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# Iterated Integrals of Jacobi Polynomials

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**Abstract**

Let $P_n^{(\alpha,\beta)}$ be the $n$-th monic Jacobi polynomial with $\alpha,\beta > -1$. Given $m$ numbers $\omega_1,\dots,\omega_m \in \mathbb{C}\setminus[-1,1]$, let $\Omega_m = (\omega_1,\dots,\omega_m)$ and $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ be the $m$-th iterated integral of $\frac{(n+m)!}{n!}\,P_n^{(\alpha,\beta)}$ normalized by the conditions

$$\frac{d^k\,\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}) = 0, \text{ for } k = 0,1,\dots,m-1.$$

The main purpose of the paper is to study the algebraic and asymptotic properties of the sequence of monic polynomials $\{\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}\}_n$. In particular, we obtain the relative asymptotic for the ratio of the sequences $\{\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}\}_n$ and $\{P_n^{(\alpha,\beta)}\}_n$. We prove that the zeros of these polynomials accumulate on a suitable ellipse.

## 1 Introduction

There is an extensive literature on the location of critical points of polynomials in terms of their zeros ([19, Part I] and [21]), whose main pillars are Rolle's Theorem, Gauss-Lucas Theorem, and their refinements. However, actual converses of these theorems have not been found yet. It is obvious that given one of the zeros of a polynomial and its critical points, the remaining zeros are uniquely determined. Nonetheless, there are few results about zero location of polynomials in terms of their critical points and a given zero, most of them contained in [19, §4.5]. In general, these follow from the Schur-Szegő composition theorem [19, Th. 3.4.1d]. Perhaps, the most relevant results in this sense are the theorems of Walsh [19, Th. 4.5.1] and Biernacki [19, Th. 4.5.2].

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Let $P_n^{(\alpha,\beta)}$ be the $n$-th monic Jacobi polynomial with parameters $\alpha, \beta \in \mathbb{R}$

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} \binom{2n+\alpha+\beta}{n}^{-1} (z-1)^k (z+1)^{n-k}, \tag{1}$$

where $2n+\alpha+\beta \neq 0, 1, \ldots, n-1$, $\binom{a}{b} = \Gamma(a+1)/(\Gamma(a-b+1)\Gamma(b+1))$ and $\Gamma(\cdot)$ is the usual Gamma function (see [22, (4.21.6) and (4.3.2)] for more details). These classical polynomials have been extensively used in mathematical analysis and practical applications (cf. [22, 20, 23]). Nowadays, there has been renewed interest in using the Jacobi polynomials in the numerical solution of differential equations. Some of these methods require explicit expressions of the integral of such polynomials and the localization of their zeros (e.g. see [4, 5]). Another area that demands this knowledge is the study of families of polynomials orthogonal in a non-standard sense, particularly the Sobolev-type orthogonality and the orthogonality with respect to a differential operator (e.g. [3, 6, 17] ).

It is well known that $P_n^{(\alpha,\beta)}$ satisfies the following differentiation relation

$$\frac{d^k P_n^{(\alpha,\beta)}}{dz^k}(z) = \frac{n!}{(n-k)!} P_{n-k}^{(\alpha+k,\beta+k)}(z), \quad 0 \leq k \leq n, \tag{2}$$

(see [22, (4.21.6)-(4.21.7)] for details). Additionally, if $\alpha, \beta > -1$, the family of polynomials $\{P_n^{(\alpha,\beta)}\}$ is orthogonal in $[-1,1]$ with respect to the weight $w(x) = (1-x)^\alpha (1+x)^\beta$.

For a fixed $m \in \mathbb{Z}_+$, let $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ be the monic polynomial of degree $n+m$ given by

$$\mathscr{P}_{n,m}^{(\alpha,\beta)} = P_{n+m}^{(\alpha-m,\beta-m)}.$$

Then,

$$\frac{d^m \mathscr{P}_{n,m}^{(\alpha,\beta)}}{dz^m}(z) = \frac{(n+m)!}{n!} P_n^{(\alpha,\beta)}(z),$$

and thus $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ is the $m$-th iterated integral (or a primitive of order $m$) of $\frac{(n+m)! P_n^{(\alpha,\beta)}}{n!}$. In what follows, we shall refer to $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ as the *$m$-th fundamental iterated integral of* $\frac{(n+m)!}{n!} P_n^{(\alpha,\beta)}$.

Given $m$ complex numbers $\omega_1, \ldots, \omega_m$, let $\Omega_k = (\omega_1, \ldots, \omega_k)$ for $1 \leq k \leq m$, and $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ be the $m$-th iterated integral of $\frac{(n+m)!}{n!} P_n^{(\alpha,\beta)}$ normalized by the conditions

$$\frac{d^k \mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}) = 0, \quad k = 0, 1, \ldots, m-1. \tag{3}$$

It is well known that there exists a unique polynomial of degree at most $m-1$, $\mathscr{A}_{n,m}(z) = \mathscr{A}_{n,m}(z; \omega_1, \ldots, \omega_m)$, satisfying the conditions

$$\frac{d^k \mathscr{A}_{n,m}}{dz^k}(\omega_{m-k}) = \frac{d^k \mathscr{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}), \quad k = 0, 1, \ldots, m-1. \tag{4}$$

The polynomial $\mathscr{A}_{n,m}$ is named the *Abel-Goncharov interpolation polynomial*, associated to the conditions (4). The existence and uniqueness of $\mathscr{A}_{n,m}$ is obvious if we observe that (4) is a triangular system of $m$ equations and $m$ unknowns (the coefficients of $\mathscr{A}_{n,m}$) whose determinant is equal to $\prod_{k=0}^{m-1} k!$. The Abel-Goncharov interpolation polynomials are a generalization of Taylor's polynomials, which correspond to the case $\omega_m = \omega_{m-1} = \cdots = \omega_1$. In section 3, you can see explicit expressions of Abel-Goncharov polynomials and some of their properties, for more details see [23, 1, 10].

Therefore, the polynomial $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ can be written as

$$\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}(z) = \mathscr{P}_{n,m}^{(\alpha,\beta)}(z) - \mathscr{A}_{n,m}(z), \tag{5}$$

and we can interpret the polynomial $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ as the polynomial solution of the Abel-Goncharov boundary value problem (see [1, §3.5])

$$\begin{cases} \frac{d^mY}{dz^m}(z) & = & \frac{(n+m)!}{n!} P_n^{(\alpha,\beta)}(z), \quad n > m, \\ \frac{d^kY}{dz^k}(\omega_{m-k}) & = & 0, \qquad\qquad k = 0,1,\ldots,m-1. \end{cases}$$

Moreover, if $\alpha,\beta > -1$ then $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ is the $(n+m)$-th monic orthogonal polynomial with respect to the discrete-continuous Sobolev bilinear form (see [3, 2]) given by

$$\langle f,g\rangle_S = \sum_{k=0}^{m-1} \frac{d^k f}{dz^k}(\omega_{m-k})\frac{d^k g}{dz^k}(\omega_{m-k}) + \int_{-1}^{1} \frac{d^m f}{dz^m}(x)\frac{d^m g}{dz^m}(x)(1-x)^{\alpha}(1+x)^{\beta}dx.$$

The main goal of this paper is to study the algebraic and asymptotic properties of the family of monic polynomials $\{\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}\}_n$, for $m \in \mathbb{Z}_+$, $\{\omega_1,\cdots,\omega_m\} \subset \mathbb{C}\setminus[-1,1]$ and $\alpha,\beta > -1$. The case $\alpha = \beta = \omega_1 = \cdots = \omega_m = 0$ was early studied in [7], where the authors wrote *"It would be interesting to obtain results, analogous to Theorem [7, Th. 2], for these polynomials"* referring to the Gegenbauer (or ultraspherical) polynomials ($\alpha = \beta > -1$). Our Theorem 4 is an extension of [7, Th. 2] for Jacobi polynomials when all the constants of integration $\omega_i$ are out of the interval $[-1,1]$.

In the next section we review some of the standard facts on Jacobi polynomials and we give the proof of some auxiliary results. The third section is devoted to study the Abel-Goncharov interpolation polynomial $\mathscr{A}_{n,m}(z)$ of the $m$-th fundamental iterated integral of Jacobi polynomials. In the last section, our main results on asymptotic behavior of the sequence of polynomials $\left\{\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}\right\}_n$ and its zeros, are stated and proved.

# 2 Fundamental iterated integrals of Jacobi polynomials

Recall that, for a fixed $m,n \in \mathbb{Z}_+$, we denote by $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ the Jacobi monic polynomial of degree $n+m$ given by $P_{n+m}^{(\alpha-m,\beta-m)}$. From [20, §135 (12) and §138 (14)-(15)] we have the next lemma.

**Lemma 2.1.** *For a fixed $m \in \mathbb{Z}_+$, let $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ be the $(n+m)$-th fundamental primitive of $n$-th monic Jacobi polynomial with parameters $\alpha, \beta \in \mathbb{R}$. Then*

$$\begin{aligned} \mathscr{P}_{n,m}^{(\alpha,\beta)}(z) &= \mathscr{P}_{n+1,m-1}^{(\alpha,\beta)}(z) + a_{n,m}^{(\alpha,\beta)}\, \mathscr{P}_{n,m-1}^{(\alpha,\beta)}(z) + b_{n,m}^{(\alpha,\beta)}\, \mathscr{P}_{n-1,m-1}^{(\alpha,\beta)}(z); \qquad (6)\\ \textit{where} \quad a_{n,m}^{(\alpha,\beta)} &= \frac{2(n+m)(\alpha-\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)},\\ b_{n,m}^{(\alpha,\beta)} &= \frac{-4(n+m)(n+m-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)^2((2n+\alpha+\beta)^2-1)} \quad \textit{and}\\ \mathscr{P}_{n,0}^{(\alpha,\beta)}(z) &= P_n^{(\alpha,\beta)}(z). \end{aligned}$$

The asymptotic behavior of the sequence of polynomials $\{\mathscr{P}_{n,m}^{(\alpha,\beta)}\}_n$, stated in the following lemma is a direct consequence of [22, Th. 8.21.7 & Eqn. (4.21.6)].

**Lemma 2.2.** *If $\alpha, \beta \in \mathbb{R}$ and $m \in \mathbb{Z}_+$, then*

*1) (Outer strong asymptotic). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathscr{P}_{n,m}^{(\alpha,\beta)}(z)}{\varphi^n(z)} &= \psi_{\alpha,\beta,m}(z)\, \sqrt{\varphi(z)}, \quad \textit{where} \qquad (7)\\ \varphi(z) &= \frac{1}{2}\left(z+\sqrt{z^2-1}\right) \textit{ with } \sqrt{z^2-1} > 0 \textit{ when } z > 1 \qquad (8)\\ \textit{and}\, \psi_{\alpha,\beta,m}(z) &= \frac{2^{2m-\alpha-\beta}\left(\sqrt{z-1}+\sqrt{z+1}\right)^{\alpha+\beta-2m}}{\sqrt[4]{(z-1)^{2(\alpha-m)+1}}\,\sqrt[4]{(z+1)^{2(\beta-m)+1}}}. \end{aligned}$$

*2) (n-th root asymptotic behavior). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\lim_{n\to\infty} \left|\mathscr{P}_{n,m}^{(\alpha,\beta)}(z)\right|^{\frac{1}{n}} = |\varphi(z)|. \qquad (9)$$

*3) (Comparative asymptotic behavior). Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$*

$$\lim_{n\to\infty} \frac{\mathscr{P}_{n,m}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)} = \left(\frac{1}{\varphi'(z)}\right)^m. \qquad (10)$$

Note that $2\varphi(z) = z+\sqrt{z^2-1}$ is the conformal mapping of $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit circle, where the ellipses $|z+\sqrt{z^2-1}| = \rho,\ \rho > 0$ are its level curves.

Furthermore, the formula (9) of $n$-th root asymptotic behavior of the fundamental iterated integral $\mathscr{P}_{n,m}^{(\alpha,\beta)}(z)$ is the same for classical Jacobi Polynomials since $m$ is fixed.

The two lemmas listed below are deduced from the well-known Rouché's Theorem (cf. [18, Th. 1.1.1]) and the Biernacki's Theorem (cf. [19, Th. 4.5.2]), respectively.

**Lemma 2.3.** *Let $f$ and $g$ be polynomials, and $\gamma$ a closed curve in the complex plane without self-intersections. If $|f(z)| < |g(z)|$ for all $z \in \gamma$, then $f+g$ and $g$ have the same number of zeros in the interior of $\gamma$.*

**Lemma 2.4.** *Let $f$ be a polynomial whose critical points lie in a compact subset $K \subset \mathbb{C}$. If there exists $\zeta \in \mathbb{C}$ such that $f(\zeta) = 0$, then the zeros of $f$ lie in the compact set $[K]_\zeta = \{z \in \mathbb{C} : \inf_{w \in K} |z - w| \leq \mathbf{d}_{K_\zeta}\}$, where $\mathbf{d}_{K_\zeta}$ is the diameter of the compact set $K_\zeta = K \cup \{\zeta\}$ (i.e. $\mathbf{d}_{K_\zeta} = \sup_{u,v \in K_\zeta} |u - v|$).*

Of course, for all $\zeta \in \mathbb{C}$ we get $K \subset K_\zeta \subset [K]_\zeta$.

We denote by $\mathbf{Z}_{n,m}^{(\alpha,\beta)}(A)$ the set of zeros of $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ on the set $A \subset \mathbb{C}$. In the next theorem, we state some aspect of interest about the asymptotic behavior of the zeros of the fundamental iterated integrals of Jacobi polynomials.

**Theorem 1.** *Let $\alpha, \beta > -1$, $m \in \mathbb{N}$ fixed and $I = (-1, 1)$, then*

1) *For each $n > 2m$, at least $(n - 2m)$ distinct zeros of $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ lie in $I$.*

2) *There exists a compact set $K \subset \mathbb{C}$, such that $(-1, 1) \subset K$ and $\bigcup_{n \geq 1} \mathbf{Z}_{k,m}^{(\alpha,\beta)}(\mathbb{C}) \subset K$.*

3) *All the roots of $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ accumulate at $[-1, 1]$.*

*Proof.*

1) From (6), for consecutive values of $m$, there exist $(2m+1)$ constants $a_0, a_1, \ldots, a_{2m}$ such that $\mathscr{P}_{n,m}^{(\alpha,\beta)}(z) = \sum_{k=0}^{2m} a_k P_{n-m+k}^{(\alpha,\beta)}(z)$. Hence, $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ is a quasi-orthogonal polynomial of order $2m$ with respect to the measure $(1-x)^\alpha (1+x)^\beta dx$ on $I$. Hence, from [9, Th. 2] we have the first assertion of the theorem.

2) If $m = 1$, all the critical points of $\mathscr{P}_{n,1}^{(\alpha,\beta)}$ lie in $(-1, 1)$ and by the first sentence of the theorem at least $n-2$ of its zeros are on $I = [-1, 1]$. Let $x_0 \in I$ such that $\mathscr{P}_{n,1}^{(\alpha,\beta)}(x_0) = 0$. Then, according to the notations in Lemma 2.4, we get that $I_{x_0} = I$ and $\mathbf{d}_{I_{x_0}} = 2$. Hence, from Lemma 2.4 we get $\left(\bigcup_{n \geq 1} \mathbf{Z}_{k,1}^{(\alpha,\beta)}(\mathbb{C})\right) \subset [I]_{\mathbf{x_0}}$.

Suppose that for a fixed $m \in \mathbb{N}$, there exists a compact set $K_m^{(\alpha,\beta)}$ such that

$$\left(\bigcup_{n \geq 1} \mathbf{Z}_{k,m}^{(\alpha,\beta)}(\mathbb{C})\right) \subset K_m^{(\alpha,\beta)}.$$

As the zeros of $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ are the critical points of $\mathscr{P}_{n,m+1}^{(\alpha,\beta)}$, from Theorem 1-1) and Lemma 2.4 we get the desired statement.

3) For a fixed $m \in \mathbb{N}$, from the Theorem 1-2) we know that the set of all zeros of $\{\mathscr{P}_{n,m}^{(\alpha,\beta)}\}$ are uniformly bounded.

Note that for all $n \in \mathbb{Z}_+$ the functions $\frac{\mathscr{P}_{n,m}^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z)}$ and $\left(\frac{1}{\varphi'(z)}\right)^m = \left(\frac{\sqrt{z^2-1}}{\varphi(z)}\right)^m$ are analytic on $\overline{\mathbb{C}} \setminus [-1, 1]$, where $\varphi$ is given by (8). Furthermore, $\left(\frac{\sqrt{z^2-1}}{\varphi(z)}\right)^m \neq 0$ if $z \in \overline{\mathbb{C}} \setminus [-1, 1]$, hence the sentence 3.- is a consequence of (10). □

In the classical Szegő's book [22, §6.72], the reader can find a full description of the distribution of the zeros of $P_{n+m}^{(\alpha-m,\beta-m)}$, i.e. $\mathscr{P}_{n,m}^{(\alpha,\beta)}$, when $\alpha,\beta \in \mathbb{R}$ and $n,m \in \mathbb{Z}_+$ are fixed. In this sense, it is convenient to cite [12], where this analysis is embedded in a more general framework, using non-standard versions of orthogonality like the so-called quasi–orthogonality.

Additionally, there is a broad literature about zero location and asymptotic behavior of classical orthogonal polynomials with varying parameters. In particular, in the case of Jacobi polynomials ($\alpha = \alpha_n$ and $\beta = \beta_n$), the reader can see [8, 11, 14, 15, 16] and references therein. The case considered in the current paper is different, because the parameters in the fundamental iterated integrals of Jacobi polynomials are constant.

# 3 Abel-Goncharov interpolation polynomials

Given $m$ complex numbers $\omega_1,\ldots,\omega_m$, and $\Omega_k$ for $1 \le k \le m$, as in (3). As we show in the first section, there exists a unique polynomial $\mathscr{A}_{n,m}$ of degree at most $m-1$, such that the equations (4) are satisfied. This polynomial is given by

$$\mathscr{A}_{n,m}(z) = \mathscr{P}_{n,m}^{(\alpha,\beta)}(\omega_m) + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{d^k \mathscr{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k})\, \mathscr{G}_{k,m}(z) \tag{11}$$

where $\mathscr{G}_{k,m}(z) = \mathscr{G}_{k,m}(z;\omega_m,\omega_{m-1},\ldots,\omega_{m-k})$ is the monic polynomial of degree $k$ , generate by the $k$-th iterated integral

$$\mathscr{G}_{k,m}(z) = k! \int_{\omega_m}^{z} \int_{\omega_{m-1}}^{s_{m-1}} \cdots \int_{\omega_{m-(k-1)}}^{s_{m-(k-1)}} ds_{m-1}\, ds_{m-2} \cdots ds_{m-k}, \tag{12}$$

see [23, §4.1.4 (15)-(16)] for more details. The polynomial $\mathscr{G}_{k,m}$ is called the $k$-th Goncharov's polynomial associated with $\{\omega_1,\ldots,\omega_m\}$.

**Examples** (Abel's polynomials )**.** *If $\omega_1,\ldots,\omega_m$ form an arithmetic progression, i.e. $\omega_{m-k} = \omega + k\vartheta$, where $\omega,\vartheta \in \mathbb{C}$ are fixed and $k = 0,1,\ldots,m-1$, it is well known that in this case the k-th Goncharov polynomials*

$$\mathscr{G}_{k,m}(z) = (z-\omega)(z-\omega-(m-k)\vartheta)^{k-1}, \tag{13}$$

*is the so called k-th Abel's polynomial.*

*If $\vartheta = 0$, we have the special case $\mathscr{G}_{k,m}(z) = (z-\omega)^k$ (Taylor's case) and then the m-th Abel-Goncharov interpolation polynomial (11) becomes the Taylor's expansion of $\mathscr{P}_{n,m}^{(\alpha,\beta)}$ in $\omega$ , as we mentioned in the introduction.*

According to (2), it follows that $\frac{1}{k!}\frac{d^k \mathscr{P}_{n,m}^{(\alpha,\beta)}}{dz^k}(\omega_{m-k}) = \binom{n+m}{k} \mathscr{P}_{n,m-k}^{(\alpha,\beta)}(\omega_{m-k})$ and replacing this formula in (11) we get

$$\mathscr{A}_{n,m}(z) = \mathscr{P}_{n,m}^{(\alpha,\beta)}(\omega_m) + \sum_{k=1}^{m-1} \binom{n+m}{k} \mathscr{P}_{n,m-k}^{(\alpha,\beta)}(\omega_{m-k})\, \mathscr{G}_{k,m}(z). \tag{14}$$

**Theorem 2.** *Given $m>0$ and $\omega_1,\ldots,\omega_m\in\mathbb{C}\setminus[-1,1]$ fixed, let $\mathscr{A}_{n,m}(z)$ be the Abel-Goncharov polynomial of interpolation associated to the conditions (4),*

$$\sigma_m=\max_{0\le k\le m-1}|\varphi(\omega_{m-k})|,\ U=\{k:|\varphi(\omega_{m-k})|=\sigma_m\}\ \text{and}\ \hat{k}=\max_{k\in U}|k|. \tag{15}$$

*Then, uniformly on compact subsets of $\overline{\mathbb{C}}$*

$$\lim_{n\to\infty}\frac{\mathscr{A}_{n,m}(z)}{n^{\hat{k}}\ \mathscr{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})} = \frac{\mathscr{G}_{\hat{k},m}(z)}{\hat{k}!}, \tag{16}$$

$$\lim_{n\to\infty}|\mathscr{A}_{n,m}(z)|^{\frac{1}{n}} = \sigma_m. \tag{17}$$

*The branch of the square root in (8) is chosen so that $|\varphi(\omega_{m-k})|>1$, for each $0\le k\le m-1$.*

*Proof.*

Let $V=\{k:|\varphi(\omega_{m-k})|<\sigma_m\}$. Obviously $U\cap V=\emptyset$ and $U\cup V=\{1,2,\cdots,m\}$. From (14), we get

$$\left(\frac{(n+m-\hat{k})!}{(n+m)!}\right)\frac{\mathscr{A}_{n,m}(z)}{\mathscr{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})} = \frac{\mathscr{G}_{\hat{k},m}(z)}{\hat{k}!}+\sum_{k\in U\setminus\{\hat{k}\}}A_{n,m,k}\,\frac{\mathscr{G}_{k,m}(z)}{k!} + \sum_{k\in V}A_{n,m,k}\,\frac{\mathscr{G}_{k,m}(z)}{k!}, \tag{18}$$

where $\quad A_{n,m,k}=\dfrac{(n+m-\hat{k})!}{(n+m-k)!}\dfrac{\mathscr{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\mathscr{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}$.

Firstly, we will prove that for all $k\in(U\cup V)\setminus\{\hat{k}\}$

$$\lim_{n\to\infty}A_{n,m,k}=0. \tag{19}$$

If $k\in V$, then $|\varphi(\omega_{m-k})|<|\varphi(\omega_{m-\hat{k}})|$,

$$A_{n,m,k}=\frac{(n+m-\hat{k})!}{(n+m-k)!}\left(\frac{\varphi(\omega_{m-k})}{\varphi(\omega_{m-\hat{k}})}\right)^n\frac{\mathscr{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\varphi^n(\omega_{m-k})}\,\frac{\varphi^n(\omega_{m-\hat{k}})}{\mathscr{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}$$

and from (7), we can assert that for $k\in V$ we get (19).

If $k\in U\setminus\{\hat{k}\}$, then $k<\hat{k}$ and $|\varphi(\omega_{m-k})|=|\varphi(\omega_{m-\hat{k}})|$. Writing $\varphi(\omega_{m-k})=|\varphi(\omega_{m-\hat{k}})|e^{i\theta}$ and $\varphi(\omega_{m-\hat{k}})=|\varphi(\omega_{m-\hat{k}})|e^{i\hat{\theta}}$, with $\theta,\hat{\theta}\in[0.2\pi)$, we get

$$A_{n,m,k}=\left(\frac{(n+m-\hat{k})!}{(n+m-k)!}\right)e^{n(\theta-\hat{\theta})i}\left(\frac{\mathscr{P}^{(\alpha,\beta)}_{n,m-k}(\omega_{m-k})}{\varphi^n(\omega_{m-k})}\right)\left(\frac{\varphi^n(\omega_{m-\hat{k}})}{\mathscr{P}^{(\alpha,\beta)}_{n,m-\hat{k}}(\omega_{m-\hat{k}})}\right)$$

and as in the previous reasoning, from (7), we can assert that if $k\in U\setminus\{\hat{k}\}$ we have (19).

Now, according to (18)-(19), we get (16). Finally, (17) is a consequence of (16) and (9). □

## 4 General primitive of Jacobi polynomials and its zeros

For $\rho \in \mathbb{R}_+$, let $\mathbf{E}_\rho$ be the ellipse $|z-1|+|z+1| = \rho+\rho^{-1}$. Obviously, $\mathbf{E}_\rho$ divides the complex plane into the following two disjoint regions

$$\begin{aligned} \overline{\mathbf{E}}_\rho &= \left\{z \in \mathbb{C} : |z-1|+|z+1| > \rho+\rho^{-1}\right\}, \\ \underline{\mathbf{E}}_\rho &= \left\{z \in \mathbb{C} : |z-1|+|z+1| \leq \rho+\rho^{-1}\right\}. \end{aligned}$$

Analogously to the notations introduced in Theorem 1, we denote

$$\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} = \left\{z \in \mathbb{C} : \mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z) = 0\right\}$$

( i.e. the set of $(n+m)$ zeros of $\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}$) and by $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m}$ the set of accumulation points of zeros of $\{\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega}\}$.

**Lemma 4.1.** *Let $\alpha,\beta > -1$, $m \in \mathbb{N}$ and $\Omega_m = (\omega_1,\ldots,\omega_m) \in \mathbb{C}^m$ fixed. Then, there exists a compact subset $K \subset \mathbb{C}$, such that $(-1,1) \subset K$ and $\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} \subset K$ for all n.*

*Proof.*
We proceed analogously to the proof of Theorem 1. If $m = 1$, for all $n \geq 1$ the critical points of $\mathscr{P}^{(\alpha,\beta)}_{n,1,\omega_1}$ are on $I = [-1,1]$. Then, from Lemma 2.4, we get $\mathbf{Z}^{(\alpha,\beta)}_{n,1,\Omega_1}$ is a subset of the compact set $[I]_{\omega_1}$, which was defined in Lemma 2.4.

Suppose that for a fixed $m \in \mathbb{N}$ there exists a compact subset $K_{m-1}$ such that $\mathbf{Z}^{(\alpha,\beta)}_{n,m-1,\Omega_{m-1}} \subset K_{m-1}$. As the zeros of $\mathscr{P}^{(\alpha,\beta)}_{n,m-1,\Omega_{m-1}}$ are the critical points of $\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}$, from Lemma 2.4 we get $\mathbf{Z}^{(\alpha,\beta)}_{n,m,\Omega_m} \subset [K_{m-1}]_{\omega_m}$. □

**Theorem 3.** *Given $m > 0$ and $\omega_1,\ldots,\omega_m \in \mathbb{C}\setminus[-1,1]$ fixed, let $\rho_m = 2\sigma_m$, where $\sigma_m$ is given by (15). Then, uniformly on compact subsets of $\overline{\mathbf{E}}_{\rho_m}$*

$$\lim_{n\to\infty} \frac{\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z)}{P^{(\alpha,\beta)}_n(z)} = \left(\frac{1}{\varphi'(z)}\right)^m. \tag{20}$$

*Furthermore, $\mathbf{Z}^{(\alpha,\beta)}_{m,\Omega_m} \subset \underline{\mathbf{E}}_{\rho_m}$.*

*Proof.* From (5) we know that

$$\frac{\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}(z)}{P^{(\alpha,\beta)}_n(z)} = \frac{\mathscr{P}^{(\alpha,\beta)}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)} - \frac{\mathscr{A}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)}.$$

The uniform limit of the first quotient in the right side is given by (10). Hence, to proof (20) it is sufficient to proof that

$$\lim_{n\to\infty} \frac{\mathscr{A}_{n,m}(z)}{P^{(\alpha,\beta)}_n(z)} = 0, \quad \text{uniformly on compact subsets of } \overline{\mathbf{E}}_{\rho_m}. \tag{21}$$

From (14), we have

$$\frac{\mathscr{A}_{n,m}(z)}{P_n^{(\alpha,\beta)}(z)} = \sum_{k=0}^{m-1} \frac{\mathscr{P}_{n,m-k}^{(\alpha,\beta)}(\omega_{m-k})}{P_n^{(\alpha,\beta)}(\omega_{m-k})} \left( \frac{(n+m)!}{(n+m-k)!} \frac{P_n^{(\alpha,\beta)}(\omega_{m-k})}{P_n^{(\alpha,\beta)}(z)} \right) \frac{\mathscr{G}_{k,m}(z)}{k!},$$

where $\mathscr{G}_m(z) \equiv 1$. For $k = 0, 1, \ldots, m-1$ we get

$$\begin{aligned} \frac{(n+m)!}{(n+m-k)!} \frac{P_n^{(\alpha,\beta)}(\omega_{m-k})}{P_n^{(\alpha,\beta)}(z)} &= \frac{(n+m)!}{(n+m-k)!} \left( \frac{\varphi(\omega_{m-k})}{\varphi(z)} \right)^n \frac{P_n^{(\alpha,\beta)}(\omega_{m-k})}{\varphi^n(\omega_{m-k})} \\ &\quad \cdot \frac{\varphi^n(z)}{P_n^{(\alpha,\beta)}(z)}. \end{aligned}$$

As $|\varphi(\omega_{m-k})| < |\varphi(z)|$ for all $z \in \overline{\mathbf{E}}_{\rho_*}$, from (10) it follows (21).

Finally, the assertion $\mathbf{Z}_{m,\Omega_m}^{(\alpha,\beta)} \subset \underline{\mathbf{E}}_{\rho_m}$ is a consequence of (20) and Lemma 4.1, using analogous argument as in the proof of 3) in Theorem 1. □

**Theorem 4.** *Assume that $m > 0$, $\omega_1, \ldots, \omega_m \in \mathbb{C} \setminus [-1,1]$ and $\hat{k} = m-1$ . Then the accumulation points of zeros of $\{\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}\}$ are located on the union of the interval $[-1,1]$ and the ellipse*

$$\mathbf{E}_{\rho_m} = \left\{ z \in \mathbb{C} : |z-1| + |z+1| = \rho_m + \rho_m^{-1} \right\}, \tag{22}$$

*where $\hat{k}$ is defined in (15), $\rho_m$ is as in Theorem 3, and the branch of the square root in (8) is chosen so that $|\varphi(\omega_k)| > 1$, for each $1 \le k \le m$.*

*Proof.* From (5), the zeros of the polynomial $\mathscr{P}_{n,m,\Omega_m}^{(\alpha,\beta)}$ satisfy the equation

$$\left| \mathscr{P}_{n,m}^{(\alpha,\beta)}(z) \right|^{\frac{1}{n}} = \left| \mathscr{A}_{n,m}(z) \right|^{\frac{1}{n}}. \tag{23}$$

Taking the limit as $n \to \infty$ on both sides of (23), from (17) and (9), we have that $\mathbf{Z}_{m,\Omega_m}^{(\alpha,\beta)} \subset \mathbf{E}_{\rho_m}$ where

$$\mathbf{E}_{\rho_m} = \left\{ z \in \mathbb{C} : |z + \sqrt{z^2-1}| = \rho_m \right\}.$$

Let $\tilde{k}$ be an index, $1 \le \tilde{k} \le m$, such that $\varphi(\omega_{\tilde{k}}) = \rho_m e^{i\tilde{\theta}}$, $0 \le \tilde{\theta} < 2\pi$. Hence, we have that $z + \sqrt{z^2-1} = \rho_m e^{i\tilde{\theta}}$, $z - \sqrt{z^2-1} = \rho_m^{-1} e^{-i\tilde{\theta}}$ and taking the difference between both we get $\sqrt{z^2-1} = (\rho_m e^{i\tilde{\theta}} + \rho_m^{-1} e^{-i\tilde{\theta}})/2$. Thus,

$$|z-1| + |z+1| = \frac{|\rho_m e^{i\tilde{\theta}} - 1|^2 + |\rho_m e^{i\tilde{\theta}} + 1|^2}{2\rho_m},$$

which is equivalent to the equation of the ellipse in (22). As the limit that we have taken is uniform on compact subsets of $\mathbb{C} \setminus [-1,1]$, the theorem is proved. □

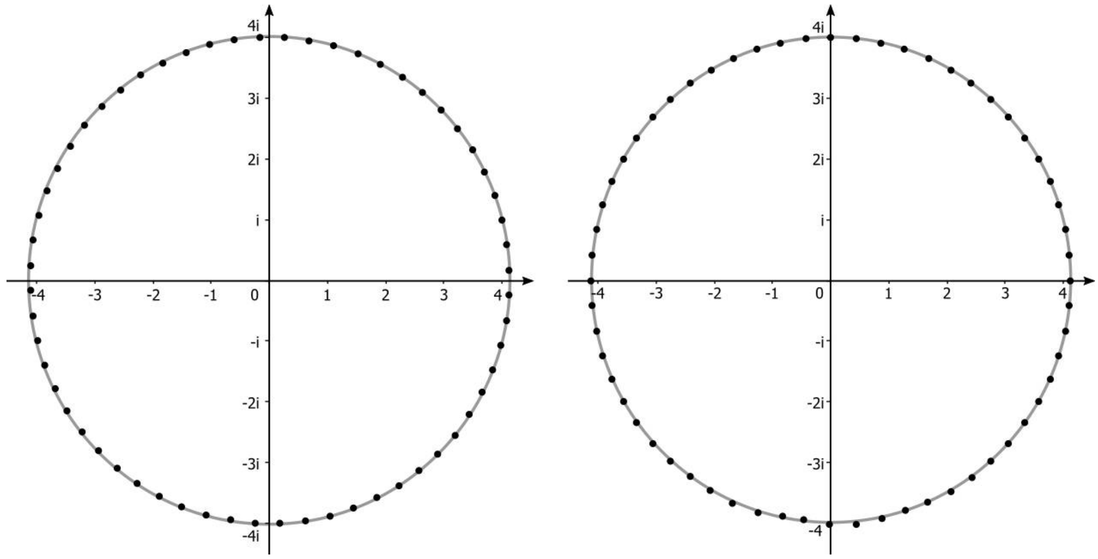


Figure 1: Zeros of $\mathscr{P}^{(-1/2,1/2)}_{60,2,\Omega_2}$ and $\mathscr{P}^{(0,0)}_{60,3,\Omega_3}$, where $\Omega_2 = (4+i,-2)$ and $\Omega_3 = (4i,-i,2)$. In each case the ellipse is given by (22)

.

In example 3, if for each $0 \le k \le m-1$ it holds that $(\omega + k\vartheta) \notin [-1,1]$, then all the zeros of the Abel's polynomials (13) are out to the interval $[-1,1]$. What is interesting for the following corollary.

**Corollary 4.1.** *Under the assumptions of theorems 2 and 4, if the zeros of the Goncharov polynomial $\mathscr{G}_{\hat{k},m}$ are outside to the interval $[-1,1]$ then the accumulation points of zeros of $\{\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}\}$ are located on the ellipse $\mathbf{E}_{\rho_m}$.*

*Proof.* Obviously, from Theorem 4 it is sufficient to prove that there does not exist an accumulation point of zeros of $\{\mathscr{P}^{(\alpha,\beta)}_{n,m,\Omega_m}\}$ located on the interval $[-1,1]$.

Let $\varepsilon \in \mathbb{R}$ such that $\omega_1, \dots, \omega_m$ and the zeros of $\mathscr{G}_{\hat{k},m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$. Thus, if $w \in \mathbf{E}_{1+\varepsilon}$, from (7) and (16) we get, for sufficiently large values of $n$,

$$\mathscr{A}_{n,m}(w) \approx \binom{n+m}{\hat{k}} \psi_{\alpha,\beta,m-\hat{k}}(\omega_{m-\hat{k}})\, \varphi^{n+\frac{1}{2}}(\omega_{m-\hat{k}})\, \mathscr{G}_{\hat{k},m}(w), \tag{24}$$

$$\mathscr{P}^{(\alpha,\beta)}_{n,m}(w) \approx \psi_{\alpha,\beta,m}(w)\, \varphi^{n+\frac{1}{2}}(w). \tag{25}$$

As the zeros of the Goncharov polynomial $\mathscr{G}_{\hat{k},m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$, then from (24), there exists $N_1 \in \mathbb{Z}_+$ such that for $n > N_1$ the zeros of the polynomial $\mathscr{A}_{n,m}$ are on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$ too. From (24)-(25)

$$\begin{aligned} |\mathscr{A}_{n,m}(w)| &\approx \binom{n+m}{\hat{k}} \left| \frac{\mathscr{G}_{\hat{k},m}(w)\, \psi_{\alpha,\beta,m-\hat{k}}(\omega_{m-\hat{k}})}{\psi_{\alpha,\beta,m}(w)} \right| \left| \frac{\varphi(\omega_{m-\hat{k}})}{\varphi(w)} \right|^{n+\frac{1}{2}} \\ &\quad \cdot \left| \psi_{\alpha,\beta,m}(w)\, \varphi^{n+\frac{1}{2}}(w) \right| \\ &\geq \left| \frac{\mathscr{G}_{\hat{k},m}(w)\, \psi_{\alpha,\beta,m}(\omega_{m-\hat{k}})}{\psi_{\alpha,\beta,m}(w)} \right| \left| \frac{\varphi(\omega_{m-\hat{k}})}{\varphi(w)} \right|^{n+\frac{1}{2}} \left| \mathscr{P}_{n,m}^{(\alpha,\beta)}(w) \right| \qquad (26) \end{aligned}$$

As it is well know from classical complex analysis (cf. [13, §51]), $\varphi(z)$ maps the ellipse $|z-1|+|z+1| = r+\frac{1}{r}$, with $r>0$, onto the circumference $|z|=r$. Hence, as each $\omega_k$ is on the exterior of the ellipse $\mathbf{E}_{1+\varepsilon}$ and $w \in \mathbf{E}_{1+\varepsilon}$, we get that $|\varphi(\omega_{m-\hat{k}})| > |\varphi(w)|$. Thus, from (26) there exists $N_2 \in \mathbb{Z}_+$ such that if $n > N_2$, then $|\mathscr{A}_{n,m}(w)| > \left|\mathscr{P}_{n,m}^{(\alpha,\beta)}(w)\right|$.

Finally, from Lemma 2.3 and Theorem 4, the corollary is proven. □

# References


[1] R.P. Agarwal, P.J.Y. Wong. Error Inequalities in Polynomial Interpolation and their Applications, Springer, Dordrecht, (1993).

[2] M. Alfaro, T. Pérez, M.A. Piñar, M.L. Rezola. Sobolev orthogonal polynomials: the discrete-continuous case. Methods Appl. Anal., **6**, 593–616, (1999).

[3] J. Bello, H. Pijeira, C. Márquez, W. Urbina. Sobolev-Gegenbauer-type orthogonality and a hydrodynamical interpretation, Integral Transform. Spec. Funct. **22**, 711–722, (2011).

[4] S. Beuchler, J. Schöberl. New shape functions for triangular $p$-FEM using integrated Jacobi polynomials, Numer. Math., **103**, 339–366, (2006).

[5] S. Beuchler, V. Pillwein. Sparse shape functions for tetrahedral $p$-FEM using integrated Jacobi Polynomials, Computing, **80**, 345–375, (2007).

[6] J. Borrego, H. Pijeira. Orthogonality with respect to a Jacobi differential operator and applications, J. Math. Anal. Appl., **404**, 491–500, (2013).

[7] P.B. Borwein, W. Chen, K. Dilcher. Zeros of iterated integrals of polynomials, Can. J. Math., **47**, 65–87, (1995).

[8] C. Bosbach, W. Gawronski, Strong asymptotics for Jacobi polynomials with varying weights, Methods Appl. Anal. **6**, 39–54 (1999).

[9] C. Brezinski, K.A. Driver, M. Redivo-Zaglia. Quasi-orthogonality with applications to some families of classical orthogonal polynomials, Appl. Numer. Math. **48**, 157–168, (2004).

[10] P.J. Davis. Interpolation and Approximation, Dover, N.Y., (1975).

[11] A.B. Kuijlaars, A. Martínez Finkelshtein. Strong asymptotics for Jacobi polynomials with varying nonstandard parameters. J. Anal. Math. **94**, 195–234, (2004).

[12] A.B. Kuijlaars, A. Martínez Finkelshtein, R. Orive. Orthogonality of Jacobi polynomials with general parameters. Electron. Trans. Numer. Anal. **19**, 1–17, (2005).

[13] A.I. Markushevich. Theory of functions of a complex variables, Vol. 1, Prentice-Hall, Inc., Englewood Cliffs, NJ, (1965).

[14] A. Martínez Finkelshtein, P. Martínez González, R. Orive. Zeros of Jacobi polynomials with varying non-classical parameters. in: Special functions (Hong Kong, 1999), World Sci. Publ., River Edge, NJ, 98–113, (2000).

[15] A. Martínez Finkelshtein, R. Orive. Riemann-Hilbert analysis of Jacobi polynomials orthogonal on a simple contour, J. Approx. Theory. **134**, 137–170, (2005).

[16] D.S. Moak, E.B. Saff, R.S. Varga. On the zeros of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}(x)$, Trans. Amer. Math. Soc. **249**, 159–162 (1979).

[17] H. Pijeira, J. Bello, W. Urbina. On polar Legendre polynomials. Rocky Mountain J. Math., **40**, 2025–2036, (2010).

[18] V.V. Prasolov. Polynomials, Springer, Berlin, (2004).

[19] Q.I. Rahman, G. Schmeisser. Analytic theory of polynomials, Oxford Univ. Press, NY, (2002).

[20] E.D. Rainville. Special Functions, Chelsea Pub. Co., NY, (1960).

[21] T. Sheil–Small. Complex Polynomials, Cambridge Univ. Press, Cambridge, (2002).

[22] G. Szegő. Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ. **23**, 4th ed., Providence, RI, (1975).

[23] A.F. Timan. Theory of approximation of functions of a real variable, Pergamon Press, Oxford, (1963).