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On Approximation Properties of Smooth Fuzzy Models

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Abstract - This paper addresses the approximation properties of the smooth fuzzy models. It is widely recognized that the fuzzy models can approximate a nonlinear function to any degree of accuracy in a convex compact region. However, in many applications, it is desirable to go beyond that and acquire a model to approximate the nonlinear function on a smooth surface to gain better performance and stability properties. Especially in the region around the steady states, when both error and change in error are approaching zero, it is much desired to avoid abrupt changes and discontinuity in the approximation of the input-output mapping. This problem has been remedied in our approach by application of the smooth compositions in the fuzzy modeling scheme. In the fuzzy decomposition stage of fuzzy modeling, we have discretized the parameters and then calculated the result through partitioning them into a dense grid. This could enable us to present the formulations by convolution and Fourier Transformation of the parameters and then obtain the approximation properties by studying the structural properties of the Fourier Transformation and convolution of the parameters. We could show that, irrespective to the shape of the membership function, one can approximate the dynamics and derivative of the continuous systems together, using the smooth fuzzy structure. The results of the paper have been tested and evaluated on a discrete event system in the hybrid and switched systems framework.

Keywords: Fuzzy Control, Fuzzy IF-THEN Systems (TSK), Smooth Compositions, Universal Approximation, Theoritecal Results.

1 Introduction

Soft computing methods have been used for identification and control of nonlinear and complex systems based on the input-output data collected from the original system [1]. There are many applications of artificial neural network and fuzzy modeling framework for the identification and model based control purpose in the industry and academia [2], [3]. Such methods show quite interesting ability in presenting the industrial processes with different types of data. The advantage of fuzzy models is that they can also include the operator information for dealing with the concept of uncertainty and handling the problisitic logics [4], [5]. The inclusion of information about the process in the generation of the methematical model makes the control task also capable of coping with the various nonlinear behaviors such as limit cycles, or where large changes in the operating conditions can be anticipated during the routine operation. From the modelling prospect, however, there is a weak point of the fuzzy modelling approach rather than neural networks that neural network can approximate an arbitrary smooth function together with its derivatives, while standard fuzzy systems cannot guarantee the accuracy in approximation of derivatives [6].

Fuzzy models are widely utilized for approximation of any nonlinear function to the desired degree of accuracy in the convex compact region [7], [8], [9]. However, there are many applications where we need to go beyond and acquire a model for approximation of the nonlinear function on a smooth surface for better performance and stability properties. The reason we are interested to acquire the smoothness property of the model is for avoiding abrupt changes, discontinuity or chattering behaviors in the approximation of the input-output mapping, particularly, in the region around the steady states, when both error and change in error are approaching zero [10], [11]. The continuity of not only the function, but also its derivatives, based on the literature, is defined as the smoothness property [12]. Generally speaking, it is more difficult to obtain smooth approximators rather than continuous approximators, while it can be more useful for the practical physical systems. Although, there has been done lots of works in fuzzy systems theory and applications, there can be found just little works on the continuity and smoothness of the fuzzy systems [6].

Primary works on fuzzy systems shows that fuzzy systems can uniformly approximate any real continuous function on a compact domain to the desired degree of accuracy [7], [8]. Then, Wang and Mendel proved that fuzzy systems, with Gaussian membership functions, product t-norm and centroid defuzzification are universal approximators [11]. Castro extended the results to Gaussian, triangular or trapezoidal membership functions, any t-norm and any practical defuzzification to be a universal approximator [13]. Then, Kreinovich further showed that fuzzy systems with Gaussian membership functions can do accurate approximation of a smooth function and its derivatives [6], however, it is unanswered that whether a fuzzy system with arbitrary continuous membership functions (not necessarily Gaussian, triangular or trapezoidal) can accurate approximation a function smoothly, i.e. not only the smooth function is approximated but also its derivatives.

Recently some new smooth compositions have been presented in [14], which have been employed for modeling static input-output mapping of dynamical systems in [15] and for the control purpose in [16]. But, there is lack of study on the approximation properties of such smooth fuzzy models.

In this paper, we will show that the practical achievement from the application of smooth compositions is that we would be able to model two (or more) different states of a discontinuous or a switched system by a single fuzzy model with the minimum variation. The other contribution will be that we can be sure that employing the smooth compositions in the design of the fuzzy models and controllers, the plant can damp the uncertainties and parameter variation and noises fast.

In fact, the uncertainty factors might appear in practice in all the various real life decision making processes and the industrial problems, e.g. the insufficient information in the knapsack problem and the network optimization [17], [18], the lack of history data for portfolio selection and return estimation in the market [19], the demand of data estimation in the changeable environment for transportation planning problem [20], supply chain design problem in the uncertain demand or variable manusfacture costs [21], and so on. Hence, it is desirable to study in which capacity smooth compositions can improve the estimation accuracy.

The rest of the manuscript is as follows. First we review mathematical smoothness and continuity properties. Then, we study the general structure of fuzzy systems. Based on the results of the two beginning sections, we formulize the smoothness property of a special class of fuzzy systems which is the main result of the paper. Following that we bring an example to demonstrate the practical functionality and properties of the obtained results and the proposed theorems. Finally we draw conclusions.

2 Preliminaries

In this section for the convenience of the readers we review some mathematical backgrounds from [11], [12].

Definition 1: A function $f(x)$ is continuous at the point c if and only if $f(x)$ is defined at c and for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ if $|x - c| < \delta$.

Definition 2: A function $f(x)$ has gap discontinuity at c if $f(c)$ is undefined.

For instance, $\frac{f_1(x)}{f_2(x)}$ has gap discontinuity at c if $f_2(c) = 0$.

Definition 3: A function $f(x)$ has jump discontinuity at c if $f(c)$ is defined and $\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$.

The function $f(x) = \begin{cases} 4, x < 0 \\ 5, x \geq 0 \end{cases}$ for example has a jump discontinuity at $x = 0$.

After review of some mathematical definitions, we continue to the main part of the manuscripts.

2.1 Structure of fuzzy systems:

We consider the multiple input single output systems to facilitate our theory development. Nevertheless, our results can be extended for the multiple - input-multiple output systems; since the multiple outputs can be decomposed readily into several single output systems.

Consider the problem of approximation for a nonlinear function of the following form:

$$f: R^n \to R \tag{1}$$

$$y = f(x_1, x_2, \cdots, x_n) \tag{2}$$

For every input variable of the system we consider an interval where there is the highest probability that the variable lies in this interval. Then, we divide the interval into 2N+1 regions and assign a membership function to each region.

Next step in constructing fuzzy system is to assign rules for the data in the different regions of the input and output domains. We consider,

$R^{(i)}$: if x_1 is M_1^i and x_2 is M_2^i and $\cdots$ and $x_n$is M_n^i then (3)

$g(x_1, \cdots, x_n)$ is b_i under the probability $\mu_i, i = 1, \cdots, r$

Here the function $g(x_1, \cdots, x_n)$ is about to approximate the function $f(x_1, \cdots, x_n)$ in the corresponding interval. The rules generated for the fuzzy system in this way, have two "if" and "then" parts. There are different ways for interpretation of the relations and making mathematical inference on the fuzzy values using the compositions of t-norm and s-norm in the fuzzy systems´ domain. The different types of the fuzzy compositions introduced in the literature [14], [22], are summarized as below,

1- Min t-norm

$$T(a, b) = \min(a, b)$$

2- Product t-norm

$$T(a, b) = ab$$

3- Max s-norm

$$T(a, b) = \max(a, b)$$

4- Probabilistic s-norm

$$S(a, b) = \mathrm{a} + \mathrm{b} - \mathrm{ab}$$

5- Lukasicwicz t-norm

$$T(a, b) = \max(a + b - 1, 0)$$

6- Lukasicwicz s-norm

$$S(a, b) = \max(a + b, 1)$$

7- Weak t-norm

$$T(a,b) = \begin{cases} \min(a,b) & \max(a,b)=1 \\ 0 & o.w \end{cases}$$

8- Strong s-norm

$$S(a,b) = \begin{cases} \max(a,b) & \min(a,b)=0 \\ 1 & o.w \end{cases}$$

9- Hamacher t-norm

$$T_H(a,b) = \frac{ab}{\gamma + (1-\gamma)\ \parallel a+b-ab \parallel} \quad \gamma \geq 0$$

10- Hamacher s-norm

$$S_H(a,b) = \frac{a+b-(2-\gamma)ab}{1-(1-\gamma)ab} \quad \gamma \geq 0$$

11- Doubois t-norm

$$T_D(a,b) = \frac{ab}{\max(a,b,\alpha)} \quad \alpha \in (0,1)$$

12- Yager t-norm

$$T_Y(a,b) = 1 - \min\left\{1, \sqrt[p]{(1-a)^p + (1-b)^p}\right\} \quad p > 0$$

13- Yager s-norm

$$S_Y(a,b) = \min\left\{1, \sqrt[p]{a^p + b^p}\right\} \quad p > 0$$

14- Smooth t-norms

$$I: T_S(a,b) = 1 - \cos(\frac{2}{\pi}\cos^{-1}(1-a)\ \cos^{-1}(1-b))$$

$$II: T_S(a,b) = \frac{4}{\pi}\tan^{-1}(\tan\left(\frac{\pi}{4}a\right)\tan(\frac{\pi}{4}b))$$

$$III: T_S(a,b) = 1 - \frac{2}{\pi}\cos^{-1}(\sin\left(\frac{\pi}{2}a\right)\sin(\frac{\pi}{2}b))$$

$$\text{IV}: T_S(a,b) = \cos(\cos^{-1}a + \cos^{-1}b - \frac{2}{\pi}\cos^{-1}a\ \cos^{-1}b) \quad (4)$$

15- Smooth s-norms

$$I: S_S(a,b) = \frac{r.d.\beta^{-\log_\beta(d) - \log_\beta(r)}}{(\beta-1)}, r = (\beta-1)a+1, s = (\beta-1)b+1, \beta \in (1,\infty)$$

$$II: S_S(a,b) = 1 - \frac{4}{\pi}\tan^{-1}(\tan\left(\frac{\pi}{4}(1-a)\right)\tan(\frac{\pi}{4}(1-b)))$$

$$III: S_S(a,b) = \frac{2}{\pi}\cos^{-1}(\cos\left(\frac{\pi}{2}a\right)\cos(\frac{\pi}{2}b))$$

$$\text{IV: } S_S(a,b) = \cos\left(\frac{2}{\pi}\cos^{-1} a \, \cos^{-1} b\right) \quad (5)$$

From the above list, the smooth t-norm and smooth s-norms are the differentiable functions of the input parameters, and we will talk about them more in the next section.

The actual output of the model is determined based on the centroid defuzification formula, which is given simply by,

$$g(\bar{x}) = \frac{\sum_{i=1}^{r} b_i \ \mu_i}{\sum_{i=1}^{r} \mu_i} \quad (6)$$

where μ_i is considered to be at the center of the region B^i at every time instant of dynamics of the system, $\bar{x} = [x_1, \cdots, x_n]$ and r is the total number of the fuzzy rules for approximation of the plant.

3 Approximation Properties

On the purpose of explaining the approximation procedure, we consider equation (4) and equation (5) as the formulation of t-norm and t-conorm under study, and denote them by smooth compositions T_{s-IV} and S_{s-IV}, respectively. The approach can be extended to other types of the smooth compositions. For the system defined by the function $f(x_1, \cdots, x_n)$ introduced above, we assume r=2, with three state variables, then, the fuzzy model will be written as,

$$g(x_1, \cdots, x_n) = \frac{N(x_1, \cdots, x_n)}{D(x_1, \cdots, x_n)} = \frac{b_1 * \mu_1 + b_2 * \mu_2}{\mu_1 + \mu_2} \quad (7)$$

where $\mu_i(\bar{x}, \alpha_i)$are the membership functions from the system state vector $\bar{x} = [a, b, c]$, $i = 1, \cdots, r$ and α is the design parameter.

$$\mu_i(\bar{x}, \alpha_i) = S_{s-IV}\left(T_{s-IV}\left(\mu_i\,(a,\cdot), \mu_i(b,\cdot), \mu_i\,(c,\cdot)\right)\right) = \quad (8)$$

$$S_{s-IV}(T_{s-IV}(T_{s-IV}(\,\mu_i(a,\cdot), \mu_i(b,\cdot)), \mu_i\,(c,\cdot)))$$

Let $\Lambda_1 = T_{s-IV}\big(\mu(a,\cdot), \mu(b,\cdot)\big)$, and $\Lambda_2 = T_{s-IV}\big(\,\Lambda_1, \mu(c,\cdot)\big)$, and upon Eq (4),

$$\Lambda_1 = \cos\left(\cos^{-1}\mu_i(a,\cdot) + \cos^{-1}\mu_i(\,b,\cdot) - \frac{2}{\pi}\cos^{-1}\ \mu_i(a,\cdot)\ \cos^{-1}\mu_i(\,b,\cdot)\right)$$

$$\Lambda_2 = \cos\left(\cos^{-1}\Lambda_1 + \cos^{-1}\mu_i(\,c,\cdot) - \frac{2}{\pi}\cos^{-1}\Lambda_1\ \ \cos^{-1}\mu_i(c,\cdot)\right).$$

Based on Eq (5), $\mu_i(\cdot, \alpha_i) = \cos\left(\frac{2}{\pi}\, cos^{-1}\Lambda_1\ cos^{-1}\Lambda_2\right)$, hence, we define,

$$\theta = \frac{2}{\pi}\, cos^{-1}\Lambda_1\ cos^{-1}\Lambda_2 \quad (9)$$

Therefore,

$\mu_i(\cdot, \alpha_i) = \cos(\theta) = real\,(\exp(j\theta)),$

where $j = \sqrt{-1}$. For the suitable selection of exponential function $G(x, \alpha_i)$ and θ we write it more simple as,

$$\mu_i\,(\cdot, \alpha_i) = G(\cdot, \alpha_i) := real\,(\exp(j\theta)) \tag{10}$$

Therefore, we can generalize the procedure and write,

$$g(x_1, \cdots, x_n) = \frac{N(x_1,\cdots,x_n)}{D(x_1,\cdots,x_n)} = \frac{\sum_i^r b_i G(\bar{x},\alpha_i)}{\sum_i^r G(\bar{x},\alpha_i)}. \tag{11}$$

If we consider a box $[-N, N], \cdots, [-N, N]$ along a dense grid with the steps $\Delta\,\alpha_1 = \cdots = \Delta\alpha_r = h$ and correspondingly $b_i = b(\alpha_i)$, we can write the summation as the integration,

$$N(\bar{x}).h^n = \int_{-N}^{N} \cdots \int_{-N}^{N} b(\bar{\alpha}).G(\bar{x}, \bar{\alpha})\ .d\bar{\alpha} \tag{12}$$

$$D(\bar{x}).h^n = \int_{-N}^{N} \cdots \int_{-N}^{N} G(\bar{x}, \bar{\alpha}).d\bar{\alpha}\,. \tag{13}$$

Now, as $h \to 0$ and $N \to \infty$, we will have the multi-dimensional integrals,

$$N_\infty(\bar{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).G(\bar{x}, \bar{\alpha})\ .d\bar{\alpha} \tag{14}$$

$$D_\infty(\bar{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(\bar{x}, \bar{\alpha}).d\bar{\alpha}\,. \tag{15}$$

The value of the last integral is independent of the system states vector $(x_1, \cdots, x_n)$. Hence, it sums up to a constant value C for $D_\infty(\bar{x})$. Therefore, to find the approximation of the function $f(\cdot)$ we just need to find the weights $b(\bar{\alpha})$, such that

$$C\ \cdot g(\bar{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).G(\bar{x}, \bar{\alpha})\ .d\bar{\alpha} \tag{16}$$

The right side of this equation is the convolution of the function $b(\bar{\alpha})$ and the real function $G(\bar{x}, \bar{\alpha})$. We can transform the convolution to the frequency domain, and use the Fourier transformation to find the weights as,

$$\hat{b}(\bar{\omega}) = \frac{C \cdot g(\bar{\omega})}{G(\bar{\omega})} \tag{17}$$

and then use the inverse Fourier Transformation to get the desired function $b(\bar{\alpha})$.

To come up to the results, here we review some theorems from the signal and system literature.

Theorem 1: Let $\mathcal{F}$ and $\mathcal{R}$ be continuous real-valued functions and assume that $\mathcal{F}$ or $\mathcal{R}$ is zero outside some bounded set. If $\mathcal{F} \in C^k$ and $\mathcal{R} \in C^l$, then $\mathcal{F} * \mathcal{R} \in C^{k+l}$.

Proof: see [23].

Theorem 2: (Derivative Theorem) If $\mathcal{F}$is a rough function, and $\mathcal{R}$ is a smooth function, then the convolution $\mathcal{F} * \mathcal{R}$ will be smoother than $\mathcal{F}$.

Proof: The theorem can be deduced from Theorem 1, see also [24].

Theorem 3: If $\mathcal{F}$is a rough function and $\mathcal{R}$ is n-times differentiable, then the convolution $\mathcal{F} * \mathcal{R}$ will be n-times differentiable.

Proof: The theorem can be deduced from Theorem 1, see also [25].

Corollary 1: The convolution $\mathcal{F} * \mathcal{R}$ is at least as smooth as the function $\mathcal{F}$ and the function $\mathcal{R}$ separately.

Theorem 4: The fuzzy model obtained by the arbitrary membership function and the smooth s-norm and t-norm compositions is continuous, n-time differentiable and smoother than a periodic cosine function.

Proof: The fuzzy model is the convolution of the function $b(\bar{\alpha})$ by the function $G(\bar{x},\bar{\alpha})$ weighted by the constant value C, according to the Eq (16). Since the function $G(\bar{x},\bar{\alpha})$ is a cosine function whatever the membership functions are, hence, based on Theorems 1-3, we can conclude Theorem 4. It is to say, the model will be smoother than cosine function, whatever the function $f(\cdot)$ is.

Remark 1: Theorem 4 applies independent of the shape and nature of the plant, according to the derivative theorem stated above. In other words, Theorem 4 applies even if the plant has a rough or discontinuous dynamics.

Remark 2: The interpretation of theorem 4 in control application will be that, the control surface which the smooth fuzzy system produces will be smooth. Even if the system has a discrete state or systematic transition, based on this theorem, the transition in the system will happen with the minimum level of abrupt changes and variations. Moreover, the control system will show a better robustness to the uncertainties and disturbances in the region around the steady state point, trying to stay on the smooth surface.

Now we look at the properties of the estimation of derivatives of the plant.

3.1 Estimation of the Dynamic System Derivatives

We first consider the first derivative of the model. Taking the first derivative we have,

$$g_1(\bar{x}) = \frac{N_1(\bar{x})}{D(\bar{x})} - \frac{N(\bar{x})D_1(\bar{x})}{D^2(\bar{x})} \tag{18}$$

$$N_\infty(\bar{x}) = \sum_i^r b_i G_1(x,\alpha_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).G_1(\bar{x},\bar{\alpha})\ .d\bar{\alpha} \tag{19}$$

$$D_\infty(\bar{x}) = \sum_i^r G_1(x,\alpha_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_1(\bar{x},\bar{\alpha}).d\bar{\alpha}\ . \tag{20}$$

Again using the same procedure, we come to,

$$C\ \cdot g_1(\bar{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).G_1(\bar{x},\bar{\alpha})\ .d\bar{\alpha}. \tag{21}$$

If we consider the m-th higher derivatives, similarly, we arrive to

$$C \cdot g_m(\bar{x}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).\, G_m(\bar{x}, \bar{\alpha}) \,.\, d\bar{\alpha}. \quad (22)$$

Theorem 5: The fuzzy model obtained by the smooth compositions is continuous and m-time differentiable.

Proof: Considering that the function $G_m(\bar{x}, \bar{\alpha})$ is a cosine function in origin and m-times differentiable, also Theorem 2, hence we always can approximate the desired function up to m-th Derivative $g_m(\bar{x})$ with the desired accuracy using the smooth fuzzy model (m is an arbitrary number).

Theorem 6: The approximation function $g(\bar{x})$ is defined everywhere in the domain of the states with the possible finite numbers of jump discontinuities.

Proof: Based on the definition, the smooth compositions are smooth with the possible number of discontinuities over their domains. Hence, the integration $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} b(\bar{\alpha}).\, G\ \ (\bar{x}, \bar{\alpha})\ d\,\bar{\alpha}$ in Eq (1) always can be calculated by the grid based sum of integration for the appropriate small h and large N.

Theorem 7: Consider the smooth fuzzy system defined above with the parameters $\bar{\alpha} = [\alpha_1, \cdots, \alpha_r]$ for the rules. The smooth fuzzy model $g(\bar{x})$ is continuous if $\exists \alpha_{i,}, i \in [1, r]$ such that $\max\big(\mu(\alpha_i)\big) > 0$, i.e. there exists at least one input fully covered by the membership functions.

Proof: We describe the case for $i = 2$ which is extendable to the cases with the higher number of rules. Consider the integral part as stated above for the case $i = 2$, when there is a discontinuity for $\alpha_2 \in [c_1, c_2]$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(\bar{\alpha}).\, G_D(\bar{x}, \bar{\alpha})\ d\,\bar{\alpha} = \int_{-\infty}^{\infty} \int_{-\infty}^{c_1} b(\alpha_1, \alpha_2).\, G_D(\bar{x}, \alpha_1, \alpha_2)\ d\,\alpha_1 d\alpha_2$$
$$+ \int_{-\infty}^{\infty} \int_{c_2}^{\infty} b(\alpha_1, \alpha_2).\, G_D(\bar{x}, \alpha_1, \alpha_2)\ \ d\,\alpha_1 d\alpha_2. \quad (23)$$

The above integrations are calculable at every point, as the last input is supposed to be fully covered by the membership functions, and is continuous.

From the Fourier Transformation viewpoint, described above, it worth mentioning that the Fourier Transformation exists only if the jump discontinuity at $\alpha_2 = c$ cannot change the value of any of the integrals, i.e. , $c_1 = \lim_{\alpha_2 \to c^-}$, $c_2 = \lim_{\alpha_2 \to c^+}$ since at this case, the inverse of the Fourier transformation will converge to the mid value level at the point of discontinuity.

Corollary 2: Consider the smooth fuzzy system with one input. The jump discontinuity of the mapping function $f(x_1, \cdots, x_n)$ will not impact on the smoothness property of the resulted smooth fuzzy model.

Proof: For the one input case, the above formulation will be,

$$\int_{-\infty}^{\infty} b(\bar{\alpha}).\, G_D(\bar{x}, \alpha_1)\ d\,\alpha_1 = \quad (24)$$

$$\int_{-N}^{c-\epsilon} b(\bar{\alpha}).G_D(\bar{x},\alpha_1)_{\epsilon\to 0}\, d\,\alpha_1 + \int_{c+\epsilon}^{+N} b(\bar{\alpha}).G_D(\bar{x},\alpha_1)_{\epsilon\to 0}\, d\,\alpha_1$$

for a suitable choice of number N. It is obvious that the value of the integral will not be affected by the point discontinuity of the system.

Theorem 8: (**Main Theorem)** Let d and n be integers, and $N > 0$ and $\epsilon > 0$ be real numbers. Assume the function $f(x_1, \cdots, x_n)$ is a D-times differentiable function on $[-N, N]^n$. Then, using the smooth fuzzy compositions, one can construct a fuzzy model $g(x_1, \cdots, x_n)$ to approximate the function $f(x_1, \cdots, x_n)$ and its derivatives up to D-th order with the desired accuracy ϵ.

Remark 3: The results we presented here compared to the earlier works by Kreinovich [6] on smoothness properties for the fuzzy models brings much lesser restrictions; As in this manuscript we have not put any restriction on shape of membership function, (to be or not be in Gaussian Form) to gain the smoothness property, compared to their work.

Now we show the effectiveness of the obtained results by an illustration.

4 Illustrative Examples

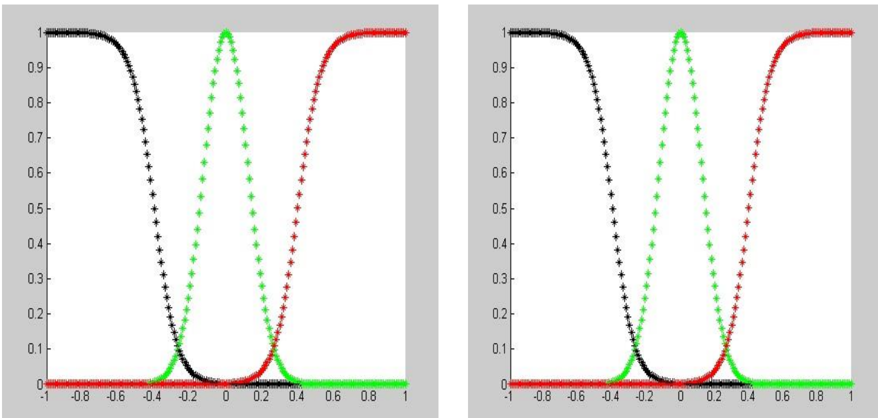
To demonstrate application of the proposed approach, we take the simple model as Table 1, where each rule consequent is shown based on the crisp number.

The table represents the logical rule that orchestrate switching between the different states of the finite state machine. Such kind of logical rules, when coupled with the controller and the plants modelled with continuous or difference equations are generally called hybrid or switched systems which have the increasing popularity for modeling and control of the devices with digital components, e.g. relays, switches, stepper motors, so on [10], [26]. Traditionally fuzzy controllers for hybrid and switched systems are designed such that every subsystem is being considered by a separate fuzzy structure. What follows is an evidence that using the smooth fuzzy schemes, it would be possible to model and control the different discrete states of the system by a single fuzzy model structure such that the augmented continuous and discrete dynamics of the system changes between the augmented continuous and discrete states of the model smoothly.

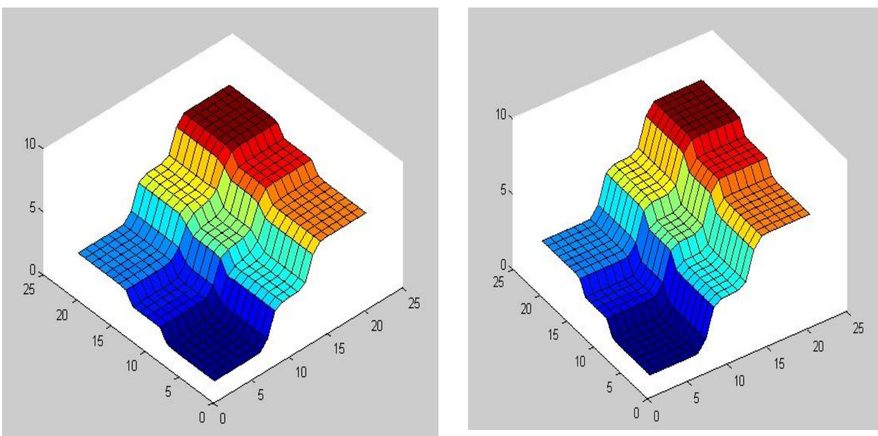
Table 1: logical rules of the switched system

$x_1 \backslash x_2$	X_{21}	X_{22}	X_{23}
X_{11}	**1**	**2**	**3**
X_{12}	**4**	**5**	**6**
X_{13}	**7**	**8**	**9**

We first consider fuzzy membership functions shown in figure 1, where they cover all the domain of the system states definition. Consequently, we have used first the conventional fuzzy inference for the fuzzy model and compared that to the smooth fuzzy structure. They are equal in the functioning for mapping the input-output relation.



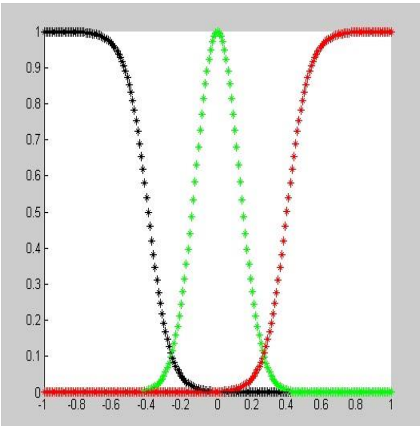
Membership function for the state x1 Membership function for the state x2



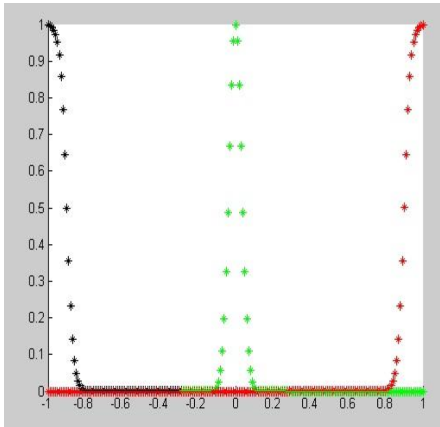
Output surface of the classical fuzzy systems Output surface of the smooth fuzzy

Figure 1: Case 1: when membership functions of both states cover the space.

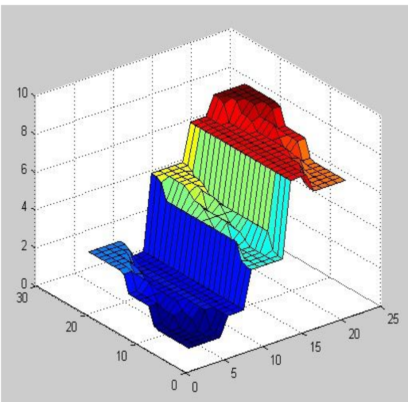
We then considered fuzzy membership functions shown in figure 2 and figure 3, where the membership function of the first state covers all the domain of system state definition, and the membership function of the second state does not cover all the second state space. Again, we have used the conventional fuzzy inference for the fuzzy model and compared that to the smooth fuzzy model. This is clear that the classical fuzzy model has great value of variation in the output. This is while the smooth fuzzy model has a minimum variation which is to say its performance is almost similar to the case 1, when the membership functions covered all the state space.



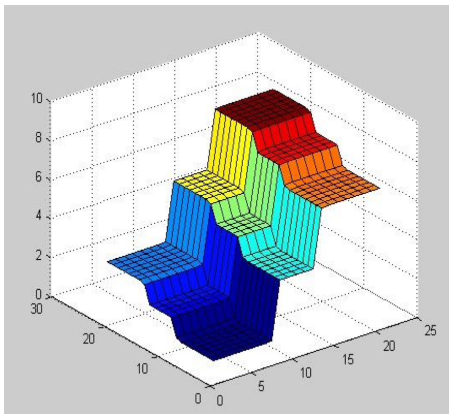
Membership function for the state x1



Membership function for the state x2



The output surface of the classical fuzzy



The output surface of the smooth fuzzy system

Figure 2: Case 2: when membership functions of just one state covers the state space

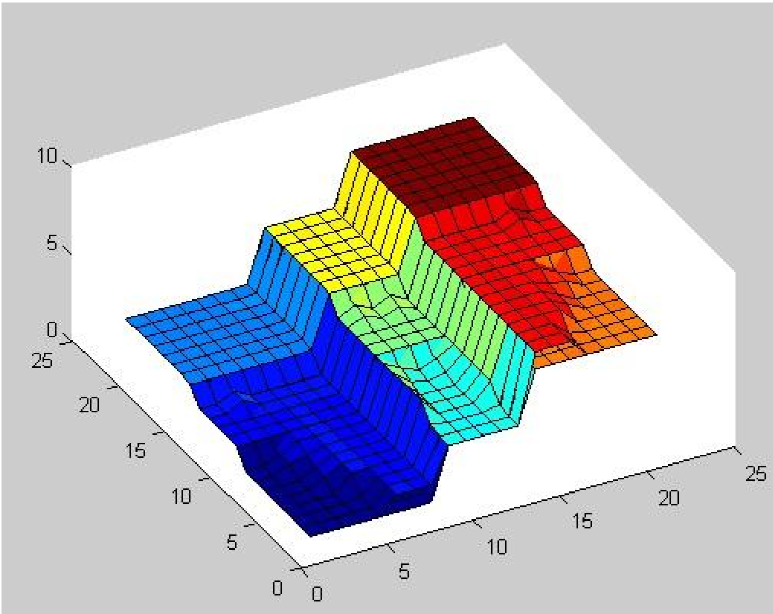
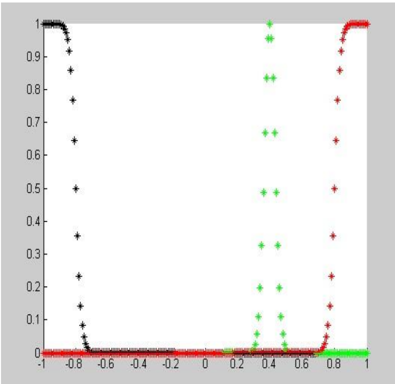


Figure 3: The comparison of the output of the fuzzy function with the smooth compositions vs the classical compositions

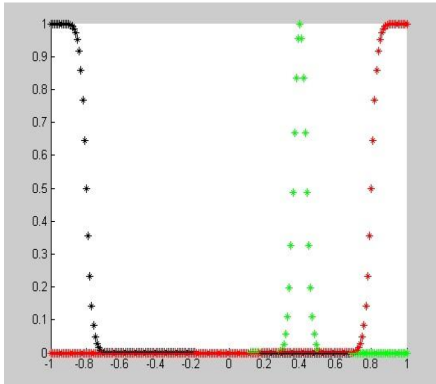
Lastly, we consider the fuzzy membership functions shown in figure 3, where none of the state spaces are covered by the membership functions. It is clear that both of the fuzzy models show high amount of variation and discontinuity.

As it is clear from the results of simulations, when the membership functions cover at least one of the state space variables, the smooth fuzzy model shows a smooth and minimum variation behavior for modeling of the input-output mapping, compared to the classical fuzzy model. In control applications, this feature can be used to damp the effect of the parameter variations and noise in the system and using the smooth compisitions one can run the system to return to the stable states after the disturbance with the minimum turbulences.

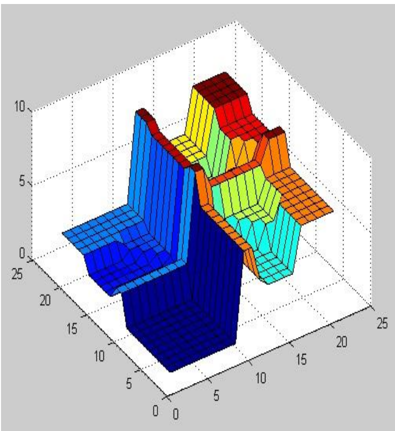
The inspection of the results in the example could clear up that i) converse to the earlier contributions, we are able to model two (or more) different states of a discontinuous or a switched system by a single smooth fuzzy model. ii) Based on the result of the case when the membership function of the second state in the simulation does not cover all the second state space, we claim that the smooth fuzzy models can uniformly approximate any real continuous function on a possible non-compact domain to the desired degree of accuracy, which is new in the fuzzy modeling literature. iii) As it has demonstrated in the simulation, for the



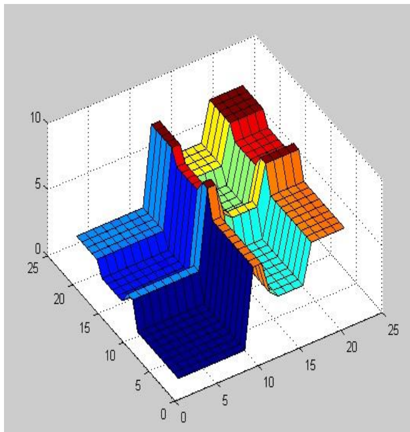
Membership function for the state x1



Membership function for the state x2



Output surface of the classical fuzzy systems



Output surface of the smooth fuzzy systems

Figure 4: Case 3: when membership functions of none of states cover the space

case when the second state in the simulation does not cover all the second state space, the error between the smooth fuzzy model and the plant, in comparison to the error value with the same definition in the classical fuzzy model of the plant, has declined much more - to the minimum possible value. Hence, we claim that employing the smooth compositions in the design of connectivist fuzzy modeling and controller schemes [27], we will not have high value of real plant-fuzzy model difference, neither the un-modeled dynamics, and therefore, will not need to restrict ourselves to the conservative methods of robust or adaptive control schemes. iv) As it is demonstrated by the simulation, the smoothness property of such fuzzy model structure could encompass the change in the discrete modes of the switched system by showing the minimum amount of variations and errors. Hence, we can generalize it and expect to be able to damp the uncertainties and parameter variations of the systems and environmental noises also very fast through the same smoothness properties of the fuzzy model. Some simulations in the earlier publications on smooth fuzzy modeling and control [15], [16], [33] have demonstrated the robustness properties of such connectivist smooth fuzzy modeling and control schemes, however, they lack providing a theoretical analysis. The current manuscript can back up their results by giving a clue that why such robustness properties exist.

Conclusion

In this paper it is shown that we can model the dynamics and derivative of the continuous systems using the smooth fuzzy structure. We did not put any limitation on shape of the membership functions, in contrast to the earlier works, where the special Gaussian membership function has been considered. As a result of this finding, what we need to do in design of smooth fuzzy systems, to approximate dynamics of the system along its derivatives, would be to stick to the common practice of choosing the centric point and do not care of shape of the membership function.

We backed up our theories by an example where each rule's consequent has been shown based on a crisp number. It can be seen as a Mamdani model with the height defuzzification, or the discrete state models of hybrid and switched systems.

Furure works

Upon the findings, the future research could focus on application of the smooth fuzzy compositions in modeling and control of the discrete states of more practical hybrid and switched systems, where in every discrete state the plant represents a continuous dynamics instead of representing a constant value. That can present a deeper systemic analysis of the materials presented in the current work and model the whole nonlinear structure by a single fuzzy model, converse to the common practice of employing a different fuzzy model for each state.

In our analysis and transformation, to run the approximation error and its derivative tends to zero we need to increase the number of partitions in the dense grid as well as the fuzzy rules. It means that in the practical applications, we will have growing numbers of fuzzy rules to make use of the smooth approximation properties. Therefore, there is a trade-off between the accuracy of the fuzzy model and the modelling complexity. Hence, It is required to think about a method for finding the minimal number of fuzzy rules for a given accuracy of the fuzzy model in the future researches. One suggestion will be to discard the rules which have weak contribution to the output.

In [28], [29] the interpolation of the fuzzy rules has been suggested for reducing the complexity in the model identification. We believe that the same procedure can be applied for lowering the number of rules in the smooth fuzzy model, such that just the rules with the essential information remain and the rest is replaced by the interpolation algorithm.

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