < \int_{\Delta_0} \frac{|P_n(x)|^p}{(x-x^*)^2} d\mu_0(x) = \int_{\Delta_0} \frac{P_n(x)}{(x-x^*)^2} \operatorname{sgn}(P_n(x)) |P_n(x)|^{p-1} d\mu_0(x) = 0.$$

Consequently, *all the zeros of P_n are simple.*

Let μ_0, μ_1 be two positive Borel measures supported on the intervals $\Delta_0 \subset \mathbb{R}$ and $\Delta_1 \subset \mathbb{R}$, respectively, where Δ_0 is an nontrivial interval. For $1 < p < \infty$, we consider on the space $\mathbb{P}$ of polynomials, the Sobolev norm

$$\|f\|_{S,p} = (\|f\|_{0,p}^p + \|f'\|_{1,p}^p)^{\frac{1}{p}} = \left(\int_{\Delta_0} |f|^p d\mu_0 + \int_{\Delta_1} |f'|^p d\mu_1. \right)^{\frac{1}{p}}. \tag{4}$$

It is not difficult to prove that (4) is a strictly convex norm and, therefore, for each $n \in \mathbb{Z}_+$ there exists a unique monic polynomial $L_n \in \mathbb{P}_n^*$ such that

$$\|L_n\|_{S,p} = \min_{Q \in \mathbb{P}_n^*} \|Q\|_{S,p}. \tag{5}$$

The polynomial L_n is called the nth *monic extremal polynomial* relative to $\|\cdot\|_{S,p}$. In Proposition 1 we give an alternative direct proof of the uniqueness of L_n. Obviously, $\|\cdot\|_{S,p}$ is not a Fejér norm because we can construct (piecewise continuously differentiable) functions such that $|f(z)| < |g(z)|, x \in \Delta_0 \cup \Delta_1$, with $|f'|$ much larger than $|g'|$, μ_1 a.e. on Δ_1, so that $\|f\|_{S,p} > \|g\|_{S,p}$. Specific examples are easy to produce.

Example 1. Let μ_0, μ_1 be probability measures supported on Δ_0 and Δ_1, respectively. Take $0 < a < c < 1$ and assume that $\Delta_0 \cup \Delta_1 \subset [-a, a]$. If $f(x) = x$ and $g(x) \equiv c$, then $|f(x)| < |g(x)|, x \in \Delta_0 \cup \Delta_1$, whereas

$$\|g\|_{S,p}^p = c^p < 1 < \int_{\Delta_0} |x|^p d\mu_0(x) + 1 = \|f\|_{S,p}^p.$$

It is well known that in the standard case of extremality with respect to the norm (1) the zeros of the extremal polynomial are all in Δ_0 and (1) is a Fejér norm, but in the Sobolev case this is not true. Althammer shows in an early example (cf. [1], where $p = 2$) that in the Sobolev case the zeros of the orthogonal polynomials may lie outside of $\Delta_0 \cup \Delta_1$. Other examples of the previous fact can be seen in [6, §2] (where $p = 2$).

However, in the numerical experiments carried out in [6, §2] (for $p = 2$), the authors found that in all the cases considered, the critical points of L_n were real numbers. Their experiments conclude with two conjectures about the location of zeros and critical points of the Sobolev-type orthogonal polynomials (see [6, Conjectures 1 and 2]). In the following theorem, we solve the problem derived from these conjectures for extremal polynomials when μ_0 and μ_1 in (4) are supported on mutually disjoint intervals.

Theorem 1. *Let $p \in (1, \infty)$ and let μ_0, μ_1 be finite positive Borel measures supported on the real line such that $\mathring{\Delta}_0 \cap \mathring{\Delta}_1 = \emptyset$ ($\mathring{A}$ denotes the interior of a real set A with the Euclidean topology of $\mathbb{R}$). Then*

(1.1) *For all $n \geq 1$, $n - 1 \leq \mathbf{N}_{\rm o}(L_n; \mathring{\Delta}_0) + \mathbf{N}_{\rm o}(L_n'; \mathring{\Delta}_1) \leq n$ and the zeros of L_n in $\mathring{\Delta}_0$ are simple, where the symbol $\mathbf{N}_{\rm o}(Q; I)$ denotes the number of zeros with odd multiplicity of the polynomial $Q \in \mathbb{P}$ on the interval $I \subset \mathbb{R}$ (i.e. points of sign change).*

(1.2) *For $n \geq 2$, the critical points of the extremal polynomial L_n are simple and contained in $\mathring{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1)$. ($\boldsymbol{Co}(A)$ denotes the convex hull of the set A).*

(1.3) *The number of zeros (or critical points) of the extremal polynomial L_n lying in $\mathring{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1) \setminus (\mathring{\Delta}_0 \cup \mathring{\Delta}_1)$ is at most one.*

(1.4) *The zeros of L_n' in $\mathring{\Delta}_0$ interlace the zeros of L_n on that set.*

The following example shows that, in general, the lower bound in inequality (1.1) of Theorem 1 cannot be improved.

Example 2. Set $\Delta_0 = [-11. -2]$, $\Delta_1 = [22.5, 36]$ and

$$\|f\|_{S,2}^2 = \int_{-11}^{-2} (f(x))^2 dx + \int_{22.5}^{36} (f'(x))^2 dx.$$

The extremal polynomial of degree $n = 2$ is $L_2(z) = z^2 - 49$ with zeros $z_1 = -7$, $z_2 = 7$ and critical point $z_3 = 0$. Then $n - 1 = 1 = \mathbf{N}_{\rm o}(L_2; \overset{\circ}{\Delta}_0) + \mathbf{N}_{\rm o}(L_2'; \overset{\circ}{\Delta}_1) = 1$.

The next result provides a natural and intrinsic characterization of the extremal polynomials defined by (5), and an extension of (3) for the Sobolev case.

Theorem 2. *L_n is the nth monic extremal polynomial with respect to $\|\cdot\|_{S,p}$ if and only if*

$$\begin{aligned}\langle Q, L_n\rangle_{S,p} := &\int_{\Delta_0} Q(x)\,\mathrm{sgn}\,(L_n(x))\,\,|L_n(x)|^{p-1} d\mu_0(x)\\ &+ \int_{\Delta_1} Q'(x)\,\mathrm{sgn}\,(L_n'(x))\,\,|L_n'(x)|^{p-1} d\mu_1(x) = 0,\end{aligned}$$

for every polynomial $Q \in \mathbb{P}_{n-1}$.

Note that unless $p = 2$, $\langle \cdot, \cdot \rangle_{S,p}$ does not define an inner product.

Theorem 2 is a corollary of Theorem 4, where (4) is replaced by a Sobolev norm with derivatives of higher order. More precisely, for $f \in \mathbb{P}$, set

$$\|f\|_{S,p,m} = \left(\sum_{k=0}^{m} \|f^{(k)}\|_{k,p}^p \right)^{\frac{1}{p}} = \left(\sum_{k=0}^{m} \int |f^{(k)}|^p d\mu_k \right)^{\frac{1}{p}}, \tag{6}$$

where m is a fixed non-negative integer, $1 < p < \infty$, μ_k is a positive Borel measure supported on $\mathbb{R}$ ($k = 0, \ldots, m$), $\mathrm{supp}\,(\mu_0)$ is an infinite set and $f^{(k)}$ denotes the kth derivative of f. When $m = 1$, (6), reduces to (4) and $\|\cdot\|_{S,p} = \|\cdot\|_{S,p,1}$.

According to (6), $L_n \in \mathbb{P}_n^*$ (the nth monic extremal polynomial with respect to (6)) is a monic polynomial that verifies

$$\|L_n\|_{S,p,m} = \min_{Q \in \mathbb{P}_n^*} \|Q\|_{S,p,m}. \tag{7}$$

When $p = 2$, and the norm (6) is given by an inner product, the corresponding Sobolev extremal polynomials (or orthogonal with respect to the associated inner product) have been extensively studied. A survey on the subject is provided in [12]. However, for $p \neq 2$ ($1 < p < \infty$) not much has been attained and the basic references are [8] for the so-called "*sequentially dominated norms*" and [10] for measures with unbounded support on the real line.

Section 2 is devoted to the study of the existence and uniqueness of the extremal polynomial with respect to the norm (6). Theorem 4, which is of independent interest, is the main tool for locating zeros and critical points. In Sec. 3, we prove Theorem 1 and Corollary 3 on the location and algebraic properties of the zeros and critical points of the extremal polynomials L_n. The rest of this paper is devoted to the study of the asymptotic distribution of the zeros and critical points of the Sobolev extremal polynomials. Next, we state the main result in this direction after introducing some needed terminology.

For any complex polynomial $Q_n(z) = c\prod_{k=1}^{n}(z - z_k)$, with $c, z_1, \ldots, z_n \in \mathbb{C}$, we denote by $\sigma(Q_n)$ the so-called *normalized zero counting measure associated with* Q_n, as

$$\sigma(Q_n) = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}, \tag{8}$$

where δ_{z_j} is the Dirac measure with mass one at the point z_j. Following the usual terminology, if $\{\mu_n\}$ is a sequence of measures on a compact set $K \subset \mathbb{C}$, we say that a measure μ is the limit of $\{\mu_n\}$ in the weak star topology of measures, if

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu,$$

for every continuous function f on K. In this case we write $w\text{-}\lim_{n\to\infty} \mu_n = \mu$.

Let μ be a finite Borel measure whose compact support $S(\mu) \subset \mathbb{C}$ has positive logarithmic capacity and let P_n be the associated monic orthogonal polynomial with respect to μ of degree n. We say that μ is regular and write $\mu \in \mathbf{Reg}$ if

$$\lim_{n\to\infty} \|P_n\|_{\mu,2}^{1/n} = \mathbf{cap}\,(S(\mu)),$$

where $\|P_n\|_{\mu,2}$ denotes the $L_2(\mu)$ norm of P_n. Theorem 3.1.1 in [18] contains several equivalent forms of defining regular measures (see also [18, Theorems 3.2.1, 3.2.3]). Recall that for any compact set $K \subset \mathbb{C}$ with $\mathbf{cap}\,(K) > 0$ there exists a unique probability measure $\mu_K, S(\mu_K) \subset K$, called the *equilibrium measure* of K, which is characterized by

$$\int \log \frac{1}{|z-x|} d\mu_K(x) \begin{cases} = \gamma, & z \in K\setminus A, \quad \mathbf{cap}\,(A) = 0, \\ \le \gamma, & z \in \mathbb{C}, \end{cases}$$

where A is a Borel set, and γ is some uniquely determined constant (actually $e^{-\gamma} = \mathbf{cap}\,(K)$).

Theorem 3. *Let $\{L_n\}$ be the sequence of monic extremal polynomials relative to* (4) *with $p \in (1,\infty)$ and $\mu_0, \mu_1 \in \mathbf{Reg}$. Then, for each integer $j > 0$*

$$\lim_{n\to\infty} \|L_n^{(j)}\|_{\Delta}^{1/n} = \mathbf{cap}\,(\Delta), \tag{9}$$

$$\underset{n\to\infty}{w\text{-}\lim}\, \sigma(L_n^{(j)}) = \mu_{\Delta}, \tag{10}$$

where $\Delta = \Delta_0 \cup \Delta_1$ and μ_Δ is the equilibrium measure on Δ.

Notice that (10) holds for $j > 0$. For $j = 0$ the zeros of the polynomials L_n can abandon Δ and their asymptotic zero distribution is governed by the balayage of μ_Δ onto a certain region which we describe later (for details, see Theorem 6).

2. The Characterization Theorem

Throughout this section, we consider the more general Sobolev norm (6) and L_n verifies (7). As $\mathbb{P}_{n-1}$ is a finite-dimensional linear space, the existence of $L_n \in \mathbb{P}_n^*$ is obvious. In addition, L_n has real coefficients, for otherwise L_n could be rewritten as $L_n = P + iQ$ where Q and P are polynomials with real coefficients, $Q \not\equiv 0$ and P is a monic polynomial of degree n satisfying

$$\|L_n\|_{S,p,m}^p = \sum_{k=0}^m \int |P^{(k)} + iQ^{(k)}|^p d\mu_k = \sum_{k=0}^m \int ((P^{(k)})^2 + (Q^{(k)})^2)^{\frac{p}{2}} d\mu_k$$

$$> \sum_{k=0}^m \int |P^{(k)}|^p d\mu_k = \|P\|_{S,p,m}^p.$$

The next proposition contains a direct proof of the uniqueness of the Sobolev extremal polynomial L_n and shows that (6) is a strictly convex norm.

Proposition 1 (Uniqueness). *Let* $\|\cdot\|_{S,p,m}$ *be the Sobolev-type norm defined by* (6)*. Then, there exists a unique monic polynomial* L_n $(\deg(L_n) = n)$ *such that* $\|L_n\|_{S,p,m} = \inf_{Q_n \in \mathbb{P}_n^*} \|Q_n\|_{S,p,m}$.

Proof. If L_n and $\widetilde{L}_n$ are two different monic extremal polynomials of degree n, from the extremality and the triangular inequality it is obvious that $\frac{1}{2}(L_n + \widetilde{L}_n)$ is also a monic extremal polynomials. Hence

$$\|L_n + \widetilde{L}_n\|_{S,p,m} = \|L_n\|_{S,p,m} + \|\widetilde{L}_n\|_{S,p,m}. \tag{11}$$

From the Minkowski inequality we obtain

$$\|L_n + \widetilde{L}_n\|_{S,p,m} = \left(\sum_{k=0}^m \|L_n^{(k)} + \widetilde{L}_n^{(k)}\|_{k,p}^p\right)^{1/p}$$

$$\le \left(\sum_{k=0}^m \left(\|L_n^{(k)}\|_{k,p} + \|\widetilde{L}_n^{(k)}\|_{k,p}\right)^p\right)^{1/p}$$

$$\le \|L_n\|_{S,p,m} + \|\widetilde{L}_n\|_{S,p,m}.$$

Therefore, by (11), the first inequality just shown is in fact the equality

$$\sum_{k=0}^m \|L_n^{(k)} + \widetilde{L}_n^{(k)}\|_{k,p}^p = \sum_{k=0}^m \left(\|L_n^{(k)}\|_{k,p} + \|\widetilde{L}_n^{(k)}\|_{k,p}\right)^p.$$

However,

$$\|L_n^{(k)} + \widetilde{L}_n^{(k)}\|_{k,p}^p \le \left(\|L_n^{(k)}\|_{k,p} + \|\widetilde{L}_n^{(k)}\|_{k,p}\right)^p, \quad k = 0, 1, \ldots, m;$$

therefore,

$$\|L_n^{(k)} + \widetilde{L}_n^{(k)}\|_{k,p} = \|L_n^{(k)}\|_{k,p} + \|\widetilde{L}_n^{(k)}\|_{k,p}, \quad k = 0, 1, \ldots, m.$$

In particular, from the Minkowski inequality for $\mathbf{L}^p(\mu_0)$, we have that there exists a constant $\alpha \geq 0$ such that $L_n = \alpha \widetilde{L}_n$ almost everywhere with respect to μ_0. But, L_n and $\widetilde{L}_n$ are monic polynomials and $\operatorname{supp}(\mu_0)$ is an infinite set, hence $L_n = \widetilde{L}_n$. □

Theorem 4 (Characterization). *Let $\|\cdot\|_{S,p,m}$ $(1 < p < \infty)$ be the Sobolev-type norm defined in* (6)*. Then, the monic polynomial L_n is the nth* monic extremal polynomial relative to $\|\cdot\|_{S,p,m}$ *if and only if*

$$\langle Q, L_n \rangle_{S,p,m} := \sum_{k=0}^{m} \int Q^{(k)} \operatorname{sgn}(L_n(x)) \, |L_n^{(k)}|^{p-1} d\mu_k = 0, \tag{12}$$

for every polynomial $Q \in \mathbb{P}_{n-1}$.

Proof. Assume that L_n is the nth *monic extremal polynomial relative to the norm* $\|\cdot\|_{S,p,m}$ and let $Q \in \mathbb{P}_{n-1}$, then

$$\|L_n\|_{S,p,m} \leq \|L_n + \alpha\, Q\|_{S,p,m} \quad \text{for all } \alpha \in \mathbb{R}. \tag{13}$$

Let $F(\alpha)$ be the auxiliary function defined for all $\alpha \in \mathbb{R}$ by the expression

$$F(\alpha) = \|L_n + \alpha Q\|_{S,p,m}^p = \sum_{k=0}^{m} \int \left| L_n^{(k)} + \alpha Q^{(k)} \right|^p d\mu_k.$$

From Proposition 1 and (13), $\alpha = 0$ is the unique minimum point of F, thus

$$0 = F'(0) = p \sum_{k=0}^{m} \int Q^{(k)} \operatorname{sgn}\big(L_n^{(k)}\big) \left| L_n^{(k)} \right|^{p-1} d\mu_k = p \, \langle Q, L_n \rangle_{S,p,m}$$

and we get (12). Now, assume that (12) takes place for every polynomial $Q \in \mathbb{P}_{n-1}$. Obviously, each monic polynomial $\widetilde{Q}$ of degree n can be written as the sum $\widetilde{Q} = L_n + Q$ where $Q \in \mathbb{P}_{n-1}$.

Let q be the conjugate exponent of p, i.e. $q = \frac{p}{p-1}$. For $k = 0, 1, \ldots, m$ we have

$$G_k = \operatorname{sgn}\big(L_n^{(k)}\big) \, |L_n^{(k)}|^{p-1} \in \mathbf{L}^q(\mu_k), \qquad \int |G_k|^q d\mu_k = \|L_n^{(k)}\|_{k,p}^p.$$

Thus

$$\|L_n\|_{S,p,m}^p = \sum_{k=0}^{m} \|L_n^{(k)}\|_{k,p}^p = \sum_{k=0}^{m} \int |G_k|^q d\mu_k = \sum_{k=0}^{m} \|G_k\|_{k,q}^q. \tag{14}$$

Let $\alpha, \beta \geq 0$, $p > 1$ and $q = \frac{p}{p-1}$. It is well known that

$$\sqrt[p]{\alpha} \cdot \sqrt[q]{\beta} \leq \frac{\alpha}{p} + \frac{\beta}{q}, \tag{15}$$

with equality if and only if $\alpha = \beta$ (cf. [13, §2.1.1 and Theorem 2]). From (12), Hölder's inequality, (15) and (14), we get

$$\begin{aligned}\|L_n\|_{S,p,m}^p &= \langle L_n, L_n\rangle_{S,p,m} = \langle L_n + Q, L_n\rangle_{S,p,m} = \langle \widetilde{Q}, L_n\rangle_{S,p,m}, \\ &= \sum_{k=0}^{m} \int \widetilde{Q}^{(k)} G_k d\mu_k, \leq \sum_{k=0}^{m} \|\widetilde{Q}^{(k)}\|_{k,p} \cdot \|G_k\|_{k,q}, \\ &\leq \sum_{k=0}^{m} \left(\frac{\|\widetilde{Q}^{(k)}\|_{k,p}^p}{p} + \frac{\|G_k\|_{k,q}^q}{q} \right) = \frac{\|\widetilde{Q}\|_{S,p,m}^p}{p} + \frac{\|L_n\|_{S,p,m}^p}{q}.\end{aligned}$$

Thus $\|L_n\|_{S,p,m} \leq \|\widetilde{Q}\|_{S,p,m}$, which completes the proof. □

Corollary 1. *Under the assumptions of Theorem* 4, *if* $n \geq 1$ *then* L_n *has at least one zero of odd multiplicity* $x_0 \in \overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0))$.

Proof. This is an immediate consequence of

$$\int \operatorname{sgn}(L_n)\ |L_n|^{p-1} d\mu_0 = \langle 1, L_n\rangle_{S,p,m} = 0.$$ □

Corollary 2. *Under the assumptions of Theorem* 4, *if* $n \geq 2$ *then* L_n' *has at least one zero of odd multiplicity in* $\overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0) \cup \operatorname{supp}(\mu_1))$.

Proof. If L_n' has no zeros of odd multiplicity on $\overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0) \cup \operatorname{supp}(\mu_1))$, then L_n is monotone on $\overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0) \cup \operatorname{supp}(\mu_1))$. From Corollary 1, L_n has exactly one zero x_0 of odd multiplicity on $\overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0))$, so

$$\operatorname{sgn}((x - x_0)L_n(x)) = \operatorname{sgn}(L_n'(x)) = c$$

is constant for all $x \in \overset{\circ}{\boldsymbol{Co}}(\operatorname{supp}(\mu_0))$ with $c = \pm 1$. Hence, by Theorem 4 we have

$$\begin{aligned}0 &= \langle c(x - x_0), L_n\rangle_{S,p,m} \\ &= \int c(x - x_0)\operatorname{sgn}(L_n)\ |L_n|^{p-1} d\mu_0 + \int c \cdot \operatorname{sgn}(L_n')\ |L_n'|^{p-1} d\mu_1 > 0\end{aligned}$$

which is a contradiction. □

3. Two Disjoint Intervals: Proof of Theorem 1

Here, we prove the results on the location of zeros and critical points announced previously. The first lemma in this section is a consequence of Biernacki's theorem [14, Theorem 4.5.2], which in turn is a converse of the Gauss–Lucas theorem.

Lemma 1 (Biernacki [14, Theorem 4.5.2]). *Let* K *be the convex hull of the critical points of a polynomial* f, *and let* $f(z_0) = 0$. *Then, the zeros of* f *lie in the* *union of all the closed disks centered at the vertices of K and radius equal to the distance from the vertex to z_0.*

Let $I \subset \mathbb{R}$ be an interval and $Q \in \mathbb{P}$. As in Theorem 1, $\mathbf{N}_{\mathrm{o}}(Q; I)$ denotes the number of zeros of Q on I with odd multiplicity (i.e. points of sign change). Additionally, $\mathbf{N}_z(Q; I)$ denotes the total number of zeros (counting multiplicities) of Q on I and for all $n \geq 1$ we write $\ell_n := \mathbf{N}_{\mathrm{o}}(L_n; \overset{\circ}{\Delta}_0) + \mathbf{N}_{\mathrm{o}}(L_n'; \overset{\circ}{\Delta}_1)$. The key to the proof of Corollary 3 is the following trivial consequence of Rolle's theorem.

Lemma 2 ([9, Lemma 2.1]). *Let I be an interval of the real line and Q a non-constant polynomial of degree n with real coefficients, then $\mathbf{N}_z(Q; I) + \mathbf{N}_z(Q'; \mathbb{C}\backslash I) \leq n$.*

Proof of Theorem 1. For $n = 1, 2$ the statements of the lemma are immediate consequences of Corollaries 1 and 2. So, in the sequel we assume that $n \geq 3$. From Lemma 2 we have $\ell_n \leq n$. The simplicity of the zeros of L_n in $\overset{\circ}{\Delta}_0$ follows directly from the inequality $\ell_n \geq n - 1$ and Lemma 2. Therefore, to complete the proof of the statement (1.1) it suffices to show that

$$\ell_n \geq n - 1. \tag{16}$$

Without loss of generality, we can assume that $\overset{\circ}{\Delta}_0 = (a, b)$, $\overset{\circ}{\Delta}_1 = (c, d)$ and $-\infty \leq a < b \leq c \leq d \leq \infty$, the case $d \leq a$ is solved similarly.

Fix $n \geq 3$ and let x_0 be the point in (a, b) closest to $[c, d]$ where L_n changes sign. This point exists due to Corollary 1. There are two possible cases, either

$$\operatorname{sgn}\left(L_n'(x_0 + \epsilon) \cdot L_n'(c + \epsilon)\right) = 1 \tag{I}$$

for all sufficiently small $\epsilon > 0$, or

$$\operatorname{sgn}\left(L_n'(x_0 + \epsilon) \cdot L_n'(c + \epsilon)\right) = -1 \tag{II}$$

for all sufficiently small $\epsilon > 0$. Let us consider each case separately.

In case I we can prove more than (16); namely,

$$\ell_n = n. \tag{17}$$

To the contrary, suppose that $\ell_n \leq n - 1$ in case I or $\ell_n \leq n - 2$ in case II. We shall see that we can find a polynomial $Q \in \mathbb{P}_{n-1}$ such that

$$Q(x) L_n(x) \geq 0, \quad x \in [a, b] \quad \text{and} \quad Q'(x) L_n'(x) \geq 0, \quad x \in [c, d]. \tag{18}$$

Suppose that (18) holds, using Theorem 2 we get

$$\begin{aligned} 0 &= \langle Q, L_n \rangle_{S,p,1} \\ &= \int_a^b Q \operatorname{sgn}(L_n)\, |L_n|^{p-1} d\mu_0 + \int_c^d Q' \operatorname{sgn}(L_n')\, |L_n'|^{p-1} d\mu_1 > 0, \end{aligned}$$

which is a contradiction and the proof of (1.1) would be complete. Therefore, it is sufficient to find such a polynomial Q.

Case I. Suppose that $\ell_n \leq n-1$ and take Q to be a polynomial of degree $\leq \ell_n$ with real coefficients, not identically equal to zero, which has a zero at each point of (a,b) where L_n changes sign and whose derivative has a zero at each point of (c,d) where L_n' changes sign. The existence of Q reduces to solving a homogeneous linear system of ℓ_n equations on ℓ_n+1 unknowns (the coefficients of Q); thus, a nontrivial solution always exists. Notice that

$$\ell_n \leq \mathbf{N}_z(Q;(a,b)) + \mathbf{N}_z(Q';(c,d))$$

with strict inequality if either Q (respectively, Q') has on (a,b) (respectively, (c,d)) zeros of multiplicity greater than one or distinct from those assigned by construction. On the other hand, because of Corollary 1 the degree of Q is at least 1; therefore, using Lemma 2, we have that

$$\ell_n \leq \mathbf{N}_z(Q;(a,b)) + \mathbf{N}_z(Q';(c,d)) \leq \deg(Q) \leq \ell_n.$$

Thus

$$\ell_n = \mathbf{N}_z(Q;(a,b)) + \mathbf{N}_z(Q';(c,d)) = \deg(Q)\,. \tag{19}$$

Hence Q (respectively, Q') has on (a,b) (respectively, (c,d)) simple zeros and has no other zero different from those given by construction. So, QL_n and $Q'L_n'$ have constant sign on $[a,b]$ and $[c,d]$, respectively. We can choose Q in such a way that $QL_n \geq 0$ on $[a,b]$ (if this was not so replace Q by $-Q$). Then, to prove (18) it remains to check that $\operatorname{sgn}(Q'(c+\epsilon)L_n'(x_0+\epsilon)) = 1$ for all ϵ sufficiently small. From Rolle's Theorem and (19) we have

$$\begin{aligned}\ell_n - 1 &= \mathbf{N}_{\mathrm{o}}(Q;(a,x_0)) + \mathbf{N}_{\mathrm{o}}(Q';(c,d))\\ &\leq \mathbf{N}_{\mathrm{o}}(Q';(a,x_0)) + \mathbf{N}_{\mathrm{o}}(Q';(c,d))\\ &\leq \ell_n - 1.\end{aligned}$$

Hence $\mathbf{N}_{\mathrm{o}}(Q';(a,x_0)) + \mathbf{N}_{\mathrm{o}}(Q';(c,d)) = \deg(Q')$ and all the zeros of Q' are contained in $(a,x_0)\cup(c,d)$. So, for all ϵ sufficiently small, we have

$$\operatorname{sgn}(Q'(x_0+\epsilon)\cdot Q'(c+\epsilon)) = 1.$$

Now, from this expression and (I), we obtain

$$\begin{aligned}\operatorname{sgn}(Q'(c+\epsilon)L_n'(c+\epsilon)) &= \operatorname{sgn}(Q'(c+\epsilon))\operatorname{sgn}(L_n'(c+\epsilon))\\ &= \operatorname{sgn}(Q'(x_0+\epsilon))\operatorname{sgn}(L_n'(x_0+\epsilon))\\ &= \operatorname{sgn}(Q(x_0+\epsilon))\operatorname{sgn}(L_n(x_0+\epsilon))\\ &= \operatorname{sgn}(Q(x_0+\epsilon)L_n(x_0+\epsilon)) = 1.\end{aligned}$$

Therefore, we get (18) and hence (17) ($\ell_n = n$).

In order to prove the remaining statements, notice that

$$\begin{aligned} n-1 = \ell_n - 1 &= \mathbf{N}_{\mathrm{o}}(L_n;(a,x_0)) + \mathbf{N}_{\mathrm{o}}(L_n';(c,d)) \\ &\le \mathbf{N}_{\mathrm{o}}(L_n';(a,x_0)) + \mathbf{N}_{\mathrm{o}}(L_n';(c,d)) + \mathbf{N}_{\mathrm{o}}(L_n';(x_0,c]) \le n-1. \end{aligned}$$

Therefore,

$$\mathbf{N}_{\mathrm{o}}(L_n';(a,d)) = n-1, \quad \mathbf{N}_{\mathrm{o}}(L_n';(x_0,c]) = 0 \quad \text{and}$$
$$\mathbf{N}_{\mathrm{o}}(L_n;(a,x_0)) = \mathbf{N}_{\mathrm{o}}(L_n';(a,x_0))$$

which proves (1.2)–(1.4).

Case II. Suppose that $\ell_n \le n-2$. The difference consists in that to the right of x_0 and c the polynomial L_n' has different signs. Here, we construct Q of degree $\le \ell_n + 1 \le n-1$ with real coefficients, not identically equal to zero, with the same interpolation conditions as above plus $Q'(c) = 0$. Following the same line of reasoning, we have

$$\ell_n + 1 = \mathbf{N}_z(Q;(a,b)) + \mathbf{N}_z(Q';[c,d)) = \deg(Q)\,.$$

Hence Q (respectively, Q') has on (a,b) (respectively, $[c,d)$) simple zeros and no other zero except those given by construction. So QL_n and $Q'L_n'$ have constant sign on $[a,b]$ and $[c,d]$, respectively. Analogous to the previous case, $\mathbf{N}_{\mathrm{o}}(Q';(a,x_0)) + \mathbf{N}_{\mathrm{o}}(Q';[c,d)) = \deg(Q')$ and all the zeros of Q' are contained in $(a,x_0) \cup [c,d)$. Now, using that Q changes sign at c and (II) we obtain

$$\begin{aligned} \operatorname{sgn}(Q'(c+\epsilon)L_n'(c+\epsilon)) &= \operatorname{sgn}(Q'(c+\epsilon))\operatorname{sgn}(L_n'(c+\epsilon)) \\ &= -\operatorname{sgn}(Q'(x_0+\epsilon))\,(-\operatorname{sgn}(L_n'(x_0+\epsilon))) \\ &= \operatorname{sgn}(Q(x_0+\epsilon))\operatorname{sgn}(L_n(x_0+\epsilon)) \\ &= \operatorname{sgn}(Q(x_0+\epsilon)L_n(x_0+\epsilon)) = 1 \end{aligned}$$

which proves that Q satisfies (18) and hence (16) is true.

Now, notice that (II) and the intermediate value theorem imply that L_n' has at least an odd zero on the interval $(x_0,c]$, thus

$$\begin{aligned} n-1 \le (\ell_n - 1) + 1 &= \mathbf{N}_{\mathrm{o}}(L_n;(a,x_0)) + \mathbf{N}_{\mathrm{o}}(L_n';(c,d)) + 1 \\ &\le \mathbf{N}_{\mathrm{o}}(L_n';(a,x_0)) + \mathbf{N}_{\mathrm{o}}(L_n';(c,d)) + \mathbf{N}_{\mathrm{o}}(L_n';(x_0,c]) \le n-1. \end{aligned}$$

Therefore,

$$\mathbf{N}_{\mathrm{o}}(L_n';(a,d)) = n-1, \quad \mathbf{N}_{\mathrm{o}}(L_n';(x_0,c]) = 1 \quad \text{and}$$
$$\mathbf{N}_{\mathrm{o}}(L_n;(a,x_0)) = \mathbf{N}_{\mathrm{o}}(L_n';(a,x_0))$$

from which (1.2)–(1.4) follow. □

Corollary 3. *Let $p \in (1, \infty)$ and let μ_0, μ_1 be finite positive Borel measures defined on the real line such that $\overset{\circ}{\Delta}_0 \cap \overset{\circ}{\Delta}_1 = \emptyset$. If we take $[a, b] = \Delta_0$ and $[c, d] = \Delta_1$, where $a,\ b,\ c,\ d \in \mathbb{R}$, then the zeros of L_n lie in*

$$D_\Delta = \begin{cases} \mathbf{D}(d, d-a) \cup \mathbf{D}(a, b-a) & \text{if } b \leq c, \\ \mathbf{D}(c, b-c) \cup \mathbf{D}(b, b-a) & \text{if } d \leq a, \end{cases} \tag{20}$$

where $\mathbf{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

Proof. For $n = 1$ the statement is certainly true because of Corollary 1. For $n \geq 2$ the result follows directly from (1.2) in Theorem 1 and the Biernacki Lemma 1. □

4. Regular Asymptotic Distribution of Critical Points

A compact set K of the complex plane is said to be *regular* if the Green function with singularity at ∞ relative to the unbounded connected component of $\mathbb{C}\backslash K$ can be extended continuously to the boundary. We refer the reader to [7, 15] and for short [18, Appendix] for this and other notions related with logarithmic potential theory. For example, the union of a finite number of bounded intervals in the real line form regular compact sets.

Suppose that μ is a finite positive Borel measure such that $S(\mu)$ is a regular compact set and $1 \leq p < \infty$. It is well known (see [18, Theorem 3.4.3]) that $\mu \in \mathbf{Reg}$ if and only if

$$\lim_{n\to\infty} \left(\frac{\|Q_n\|_{S(\mu)}}{\|Q_n\|_{\mu,p}} \right)^{1/n} = 1, \tag{21}$$

where $\{Q_n\}, n \in \mathbb{Z}_+$, is any sequence of polynomials such that $\deg(Q_n) = n$ and $\|\cdot\|_{S(\mu)}$ denotes the usual sup norm on $S(\mu)$. Using Cauchy's integral theorem for the derivative of a holomorphic function, with the same hypothesis on $S(\mu)$ it is easy to show (see [11, Lemma 3]) that for all $j \in \mathbb{Z}_+$

$$\limsup_{n\to\infty} \left(\frac{\|Q_n^{(j)}\|_{S(\mu)}}{\|Q_n\|_{S(\mu)}} \right)^{1/n} \leq 1. \tag{22}$$

One last result that we will used is contained in [2, Theorem 2.1, Corollary 2.1]. Let K be a compact subset of the real line with $\mathbf{cap}\,(K) > 0$ and let $\{Q_n\}$ be a sequence of monic polynomials, $\deg(Q_n) = n$. Then

$$\limsup_{n\to\infty} \|Q_n\|_K^{1/n} = \mathbf{cap}\,(K) \Rightarrow w\text{-}\lim_{n\to\infty} \sigma(Q_n) = \mu_K. \tag{23}$$

Proof of Theorem 3. From (23), (10) follows from (9). Therefore, let us prove (9).

Let T_n denote the nth monic Chebyshev polynomial with respect to Δ; that is, $\|T_n\|_\Delta \le \|Q_n\|_\Delta$ for any monic polynomial Q_n of degree n. In particular,

$$\begin{aligned}\liminf_{n\to\infty} \|L_n^{(j)}\|_\Delta^{1/n} &\ge \liminf_{n\to\infty} \left(\frac{n!}{(n-j)!}\|T_{n-j}\|_\Delta\right)^{1/n} \\ &\ge \liminf_{n\to\infty} \|T_{n-j}\|_\Delta^{1/n} = \mathbf{cap}\,(\Delta)\,,\end{aligned}$$

since it is well known that $\lim_{n\to\infty} \|T_n\|_\Delta^{1/n} = \mathbf{cap}\,(\Delta)$ (see, for example [15, Corollary 5.5.5]). Hence, we only need to prove that

$$\limsup_{n\to\infty} \|L_n^{(j)}\|_\Delta^{1/n} \le \mathbf{cap}\,(\Delta)\,. \tag{24}$$

Since $\mu_0, \mu_1 \in \mathbf{Reg}$ and Δ_0, Δ_1 are regular compact sets, from (21) and (22) we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|L_n^{(j)}\|_{\Delta_0}^{p/n} &\le \limsup_{n\to\infty} \|L_n\|_{\Delta_0}^{p/n} \le \limsup_{n\to\infty} \|L_n\|_{\mu_0,p}^{p/n} \\ &\le \limsup_{n\to\infty} \|L_n\|_{S,p}^{p/n}\,,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \|L_n^{(j)}\|_{\Delta_1}^{p/n} &\le \limsup_{n\to\infty} \|L_n'\|_{\Delta_1}^{p/n} \le \limsup_{n\to\infty} \|L_n'\|_{\mu_1,p}^{p/n} \\ &\le \limsup_{n\to\infty} \|L_n\|_{S,p}^{p/n}.\end{aligned}$$

The conjunction of these two relations imply that

$$\limsup_{n\to\infty} \|L_n^{(j)}\|_\Delta^{1/n} \le \limsup_{n\to\infty} \|L_n\|_{S,p}^{1/n}. \tag{26}$$

Now, from the extremality of L_n in the Sobolev norm, we get

$$\begin{aligned}\|L_n\|_{S,p}^p \le \|T_n\|_{S,p}^p &= \|T_n\|_{\mu_0,p}^p + \|T_n'\|_{\mu_1,p}^p \\ &\le \mu_0(\Delta_0)\,\|T_n\|_{\Delta_0}^p + \mu_1(\Delta_1)\,\|T_n'\|_{\Delta_1}^p \\ &\le \mu_0(\Delta_0)\,\|T_n\|_{\Delta}^p + \mu_1(\Delta_1)\,\|T_n'\|_{\Delta}^p.\end{aligned}$$

On the other hand, using again (21) and (22) it follows that

$$\limsup_{n\to\infty} \left(\frac{\|L_n\|_{S,p}}{\|T_n\|_\Delta}\right)^{1/n} \le \limsup_{n\to\infty} \left(\mu_0(\Delta_0) + \mu_1(\Delta_1)\frac{\|T_n'\|_\Delta^p}{\|T_n\|_\Delta^p}\right)^{1/pn} \le 1,$$

whence

$$\limsup_{n\to\infty} \|L_n\|_{S,p}^{1/n} \le \mathbf{cap}\,(\Delta)\,. \tag{27}$$

Now, (26) and (27) give (24) and we are done. ☐

Remark 1. If $\Delta_0 = [a, b]$ and $\Delta_1 = [c, d]$ are bounded and nontrivial intervals of $\mathbb{R}$, for the equilibrium measure μ_Δ in Theorem 3, we have two possibilities:

- If $\Delta_0 \cap \Delta_1 \neq \emptyset$, then $\Delta = [\alpha, \beta]$ (an interval) and μ_Δ is the arcsine measure

$$d\mu_\Delta = \frac{dx}{\pi\sqrt{|(x-\alpha)(x-\beta)|}}, \quad x \in (\alpha, \beta). \tag{28}$$

- If $\Delta_0 \cap \Delta_1 = \emptyset$ then there exists $x_* \in \Delta_g = \overset{\circ}{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1) \setminus (\Delta_0 \cup \Delta_1)$ such that

$$\int_{\Delta_g} \frac{(x - x_*)\, dx}{\sqrt{|(x-a)(x-b)(x-c)(x-d)|}} = 0, \tag{29}$$

and (see [18, Lemma 4.4.1])

$$d\mu_\Delta = \frac{|x - x_*|\, dx}{\pi\sqrt{|(x-a)(x-b)(x-c)(x-d)|}}, \quad x \in (a,b) \cup (c,d). \tag{30}$$

Combining Theorems 1 and 3 we obtain the following theorem.

Theorem 5. *Let $\mu_0, \mu_1 \in$ **Reg** be such that Δ_0 and Δ_1 are bounded and nontrivial intervals of $\mathbb{R}$ satisfying $\overset{\circ}{\Delta}_0 \cap \overset{\circ}{\Delta}_1 = \emptyset$. Denote by $\{L_n\}$ the sequence of monic extremal polynomials relative to the corresponding Sobolev norm (4) with $p \in (1, \infty)$. Then, for all $j > 0$*

$$\limsup_{n\to\infty} |L_n^{(j)}(z)|^{1/n} = \mathbf{cap}\,(\Delta)\, e^{g_\Omega(z;\infty)}, \quad z \in \mathbb{C} \tag{31}$$

except on a set of zero capacity, where $g_\Omega(z;\infty)$ is the Green's function for $\Omega = \overline{\mathbb{C}} \setminus \Delta$ with singularity at infinity (cf. [18, Appendix V]).

Moreover, uniformly on compact subset of $\widehat{\Omega} = \mathbb{C} \setminus \boldsymbol{Co}(\Delta_0 \cup \Delta_1)$

$$\lim_{n\to\infty} \left|L_n^{(j)}(z)\right|^{1/n} = \mathbf{cap}\,(\Delta)\, e^{g_\Omega(z;\infty)}, \tag{32}$$

and

$$\lim_{n\to\infty} \frac{L_n^{(j+1)}(z)}{nL_n^{(j)}(z)} = \int \frac{d\mu_\Delta(x)}{z - x}, \tag{33}$$

where $d\mu_\Delta$ is given by (30).

Remark 2. To give an explicit expressions for the function on the right side of (31) and (32), we assume without loss of generality that $\Delta_0 = [a,b]$ and $\Delta_1 = [c,d]$ with $-\infty < a < b \leq c < d < \infty$. As we show below, there are closed formulas for Green's function $g_\Omega(z;\infty)$ and the logarithmic capacity of Δ. When both segments are symmetrical, $g_\Omega(z;\infty)$ and $\mathbf{cap}\,(\Delta)$ are given by elementary formulas (see (37)). In general, (formulas (35) and (36)), they can be expressed in terms of theta-functions (see [16, 17]) defined by

$$\theta(u,\tau,r,s) = \sum_{k\in\mathbb{Z}} e^{2\pi i((k+r)^2\frac{\tau}{2}+(k+r)(u+s))},$$

where $u,\tau \in \mathbb{C}$, $\mathrm{Im}\,(\tau) > 0$ and $r,s \in \mathbb{R}$. Here, we will be particularly interested in the functions

$$\vartheta(u,\tau) = \theta\left(u,\tau,\frac{1}{2},\frac{1}{2}\right) \quad \text{and} \quad \vartheta_0(\tau) = \theta\left(0,\tau,0,\frac{1}{2}\right). \tag{34}$$

Following the procedure for computing the logarithmic capacity of two segments given in [5, §1.3.3; 16, Chap. 2], let $\Psi(z)$ be the function

$$\Psi(z) = \sqrt{\frac{(z-a)(d-b)}{(z-b)(d-a)}},$$

where $\sqrt{z} > 0$ for $z > 0$, and $\Upsilon(w)$ is the elliptic integral

$$\Upsilon(z) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-\upsilon^2x^2)}}, \quad \text{where } \upsilon = \Psi(c)^{-1}.$$

Putting $\Phi(z) = \Upsilon(\Psi(z))$ and $\tau = \frac{\Phi(c)}{\Phi(d)}$, we obtain

$$g_\Omega(z;\infty) = -\log\left|\frac{\vartheta((2\Phi(d))^{-1}(\Phi(z)-\Phi(\infty)),\tau)}{\vartheta((2\Phi(d))^{-1}(\Phi(z)+\overline{\Phi(\infty)}),\tau)}\right|, \tag{35}$$

and

$$\mathbf{cap}\,(\Delta) = \left|\frac{\vartheta_0(\tau)\ \sqrt[4]{(c-a)(c-b)(d-a)(d-b)}}{2\ \vartheta(\Phi^{-1}(d)\mathrm{Re}\,(\Phi(\infty))\,,\tau)}\right|. \tag{36}$$

We recall that $a < b < c < d$. Hence if $-a = d$ and $-b = c$ for Green's function and the logarithmic capacity, we obtain

$$g_\Omega(z;\infty) = \frac{1}{2}\log^+\left|\frac{\sqrt{z^2-b^2}+\sqrt{z^2-a^2}}{\sqrt{z^2-b^2}-\sqrt{z^2-a^2}}\right| \quad \text{and} \quad \mathbf{cap}\,(\Delta) = \frac{1}{2}\sqrt{a^2-b^2}, \tag{37}$$

where

$$\log^+|z| = \begin{cases} \log|z| & \text{if } |z| > 1, \\ 0 & \text{if } |z| \le 1. \end{cases}$$

Proof of Theorem 5. From Theorem 1, we have that for all $n \ge 2$, the critical points of the extremal polynomial L_n are simple and contained in $\mathring{\boldsymbol{Co}}\,(\Delta_0 \cup \Delta_1)$. Now, Rolle's theorem implies that the zeros of all derivatives of higher order of L_n lie in the convex hull of the set of its critical points. Therefore, for all $n \ge 2$ and $j \ge 1$, the $(n-j)$ zeros $\{x_k^{(j)}\}$ of $L_n^{(j)}$ lie on $\mathring{\boldsymbol{Co}}\,(\Delta_0 \cup \Delta_1)$. Thus, for each fixed

$j \geq 1$ the measure $\sigma(L_n^{(j)})$ has its support contained in $\overset{\circ}{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1)$. From the lower envelope theorem (cf. [18, Appendix III]) and (10), we get

$$\liminf_{n \to \infty} \int \log \frac{1}{|z-x|} \, d\sigma(L_n^{(j)})(x) = \int \log \frac{1}{|z-x|} \, d\mu_\Delta(x), \tag{38}$$

for all $z \in \mathbb{C}$ except on a set of zero capacity. But, from [18, Chap. 1-(2.3)]

$$\int \log \frac{1}{|z-x|} \, d\mu_\Delta(x) = \log \frac{1}{\mathbf{cap}\,(\Delta)} - g_\Omega(z; \infty),$$

hence (38) is equivalent to (31).

In order to prove (32), notice that for each fixed $j \geq 1$, the family of functions

$$\left\{ \int \log \frac{1}{|z-x|} \, d\sigma(L_n^{(j)})(x) \right\}, \quad n \in \mathbb{Z}_+,$$

is harmonic and uniformly bounded on compact subsets of $\widehat{\Omega}$. From (31), any subsequence which converges uniformly on compact subsets of $\widehat{\Omega}$ must tend to $\int \log |z-x|^{-1} \, d\mu_\Delta(x)$ (independent of the convergent subsequence chosen). Therefore, due to the uniform boundedness of the sequence of functions, the whole sequence converges uniformly on compact subsets of $\widehat{\Omega}$ to this function. This is equivalent to (32).

Finally, expanding the rational function in the right-hand side of (33) in partial fractions, we get

$$\frac{L_n^{(j+1)}(z)}{nL_n^{(j)}(z)} = \frac{1}{n} \sum_{k=1}^{n-j} \frac{1}{z - x_k^{(j)}} = \frac{n-j}{n} \int \frac{d\sigma(L_n^{(j)})(x)}{z-x}.$$

Hence, it is straightforward that for each fixed $j \geq 1$, the sequence of rational functions $\{L_n^{(j+1)}(z)/nL_n^{(j)}(z)\}$ is uniformly bounded on each compact subset of $\widehat{\Omega}$.

As, all the measures $\sigma(L_n^{(j)})$, are supported in $\overset{\circ}{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1)$ then for a fixed $z \in \widehat{\Omega}$, the function $f_z(x) = (z-x)^{-1}$ is continuous on $\overset{\circ}{\boldsymbol{Co}}(\Delta_0 \cup \Delta_1)$. Therefore, from (10), we find that any subsequence of $\{L_n^{(j+1)}(z)/nL_n^{(j)}(z)\}$ which converges uniformly on compact subsets of $\widehat{\Omega}$ converges pointwise to

$$\int \frac{d\mu_\Delta(x)}{z-x}. \tag{39}$$

Thus, the whole sequence converges uniformly on compact subsets of $\widehat{\Omega}$ to this function as stated in (33). The expression of μ_Δ is given in (30).

To discuss the zeros of Sobolev extremal polynomials L_n, we need to introduce some notation. Recall that $\Omega = \overline{\mathbb{C}} \setminus \Delta$ and that in the case we are now considering $g_\Omega(z, \infty)$ is given by (35). For $\rho > 0$, consider the set $\{z \in \mathbb{C} : g_\Omega(z, \infty) < \rho\}$. Depending on ρ this set has either one connected component (when ρ is large enough or two connected components one of which contains Δ_0 and the other Δ_1.

Let G_ρ be the connected components of $\{z \in \mathbb{C} : g_\Omega(z,\infty) < \rho\}$ which is disjoint from Δ_0. Therefore, $G_\rho = \emptyset$ for all $\rho > \rho_0 > 0$. Set $G = \bigcup_{\rho>0} G_\rho$ and

$$\Gamma = \partial G \cup \Delta_0,$$

where ∂G is the boundary of G. In the sequel, $[\mu]_\Gamma$ denotes the balayage of a measure μ onto Γ. See [18, Appendix A:VII] for a brief introduction to the notion of balayage of a measure and for more details we refer to [7, Chap. IV]. □

Theorem 6. *Under the hypotheses of Theorem* 5, *let* σ *be a limit of a subsequence of* $\{\sigma(L_n)\}$ *in the sense of the weak star topology of measures. Then* $S(\sigma) \subset \Delta \cup \overline{G}$ *and* $[\sigma]_\Gamma = [\mu_\Delta]_\Gamma$.

Proof. The proof of this result is similar to that of [6, Theorem 2] with $p = 2$. The main tool in the proof of [6, Theorem 2] is [6, Theorem 5] on the distribution of zeros of certain family of weighted polynomials. For the application of [6, Theorem 5] it is necessary to proof that if $z \in \mathbb{C}\backslash G$ then

$$\limsup_{n\to\infty} |L_n(z)|^{1/n} \leq \mathbf{cap}\,(\Delta)\ e^{g_\Omega(z,\infty)}. \tag{40}$$

Replacing in the proof of [6, Lemma 8] the expressions [6, (4.1), (4.2)] by (9), (25) and (27), it is straightforward to deduce (40). □

It is to be expected (and numerical experiments seem to indicate) that the accumulation points of the zeros of the polynomials L_n draw Γ.

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