LINEARIZATIONS OF RATIONAL MATRICES

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“Knowing is not enough, we must apply.
Willing is not enough, we must do.”
– Bruce Lee

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CONTENIDOS PUBLICADOS Y PRESENTADOS

Los resultados originales que aparecen en el Capítulo 3 de esta tesis han sido publicados en

1 – F. M. Dopico, S. Marcaida, M. C. Quintana, *Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis*, Linear Algebra and its Applications, 570 (2019) 1–45. DOI: 10.1016/j.laa.2019.02.003

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Los resultados originales que aparecen en los Capítulos 4 y 5 de esta tesis han sido publicados o enviados a publicar en

2 – F. M. Dopico, S. Marcaida, M. C. Quintana, P. Van Dooren, *Local linearizations of rational matrices with application to rational approximations of nonlinear eigenvalue problems*, Linear Algebra and its Applications, 604 (2020) 441 – 475. DOI: 10.1016/j.laa.2020.07.004

3 – F. M. Dopico, S. Marcaida, M. C. Quintana, P. Van Dooren, *Block full rank linearizations of rational matrices*, submitted. URL: arXiv:2011.00955v1

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El Capítulo 9 está completamente incluido en la publicación

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Abstract

This PhD thesis belongs to the area of Numerical Linear Algebra. Specifically, to the numerical solution of the Rational Eigenvalue Problem (REP). This is a type of eigenvalue problem associated with rational matrices, which are matrices whose entries are rational functions. REPs appear directly from applications or as approximations to arbitrary Nonlinear Eigenvalue Problems (NLEPs). Rational matrices also appear in linear systems and control theory, among other applications. Nowadays, a competitive method for solving REPs is via linearization. This is due to the fact that there exist backward stable and efficient algorithms to solve the linearized problem, which allows to recover the information of the original rational problem. In particular, linearizations transform the REP into a generalized eigenvalue problem in such a way that the pole and zero information of the corresponding rational matrix is preserved. To recover the pole and zero information of rational matrices, it is fundamental the notion of polynomial system matrix, introduced by Rosenbrock in 1970, and the fact that rational matrices can always be seen as transfer functions of polynomial system matrices.

This thesis addresses different topics regarding the problem of linearizing REPs. On the one hand, one of the main objectives has been to develop a theory of linearizations of rational matrices to study the properties of the linearizations that have appeared so far in the literature in a general framework. For this purpose, a definition of local linearization of rational matrix is introduced, by developing as starting point the extension of Rosenbrock's minimal polynomial system matrices to a local scenario. This new theory of local linearizations captures and explains rigorously the properties of all the different linearizations that have been used from the 1970's for computing zeros, poles and eigenvalues of rational matrices. In particular, this theory has been applied to a number of pencils that have appeared in some influential papers on solving numerically NLEPs through rational approximation.

On the other hand, the work has focused on the construction of linearizations of rational matrices taking into account different aspects. In some cases, we focus on preserving particular structures of the corresponding rational matrix in the linearization. The structures considered are symmetric (Hermitian), skew-symmetric (skew-Hermitian), among others. In other cases, we focus on the direct construction of the linearizations from the original representation of the rational matrix. The representations considered are rational matrices expressed as the sum of their polynomial and strictly proper parts, rational matrices written as general transfer function matrices, and rational matrices expressed by their Laurent expansion around the point at infinity. In addition, we describe the recovery rules of the information of the original rational matrix from the information of the new linearizations, including in some cases not just the zero and pole information but also the information about the minimal indices.

Finally, in this dissertation we tackle one of the most important open problems related to linearizations of rational matrices. That is the analysis of the backward stability for solving REPs by running a backward stable algorithm on a linearization. On this subject, a global backward error analysis has been developed by considering the linearizations in the family of "block Kronecker linearizations". An analysis of this type had not been developed before in the literature.

Contents

Chapter 1

Introduction, motivation and summary of main results

The main objects of study in this PhD thesis are rational matrices. A rational matrix $R(\lambda)$ is a matrix whose entries are quotients of polynomials in the scalar variable λ, i.e., rational functions. Rational matrices have received a lot of attention since the 1950s, as a consequence of their fundamental role in linear systems and control theory [67, 68]. Classical references on rational matrices and their applications to these areas are [55, 78].

Rational matrices can have poles and zeros and have rational right and left null spaces, which can be trivial, i.e., equal to $\{0\}$. Via the notion of the Smith-McMillan form, one can associate partial multiplicities to the poles and zeros, and via the notion of minimal polynomial bases for rational vector spaces, one can associate the so called right and left minimal indices to the right and left null spaces, which exist only when the rational matrix is singular, i.e., rectangular or square with identically zero determinant. All these quantities are among the most relevant structural data of a rational matrix [55, 68].

Many classic problems in linear systems and control theory can be posed in terms of rational matrices [55, 78, 90] and are related to the computation of their zeros and poles [84]. For that, it is fundamental the key concept introduced by Rosenbrock [78] in 1970 of polynomial system matrices of rational matrices. This notion allows us, among other things, to include simultaneously all the information about the zeros and the poles of a rational matrix into a polynomial matrix. More precisely, polynomial system matrices $P(\lambda)$ are block partitioned polynomial matrices of the form

$$P(\lambda) := \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix},$$

where $A(\lambda)$ is assumed to be regular. Rosenbrock showed that the finite pole and zero structure of its transfer function matrix

$$R(\lambda) := D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$$

can be retrieved from the polynomial matrices $A(\lambda)$ and $P(\lambda)$, respectively, provided $P(\lambda)$ is minimal, meaning that the polynomial matrices

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix},$$

have, respectively, full row and column rank for all finite λ.

Currently, the computation of the zeros of rational matrices is also playing a fundamental role in the very active area of Nonlinear Eigenvalue Problems (NLEPs) [46], either because they appear directly in rational eigenvalue problems (REPs) modeling real-life problems [69] or because other NLEPs are approximated by REPs [46, 47, 60, 61, 35, 80]. Given a rational matrix $R(\lambda)$, the REP consists of finding scalars λ_0 such that λ_0 is not a pole of $R(\lambda)$, i.e., $R(\lambda_0)$ has finite entries, and that there exist nonzero constant vectors x and y satisfying

$$R(\lambda_0)x = 0 \quad \text{and} \quad y^T R(\lambda_0) = 0,$$

under the assumption that $R(\lambda)$ is regular, i.e., $R(\lambda)$ is square and its determinant is not identically equal to zero. The scalar λ_0 is said to be an eigenvalue of $R(\lambda)$ and the vectors x and y^T are called, respectively, right and left eigenvectors associated with λ_0. A non-regular rational matrix $R(\lambda)$ is also called singular. In general, regardless of whether $R(\lambda)$ is regular or singular, a scalar λ_0 is said to be an eigenvalue of $R(\lambda)$ if λ_0 is not a pole of $R(\lambda)$ and

$$\operatorname{rank} R(\lambda_0) < \operatorname{nrank} R(\lambda),$$

where $\operatorname{nrank} R(\lambda)$ denotes the normal rank of $R(\lambda)$, i.e., the rank of $R(\lambda)$ over the field of rational functions in λ. Then, the problem of finding the eigenvalues of a rational matrix can also be seen as the problem of finding the zeros of $R(\lambda)$ that are not poles. If λ_0 is a pole of $R(\lambda)$ and there exists a polynomial vector $v(\lambda)$ such that $v(\lambda_0) \neq 0$ and that $\lim_{\lambda \to \lambda_0} R(\lambda)v(\lambda) = 0$ then λ_0 is said to be an eigenpole of $R(\lambda)$ [2]. A couple of examples of REPs and associated rational matrices are:

- The loaded elastic string problem in [13], whose corresponding rational matrix is of the form:

  $$R(\lambda) = A - \lambda B + \frac{\lambda}{\lambda - \sigma} E,$$

  where $A, B \in \mathbb{R}^{n \times n}$ are symmetric tridiagonal matrices, E has only one nonzero entry in (n, n) position and n is large.

- The damped vibration of a viscoelastic structure problem in [69], whose corresponding rational matrix is of the form:

  $$R(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + b_i \lambda} G_i,$$

  where $M, K \in \mathbb{R}^{n \times n}$ are positive definite and n is large.

1.1 Linearizations of rational matrices

In the 1970s the first numerical algorithms for computing the structural data of rational matrices were developed, see [84] and the references therein. The most reliable algorithms were based on constructing a matrix pencil, i.e., a matrix polynomial of degree 1, containing the information about the structural data of the considered rational matrix [84, 92]. These pencils are among the first examples of linearizations of rational matrices and are, in fact, particular instances of minimal polynomial system matrices. Then, backward stable algorithms developed also in the 1970s, for computing the eigenvalues and/or other structural data of general pencils [71, 83], were applied to these matrix pencils. Nowadays, given a matrix pencil $L(\lambda)$ linearizing a rational matrix, one can apply to $L(\lambda)$ the backward stable eigenvalue algorithms developed in [71, 83] for problems of moderate size, or Krylov methods adapted to the structure of $L(\lambda)$ in the large-scale setting [24, 47].

As we explained in the previous paragraph, the approach of constructing a linear polynomial matrix containing information about the structural data of rational matrices was first introduced in the 1970s. However, a formal definition of linearization of rational matrices was not given at that time. The term linearization of rational matrix was also used in the reference [79] and in works on NLEPs [47, 60, 35] without referring to a formal definition of linearization. A first formal definition of linearization of a rational matrix was proposed in [2]. Then, a different definition was introduced in [6], together with the first formal definition of strong linearization, i.e., a pencil that allows to recover both the finite and infinite pole and zero structure of $R(\lambda)$. However, the pencils considered for linearizing NLEPs do not satisfy the definitions of linearizations given in [2, 6]. This was our motivation to develop a more general theory of linearizations of rational matrices in Chapter 4.

In addition to formal definitions, some works on linearizations of rational matrices have introduced families of strong linearizations that are constructed from the fact that any rational matrix $R(\lambda)$ can be written as

$$R(\lambda) = D(\lambda) + C(I_n\lambda - A)^{-1}B \tag{1.1}$$

where $D(\lambda)$ is a polynomial matrix, called the polynomial part of $R(\lambda)$, and $C(I_n\lambda - A)^{-1}B$ is a minimal state-space realization [48] of the strictly proper part of $R(\lambda)$ (see (2.1)). Thanks to this property, strong linearizations of rational matrices are constructed from strong linearizations of the polynomial part $D(\lambda)$ combined with minimal state-space realizations of the strictly proper part. In addition, the study of the recovery properties from these families of linearizations has received considerable attention. References in these lines include [2, 3, 4, 6, 8, 17, 18, 19, 27].

Among the new classes of strong linearizations, we mention the family of "strong block minimal bases linearizations" of rational matrices introduced in [6, Theorem 5.11], as a wide family of strong linearizations constructed by considering "strong block minimal bases pencils" associated to their polynomial parts, see [26]. They include as particular cases the Fiedler-like linearizations (modulo permutations) [8, Section 8] and are valid for general rectangular rational matrices. Chapter 9 is devoted to analyse the backward stability when running a backward stable algorithm for computing the eigenvalues of a particular type of the strong block minimal bases linearization of rational matrices developed in [6], which are called "block Kronecker linearizations". For that, we generalize in a nontrivial way the analysis of backward errors of the "block Kronecker linearizations" of polynomial matrices developed in [26] in combination with the theory of strong linearizations for rational matrices in [6].

The question whether or not other strong linearizations of rational matrices can be constructed based on other kinds of strong linearizations of the polynomial parts arises naturally. For answering this question, we construct in Chapter 3 strong linearizations of a rational matrix by using strong linearizations of its polynomial part $D(\lambda)$ that belong to other important family of strong linearizations of polynomial matrices, i.e., the so-called vector spaces of linearizations, originally introduced in [63], further studied in [21, 72], and extended in [36]. In particular, we consider strong linearizations of $D(\lambda)$ that belong to the ansatz spaces $\mathbb{M}_1(D)$ or $\mathbb{M}_2(D)$, developed by Faßbender and Saltenberger in [36], where polynomial matrices are expressed in terms of polynomial bases other than the monomial basis. Another motivation of the results in Chapter 3 is that, in order to compute the eigenvalues of polynomial matrices from linearizations, the work [57] shows that, for polynomial matrices of large degree, the use of the monomial basis to express the matrix leads to numerical instabilities due to the ill-conditioning of the eigenvalues in certain situations. According to the algorithms in [24, 79, 81], it is expected that this instability appears also while computing eigenvalues of REPs when the polynomial part of the rational matrix has large degree and is expressed in terms of the monomial basis. For that reason, it is of interest to consider rational matrices with polynomial parts expressed in other bases as the Chebyshev basis. As a consequence of the results in Chapter 3, we can conclude that the combination of the results in Chapter 3 and those in [6] allows us to construct very easily infinitely many strong linearizations of rational matrices via the following three-step strategy: (1) express the rational matrix as the sum of its polynomial and strictly proper parts; (2) construct any of the strong linearizations of the polynomial part known so far; and (3) combine adequately that strong linearization with a minimal state-space realization of the strictly proper part.

Despite the intense activity described in the previous paragraph, there are pencils that have been used in influential references as [47, 60] for solving numerically REPs that approximate NLEPs which do not satisfy the definitions of linearization of rational matrices given in [2, 6]. The reason is that these definitions focus on pencils that allow to recover the complete pole and zero structure of rational matrices, while in [47, 60] only the eigenvalue information in a certain subset of the complex plane is necessary. This was our motivation to the development in Chapter 4 of a new theory of linearizations of rational matrices in a local sense. These linearizations are pencils that preserve the structure of zeros and poles of the corresponding rational matrix in a particular subset of the underlying field, in the whole underlying field and/or at infinity. Apart from a new definition, a specific family of local linearizations of rational matrices is also introduced in Chapter 4, that are called block full rank linearizations, as a template that covers many of the pencils, available in the literature, for linearizing rational matrices. In Chapter 5 we study the properties of the linearizations for rational approximations of NLEPs in [47, 60] by using the theory in Chapter 4.

As we explained, there exist different methods for constructing linearizations of rational matrices when the corresponding rational matrix is expressed as in (1.1). Furthermore, if the rational matrix is not in the form (1.1), there exist procedures for obtaining such a representation [55, 78, 90]. However, these procedures are not simple, and may introduce errors that were not present in the original problem. Motivated by this fact, we construct in Chapter 6 linearizations for rational matrices from more general representations. In particular, we will show how to construct linearizations of rational matrices that are written in the general form

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda), \tag{1.2}$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are polynomial matrices, possibly non linear and possibly expressed in different bases. Representations of the form (1.2) arise naturally, for example, when solving REPs of the form

$$R(\lambda)x = \left(P(\lambda) + \sum_{i=1}^{m} \frac{n_i(\lambda)}{d_i(\lambda)}(A_i\lambda - B_i) \right) x = 0,$$

where $P(\lambda)$ is a polynomial matrix, $\frac{n_i(\lambda)}{d_i(\lambda)}$ are scalar rational functions, and A_i and B_i are constant matrices.

1.1.1 Strongly minimal linearizations: preserving structures

In Chapter 7 we consider a particular type of the strong linearizations for rational matrices defined in Chapter 4, which we called strongly minimal linearizations. Such strongly minimal linearizations of a rational matrix $R(\lambda)$ are linear polynomial system matrices of the form

$$L(\lambda) := \begin{bmatrix} \lambda A_1 - A_0 & \lambda B_1 - B_0 \\ -\lambda C_1 + C_0 & \lambda D_1 - D_0 \end{bmatrix},$$

where $R(\lambda)$ is the transfer function matrix of $L(\lambda)$ and such that the pencils

$$\begin{bmatrix} \lambda A_1 - A_0 & \lambda B_1 - B_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda A_1 - A_0 \\ -\lambda C_1 + C_0 \end{bmatrix}$$

have no finite or infinite eigenvalues. We will see that the strong minimality conditions imply the strong irreducibility conditions in [91], and the former are easier to test. In addition, we will also show that when the strong minimality conditions are not satisfied, we can reduce the system matrix to one where they are satisfied without modifying the corresponding transfer function matrix $R(\lambda)$.

One important property of strongly minimal linearizations is that they can preserve many different structures of the original rational matrix without imposing any restriction on such a matrix, as we explain in Chapter 8. This result is in stark contrast with previous results existing in the literature on linearizations that preserve structures, which impose conditions on the rational matrix as a consequence of using other definitions of linearizations. This is one of the most important results in this thesis, and for proving it, we will use some ideas developed in [27] (Chapter 3), where we also construct structured linearizations for symmetric and Hermitian rational matrices. In contrast, in Chapter 8 we use the notion of strongly minimal linearization instead of the notion of strong linearization for rational matrices in [6].

It is known that structured polynomial and rational matrices have symmetries in their spectra [40, 51, 59, 64], and these spectral symmetries reflect specific physical properties, as they originate usually from the physical symmetries of the underlying applications [66, 64]. Such special structures occur in numerous applications in engineering, mechanics, control, and linear systems theory. Some of the most common algebraic structures that appear in applications are the (skew-)symmetric [51], and alternating structures [64]. Symmetric (or Hermitian) matrix polynomials arise in the classical problem of vibration analysis [41], and alternating matrix polynomials find applications, for instance, in the study of corner singularities in anisotropic elastic materials [70], in the study of gyroscopic systems [59], in the continuous-time linear-quadratic optimal control problem and in the spectral factorization problem [85, 40].

Because of the numerous applications where structured polynomial and rational matrices occur, there have been several attempts to construct linearizations for them that display the same structure [1, 19, 27, 37, 40, 51, 58]. But these earlier attempts impose certain conditions on the corresponding polynomial or rational matrix for the construction of the linearization to apply, such as regularity, strict properness or invertibility of certain matrix coefficients. In Chapter 8 we give a construction of structured linearizations for structured polynomial and rational matrices without imposing any conditions, by using the notion of strongly minimal linearization. Moreover, the proof used for this construction is different from these earlier papers, and we claim it to be simpler as well.

We give more details on the structure of the thesis in the following section.

1.2 Structure and organization of the thesis

This dissertation is organized as follows:

Chapter 2 presents some preliminaries and basic notions on rational matrices that will be used throughout the dissertation. In particular, we introduce the definitions of the structural data of rational matrices and the key concept of polynomial system matrix of a transfer function matrix introduced by Rosenbrock [78], as well as some important properties related to it.

Chapter 3 is devoted to the construction of strong linearizations of rational matrices by using the definition of strong linearization in [6] and writing the corresponding rational matrix as the sum of its polynomial and strictly proper parts. In particular, in Sections 3.2 and 3.3, we construct strong linearizations of rational matrices with polynomial parts expressed in terms of a basis that satisfies a three term recurrence relation. In addition, in Section 3.8, we briefly discuss how to construct strong linearizations when the polynomial part is expressed in other bases. We emphasize that the construction of these new strong linearizations is a consequence of the theory of strong linearizations developed in [6] together with Lemma 3.1.2. More precisely, given a strong linearization of a rational matrix, Lemma 3.1.2 allows to obtain infinitely many strong linearizations of the rational matrix by using strict equivalence with a certain structure. The rest of this chapter is organized as follows. In Section 3.4, we show how to recover the eigenvectors of the rational matrix from those of its strong linearizations constructed in Sections 3.2 and 3.3. Moreover, given a symmetric rational matrix, in Section 3.6 we construct strong linearizations that preserve its symmetric structure by using symmetric realizations of the strictly proper part, which are introduced in Section 3.5, and strong linearizations in the double ansatz space $\mathbb{DM}(D)$ [36] of the polynomial part. Finally, in Section 3.7, we present analogous results for Hermitian rational matrices.

Chapter 4 presents a rigorous theory of local linearizations of rational matrices. For that, we first extend the concept of Rosenbrock's minimal polynomial system matrices to a local sense. Local minimal polynomial system matrices are defined and studied in Section 4.1. Section 4.2 presents the main definitions and properties of local linearizations of rational matrices. Sections 4.3 and 4.4 introduce the very general families of block full rank pencils and linearizations, as templates that cover many of the pencils available in the literature for linearizing rational matrices. Then, in **Chapter 5**, the theory of local linearizations is applied to a number of pencils that have appeared in the influential papers [47, 60] on solving numerically NLEPs by combining rational approximations and linearizations of the resulting rational matrices. It will be emphasized throughout the chapter that the theory of local linearizations allows us to view these pencils, and to explain their properties, from rather different perspectives. In particular, the pencils introduced in [47] are analysed and studied in Section 5.1, and those in [60] in Section 5.2.

In **Chapter 6**, by using the theory of local linearizations in Chapter 4, we construct a new family of linearizations for rational matrices from general representations. These linearizations are given in Sections 6.1 and 6.2. In Section 6.3, we present an example that highlights the difference between our approach and the previous approaches to the problem of linearizing rational matrices. In Section 6.4 we study how to recover minimal bases, minimal indices and eigenvectors of rational matrices from those of their linearizations constructed in Sections 6.1 and 6.2. Finally, we apply the new linearizations to solve scalar rational equations in Section 6.5.

Chapter 7 recalls the notion of strong minimality of polynomial system matrices introduced in Chapter 4 and also the definition of strongly irreducible polynomial system matrix in [91]. These notions are given in Section 7.2 where, in addition, we establish the relation between them for the case of linear polynomial system matrices. In Subsection 7.2.2, we study the relation of strongly minimal linearizations with other classes of linearizations for polynomial and rational matrices in the literature. We then give, in Section 7.3, an algorithm to construct a strongly minimal linear system matrix from an arbitrary one, and we discuss the computational aspects in Section 7.4. Finally, we show some numerical experiments in Section 7.5.

Chapter 8 is devoted to the construction of strongly minimal linearizations for arbitrary and structured rational matrices. For that, in Section 8.1, we first show how to construct strongly minimal linearizations of arbitrary polynomial matrices and, in Section 8.2, we extend this construction to structured strongly minimal linearizations of structured polynomial matrices. Then, in Sections 8.3 and 8.4, we develop the same results for strictly proper rational matrices. That is, we build strongly minimal linearizations for arbitrary and structured strictly proper rational matrices, respectively. In Section 8.5, we combine the results in the previous sections of the chapter to construct strongly minimal linearizations for arbitrary and structured rational matrices. Finally, in Section 8.6, we comment some algorithmic aspects.

Chapter 9 studies the backward stability of running a backward stable algorithm to compute the eigenvalues on a pencil $S(\lambda)$ that is a strong linearization of a rational matrix of block Kronecker type. We describe in Section 9.2 the basic systems of matrix equations we will use, and, in Section 9.3, some bounds for the singular values of certain matrices related to these systems of matrix equations. In Section 9.4 we explain how to restore the structure of block Kronecker linearizations of rational matrices after they suffer sufficiently small perturbations. Then, in Section 9.5, we give sufficient conditions on the pencil $S(\lambda)$ and on the corresponding rational matrix that guarantee structural backward stability for (regular or singular) REPs solved via block Kronecker linearizations. In Section 9.6 we state the results for rational matrices having a linear polynomial part, since those in previous sections are developed for rational matrices with polynomial parts of degree greater than 1. In Section 9.7 we derive a scaling technique that allows to guarantee structural backward stability, taking into account the conditions in Section 9.5. We conclude the chapter, in Section 9.8, by presenting a number of numerical results illustrating our theoretical bounds.

Finally, **Chapter 10** summarizes the main conclusions of this dissertation (see Section 10.1). In addition, we give a list of papers (published or in progress) where the original contributions of this thesis are contained (Section 10.2) and a list of conferences where many of the results have been presented (Section 10.3). Moreover, in Section 10.4, some open problems related to the results of this dissertation are proposed.

1.3 Notation

Throughout this dissertation, $\mathbb{F}$ denotes an arbitrary field. The algebraic closure of $\mathbb{F}$ is denoted by $\overline{\mathbb{F}}$. In some chapters, $\mathbb{F}$ will be considered to be an algebraically closed field, that is, $\mathbb{F} = \overline{\mathbb{F}}$.

The ring of univariate polynomials in the variable λ with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[\lambda]$, whose elements are of the form

$$p(\lambda) := a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \quad \text{with} \quad a_i \in \mathbb{F} \quad \text{for} \quad i = 0, \ldots, n.$$

If $\mathbb{F}$ is algebraically closed then every non-constant polynomial in $\mathbb{F}[\lambda]$ has a root in $\mathbb{F}$. If $a_n \neq 0$ then $p(\lambda)$ is said to have degree n and is denoted by $\deg(p(\lambda)) = n$. The field of rational functions over $\mathbb{F}$, denoted by $\mathbb{F}(\lambda)$, is the quotient field of the polynomial ring $\mathbb{F}[\lambda]$. Namely,

$$\mathbb{F}(\lambda) = \left\{ \frac{p(\lambda)}{q(\lambda)} \, : \, p(\lambda), q(\lambda) \in \mathbb{F}[\lambda], q(\lambda) \neq 0 \right\},$$

with the equivalence relation $\dfrac{p_1(\lambda)}{q_1(\lambda)} = \dfrac{p_2(\lambda)}{q_2(\lambda)}$ if $p_1(\lambda)q_2(\lambda) = q_1(\lambda)p_2(\lambda)$.

The sets of $p \times m$ matrices with elements in $\mathbb{F}(\lambda)$, $\mathbb{F}[\lambda]$ and $\mathbb{F}$ are denoted by $\mathbb{F}(\lambda)^{p\times m}$, $\mathbb{F}[\lambda]^{p\times m}$ and $\mathbb{F}^{p\times m}$, respectively. The elements of $\mathbb{F}[\lambda]^{p\times m}$ are called polynomial matrices. The degree of a polynomial matrix $P(\lambda)$ is the maximum degree of its entries and is denoted by $\deg(P(\lambda))$ or $\deg P(\lambda)$. If $\deg(P(\lambda))$ is equal to 1 or 0, then $P(\lambda)$ is said to be a pencil. The elements of $\mathbb{F}(\lambda)^{p\times m}$ are called rational matrices. The normal rank of a polynomial or rational matrix $R(\lambda)$ is the size of its largest non identically zero minor and is denoted by $\operatorname{nrank} R(\lambda)$.

Given a constant matrix $A \in \mathbb{F}^{p\times m}$, the rank of A is the size of its largest non zero minor and is denoted by $\operatorname{rank} A$. The transpose matrix of A is denoted by A^T. If $\mathbb{F}$ is the field of complex numbers $\mathbb{C}$, A^* denotes the conjugate transpose of A. The Kronecker product of two constant matrices A and B is denoted by $A \otimes B$ (see [54, Chapter 4]). Diagonal matrices, with diagonal entries $d_1, \ldots, d_m$, are often denoted by $\operatorname{diag}(d_1, \ldots, d_m)$.

Chapter 2

Preliminaries on rational matrices

In this chapter, we introduce preliminaries and basic notions on rational matrices. Some results presented here are original contributions of the author. In particular, such results are Proposition 2.1.5, Lemma 2.4.5, Propositions 2.4.7 and 2.4.8 and Lemma 2.4.10.

We consider an arbitrary field $\mathbb{F}$, e.g., the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. Recall that rational matrices are matrices whose entries are rational functions, and a rational function $r(\lambda)$ is a ratio such that both numerator $n(\lambda)$ and denominator $d(\lambda)$ are polynomials. Namely,

$$r(\lambda) = \frac{n(\lambda)}{d(\lambda)} \in \mathbb{F}(\lambda).$$

Regarding the degrees of the numerator and the denominator, the rational function $r(\lambda)$ is said to be *proper* if $\deg(n(\lambda)) \leq \deg(d(\lambda))$, and *strictly proper* if $\deg(n(\lambda)) < \deg(d(\lambda))$. We denote by $\mathbb{F}_{pr}(\lambda)$ the ring of proper rational functions. The units of $\mathbb{F}_{pr}(\lambda)$ are called *biproper rational functions*, i.e., rational functions having the same degree of numerator and denominator. $\mathbb{F}_{pr}(\lambda)^{p\times m}$ denotes the set of $p \times m$ matrices with entries in $\mathbb{F}_{pr}(\lambda)$, which are called *proper matrices*. A *biproper matrix* is a square proper matrix whose determinant is a biproper rational function.

By the division algorithm for polynomials, any rational function $r(\lambda)$ can be uniquely written as $r(\lambda) = p(\lambda) + r_{sp}(\lambda)$, where $p(\lambda)$ is a polynomial and $r_{sp}(\lambda)$ a strictly proper rational function. Therefore, any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be uniquely written as

$$R(\lambda) = D(\lambda) + R_{sp}(\lambda) \tag{2.1}$$

where $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is a polynomial matrix and $R_{sp}(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times m}$ is a *strictly proper rational matrix*, i.e., the entries of $R_{sp}(\lambda)$ are strictly proper rational functions. Then, $D(\lambda)$ is called the *polynomial part* of $R(\lambda)$ and $R_{sp}(\lambda)$ is called the *strictly proper part* of $R(\lambda)$.

A rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is said to be *regular or nonsingular* if $R(\lambda)$ is square (i.e., $p = m$) and $\det R(\lambda) \not\equiv 0$. Otherwise, $R(\lambda)$ is said to be *singular*. Square polynomial matrices with nonzero constant determinant are said to be *unimodular*, i.e., matrices $U(\lambda)$ in $\mathbb{F}[\lambda]^{m\times m}$ such that $\det U(\lambda)$ is a nonzero constant. Equivalently, a polynomial matrix $U(\lambda)$ is unimodular if and only if the inverse of $U(\lambda)$ is also polynomial. A diagonal form for rational matrices $R(\lambda)$ is the so-called Smith-McMillan form, introduced by McMillan in 1952 [68], which uses unimodular equivalences. That is, transformations of the form $U_1(\lambda)R(\lambda)U_2(\lambda)$, where both $U_1(\lambda)$ and $U_2(\lambda)$ are unimodular matrices.

Definition 2.0.1 ((Finite or global) Smith–McMillan form)**.** *For any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ there exist unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that*

$$U_1(\lambda)R(\lambda)U_2(\lambda) = \begin{bmatrix} \operatorname{diag}\left(\dfrac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \dfrac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix} \tag{2.2}$$

where $r = \operatorname{nrank} R(\lambda)$ and, for $i = 1, \dots, r$, $\dfrac{\epsilon_i(\lambda)}{\psi_i(\lambda)}$ are nonzero irreducible rational functions with $\epsilon_i(\lambda)$ and $\psi_i(\lambda)$ monic polynomials (i.e., with leading coefficient equal to 1) and $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ while $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$, where $\mid$ stands for divisibility. The diagonal matrix on the right hand side in (2.2) is called the (finite or global) Smith–McMillan form of $R(\lambda)$.

Other more recent references for the Smith-McMillan form of a rational matrix are [55, 78, 90]. The rational functions $\dfrac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \dfrac{\epsilon_r(\lambda)}{\psi_r(\lambda)}$ in (2.2) are called the *(finite) invariant rational functions* of $R(\lambda)$ and the *finite poles* and *zeros* of $R(\lambda)$ are the roots in $\overline{\mathbb{F}}$ of the denominators and numerators of the invariant rational functions, respectively. We give more details about zeros and poles in the following section. Notice that the Smith–McMillan form of a rational matrix is invariant under unimodular equivalence. If $R(\lambda)$ is polynomial then $\psi_1(\lambda) = \cdots = \psi_r(\lambda) = 1$, $\epsilon_1(\lambda), \dots, \epsilon_r(\lambda)$ are called the *invariant polynomials* of $R(\lambda)$, and the diagonal matrix in (2.2) is called the *Smith normal form* of $R(\lambda)$.

2.1 Zeros and poles: Local Smith–McMillan form

As we explained, the finite poles and zeros of a rational matrix are the roots in $\overline{\mathbb{F}}$ of the polynomials that appear on the denominators and numerators, respectively, in its (global) Smith-McMillan form. In this section, we introduce a local definition of the Smith-McMillan form of rational matrices, and more notions related to their poles and zeros. For that, we first introduce some definitions and equivalence transformations on rational matrices.

Definition 2.1.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let $\lambda_0 \in \mathbb{F}$, and $\Sigma \subseteq \mathbb{F}$ be nonempty.*

(i) $R(\lambda)$ is defined at λ_0 if $R(\lambda_0) \in \mathbb{F}^{p\times m}$.

(ii) $R(\lambda)$ is defined in Σ if $R(\lambda)$ is defined at each $\lambda_0 \in \Sigma$.

(iii) $R(\lambda)$ is defined at ∞ if $R(1/\lambda)$ is defined at 0.

Definition 2.1.2. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$. Let $\lambda_0 \in \mathbb{F}$, and $\Sigma \subseteq \mathbb{F}$ be nonempty.*

(i) $R(\lambda)$ is invertible at λ_0 if it is defined at λ_0 and $\det R(\lambda_0) \neq 0$.

(ii) $R(\lambda)$ is invertible in Σ if it is invertible at each $\lambda_0 \in \Sigma$.

(iii) $R(\lambda)$ is invertible at ∞ if $R(1/\lambda)$ is invertible at 0.

Notice that a rational matrix $R(\lambda)$ is defined at a point λ_0 if and only if λ_0 is not a pole of $R(\lambda)$. In addition, a rational matrix is unimodular if and only if it is invertible in $\overline{\mathbb{F}}$, and is biproper if and only if it is invertible at infinity. See [55] and [90] for more information on these and other concepts related to rational matrices.

In regard to the previous definitions, we introduce some equivalence relations defined in the set of rational matrices [9, 10, 42].

Definition 2.1.3 (Equivalences)**.** *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. Two rational matrices $R_1(\lambda), R_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are said to be equivalent in Σ if there exist rational matrices $G_1(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ and $G_2(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ both invertible in Σ such that*

$$G_1(\lambda)R_1(\lambda)G_2(\lambda) = R_2(\lambda).$$

This is denoted by

$$R_1(\lambda) \sim_\Sigma R_2(\lambda).$$

When $\Sigma = \{\lambda_0\}$, we have local equivalence at λ_0 and is denoted by $R_1(\lambda) \sim_{\lambda_0} R_2(\lambda)$. If $G_1(\lambda)$ and $G_2(\lambda)$ are biproper then $R_1(\lambda)$ and $R_2(\lambda)$ are said to be equivalent at ∞, which is denoted by $R_1(\lambda) \sim_\infty R_2(\lambda)$.

If, in Definition 2.1.3, the rational matrices $G_1(\lambda)$ and $G_2(\lambda)$ are both unimodular, then the standard definition of unimodular equivalence is recovered. In this case, $R_1(\lambda)$ and $R_2(\lambda)$ are said to be unimodularly equivalent.

We can now present the notion of the local Smith-McMillan form (see [10, 88]). Since poles and zeros of a rational matrix belong to the algebraic closure $\overline{\mathbb{F}}$, we consider $\overline{\mathbb{F}}$ instead of $\mathbb{F}$ in order to define the local Smith-McMillan form.

Definition 2.1.4 (Local Smith–McMillan form)**.** *Let $\lambda_0 \in \overline{\mathbb{F}}$ and let $R(\lambda) \in \overline{\mathbb{F}}(\lambda)^{p\times m}$ be a rational matrix, with $r = \operatorname{nrank} R(\lambda)$. $R(\lambda)$ admits a representation of the form:*

$$R(\lambda) \sim_{\lambda_0} \begin{bmatrix} \operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1}, \ldots, (\lambda-\lambda_0)^{\nu_r}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix} \tag{2.3}$$

for some integers $\nu_1, \ldots, \nu_r$ with $\nu_1 \leq \ldots \leq \nu_r$. The diagonal matrix on the right hand side in (2.3) *is called the local Smith–McMillan form of $R(\lambda)$ at λ_0.*

The integers $\nu_1, \ldots, \nu_r$ are uniquely determined by $R(\lambda)$ and λ_0, and are called the *invariant orders or structural indices* at λ_0 of $R(\lambda)$. In order to define zeros and poles, together with partial multiplicities, we need to distinguish between positive and negative invariant orders [55, 90]:

(a) If $\nu_i > 0$, for some $i = 1, \ldots, r$, then λ_0 is said to be a *zero* of $R(\lambda)$ with *partial multiplicity* ν_i; and the factor $(\lambda - \lambda_0)^{\nu_i}$ is called a *zero elementary divisor* of $R(\lambda)$ at λ_0.

(b) If $\nu_i < 0$, for some $i = 1, \ldots, r$, then λ_0 is said to be a *pole* of $R(\lambda)$ with *partial multiplicity* $-\nu_i$; and the factor $(\lambda - \lambda_0)^{-\nu_i}$ is called a *pole elementary divisor* of $R(\lambda)$ at λ_0.

The zero and pole elementary divisors of $R(\lambda)$ in a nonempty subset $\Sigma \subseteq \overline{\mathbb{F}}$ are the zero and pole elementary divisors of $R(\lambda)$ for all $\lambda_0 \in \Sigma$, respectively. If $R(\lambda)$ is a polynomial matrix then the nonzero integers ν_i are all positive and are called the *partial multiplicities* of $R(\lambda)$ at λ_0, and the factors $(\lambda - \lambda_0)^{\nu_i}$ with $\nu_i \neq 0$ are called *elementary divisors* of $R(\lambda)$ at λ_0.

The *invariant orders at infinity* $q_1 \leq \cdots \leq q_r$ of a rational matrix $R(\lambda)$ are defined as the invariant orders at $\lambda_0 = 0$ of $R(1/\lambda)$, and the *Smith–McMillan form of* $R(\lambda)$ *at* ∞ is of the form

$$R(\lambda) \sim_\infty \begin{bmatrix} \operatorname{diag}\left(\frac{1}{\lambda^{q_1}}, \ldots, \frac{1}{\lambda^{q_r}}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix}. \tag{2.4}$$

For $q_i < 0$, $-q_i$ are the *partial multiplicities of* ∞ *as pole* while, for $q_i > 0$, q_i are the *partial multiplicities of* ∞ *as zero* (see [55]).

By using the local Smith–McMillan form, the next result in [28] shows that the equivalence of rational matrices in nonempty sets is a local property.

Proposition 2.1.5. *Let* $\Sigma \subseteq \overline{\mathbb{F}}$ *be nonempty. Two rational matrices of the same size are equivalent in* Σ *if and only if they are equivalent at each* $\lambda_0 \in \Sigma$.

Proof. If two rational matrices are equivalent in Σ then, by Definition 2.1.3, it is straightforward that they are equivalent at each $\lambda_0 \in \Sigma$. For the converse, suppose that $G(\lambda) \sim_{\lambda_0} H(\lambda)$ for all $\lambda_0 \in \Sigma$. Then, $G(\lambda)$ and $H(\lambda)$ have the same local Smith–McMillan forms at each $\lambda_0 \in \Sigma$. Let us consider $M_G(\lambda)$ and $M_H(\lambda)$ as the global Smith–McMillan forms of $G(\lambda)$ and $H(\lambda)$, respectively. Thus, there exist unimodular matrices $U_i^G(\lambda)$, $U_i^H(\lambda)$, for $i = 1, 2$, such that $G(\lambda) = U_1^G(\lambda)M_G(\lambda)U_2^G(\lambda)$, $H(\lambda) = U_1^H(\lambda)M_H(\lambda)U_2^H(\lambda)$, and we can write

$$M_G(\lambda) = \operatorname{diag}\left(f_1(\lambda)g_1(\lambda), \ldots, f_r(\lambda)g_r(\lambda), 0_{(p-r)\times(m-r)}\right), \text{ and}$$
$$M_H(\lambda) = \operatorname{diag}\left(f_1(\lambda)h_1(\lambda), \ldots, f_r(\lambda)h_r(\lambda), 0_{(p-r)\times(m-r)}\right),$$

where $f_i(\lambda)$ are rational functions which are either equal to one or have poles or zeros in Σ, while $g_i(\lambda)$ and $h_i(\lambda)$ are rational functions that do not have any poles or zeros in Σ. Let us define $R(\lambda) := \operatorname{diag}\left(\frac{h_1(\lambda)}{g_1(\lambda)}, \ldots, \frac{h_r(\lambda)}{g_r(\lambda)}, I_{m-r}\right)$. Hence, $M_H(\lambda) = M_G(\lambda)R(\lambda)$. Therefore, $H(\lambda) = U_1^H(\lambda)U_1^G(\lambda)^{-1}G(\lambda)U_2^G(\lambda)^{-1}R(\lambda)U_2^H(\lambda)$, and $G(\lambda) \sim_\Sigma H(\lambda)$ since the matrices $U_1^H(\lambda)U_1^G(\lambda)^{-1}$ and $U_2^G(\lambda)^{-1}R(\lambda)U_2^H(\lambda)$ are invertible in Σ. ■

2.2 The rational eigenvalue problem

A *finite eigenvalue* of a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is any $\lambda_0 \in \overline{\mathbb{F}}$ such that

$$\operatorname{rank} R(\lambda_0) < \operatorname{nrank} R(\lambda), \text{ with } R(\lambda_0) \in \overline{\mathbb{F}}^{p\times m}.$$

That is, λ_0 is a finite zero of $R(\lambda)$ but not a pole. The *Rational Eigenvalue Problem* (REP) consists of finding the eigenvalues of $R(\lambda)$. If $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ is regular, which is the most common case in applications of REPs, the REP is equivalent to the problem of finding scalars $\lambda_0 \in \overline{\mathbb{F}}$ such that there exist nonzero constant vectors $x \in \overline{\mathbb{F}}^{m\times 1}$ and $y \in \overline{\mathbb{F}}^{m\times 1}$ satisfying

$$R(\lambda_0)x = 0 \quad \text{and} \quad y^T R(\lambda_0) = 0,$$

respectively. The vectors x are called *right eigenvectors associated with* λ_0, and the vectors y *left eigenvectors*. Given $\lambda_0 \in \overline{\mathbb{F}}$ and $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, we define the following vector spaces over $\overline{\mathbb{F}}$:

$$\begin{aligned} \mathcal{N}_r(R(\lambda_0)) &= \{x \in \overline{\mathbb{F}}^{m\times 1} : R(\lambda_0)x = 0\}, \text{ and} \\ \mathcal{N}_\ell(R(\lambda_0)) &= \{y^T \in \overline{\mathbb{F}}^{1\times p} : y^T R(\lambda_0) = 0\}, \end{aligned}$$

which are called, respectively, the *right and left nullspaces over* $\overline{\mathbb{F}}$ *of* $R(\lambda_0)$. If λ_0 is an eigenvalue of $R(\lambda)$, then $\mathcal{N}_r(R(\lambda_0))$ and $\mathcal{N}_\ell(R(\lambda_0))$ are non trivial and contain, respectively, the right and left eigenvectors of $R(\lambda)$ associated with λ_0.

Rational matrices may also have infinite eigenvalues. In order to define them, we need the notion of reversal.

Definition 2.2.1 (Reversal of a rational matrix)**.** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix expressed in the form* (2.1)*. We define the reversal of* $R(\lambda)$ *as the rational matrix*

$$\operatorname{rev} R(\lambda) = \lambda^d R\left(\frac{1}{\lambda}\right)$$

where $d = \deg(D(\lambda))$ *if* $R(\lambda)$ *is not strictly proper, and* $d = 0$ *otherwise.*

This definition of reversal extends the definition of reversal for polynomial matrices (see [22, Definition 2.12] or [63, Definition 2.2]). Following the usual definition in polynomial matrices [63, Definition 2.3], we say that $R(\lambda)$ has an *eigenvalue at* *infinity* if rev $R(\lambda)$ has an eigenvalue at $\lambda = 0$. If $R(\lambda)$ has an eigenvalue at infinity, we say that z is a *right* (respectively *left*) *eigenvector associated with infinity* if z is a right (respectively left) eigenvector associated with 0 of rev $R(\lambda)$.

Remark 2.2.2. Recall that, for finite points, eigenvalues are defined as those zeros that are not poles. However, if we define eigenvalues at infinity in this way any non-proper $R(\lambda)$ would not have eigenvalues at infinity. This is due to the fact that if $R(\lambda)$ is not proper, i.e., $\deg(D(\lambda)) \geq 1$, $R(\lambda)$ has always a pole at ∞ (see [10]). In particular, this would happen if $R(\lambda)$ is a polynomial matrix. Therefore, as in the polynomial case in [44], we consider rev $R(\lambda)$ in order to define eigenvalues at infinity.

Remark 2.2.3. The standard literature on polynomial matrices [39, 44] uses only the term eigenvalues instead of zeros and poles and the eigenvalue structure at infinity is defined through the notion of the reversal. We discuss the relation between finite and infinite zeros and poles and eigenvalues of polynomial matrices in this remark. Note first that a polynomial matrix $P(\lambda)$ does not have finite poles, i.e., all the invariant orders ν_i in (2.3) are nonnegative for any finite λ_0. Then, the finite eigenvalues of $P(\lambda)$ and their partial multiplicities [44] are exactly the same as the finite zeros of $P(\lambda)$ and their partial multiplicities. However, a polynomial matrix $P(\lambda)$ of degree d and normal rank r is said to have an eigenvalue at ∞ with partial multiplicities $t_q \leq \cdots \leq t_r$ if the reversal rev $P(\lambda) := \lambda^d P(1/\lambda)$ has an eigenvalue at 0 with partial multiplicities $t_q \leq \cdots \leq t_r$. In this situation the invariant orders of $P(\lambda)$ at ∞ (i.e., the invariant orders of $P(1/\lambda)$ at 0) are

$$(q_1, q_2, \ldots, q_r) = (\underbrace{0, \ldots, 0}_{q-1}, t_q, \ldots, t_r) - (d, d, \ldots, d). \tag{2.5}$$

Thus, the pole-zero and eigenvalue structures at infinity are different but easily related through (2.5).

2.3 Minimal bases and minimal indices

In this section, we review the notions of minimal bases and minimal indices of rational subspaces [38] and rational matrices.

It is known that every rational vector subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ over the field $\mathbb{F}(\lambda)$ has bases consisting of polynomial vectors. We refer to such bases as *polynomial bases.* Among them some are minimal in the following sense introduced by Forney [38]: a *minimal basis* of $\mathcal{V}$ is a polynomial basis of $\mathcal{V}$ consisting of polynomial vectors whose sum of degrees is minimal among all polynomial bases of $\mathcal{V}$. Minimal bases are not unique, but the ordered list of degrees of the polynomial vectors in any minimal basis of $\mathcal{V}$ is always the same. Hence, these degrees are uniquely determined by $\mathcal{V}$ and are called the *minimal indices* of $\mathcal{V}$.

Let now $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix. We consider the rational vector subspaces:

$$\mathcal{N}_r(R) = \{x(\lambda) \in \mathbb{F}(\lambda)^{m\times 1} : R(\lambda)x(\lambda) = 0\}, \text{ and}$$
$$\mathcal{N}_\ell(R) = \{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times p} : y(\lambda)^T R(\lambda) = 0\},$$

which are called the *right* and *left null-spaces over* $\mathbb{F}(\lambda)$ *of* $R(\lambda)$, respectively. If $R(\lambda)$ is singular at least one of these null-spaces is non-trivial. If $\mathcal{N}_r(R)$ (resp. $\mathcal{N}_\ell(R)$) is non trivial, it has minimal bases and minimal indices, which are called the right (resp. left) minimal bases and *right (resp. left) minimal indices of* $R(\lambda)$. By the rank-nullity theorem,

$$\dim \mathcal{N}_\ell(R) = p - \operatorname{nrank} R(\lambda), \text{ and } \dim \mathcal{N}_r(R) = m - \operatorname{nrank} R(\lambda).$$

Therefore, if $r = \operatorname{nrank} R(\lambda)$, then $R(\lambda)$ has $p - r$ left minimal indices and $m - r$ right minimal indices.

Minimal bases appearing in this dissertation are arranged as the columns or rows of polynomial matrices. With a slight abuse of notation, we say that a $p \times m$ polynomial matrix with $p > m$ (resp. $p < m$) is a *minimal basis* if its columns (resp. rows) form a minimal basis of the rational subspace they span. The following definitions are useful for characterizing minimal bases.

Definition 2.3.1. *The ith column (resp. row) degree of a matrix polynomial $B(\lambda)$ is the degree of the ith column (resp. row) of $B(\lambda)$.*

Definition 2.3.2. *Let $B(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a polynomial matrix with column (resp. row) degrees $d_1, d_2, \ldots, d_m$ (resp. $d_1, d_2, \ldots, d_p$). The highest column (resp. row) degree coefficient matrix of $B(\lambda)$, denoted by B_{hcd} (resp. B_{hrd}), is the $p\times m$ constant matrix whose jth column (resp. row) is the coefficient of λ^{d_j} in the jth column (resp. row) of $B(\lambda)$. The polynomial matrix $B(\lambda)$ is called column (resp. row) reduced if B_{hcd} (resp. B_{hrd}) has full colum (resp. row) rank.*

Theorem 2.3.3 states one of the most useful characterizations of minimal bases (see [38, Main Theorem] or [26, Theorem 2.2]).

Theorem 2.3.3. *The columns (resp. rows) of a polynomial matrix $B(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with $p > m$ (resp. $p < m$) are a minimal basis of the subspace they span if and only if $B(\lambda)$ is column (resp. row) reduced and $B(\lambda_0)$ has full column (resp. row) rank for all $\lambda_0 \in \overline{\mathbb{F}}$.*

Associated with minimal bases, the notion of dual minimal basis [26, Definition 2.5] is also considered.

Definition 2.3.4. *Let $K(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a minimal basis with $p < m$. Another minimal basis $N(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ is said to be dual to $K(\lambda)$ if $p + q = m$ and $K(\lambda)N(\lambda)^T = 0$. Then $N(\lambda)$ is said to be a dual minimal basis of $K(\lambda)$.*

2.4 Polynomial system matrices

Polynomial system matrices are a classical tool for studying rational matrices. They were introduced by Rosenbrock and are analyzed in detail in [78]. Among them, minimal polynomial system matrices have been used in many problems dealing with rational matrices because they allow to extract all the information about finite poles and zeros.

Definition 2.4.1 (Polynomial system matrix and transfer function)**.** *Any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be written as*

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) \tag{2.6}$$

for some polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with $A(\lambda)$ regular if $n > 0$. Then, a polynomial matrix of the form

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{2.7}$$

is called a polynomial system matrix of $R(\lambda)$. That is, $R(\lambda)$ is the Schur complement of $A(\lambda)$ in $P(\lambda)$ and is called the transfer function matrix of $P(\lambda)$.

If $n = 0$ in (2.6), we assume that the matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are empty, and $R(\lambda) = D(\lambda)$ is a polynomial matrix. But polynomial matrices can also be represented as in (2.6) with $n > 0$. For instance, this can be obtained if $A(\lambda)$ is unimodular. In general, representations of the form (2.6) always exist for any rational matrix $R(\lambda)$ and are not unique. They are called *realizations* of $R(\lambda)$. We refer to $A(\lambda)$ as the *state matrix* of $P(\lambda)$ and the integer $\deg(\det A(\lambda))$ is called the *order* of $P(\lambda)$. Moreover, $P(\lambda)$ is said to have *least order*, or to be *minimal*, if its order is the smallest integer for which polynomial matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ satisfying (2.6) exist. Another equivalent definition for minimality is the following [78].

Definition 2.4.2 (Minimal polynomial system matrix)**.** *The polynomial system matrix $P(\lambda)$ in* (2.7)*, with $n > 0$, is said to have least order, or to be minimal or irreducible, if the matrices*

$$\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \tag{2.8}$$

have no eigenvalues in $\overline{\mathbb{F}}$. In such a case, the realization in (2.6) *is also said to be minimal.*

The least order is uniquely determined by $R(\lambda)$ and is denoted by $\nu(R(\lambda))$. It is also called the *least order* of $R(\lambda)$ ([78, Chapter 3, Section 5.1] or [90, Section 1.10]). From [78, Chapter 3, Theorem 4.1], it can be deduced that $\nu(R(\lambda))$ is the degree of the polynomial that results by making the product of the denominators in the (finite) Smith–McMillan form of $R(\lambda)$.

Remark 2.4.3. Notice that the definition of polynomial system matrix $P(\lambda)$ in (2.7) includes a specific partition. However, the state matrix $A(\lambda)$ might be a submatrix of $P(\lambda)$ different from the $(1,1)$-block. Throughout this dissertation, polynomial matrices are partitioned in different ways giving rise to different polynomial system matrices of (possibly) different transfer functions. In such cases, we often use expressions as "$P(\lambda)$ is a polynomial system matrix with state matrix $A(\lambda)$" in order to avoid ambiguities. In the case $n = 0$, we use the expression "$P(\lambda)$ is a polynomial system matrix with empty state matrix".

The main property of a polynomial system matrix $P(\lambda)$ being minimal, or of least order, is that the finite pole and zero information of its transfer function matrix is contained in $P(\lambda)$ [78].

Theorem 2.4.4. *Let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a minimal polynomial system matrix, with state matrix $A(\lambda)$, whose transfer function matrix is $R(\lambda)$. Then the finite elementary divisors of $A(\lambda)$ are the finite pole elementary divisors of $R(\lambda)$, and the finite elementary divisors of $P(\lambda)$ are the finite zero elementary divisors of $R(\lambda)$.

A polynomial system matrix $P(\lambda)$ of $R(\lambda)$ is said to be a polynomial system matrix in *state-space form* if $A(\lambda) = \lambda I_n - A$, $B(\lambda) = B$ and $C(\lambda) = C$ for some constant matrices $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$ and $C \in \mathbb{F}^{p\times n}$. It is known that any strictly proper rational matrix admits *state-space realizations* (see [78] or [55]). Thus, if we consider $R(\lambda)$ as in (2.1), this means that for some positive integer n there exist constant matrices $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$ and $C \in \mathbb{F}^{p\times n}$ such that $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ and

$$\begin{bmatrix} \lambda I_n - A & B \\ -C & D(\lambda) \end{bmatrix}$$

is a polynomial system matrix of $R(\lambda)$. Therefore $R(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$. In addition, the state-space realization may always be taken of least order, or minimal (i.e., such that the polynomial system matrix in state-space form is of least order).

Notice that any polynomial system matrix $P(\lambda)$ can be written as the following block LDU factorization:

$$P(\lambda) = \begin{bmatrix} I_n & 0 \\ -C(\lambda)A(\lambda)^{-1} & I_p \end{bmatrix} \begin{bmatrix} A(\lambda) & 0 \\ 0 & R(\lambda) \end{bmatrix} \begin{bmatrix} I_n & A(\lambda)^{-1}B(\lambda) \\ 0 & I_m \end{bmatrix}.$$

Then the relation between the normal ranks of $P(\lambda)$ and its transfer function matrix $R(\lambda)$ is

$$\operatorname{nrank} P(\lambda) = n + \operatorname{nrank} R(\lambda). \tag{2.9}$$

Using the factorization above and the rank property in (2.9), the following Lemma 2.4.5 included in [75] establishes a linear map between the right (resp. left) nullspace of a rational matrix $R(\lambda)$ and the right (resp. left) nullspace of a polynomial system matrix of $R(\lambda)$.

Lemma 2.4.5. *Let $P(\lambda)$ be a polynomial system matrix as in (2.7), and let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be its transfer function matrix. Then, the following statements hold:*

(a) *The linear map*

$$\begin{aligned} T_r \,:\, \mathcal{N}_r(R) &\longrightarrow \mathcal{N}_r(P) \\ x(\lambda) &\longmapsto \begin{bmatrix} -A(\lambda)^{-1}B(\lambda) \\ I_m \end{bmatrix} x(\lambda) \end{aligned}$$

is a bijection between the right nullspaces of $R(\lambda)$ and $P(\lambda)$.

(b) *The linear map*

$$\begin{aligned} T_\ell \,:\, \mathcal{N}_\ell(R) &\longrightarrow \mathcal{N}_\ell(P) \\ y(\lambda)^T &\longmapsto y(\lambda)^T \begin{bmatrix} C(\lambda)A(\lambda)^{-1} & I_p \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces of $R(\lambda)$ and $P(\lambda)$.

Proof. We only prove part (a) since part (b) can be proved in a similar way. First, we observe that the map T_r is linear. Second, we notice that for any vector $x(\lambda)$, we have

$$\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} -A(\lambda)^{-1}B(\lambda)x(\lambda) \\ x(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ R(\lambda)x(\lambda) \end{bmatrix},$$

which shows that T_r maps vectors in the right nullspace of $R(\lambda)$ to vectors in the right nullspace of $P(\lambda)$. Finally, by (2.9) and the rank-nullity theorem, we have

$$\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(R). \tag{2.10}$$

Since the right nullspaces of $P(\lambda)$ and $R(\lambda)$ have the same dimension and the linear map T_r is clearly injective, we conclude that the map T_r is bijective. ■

Remark 2.4.6. Since the maps in Lemma 2.4.5 are bijections, they preserve linear independence. Hence, one can recover a basis of the right (resp. left) nullspace of $R(\lambda)$ from a basis of the right (resp. left) nullspace of $P(\lambda)$, and conversely. For instance, from part (a) in Lemma 2.4.5, we obtain that if $\{x_i(\lambda)\}_{i=1}^t$ is a basis of $\mathcal{N}_r(R)$, then $\left\{ \begin{bmatrix} -A(\lambda)^{-1}B(\lambda)x_i(\lambda) \\ x_i(\lambda) \end{bmatrix} \right\}_{i=1}^t$ is a basis of $\mathcal{N}_r(P)$. Conversely, if $\left\{ \begin{bmatrix} y_i(\lambda) \\ x_i(\lambda) \end{bmatrix} \right\}_{i=1}^t$ is a basis of $\mathcal{N}_r(P)$ then $\{x_i(\lambda)\}_{i=1}^t$ is a basis of $\mathcal{N}_r(R)$.

To recover minimal bases and minimal indices of rational matrices from polynomial system matrices $P(\lambda)$ one has to assume extra conditions on $P(\lambda)$. For results about recovery of minimal bases and minimal indices of rational matrices from polynomial system matrices we refer to [8, 91, 92].

2.4.1 Eigenvectors of polynomial system and transfer function matrices

We know from [7, Proposition 3.1] how to recover right eigenvectors of a polynomial system matrix $P(\lambda)$ from those of its transfer function $R(\lambda)$, and conversely. In Proposition 2.4.7 we state a extended version of [7, Proposition 3.1], that includes a result about bases of the right null-spaces of $P(\lambda)$ and $R(\lambda)$ evaluated at the eigenvalue of interest, that was introduced in [27, Proposition 5.1]. We state without proof the analogous result for left eigenvectors and null-spaces in Proposition 2.4.8, introduced in [27, Proposition 5.2].

For the sake of brevity, the following nomenclature is adopted: "(λ_0, x_0) is a solution of the REP $R(\lambda)x = 0$" means that λ_0 is a finite eigenvalue of $R(\lambda)$ and x_0 is a right eigenvector corresponding to λ_0, and "(λ_0, x_0) is a solution of the REP $x^T R(\lambda) = 0$" means that λ_0 is a finite eigenvalue of $R(\lambda)$ and x_0 is a left eigenvector corresponding to λ_0. Although it is not common in the literature, if $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is singular, we call right and left eigenvectors of $R(\lambda)$ associated with an eigenvalue λ_0 to any nonzero vectors x and y satisfying $R(\lambda_0)x = 0$ and $y^T R(\lambda_0) = 0$, respectively.

In what follows, we assume that eigenvectors of the form $\begin{bmatrix} y \\ x \end{bmatrix}$ are partitioned conformable to the corresponding polynomial system matrix.

Proposition 2.4.7. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be any polynomial system matrix with $R(\lambda)$ as transfer function matrix.

a) *If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the PEP $P(\lambda)z = 0$ such that $\det A(\lambda_0) \neq 0$, then (λ_0, x_0) is a solution of the REP $R(\lambda)x = 0$.*

b) *Moreover, if $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$, with $\det A(\lambda_0) \neq 0$, then $\{x_1, \dots, x_t\}$ is a basis of $\mathcal{N}_r(R(\lambda_0))$.*

c) *Conversely, if (λ_0, x_0) is a solution of the REP $R(\lambda)x = 0$ such that $\det A(\lambda_0) \neq 0$ and y_0 is defined as the unique solution of $A(\lambda_0)y_0 + B(\lambda_0)x_0 = 0$, then $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the PEP $P(\lambda)z = 0$.*

d) *Moreover, if $\{x_1, \dots, x_t\}$ is a basis of $\mathcal{N}_r(R(\lambda_0))$, with $\det A(\lambda_0) \neq 0$, and, for $i = 1, \dots, t$, y_i is defined as the unique solution of $A(\lambda_0)y_i + B(\lambda_0)x_i = 0$, then $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$.*

Proof. The statements *a*) and *c*) are the results in [7, Proposition 3.1] stated here for a rectangular matrix $R(\lambda)$. The proofs are exactly the same as in [7] and, therefore, are omitted. To prove *b*) and *d*) we write

$$\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \\ -C(\lambda_0) & D(\lambda_0) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -C(\lambda_0)A(\lambda_0)^{-1} & I_p \end{bmatrix} \begin{bmatrix} A(\lambda_0) & 0 \\ 0 & R(\lambda_0) \end{bmatrix} \begin{bmatrix} I_n & A(\lambda_0)^{-1}B(\lambda_0) \\ 0 & I_m \end{bmatrix}.$$

Since $\det A(\lambda_0) \neq 0$, $\text{rank} P(\lambda_0) = n + \text{rank} R(\lambda_0)$. Therefore

$$\dim \mathcal{N}_r(P(\lambda_0)) = \dim \mathcal{N}_r(R(\lambda_0)). \tag{2.11}$$

Then *b*) and *d*) are obtained by using *a*) and *c*), respectively, taking (2.11) and the linear independence of the considered sets into account, and observing that $P(\lambda_0) \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = 0$ if and only if $y_0 = -A(\lambda_0)^{-1}B(\lambda_0)x_0$ and $R(\lambda_0)x_0 = 0$. ■

Proposition 2.4.8 is an analogous result to Proposition 2.4.7 for left eigenvectors and left null-spaces, and it can be proved in a similar way. It was introduced in [27, Proposition 5.2].

Proposition 2.4.8. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix and*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be any polynomial system matrix with $R(\lambda)$ *as transfer function matrix.*

a) If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the PEP* $z^T P(\lambda) = 0$ *such that* $\det A(\lambda_0) \neq 0$, *then* (λ_0, x_0) *is a solution of the REP* $x^T R(\lambda) = 0$.

b) Moreover, if $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_q \\ x_q \end{bmatrix}\right\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$, *with* $\det A(\lambda_0) \neq 0$, *then* $\{x_1, \dots, x_q\}$ *is a basis of* $\mathcal{N}_\ell(R(\lambda_0))$.

c) Conversely, if (λ_0, x_0) *is a solution of the REP* $x^T R(\lambda) = 0$ *such that* $\det A(\lambda_0) \neq 0$, *and* y_0 *is defined as the unique solution of* $y_0^T A(\lambda_0) - x_0^T C(\lambda_0) = 0$, *then* $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the PEP* $z^T P(\lambda) = 0$.

d) Moreover, if $\{x_1, \dots, x_q\}$ *is a basis of* $\mathcal{N}_\ell(R(\lambda_0))$, *with* $\det A(\lambda_0) \neq 0$, *and, for* $i = 1, \dots, q$, y_i *is defined as the unique solution of* $y_i^T A(\lambda_0) - x_i^T C(\lambda_0) = 0$, *then* $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_q \\ x_q \end{bmatrix}\right\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

Remark 2.4.9. If $R(\lambda)$ is singular, then for any $\lambda_0 \in \overline{\mathbb{F}}$ that is not a pole of $R(\lambda)$, including those λ_0 that are not eigenvalues of $R(\lambda)$, $\mathcal{N}_r(R(\lambda_0)) \neq \{0\}$ or $\mathcal{N}_\ell(R(\lambda_0)) \neq \{0\}$. The reader can check easily that Propositions 2.4.7 and 2.4.8 remain valid for any $\lambda_0 \in \overline{\mathbb{F}}$ that is not a pole of $R(\lambda)$ in the case $R(\lambda)$ is singular.

As corollary, we obtain the following Lemma 2.4.10 that establishes bijections relating the right and left null spaces of a polynomial system matrix and those of its transfer function matrix, evaluated at a finite λ_0. This result is analogous to that in Lemma 2.4.5 and was introduced in [29].

Lemma 2.4.10. *Let $P(\lambda)$ be a polynomial system matrix as in* (2.7)*, and let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be its transfer function matrix. Let $\lambda_0 \in \overline{\mathbb{F}}$ such that* $\det A(\lambda_0) \neq 0$*. Then, the following statements hold:*

(a) *The linear map*

$$\begin{aligned} F_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(P(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0) \\ I_m \end{bmatrix} x \end{aligned}$$

is a bijection between the right nullspaces over $\overline{\mathbb{F}}$ of $R(\lambda_0)$ and $P(\lambda_0)$.

(b) *The linear map*

$$\begin{aligned} F_\ell \,:\, \mathcal{N}_\ell(R(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(P(\lambda_0)) \\ y^T &\longmapsto y^T \begin{bmatrix} C(\lambda_0)A(\lambda_0)^{-1} & I_p \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces over $\overline{\mathbb{F}}$ of $R(\lambda_0)$ and $P(\lambda_0)$.

2.5 Definitions of strong linearizations of polynomial and rational matrices in the literature

The standard method of dealing with the rational (and polynomial) eigenvalue problem consists of linearizing. That is, reformulating the corresponding rational matrix into a linear polynomial matrix in such a way that the eigenstructure can be exactly recovered, i.e., the zero structure, the polar structure, and the left and right null space structure.

2.5.1 Strong linearizations of polynomial matrices

In this subsection, we recall the classical definitions of linearization and strong linearization of polynomial matrices [42, 44].

Let $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a polynomial matrix. A linear polynomial matrix $L(\lambda) = B\lambda + A \in \mathbb{F}[\lambda]^{(p+s)\times(m+s)}$ is a *linearization* of $P(\lambda)$ in the sense of Gohberg, Lancaster and Rodman [44], or in the GLR-sense for short, if there exist unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ such that

$$U_1(\lambda)L(\lambda)U_2(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix},$$

where I_s denotes the identity matrix of size any integer $s \geq 0$. The key property of a GLR-linearization is that it has the same finite eigenvalues with the same partial multiplicities as $P(\lambda)$. Furthermore, a linearization $L(\lambda)$ is said to be a *strong linearization* in the GLR-sense if $\operatorname{rev} L(\lambda) = A\lambda + B$ is a linearization of the polynomial matrix $\operatorname{rev}_g P(\lambda) = \lambda^g P(1/\lambda)$, where g is an integer greater than or equal to the degree of $P(\lambda)$, i.e., $g \geq \deg(P(\lambda))$, and is called *grade* of $P(\lambda)$ (see, for instance, [22]). Then, not only finite but infinite eigenvalues can be recovered from strong linearizations in the GLR-sense, together with their partial multiplicities. However, the minimal indices of a (strong) linearization $L(\lambda)$ in the GLR-sense may be completely unrelated to those of $P(\lambda)$ [22, Section 4], except for the fact that the number of left (resp. right) minimal indices of $L(\lambda)$ and $P(\lambda)$ are equal. Nevertheless, the strong linearizations in the GLR-sense that are used in practice have minimal indices that are simply related to those of the polynomial matrix through addition of a constant shift (see [26] and the references therein).

In [26] a wide family of (strong) linearizations for polynomial matrices is constructed, which are called (strong) block minimal bases pencils [26, Definition 3.1] and that uses the notion of minimal basis (recall Section 2.3). We introduce the family of (strong) block minimal bases pencils in the following definition, as it will be useful throughout this dissertation.

Definition 2.5.1. *((Strong) block minimal bases pencil) A block minimal bases pencil is a linear polynomial matrix over $\mathbb{F}$ with the following structure*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}, \tag{2.12}$$

where $K_1(\lambda)$ and $K_2(\lambda)$ are both minimal bases. Moreover, given a polynomial matrix $P(\lambda)$, it is said that $L(\lambda)$ is associated with $P(\lambda)$ if

$$N_2(\lambda)M(\lambda)N_1(\lambda)^T = P(\lambda),$$

where $N_1(\lambda)$ and $N_2(\lambda)$ are minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. In addition, if $K_1(\lambda)$ (resp. $K_2(\lambda)$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda)$ (resp. $N_2(\lambda)$) dual to $K_1(\lambda)$ (resp. $K_2(\lambda)$) all equal, then $L(\lambda)$ is called a strong block minimal bases pencil.

Theorem 2.5.2. [26] *A block minimal bases pencil* $L(\lambda)$ *as in* (2.12) *associated with a polynomial matrix* $P(\lambda)$ *is a linearization of* $P(\lambda)$*. Moreover, if the block minimal bases pencil* $L(\lambda)$ *is strong, then* $L(\lambda)$ *is a strong linearization of* $P(\lambda)$ *considered as a polynomial matrix of grade* $\deg N_1(\lambda) + \deg N_2(\lambda) + 1$.

2.5.2 Strong linearizations of rational matrices

Next we present the definition of strong linearization for a rational matrix given in [6]. This definition contains the notion of first invariant order at infinity q_1 of a rational matrix $R(\lambda)$. For any non strictly proper rational matrix this number is $-\deg(D(\lambda))$ where $D(\lambda)$ is the polynomial part of $R(\lambda)$ in the expression (2.1); otherwise, $q_1 > 0$. More information can be found in [10, 6, 90].

Definition 2.5.3. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$*. Let* q_1 *be its first invariant order at infinity and* $g = \min(0, q_1)$*. Let* $n = \nu(R(\lambda))$*. A strong linearization of* $R(\lambda)$ *is a linear polynomial matrix*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{2.13}$$

such that the following conditions hold:

(a) if $n > 0$ *then* $\det(A_1\lambda + A_0) \neq 0$*, and*

(b) if $\widehat{R}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$*,* $\widehat{q}_1$ *is its first invariant order at infinity and* $\widehat{g} = \min(0, \widehat{q}_1)$ *then:*

  (i) there exist nonnegative integers s_1, s_2*, with* $s_1 - s_2 = q - p = r - m$*, and unimodular matrices* $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ *and* $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ *such that*

$$U_1(\lambda)\operatorname{diag}(R(\lambda), I_{s_1})U_2(\lambda) = \operatorname{diag}(\widehat{R}(\lambda), I_{s_2}), \text{ and}$$

  (ii) there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+s_1)\times(p+s_1)}$ *and* $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+s_1)\times(m+s_1)}$ *such that*

$$B_1(\lambda)\operatorname{diag}(\lambda^g R(\lambda), I_{s_1})B_2(\lambda) = \operatorname{diag}(\lambda^{\widehat{g}}\widehat{R}(\lambda), I_{s_2}).$$

If, instead of $n = \nu(R(\lambda))$, in Definition 2.5.3 $L(\lambda)$ is considered to be a minimal polynomial system matrix and only condition *(i)* in part (b) holds, then $\mathcal{L}(\lambda)$ is just called *linearization* of $R(\lambda)$ in [6]. In addition, it is known [6] that if condition *(i)* in Definition 2.5.3 holds, then condition *(ii)* is equivalent to the existence of unimodular matrices $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$W_1(\lambda)\operatorname{diag}\left(\frac{1}{\lambda^g}R\left(\frac{1}{\lambda}\right), I_{s_1}\right)W_2(\lambda) = \operatorname{diag}\left(\frac{1}{\lambda^{\widehat{g}}}\widehat{R}\left(\frac{1}{\lambda}\right), I_{s_2}\right). \tag{2.14}$$

In Definition 2.5.3 it can always be taken $s_1 = 0$ or $s_2 = 0$, according to $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$. We now consider $s_1 \geq 0$ and $s_2 = 0$. Notice that with this choice and with the notion of reversal given in Definition 2.2.1, (2.14) is equivalent to

$$W_1(\lambda)\,\mathrm{diag}\,(\mathrm{rev}\,R(\lambda), I_{s_1})\,W_2(\lambda) = \mathrm{rev}\,\widehat{R}(\lambda). \tag{2.15}$$

Remark 2.5.4. Notice that Definition 2.5.3 extends the notion of strong linearization of polynomial matrices in the usual GLR-sense [63, Definition 2.5]. In particular, if $R(\lambda)$ is a polynomial matrix, then $n = \nu(R(\lambda)) = 0$. Therefore, a strong linearization $\mathcal{L}(\lambda)$ of $R(\lambda)$ is of the form $\mathcal{L}(\lambda) = D_1\lambda + D_0$, with $\widehat{R}(\lambda) = \mathcal{L}(\lambda)$, $g = q_1 = -\deg(R(\lambda))$ and $\widehat{g} = \widehat{q}_1 = -\deg(\mathcal{L}(\lambda))$.

By considering rational matrices expressed as in (2.1) and strong block minimal bases pencils associated to their polynomial parts, in [6, Theorem 5.11] is given a method to construct strong linearizations for rational matrices in the sense of Definition 2.5.3, which are called strong block minimal bases linearizations.

Theorem 2.5.5. *[6] Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix, let $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and its strictly proper part $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, and let $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal state-space realization of $R_{sp}(\lambda)$, where $n = \nu(R(\lambda)) = \nu(R_{sp}(\lambda))$. Assume that* $\deg(D(\lambda)) \geq 2$ *and let*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

be a strong block minimal bases pencil associated to $D(\lambda)$. Let $N_1(\lambda)$ and $N_2(\lambda)$ be minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, such that $N_2(\lambda)M(\lambda)N_1(\lambda)^T = D(\lambda)$ and $\deg(N_1(\lambda)) + \deg(N_2(\lambda)) + 1 = \deg(D(\lambda))$. *Let $\widehat{K}_1$ and $\widehat{K}_2$ be constant matrices such that the matrices*

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix},$$

are unimodular for $i = 1, 2$. Then, for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n\times n}$, the linear polynomial matrix

$$\mathcal{L}(\lambda) := \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right],$$

is a strong linearization of $R(\lambda)$.

We will introduce a definition of local linearization of rational matrices in Chapter 4 that includes the above definitions of linearizations and strong linearizations of polynomial and rational matrices. In particular, we introduce in Chapter 4 linearizations satisfying minimality conditions that preserve pole and zero structures in a particular target set and/or at infinity. Then, in Chapter 4 a linearization will be said to be strong if the minimality conditions are satisfied in the whole underlying field and also at infinity. However, we will make a different and more flexible treatment of the conditions for a pencil to be a linearization in the strong sense, which allows us to construct linearizations for polynomial and rational matrices that can not be constructed with the existing definitions of strong linearizations in the literature. Examples of such linearizations are the linearizations introduced in Chapter 6, for arbitrary rational matrices from general representations, and the linearizations for polynomial and rational matrices preserving structures in Chapter 8.

Chapter 3

Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis

All the results in this chapter were introduced in [27], where a new family of strong linearizations of rational matrices is constructed according to Definition 2.5.3. For that, the corresponding rational matrix is considered as the sum of its polynomial part and its strictly proper part, and the polynomial part is expressed in a basis that satisfies a three term recurrence relation. Then, we combine the theory developed in [7] and the linearizations of polynomial matrices introduced in [36]. We also show how to recover eigenvectors of a rational matrix from those of its linearizations in this family. In addition, we present strong linearizations that preserve the structure of symmetric or Hermitian rational matrices. We complete the chapter by discussing how to extend the results in this chapter when the polynomial part is expressed in other bases. After the publication of [27], the recovery of minimal bases and minimal indices of the linearizations developed here were studied by other authors in [8].

A conclusion of the results presented in this chapter is that the combination of them with those in [7], allows us to use essentially all the strong linearizations of polynomial matrices developed in the literature to construct strong linearizations of any rational matrix in the sense of Definition 2.5.3, by expressing such a matrix in terms of its polynomial and strictly proper parts.

3.1 Some preliminaries

In the definition of strong linearization in [7] (recall Definition 2.5.3), it may seem that the least order $\nu(R(\lambda))$ has to be previously known in order to verify that a linear polynomial matrix as in (2.13) is a strong linearization of the rational matrix $R(\lambda)$. However, there are conditions to ensure that the size of $A_1\lambda + A_0$ is

$n = \nu(R(\lambda))$. We state them in Proposition 3.1.1.

Proposition 3.1.1. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

be a linear polynomial matrix with $n > 0$ *and* $\det(A_1\lambda + A_0) \neq 0$. *Assume that there exist nonnegative integers* s_1, s_2, *with* $s_1 - s_2 = q - p = r - m$, *and unimodular matrices* $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ *and* $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ *such that*

$$U_1(\lambda)\operatorname{diag}(R(\lambda), I_{s_1})U_2(\lambda) = \operatorname{diag}(\widehat{R}(\lambda), I_{s_2}), \tag{3.1}$$

where $\widehat{R}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$. *Then* $n = \nu(R(\lambda))$ *if and only if the following conditions hold:*

a) A_1 *is invertible, and*

b) $\operatorname{rank}\begin{bmatrix} A_1\mu + A_0 \\ C_1\mu + C_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1\mu + A_0 & B_1\mu + B_0 \end{bmatrix} = n$ *for all* $\mu \in \overline{\mathbb{F}}$.

Proof. Condition $b)$ is equivalent to $\mathcal{L}(\lambda)$ being a minimal polynomial system matrix, since $\det(A_1\lambda + A_0) \neq 0$, see [78, Chapters 2 and 3]. By condition (3.1) and [6, Lemma 2.1], we have that $\nu(R(\lambda)) = \nu(\widehat{R}(\lambda))$. Assume that $n = \nu(R(\lambda))$. Thus, $\nu(\widehat{R}(\lambda)) = n$, and $\deg(\det(A_1\lambda + A_0)) \geq \nu(\widehat{R}(\lambda)) = n$. However, $\deg(\det(A_1\lambda + A_0)) \leq n$. Therefore, $\deg(\det(A_1\lambda + A_0)) = n$ and $\deg(\det(A_1\lambda + A_0)) = \nu(\widehat{R}(\lambda))$, which imply conditions $a)$ and $b)$, respectively. We assume now that conditions $a)$ and $b)$ hold. On the one hand, A_1 being invertible implies that $\deg(\det(A_1\lambda+A_0)) = n$. On the other hand, $\mathcal{L}(\lambda)$ being a minimal polynomial system matrix means that $\deg(\det(A_1\lambda + A_0)) = \nu(\widehat{R}(\lambda))$. Therefore, $n = \nu(\widehat{R}(\lambda)) = \nu(R(\lambda))$. ■

Key Lemma Lemma 3.1.2 follows from Definition 2.5.3. It shows an easy way to obtain strong linearizations for a rational matrix $R(\lambda)$ from a particular strong linearization $\mathcal{L}(\lambda)$ by multiplying $\mathcal{L}(\lambda)$ by some appropriate matrices. This simple result is fundamental in this work, and we conjecture that it will be fundamental for constructing other families of strong linearizations of rational matrices.

Lemma 3.1.2. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix, and let*

$$\mathcal{L}_1(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a strong linearization of $R(\lambda)$. *Consider* $Q_1, Q_3 \in \mathbb{F}^{n\times n}$, $Q_2 \in \mathbb{F}^{(p+s)\times(p+s)}$, $Q_4 \in \mathbb{F}^{(m+s)\times(m+s)}$ *nonsingular matrices,* $W \in \mathbb{F}^{(p+s)\times n}$, *and* $Z \in \mathbb{F}^{n\times(m+s)}$. *Then the linear polynomial matrix*

$$\mathcal{L}_2(\lambda) = \begin{bmatrix} Q_1 & 0 \\ W & Q_2 \end{bmatrix} \mathcal{L}_1(\lambda) \begin{bmatrix} Q_3 & Z \\ 0 & Q_4 \end{bmatrix}$$

is a strong linearization of $R(\lambda)$.

Proof. Let us write

$$\mathcal{L}_2(\lambda) = \begin{bmatrix} A_2\lambda + \tilde{A}_0 & B_2\lambda + \tilde{B}_0 \\ -(C_2\lambda + \tilde{C}_0) & D_2\lambda + \tilde{D}_0 \end{bmatrix}.$$

We have $\det(A_2\lambda + \tilde{A}_0) \neq 0$ if $n > 0$, since $A_2\lambda + \tilde{A}_0 = Q_1(A_1\lambda + A_0)Q_3$. Let us consider the transfer functions $\widehat{R}_1(\lambda)$, $\widehat{R}_2(\lambda)$ of $\mathcal{L}_1(\lambda)$, $\mathcal{L}_2(\lambda)$, respectively. They satisfy $\widehat{R}_2(\lambda) = Q_2\widehat{R}_1(\lambda)Q_4$. Let q_1 be the first invariant order at infinity of $R(\lambda)$ and $g = \min(0, q_1)$. For $i = 1, 2$, let $\widehat{g}_i = \min(0, \widehat{q}_i)$, where $\widehat{q}_i$ is the first invariant order at infinity of $\widehat{R}_i(\lambda)$. Since $\mathcal{L}_1(\lambda)$ is a strong linearization of $R(\lambda)$, there exist unimodular matrices $U_1(\lambda)$ and $U_2(\lambda)$ such that $U_1(\lambda)\operatorname{diag}(R(\lambda), I_s)U_2(\lambda) = \widehat{R}_1(\lambda)$, and biproper matrices $B_1(\lambda)$ and $B_2(\lambda)$ such that $B_1(\lambda)\operatorname{diag}(\lambda^g R(\lambda), I_s)B_2(\lambda) = \lambda^{\widehat{g}_1}\widehat{R}_1(\lambda)$. By using the equality $\widehat{R}_2(\lambda) = Q_2\widehat{R}_1(\lambda)Q_4$, we have that $\widehat{g}_1 = \widehat{g}_2$, and by the same equality, we get

$$Q_2U_1(\lambda)\operatorname{diag}(R(\lambda), I_s)U_2(\lambda)Q_4 = \widehat{R}_2(\lambda),$$

and

$$Q_2B_1(\lambda)\operatorname{diag}(\lambda^g R(\lambda), I_s)B_2(\lambda)Q_4 = \lambda^{\widehat{g}_2}\widehat{R}_2(\lambda).$$

Then we obtain that conditions (a) and (b) in Definition 2.5.3 hold for $\mathcal{L}_2(\lambda)$. ■

By using Lemma 3.1.2, strong linearizations of a rational matrix $R(\lambda)$ expressed in the form (2.1) can be constructed from combining minimal state-space realizations of the strictly proper part $R_{sp}(\lambda)$ and strong linearizations of its polynomial part $D(\lambda)$. In particular, strong block minimal bases pencils associated to $D(\lambda)$ [26] can be used to construct a strong linearization for $R(\lambda)$, by using Theorem 2.5.5. Then one can use Lemma 3.1.2 to construct other strong linearizations for $R(\lambda)$ taking into account that strong block minimal bases linearizations of polynomial matrices unify many of the strong linearizations for polynomial matrices existing in the literature, in the sense that most of them are strictly equivalent to strong block minimal bases linearizations [26].

Remark 3.1.3. A first application of the key Lemma 3.1.2 is to construct strong linearizations of a rational matrix $R(\lambda)$ from any Fiedler-like strong linearization $L_F(\lambda)$ of its polynomial part $D(\lambda)$. For this purpose, note that [15, Theorems 3.8, 3.15, 3.16] guarantee that there exist permutation matrices Π_1 and Π_2 and a strong block minimal bases pencil $L(\lambda)$ associated to $D(\lambda)$ such that $L_F(\lambda) = \Pi_1 L(\lambda)\Pi_2$. In addition, Theorem 2.5.5 explains how to construct a strong linearization $\mathcal{L}(\lambda)$ of $R(\lambda)$ from $L(\lambda)$. Thus, according to Lemma 3.1.2, $\operatorname{diag}(I_n, \Pi_1)\mathcal{L}(\lambda)\operatorname{diag}(I_n, \Pi_2)$ is a strong linearization of $R(\lambda)$ based on $L_F(\lambda)$. This idea is used in [8], where it is proved that the families of Fiedler-like linearizations of rational matrices introduced in [2, 4, 17, 19] are, modulo permutations, particular instances of strong block minimal bases linearizations.

3.2 $\mathbb{M}_1$-strong linearizations

In this section and in Section 3.3, we present strong linearizations of square rational matrices $R(\lambda)$ with polynomial part $D(\lambda)$ expressed in an orthogonal basis. More precisely, we consider strong linearizations of $D(\lambda)$ that belong to the ansatz spaces $\mathbb{M}_1(D)$ or $\mathbb{M}_2(D)$, developed by H. Faßbender and P. Saltenberger in [36], and based on them, we construct strong linearizations of $R(\lambda)$ by using Lemma 3.1.2 and the strong linearizations presented in [6, Section 5.2].

We consider an arbitrary field $\mathbb{F}$ throughout this chapter, although the results in [36] are stated only for the real field $\mathbb{R}$. Nevertheless, the results of [36] that are used in this chapter are also valid for any field $\mathbb{F}$. We consider a polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ of $\mathbb{F}[\lambda]$, viewed as an $\mathbb{F}$-vector space, with $\phi_j(\lambda)$ a polynomial of degree j, that satisfies the following three-term recurrence relation:

$$\alpha_j \phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j \phi_{j-1}(\lambda) \quad j \geq 0 \tag{3.2}$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$. Let $P(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ be a polynomial matrix of degree k written in terms of this basis as follows

$$P(\lambda) = P_k \phi_k(\lambda) + P_{k-1}\phi_{k-1}(\lambda) + \cdots + P_1 \phi_1(\lambda) + P_0 \phi_0(\lambda). \tag{3.3}$$

We define $\Phi_k(\lambda) = [\phi_{k-1}(\lambda) \cdots \phi_1(\lambda)\ \phi_0(\lambda)]^T$ and $V_P = \{v \otimes P(\lambda) : v \in \mathbb{F}^k\}$, and we consider the set of pencils

$$\mathbb{M}_1(P) = \{L(\lambda) = \lambda X + Y : X, Y \in \mathbb{F}^{km \times km},\ L(\lambda)(\Phi_k(\lambda) \otimes I_m) \in V_P\}.$$

A pencil $L(\lambda) \in \mathbb{M}_1(P)$, which verifies $L(\lambda)(\Phi_k(\lambda) \otimes I_m) = v \otimes P(\lambda)$ for some vector $v \in \mathbb{F}^k$, is said to have *right ansatz vector* v. A particular pencil in $\mathbb{M}_1(P)$ introduced in [36, page 63] is

$$F_\Phi^P(\lambda) = \begin{bmatrix} m_\Phi^P(\lambda) \\ M_\Phi(\lambda) \otimes I_m \end{bmatrix} \in \mathbb{F}[\lambda]^{km \times km}, \tag{3.4}$$

where

$$m_\Phi^P(\lambda) = \begin{bmatrix} \frac{(\lambda - \beta_{k-1})}{\alpha_{k-1}} P_k + P_{k-1} & P_{k-2} - \frac{\gamma_{k-1}}{\alpha_{k-1}} P_k & P_{k-3} & \cdots & P_1 & P_0 \end{bmatrix},$$

and

$$M_\Phi(\lambda) = \begin{bmatrix} -\alpha_{k-2} & (\lambda - \beta_{k-2}) & -\gamma_{k-2} & & & \\ & -\alpha_{k-3} & (\lambda - \beta_{k-3}) & -\gamma_{k-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 & (\lambda - \beta_1) & -\gamma_1 \\ & & & & -\alpha_0 & (\lambda - \beta_0) \end{bmatrix}.$$

Since $m_\Phi^P(\lambda)(\Phi_k(\lambda)\otimes I_m) = P(\lambda)$ and $(M_\Phi(\lambda)\otimes I_m)(\Phi_k(\lambda)\otimes I_m) = 0$, we get that $F_\Phi^P(\lambda)(\Phi_k(\lambda)\otimes I_m) = e_1\otimes P(\lambda)$, where e_1 is the first canonical vector of $\mathbb{F}^k$. Therefore, $F_\Phi^P(\lambda) \in \mathbb{M}_1(P)$ with right ansatz vector $e_1 \in \mathbb{F}^k$. This particular example is very important because, by using it, we can obtain all the elements in $\mathbb{M}_1(P)$. This follows from the next theorem.

Theorem 3.2.1. ***[36, Theorem 1]*** *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix with degree $k \geq 2$. Then $L(\lambda) \in \mathbb{M}_1(P)$ with right ansatz vector $v \in \mathbb{F}^k$ if and only if*

$$L(\lambda) = [v\otimes I_m \quad H]F_\Phi^P(\lambda)$$

for some matrix $H \in \mathbb{F}^{km\times(k-1)m}$.

Remark 3.2.2. For the monomial basis $\{\phi_j(\lambda) = \lambda^j\}_{j=0}^\infty$ the space $\mathbb{M}_1(P)$ is denoted $\mathbb{L}_1(P)$ (see [63]). In this case $\alpha_j = 1$ and $\beta_j = \gamma_j = 0$ for all $j \geq 0$ in (3.2) and the matrix $F_\Phi^P(\lambda)$ is the first companion form of $P(\lambda)$. ■

It is known that $F_\Phi^P(\lambda)$ is a strong linearization of $P(\lambda)$ (see [5, Theorem 2] for regular polynomial matrices $P(\lambda)$, and [21, Section 7] for singular), but we can obtain this property as an immediate corollary of the next result.

Lemma 3.2.3. *$F_\Phi^P(\lambda)$ is a strong block minimal bases pencil with only one block column associated to $P(\lambda)$ with sharp degree. Moreover, $\Phi_k(\lambda)^T\otimes I_m$ is a minimal basis dual to the minimal basis $M_\Phi(\lambda)\otimes I_m$.*

Proof. Let us denote $M(\lambda) = m_\Phi^P(\lambda)$ and $K(\lambda) = M_\Phi(\lambda)\otimes I_m$. We consider

$$F_\Phi^P(\lambda) = \begin{bmatrix} M(\lambda) \\ K(\lambda) \end{bmatrix}.$$

Note that $M_\Phi(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$ because $\alpha_i \neq 0$ for all $i \geq 0$. Also note that $M_\Phi(\lambda)$ is row reduced because its highest row degree coefficient matrix

$$[M_\Phi]_{hr} = \begin{bmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 1 & 0 \\ & & & & 0 & 1 \end{bmatrix}$$

has full row rank. We conclude that $M_\Phi(\lambda)$ is a minimal basis, and therefore, $K(\lambda) = M_\Phi(\lambda)\otimes I_m$ is also a minimal basis [26, Corollary 2.4]. Let us denote $N(\lambda) = \Phi_k(\lambda)^T\otimes I_m$. Note that $\Phi_k(\lambda)^T$ is a minimal basis because $\phi_0(\lambda) = 1$, so $\Phi_k(\lambda_0)$ has rank 1 for all $\lambda_0 \in \overline{\mathbb{F}}$, and

$$[\Phi_k^T]_{hr} = \begin{bmatrix} \frac{1}{\alpha_0\alpha_1\cdots\alpha_{k-2}} & 0 & \cdots & 0 \end{bmatrix}$$

has also rank 1. Therefore, $N(\lambda) = \Phi_k(\lambda)^T \otimes I_m$ is also a minimal basis. Since $K(\lambda)N(\lambda)^T = (M_\Phi(\lambda) \otimes I_m)(\Phi_k(\lambda) \otimes I_m) = 0$ and $\begin{bmatrix} K(\lambda) \\ N(\lambda) \end{bmatrix}$ is a square matrix, we have that $K(\lambda)$ and $N(\lambda)$ are dual minimal bases. In addition, it is obvious that all the row degrees of $K(\lambda)$ are equal to 1 and all the row degrees of $\Phi_k(\lambda)^T \otimes I_m$ are equal to $k-1$. Hence, $F_\Phi^P(\lambda)$ is a strong block minimal bases pencil associated to the polynomial matrix $M(\lambda)N(\lambda)^T = m_\Phi^P(\lambda)(\Phi_k(\lambda) \otimes I_m) = P(\lambda)$ and $\deg(P(\lambda)) = 1 + \deg(N(\lambda))$, which means that $F_\Phi^P(\lambda)$ has sharp degree. ■

Since every strong block minimal bases pencil is a strong linearization (see [26, Theorem 3.3]), the following corollary is straightforward.

Corollary 3.2.4. *$F_\Phi^P(\lambda)$ is a strong linearization for $P(\lambda)$.*

The proof of the next result is trivial because if $L(\lambda) = [v \otimes I_m \quad H]F_\Phi^P(\lambda)$ with $[v \otimes I_m \quad H]$ nonsingular then $L(\lambda)$ is strictly equivalent to $F_\Phi^P(\lambda)$.

Corollary 3.2.5. *[36, Corollary 2.1] Let $L(\lambda) = [v \otimes I_m \quad H]F_\Phi^P(\lambda) \in \mathbb{M}_1(P)$. If $[v \otimes I_m \quad H]$ is nonsingular then $L(\lambda)$ is a strong linearization for $P(\lambda)$.*

Remark 3.2.6. Although $F_\Phi^P(\lambda)$ is a strong block minimal bases pencil associated to $P(\lambda)$ this structure is not preserved in general when we multiply on the left by a nonsingular matrix $[v \otimes I_m \quad H]$. For example, consider the polynomial matrix $P(\lambda) = I\lambda^3 + 2I\lambda^2 + I\lambda + S \in \mathbb{R}[\lambda]^{2\times 2}$ expressed in the monomial basis, where $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and I stands for I_2. In this case, the matrix $F_\Phi^P(\lambda)$ is $F_\Phi^P(\lambda) = \begin{bmatrix} \lambda I + 2I & I & S \\ -I & \lambda I & 0 \\ 0 & -I & \lambda I \end{bmatrix}$. Let $v = [1 \quad 1 \quad 0]^T$ and $H = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$ and let $L(\lambda) = [v \otimes I \quad H]F_\Phi^P(\lambda) = \begin{bmatrix} \lambda I + 2I & I & S \\ \lambda I + I & \lambda I + I & S \\ 0 & -I & \lambda I \end{bmatrix}$. Notice that if $L(\lambda)$ were a strong block minimal bases pencil associated to $P(\lambda)$, one of these two different situations would happen in (2.12):

1. $M(\lambda) = [\lambda I + 2I \quad I \quad S]$, $K_1(\lambda) = \begin{bmatrix} \lambda I + I & \lambda I + I & S \\ 0 & -I & \lambda I \end{bmatrix}$ and $K_2(\lambda)$ empty.

2. $M(\lambda) = \begin{bmatrix} \lambda I + 2I \\ \lambda I + I \\ 0 \end{bmatrix}$, $K_2(\lambda)^T = \begin{bmatrix} I & S \\ \lambda I + I & S \\ -I & \lambda I \end{bmatrix}$ and $K_1(\lambda)$ empty.

In the first case, the matrix $K_1(\lambda)$ does not have full row rank for $\lambda = -1$. In the second case, the matrix $K_2(\lambda)$ does not have full row rank for $\lambda = 0$. Therefore, $L(\lambda)$ is not a strong block minimal bases pencil associated to $P(\lambda)$. ■

From the fact that $F_\Phi^P(\lambda)$ is a strong block minimal bases pencil, we can obtain strong linearizations for rational matrices by applying Theorem 5.11 in [6] (see Theorem 2.5.5). For this purpose, we prove first the following lemma.

Lemma 3.2.7. *The matrix*

$$U(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \otimes I_m \\ e_k^T \otimes I_m \end{bmatrix} = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix} \otimes I_m$$

is unimodular, and its inverse has the form $U(\lambda)^{-1} = [\widehat{\Phi}_k(\lambda) \quad \Phi_k(\lambda) \otimes I_m]$ *with* $\widehat{\Phi}_k(\lambda) \in \mathbb{F}[\lambda]^{km\times(k-1)m}$.

Proof. Let us consider the matrix

$$\tilde{U}(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix} = \begin{bmatrix} -\alpha_{k-2} & (\lambda-\beta_{k-2}) & -\gamma_{k-2} & & & \\ & -\alpha_{k-3} & (\lambda-\beta_{k-3}) & -\gamma_{k-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 & (\lambda-\beta_1) & -\gamma_1 \\ & & & & -\alpha_0 & (\lambda-\beta_0) \\ 0 & & \cdots & & 0 & 1 \end{bmatrix}.$$

Since $\tilde{U}(\lambda)$ is upper triangular, its determinant is $(-\alpha_{k-2})\cdots(-\alpha_0)$, i.e., a constant different from zero. Therefore, $\tilde{U}(\lambda)$ is unimodular. Finally, note that $\tilde{U}(\lambda)\Phi_k(\lambda) = e_k \in \mathbb{F}^k$. Thus $\Phi_k(\lambda)$ is the last column of $\tilde{U}(\lambda)^{-1}$. ■

Theorem 3.2.8. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ *be a rational matrix, let* $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ *be its unique decomposition into its polynomial part* $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ *and its strictly proper part* $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$, *and let* $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ *be a minimal state-space realization of* $R_{sp}(\lambda)$. *Assume that* $\deg(D(\lambda)) \geq 2$. *Write* $D(\lambda)$ *in terms of the polynomial basis* $\{\phi_j(\lambda)\}_{j=0}^\infty$ *satisfying the three-term recurrence relation* (3.2), *as*

$$D(\lambda) = D_k\phi_k(\lambda) + D_{k-1}\phi_{k-1}(\lambda) + \cdots + D_1\phi_1(\lambda) + D_0\phi_0(\lambda) \tag{3.5}$$

with $D_k \neq 0$, *and let* $F_\Phi^D(\lambda)$ *be the matrix pencil defined as in* (3.4). *Then, for any nonsingular matrices* $X, Y \in \mathbb{F}^{n\times n}$ *the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} & XB \\ \hline -CY & \multicolumn{2}{c}{\multirow{2}{*}{$F_\Phi^D(\lambda)$}} \\ 0_{(k-1)m\times n} & & \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

Proof. Lemmas 3.2.3 and 3.2.7 allow us to apply [6, Theorem 5.11], with $K_1(\lambda) = M_\Phi(\lambda) \otimes I_m$, $\widehat{K}_1 = e_k^T \otimes I_m$, $K_2(\lambda)^T$ empty and $\widehat{K}_2^T = I_m$. ■

Then, from combining Lemma 3.1.2 and Theorem 3.2.8 we obtain strong linearizations of a rational matrix from strong linearizations in $\mathbb{M}_1(D)$ of its polynomial part.

Theorem 3.2.9. *Under the same assumptions as in Theorem 3.2.8, let* $v \in \mathbb{F}^k$, $H \in \mathbb{F}^{km\times(k-1)m}$ *with* $[v \otimes I_m \quad H]$ *nonsingular and let* $L(\lambda) = [v \otimes I_m \quad H]F_\Phi^D(\lambda) \in \mathbb{M}_1(D)$. *Then, the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} \quad XB \\ \hline -(v \otimes I_m)CY & L(\lambda) \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

Proof. Set $K = [v\otimes I_m \quad H]$. If K is nonsingular then, by Lemma 3.1.2 and Theorem 3.2.8,

$$\begin{aligned} \mathcal{L}(\lambda) &= \begin{bmatrix} I_n & 0 \\ 0 & K \end{bmatrix} \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} \quad XB \\ \hline \begin{array}{c} -CY \\ 0_{(k-1)m\times n} \end{array} & F_\Phi^D(\lambda) \end{array}\right] \\ &= \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} \quad XB \\ \hline -(v \otimes I_m)CY & L(\lambda) \end{array}\right]. \end{aligned}$$

is a strong linearization of $R(\lambda)$. ∎

The strong linearizations of square rational matrices constructed in Theorem 3.2.9 will be called $\mathbb{M}_1$*-strong linearizations.*

3.3 $\mathbb{M}_2$-strong linearizations

In this section we obtain strong linearizations of a square rational matrix from the transposed version of $\mathbb{M}_1(P)$, where $P(\lambda)$ is the polynomial matrix in (3.3). Since the proofs of the results are similar to those in Section 3.2, they are omitted for brevity. We define $W_P = \{w^T \otimes P(\lambda) : w \in \mathbb{F}^k\}$, and we consider the set of pencils

$$\mathbb{M}_2(P) = \{L(\lambda) = \lambda X + Y : X, Y \in \mathbb{F}^{km\times km},\ (\Phi_k(\lambda)^T \otimes I_m)L(\lambda) \in W_P\}.$$

A pencil $L(\lambda) \in \mathbb{M}_2(P)$, which verifies $(\Phi_k(\lambda)^T \otimes I_m)L(\lambda) = w^T \otimes P(\lambda)$ for some vector $w \in \mathbb{F}^k$, is said to have *left ansatz vector* w. Pencils in $\mathbb{M}_2(P)$ are characterized in [36, Theorem 2]. We need the definition of the block-transpose of a $km\times lm$ pencil $L(\lambda)$. If we express $L(\lambda)$ as $L(\lambda) = \sum_{i=1}^{k}\sum_{j=1}^{l} e_i e_j^T \otimes L_{ij}(\lambda)$ for certain $m \times m$ pencils

$L_{ij}(\lambda)$, where e_i denotes the ith canonical vector in $\mathbb{F}^k$, and e_j the jth canonical vector in $\mathbb{F}^l$, $L(\lambda)^{\mathcal{B}} = \sum_{i=1}^{k}\sum_{j=1}^{l} e_j e_i^T \otimes L_{ij}(\lambda)$ is called the *block-transpose* of $L(\lambda)$ (see [53]). Notice that $F_\Phi^P(\lambda)^{\mathcal{B}} = [m_\Phi^P(\lambda)^{\mathcal{B}} \quad M_\Phi(\lambda)^T \otimes I_m]$.

Theorem 3.3.1. ***[36, Theorem 2]*** *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix with degree $k \geq 2$. Then $L(\lambda) \in \mathbb{M}_2(P)$ with left ansatz vector $w \in \mathbb{F}^k$ if and only if*

$$L(\lambda) = F_\Phi^P(\lambda)^{\mathcal{B}} \begin{bmatrix} w^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix}$$

for some matrix $H \in \mathbb{F}^{km\times(k-1)m}$ partitioned into $k \times (k-1)$ blocks each of size $m \times m$.

The vector space $\mathbb{M}_2(P)$ reduces to the well-known space $\mathbb{L}_2(P)$ when $\{\phi_k(\lambda)\}_{k=0}^\infty$ is the monomial basis, see [63]. Lemma 3.3.2 is for $\mathbb{M}_2(P)$ the counterpart of Lemma 3.2.3 for $\mathbb{M}_1(P)$ and can be used to proceed with $\mathbb{M}_2(P)$ analogously as we did with $\mathbb{M}_1(P)$.

Lemma 3.3.2. *$F_\Phi^P(\lambda)^{\mathcal{B}}$ is a strong block minimal bases pencil with only one block row associated to $P(\lambda)$ with sharp degree.*

In particular, Lemma 3.3.2 allows us to apply [6, Theorem 5.11] to the strong linearization $F_\Phi^D(\lambda)^{\mathcal{B}}$ of the polynomial part of a square rational matrix, with $K_2(\lambda) = M_\Phi(\lambda)\otimes I_m$, $\widehat{K}_2 = e_k^T \otimes I_m$, $K_1(\lambda)$ empty and $\widehat{K}_1 = I_m$. Thus, we get the following results to obtain strong linearizations of a square rational matrix $R(\lambda) = D(\lambda)+R_{sp}(\lambda)$ expressed as in (2.1) from strong linearizations in $\mathbb{M}_2(D)$.

Theorem 3.3.3. *Under the same assumptions as in Theorem 3.2.8, the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB & 0_{n\times(k-1)m} \\ \hline \begin{matrix} 0_{(k-1)m\times n} \\ -CY \end{matrix} & \multicolumn{2}{c}{F_\Phi^D(\lambda)^{\mathcal{B}}} \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

Theorem 3.3.4. *Under the same assumptions as in Theorem 3.2.8, let $w \in \mathbb{F}^k$, $H \in \mathbb{F}^{km\times(k-1)m}$ with $\begin{bmatrix} w^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix}$ nonsingular and let $L(\lambda) = F_\Phi^D(\lambda)^{\mathcal{B}} \begin{bmatrix} w^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix} \in \mathbb{M}_2(D)$. Then the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T \otimes I_m) \\ \hline \begin{matrix} 0_{(k-1)m\times n} \\ -CY \end{matrix} & L(\lambda) \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

Proof. We apply Lemma 3.1.2 by multiplying on the right the matrix $\mathcal{L}(\lambda)$ in Theorem 3.3.3 by the matrix $\begin{bmatrix} I_n & 0 \\ 0 & K \end{bmatrix}$ with $K = \begin{bmatrix} w^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix}$ nonsingular. ■

The strong linearizations of rational matrices constructed in Theorem 3.3.4 will be called $\mathbb{M}_2$-*strong linearizations.*

3.4 Recovering eigenvectors from $\mathbb{M}_1$- and $\mathbb{M}_2$-strong linearizations of rational matrices

In this section we will recover right and left eigenvectors of a rational matrix. These eigenvectors will be obtained without essentially computational cost from the right and left eigenvectors of the strong linearizations that we have constructed in Theorems 3.2.9 and 3.3.4.

3.4.1 Eigenvectors from $\mathbb{M}_1$-strong linearizations

We consider in this subsection the linearizations that we have constructed in Theorem 3.2.9, which we called $\mathbb{M}_1$-strong linearizations. We will recover the eigenvectors of a rational matrix $R(\lambda)$ from those of its $\mathbb{M}_1$-strong linearizations, and conversely. Lemma 3.4.1 will be used for this purpose.

Lemma 3.4.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} & XB \\ \hline -(v \otimes I_m)CY & \multicolumn{2}{c}{L(\lambda)} \end{array}\right]$$

be an $\mathbb{M}_1$-strong linearization of $R(\lambda)$, and let $\widehat{R}(\lambda)$ be the transfer function of $\mathcal{L}(\lambda)$. Then

$$\widehat{R}(\lambda)(\Phi_k(\lambda) \otimes I_m) = v \otimes R(\lambda). \tag{3.6}$$

Proof. We consider the transfer function of the matrix $\mathcal{L}(\lambda)$,

$$\widehat{R}(\lambda) = L(\lambda) + \begin{bmatrix} 0_{km\times(k-1)m} & (v \otimes I_m)C(\lambda I_n - A)^{-1}B \end{bmatrix}.$$

Let $D(\lambda)$ be the polynomial part of $R(\lambda)$. Since $L(\lambda)$ belongs to $\mathbb{M}_1(D)$, $L(\lambda)(\Phi_k(\lambda)\otimes I_m) = v \otimes D(\lambda) = (v \otimes I_m)D(\lambda)$. Therefore, we obtain

$$\begin{aligned} \widehat{R}(\lambda)(\Phi_k(\lambda) \otimes I_m) &= (L(\lambda) + \begin{bmatrix} 0_{km\times(k-1)m} & (v \otimes I_m)C(\lambda I_n - A)^{-1}B \end{bmatrix})(\Phi_k(\lambda) \otimes I_m) \\ &= (v \otimes I_m)D(\lambda) + (v \otimes I_m)C(\lambda I_n - A)^{-1}B \\ &= (v \otimes I_m)R(\lambda). \end{aligned}$$

■

Remark 3.4.2. Since $\mathcal{L}(\lambda)$ is a strong linearization of the rational matrix $R(\lambda)$ we have, by Definition 2.5.3, that there are unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{km\times km}$ such that

$$U(\lambda)\widehat{R}(\lambda)V(\lambda) = \operatorname{diag}(R(\lambda), I_{(k-1)m}). \tag{3.7}$$

Thus, if we consider a finite eigenvalue λ_0 of $R(\lambda)$ then it is also an eigenvalue of the transfer function $\widehat{R}(\lambda)$ and

$$\dim \mathcal{N}_r(R(\lambda_0)) = \dim \mathcal{N}_r(\widehat{R}(\lambda_0)). \tag{3.8}$$

By [6, Theorem 3.10], $\det(\lambda_0 I_n - A) \neq 0$. Thus, by Proposition 2.4.7,

$$\dim \mathcal{N}_r(\widehat{R}(\lambda_0)) = \dim \mathcal{N}_r(\mathcal{L}(\lambda_0)). \tag{3.9}$$

By (3.7) and Proposition 2.4.8, we have the same equalities for the dimensions of the left null-spaces, i.e.,

$$\dim \mathcal{N}_\ell(R(\lambda_0)) = \dim \mathcal{N}_\ell(\widehat{R}(\lambda_0)) \quad \text{and} \quad \dim \mathcal{N}_\ell(\widehat{R}(\lambda_0)) = \dim \mathcal{N}_\ell(\mathcal{L}(\lambda_0)). \tag{3.10}$$

Moreover, notice that since $R(\lambda)$ is square, $\dim \mathcal{N}_r(R(\lambda_0)) = \dim \mathcal{N}_\ell(R(\lambda_0))$. ■

A consequence of Lemma 3.4.1 is that we can recover very easily right eigenvectors of a rational matrix $R(\lambda)$ from the eigenvectors of the transfer function $\widehat{R}(\lambda)$ of any $\mathbb{M}_1$-strong linearization of $R(\lambda)$. We state that in Theorem 3.4.3, and we emphasize that this result is in the spirit of the one presented in [36, Proposition 3.1] for polynomial matrices $P(\lambda)$ and their strong linearizations in $\mathbb{M}_1(P)$.

Theorem 3.4.3. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, and let $\widehat{R}(\lambda)$ be the transfer function of an $\mathbb{M}_1$-strong linearization*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} \quad XB \\ \hline -(v\otimes I_m)CY & L(\lambda) \end{array}\right]$$

of $R(\lambda)$. Let λ_0 be a finite eigenvalue of $R(\lambda)$. Then, $u \in \mathcal{N}_r(R(\lambda_0))$ if and only if $\Phi_k(\lambda_0)\otimes u \in \mathcal{N}_r(\widehat{R}(\lambda_0))$. Moreover, $\{u_1,\ldots,u_t\}$ is a basis of $\mathcal{N}_r(R(\lambda_0))$ if and only if $\{\Phi_k(\lambda_0)\otimes u_1,\ldots,\Phi_k(\lambda_0)\otimes u_t\}$ is a basis of $\mathcal{N}_r(\widehat{R}(\lambda_0))$.

Proof. By Lemma 3.4.1, $\widehat{R}(\lambda_0)(\Phi_k(\lambda_0)\otimes I_m) = v\otimes R(\lambda_0)$. Thus, it is easy to see that $u \in \mathcal{N}_r(R(\lambda_0))$ if and only if $\Phi_k(\lambda_0)\otimes u \in \mathcal{N}_r(\widehat{R}(\lambda_0))$. Consider $\{u_1,\ldots,u_t\}$ a basis of $\mathcal{N}_r(R(\lambda_0))$. Therefore, as $\dim \mathcal{N}_r(R(\lambda_0)) = \dim \mathcal{N}_r(\widehat{R}(\lambda_0))$, an immediate linear independence argument proves that $\{\Phi_k(\lambda_0)\otimes u_1,\ldots,\Phi_k(\lambda_0)\otimes u_t\}$ is a basis of $\mathcal{N}_r(\widehat{R}(\lambda_0))$, and conversely. ■

In addition, by using Proposition 2.4.7, we can recover the right eigenvectors of the transfer function $\widehat{R}(\lambda)$ from the right eigenvectors of the linearization $\mathcal{L}(\lambda)$, and conversely. In particular, if $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the polynomial eigenvalue problem $\mathcal{L}(\lambda)z = 0$ such that $\det(\lambda_0 I_n - A) \neq 0$, then (λ_0, x_0) is a solution of the rational eigenvalue problem $\widehat{R}(\lambda)x = 0$.

In what follows, if we have a vector $\begin{bmatrix} y \\ x \end{bmatrix}$, with $y \in \overline{\mathbb{F}}^{n\times 1}$ and $x \in \overline{\mathbb{F}}^{km\times 1}$, we will consider the vector x partitioned as $x = \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(k)} \end{bmatrix}^T$ with $x^{(j)} \in \overline{\mathbb{F}}^{m\times 1}$ for $j = 1, \ldots, k$. Recall also in Theorem 3.4.4 that, as we have explained in Remark 3.4.2, if $\lambda_0 \in \overline{\mathbb{F}}$ is a finite eigenvalue of $R(\lambda)$ then $\det(\lambda_0 I_n - A) \neq 0$. However, if λ_0 is an eigenvalue of $\mathcal{L}(\lambda)$, then, according to [6, Theorem 3.10], λ_0 might be a zero of $R(\lambda)$ that is simultaneously a pole and, therefore, $\det(\lambda_0 I_n - A) = 0$, and λ_0 is not an eigenvalue of $R(\lambda)$. This is the reason why the condition $\det(\lambda_0 I_n - A) \neq 0$ is assumed in parts *a)* and *b)* of Theorem 3.4.4.

Theorem 3.4.4. ***(Recovery of right eigenvectors from $\mathbb{M}_1$-strong linearizations)*** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, and let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} & XB \\ \hline -(v \otimes I_m)CY & \multicolumn{2}{c}{L(\lambda)} \end{array}\right]$$

be an $\mathbb{M}_1$-strong linearization of $R(\lambda)$.

a) If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $\mathcal{L}(\lambda)z = 0$ such that $\det(\lambda_0 I_n - A) \neq 0$, *then $(\lambda_0, x_0^{(k)})$ is a solution of the REP $R(\lambda)x = 0$.*

b) Moreover, if $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ is a basis of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$, with $\det(\lambda_0 I_n - A) \neq 0$, *then $\{x_1^{(k)}, \ldots, x_t^{(k)}\}$ is a basis of $\mathcal{N}_r(R(\lambda_0))$.*

c) Conversely, if (λ_0, u_0) is a solution of the REP $R(\lambda)x = 0$, $x_0 = \Phi_k(\lambda_0) \otimes u_0$ and y_0 is defined as the unique solution of $(\lambda_0 I_n - A)Y y_0 + B u_0 = 0$, then $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $\mathcal{L}(\lambda)z = 0$.

d) Moreover, if $\{u_1, \ldots, u_t\}$ is a basis of $\mathcal{N}_r(R(\lambda_0))$ and, for $i = 1, \ldots, t$, $x_i = \Phi_k(\lambda_0) \otimes u_i$ and y_i is defined as the unique solution of $(\lambda_0 I_n - A)Y y_i + B u_i = 0$, then $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ is a basis of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$.

Proof. By Proposition 2.4.7, if $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $\mathcal{L}(\lambda)z = 0$ such that $\det(\lambda_0 I_n - A) \neq 0$, then (λ_0, x_0) is a solution of the REP $\widehat{R}(\lambda)x = 0$, where

$\widehat{R}(\lambda)$ is the transfer function matrix of $\mathcal{L}(\lambda)$. By Theorem 3.4.3, x_0 has the form $x_0 = \Phi_k(\lambda_0) \otimes u$ for some $u \in \mathcal{N}_r(R(\lambda_0))$. Since $\phi_0(\lambda) = 1$ we have that $u = x_0^{(k)}$, which proves *a*). The converse *c*) is proved analogously. The implications *b*) and *d*) are consequences of (3.8), (3.9), basic arguments of linear independence, and the fact that $\mathcal{L}(\lambda_0)\begin{bmatrix} y_0 \\ x_0 \end{bmatrix} = 0$ if and only if $(\lambda_0 I_n - A)Yy_0 + XBx_0^{(k)} = 0$ and $\widehat{R}(\lambda_0)x_0 = 0$. ■

Next, we pay attention to the recovery of left eigenvectors.

Theorem 3.4.5. ***(Recovery of left eigenvectors from $\mathbb{M}_1$-strong linearizations)***
Let $R(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, let

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & 0_{n \times (k-1)m} & XB \\ \hline -(v \otimes I_m)CY & \multicolumn{2}{c}{L(\lambda)} \end{array}\right]$$

be an $\mathbb{M}_1$-strong linearization of $R(\lambda)$, and let $\widehat{R}(\lambda)$ be the transfer function of $\mathcal{L}(\lambda)$.

a) If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $z^T\mathcal{L}(\lambda) = 0$ such that $\det(\lambda_0 I_n - A) \neq 0$, then $(\lambda_0, (v^T \otimes I_m)x_0)$ is a solution of the REP $x^T R(\lambda) = 0$.

b) Moreover, if $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}\right\}$ is a basis of $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$, with $\det(\lambda_0 I_n - A) \neq 0$, then $\{(v^T \otimes I_m)x_1, \ldots, (v^T \otimes I_m)x_t\}$ is a basis of $\mathcal{N}_\ell(R(\lambda_0))$.

c) Conversely, if (λ_0, u_0) is a solution of the REP $x^T R(\lambda) = 0$, then there exists $x_0 \in \mathcal{N}_\ell(\widehat{R}(\lambda_0))$ such that $u_0 = (v^T \otimes I_m)x_0$ and if y_0 is defined as the unique solution of $y_0^T X(\lambda_0 I_n - A) - u_0^T C = 0$, then $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $z^T\mathcal{L}(\lambda) = 0$.

d) Moreover, if $\{u_1, \ldots, u_t\}$ is a basis of $\mathcal{N}_\ell(R(\lambda_0))$ then, for $i = 1, \ldots, t$, there exists $x_i \in \mathcal{N}_\ell(\widehat{R}(\lambda_0))$ such that $u_i = (v^T \otimes I_m)x_i$ and if y_i is defined as the unique solution of $y_i^T X(\lambda_0 I_n - A) - u_i^T C = 0$, then $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}\right\}$ is a basis of $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$.

Proof. We consider the transfer function of $\mathcal{L}(\lambda)$, $\widehat{R}(\lambda) = L(\lambda) + [0_{km \times (k-1)m} \quad (v \otimes I_m)C(\lambda I_n - A)^{-1}B]$. If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $z^T\mathcal{L}(\lambda) = 0$ such that $\det(\lambda_0 I_n - A) \neq 0$, by using Proposition 2.4.8 *a*) applied to $\mathcal{L}(\lambda)$, we get

$$x_0^T \widehat{R}(\lambda_0) = x_0^T L(\lambda_0) + [0_{1 \times (k-1)m} \quad x_0^T (v \otimes I_m)C(\lambda_0 I_n - A)^{-1}B] = 0, \tag{3.11}$$

where $x_0 \neq 0$ since (λ_0, x_0) is a solution of the REP $x^T \widehat{R}(\lambda) = 0$ [1]. In addition, by Lemma 3.4.1, $x_0^T \widehat{R}(\lambda_0)(\Phi_k(\lambda_0) \otimes I_m) = x_0^T(v \otimes I_m)R(\lambda_0)$. Therefore $x_0^T(v \otimes I_m)R(\lambda_0) = 0$. To see that $(v^T \otimes I_m)x_0$ is a left eigenvector of $R(\lambda_0)$, we only need to prove that $x_0^T(v \otimes I_m) \neq 0$. Let us suppose that $x_0^T(v \otimes I_m) = 0$, and let us get a contradiction. In this case $x_0^T(v \otimes I_m)C(\lambda_0 I_n - A)^{-1}B = 0$ and, therefore, $x_0^T L(\lambda_0) = x_0^T[v \otimes I_m \quad H]F_\Phi^D(\lambda_0) = 0$ by (3.11). We call $w^T = x_0^T[v \otimes I_m \quad H]$ and we consider w partitioned as $w = (w_i)_{i=1}^k$ with $w_i \in \overline{\mathbb{F}}^{m \times 1}$. We have that $w_1^T = x_0^T(v \otimes I_m) = 0$. Therefore $[0 \quad w_2^T \quad \cdots \quad w_k^T]F_\Phi^D(\lambda_0) = 0$. This implies $-\alpha_{k-2}w_2^T = 0$ and thus $w_2 = 0$, since $\alpha_{k-2} \neq 0$. Therefore $[0 \quad 0 \quad w_3^T \quad \cdots \quad w_k^T]F_\Phi^D(\lambda_0) = 0$ and $w_3 = 0$. Proceeding in this way it is easy to prove that $w_i = 0$ for $i = 2, \cdots, k$. Thus $x_0^T[v \otimes I_m \quad H] = 0$ which is a contradiction because $[v \otimes I_m \quad H]$ is assumed to be regular and $x_0 \neq 0$. This proves a).

The implication b) is proved as follows. From a), the vectors $(v^T \otimes I_m)x_1, \ldots, (v^T \otimes I_m)x_t$ belong to $\mathcal{N}_\ell(R(\lambda_0))$. Therefore, as a consequence of (3.10), if we prove that $\{(v^T \otimes I_m)x_1, \ldots, (v^T \otimes I_m)x_t\}$ is linearly independent, then b) is proved. For this purpose, let $\alpha_1, \ldots, \alpha_t \in \overline{\mathbb{F}}$ be arbitrary scalars such that at least one is different from zero. Thus $0 \neq \begin{bmatrix} \alpha_1 y_1 + \cdots + \alpha_t y_t \\ \alpha_1 x_1 + \cdots + \alpha_t x_t \end{bmatrix} \in \mathcal{N}_\ell(\mathcal{L}(\lambda_0))$, and, from part a), $x = (v^T \otimes I_m)(\alpha_1 x_1 + \cdots + \alpha_t x_t) \neq 0$ and $x \in \mathcal{N}_\ell(R(\lambda_0))$.

For proving c), we prove first that there exists a basis of $\mathcal{N}_\ell(R(\lambda_0))$ of the form $\{(v^T \otimes I_m)x_1, \ldots, (v^T \otimes I_m)x_t\}$, where $\{x_1, \ldots, x_t\}$ is a basis of $\mathcal{N}_\ell(\widehat{R}(\lambda_0))$. To this purpose, let $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ be a basis of $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$. Then, Proposition 2.4.8 b) applied to $\mathcal{L}(\lambda)$ implies that $\{x_1, \ldots, x_t\}$ is a basis of $\mathcal{N}_\ell(\widehat{R}(\lambda_0))$ and Theorem 3.4.5 b) that $\{(v^T \otimes I_m)x_1, \ldots, (v^T \otimes I_m)x_t\}$ is a basis of $\mathcal{N}_\ell(R(\lambda_0))$. Then, if (λ_0, u_0) is a solution of the REP $x^T R(\lambda) = 0$, u_0 can be written as $u_0 = (v^T \otimes I_m) \sum_{i=1}^t a_i x_i$ with $a_i \in \overline{\mathbb{F}}$, and we define $x_0 = \sum_{i=1}^t a_i x_i \in \mathcal{N}_\ell(\widehat{R}(\lambda_0))$. Finally, Proposition 2.4.8 c) applied to the solution (λ_0, x_0) of the REP $x^T \widehat{R}(\lambda) = 0$ and to $\mathcal{L}(\lambda)$, and the fact that $\det(\lambda_0 I_n - A) \neq 0$ imply that if y_0 is the unique solution of $y_0^T X(\lambda_0 I_n - A) - x_0^T(v \otimes I_m)C = 0$, which is equivalent to $y_0^T X(\lambda_0 I_n - A) - u_0^T C = 0$, then $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ is a solution of the LEP $z^T \mathcal{L}(\lambda) = 0$.

Finally, the proof of d) proceeds as follows. From part c), we obtain that the vectors $x_1, \ldots, x_t$ satisfying $u_i = (v^T \otimes I_m)x_i$ exist, and that the vectors $\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}$

[1]With the notation of Proposition 2.4.8, it is easy to see that $[y_0^T \ x_0^T]P(\lambda_0) = 0$ if and only if $y_0^T A(\lambda_0) - x_0^T C(\lambda_0) = 0$ and $x_0^T R(\lambda_0) = 0$. Thus, $x_0 = 0$ and $\det A(\lambda_0) \neq 0$ imply $y_0 = 0$. Therefore, any left eigenvector of $P(\lambda)$ corresponding to the finite eigenvalue λ_0 must have $x_0 \neq 0$.

belong to $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$. Therefore, taking (3.10) into account, it only remains to prove that $\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}$ are linearly independent. This is easily proved by contradiction: If $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ is linearly dependent, then $\{x_1, \dots, x_t\}$ is linearly dependent, and $\{u_1, \dots, u_t\}$ is linearly dependent, which is a contradiction since $\{u_1, \dots, u_t\}$ is a basis. ■

Remark 3.4.6. Analogously to Remark 2.4.9, if $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ is singular, then the results on null-spaces proved so far in Section 3.4.1 are valid for any $\lambda_0 \in \overline{\mathbb{F}}$ that satisfies $\det(\lambda_0 I_n - A) \neq 0$.

Finally, we study the recovery of the eigenvectors corresponding to the infinite eigenvalue from $\mathbb{M}_1$-strong linearizations.

Theorem 3.4.7. ***(Recovery of eigenvectors associated to infinity from $\mathbb{M}_1$-strong linearizations)*** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n\times(k-1)m} \quad XB \\ \hline -(v \otimes I_m)CY & L(\lambda) \end{array} \right]$$

be an $\mathbb{M}_1$-strong linearization of $R(\lambda)$, and let D_k be the leading matrix coefficient of the polynomial part of $R(\lambda)$ as in (3.5)*. Then the following statements hold:*

a) $\mathcal{N}_r(\operatorname{rev} R(0)) = \mathcal{N}_r(D_k)$ *and* $x_0 \in \mathcal{N}_r(D_k)$ *if and only if* $\begin{bmatrix} 0 \\ e_1 \otimes x_0 \end{bmatrix} \in \mathcal{N}_r(\operatorname{rev} \mathcal{L}(0))$. *Moreover,* $\{x_1, \dots, x_q\}$ *is a basis of* $\mathcal{N}_r(\operatorname{rev} R(0))$ *if and only if* $\left\{ \begin{bmatrix} 0 \\ e_1 \otimes x_1 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ e_1 \otimes x_q \end{bmatrix} \right\}$ *is a basis of* $\mathcal{N}_r(\operatorname{rev} \mathcal{L}(0))$.

b) $\mathcal{N}_\ell(\operatorname{rev} R(0)) = \mathcal{N}_\ell(D_k)$ *and* $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} \in \mathcal{N}_\ell(\operatorname{rev} \mathcal{L}(0))$ *if and only if* $(v^T \otimes I_m)x_0 \in \mathcal{N}_\ell(D_k)$. *Moreover,* $\left\{ \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ x_q \end{bmatrix} \right\}$ *is a basis of* $\mathcal{N}_\ell(\operatorname{rev} \mathcal{L}(0))$ *if and only if* $\left\{ (v^T \otimes I_m)x_1, \dots, (v^T \otimes I_m)x_q \right\}$ *is a basis of* $\mathcal{N}_\ell(\operatorname{rev} R(0))$.

Proof. Notice that from (3.4),

$$F_\Phi^D(\lambda) = \lambda \begin{bmatrix} \alpha_{k-1}^{-1} D_k & 0 \\ 0 & I_{(k-1)m} \end{bmatrix} + F_\Phi^D(0).$$

We consider

$$L(\lambda) = [v \otimes I_m \quad H] F_\Phi^D(\lambda) = [\alpha_{k-1}^{-1}(v \otimes D_k) \quad H]\lambda + L(0) =: L_1\lambda + L_0$$

and let $\widehat{R}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. We have that $\operatorname{rev}\mathcal{L}(0) = \left[\begin{array}{c|c} XY & 0 \\ \hline 0 & L_1 \end{array}\right]$ and $\operatorname{rev}\widehat{R}(0) = \operatorname{rev} L(0) = L_1$. Moreover, $\operatorname{rev} R(0) = \alpha_0^{-1}\alpha_1^{-1}\cdots\alpha_{k-1}^{-1}D_k$, that is, the coefficient of λ^k in $D(\lambda)$. Therefore, $\mathcal{N}_r(\operatorname{rev} R(0)) = \mathcal{N}_r(D_k)$, $\mathcal{N}_\ell(\operatorname{rev} R(0)) = \mathcal{N}_\ell(D_k)$ and ∞ is an eigenvalue of $R(\lambda)$ if and only if D_k is singular. In addition, every right (respectively left) eigenvector w of $\operatorname{rev}\mathcal{L}(0)$ has the form $w = \begin{bmatrix} 0 \\ x_0 \end{bmatrix}$ for some $x_0 \in \mathcal{N}_r(L_1)$ (respectively $x_0 \in \mathcal{N}_\ell(L_1)$). By Lemma 3.4.1, we have

$$\lambda\widehat{R}\left(\frac{1}{\lambda}\right)\left(\lambda^{k-1}\Phi_k\left(\frac{1}{\lambda}\right)\otimes I_m\right) = v\otimes\lambda^k R\left(\frac{1}{\lambda}\right).$$

Therefore,

$$\operatorname{rev}\widehat{R}(0)(\operatorname{rev}\Phi_k(0)\otimes I_m) = (v\otimes I_m)\operatorname{rev} R(0).$$

Since $\operatorname{rev}\Phi_k(0) = \alpha_0^{-1}\alpha_1^{-1}\cdots\alpha_{k-2}^{-1}e_1$, we obtain

$$\alpha_0^{-1}\alpha_1^{-1}\cdots\alpha_{k-2}^{-1}\operatorname{rev}\widehat{R}(0)(e_1\otimes I_m) = (v\otimes I_m)\operatorname{rev} R(0).$$

In addition, by (2.15), there exist unimodular matrices $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$W_1(0)\operatorname{diag}\left(\operatorname{rev} R(0), I_{(k-1)m}\right)W_2(0) = \operatorname{rev}\widehat{R}(0),$$

which implies that $\dim\mathcal{N}_r(\operatorname{rev} R(0)) = \dim\mathcal{N}_r(\operatorname{rev}\widehat{R}(0))$ and $\dim\mathcal{N}_\ell(\operatorname{rev} R(0)) = \dim\mathcal{N}_\ell(\operatorname{rev}\widehat{R}(0))$. Finally *a*) and *b*) follow from the results above by using similar arguments to the ones we used in the recovery of eigenvectors associated to finite eigenvalues. ■

3.4.2 Eigenvectors from $\mathbb{M}_2$-strong linearizations

If we proceed analogously as we did with $\mathbb{M}_1$-strong linearizations, and we use Lemma 3.4.8, then we get Theorems 3.4.9, 3.4.10 and 3.4.12 to recover right and left eigenvectors of a rational matrix from those of its $\mathbb{M}_2$-strong linearizations. The proofs are essentially the same as those in Section 3.4.1 by interchanging the roles of left and right eigenvectors, and they are omitted for brevity.

Lemma 3.4.8. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix with polynomial part of degree $k \geq 2$, let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T\otimes I_m) \\ \hline \begin{matrix} 0_{(k-1)m\times n} \\ -CY \end{matrix} & L(\lambda) \end{array}\right]$$

be an $\mathbb{M}_2$-strong linearization of $R(\lambda)$, and let $\widehat{R}(\lambda)$ be the transfer function of $\mathcal{L}(\lambda)$. Then

$$(\Phi_k(\lambda)^T\otimes I_m)\widehat{R}(\lambda) = w^T\otimes R(\lambda). \tag{3.12}$$

Theorem 3.4.9. ***(Recovery of right eigenvectors from*** $\mathbb{M}_2$***-strong linearizations)*** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ *be a rational matrix with polynomial part of degree* $k \geq 2$*, let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T \otimes I_m) \\ \hline \begin{matrix} 0_{(k-1)m\times n} \\ -CY \end{matrix} & L(\lambda) \end{array}\right]$$

be an $\mathbb{M}_2$*-strong linearization of* $R(\lambda)$*, and let* $\widehat{R}(\lambda)$ *be the transfer function of* $\mathcal{L}(\lambda)$*.*

a) If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the LEP* $\mathcal{L}(\lambda)z = 0$ *such that* $\det(\lambda_0 I_n - A) \neq 0$ *then,* $(\lambda_0, (w^T \otimes I_m)x_0)$ *is a solution of the REP* $R(\lambda)x = 0$*.*

b) Moreover, if $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}\right\}$ *is a basis of* $\mathcal{N}_r(\mathcal{L}(\lambda_0))$*, with* $\det(\lambda_0 I_n - A) \neq 0$*, then* $\{(w^T \otimes I_m)x_1, \ldots, (w^T \otimes I_m)x_t\}$ *is a basis of* $\mathcal{N}_r(R(\lambda_0))$*.*

c) Conversely, if (λ_0, u_0) *is a solution of the REP* $R(\lambda)x = 0$*, then there exists* $x_0 \in \mathcal{N}_r(\widehat{R}(\lambda_0))$ *such that* $u_0 = (w^T \otimes I_m)x_0$ *and if* y_0 *is defined as the unique solution of* $(\lambda_0 I_n - A)Yy_0 + Bu_0 = 0$*, then* $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the LEP* $\mathcal{L}(\lambda)z = 0$*.*

d) Moreover, if $\{u_1, \ldots, u_t\}$ *is a basis of* $\mathcal{N}_r(R(\lambda_0))$ *then, for* $i = 1, \ldots, t$*, there exists* $x_i \in \mathcal{N}_r(\widehat{R}(\lambda_0))$ *such that* $u_i = (w^T \otimes I_m)x_i$ *and if* y_i *is defined as the unique solution of* $(\lambda_0 I_n - A)Yy_i + Bu_i = 0$*, then* $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}\right\}$ *is a basis of* $\mathcal{N}_r(\mathcal{L}(\lambda_0))$*.*

Theorem 3.4.10. ***(Recovery of left eigenvectors from*** $\mathbb{M}_2$***-strong linearizations)*** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ *be a rational matrix with polynomial part of degree* $k \geq 2$*, and let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T \otimes I_m) \\ \hline \begin{matrix} 0_{(k-1)m\times n} \\ -CY \end{matrix} & L(\lambda) \end{array}\right]$$

be an $\mathbb{M}_2$*-strong linearization of* $R(\lambda)$*.*

a) If $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the LEP* $z^T\mathcal{L}(\lambda) = 0$ *such that* $\det(\lambda_0 I_n - A) \neq 0$*, then* $(\lambda_0, x_0^{(k)})$ *is a solution of the REP* $x^T R(\lambda) = 0$*.*

b) Moreover, if $\left\{\begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix}\right\}$ *is a basis of* $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$*, with* $\det(\lambda_0 I_n - A) \neq 0$*, then* $\{x_1^{(k)}, \ldots, x_t^{(k)}\}$ *is a basis of* $\mathcal{N}_\ell(R(\lambda_0))$*.*

c) Conversely, if (λ_0, u_0) *is a solution of the REP* $x^T R(\lambda) = 0$, $x_0 = \Phi_k(\lambda_0) \otimes u_0$ *and* y_0 *is defined as the unique solution of* $y_0^T X(\lambda_0 I_n - A) - u_0^T C = 0$, *then* $\left(\lambda_0, \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}\right)$ *is a solution of the LEP* $z^T \mathcal{L}(\lambda) = 0$.

d) Moreover, if $\{u_1, \ldots, u_t\}$ *is a basis of* $\mathcal{N}_\ell(R(\lambda_0))$ *and, for* $i = 1, \ldots, t$, $x_i = \Phi_k(\lambda_0) \otimes u_i$ *and* y_i *is defined as the unique solution of* $y_i^T X(\lambda_0 I_n - A) - u_i^T C = 0$, *then* $\left\{ \begin{bmatrix} y_1 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} y_t \\ x_t \end{bmatrix} \right\}$ *is a basis of* $\mathcal{N}_\ell(\mathcal{L}(\lambda_0))$.

Remark 3.4.11. Analogously to Remarks 2.4.9 and 3.4.6, if $R(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$ is singular, then the results on null-spaces in Theorems 3.4.9 and 3.4.10 hold for any $\lambda_0 \in \overline{\mathbb{F}}$ such that $\det(\lambda_0 I_n - A) \neq 0$. ■

Theorem 3.4.12. ***(Recovery of eigenvectors associated to infinity from*** $\mathbb{M}_2$***-strong linearizations)*** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$ *be a rational matrix with polynomial part of degree* $k \geq 2$, *let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T \otimes I_m) \\ \hline \begin{matrix} 0_{(k-1)m \times n} \\ -CY \end{matrix} & L(\lambda) \end{array} \right]$$

be an $\mathbb{M}_2$*-strong linearization of* $R(\lambda)$, *and let* D_k *be the leading matrix coefficient of the polynomial part of* $R(\lambda)$ *as in* (3.5)*. Then the following statements hold:*

a) $\mathcal{N}_r(\operatorname{rev} R(0)) = \mathcal{N}_r(D_k)$ *and* $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} \in \mathcal{N}_r(\operatorname{rev} \mathcal{L}(0))$ *if and only if* $(w^T \otimes I_m)x_0 \in \mathcal{N}_r(D_k)$*. Moreover,* $\left\{ \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ x_q \end{bmatrix} \right\}$ *is a basis of* $\mathcal{N}_r(\operatorname{rev} \mathcal{L}(0))$ *if and only if* $\left\{(w^T \otimes I_m)x_1, \ldots, (w^T \otimes I_m)x_q\right\}$ *is a basis of* $\mathcal{N}_r(\operatorname{rev} R(0))$.

b) $\mathcal{N}_\ell(\operatorname{rev} R(0)) = \mathcal{N}_\ell(D_k)$ *and* $x_0 \in \mathcal{N}_\ell(D_k)$ *if and only if* $\begin{bmatrix} 0 \\ e_1 \otimes x_0 \end{bmatrix} \in \mathcal{N}_\ell(\operatorname{rev} \mathcal{L}(0))$*. Moreover,* $\{x_1, \ldots, x_q\}$ *is a basis of* $\mathcal{N}_\ell(\operatorname{rev} R(0))$ *if and only if* $\left\{ \begin{bmatrix} 0 \\ e_1 \otimes x_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ e_1 \otimes x_q \end{bmatrix} \right\}$ *is a basis of* $\mathcal{N}_\ell(\operatorname{rev} \mathcal{L}(0))$.

3.5 Symmetric realizations of symmetric rational matrices

In this section and in the next one our aim is to obtain a strong linearization of a *symmetric rational matrix* $R(\lambda) \in \mathbb{F}(\lambda)^{m \times m}$, i.e., $R(\lambda)^T = R(\lambda)$, that preserves its symmetric structure. We write $R(\lambda)$ as

$$R(\lambda) = D(\lambda) + R_{sp}(\lambda) \tag{3.13}$$

with $D(\lambda)$ its polynomial part and $R_{sp}(\lambda)$ its strictly proper part. Since (3.13) is a unique decomposition we obtain the following result just by taking transposes.

Proposition 3.5.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric rational matrix. Then the matrices $D(\lambda)$ and $R_{sp}(\lambda)$ in (3.13) are also symmetric.*

Proposition 3.5.5 is the main result in this section and shows that any symmetric strictly proper rational matrix admits a state-space realization that reveals transparently the symmetry. In order to state concisely Proposition 3.5.5, we will use the following definition.

Definition 3.5.2. *Let $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric strictly proper rational matrix and let $n = \nu(R_{sp}(\lambda))$ be the least order of $R_{sp}(\lambda)$. A symmetric minimal state-space realization of $R_{sp}(\lambda)$ is an expression of the form*

$$R_{sp}(\lambda) = W(S_1\lambda - S_2)^{-1}W^T$$

where $S_1, S_2 \in \mathbb{F}^{n\times n}$ are symmetric matrices with S_1 nonsingular and $W \in \mathbb{F}^{m\times n}$.

We remark that the realization described in Definition 3.5.2 is equivalent to [11, Definition 4.44] for a minimal state-space realization. However, in Definition 3.5.2 we express strictly proper matrices in a form more convenient for the goals of this chapter. In particular, we will see in Section 3.6 that by combining a symmetric minimal state-space realization of the matrix $R_{sp}(\lambda)$ in (3.13) and a symmetric strong block minimal bases pencil associated to $D(\lambda)$, we can construct symmetric strong linearizations of $R(\lambda)$. The next technical lemma is used in the proof of Proposition 3.5.5.

Lemma 3.5.3. *Let $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric strictly proper rational matrix and let $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal state-space realization of $R_{sp}(\lambda)$. Then there exists a unique nonsingular and symmetric matrix $S \in \mathbb{F}^{n\times n}$ such that $A^T = S^{-1}AS$ and $C^T = S^{-1}B$.*

Proof. As $R_{sp}(\lambda)$ is symmetric, $R_{sp}(\lambda) = B^T(\lambda I_n - A^T)^{-1}C^T$ is also a minimal state-space realization of $R_{sp}(\lambda)$ since both have the same minimal order n. Therefore, by [48, Proposition 3.3.2], the realizations (A, B, C) and (A^T, C^T, B^T) are similar and there exists a unique nonsingular matrix $S \in \mathbb{F}^{n\times n}$ such that

$$A^T = S^{-1}AS, \quad C^T = S^{-1}B, \quad B^T = CS. \tag{3.14}$$

The fact that (A, B, C) is a minimal realization of $R_{sp}(\lambda)$ is equivalent to that (A, B) and (A, C) are controllable and observable, respectively (see [78, Chapter

3]). That means that the controllability matrix of (A,B) and the observability matrix of (A,C), i.e.,

$$\mathcal{C}(A,B) = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B] \quad \text{and} \quad \mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix},$$

have both rank n. From the equalities in (3.14) it is easy to see that $S^{-1}\mathcal{C}(A,B) = \mathcal{O}(A,C)^T$, and $S^{-T}\mathcal{C}(A,B) = \mathcal{O}(A,C)^T$. As $\mathcal{C}(A,B)$ has full row rank, we deduce that $S = S^T$. ■

Remark 3.5.4. Notice that the system similarity matrix S between the realizations in Lemma 3.5.3 is given by $S = \mathcal{O}(A,C)^+\mathcal{C}(A,B)^T = \mathcal{C}(A,B)(\mathcal{O}(A,C)^T)^\dagger$ where $+$ denotes any left inverse and $\dagger$ denotes any right inverse. Notice also that these left and right inverses exist because (A,B,C) is a minimal realization of $R_{sp}(\lambda)$ and that they can be taken to be the Moore–Penrose inverses. Thus S can be efficiently computed when $\mathbb{F} = \mathbb{R}, \mathbb{C}$. ■

Proposition 3.5.5. *Any symmetric strictly proper rational matrix has a symmetric minimal state-space realization.*

Proof. As said in Section 2.4, any strictly proper rational matrix $R_{sp}(\lambda)$ admits a minimal state-space realization, that is, $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ [78]. By Lemma 3.5.3, there exists a unique nonsingular and symmetric matrix S such that $A^T = S^{-1}AS$ and $C^T = S^{-1}B$. Thus, $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}SC^T = C(\lambda S^{-1} - S^{-1}A)^{-1}C^T$, and $S^{-1}A$ is symmetric, as $(S^{-1}A)^T = A^TS^{-1} = S^{-1}A$. ■

Remark 3.5.6. We can construct a symmetric minimal state-space realization of a symmetric strictly proper rational matrix $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ without previously considering a non-symmetric minimal state-space realization of $R_{sp}(\lambda)$, in contrast to what we have done in the proof of Proposition 3.5.5. For this purpose we require $\mathbb{F}$ not to be a field of characteristic 2. Let $R_{sp}(\lambda) = G_1\lambda^{-1} + G_2\lambda^{-2} + \cdots$ be the Laurent series of $R_{sp}(\lambda)$, which converges for $|\lambda|$ large enough. Let $n = \nu(R_{sp}(\lambda))$ be the least order of $R_{sp}(\lambda)$. We consider the block Hankel matrix

$$H_n = \begin{bmatrix} G_1 & G_2 & \cdots & G_n \\ G_2 & G_3 & \cdots & G_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_n & G_{n+1} & \cdots & G_{2n-1} \end{bmatrix} \tag{3.15}$$

and follow in a symmetric way the three steps of the algorithm in [48, Section 3.4] to get a symmetric minimal state-space realization from the Hankel matrix. Notice that the Hankel matrix is symmetric since $R_{sp}(\lambda)$ is symmetric, which implies $G_i = G_i^T$ for all $i \geq 1$, and $\operatorname{rank} H_n = n$ by [48, Proposition 3.3.2]. Therefore we can write

$$H_n = X \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} X^T = X \begin{bmatrix} K \\ 0 \end{bmatrix} [I_n \quad 0] X^T$$

with X nonsingular and $K \in \mathbb{F}^{n \times n}$ diagonal (see [62, Theorem 34.1]). Let us denote

$$\Gamma = X \begin{bmatrix} K \\ 0 \end{bmatrix} \text{ and } \Lambda = [I_n \quad 0] X^T.$$

We have that $H_n = \Gamma\Lambda$. We write $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, where $X_1 = [X_{i1}]_{i=1}^n$ with $X_{i1} \in \mathbb{F}^{m \times n}$ for $i = 1, \ldots, n$. Thus

$$\Gamma = \begin{bmatrix} X_{11}K \\ \vdots \\ X_{n1}K \end{bmatrix} \text{ and } \Lambda = [X_{11}^T \quad \cdots \quad X_{n1}^T].$$

We define

$$R = \begin{bmatrix} G_2 & G_3 & \cdots & G_{n+1} \\ G_3 & G_4 & \cdots & G_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n+1} & G_{n+2} & \cdots & G_{2n} \end{bmatrix}$$

and we set $C = X_{11}K$, $B = X_{11}^T$ and $A = \Gamma^+ R \Lambda^+$, with $\Gamma^+ = [K^{-1} \quad 0] X^{-1}$ and $\Lambda^+ = X^{-T} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. Thus $A = [K^{-1} \quad 0] X^{-1} R X^{-T} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ and, by [48, Theorem 3.4.1], (A, B, C) is a minimal realization for $R_{sp}(\lambda)$. Therefore

$$\begin{aligned} R_{sp}(\lambda) &= X_{11}K \left(\lambda I_n - [K^{-1} \quad 0] X^{-1} R X^{-T} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right)^{-1} X_{11}^T \\ &= X_{11} \left(\lambda K^{-1} - [K^{-1} \quad 0] X^{-1} R X^{-T} \begin{bmatrix} K^{-1} \\ 0 \end{bmatrix} \right)^{-1} X_{11}^T. \end{aligned}$$

Finally we set $W = X_{11}$, $S_1 = K^{-1}$ and $S_2 = [K^{-1} \quad 0] X^{-1} R X^{-T} \begin{bmatrix} K^{-1} \\ 0 \end{bmatrix}$, and we obtain a symmetric minimal state-space realization of $R_{sp}(\lambda)$.

In the particular case $R_{sp}(\lambda) \in \mathbb{R}(\lambda)^{m \times m}$, which is of significant importance in applications, the Hankel matrix H_n is real and symmetric. Therefore, we can write

$$H_n = P \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} P^T$$

with P *orthogonal*, i.e., $P^{-1} = P^T$, and K a diagonal matrix that has the eigenvalues different from zero of H_n at the diagonal elements. In this case, let $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, where $P_1 = [P_{i1}]_{i=1}^n$ with $P_{i1} \in \mathbb{F}^{m\times n}$ for $i = 1, \dots, n$. Then we obtain $R_{sp}(\lambda) = P_{11}(\lambda K^{-1} - K^{-1}P_1^T R P_1 K^{-1})^{-1} P_{11}^T$. That is, $R_{sp}(\lambda)$ has a symmetric minimal state-space realization $R_{sp}(\lambda) = W(\lambda S_1 - S_2)^{-1} W^T$ where $W = P_{11}$, $S_1 = K^{-1}$ and $S_2 = K^{-1}P_1^T R P_1 K^{-1}$. ■

From Proposition 3.5.5 and Remark 3.5.6 we know how to write the strictly proper part $R_{sp}(\lambda)$ of a symmetric rational matrix $R(\lambda)$ as a symmetric minimal state-space realization with or without having in advance a particular non-symmetric minimal state-space realization of $R_{sp}(\lambda)$. Moreover, it is worth to emphasize that in many applications of symmetric REPs, this can be done very easily from the data of the model without any computational cost (see [6, Section 5.3] or [79, Section 4]).

3.6 Symmetric strong linearizations for symmetric rational matrices

In this section symmetric strong linearizations for symmetric rational matrices will be constructed. We start with Example 3.6.1 in which we construct a symmetric strong linearization of a symmetric rational matrix when the polynomial part has odd degree. We will use Proposition 3.5.5 and a particular symmetric strong block minimal bases pencil associated to its polynomial part with sharp degree. After that, we present symmetric strong linearizations for symmetric rational matrices in which the polynomial part may have even or odd degree but the leading coefficient must be nonsingular. In order to get these results, we need to study symmetric strong linearizations in the polynomial case.

Example 3.6.1. Let $R(\lambda) = D(\lambda) + R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric rational matrix. Consider the polynomial part $D(\lambda)$ written in terms of the monomial basis $D(\lambda) = D_k\lambda^k + D_{k-1}\lambda^{k-1} + \cdots + D_0 \in \mathbb{F}[\lambda]^{m\times m}$, with $k > 1$ and $D_k \neq 0$, and the matrices

$$L_p(\lambda) = \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{p\times(p+1)}, \tag{3.16}$$

and

$$\Lambda_p(\lambda)^T = \begin{bmatrix} \lambda^p & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(p+1)}. \tag{3.17}$$

A *block Kronecker linearization* of $D(\lambda)$ is a pencil

$$L(\lambda) = \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_m \\ \hline L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right] \begin{array}{l} \}\,{\scriptstyle (\eta+1)m} \\ \}\,{\scriptstyle \varepsilon m} \end{array} \tag{3.18}$$

$$\underbrace{\qquad\qquad}_{(\varepsilon+1)m} \quad \underbrace{\qquad\qquad}_{\eta m}$$

such that $D(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_m)\, M(\lambda)\, (\Lambda_\varepsilon(\lambda) \otimes I_m)$ (see [26, Definition 4.1]). Recall that block Kronecker linearizations are particular cases of strong block minimal bases pencils [26]. If the polynomial part $D(\lambda)$ has odd degree $k = 2q+1$ we can consider the symmetric block Kronecker linearization in which

$$M(\lambda) = \begin{bmatrix} D_{2q+1}\lambda + D_{2q} & & & \\ & D_{2q-1}\lambda + D_{2q-2} & & \\ & & \ddots & \\ & & & D_1\lambda + D_0 \end{bmatrix}$$

and $\varepsilon = \eta = q$. Proposition 3.5.5 allows us to write $R_{sp}(\lambda) = W(\lambda S_1 - S_2)^{-1}W^T$ with S_1 and S_2 symmetric and S_1 nonsingular. Applying [6, Theorem 5.11] with $Y = -S_1 X^T$ for any nonsingular matrix $X \in \mathbb{F}^{n\times n}$, $C = WS_1^{-1}$, $A = S_2 S_1^{-1}$, $B = W^T$, and $\widehat{K}_1 = \widehat{K}_2 = e_{q+1}^T \otimes I_m$, we obtain that the linear polynomial matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc:c} X(S_2 - \lambda S_1)X^T & 0 & XW^T & 0 \\ \hline \begin{matrix} 0 \\ WX^T \end{matrix} & \multicolumn{2}{c:}{M(\lambda)} & L_q(\lambda)^T \otimes I_m \\ \hdashline 0 & \multicolumn{2}{c:}{L_q(\lambda) \otimes I_m} & 0 \end{array}\right]$$

is a symmetric strong linearization of $R(\lambda)$.

Remark 3.6.2. The approach in Example 3.6.1 can be extended to other symmetric strong block minimal bases pencils of the symmetric polynomial part $D(\lambda)$ of $R(\lambda) = R(\lambda)^T$ to construct other symmetric strong linearizations of $R(\lambda)$, as long as $D(\lambda)$ has odd-degree. See, for instance, the pencils considered in [30]. However, the linearization in Example 3.6.1 is particularly simple and, in view of the results in [14], we expect that it will have favourable numerical properties. ■

Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix of degree k. A $km \times km$ pencil $L(\lambda)$ is called *block-symmetric* if $L(\lambda) = L(\lambda)^{\mathcal{B}}$, where $L(\lambda)$ is viewed as a block partitioned pencil with $k \times k$ blocks each of them of size $m \times m$. Notice that a pencil $L(\lambda)$ satisfies $L(\lambda)(\Phi_k(\lambda) \otimes I_m) = v \otimes P(\lambda)$ for some vector $v \in \mathbb{F}^k$ if and only if $L(\lambda)^{\mathcal{B}}$ satisfies $(\Phi_k(\lambda)^T \otimes I_m)L(\lambda)^{\mathcal{B}} = v^T \otimes P(\lambda)$. Thus, if $L(\lambda) \in \mathbb{M}_1(P)$ is block-symmetric, then $L(\lambda) \in \mathbb{M}_1(P) \cap \mathbb{M}_2(P)$. This intersection space was introduced in [36], it is called *double generalized ansatz space*, and it is denoted by

$$\mathbb{DM}(P) = \mathbb{M}_1(P) \cap \mathbb{M}_2(P).$$

If $\{\phi_j(\lambda)\}_{j=0}^\infty$ is the monomial basis, the space $\mathbb{DM}(P)$ is denoted $\mathbb{DL}(P)$ and was introduced originally in [63]. In [36, Corollary 6] it is shown that if a pencil $L(\lambda)$ belongs to $\mathbb{DM}(P)$ then its right and left ansatz vectors are the same, which is called simply *ansatz vector*, and that

$$\mathbb{DM}(P) = \{L(\lambda) \in \mathbb{M}_1(P) : L(\lambda) = L(\lambda)^{\mathcal{B}}\}.$$

In fact, if $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ is a symmetric polynomial matrix we obtain that any pencil in $\mathbb{DM}(P)$ must be symmetric. This result is not in [36], and we state it in Theorem 3.6.5. For its proof, we use Lemmas 3.6.3 and 3.6.4.

Lemma 3.6.3. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix of degree $k \geq 2$ and let $L(\lambda) \in \mathbb{DM}(P)$ with ansatz vector $0 \in \mathbb{F}^k$. Then $L(\lambda) = 0$.*

Proof. Notice that from (3.4),

$$F_\Phi^P(\lambda) = \lambda \begin{bmatrix} \alpha_{k-1}^{-1} P_k & 0 \\ 0 & I_{(k-1)m} \end{bmatrix} + F_\Phi^P(0), \tag{3.19}$$

where $P(\lambda)$ is expressed as in (3.3). From [36, Corollary 6], $L(\lambda)$ must have the form

$$L(\lambda) = [0_{km\times m} \quad H] F_\Phi^P(\lambda) = [0 \quad H]\lambda + [0 \quad H] F_\Phi^P(0) = \begin{bmatrix} 0 \\ H^{\mathcal{B}} \end{bmatrix} \lambda + F_\Phi^P(0)^{\mathcal{B}} \begin{bmatrix} 0 \\ H^{\mathcal{B}} \end{bmatrix}.$$

As $L(\lambda)$ is block symmetric, H must have the form $H = \begin{bmatrix} 0 \\ W \end{bmatrix}$ where W is a $(k-1)m \times (k-1)m$ block symmetric matrix. Let $W = [W_{ij}]_{i,j=1}^{k-1}$ with $W_{ij} \in \mathbb{F}^{m\times m}$. Then

$$[0_{km\times m} \quad H] F_\Phi^P(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -\alpha_{k-2} W_{11} & * & \cdots & * \\ -\alpha_{k-2} W_{21} & * & \cdots & * \\ \vdots & & & \\ -\alpha_{k-2} W_{(k-1)1} & * & \cdots & * \end{bmatrix}.$$

Notice that $[0_{km\times m} \quad H] F_\Phi^P(0)$ is also block symmetric because of the block symmetry of $L(\lambda)$. Then, we obtain that $W_{i1} = W_{1i} = 0$ with $i = 1, \ldots, k-1$. Next, we proceed by induction. Let $j \in \{2, \ldots, k-1\}$ and suppose that $W_{it} = W_{ti} = 0$ for all $i = 1, \ldots, k-1$ and $t = 1, \ldots, j-1$. Then,

$$[0_{km\times m} \quad H] F_\Phi^P(0) = \begin{array}{c} \\ \\ \\ (j) \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & \cdots & \overset{(j-1)}{0} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_{k-(j+1)} W_{jj} & * & \cdots & * \\ 0 & \cdots & 0 & -\alpha_{k-(j+1)} W_{(j+1)j} & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_{k-(j+1)} W_{(k-1)j} & * & \cdots & * \end{bmatrix}.$$

Therefore, $W_{ij} = W_{ji} = 0$ with $i = 1, \ldots, k-1$. By induction, $H = 0$ and $L(\lambda) = 0$. ■

Theorem 3.4 in [53] states that for each $v \in \mathbb{F}^k$ there is a uniquely determined pencil in $\mathbb{DL}(P)$ with ansatz vector v. We show this result extended to the space $\mathbb{DM}(P)$ in the following lemma.

Lemma 3.6.4. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix of degree $k \geq 2$. For each $v \in \mathbb{F}^k$ there is only one pencil in $\mathbb{DM}(P)$ with ansatz vector v.*

Proof. We consider the linear map $\mathbb{DM}(P) \longrightarrow \mathbb{F}^k$ that associates to any pencil $L(\lambda)$ in $\mathbb{DM}(P)$ its ansatz vector $v \in \mathbb{F}^k$. By Lemma 3.6.3 this map is injective and by [36, Corollary 7] $\dim(\mathbb{DM}(P)) = k$. Therefore, the map is bijective. ■

Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a symmetric polynomial matrix, and let us define the set

$$\mathbb{S}(P) = \{L(\lambda) \in \mathbb{M}_1(P) : L(\lambda) = L(\lambda)^T\}.$$

The elements in $\mathbb{S}(P)$ are in $\mathbb{DM}(P)$ because if $L(\lambda) = [v \otimes I_m \quad H]F_\Phi^P(\lambda) \in \mathbb{S}(P)$ then $L(\lambda)^T = F_\Phi^P(\lambda)^{\mathcal{B}} \begin{bmatrix} v^T \otimes I_m \\ H^T \end{bmatrix} \in \mathbb{M}_2(P)$, since in the case $P(\lambda)$ is symmetric $F_\Phi^P(\lambda)^T = F_\Phi^P(\lambda)^{\mathcal{B}}$, and $L(\lambda) = L(\lambda)^T$. Moreover, Theorem 3.6.5 shows that $\mathbb{S}(P)$ and $\mathbb{DM}(P)$ are equal.

Theorem 3.6.5. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a symmetric polynomial matrix of degree $k \geq 2$. Then*

$$\mathbb{DM}(P) = \mathbb{S}(P).$$

Proof. We have already seen that $\mathbb{S}(P) \subseteq \mathbb{DM}(P)$. To see the other inclusion we only have to use Lemma 3.6.4 and [36, Corollary 6], and notice that if $L(\lambda) \in \mathbb{DM}(P)$ with $P(\lambda)$ symmetric then

$$L(\lambda) = [v \otimes I_m \quad H]F_\Phi^P(\lambda) = F_\Phi^P(\lambda)^{\mathcal{B}} \begin{bmatrix} v^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix},$$

and

$$L(\lambda)^T = [v \otimes I_m \quad (H^{\mathcal{B}})^T]F_\Phi^P(\lambda) = F_\Phi^P(\lambda)^{\mathcal{B}} \begin{bmatrix} v^T \otimes I_m \\ H^T \end{bmatrix},$$

which implies that $L(\lambda)^T \in \mathbb{DM}(P)$ and that $L(\lambda)$ and $L(\lambda)^T$ have the same ansatz vector. So, by Lemma 3.6.4, $L(\lambda) = L(\lambda)^T$ and $L(\lambda) \in \mathbb{S}(P)$. ■

Therefore, if $P(\lambda)$ is a symmetric polynomial matrix all the pencils in $\mathbb{DM}(P)$ are also symmetric. In order to find linearizations in $\mathbb{DM}(P)$ we have to consider only regular polynomials $P(\lambda)$ because by [36, Theorem 7] if $P(\lambda)$ is a singular polynomial matrix then none of the pencils in $\mathbb{DM}(P)$ is a linearization for $P(\lambda)$.

In Theorem 3.6.9, we construct symmetric strong linearizations for a symmetric rational matrix from a particular symmetric strong linearization of its polynomial part $D(\lambda)$ when the leading coefficient D_k of $D(\lambda)$ is nonsingular. This particular strong linearization is the pencil in $\mathbb{DM}(D)$ with ansatz vector e_k, i.e., the last vector in the canonical basis of $\mathbb{F}^k$. Some properties of this pencil are studied in Lemma 3.6.6 and Corollary 3.6.7.

Lemma 3.6.6. *Let $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix with degree $k \geq 2$ and let $L(\lambda) = [e_k \otimes I_m \quad H]F_\Phi^D(\lambda) \in \mathbb{DM}(D)$. Then $[e_k \otimes I_m \quad H]$ is nonsingular if and only if the leading matrix coefficient D_k of $D(\lambda)$ is nonsingular.*

Proof. Let $L(\lambda) = [e_k \otimes I_m \quad H]F_\Phi^D(\lambda) \in \mathbb{DM}(D)$. We write, by using (3.19) and [36, Corollary 6],

$$\begin{aligned} L(\lambda) = [e_k \otimes I_m \quad H]F_\Phi^D(\lambda) &= [e_k \otimes \alpha_{k-1}^{-1} D_k \quad H]\lambda + [e_k \otimes I_m \quad H]F_\Phi^D(0) \\ &= \begin{bmatrix} e_k^T \otimes \alpha_{k-1}^{-1} D_k \\ H^{\mathcal{B}} \end{bmatrix} \lambda + F_\Phi^D(0)^{\mathcal{B}} \begin{bmatrix} e_k^T \otimes I_m \\ H^{\mathcal{B}} \end{bmatrix}. \end{aligned}$$

Then $H = \begin{bmatrix} 0_{m\times(k-2)m} & \alpha_{k-1}^{-1} D_k \\ \multicolumn{2}{c}{H'} \end{bmatrix}$ for some $(k-1)m \times (k-1)m$ block symmetric matrix H'. Let $H' = [H'_{ij}]_{i,j=1}^{k-1}$ with $H'_{ij} \in \mathbb{F}^{m\times m}$. If we calculate the first block row and block column of the product $[e_k \otimes I_m \quad H]F_\Phi^D(0)$ we obtain

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{\alpha_0}{\alpha_{k-1}} D_k & -\frac{\beta_0}{\alpha_{k-1}} D_k \\ -\alpha_{k-2} H'_{11} & * & \cdots & * & * & * \\ -\alpha_{k-2} H'_{21} & * & \cdots & * & * & * \\ \vdots & & & & & \\ -\alpha_{k-2} H'_{(k-2)1} & * & \cdots & * & * & * \\ -\frac{\beta_{k-1}}{\alpha_{k-1}} D_k + D_{k-1} - \alpha_{k-2} H'_{(k-1)1} & * & \cdots & * & * & * \end{bmatrix}.$$

Since $[e_k \otimes I_m \quad H]F_\Phi^D(0)$ is block symmetric we obtain

$$H'_{1i} = H'_{i1} = 0 \text{ for } i = 1, \ldots, k-3 \tag{3.20}$$

and

$$-\alpha_{k-2} H'_{(k-2)1} = -\frac{\alpha_0}{\alpha_{k-1}} D_k.$$

Thus,

$$H'_{(k-2)1} = H'_{1(k-2)} = \frac{\alpha_0}{\alpha_{k-1}\alpha_{k-2}} D_k. \tag{3.21}$$

Using (3.20) and (3.21) and calculating the second block row and block column of the product $[e_k \otimes I_m \quad H]F_\Phi^D(0)$ as before, we obtain

$$H'_{2i} = H'_{i2} = 0 \text{ for } i = 1, \ldots, k-4$$

and

$$-\alpha_{k-3}H'_{(k-3)2} = -\alpha_1 H'_{1(k-2)}.$$

Thus,

$$H'_{(k-3)2} = H'_{2(k-3)} = \frac{\alpha_0\alpha_1}{\alpha_{k-1}\alpha_{k-2}\alpha_{k-3}}D_k.$$

In general, an induction argument proves that

$$H'_{(k-j)i} = H'_{i(k-j)} = \frac{\alpha_0\alpha_1\cdots\alpha_{i-1}}{\alpha_{k-1}\alpha_{k-2}\cdots\alpha_{k-j}}D_k \text{ for } j-i=1,$$

and the matrix $[e_k \otimes I_m \quad H]$ has the following block anti-triangular form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \alpha_{k-1}^{-1}D_k \\ 0 & 0 & \cdots & 0 & 0 & \frac{\alpha_0}{\alpha_{k-1}\alpha_{k-2}}D_k & * \\ 0 & 0 & \cdots & 0 & \frac{\alpha_0\alpha_1}{\alpha_{k-1}\alpha_{k-2}\alpha_{k-3}}D_k & * & * \\ \vdots & & & \iddots & & & \\ 0 & 0 & \frac{\alpha_0\alpha_1}{\alpha_{k-1}\alpha_{k-2}\alpha_{k-3}}D_k & * & * & * & * \\ 0 & \frac{\alpha_0}{\alpha_{k-1}\alpha_{k-2}}D_k & * & * & * & * & * \\ I_m & * & * & * & * & * & * \end{bmatrix}.$$

Therefore $[e_k \otimes I_m \quad H]$ is nonsingular if and only if D_k is nonsingular. ■

Corollary 3.6.7. *Let $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ be a polynomial matrix with degree $k \geq 2$ and leading matrix coefficient D_k, and let $L(\lambda) = [e_k \otimes I_m \quad H]F_\Phi^D(\lambda) \in \mathbb{DM}(D)$. Then the following statements hold:*

1. *$L(\lambda)$ is a strong linearization of $D(\lambda)$ if D_k is nonsingular.*
2. *If $D(\lambda)$ is regular, $L(\lambda)$ is a strong linearization of $D(\lambda)$ if and only if D_k is nonsingular.*

Proof. Item 1. follows from Lemma 3.6.6 and Corollary 3.2.5. Item 2. follows from Lemma 3.6.6 and [36, Theorem 3]. ■

Computing the pencil in $\mathbb{DM}(P)$ with ansatz vector e_k, or with any other ansatz vector v, may be difficult. In general, one can follow the procedure in [36, Section 7] or use the MATLAB code in [72, Subsection 7.1]. However, if the recurrence relation (3.2) is simple and k is low, then the computation can be performed easily by hand, as we illustrate in Example 3.6.8.

Example 3.6.8. For a second degree polynomial matrix $D(\lambda) = D_2\phi_2(\lambda)+D_1\phi_1(\lambda)+D_0\phi_0(\lambda)$ expressed in terms of a polynomial basis satisfying (3.2), the pencil $L(\lambda) \in \mathbb{DM}(D)$ with ansatz vector e_2 is

$$L(\lambda) = \begin{bmatrix} -\frac{\alpha_0}{\alpha_1}D_2 & \frac{\lambda-\beta_0}{\alpha_1}D_2 \\ \frac{\lambda-\beta_0}{\alpha_1}D_2 & \left(\frac{\beta_0-\beta_1}{\alpha_0\alpha_1}(\lambda-\beta_0)-\frac{\gamma_1}{\alpha_1}\right)D_2+\frac{\lambda-\beta_0}{\alpha_0}D_1+D_0 \end{bmatrix}.$$

This can be obtained, for instance, by computing the matrix H' as in the proof of Lemma 3.6.6. For example, Chebyshev polynomials of the first kind $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ satisfy the following three-term recurrence relation:

$$\frac{1}{2}\phi_{j+1}(\lambda) = \lambda\phi_j(\lambda) - \frac{1}{2}\phi_{j-1}(\lambda) \quad j \geq 1 \tag{3.22}$$

where $\phi_{-1}(\lambda) = 0$, $\phi_0(\lambda) = 1$ and $\phi_1(\lambda) = \lambda$. Therefore, $\alpha_0 = 1$, $\alpha_j = \gamma_j = \frac{1}{2}$ for $j \geq 1$, $\beta_j = 0$ $j \geq 0$ and

$$L(\lambda) = \begin{bmatrix} -2D_2 & 2\lambda D_2 \\ 2\lambda D_2 & \lambda D_1 + D_0 - D_2 \end{bmatrix}.$$

Chebyshev polynomials of the second kind satisfy the same recurrence relation with $\phi_1(\lambda) = 2\lambda$. Thus, $\alpha_j = \gamma_j = \frac{1}{2}$, $\beta_j = 0$ for $j \geq 0$ and

$$L(\lambda) = \begin{bmatrix} -D_2 & 2\lambda D_2 \\ 2\lambda D_2 & 2\lambda D_1 + D_0 - D_2 \end{bmatrix}.$$

For a cubic polynomial matrix $D(\lambda) = D_3\phi_3(\lambda) + D_2\phi_2(\lambda) + D_1\phi_1(\lambda) + D_0\phi_0(\lambda)$ expressed in terms of Chebyshev polynomials of the first kind, the pencil $L(\lambda) \in \mathbb{DM}(D)$ with ansatz vector e_3 is

$$L(\lambda) = \begin{bmatrix} 0 & -2D_3 & 2\lambda D_3 \\ -2D_3 & 4\lambda D_3 - 2D_2 & 2\lambda D_2 - 2D_3 \\ 2\lambda D_3 & 2\lambda D_2 - 2D_3 & \lambda(D_1 + D_3) + D_0 - D_2 \end{bmatrix}.$$

If $D(\lambda)$ is expressed in terms of Chebyshev polynomials of the second kind we obtain

$$L(\lambda) = \begin{bmatrix} 0 & -D_3 & 2\lambda D_3 \\ -D_3 & 2\lambda D_3 - D_2 & 2\lambda D_2 - D_3 \\ 2\lambda D_3 & 2\lambda D_2 - D_3 & 2\lambda D_1 + D_0 - D_2 \end{bmatrix}.$$

By using Theorems 3.2.9 or 3.3.4, Theorem 3.6.5, Lemma 3.6.6 and Proposition 3.5.5, we obtain in Theorem 3.6.9 symmetric strong linearizations of a symmetric rational matrix when the leading coefficient of its polynomial part is nonsingular as we announced.

Theorem 3.6.9. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric rational matrix and let $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and its strictly proper part $R_{sp}(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$. Assume that* $\deg(D(\lambda)) = k \geq 2$ *and let $n = \nu(R(\lambda))$. Consider a symmetric minimal state-space realization of $R_{sp}(\lambda)$, i.e., $R_{sp}(\lambda) = W(\lambda S_1 - S_2)^{-1}W^T$ as in Definition 3.5.2, and $L(\lambda) \in \mathbb{DM}(D)$ with ansatz vector e_k. If the leading matrix coefficient D_k of $D(\lambda)$ is nonsingular then, for any nonsigular matrix $Z \in \mathbb{F}^{n\times n}$, the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} Z(S_2 - \lambda S_1)Z^T & 0_{n\times(k-1)m} & ZW^T \\ \hline 0_{(k-1)m\times n} & \multicolumn{2}{c}{\multirow{2}{*}{$L(\lambda)$}} \\ WZ^T & & \end{array}\right] \tag{3.23}$$

is a symmetric strong linearization of $R(\lambda)$.

Proof. Let $L(\lambda) = [e_k \otimes I_m \quad H]F^D_\Phi(\lambda)$ be the pencil in $\mathbb{DM}(D)$ with ansatz vector e_k. Since D_k is nonsingular, the matrix $[e_k \otimes I_m \quad H]$ is also nonsingular by using Lemma 3.6.6. Notice that if $R_{sp}(\lambda) = W(\lambda S_1 - S_2)^{-1}W^T$ is a symmetric minimal state-space realization of $R_{sp}(\lambda)$ then $R_{sp}(\lambda) = W(\lambda I_n - S_1^{-1}S_2)^{-1}S_1^{-1}W^T$ is a minimal state-space realization. It only remains to consider Theorem 3.2.9 with $X = ZS_1$ and $Y = -Z^T$. Equivalently, we can consider Theorem 3.3.4 with $X = ZS_1$ and $Y = -Z^T$. ■

Example 3.6.10. Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a symmetric rational matrix and write $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ as the sum of its polynomial part and its strictly proper part. Suppose that

$$D(\lambda) = D_k\lambda^k + D_{k-1}\lambda^{k-1} + \cdots + D_1\lambda + D_0,$$

with $k \geq 2$ and D_k nonsingular, and write $R_{sp}(\lambda) = W(\lambda S_1 - S_2)^{-1}W^T$ as a symmetric minimal state-space realization. For the monomial basis we obtain by [53, Theorem 3.5] that the pencil $L(\lambda) \in \mathbb{DL}(D)$ with ansatz vector e_k is

$$L(\lambda) = \lambda \begin{bmatrix} & & & D_k \\ & & \iddots & D_{k-1} \\ & \iddots & \iddots & \vdots \\ \iddots & \iddots & & D_2 \\ D_k & D_{k-1} & \cdots & D_2 & D_1 \end{bmatrix} - \begin{bmatrix} & & D_k & \\ & \iddots & D_{k-1} & \\ \iddots & \iddots & \vdots & \\ D_k & D_{k-1} & \cdots & D_2 & \\ & & & & -D_0 \end{bmatrix}.$$

Then, by Theorem 3.6.9 with $Z = I_n$, the linear polynomial matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} S_2 - \lambda S_1 & 0_{n\times(k-1)m} & W^T \\ \hline 0_{(k-1)m\times n} & \multicolumn{2}{c}{\multirow{2}{*}{$L(\lambda)$}} \\ W & & \end{array}\right]$$

is a symmetric strong linearization of $R(\lambda)$.

We can obtain infinitely many symmetric strong linearizations by using Theorem 3.6.9 and Lemma 3.1.2.

Corollary 3.6.11. *Under the same assumptions as in Theorem 3.6.9, consider the symmetric strong linearization* $\mathcal{L}(\lambda)$ *in* (3.23). *Let* $Q \in \mathbb{F}^{n\times n}$, $P \in \mathbb{F}^{km\times km}$ *be nonsingular matrices and* $R \in \mathbb{F}^{km\times n}$. *Then*

$$\widehat{\mathcal{L}}(\lambda) = \begin{bmatrix} Q & 0 \\ R & P \end{bmatrix} \mathcal{L}(\lambda) \begin{bmatrix} Q^T & R^T \\ 0 & P^T \end{bmatrix}$$

is a symmetric strong linearization of $R(\lambda)$.

3.7 Hermitian strong linearizations for Hermitian rational matrices

In this section we extend the results in Sections 3.5 and 3.6 from symmetric to Hermitian rational matrices. Since most of the arguments are similar to those in the symmetric case, we limit ourselves to state the main results, and most of the proofs are omitted. We consider the ring of polynomials $\mathbb{C}[\lambda]$ and a polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ that satisfies the three-term recurrence relation:

$$\alpha_j\phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j\phi_{j-1}(\lambda) \quad j \geq 0$$

as in (3.2), with $\alpha_j, \beta_j, \gamma_j \in \mathbb{R}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$. Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ be a polynomial matrix of degree k written in terms of this basis, i.e., $P(\lambda) = \sum_{i=0}^{k} P_i\phi_i(\lambda)$ with $P_i \in \mathbb{C}^{m\times m}$. Suppose that $P(\lambda)$ is *Hermitian*, i.e., $P(\lambda)^* = P(\overline{\lambda})$ or, equivalently, $P^*(\lambda) = P(\lambda)$, where $P^*(\lambda)$ is defined as $P^*(\lambda) = \sum_{i=0}^{k} P_i^*\phi_i(\lambda)$ with P_i^* the conjugate transpose of $P_i \in \mathbb{C}^{m\times m}$. We also consider the set of pencils

$$\mathbb{H}(P) = \{\lambda X + Y \in \mathbb{M}_1(P) : X^* = X, Y^* = Y\}.$$

That is, $\mathbb{H}(P)$ is the set of pencils in $\mathbb{M}_1(P)$ that are Hermitian. Theorem 3.7.1 shows that the elements of $\mathbb{H}(P)$ are in $\mathbb{DM}(P)$, and that, in fact, they are the pencils in $\mathbb{DM}(P)$ with real ansatz vector. The proof of Theorem 3.7.1 is ommitted for brevity since it is similar to the proof of [53, Theorem 6.1], which is Theorem 3.7.1 in the particular case $\phi_j(\lambda) = \lambda^j$ for $j \geq 0$.

Theorem 3.7.1. *Let* $P(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ *be a Hermitian polynomial matrix. Then* $\mathbb{H}(P)$ *is the subset of all pencils in* $\mathbb{DM}(P)$ *with real ansatz vector.*

Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ be a Hermitian rational matrix, i.e., a rational matrix satisfying $R(\lambda)^* = R(\overline{\lambda})$. Consider $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ as in (2.1). Then $D(\lambda)$ and $R_{sp}(\lambda)$ are also Hermitian. For Hermitian strictly proper rational matrices we introduce the notion of Hermitian minimal state-space realizations, in the spirit of Definition 3.5.2.

Definition 3.7.2. *Let $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ be a Hermitian strictly proper rational matrix and let $n = \nu(R_{sp}(\lambda))$. A Hermitian minimal state-space realization of $R_{sp}(\lambda)$ is an expression of the form*

$$R_{sp}(\lambda) = W(\lambda H_1 - H_2)^{-1}W^*$$

where $H_1, H_2 \in \mathbb{C}^{n\times n}$ are Hermitian matrices, with H_1 nonsingular, and $W \in \mathbb{C}^{m\times n}$.

Following arguments similar to those in Lemma 3.5.3 and Proposition 3.5.5, it is easy to see that the strictly proper part of a Hermitian rational matrix has a Hermitian minimal state-space realization.

Proposition 3.7.3. *Any Hermitian strictly proper rational matrix has a Hermitian minimal state-space realization.*

Proof. In order to obtain a Hermitian minimal state-space realization of $R_{sp}(\lambda)$, we can consider a minimal state-space realization $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$. We prove analogously to Lemma 3.5.3 that there exists a unique nonsingular and Hermitian matrix $H \in \mathbb{C}^{n\times n}$ such that $A^* = H^{-1}AH$ and $C^* = H^{-1}B$. Therefore, $R_{sp}(\lambda) = C(\lambda H^{-1} - H^{-1}A)^{-1}C^*$ is a Hermitian minimal state-space realization of $R_{sp}(\lambda)$. ■

Remark 3.7.4. Another constructive way to prove Proposition 3.7.3 is to consider the Hankel matrix H_n of $R_{sp}(\lambda)$ defined in (3.15), that is also Hermitian, and write

$$H_n = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*$$

with U *unitary*, i.e., $U^{-1} = U^*$, and K a diagonal matrix that has the eigenvalues different from zero of H_n at the diagonal elements. Then proceed as in the last paragraph of Remark 3.5.6 to get a Hermitian minimal state-space realization. Notice that K is Hermitian because the eigenvalues of H_n are real. ■

By using Proposition 3.7.3 and Theorem 3.7.1, we obtain in Theorem 3.7.5 Hermitian strong linearizations of a Hermitian rational matrix when the leading coefficient of its polynomial part is nonsingular, analogously as we did in Theorem 3.6.9 for the symmetric case.

Theorem 3.7.5. *Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ be a Hermitian rational matrix and let $R(\lambda) = D(\lambda) + R_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ and its strictly proper part $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$. Assume that* $\deg(D(\lambda)) =$

$k \geq 2$ and let $n = \nu(R(\lambda))$. Consider a Hermitian minimal state-space realization of $R_{sp}(\lambda)$, i.e., $R_{sp}(\lambda) = W(\lambda H_1 - H_2)^{-1}W^*$ as in Definition 3.7.2, and $L(\lambda) \in \mathbb{DM}(D)$ with ansatz vector e_k. If the leading matrix coefficient D_k of $D(\lambda)$ is nonsingular then, for any nonsigular matrix $Z \in \mathbb{C}^{n\times n}$, the linear polynomial matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} Z(H_2 - \lambda H_1)Z^* & 0_{n\times(k-1)m} & ZW^* \\ \hline \begin{array}{c} 0_{(k-1)m\times n} \\ WZ^* \end{array} & \multicolumn{2}{c}{L(\lambda)} \end{array}\right] \tag{3.24}$$

is a Hermitian strong linearization of $R(\lambda)$.

As in Corollary 3.6.11, we can obtain infinitely many Hermitian strong linearizations by using Theorem 3.7.5 and Lemma 3.1.2.

Corollary 3.7.6. *Under the same assumptions as in Theorem 3.7.5, consider the Hermitian strong linearization $\mathcal{L}(\lambda)$ in (3.24). Let $Q \in \mathbb{C}^{n\times n}$, $P \in \mathbb{C}^{km\times km}$ be nonsingular matrices and $R \in \mathbb{C}^{km\times n}$. Then*

$$\widehat{\mathcal{L}}(\lambda) = \begin{bmatrix} Q & 0 \\ R & P \end{bmatrix} \mathcal{L}(\lambda) \begin{bmatrix} Q^* & R^* \\ 0 & P^* \end{bmatrix}$$

is a Hermitian strong linearization of $R(\lambda)$.

3.8 Strong linearizations of rational matrices with polynomial part expressed in other polynomial bases

Polynomial bases $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ satisfying a three-term recurrence relation as in (3.2) are by far the most useful in applications. However, from a theoretical point of view, a natural question is whether or not the results in this chapter can be extended to other polynomial bases. The goal of this section is to show that this can be done by using exactly the same tools that we have used in previous sections, that is, [6, Theorem 5.11], our key Lemma 3.1.2, and the results in [36]. Since the arguments in this section are very similar to the ones previously used, we will simply sketch the main ideas.

Let $D(\lambda)$ be the polynomial part of a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$, with $\deg(D(\lambda)) = k \geq 2$. Let us consider, motivated by (3.4) and its properties, a polynomial basis $\{\psi_j(\lambda)\}_{j=0}^{\infty}$ of $\mathbb{F}[\lambda]$, with $\psi_j(\lambda)$ a polynomial of degree j, that satisfies a linear relation:

$$M_\Psi(\lambda)\Psi_k(\lambda) = 0, \tag{3.25}$$

where $M_\Psi(\lambda) \in \mathbb{F}[\lambda]^{(k-1)\times k}$ is a minimal basis with all its row degrees equal to 1, and $\Psi_k(\lambda) = [\psi_{k-1}(\lambda) \ \cdots \ \psi_1(\lambda) \ \psi_0(\lambda)]^T$ with $\Psi_k(\lambda_0) \neq 0$ for all $\lambda_0 \in \overline{\mathbb{F}}$. Then there exists a vector $w \in \mathbb{F}^k$ such that

$$U(\lambda) = \begin{bmatrix} M_\Psi(\lambda) \otimes I_m \\ w^T \otimes I_m \end{bmatrix} \tag{3.26}$$

is unimodular, and its inverse has the form $U(\lambda)^{-1} = [\widehat{\Psi}_k(\lambda) \quad \Psi_k(\lambda) \otimes I_m]$ with $\widehat{\Psi}_k(\lambda) \in \mathbb{F}[\lambda]^{km\times(k-1)m}$ (see [6, Lemma 5.5]). Let

$$F_\Psi^D(\lambda) = \begin{bmatrix} m_\Psi^D(\lambda) \\ M_\Psi(\lambda) \otimes I_m \end{bmatrix} \in \mathbb{F}[\lambda]^{km\times km} \tag{3.27}$$

be a pencil such that $m_\Psi^D(\lambda)(\Psi_k(\lambda)\otimes I_m) = D(\lambda)$. Then, $F_\Psi^D(\lambda)$ is a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree which verifies $F_\Psi^D(\lambda)(\Psi_k(\lambda)\otimes I_m) = e_1 \otimes D(\lambda)$. Thus, we can apply [6, Theorem 5.11] and Lemma 3.1.2 in order to construct strong linearizations of $R(\lambda)$ from pencils of the form $L(\lambda) = [v \otimes I_m \quad H]F_\Psi^D(\lambda)$ with $v \in \mathbb{F}^k$ and $[v \otimes I_m \quad H]$ nonsingular. Notice that pencils $L(\lambda)$ of this form verify the ansatz relation $L(\lambda)(\Psi_k(\lambda) \otimes I_m) = v \otimes D(\lambda)$. In summary, with those arguments, we obtain the following result that is the generalization of Theorem 3.2.9 for polynomial bases as in (5.19).

Theorem 3.8.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix written as in* (2.1)*, and let $R_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal state-space realization of $R_{sp}(\lambda)$. Assume that* $\deg(D(\lambda)) \geq 2$ *and write $D(\lambda)$ in terms of a polynomial basis $\{\psi_j(\lambda)\}_{j=0}^\infty$ satisfying* (5.19)*, as*

$$D(\lambda) = D_k\psi_k(\lambda) + D_{k-1}\psi_{k-1}(\lambda) + \cdots + D_1\psi_1(\lambda) + D_0\psi_0(\lambda) \tag{3.28}$$

with $D_k \neq 0$. Let $L(\lambda) = [v \otimes I_m \quad H]F_\Psi^D(\lambda)$ with $[v \otimes I_m \quad H]$ nonsingular and $F_\Psi^D(\lambda)$ as in (3.27)*. Let $w \in \mathbb{F}^k$ be the vector in* (3.26)*. Then, for any nonsingular matrices $X, Y \in \mathbb{F}^{n\times n}$ the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & XB(w^T \otimes I_m) \\ \hline -(v \otimes I_m)CY & L(\lambda) \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

In a similar manner, the results in Sections 3.3 and 3.4 can be extended to square rational matrices with polynomial parts expressed in terms of polynomial bases as in (5.19).

In Example 3.8.2 we consider degree-graded polynomial bases presented in [36, Section 9], and we construct strong linearizations of square rational matrices by expressing the polynomial parts in terms of these bases and using Theorem 3.8.1.

Example 3.8.2. Let $\{\psi_j(\lambda)\}_{j=0}^{\infty}$ be a degree-graded polynomial basis of $\mathbb{F}[\lambda]$ that satisfies the following recurrence relation:

$$\psi_j(\lambda) = (\lambda - \alpha_j)\psi_{j-1}(\lambda) + \sum_{i=0}^{j-2} \beta_j^i \psi_i(\lambda) \quad j \geq 1$$

where $\alpha_j \in \mathbb{F}$ for $j \geq 1$, $\beta_j^i \in \mathbb{F}$ for $j \geq 2$, $0 \leq i \leq j-2$ and $\psi_0(\lambda) = 1$. Let $R(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$ be an $m \times m$ rational matrix written as in Theorem 3.8.1. We express the polynomial part $D(\lambda)$ in terms of the polynomial basis $\{\psi_j(\lambda)\}_{j=0}^{\infty}$, as in (3.28). Let us denote $\Psi_k(\lambda) = [\psi_{k-1}(\lambda) \; \cdots \; \psi_1(\lambda) \; \psi_0(\lambda)]^T$ and consider the following pencil $G_\Psi^D(\lambda)$ introduced in [36, Section 9]:

$$G_\Psi^D(\lambda) = \begin{bmatrix} m_\Psi^D(\lambda) \\ M_\Psi(\lambda) \otimes I_m \end{bmatrix} \in \mathbb{F}[\lambda]^{km \times km},$$

where

$$m_\Psi^D(\lambda) = \begin{bmatrix} (\lambda - \alpha_k)D_k + D_{k-1} & \beta_k^{k-2}D_k + D_{k-2} & \cdots & \beta_k^1 D_k + D_1 & \beta_k^0 D_k + D_0 \end{bmatrix},$$

and

$$M_\Psi(\lambda) = \begin{bmatrix} -1 & (\lambda - \alpha_{k-1}) & \beta_{k-1}^{k-3} & \beta_{k-1}^{k-4} & \cdots & \beta_{k-1}^2 & \beta_{k-1}^1 & \beta_{k-1}^0 \\ & -1 & (\lambda - \alpha_{k-2}) & \beta_{k-2}^{k-4} & \cdots & \beta_{k-2}^2 & \beta_{k-2}^1 & \beta_{k-2}^0 \\ & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & & -1 & (\lambda - \alpha_2) & \beta_2^0 \\ & & & & & & -1 & (\lambda - \alpha_1) \end{bmatrix}.$$

The matrix $G_\Psi^D(\lambda)$ verifies that $G_\Psi^D(\lambda)(\Psi_k(\lambda) \otimes I_m) = e_1 \otimes D(\lambda)$. Moreover, $G_\Psi^D(\lambda)$ is a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree. It can be proved, as in [36, Theorem 1], that any pencil $L(\lambda)$ that verifies $L(\lambda)(\Psi_k(\lambda) \otimes I_m) = v \otimes D(\lambda)$ for some vector $v \in \mathbb{F}^k$ can be written as $L(\lambda) = [v \otimes I_m \quad H]G_\Psi^D(\lambda)$ for some matrix $H \in \mathbb{F}^{km \times (k-1)m}$. If we consider a pencil $L(\lambda)$ of this form with $[v \otimes I_m \quad H]$ nonsingular we can obtain strong linearizations for $R(\lambda)$. In particular, we have that conditions in Theorem 3.8.1 hold, and we can apply it with $w = e_k$. Then, we have that for any nonsingular matrices $X, Y \in \mathbb{F}^{n \times n}$ the linear polynomial matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} X(\lambda I_n - A)Y & 0_{n \times (k-1)m} \quad XB \\ \hline -(v \otimes I_m)CY & L(\lambda) \end{array}\right]$$

is a strong linearization of $R(\lambda)$.

Chapter 4

Local linearizations of rational matrices

In this chapter, we present a definition of local linearizations of rational matrices and study their properties. This new theory of local linearizations captures and explains rigorously the properties of all the different pencils that have been constructed from the 1970's in the literature for computing zeros, poles and eigenvalues of rational matrices. The results in this chapter appear in [28] and [29].

Local linearizations are pencils associated to a rational matrix that preserve its structure of zeros and poles in subsets of any algebraically closed field $\mathbb{F}$, in the whole $\mathbb{F}$ and also at infinity. In practice, one is often interested in studying the pole and zero structure of rational matrices not in the whole space $\mathbb{F} \cup \{\infty\}$ but in a particular region (see [46, 47, 60, 35]). For instance, this happens when a REP arises from approximating a NLEP, since the approximation is usually reliable only in a target region. As a consequence, the eigenvalues (those zeros that are not poles) of the approximating REP need to be computed only in that region. In this scenario, one can use local linearizations of the corresponding rational matrix which contain the information about the poles and zeros in the target region, but possibly not in the whole space $\mathbb{F} \cup \{\infty\}$. In general, the pencils in [46, 47, 60, 35] do not satisfy the definitions of linearizations and strong linearizations of rational matrices introduced in [6]. Thereby local linearizations provide extra flexibility in solving NLEPs. We assume throughout this chapter that $\mathbb{F}$ is an algebraically closed field.

4.1 Polynomial system matrices minimal in a set and at infinity

In this section, we extend the concept of minimal polynomial system matrices from the classical scenario to a local notion. Some of the definitions in this section can also be found in [16] expressed in an abstract algebraic language.

4.1.1 Polynomial system matrices minimal in a set

Consider a polynomial system matrix

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{4.1}$$

with transfer function matrix $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$. The next Definition 4.1.1 extends to subsets of $\mathbb{F}$ the classical definition of minimal, or with least order, polynomial system matrices introduced by Rosenbrock in [78].

Definition 4.1.1 (Polynomial system matrix minimal in a subset of $\mathbb{F}$)**.** *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. The polynomial system matrix $P(\lambda)$ in* (4.1)*, with $n > 0$, is said to be minimal in Σ if, for each $\lambda_0 \in \Sigma$, the following condition holds:*

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix} = n. \tag{4.2}$$

Rosenbrock's definition coincides with Definition 4.1.1 when $\Sigma = \mathbb{F}$.

Remark 4.1.2. Notice that $\operatorname{nrank} \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} = \operatorname{nrank} \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} = n$ since $A(\lambda)$ is regular. Thus, the rank condition (4.2) holds if and only if λ_0 is neither an eigenvalue of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ nor of $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$.

Remark 4.1.3. For convenience, if $n = 0$ in (4.1), we adopt the agreement that $P(\lambda)$ is minimal at every point $\lambda_0 \in \mathbb{F}$.

In the next example, we illustrate Definition 4.1.1 with a rational matrix and a polynomial system matrix taken from the reference [35] dealing with numerical algorithms for solving NLEPs via rational approximation. We advance that we will use Example 4.1.4 several times for illustrating different concepts. In this respect, we emphasize that [35] does not mention polynomial system matrices at all.

Example 4.1.4. Let $R(\lambda)$ be a rational matrix of the form

$$R(\lambda) = -B_0 + \lambda A_0 + \frac{B_1}{\lambda - \sigma_1} + \cdots + \frac{B_s}{\lambda - \sigma_s} \in \mathbb{C}(\lambda)^{p \times p}, \tag{4.3}$$

with $A_0, B_0, \ldots, B_s \in \mathbb{C}^{p\times p}$, $\sigma_1, \ldots, \sigma_s \in \mathbb{C}$, and $\sigma_i \neq \sigma_j$ if $i \neq j$. Let us consider the linear polynomial matrix

$$P(\lambda) = \left[\begin{array}{ccc|c} (\lambda - \sigma_1)I & & & I \\ & \ddots & & \vdots \\ & & (\lambda - \sigma_s)I & I \\ \hline -B_1 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right].$$

These matrices are introduced in [35] to tackle a NLEP $T(\lambda)v = 0$, in a certain region $\Omega \subseteq \mathbb{C}$, where the matrix $T(\lambda)$ is of the form $T(\lambda) = -B_0 + \lambda A_0 + f_1(\lambda)A_1 + \cdots + f_q(\lambda)A_q$, with $A_0, A_1, \ldots, A_q \in \mathbb{C}^{p\times p}$ and $f_i : \Omega \subseteq \mathbb{C} \longrightarrow \mathbb{C}$, $i = 1, \ldots, q$, being scalar functions nonlinear in the variable λ and holomorphic in Ω. For solving a NLEP of this form, the nonlinear matrix $T(\lambda)$ is approximated in Ω by a rational matrix $R(\lambda)$ as in (4.3), and $P(\lambda)$ is considered to linearize $R(\lambda)$. It is easy to see that $P(\lambda)$ is, in fact, a linear polynomial system matrix of $R(\lambda)$, by setting the matrix $\operatorname{diag}((\lambda-\sigma_1)I, \ldots, (\lambda-\sigma_s)I)$ as state matrix $A(\lambda)$ in (4.1). Moreover, without any assumption, $P(\lambda)$ is minimal in $\Sigma := \mathbb{C} \setminus \{\sigma_1, \ldots, \sigma_s\}$. In particular, and according to [35], Ω is a subset of Σ. Therefore, $P(\lambda)$ is minimal in the target set Ω. For completeness, notice that a polynomial system matrix as $P(\lambda)$ is minimal in $\mathbb{C}$ if and only if all the matrices $B_1, \ldots, B_s$ are nonsingular.

The next result provides the pole and zero elementary divisors of a rational matrix $R(\lambda)$ in a subset Σ from any polynomial system matrix of $R(\lambda)$ minimal in Σ. This result is the counterpart of [78, Chapter 3, Theorem 4.1] for polynomial system matrices minimal in a particular subset instead of polynomial system matrices of least order.

Theorem 4.1.5. *Let $\Sigma \subseteq \mathbb{F}$ be nonempty. Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix minimal in Σ whose transfer function matrix is $R(\lambda)$. Then the elementary divisors of $A(\lambda)$ in Σ are the pole elementary divisors of $R(\lambda)$ in Σ, and the elementary divisors of $P(\lambda)$ in Σ are the zero elementary divisors of $R(\lambda)$ in Σ.

Proof. We give the proof for a finite point $\lambda_0 \in \Sigma$. Then, the result can be extended to Σ in a natural way. Let us consider the Smith normal form of $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$. Namely, $U(\lambda) \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} V(\lambda) = \begin{bmatrix} S(\lambda) & 0 \end{bmatrix}$ with $U(\lambda)$ and $V(\lambda)$ unimodular matrices. Observe that $S(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular since $\operatorname{nrank} \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} = n$. We set $H_1(\lambda) := S(\lambda)^{-1}U(\lambda)$. Since $P(\lambda)$ is minimal at λ_0, $S(\lambda)$ has no zeros at λ_0. Therefore, $H_1(\lambda)$ is invertible at λ_0. Moreover, $\begin{bmatrix} H_1(\lambda)A(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$ is a polynomial matrix, as it is equal to $\begin{bmatrix} I_n & 0 \end{bmatrix} V(\lambda)^{-1}$, has full row normal rank, and has no zeros in $\mathbb{F}$. Now, let us consider the Smith normal form of the polynomial matrix $\begin{bmatrix} H_1(\lambda)A(\lambda) \\ -C(\lambda) \end{bmatrix}$. Namely, $\widetilde{U}(\lambda) \begin{bmatrix} H_1(\lambda)A(\lambda) \\ -C(\lambda) \end{bmatrix} \widetilde{V}(\lambda) = \begin{bmatrix} \widetilde{S}(\lambda) \\ 0 \end{bmatrix}$ with $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ unimodular matrices. Observe that $\widetilde{S}(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular since $H_1(\lambda)$ is regular and $\operatorname{nrank} \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} = n$. We set $H_2(\lambda) := \widetilde{V}(\lambda)\widetilde{S}(\lambda)^{-1}$. Moreover, the matrix $\begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) \\ -C(\lambda)H_2(\lambda) \end{bmatrix}$ is also polynomial, as it is equal to $\widetilde{U}(\lambda)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, has full column normal rank, and has no zeros in $\mathbb{F}$. Since $P(\lambda)$ is minimal at λ_0 and $H_1(\lambda)$ is invertible at λ_0, $\widetilde{S}(\lambda)$ has no zeros at λ_0. Therefore, $H_2(\lambda)$ is invertible at λ_0. Let us define now the polynomial system matrix

$$\widetilde{P}(\lambda) := \begin{bmatrix} H_1(\lambda) & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} H_2(\lambda) & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) & H_1(\lambda)B(\lambda) \\ -C(\lambda)H_2(\lambda) & D(\lambda) \end{bmatrix}.$$

We claim that $\widetilde{P}(\lambda)$ is a minimal polynomial system matrix in $\mathbb{F}$ or in the classical sense of [78]. For that, it remains to prove that the matrix

$$Z(\lambda) := \begin{bmatrix} H_1(\lambda)A(\lambda)H_2(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$$

has full row rank for all $\lambda \in \mathbb{F}$. Let us suppose that there exists $\lambda_1 \in \mathbb{F}$ such that $\operatorname{rank} Z(\lambda_1) < n$. We know that $\operatorname{rank} \begin{bmatrix} H_1(\lambda_1)A(\lambda_1)\widetilde{V}(\lambda_1) & H_1(\lambda_1)B(\lambda_1) \end{bmatrix} = n$ since the Smith normal form of $\begin{bmatrix} H_1(\lambda)A(\lambda) & H_1(\lambda)B(\lambda) \end{bmatrix}$ is equal to $\begin{bmatrix} I_n & 0 \end{bmatrix}$ and $\widetilde{V}(\lambda)$ is unimodular. On the other hand, we have that

$$\operatorname{rank} \begin{bmatrix} H_1(\lambda_1)A(\lambda_1)\widetilde{V}(\lambda_1) & H_1(\lambda_1)B(\lambda_1) \end{bmatrix} = \operatorname{rank} \left(Z(\lambda_1) \begin{bmatrix} \widetilde{S}(\lambda_1) & 0 \\ 0 & I_m \end{bmatrix} \right)$$

and $\operatorname{rank} \left(Z(\lambda_1) \begin{bmatrix} \widetilde{S}(\lambda_1) & 0 \\ 0 & I_m \end{bmatrix} \right) \leq \operatorname{rank} Z(\lambda_1) < n$, which is a contradiction. Therefore, $\widetilde{P}(\lambda)$ is a minimal polynomial system matrix. Its transfer function matrix is $R(\lambda)$. Then, by [78, Chapter 3, Theorem 4.1], we know that the zero elementary divisors of $R(\lambda)$ are the elementary divisors of $\widetilde{P}(\lambda)$, and that the pole elementary divisors of $R(\lambda)$ are the elementary divisors of $H_1(\lambda)A(\lambda)H_2(\lambda)$. Finally, the result follows by taking into account that $P(\lambda) \sim_{\lambda_0} \widetilde{P}(\lambda)$ and $A(\lambda) \sim_{\lambda_0} H_1(\lambda)A(\lambda)H_2(\lambda)$, since $H_1(\lambda)$ and $H_2(\lambda)$ are both invertible at λ_0. ■

Example 4.1.6. If Theorem 4.1.5 is applied in Example 4.1.4, we obtain that (without any hypothesis) the eigenvalues of $P(\lambda)$ in Σ coincide exactly with the zeros of $R(\lambda)$ in Σ, with exactly the same multiplicities (geometric, algebraic and partial). In addition, all the zeros of $R(\lambda)$ in Σ are, in fact, eigenvalues of $R(\lambda)$ because the only potential poles of $R(\lambda)$ are $\sigma_1, \ldots, \sigma_s$. This result is stronger than Lemma 3.1 and Corollary 3.2 in [35] from two perspectives: [35] deals with determinants and, so, only gives information on algebraic multiplicities, and the requirements in [35] impose the additional hypothesis that A_0 is nonsingular. Note that, under the assumption that all the matrices $B_1, \ldots, B_s$ are nonsingular, $P(\lambda)$ and $A(\lambda)$ allow us to obtain the complete information on finite zeros and poles (including all the multiplicities) of $R(\lambda)$ in $\mathbb{C}$.

4.1.2 Polynomial system matrices minimal at infinity

Theorem 4.1.5 characterizes polynomial system matrices that contain the information of the invariant orders at finite points of their transfer functions. The extension of these results for including the information at infinity is an old problem that has been considered in classical papers as, for instance, in [91, 92]. However, a satisfactory solution has been found, so far, only for polynomial system matrices with state matrix $A(\lambda)$ being a linear polynomial matrix and the other blocks $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ being constant matrices. In other cases, recovering the information at infinity requires to embed the polynomial system matrix into a larger matrix. In this section, we propose a new approach for obtaining a counterpart of Theorem 4.1.5 at infinity.

First, we introduce the notion of g-reversal of a rational matrix in Definition 4.1.7, where g is any integer.

Definition 4.1.7 (g-reversal of a rational matrix)**.** *Let $R(\lambda)$ be a rational matrix, and let g be an integer. We define the g-reversal of $R(\lambda)$ as the rational matrix*

$$\operatorname{rev}_g R(\lambda) := \lambda^g R\left(\frac{1}{\lambda}\right).$$

Consider now $R(\lambda)$ expressed as the sum of its polynomial and strictly proper parts. Namely,

$$R(\lambda) = D(\lambda) + R_{sp}(\lambda)$$

where $D(\lambda)$ is a polynomial matrix and $R_{sp}(\lambda)$ is a strictly proper rational matrix. If $g = \deg(D(\lambda))$, whenever $R(\lambda)$ is not strictly proper, or $g = 0$, if $R(\lambda)$ is strictly proper, then the g-reversal is called the *reversal of* $R(\lambda)$ and is denoted by just $\operatorname{rev} R(\lambda)$ (recall Definition 2.2.1).

Definition 4.1.7 extends the definition of g-reversal for polynomial matrices (see, for instance, [22, Definition 2.12]). However, in the definition of g-reversal of a polynomial matrix considered previously in the literature, g is always taken larger than or equal to the degree of the polynomial matrix, while in Definition 4.1.7 we only ask for g to be an integer.

Given a polynomial system matrix $P(\lambda)$ as in (4.1) of degree d, we have that

$$\operatorname{rev} P(\lambda) = \begin{bmatrix} \operatorname{rev}_d A(\lambda) & \operatorname{rev}_d B(\lambda) \\ -\operatorname{rev}_d C(\lambda) & \operatorname{rev}_d D(\lambda) \end{bmatrix}$$

is also a polynomial matrix. Moreover, $\operatorname{rev}_d A(\lambda)$ is regular since $A(\lambda)$ is regular. Therefore, $\operatorname{rev} P(\lambda)$ is also a polynomial system matrix. By using the notion of reversal, we introduce the concept of minimality at infinity of a polynomial system matrix.

Definition 4.1.8 (Polynomial system matrix minimal at infinity)**.** *The polynomial system matrix $P(\lambda)$ in* (4.1) *is minimal at ∞ if* $\operatorname{rev} P(\lambda)$ *is minimal at* 0.

Example 4.1.9. The polynomial system matrix in Example 4.1.4 is minimal at ∞ since $\text{rev}P(\lambda)$ is, obviously, minimal at 0.

Remark 4.1.10. A polynomial system matrix $P(\lambda)$ as in (4.1), with $\deg(P(\lambda)) = d$ and $n > 0$, is minimal at ∞ if and only if

$$\text{rank} \begin{bmatrix} \text{rev}_d\, A(0) \\ \text{rev}_d\, C(0) \end{bmatrix} = \text{rank} \begin{bmatrix} \text{rev}_d\, A(0) & \text{rev}_d\, B(0) \end{bmatrix} = n.$$

More precisely, let A_d, B_d and C_d be the matrix coefficients of λ^d in $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$, respectively. Then $P(\lambda)$ is minimal at ∞ if and only if

$$\text{rank} \begin{bmatrix} A_d \\ C_d \end{bmatrix} = \text{rank} \begin{bmatrix} A_d & B_d \end{bmatrix} = n.$$

Notice that if $d = 0$ then $P(\lambda)$ is a constant polynomial system matrix, and A_0 must be invertible. Therefore, in this case, the rank condition above is automatically satisfied, and $P(\lambda)$ is minimal at ∞.

Theorem 4.1.11 is essentially the counterpart of Theorem 4.1.5 at infinity. We state it in terms of reversals and their elementary divisors at 0 as we only have defined elementary divisors for finite points. The implications of Theorem 4.1.11 on the structure at infinity are made explicit in Theorem 4.1.13.

Theorem 4.1.11. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of degree d *minimal at* ∞ *whose transfer function matrix is* $R(\lambda)$*. Then the elementary divisors of* $\text{rev}_d\, A(\lambda)$ *at* 0 *are the pole elementary divisors of* $\text{rev}_d\, R(\lambda)$ *at* 0*, and the elementary divisors of* $\text{rev}\, P(\lambda)$ *at* 0 *are the zero elementary divisors of* $\text{rev}_d\, R(\lambda)$ *at* 0.

Proof. It can be easily proved that the transfer function matrix of $\text{rev}\, P(\lambda)$ is $\text{rev}_d\, R(\lambda)$. The theorem then follows by applying Theorem 4.1.5, since $\text{rev}\, P(\lambda)$ is minimal at 0. ∎

Once we have obtained the elementary divisors of the d-reversal of a rational matrix at 0, from one of its polynomial system matrices of degree d minimal at ∞, we can then obtain its invariant orders at infinity as we state in Theorem 4.1.13. For proving that, we use Lemma 4.1.12.

Lemma 4.1.12. Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with $\text{nrank}\, R(\lambda) = r$, and let g be an integer. Let $e_1, \ldots, e_r$ be the invariant orders of $\text{rev}_g\, R(\lambda)$ at 0, and let $q_1, \ldots, q_r$ be the invariant orders at infinity of $R(\lambda)$. Then

$$e_i = q_i + g \quad i = 1, \ldots, r. \tag{4.4}$$

Proof. From the local Smith–McMillan form at infinity of $R(\lambda)$, there exist biproper rational matrices $B_1(\lambda)$ and $B_2(\lambda)$ such that

$$R(\lambda) = B_1(\lambda)\,\mathrm{diag}\left((1/\lambda)^{q_1}, \ldots, (1/\lambda)^{q_r}, 0_{(p-r)\times(m-r)}\right) B_2(\lambda).$$

Let us perform the transformation $\lambda \longmapsto 1/\lambda$ on the variable of the equation above. Thus,

$$R(1/\lambda) = B_1(1/\lambda)\,\mathrm{diag}\left(\lambda^{q_1}, \ldots, \lambda^{q_r}, 0_{(p-r)\times(m-r)}\right) B_2(1/\lambda).$$

By [10, Lemma 6.9], $B_1(1/\lambda)$ and $B_2(1/\lambda)$ are invertible at 0. We now multiply the previous equation by λ^g, and we get that $q_i + g$ for $i = 1, \ldots, r$ are the invariant orders of $\mathrm{rev}_g\, R(\lambda)$ at 0. ■

Theorem 4.1.13. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with* $\mathrm{nrank}\, R(\lambda) = r$ *and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of degree d minimal at ∞ whose transfer function matrix is $R(\lambda)$. Let $e_1 \le \cdots \le e_s$ be the partial multiplicities of $\mathrm{rev}_d\, A(\lambda)$ *at* 0 *and let $\widetilde{e}_1 \le \cdots \le \widetilde{e}_u$ be the partial multiplicities of* $\mathrm{rev}\, P(\lambda)$ *at* 0. *Then the invariant orders at infinity $q_1 \le \cdots \le q_r$ of $R(\lambda)$ are*

$$(q_1, q_2, \ldots, q_r) = (-e_s, -e_{s-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-s-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (d, d, \ldots, d).$$

Proof. By Theorem 4.1.11, we know that e_i and $\widetilde{e}_j$ with $i = 1, \ldots, s$ and $j = 1, \ldots, u$ are the pole and zero partial multiplicities of $\mathrm{rev}_d\, R(\lambda)$ at 0, respectively. Thus, the invariant orders of $\mathrm{rev}_d\, R(\lambda)$ at 0 are

$$-e_s \le -e_{s-1} \le \cdots \le -e_1 < \underbrace{0 = \cdots = 0}_{r-s-u} < \widetilde{e}_1 \le \cdots \le \widetilde{e}_u.$$

Then the use of Lemma 4.1.12 completes the proof. ■

Example 4.1.14. By combining Theorem 4.1.13 and Example 4.1.9, we see that $P(\lambda)$ contains the complete information about the invariant orders at ∞ of $R(\lambda)$ (without imposing any hypothesis). Note that, in this case, $d = 1$ and that the 1-reversal of the state matrix, i.e., $\mathrm{rev}_1\, A(\lambda) = \mathrm{diag}((1 - \lambda\sigma_1)I, \ldots, (1 - \lambda\sigma_s)I)$, has no partial multiplicities at 0. This result on the relationship between the infinite structure of $R(\lambda)$ and the reversal of $P(\lambda)$ is not mentioned in [35].

For polynomial system matrices that are minimal at infinity and, also, at every finite point, we state Definition 4.1.15 about strong minimality. This definition was introduced in [32, Definition 3.3]. However, in [32] the definition is given in terms of eigenvalues instead of minimality at every point, but both definitions are equivalent. We introduce the definition in [32] in Chapter 7.

Definition 4.1.15 (Strongly minimal polynomial system matrix). *The polynomial system matrix $P(\lambda)$ in* (4.1) *is strongly minimal if it is minimal at each point of* $\mathbb{F} \cup \{\infty\}$.

We emphasize that, as a consequence of Theorems 4.1.5 and 4.1.13, strongly minimal polynomial system matrices contain all the information about the invariant orders of their transfer function matrices, both at finite points and at infinity.

4.2 Local linearizations of rational matrices

In this section, we give separately the definitions of linearizations of rational matrices in subsets of $\mathbb{F}$ and at infinity. These linearizations will be useful in order to study the pole and zero structure of rational matrices in different sets containing infinity or not. In particular, and as an application of these definitions, we will study in Chapter 5 the structure of the linearizations that appear in [47] and [60].

4.2.1 Linearizations in a set

In this subsection we introduce the definition of linearization of a rational matrix in a set not containing infinity and study some of its properties.

Definition 4.2.1 (Linearization in a subset of $\mathbb{F}$). *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\Sigma \subseteq \mathbb{F}$ be nonempty. Let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{4.5}$$

be a linear polynomial system matrix with state matrix $A_1\lambda + A_0$ and let

$$\widehat{R}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0) \in \mathbb{F}(\lambda)^{q\times r}$$

be its transfer function matrix. $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Σ if the following conditions hold:

(a) *$\mathcal{L}(\lambda)$ is minimal in Σ, and*

(b) *there exist nonnegative integers s_1, s_2 satisfying $s_1 - s_2 = q - p = r - m$, such that*

$$\operatorname{diag}(R(\lambda), I_{s_1}) \sim_\Sigma \operatorname{diag}(\widehat{R}(\lambda), I_{s_2}). \tag{4.6}$$

Linearizations of rational matrices are polynomial system matrices and their definition includes a specific partition. Thus, a fixed linear polynomial matrix may be partitioned in different ways giving rise to different linearizations of the same or of different rational matrices, or in different subsets. To deal with different partitions, we will use expressions as "$\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Σ with state matrix $A_1\lambda + A_0$" when it is necessary for avoiding any ambiguity. The expression "$\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Σ with empty state matrix" will cover the case $n = 0$ in (4.5), which does not give us pole information since the pole information is contained in the state matrix (see Remark 4.2.7).

Remark 4.2.2. We remark the following extreme cases since they are important in applications and make Definition 4.2.1 very general:

1. $\widehat{R}(\lambda) = R(\lambda)$. That is, the transfer function matrix of $\mathcal{L}(\lambda)$ is the desired rational matrix $R(\lambda)$. Then we just have to check condition (a). It follows that any linear polynomial system matrix $\mathcal{L}(\lambda)$ is a linearization of its transfer function matrix in the sets where $\mathcal{L}(\lambda)$ is minimal.

2. $n = 0$. Then it is not necessary to take into account condition (a) (it is automatically satisfied by the agreement in Remark 4.1.3) and, therefore, we just have to check condition (b) with $\widehat{R}(\lambda) = D_1\lambda + D_0 = \mathcal{L}(\lambda)$. That is,

$$\operatorname{diag}(R(\lambda), I_{s_1}) \sim_\Sigma \operatorname{diag}(\mathcal{L}(\lambda), I_{s_2}).$$

   Notice that if we want a linearization of $R(\lambda)$ in $\Sigma = \mathbb{F}$ we cannot consider the case $n = 0$ unless $R(\lambda)$ is polynomial.

In condition (4.6), one can always take $s_1 = 0$ or $s_2 = 0$, according to $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$, respectively. This is a consequence of the local Smith–McMillan forms of $\operatorname{diag}(R(\lambda), I_{s_1})$ and $\operatorname{diag}(\widehat{R}(\lambda), I_{s_2})$ being equivalent to each other in Σ. In the rest of the results of this subsection, we will consider $s := s_1 \geq 0$ and $s_2 = 0$, since it corresponds to the most interesting situation in applications.

Remark 4.2.3. If we have a linearization of $R(\lambda)$ in a set Σ then, for each point $\mu \in \Sigma$, there exist rational matrices $H_1^\mu(\lambda)$ and $H_2^\mu(\lambda)$ invertible at μ such that $H_1^\mu(\lambda)\operatorname{diag}(R(\lambda), I_s)H_2^\mu(\lambda) = \widehat{R}(\lambda)$. In principle, for different values of $\mu \in \Sigma$, the rational matrices $H_1^\mu(\lambda)$ (respectively, $H_2^\mu(\lambda)$) may be different from each other, that is, $H_1^\mu(\lambda)$ (resp., $H_2^\mu(\lambda)$) depends on μ. However, Proposition 2.1.5 implies that the existence of $H_1^\mu(\lambda)$ and $H_2^\mu(\lambda)$ for each $\mu \in \Sigma$ is equivalent to the existence of two rational matrices $H_1(\lambda)$ and $H_2(\lambda)$ both invertible in Σ (and independent of μ) such that $H_1(\lambda)\operatorname{diag}(R(\lambda), I_s)H_2(\lambda) = \widehat{R}(\lambda)$.

Remark 4.2.4. When $\Sigma = \mathbb{F}$, in Definition 4.2.1, condition (4.6) is satisfied with unimodular equivalence. Therefore, a linearization in $\mathbb{F}$, or at every point of $\mathbb{F}$, is a linearization in the sense of [7, Definition 3.2] and vice versa.

The next result gives the relation between the Smith–McMillan forms at a finite point of the rational matrices $R(\lambda)$ and $\operatorname{diag}(R(\lambda), I_s)$, with $s > 0$. It is motivated by (4.6).

Lemma 4.2.5. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let* $\operatorname{diag}((\lambda-\lambda_0)^{\nu_1}, \dots, (\lambda-\lambda_0)^{\nu_k}, (\lambda-\lambda_0)^{\nu_{k+1}}, \dots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)})$ *be the Smith–McMillan form at* $\lambda_0 \in \mathbb{F}$ *of* $R(\lambda)$*, with* $\nu_i \leq 0$ *for* $i = 1, \dots, k$ *and* $\nu_i > 0$ *for* $i = k+1, \dots, r$*. Then the Smith–McMillan form at* λ_0 *of* $\operatorname{diag}(R(\lambda), I_s)$ *is* $\operatorname{diag}((\lambda-\lambda_0)^{\nu_1}, \dots, (\lambda-\lambda_0)^{\nu_k}, I_s, (\lambda-\lambda_0)^{\nu_{k+1}}, \dots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)})$.

Proof. Let $M(\lambda) := \operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1}, \dots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)}\right)$ be the local Smith–McMillan form of $R(\lambda)$ at λ_0. Then, $R(\lambda) = R_1(\lambda)M(\lambda)R_2(\lambda)$ for some rational matrices $R_1(\lambda)$ and $R_2(\lambda)$ invertible at λ_0. Moreover,

$$\operatorname{diag}(R(\lambda), I_s) = \operatorname{diag}(R_1(\lambda), I_s)\operatorname{diag}(M(\lambda), I_s)\operatorname{diag}(R_2(\lambda), I_s).$$

Therefore, since the matrices $\operatorname{diag}(R_1(\lambda), I_s)$ and $\operatorname{diag}(R_2(\lambda), I_s)$ are invertible at λ_0, the local Smith–McMillan form of $\operatorname{diag}(R(\lambda), I_s)$ at λ_0 is $\operatorname{diag}(M(\lambda), I_s)$ up to a permutation. ∎

Theorem 4.2.6 states the spectral information that one can obtain from local linearizations in the spirit of [7, Theorem 3.10].

Theorem 4.2.6 (Spectral characterization of linearizations in a subset of $\mathbb{F}$)**.** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, $\Sigma \subseteq \mathbb{F}$ *nonempty and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$*, minimal in* Σ*. Then* $\mathcal{L}(\lambda)$ *is a linearization of* $R(\lambda)$ *in* Σ *if and only if the following conditions hold:*

(a) $\operatorname{nrank}\mathcal{L}(\lambda) = \operatorname{nrank}R(\lambda) + n + s$,

(b) *the pole elementary divisors of* $R(\lambda)$ *in* Σ *are the elementary divisors of* $A_1\lambda + A_0$ *in* Σ*, and the zero elementary divisors of* $R(\lambda)$ *in* Σ *are the elementary divisors of* $\mathcal{L}(\lambda)$ *in* Σ*.*

Proof. We give the proof for a point $\lambda_0 \in \Sigma$. Then, the result can be extended to Σ in a natural way. Let $\widehat{R}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. First, assume that $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at λ_0. By (2.9), $\operatorname{nrank}\widehat{R}(\lambda) = \operatorname{nrank}\mathcal{L}(\lambda) - n$. And, by Lemma 4.2.5, $\operatorname{nrank}\widehat{R}(\lambda) = \operatorname{nrank}R(\lambda) + s$. Then, $\operatorname{nrank}\mathcal{L}(\lambda) = \operatorname{nrank}R(\lambda) + n + s$. By Lemma 4.2.5, we also have that $R(\lambda)$ and $\widehat{R}(\lambda)$ have the same pole and zero elementary divisors at λ_0. Then (b) follows from Theorem 4.1.5, since the pole elementary divisors of $\widehat{R}(\lambda)$ at λ_0 are the elementary divisors of $A_1\lambda + A_0$ at λ_0, and the zero elementary divisors of $\widehat{R}(\lambda)$ at λ_0 are the elementary divisors of $\mathcal{L}(\lambda)$ at λ_0. For the converse, suppose that $\operatorname{diag}((\lambda-\lambda_0)^{\nu_1}, \dots, (\lambda-\lambda_0)^{\nu_k}, (\lambda-\lambda_0)^{\nu_{k+1}}, \dots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)})$ is the Smith–McMillan form at λ_0 of $R(\lambda)$, with $\nu_i \leq 0$ for

$i = 1, \dots, k$ and $\nu_i > 0$ for $i = k+1, \dots, r$. From (b) and Theorem 4.1.5, the pole and zero elementary divisors of $R(\lambda)$ and $\widehat{R}(\lambda)$ are the same. Moreover, by (2.9) and (a), $\operatorname{nrank} \widehat{R}(\lambda) = \operatorname{nrank} R(\lambda) + s$. Therefore, the Smith–McMillan form at λ_0 of $\widehat{R}(\lambda)$ must be $\operatorname{diag}\big((\lambda-\lambda_0)^{\nu_1}, \dots, (\lambda-\lambda_0)^{\nu_k}, I_s, (\lambda-\lambda_0)^{\nu_{k+1}}, \dots, (\lambda-\lambda_0)^{\nu_r}, 0_{(p-r)\times(m-r)}\big)$. This is also the Smith–McMillan form at λ_0 of $\operatorname{diag}(R(\lambda), I_s)$, as stated in the previous lemma. Thus, $\operatorname{diag}(R(\lambda), I_s) \sim_{\lambda_0} \widehat{R}(\lambda)$. ■

Remark 4.2.7. Notice that if $n = 0$ in Theorem 4.2.6 then we can not obtain pole information in Σ from the linearization $\mathcal{L}(\lambda)$ since the state matrix is empty.

Example 4.2.8. Consider Example 4.1.4. By combining the discussion in that example with Remark 4.2.2(case 1), we immediately obtain that $P(\lambda)$ is a linearization of $R(\lambda)$ in Σ. With a bit more effort, it is also easy to obtain the following stronger result: $P(\lambda)$ is a linearization of $R(\lambda)$ in $\mathbb{C}\backslash\Pi$ where $\Pi := \{\sigma_i : B_i \text{ is singular for } 1 \leq i \leq s\}$.

4.2.2 Linearizations at infinity and in sets containing infinity

Our definition of linearization of a rational matrix at infinity is based on the notion of g-reversal of a rational matrix introduced in Definition 4.1.7.

Definition 4.2.9 (Linearization at infinity of grade g)**.** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{4.7}$$

be a linear polynomial system matrix with state matrix $A_1\lambda + A_0$ and let

$$\widehat{R}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0) \in \mathbb{F}(\lambda)^{q\times r}$$

be its transfer function matrix. Let g be an integer. $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade g if the following conditions hold:

(a) $\operatorname{rev} \mathcal{L}(\lambda)$ *is minimal at* 0, *and*

(b) *there exist nonnegative integers s_1, s_2, with $s_1 - s_2 = q - p = r - m$, such that*

$$\operatorname{diag}(\operatorname{rev}_g R(\lambda), I_{s_1}) \sim_0 \operatorname{diag}(\operatorname{rev}_\ell \widehat{R}(\lambda), I_{s_2}), \tag{4.8}$$

where $\ell = \deg(\mathcal{L}(\lambda))$.

Observe that Definition 4.2.9 allows, for completeness, the possibility of $\ell = \deg(\mathcal{L}(\lambda))$ being equal to 0. We admit that this case has a very limited interest in applications, since it corresponds to $\mathcal{L}(\lambda)$ and $\operatorname{rev}_\ell \widehat{R}(\lambda) = \widehat{R}(\lambda)$ being constant matrices. However, it includes linearizations at ∞ of rational matrices $R(\lambda)$ such that, for some integer g, $\operatorname{rev}_g R(\lambda)$ has all its invariant orders at 0 equal to zero. Moreover, notice that, in any case, $\operatorname{rev} \mathcal{L}(\lambda)$ is also a linear polynomial system matrix since $\operatorname{rev}_\ell(A_1\lambda + A_0)$ is nonsingular. We then have the following characterization of linearizations at infinity, which follows from Definition 4.2.1 and the fact that $\operatorname{rev}_\ell \widehat{R}(\lambda)$ with $\ell = \deg(\mathcal{L}(\lambda))$ is the transfer function matrix of $\operatorname{rev} \mathcal{L}(\lambda)$.

Proposition 4.2.10. *A linear polynomial system matrix $\mathcal{L}(\lambda)$ as in (4.7) is a linearization of a rational matrix $R(\lambda)$ at ∞ of grade g if and only if $\operatorname{rev} \mathcal{L}(\lambda)$ is a linearization of $\operatorname{rev}_g R(\lambda)$ at 0.*

Conditions (a) and (b) in Definition 4.2.9 can be stated in a different way as we show in Remarks 4.2.11 and 4.2.13, respectively.

Remark 4.2.11. As a particular case of what is discussed in Remark 4.1.10, condition (a) in Definition 4.2.9 is equivalent to

$$\operatorname{rank} \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & B_1 \end{bmatrix} = n, \tag{4.9}$$

if $\mathcal{L}(\lambda)$ is nonconstant, i.e., if $\ell = 1$. If $\mathcal{L}(\lambda)$ is constant, i.e., $\ell = 0$, condition (a) is automatically satisfied since $\mathcal{L}(\lambda)$ is a polynomial system matrix and, therefore, A_0 is invertible. We emphasize that when a nonconstant linear polynomial system matrix $\mathcal{L}(\lambda)$ as in (4.7) satisfies condition (4.9) then $\mathcal{L}(\lambda)$ is a linearization of its transfer function matrix $\widehat{R}(\lambda)$ at ∞ of grade 1. If $\mathcal{L}(\lambda)$ is constant then $\mathcal{L}(\lambda)$ is a linearization of $\widehat{R}(\lambda)$ at ∞ of grade 0.

Example 4.2.12. Consider the matrices in Example 4.1.4. By Remark 4.2.11, the linear polynomial system matrix $P(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade 1.

Remark 4.2.13. By performing the transformation $\lambda \mapsto 1/\lambda$, condition (b) in Definition 4.2.9 is equivalent to $\operatorname{diag}((1/\lambda)^g R(\lambda), I_{s_1}) \sim_\infty \operatorname{diag}((1/\lambda)^\ell \widehat{R}(\lambda), I_{s_2})$.

We state in Theorem 4.2.14 a characterization of linearizations at ∞ analogous to the one in Theorem 4.2.6 for linearizations at finite points. In this characterization, we consider the most usual situation $s_1 := s \geq 0$ and $s_2 = 0$, assuming $q \geq p$ and $r \geq m$. Its proof is omitted since it follows immediately from Theorem 4.2.6 and Proposition 4.2.10.

Theorem 4.2.14 (Spectral characterization of linearizations at infinity)**.** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$, such that $\operatorname{rev} \mathcal{L}(\lambda)$ is minimal at 0. Let $\ell = \deg(\mathcal{L}(\lambda))$. Then $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade g if and only if the following conditions hold:

(a) $\operatorname{nrank} \mathcal{L}(\lambda) = \operatorname{nrank} R(\lambda) + n + s$,

(b) *the pole elementary divisors of* $\operatorname{rev}_g R(\lambda)$ *at* 0 *are the elementary divisors of* $\operatorname{rev}_\ell(A_1\lambda + A_0)$ *at* 0, *and the zero elementary divisors of* $\operatorname{rev}_g R(\lambda)$ *at* 0 *are the elementary divisors of* $\operatorname{rev} \mathcal{L}(\lambda)$ *at* 0.

Next, we study in Proposition 4.2.15 how to recover the invariant orders at infinity of rational matrices from linearizations at infinity of grade g. Its proof is analogous to the one for Theorem 4.1.13. It follows from combining Theorem 4.2.14 and Lemma 4.1.12.

Proposition 4.2.15. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *with* $\operatorname{nrank} R(\lambda) = r$, *and let*

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a linearization at infinity of grade g *of* $R(\lambda)$ *with* $\ell = \deg(\mathcal{L}(\lambda))$. *Let* $e_1 \leq \cdots \leq e_t$ *be the partial multiplicities of* $\operatorname{rev}_\ell(A_1\lambda + A_0)$ *at* 0, *and let* $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ *be the partial multiplicities of* $\operatorname{rev} \mathcal{L}(\lambda)$ *at* 0. *Then the invariant orders at infinity* $q_1 \leq q_2 \leq \cdots \leq q_r$ *of* $R(\lambda)$ *are*

$$(q_1, q_2, \ldots, q_r) = (-e_t, -e_{t-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-t-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (g, g, \ldots, g).$$

A linear polynomial system matrix that satisfies Definition 4.2.1 in $\mathbb{F}$ and Definition 4.2.9, for a certain grade g, allows us to recover the complete information about the poles and zeros of the corresponding rational matrix, finite and at infinity. This is due to Theorem 4.2.6 and Proposition 4.2.15. This important case leads us to introduce the following definition.

Definition 4.2.16 (g-strong linearization)**.** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let* g *be an integer. A linear polynomial system matrix* $\mathcal{L}(\lambda)$ *is said to be a strong linearization of grade* g, *or a* g*-strong linearization, of* $R(\lambda)$ *if* $\mathcal{L}(\lambda)$ *is a linearization of* $R(\lambda)$ *in* $\mathbb{F}$ *and also at* ∞ *of grade* g.

Example 4.2.17. Consider again the matrices in Example 4.1.4. Then the linear polynomial system matrix $P(\lambda)$ is a 1-strong linearization of $R(\lambda)$ if and only if all the matrices $B_1, \ldots, B_s$ are nonsingular.

In Example 4.2.18 we consider a linear polynomial system matrix $\mathcal{L}(\lambda)$ that is a linearization of a rational matrix $R(\lambda)$ in $\mathbb{F} \cup \{\infty\}$. However, it is not a strong linearization in the sense of [7, Definition 3.4]. In particular, the grade of $\mathcal{L}(\lambda)$ as linearization at ∞ is not equal to the degree of the polynomial part of $R(\lambda)$. Actually, the grade is less than the degree of the polynomial part.

Example 4.2.18. Let us consider the rational matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda^2+\lambda-1}{\lambda} & -\frac{1}{\lambda} \\ -1 & -\lambda^2+\lambda-2 \end{bmatrix}.$$

It can be easily proved that

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|cc} \lambda & 0 & 1 & 1 \\ 0 & 1 & 0 & \lambda \\ \hline 1 & 0 & \lambda+1 & 0 \\ \lambda & \lambda & 0 & \lambda-1 \end{array}\right] := \left[\begin{array}{c|c} A_1\lambda + A_0 & B_1\lambda + B_0 \\ \hline -\left(C_1\lambda + C_0\right) & D_1\lambda + D_0 \end{array}\right]$$

is a linear polynomial system matrix of $R(\lambda)$, with state matrix $A_1\lambda+A_0$. Moreover, note that $\mathcal{L}(\lambda)$ is minimal for all $\lambda_0 \in \mathbb{F}$. Therefore, by Remark 4.2.2(case 1), $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in $\mathbb{F}$. By Remark 4.2.11, $\mathcal{L}(\lambda)$ is also a linearization of $R(\lambda)$ at ∞ of grade 1 since

$$\operatorname{rank}\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 & B_1 \end{bmatrix} = 2.$$

Thus, $\mathcal{L}(\lambda)$ is a 1-strong linearization of $R(\lambda)$, according to Definition 4.2.16. However, $\mathcal{L}(\lambda)$ is not a strong linearization according to [7, Definition 3.4] since A_1 is singular. Nevertheless, we can recover easily the invariant orders at ∞ from $\mathcal{L}(\lambda)$ by applying Proposition 4.2.15 with $g = 1$. For this purpose, note that $\operatorname{rev}\mathcal{L}(\lambda)$ does not have elementary divisors at 0, since $\operatorname{rev}\mathcal{L}(\lambda)$ is invertible at 0. Moreover, the only elementary divisor at 0 of $A_1 + A_0\lambda$ is λ. Therefore, the invariant orders at infinity of $R(\lambda)$ are -2 and -1 by Proposition 4.2.15. The invariant orders of $R(\lambda)$ at any finite point can be recovered from $\mathcal{L}(\lambda)$ by using Theorem 4.2.6. It is worthwhile to emphasize that the grade of $\mathcal{L}(\lambda)$ as linearization at ∞ of $R(\lambda)$ is different from the degree of the polynomial part of $R(\lambda)$.

4.3 Block full rank pencils

In this section, we introduce a wide family of pencils that are linearizations with empty state matrix of rational matrices. Thus, they give us information about zeros locally, i.e., in subsets of $\mathbb{F}$ and/or at ∞ under some conditions. These pencils will be called block full rank pencils, since they generalize the block minimal bases pencils introduced in [26, Definition 3.1]. The definition of block full rank pencils is motivated by the fact that most of the linearizations for rational approximations of NLEPs that have been constructed so far are pencils of this type. The structure of block full rank pencils is extended in Section 4.4 to introduce a wide family of pencils that give us information not only about zeros but also about poles of rational matrices locally. They will be called block full rank linearizations.

The key results in this section are Theorems 4.3.5 and 4.3.7, which will be applied in the following section to establish rigorously and very easily properties of the linearizations used in [47] and [60]. Note that, according to Theorem 4.2.6, the results in this section are not useful for studying, or computing, the finite poles of rational matrices because the considered linearizations have empty state matrix. This may be a drawback in certain situations, but we emphasize again that it is not in the development of algorithms for solving large-scale NLEPs via rational approximations [46, 47, 60, 35]. This is due to the fact that, in those cases, the poles of the rational matrix are chosen for constructing the approximation and/or are located outside the corresponding target set.

Definition 4.3.1. (Block full rank pencil) *A block full rank pencil is a linear polynomial matrix over* $\mathbb{F}$ *with the following structure*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \tag{4.10}$$

where $K_1(\lambda)$ *and* $K_2(\lambda)$ *are pencils with full row normal rank.*

Definition 4.3.1 includes the cases when $K_1(\lambda)$ or $K_2(\lambda)$ are empty matrices. That is, when $L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \end{bmatrix}$, $L(\lambda) = \begin{bmatrix} M(\lambda) \\ K_1(\lambda) \end{bmatrix}$ or $L(\lambda) = M(\lambda)$.

We introduce some auxiliary concepts and results before establishing the most important properties of block full rank pencils in Theorems 4.4.1 and 4.3.7. We will say that a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ has *full row rank in* $\Sigma \subseteq \mathbb{F}$ if, for all $\lambda_0 \in \Sigma$, $R(\lambda_0) \in \mathbb{F}^{p\times m}$, i.e., $R(\lambda)$ is defined at λ_0, and $\operatorname{rank} R(\lambda_0) = p$. Observe that this implies that $R(\lambda)$ has no poles in Σ. The following lemma connects rational matrices with full row rank in a set Σ with minimal bases, and establishes other properties that will be used later.

Lemma 4.3.2. *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix with full row normal rank and let* $T(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ *be a minimal basis of the row space of* $R(\lambda)$. *Then the following statements hold:*

(a) *There exists a unique regular rational matrix* $S(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ *such that* $R(\lambda) = S(\lambda)T(\lambda)$.

(b) $R(\lambda)$ *has full row rank in* $\Sigma \subseteq \mathbb{F}$ *if and only if* $S(\lambda)$ *in* (a) *is invertible in* Σ.

(c) $R(\lambda)$ *is a polynomial matrix if and only if* $S(\lambda)$ *in* (a) *is a polynomial matrix.*

(d) *If* $R(\lambda)$ *is a matrix pencil, then* $S(\lambda)$ *in* (a) *and* $T(\lambda)$ *are both matrix pencils.*

Proof. Part (a). Each row of $S(\lambda)$ is uniquely defined because its entries are the unique rational coefficients that allow us to express the corresponding row of $R(\lambda)$ as a unique linear combination of the rows of $T(\lambda)$. Moreover, $S(\lambda)$ must be regular since, otherwise, there would exist a nonzero vector $y(\lambda) \in \mathbb{F}(\lambda)^{p\times 1}$ such that $y(\lambda)^T S(\lambda) = 0$. So, $y(\lambda)^T R(\lambda) = 0$, which contradicts that $\operatorname{rank} R(\lambda) = p$.

Part (b). It is obvious that if $S(\lambda)$ is invertible in Σ, then $R(\lambda)$ has full row rank in Σ, because $T(\lambda)$ is defined in Σ, as $T(\lambda)$ is a polynomial matrix, and $T(\lambda)$ has full row rank in Σ, since $T(\lambda)$ is a minimal basis. The proof of the converse implication starts by proving that if $R(\lambda)$ has full row rank in Σ, then $S(\lambda)$ is defined in Σ. To see this, note that the Smith form of $T(\lambda)$ is $[I_p \;\; 0]$, because $T(\lambda)$ is a minimal basis and, therefore, does not have finite zeros. Thus, there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that $T(\lambda) = U(\lambda)\,[I_p \;\; 0]\,V(\lambda)$, and $R(\lambda)V(\lambda)^{-1} = [S(\lambda)U(\lambda) \;\; 0]$. This shows that $C(\lambda) := S(\lambda)U(\lambda)$ is defined in Σ, because $R(\lambda)$ and $V(\lambda)^{-1}$ are both defined in Σ ($R(\lambda)$ by hypothesis and $V(\lambda)^{-1}$ because is unimodular and so a polynomial matrix). Therefore, $S(\lambda) = C(\lambda)U(\lambda)^{-1}$ is defined in Σ. This implies that we can write $R(\lambda_0) = S(\lambda_0)T(\lambda_0)$ for each $\lambda_0 \in \Sigma$, which in turns implies that $S(\lambda_0)$ is invertible because $R(\lambda_0)$ has full row rank.

Part (c). It follows directly from [38, Main Theorem, part 4].

Part (d). From [38, Main Theorem, part 4], we have that

$$\deg(\mathrm{row}_i\,(R(\lambda))) = \max_{1\le j\le p} (\deg(s_{ij}(\lambda)) + \deg(\mathrm{row}_j\,(T(\lambda)))) \le 1, \qquad \text{for } 1 \le i \le p, \tag{4.11}$$

where $\mathrm{row}_i\,(R(\lambda))$ denotes the ith row of $R(\lambda)$ and the maximum is taken over the nonzero entries $s_{ij}(\lambda)$ of $S(\lambda)$. Since all the rows of $T(\lambda)$ are different from zero, (4.11) implies that $\deg(s_{ij}(\lambda)) \le 1$ for each nonzero entry of $S(\lambda)$. Moreover, each column of $S(\lambda)$ has at least one nonzero entry, because $S(\lambda)$ is regular, which, combined with (4.11), implies that $\deg(\mathrm{row}_j\,(T(\lambda))) \le 1$, for each $j = 1, \dots, p$. ■

The last concepts we need before stating and proving the main Theorem 4.3.5 are those of rational basis and dual rational bases.

Definition 4.3.3 (Rational basis)**.** *A rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ (with $p < m$) is said to be a rational basis if it is a basis of the rational subspace spanned by its rows, i.e., if it has full row normal rank.*

Definition 4.3.4 (Dual rational bases)**.** *Two rational bases $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $H(\lambda) \in \mathbb{F}(\lambda)^{q\times m}$ are said to be dual if $p + q = m$ and $R(\lambda)\,H(\lambda)^T = 0$.*

Theorem 4.3.5. *Let $\Omega \subseteq \mathbb{F}$ be nonempty. Consider a block full rank pencil*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix},$$

as in (4.10)*, and let* $N_1(\lambda)$ *and* $N_2(\lambda)$ *be any rational bases dual to* $K_1(\lambda)$ *and* $K_2(\lambda)$*, respectively. If* $K_i(\lambda)$ *and* $N_i(\lambda)$ *have full row rank in* Ω*, for* $i = 1, 2$*, then* $L(\lambda)$ *is a linearization of the rational matrix*

$$R(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

in Ω *with empty state matrix.*

Proof. In order to simplify the notation, throughout this proof we do not specify the sizes of different identity matrices and all of them are denoted by I. Let $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), \widetilde{N}_1(\lambda)$ and $\widetilde{N}_2(\lambda)$ be minimal bases of the row spaces of $K_1(\lambda)$, $K_2(\lambda)$, $N_1(\lambda)$ and $N_2(\lambda)$, respectively. Then, Lemma 4.3.2 implies that there exist regular rational matrices $S_1(\lambda)$, $S_2(\lambda)$, $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$\begin{aligned} K_i(\lambda) &= S_i(\lambda)\widetilde{K}_i(\lambda), \quad \text{and } S_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \\ N_i(\lambda) &= W_i(\lambda)\widetilde{N}_i(\lambda), \quad \text{and } W_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \end{aligned}$$

Moreover, $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), S_1(\lambda)$ and $S_2(\lambda)$ are all matrix pencils. Then, $L(\lambda)$ can be factorized as follows,

$$L(\lambda) = \begin{bmatrix} I & 0 \\ 0 & S_1(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda) & \widetilde{K}_2(\lambda)^T \\ \widetilde{K}_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_2(\lambda)^T \end{bmatrix}, \tag{4.12}$$

where the first and third factors are invertible in Ω. Note that the factor in the middle is a block minimal bases pencil (see [26, Definition 3.1]) associated with the polynomial matrix $\widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T$, since the regularity of $S_i(\lambda)$ and $W_i(\lambda)$ implies that $\widetilde{K}_i(\lambda)$ and $\widetilde{N}_i(\lambda)$ are dual minimal bases for $i = 1, 2$. Then, there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that

$$\begin{aligned} \begin{bmatrix} M(\lambda) & \widetilde{K}_2(\lambda)^T \\ \widetilde{K}_1(\lambda) & 0 \end{bmatrix} &= U(\lambda) \begin{bmatrix} \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ 0 & I \end{bmatrix} V(\lambda) \\ &= U(\lambda) \begin{bmatrix} W_2(\lambda)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R(\lambda) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_1(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix} V(\lambda), \end{aligned} \tag{4.13}$$

where $U(\lambda)\,\mathrm{diag}(W_2(\lambda)^{-1}, I)$ and $\mathrm{diag}(W_1(\lambda)^{-T}, I)V(\lambda)$ are invertible in Ω. From combining (4.12) and (4.13), we obtain that $L(\lambda) \sim_\Omega \mathrm{diag}(R(\lambda), I)$. ■

Remark 4.3.6. Under the conditions of Theorem 4.3.5, we will say for brevity that "$L(\lambda)$ is a block full rank pencil associated with $R(\lambda)$ in Ω". We emphasize that this "association" is not one-to-one because there are infinitely many rational bases $N_i(\lambda)$ dual to $K_i(\lambda)$. If $K_1(\lambda)$ (resp. $K_2(\lambda)$) is an empty matrix, we can take any rational matrix $N_1(\lambda) \in \mathbb{F}(\lambda)^{s_1 \times s_1}$ (resp. $N_2(\lambda) \in \mathbb{F}(\lambda)^{s_2 \times s_2}$) invertible in Ω, where s_1 (resp. s_2) is the number of colums (resp. rows) of $M(\lambda)$. The standard choices are $N_1(\lambda) = I_{s_1}$ and $N_2(\lambda) = I_{s_2}$.

In the scenario of Theorem 4.3.5, Theorem 4.2.6 guarantees that the elementary divisors of $L(\lambda)$ in Ω coincide exactly with the zero elementary divisors of $R(\lambda)$ in Ω. Moreover, it is clear from the expression $R(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ that $R(\lambda)$ does not have poles in Ω, since the matrices $N_i(\lambda)$ must be defined in Ω but they are not defined at the poles of $R(\lambda)$. Thus, $R(\lambda)$ has only eigenvalues in Ω, and all the information about them, i.e., geometric, algebraic and partial multiplicities, is contained in $L(\lambda)$.

Next, we present sufficient conditions for a block full rank pencil to be a linearization of $R(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$ at ∞ of a certain grade g. In order to avoid cases with limited interest in applications, in Theorem 4.3.7 we assume $\deg(L(\lambda)) = 1$.

Theorem 4.3.7. *Consider a block full rank pencil*

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix},$$

as in (4.10)*, with* $\deg(L(\lambda)) = 1$*, and let* $N_1(\lambda)$ *and* $N_2(\lambda)$ *be rational bases dual to* $K_1(\lambda)$ *and* $K_2(\lambda)$*, respectively. If, for* $i = 1, 2$*,* $\operatorname{rev}_1 K_i(\lambda)$ *has full row rank at 0, and there exists an integer number* t_i *such that* $\operatorname{rev}_{t_i} N_i(\lambda)$ *has full row rank at 0, then* $L(\lambda)$ *is a linearization of the rational matrix*

$$R(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

at ∞ *of grade* $1 + t_1 + t_2$ *with empty state matrix.*

Proof. Note that $\operatorname{rev} L(\lambda) = \begin{bmatrix} \operatorname{rev}_1 M(\lambda) & \operatorname{rev}_1 K_2(\lambda)^T \\ \operatorname{rev}_1 K_1(\lambda) & 0 \end{bmatrix}$ is a block full rank pencil. Moreover, for $i = 1, 2$, $\operatorname{rev}_{t_i} N_i(\lambda)$ has full row normal rank, and $K_i(\lambda)\, N_i(\lambda)^T = 0$ implies $(\operatorname{rev}_1 K_i(\lambda))\,(\operatorname{rev}_{t_i} N_i(\lambda))^T = 0$. Therefore, $\operatorname{rev}_{t_i} N_i(\lambda)$ is a rational basis dual to $\operatorname{rev}_1 K_i(\lambda)$. Then, Theorem 4.3.5 applied to $\operatorname{rev} L(\lambda)$ proves that $\operatorname{rev} L(\lambda)$ is a linearization at 0 of $(\operatorname{rev}_{t_2} N_2(\lambda))\,(\operatorname{rev}_1 M(\lambda))\,(\operatorname{rev}_{t_1} N_1(\lambda)^T) = \operatorname{rev}_{1+t_1+t_2} R(\lambda)$, with empty state matrix, which combined with Proposition 4.2.10 proves the result. ■

As a consequence of Theorems 4.3.5 and 4.3.7, we obtain Corollary 4.3.8. It generalizes the structure of most of the linearizations of rational approximations of NLEPs that appear in the literature in a constructive way. Moreover, it is very useful in order to characterize easily some pencils as linearizations of rational matrices when only the information about the zeros in subsets not containing poles is of interest.

Corollary 4.3.8. *Let*

$$R(\lambda) = (A_0 - \lambda B_0)R_0(\lambda) + (A_1 - \lambda B_1)R_1(\lambda) + \cdots + (A_N - \lambda B_N)R_N(\lambda)$$

be a $p \times m$ *rational matrix written in terms of some matrix pencils* $A_i - \lambda B_i \in \mathbb{F}[\lambda]^{p \times n_i}$ *and rational matrices* $R_i(\lambda) \in \mathbb{F}(\lambda)^{n_i \times m}$*. Define*

$$M(\lambda) := [(A_0 - \lambda B_0) \quad (A_1 - \lambda B_1) \quad \cdots \quad (A_N - \lambda B_N)] \text{ and}$$

$$N_1(\lambda) := \begin{bmatrix} R_0(\lambda)^T & R_1(\lambda)^T & \cdots & R_N(\lambda)^T \end{bmatrix},$$

and assume that $N_1(\lambda)$ *has full row normal rank. Let* $L(\lambda) = \left[\begin{array}{c} M(\lambda) \\ \hdashline K_1(\lambda) \end{array}\right]$ *be a block full rank pencil of degree* 1 *with only one block column and such that* $K_1(\lambda)$ *and* $N_1(\lambda)$ *are dual rational bases. Let* $\Omega \subseteq \mathbb{F}$ *be nonempty. Then the following statements hold:*

(a) *If* $K_1(\lambda)$ *and* $N_1(\lambda)$ *have full row rank in* Ω *then* $L(\lambda)$ *is a linearization of* $R(\lambda)$ *in* Ω *with empty state matrix.*

(b) *If* $\operatorname{rev}_1 K_1(\lambda)$ *has full row rank at* 0, *and there exists an integer* t *such that* $\operatorname{rev}_t N_1(\lambda)$ *has full row rank at* 0, *then* $L(\lambda)$ *is a linearization of* $R(\lambda)$ *at* ∞ *of grade* $1 + t$ *with empty state matrix.*

Remark 4.3.9. We emphasize that in some relevant applications the rational matrices $R_i(\lambda)$ of Corollary 4.3.8 are just of the form $R_i(\lambda) = r_i(\lambda)I_m$, where $r_i(\lambda)$ are scalar rational functions, and/or most of the pencils $A_i - \lambda B_i$ are constant matrices or a linear scalar function times a constant matrix. Moreover, in some other applications a low rank structure is present in $R(\lambda)$, that is, some of the terms in $R(\lambda)$ have a rank much smaller than $\min\{p, m\}$, and the corresponding rational matrices are written in the form $R_i(\lambda) = r_i(\lambda)R_i$, where $R_i \in \mathbb{F}^{n_i \times m}$ is a constant matrix with $n_i \ll m$.

In the next example, we revisit the pencil introduced in Example 4.1.4 from the perspective of the block full rank pencils. This example illustrates how the theory of block full rank pencils may simplify the analysis of the properties of important linearizations of rational matrices when one is not interested in the information about the poles.

Example 4.3.10. Consider Example 4.1.4. We partition $P(\lambda)$ as follows:

$$P(\lambda) = \left[\begin{array}{cccc} (\lambda - \sigma_1)I & & & I \\ & \ddots & & \vdots \\ & & (\lambda - \sigma_s)I & I \\ \hdashline -B_1 & \cdots & -B_s & \lambda A_0 - B_0 \end{array}\right] =: \left[\begin{array}{c} K_1(\lambda) \\ \hdashline M(\lambda) \end{array}\right].$$

Observe that, in the above partition, we are considering a permuted version of the structure of the pencil $L(\lambda)$ in Corollary 4.3.8. Note now that $K_1(\lambda)$ has full row rank in $\mathbb{C}$, and

$$N_1(\lambda) := \begin{bmatrix} \frac{1}{\sigma_1 - \lambda} I & \cdots & \frac{1}{\sigma_s - \lambda} I & I \end{bmatrix}$$

is a rational basis dual to $K_1(\lambda)$ with full row rank in $\Sigma := \mathbb{C} \setminus \{\sigma_1, \ldots, \sigma_s\}$. Then, by Corollary 4.3.8(a), $P(\lambda)$ is a linearization of $R(\lambda)$ in Σ with empty state matrix. Moreover, note that $\operatorname{rev}_1 K_1(\lambda)$ and $\operatorname{rev}_0 N_1(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda\sigma_1 - 1} I & \cdots & \frac{\lambda}{\lambda\sigma_s - 1} I & I \end{bmatrix}$ both have full row rank at 0. Thus, by Corollary 4.3.8(b), $P(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade 1 with empty state matrix.

4.4 Block full rank linearizations of rational matrices

Block full rank pencils in the above section are local linearizations of rational matrices that contain information about their zeros. In this section, we introduce in Theorems 4.4.1 and 4.4.10 a wide family of pencils that give us information about both zeros and poles of rational matrices locally, at finite points and/or at infinity. They will be called block full rank linearizations, where we use a name similar to that of block full rank pencils for emphasizing the connection between both concepts.

4.4.1 Block full rank linearizations at finite points

In Theorem 4.4.1, we generalize Theorem 4.3.5 (or [28, Theorem 5.3]) in order to obtain local linearizations that give us not only information about the zeros but also about the poles of the corresponding rational matrix.

Theorem 4.4.1. *Consider a nonconstant linear polynomial system matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -\,C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{4.14}$$

with $n > 0$ and state matrix $A(\lambda)$. Let $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ be a block full rank pencil, and let $N_1(\lambda)$ and $N_2(\lambda)$ be any rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Let Ω be a nonempty subset of $\mathbb{F}$ such that $K_i(\lambda)$ and $N_i(\lambda)$ have full row rank in Ω for $i = 1, 2$. If

$$\operatorname{rank}\begin{bmatrix} A(\lambda_0) \\ -N_2(\lambda_0)C(\lambda_0) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A(\lambda_0) & B(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} = n \tag{4.15}$$

for all $\lambda_0 \in \Omega$ then $\mathcal{L}(\lambda)$ is a linearization of the rational matrix

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T \tag{4.16}$$

in Ω with state matrix $A(\lambda)$.

A pencil of the form (4.14) satisfying the hypotheses in Theorem 4.4.1 is called a *block full rank linearization.* In particular, $\mathcal{L}(\lambda)$ is said to be a *block full rank linearization of $R(\lambda)$ in Ω with state matrix $A(\lambda)$.*

Remark 4.4.2. The extreme case of $n = 0$ in the linear polynomial system matrix (4.14) was studied in Theorem 4.3.5 ([28, Theorem 5.3]). It states that the block full rank pencil $L(\lambda)$ in Theorem 4.4.1 is a linearization of the rational matrix

$$G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

in Ω with empty state matrix. In this case, $L(\lambda)$ is said to be a *block full rank linearization of $G(\lambda)$ in Ω with empty state matrix.*

Proof of Theorem 4.4.1. In order to simplify the notation, throughout this proof we do not specify the sizes of different identity matrices and all of them are denoted by I. Let $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), \widetilde{N}_1(\lambda)$ and $\widetilde{N}_2(\lambda)$ be minimal bases of the row spaces of $K_1(\lambda)$, $K_2(\lambda)$, $N_1(\lambda)$ and $N_2(\lambda)$, respectively. Then, by Lemma 4.3.2, there exist rational matrices $S_1(\lambda)$, $S_2(\lambda)$, $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$\begin{aligned} K_i(\lambda) &= S_i(\lambda)\widetilde{K}_i(\lambda), \quad \text{and } S_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2, \\ N_i(\lambda) &= W_i(\lambda)\widetilde{N}_i(\lambda), \quad \text{and } W_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \end{aligned}$$

Moreover, $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), S_1(\lambda)$ and $S_2(\lambda)$ are all matrix pencils. We consider the linear polynomial system matrix

$$\widetilde{\mathcal{L}}(\lambda) := \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -\,C(\lambda) & M(\lambda) & \widetilde{K}_2(\lambda)^T \\ 0 & \widetilde{K}_1(\lambda) & 0 \end{array}\right], \tag{4.17}$$

which is equivalent in Ω to $\mathcal{L}(\lambda)$, since $\begin{bmatrix} I & 0 \\ 0 & S_1(\lambda) \end{bmatrix} \widetilde{\mathcal{L}}(\lambda) \begin{bmatrix} I & 0 \\ 0 & S_2(\lambda)^T \end{bmatrix} = \mathcal{L}(\lambda)$. For $i = 1, 2$, there exist unimodular matrices

$$U_i(\lambda) = \begin{bmatrix} \widetilde{K}_i(\lambda) \\ \widehat{K}_i(\lambda) \end{bmatrix}, \text{ and } \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & \widetilde{N}_i(\lambda)^T \end{bmatrix}$$

as in [26, Theorem 2.10]. Consider now the unimodular matrices

$$V_1(\lambda) := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \widehat{N}_1(\lambda)^T & \widetilde{N}_1(\lambda)^T & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & -X(\lambda) & 0 & I \end{bmatrix}, \text{ and}$$

$$V_2(\lambda) := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & -Y(\lambda) \\ 0 & 0 & 0 & I \\ 0 & I & 0 & -Z(\lambda) \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widehat{N}_2(\lambda) & 0 \\ 0 & \widetilde{N}_2(\lambda) & 0 \\ 0 & 0 & I \end{bmatrix},$$

where

$Z(\lambda) := \widehat{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T, X(\lambda) := \widehat{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T, Y(\lambda) := \widetilde{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T.$

We obtain that

$$V_2(\lambda)\widetilde{\mathcal{L}}(\lambda)V_1(\lambda) = \left[\begin{array}{c|ccc} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T & B(\lambda)\widehat{N}_1(\lambda)^T & 0 \\ \hline -\,\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 & 0 \\ 0 & 0 & I & 0 \\ -\widehat{N}_2(\lambda)C(\lambda) & 0 & 0 & I \end{array}\right],$$

which is, in addition, unimodularly equivalent to

$$\left[\begin{array}{c|cc} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ \hline -\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ 0 & 0 & I \end{array}\right] := P(\lambda).$$

By condition (4.15), the polynomial matrix

$$H(\lambda) := \left[\begin{array}{c|c} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T \\ \hline -\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T \end{array}\right]$$

is minimal in Ω, and its transfer function matrix is

$$\widetilde{R}(\lambda) = \widetilde{N}_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]\widetilde{N}_1(\lambda)^T.$$

Moreover, $W_2(\lambda)\widetilde{R}(\lambda)W_1(\lambda)^T = R(\lambda)$, and, thus, $\widetilde{R}(\lambda)$ and $R(\lambda)$ are equivalent in Ω. Therefore, the zero elementary divisors of $H(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω, and the zero elementary divisors of $A(\lambda)$ in Ω are the pole elementary divisors of $R(\lambda)$ in Ω. In addition, $P(\lambda) = \begin{bmatrix} H(\lambda) & 0 \\ 0 & I \end{bmatrix}$ is unimodularly equivalent to $\widetilde{\mathcal{L}}(\lambda)$, which is equivalent in Ω to $\mathcal{L}(\lambda)$. Therefore, the zero elementary divisors of $\mathcal{L}(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω. By Theorem 4.2.6, $\mathcal{L}(\lambda)$ is a linearization in Ω of $R(\lambda)$, since it is immediate to check that the rank condition in Theorem 4.2.6(a) is satisfied. ■

Remark 4.4.3. The linear polynomial system matrix in (4.14) generalizes the structure of the strong block minimal bases linearizations of rational matrices presented in [7, Theorem 5.11] (recall Theorem 2.5.5) from three perspectives: general pencils $B(\lambda)$ and $C(\lambda)$ are allowed, while those in [7] have a very particular structure; $A(\lambda)$ can be any regular pencil, while in [7] its coefficient in λ must be invertible; and $\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ is an arbitrary block full rank pencil (4.10), while in [7] strong block minimal bases pencils (2.12) are considered. We give more details in Subsection 4.4.3.

Remark 4.4.4. Notice that $\mathcal{L}(\lambda)$ being minimal in Ω is a necessary condition, but not sufficient, in order the rank condition (4.15) to be satisfied.

Under the conditions of Theorem 4.4.1, Theorem 4.2.6 guarantees that the elementary divisors of $\mathcal{L}(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω, and that the elementary divisors of $A(\lambda)$ in Ω are the pole elementary divisors of $R(\lambda)$ in Ω.

We would like to emphasize the fact that Theorem 4.4.1 gives very simple conditions to determine if a pencil is a linearization (in a target set) and, in addition, to determine the rational matrices associated with it by considering rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$. Notice that this "association" is not one-to-one because there are infinitely many rational bases $N_1(\lambda)$ and $N_2(\lambda)$ dual to $K_1(\lambda)$ and $K_2(\lambda)$. Thus, Theorem 4.4.1 is important, for instance, if a linearization gets perturbed and one wants to know which rational matrix is associated with the new perturbed pencil. Then, if the perturbation is small enough that the rank conditions in Theorem 4.4.1 are still satisfied and after restoring the zero blocks of the block full rank linearization, we will get a perturbed block full rank linearization and one can then reconstruct the new rational matrix associated with it.

Remark 4.4.5. If in Theorem 4.4.1, $K_1(\lambda)$ (resp. $K_2(\lambda)$) is an empty matrix, we can take the dual rational basis $N_1(\lambda)$ (resp. $N_2(\lambda)$) as any rational matrix invertible in Ω of size the number of colums (resp. rows) of $M(\lambda)$.

Example 4.4.6. Let us see a simple example that illustrates Remark 4.4.5. For instance, for constructing a linearization of the rational matrix

$$R(\lambda) = \frac{\lambda-2}{\lambda+2}\begin{bmatrix} \frac{-\lambda+3}{\lambda^2-1} & \frac{1}{\lambda(\lambda-1)} \end{bmatrix} \text{ in the set } \Omega := \mathbb{F} - \{-1, 0, 1\},$$

we can consider $K_1(\lambda)$ and $K_2(\lambda)$ as empty matrices and dual rational bases $N_1(\lambda)^T :=$ $\operatorname{diag}\left(\frac{1}{\lambda^2-1}, \frac{1}{(\lambda-1)\lambda}\right)$ and $N_2(\lambda) := 1$, both invertible in Ω. Then, by Theorem 4.4.1, the following linear polynomial system matrix

$$\mathcal{L}(\lambda) := \left[\begin{array}{c|cc} \lambda+2 & -\lambda+3 & 1 \\ \hline -\lambda+2 & 0 & 0 \end{array}\right] := \left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline -C(\lambda) & M(\lambda) \end{array}\right]$$

is a linearization of $R(\lambda)$ in Ω with state matrix $\lambda+2$, since

$$\operatorname{rank}\begin{bmatrix} \lambda+2 \\ -\lambda+2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda+2 & \frac{-\lambda+3}{\lambda^2-1} & \frac{1}{\lambda(\lambda-1)} \end{bmatrix} = 1$$

for all $\lambda \in \Omega$. Therefore, we can recover from $\mathcal{L}(\lambda)$ the pole and zero structure of $R(\lambda)$ in Ω. More precisely, -2 is the only zero of the state matrix in Ω and, thus, is the only pole of $R(\lambda)$ in Ω. Moreover, 2 is the unique zero of $\mathcal{L}(\lambda)$ in Ω and, thus, is the unique zero of $R(\lambda)$ in Ω.

Remark 4.4.7. We notice that, although the state matrix $A(\lambda)$ appears in the (1,1) block in (4.14), in practice, it can be any regular submatrix of $\mathcal{L}(\lambda)$. In particular, in some applications [47, 60] we have found pencils with the structure of block full rank linearizations of the form

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|c} M(\lambda) & K_2(\lambda)^T & -C(\lambda) \\ K_1(\lambda) & 0 & 0 \\ \hline B(\lambda) & 0 & A(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(q+n)\times(r+n)}. \tag{4.18}$$

4.4.2 Block full rank linearizations at infinity

We now study the counterpart of Theorem 4.4.1 at infinity. First, we define the notion of degree of a rational matrix. For the scalar case, we define the degree of a rational function $r(\lambda) = \dfrac{n(\lambda)}{d(\lambda)}$ as

$$\deg(r(\lambda)) := \deg(n(\lambda)) - \deg(d(\lambda)). \tag{4.19}$$

Then, for rational matrices we consider the following definition (see, for instance, [90, p.10]).

Definition 4.4.8. *Let $R(\lambda) = [r_{ij}(\lambda)] \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix with entries $r_{ij}(\lambda) = \dfrac{n_{ij}(\lambda)}{d_{ij}(\lambda)}$. The degree of $R(\lambda)$ is then defined as*

$$\deg(R(\lambda)) := \max_{\substack{i=1,\dots,p\\ j=1,\dots,m}} \{\deg(r_{ij}(\lambda))\}.$$

Notice that this notion of degree of a rational matrix generalizes the notion of degree of a polynomial matrix. In what follows, we call the degrees of each row of $R(\lambda)$, the row degrees of $R(\lambda)$.

Lemma 4.4.9. *Let $R(\lambda)$ be a rational matrix. Then there exists an integer t such that all the rows of $\operatorname{rev}_t R(\lambda)$ are defined at 0 and are all different from zero at 0 if and only if all the row degrees of $R(\lambda)$ are equal to t.*

Proof. First, we consider a rational function $r(\lambda) = \dfrac{a(\lambda)}{b(\lambda)}$ such that the numerator $a(\lambda)$ has degree n, and that the denominator $b(\lambda)$ has degree m. We assume that there exists an integer t for which $\operatorname{rev}_t r(0)$ is defined and is different from 0. We can write

$$\operatorname{rev}_t r(\lambda) = \lambda^{t+m-n} \frac{\operatorname{rev}_n a(\lambda)}{\operatorname{rev}_m b(\lambda)} := \lambda^{t+m-n} h(\lambda), \tag{4.20}$$

where 0 is not a pole nor a zero of $h(\lambda)$ since $\operatorname{rev}_n a(0) \neq 0$ and $\operatorname{rev}_m b(0) \neq 0$. That is, $h(\lambda)$ is defined and is different from 0 at 0. Therefore, $t + m - n = 0$, so that $\operatorname{rev}_t r(\lambda)$ is also defined and is different from 0 at 0. Then, $t = n - m = \deg(r(\lambda))$. Now, we assume that we have a rational row vector $v(\lambda) = [r_1(\lambda) \quad \cdots \quad r_s(\lambda)]$ such that, for some integer t, $\operatorname{rev}_t v(0)$ is defined and is different from 0. Then, it must be $t = \max_{i=1,\dots,s} \{\deg(r_i(\lambda))\}$. That is, $t = \deg(v(\lambda))$. Finally, consider a rational matrix $R(\lambda) = [v_1(\lambda)^T \quad \cdots \quad v_q(\lambda)^T]^T$ where $v_i(\lambda)$ are rational row vectors. Assume that there exists an integer t such that all the rows of $\operatorname{rev}_t R(\lambda)$ are defined at 0 and are different from zero at 0. Then, for each row $v_i(\lambda)$, it must hold that $t = \deg(v_i(\lambda))$.

The converse is trivial taking into account equation (4.20) for each entry of $R(\lambda)$. ■

Theorem 4.4.10. *Consider a nonconstant linear polynomial system matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{4.21}$$

with $n > 0$ and state matrix $A(\lambda)$. Let $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ be a block full rank pencil and let $N_1(\lambda)$ (resp. $N_2(\lambda)$) be any rational basis dual to $K_1(\lambda)$ (resp. $K_2(\lambda)$) with its row degrees all equal to an integer t_1 (resp. t_2). If $\operatorname{rev}_1 K_i(\lambda)$ *and* $\operatorname{rev}_{t_i} N_i(\lambda)$ *have full row rank at zero for $i = 1, 2$ and*

$$\operatorname{rank}\begin{bmatrix} \operatorname{rev}_1 A(0) \\ -\operatorname{rev}_{t_2} N_2(0)\operatorname{rev}_1 C(0) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \operatorname{rev}_1 A(0) & \operatorname{rev}_1 B(0)\operatorname{rev}_{t_1} N_1(0)^T \end{bmatrix} = n \tag{4.22}$$

then $\mathcal{L}(\lambda)$ is a linearization of the rational matrix

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$$

at ∞ of grade $1 + t_1 + t_2$ *with state matrix $A(\lambda)$.*

A pencil of the form (4.21) satisfying the hypotheses in Theorem 4.4.10 is called *block full rank linearization at infinity.* In particular, $\mathcal{L}(\lambda)$ is said to be a block full rank linearization of $R(\lambda)$ at ∞ of grade $1 + t_1 + t_2$ with state matrix $A(\lambda)$. In general, a block full rank linearization is said to be *strong* if it is a linearization in $\mathbb{F} \cup \{\infty\}$.

Remark 4.4.11. The extreme case of $n = 0$ in the linear polynomial system matrix (4.21) was studied in Theorem 4.3.7 ([28, Theorem 5.5]). It states that the block full rank pencil $L(\lambda)$ in Theorem 4.4.10 is a linearization of the rational matrix

$$G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

at ∞ of grade $1 + t_1 + t_2$ with empty state matrix. In this case, $L(\lambda)$ is said to be a block full rank linearization of $G(\lambda)$ at ∞ of grade $1 + t_1 + t_2$ with empty state matrix. We notice that, by Lemma 4.4.9, the integers t_1 and t_2 appearing in Theorem 4.3.7 are the row degrees of the dual bases $N_1(\lambda)$ and $N_2(\lambda)$, respectively.

Proof of Theorem 4.4.10. The result follows by applying Theorem 4.4.1 to $\operatorname{rev}_1 \mathcal{L}(\lambda)$ at 0. Let $g := 1 + t_1 + t_2$, then $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is a linearization at 0 of

$$\operatorname{rev}_{t_2} N_2(\lambda)[\operatorname{rev}_1 M(\lambda) + \operatorname{rev}_1 C(\lambda)(\operatorname{rev}_1 A(\lambda))^{-1}\operatorname{rev}_1 B(\lambda)]\operatorname{rev}_{t_1} N_1(\lambda)^T = \operatorname{rev}_g R(\lambda).$$

■

Example 4.4.12. We now consider the rational matrix

$$R(\lambda) := \sum_{k=0}^{2} A_k \frac{\lambda^k}{(\lambda-\epsilon)^2} + I_n \frac{1}{\lambda} \in \mathbb{F}(\lambda)^{n\times n} \text{ and the set } \Omega := \mathbb{F} - \{\epsilon\},$$

for some $\epsilon \in \mathbb{F}$. Then we define the following linear polynomial system matrix

$$\mathcal{L}(\lambda) := \left[\begin{array}{c|ccc|c} -\lambda I_n & 0 & (\lambda-\epsilon)I_n & 0 \\ \hline 0 & A_2 & 0 & -I_n \\ (\lambda-\epsilon)I_n & 0 & \lambda A_1 + A_0 & \lambda I_n \\ \hline 0 & -I_n & \lambda I_n & 0 \end{array}\right] =: \left[\begin{array}{c|c|c} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ \hline 0 & K_1(\lambda) & 0 \end{array}\right],$$

with state matrix $A(\lambda) = \lambda I_n$. We consider the dual rational bases $N_1(\lambda) := N_2(\lambda) := \begin{bmatrix} \frac{\lambda I_n}{\lambda-\epsilon} & \frac{I_n}{\lambda-\epsilon} \end{bmatrix}$, which have row degrees $t_1 = t_2 = 0$. Then $R(\lambda)$ can be written as $R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$. Notice that $\operatorname{rev}_{t_i} N_i(\lambda)$ and $\operatorname{rev}_1 K_i(\lambda)$ have both full row rank at 0, for $i = 1, 2$, and that condition (4.22) is satisfied since $\operatorname{rev}_1 A(0) = I_n$. Thus, by Theorem 4.4.10, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade $1+t_1+t_2 = 1$ with state matrix $A(\lambda)$. In addition, by Theorem 4.4.1, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω, since $N_i(\lambda)$ and $K_i(\lambda)$ have both full row rank in Ω, for $i = 1, 2$, and condition (4.15) is satisfied in Ω. Observe that, if $R(\lambda)$ has symmetric coefficients, $\mathcal{L}(\lambda)$ preserves the symmetry.

Remark 4.4.13. If we want a linearization as in (4.21) to be a linearization at all finite and infinite points we need, besides minimality conditions, the matrices $K_1(\lambda)$ and $K_2(\lambda)$ being minimal bases with all their row degrees equal to 1. Notice that if a pencil $K(\lambda)$ has full row rank in $\mathbb{F}$ and, in addition, $\operatorname{rev}_1 K(\lambda)$ has full row rank at 0 then $K(\lambda)$ is a minimal basis. Conversely, if $K(\lambda)$ is a minimal basis with all its row degrees equal to one then $K(\lambda)$ has full row rank in $\mathbb{F}$ and $\operatorname{rev}_1 K(\lambda)$ has full row rank at 0.

4.4.3 Strong block minimal bases linearizations as block full rank linearizations

By using strong block minimal bases pencils, in [7, Theorem 5.11] (see Theorem 2.5.5) (strong) linearizations are constructed that contain the complete spectral information of rational matrices, finite and infinite, as well as the information about their minimal bases and indices [8], when the corresponding rational matrix $R(\lambda)$ is expressed in the form $R(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$, where $D(\lambda)$ is the polynomial part of $R(\lambda)$ with $\deg(D(\lambda)) > 1$, and $C(\lambda I_n - A)^{-1}B$ is a minimal state-space realization of the strictly proper part of $R(\lambda)$. We will see that such linearizations satisfy Theorem 4.4.1, with $\Omega = \mathbb{F}$, and Theorem 4.4.10.

For the construction, in [7] $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ is considered to be a strong block minimal bases pencil as in (2.12) associated with $D(\lambda)$ with sharp degree, that is, $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$, where $N_1(\lambda)$ (respectively $N_2(\lambda)$) is a minimal basis dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$). Then, constant matrices $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$ and $\widehat{K}_2 \in \mathbb{F}^{p\times(p+\widehat{p})}$ and matrices $\widehat{N}_1(\lambda)^T \in \mathbb{F}[\lambda]^{(m+\widehat{m})\times\widehat{m}}$ and $\widehat{N}_2(\lambda)^T \in \mathbb{F}[\lambda]^{(p+\widehat{p})\times\widehat{p}}$ exist such that, for $i = 1, 2$,

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

are unimodular. Finally, the following linear polynomial matrix is constructed

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right], \tag{4.23}$$

where $X, Y \in \mathbb{F}^{n\times n}$ are any nonsingular constant matrices. With these assumptions, $\mathcal{L}(\lambda)$ is a strong linearization of $R(\lambda)$ [7, Theorem 5.11].

This result follows as a corollary of Theorems 4.4.1 and 4.4.10 as well. First, since $\widehat{K}_i N_i(\lambda)^T = I$, notice that $R(\lambda)$ can be written as in (4.16):

$$N_2(\lambda)[M(\lambda) + \widehat{K}_2^T CYY^{-1}(\lambda I_n - A)^{-1}X^{-1}XB\widehat{K}_1]N_1(\lambda)^T = D(\lambda) + C(\lambda I_n - A)^{-1}B,$$

and, in addition, conditions (4.15), with $\Omega = \mathbb{F}$, and (4.22) are satisfied. More precisely, we have that

$$\operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y \\ -N_2(\lambda)\widehat{K}_2^T CY \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y \\ -CY \end{bmatrix} = n, \text{ and}$$

$$\operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y & XB\widehat{K}_1 N_1(\lambda)^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y & XB \end{bmatrix} = n$$

for all $\lambda \in \mathbb{F}$, since X and Y are nonsingular, and the realization $C(\lambda I_n - A)^{-1}B$ is minimal. Therefore, condition (4.15) is satisfied and, thus, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in $\mathbb{F}$ by Theorem 4.4.1. Moreover, we have that condition (4.22) is satisfied since $\operatorname{rev}_1(\lambda I_n - A)$ evaluated at 0 is just I_n. Then, by Theorem 4.4.10, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at infinity of grade $\deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1 = \deg(D(\lambda))$.

4.4.4 Recovery of eigenvectors

In this section, we show how to recover right and left eigenvectors of a regular rational matrix $R(\lambda)$, associated with the eigenvalues in a set Ω, from those of a block full rank linearization of $R(\lambda)$ in Ω. For that, in Theorem 4.4.15 we study the relation between their right and left nullspaces. We will use the following Lemma 4.4.14.

Lemma 4.4.14. *Let $K(\lambda)$ be a rational basis and let $N(\lambda)$ be any rational basis dual to $K(\lambda)$. Let Ω be a nonempty subset of $\mathbb{F}$ such that $K(\lambda)$ and $N(\lambda)$ have full row rank in Ω. Then there exist rational matrices of the form*

$$V(\lambda) = \begin{bmatrix} K(\lambda) \\ \bar{K}(\lambda) \end{bmatrix} \quad \text{and} \quad V(\lambda)^{-1} = \begin{bmatrix} \bar{N}(\lambda)^T & N(\lambda)^T \end{bmatrix} \tag{4.24}$$

that are invertible in Ω.

Proof. By Lemma 4.3.2, there exist rational matrices $S(\lambda)$ and $W(\lambda)$, both invertible in Ω, such that $K(\lambda) = S(\lambda)\widetilde{K}(\lambda)$ and $N(\lambda) = W(\lambda)\widetilde{N}(\lambda)$, where $\widetilde{K}(\lambda)$ and $\widetilde{N}(\lambda)$ are minimal bases of the row spaces of $K(\lambda)$ and $N(\lambda)$, respectively. Then, by [26, Theorem 2.10], there exist unimodular matrices of the form

$$U(\lambda) = \begin{bmatrix} \widetilde{K}(\lambda) \\ \widehat{K}(\lambda) \end{bmatrix} \quad \text{and} \quad U(\lambda)^{-1} = \begin{bmatrix} \widehat{N}(\lambda)^T & \widetilde{N}(\lambda)^T \end{bmatrix}.$$

Then we consider the rational matrix

$$V(\lambda) := \begin{bmatrix} S(\lambda) & 0 \\ 0 & W(\lambda)^{-T} \end{bmatrix} U(\lambda) = \begin{bmatrix} K(\lambda) \\ \bar{K}(\lambda) \end{bmatrix},$$

with $\bar{K}(\lambda) := W(\lambda)^{-T}\widehat{K}(\lambda)$. We have that $V(\lambda)$ is invertible in Ω since $S(\lambda)$ and $W(\lambda)$ are invertible in Ω and $U(\lambda)$ is unimodular. Finally, we note that $V(\lambda)^{-1} = U(\lambda)^{-1}\begin{bmatrix} S(\lambda)^{-1} & 0 \\ 0 & W(\lambda)^T \end{bmatrix} = \begin{bmatrix} \widehat{N}(\lambda)^T S(\lambda)^{-1} & \widetilde{N}(\lambda)^T W(\lambda)^T \end{bmatrix} = \begin{bmatrix} \bar{N}(\lambda)^T & N(\lambda)^T \end{bmatrix}$, with $\bar{N}(\lambda)^T := \widehat{N}(\lambda)^T S(\lambda)^{-1}$. ∎

Let us now consider the rational bases $K_1(\lambda)$ and $K_2(\lambda)$ of the block full rank linearization in Theorem 4.4.1 and their dual rational bases $N_1(\lambda)$ and $N_2(\lambda)$. It then follows that, since they have full row rank in a nonempty set Ω, there exist rational matrices of the form

$$V_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \bar{K}_i(\lambda) \end{bmatrix} \quad \text{and} \quad V_i(\lambda)^{-1} = \begin{bmatrix} \bar{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix} \tag{4.25}$$

that are invertible in Ω, for $i = 1, 2$, by Lemma 4.4.14. We use these matrices in the following Theorem 4.4.15.

Theorem 4.4.15. *Let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

be a block full rank linearization in a nonempty set Ω *of the rational matrix*

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T,$$

as in Theorem 4.4.1. Consider the rational matrices $\bar{N}_1(\lambda)$ *and* $\bar{N}_2(\lambda)$ *in* (4.25) *and denote* $S(\lambda) := M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$*. Let* $\lambda_0 \in \Omega$ *such that* $\det A(\lambda_0) \neq 0$*. Then, the following statements hold:*

(a) *The linear map*

$$\begin{aligned} E_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(\mathcal{L}(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0)N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)^T S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x \end{aligned}$$

is a bijection between the right nullspaces over $\mathbb{F}$ *of* $R(\lambda_0)$ *and* $\mathcal{L}(\lambda_0)$*.*

(b) *The linear map*

$$\begin{aligned} E_\ell \,:\, \mathcal{N}_\ell(R(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(\mathcal{L}(\lambda_0)) \\ y^T &\longmapsto y^T \begin{bmatrix} N_2(\lambda_0)C(\lambda_0)A(\lambda_0)^{-1} & N_2(\lambda_0) & -N_2(\lambda_0)S(\lambda_0)\bar{N}_1(\lambda_0)^T \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces over $\mathbb{F}$ *of* $R(\lambda_0)$ *and* $\mathcal{L}(\lambda_0)$*.*

Proof. We only prove (a) since (b) is analogous. We will see that E_r is a composition of maps, $E_r = G_r \circ H_r$, where G_r and H_r are linear bijections. For that, we consider the transfer function of $\mathcal{L}(\lambda)$. That is, the rational matrix

$$\widehat{R}(\lambda) = \begin{bmatrix} S(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \in \mathbb{F}(\lambda)^{q \times r}. \tag{4.26}$$

By Lemma 2.4.10, the linear map

$$\begin{aligned} G_r \,:\, \mathcal{N}_r(\widehat{R}(\lambda_0)) &\longrightarrow \mathcal{N}_r(\mathcal{L}(\lambda_0)) \\ \hat{x} &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1} \begin{bmatrix} B(\lambda_0) & 0 \end{bmatrix} \\ I_r \end{bmatrix} \hat{x} \end{aligned}$$

is a bijection between the right nullspaces of $\widehat{R}(\lambda_0)$ and $\mathcal{L}(\lambda_0)$, since $\widehat{R}(\lambda)$ is the transfer function of $\mathcal{L}(\lambda)$ and $\det A(\lambda_0) \neq 0$. Now, we consider the linear map

$$\begin{aligned} H_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(\widehat{R}(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)^T S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x. \end{aligned}$$

To see that H_r is well defined, let us consider the rational matrix $V_2(\lambda)^{-1} = \begin{bmatrix} \bar{N}_2(\lambda)^T & N_2(\lambda)^T \end{bmatrix}$ in (4.25). Then, we have that

$$\begin{bmatrix} V_2(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} N_1(\lambda)^T \\ -\bar{N}_2(\lambda)^T S(\lambda) N_1(\lambda)^T \end{bmatrix} = \begin{bmatrix} 0 \\ R(\lambda) \\ 0 \end{bmatrix}. \qquad (4.27)$$

By (4.27), H_r is well defined since the matrix $\begin{bmatrix} V_2(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix}$ is invertible in Ω, by Lemma 4.4.14. In addition, since $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω, we have that $\widehat{R}(\lambda)$ and $\text{diag}(R(\lambda), I_s)$ are equivalent in Ω for some $s > 0$ (see Definition 4.2.1). Therefore, $\dim \mathcal{N}_r(R(\lambda_0)) = \dim \mathcal{N}_r(\widehat{R}(\lambda_0))$. Thus, to see that H_r is a bijection we only have to prove that H_r is injective. Assume that $\begin{bmatrix} N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)^T S(\lambda_0) N_1(\lambda_0)^T \end{bmatrix} x = 0$ for some $x \in \mathcal{N}_r(R(\lambda_0))$. In particular, $N_1(\lambda_0)^T x = 0$. Since $N_1(\lambda_0)^T$ has full column rank, $x = 0$. Therefore, H_r is a bijection. Finally, note that $E_r = G_r \circ H_r$. ■

Remark 4.4.16. We recall that, since the linear maps E_r and E_ℓ are bijections, one can recover a basis of the right (resp. left) nullspace of $R(\lambda_0)$ from a basis of the right (resp. left) nullspace of $\mathcal{L}(\lambda_0)$, and conversely. For instance, if $\{x_i\}_{i=1}^t$ is a basis of $\mathcal{N}_r(R(\lambda_0))$ then $\left\{ \begin{bmatrix} -A(\lambda_0)^{-1} B(\lambda_0) N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)^T S(\lambda_0) N_1(\lambda_0)^T \end{bmatrix} x_i \right\}_{i=1}^t$ is a basis of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$. And, given a basis $\{y_i\}_{i=1}^t$ of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$, we can recover a basis of $\mathcal{N}_r(R(\lambda_0))$ taking into account that the elements will be of the form $y_i = \begin{bmatrix} -A(\lambda_0)^{-1} B(\lambda_0) N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)^T S(\lambda_0) N_1(\lambda_0)^T \end{bmatrix} x_i$ for some basis $\{x_i\}_{i=1}^t$ of $\mathcal{N}_r(R(\lambda_0))$. Then, to recover $\{x_i\}_{i=1}^t$ we can use that $\bar{K}_1(\lambda_0) N_1(\lambda_0)^T x_i = x_i$. However, in practice, it can usually be recovered from the structure of $N_1(\lambda_0)^T x_i$, without computing $\bar{K}_1(\lambda_0)$.

Chapter 5

Application of the local linearization theory to linearizations of rational approximations of nonlinear eigenvalue problems

In this chapter we study in depth the pencils introduced in the influential references [47] and [60] for linearizing rational matrices obtained from approximating NonLinear Eigenvalue Problems (NLEPs). In particular, the reference [47] presents one of the first systematic approaches for solving large scale NLEPs. The results in this chapter are valid and are stated in any algebraically closed field $\mathbb{F}$. Note, however, that references [47] and [60] consider only the complex field $\mathbb{C}$ and that this restriction is important in the approximation phase of the NLEP. We start by defining a NLEP [46]. Given a non-empty open set $\Omega \subseteq \mathbb{F}$ and an analytic matrix-valued function

$$\begin{array}{rccc} F: & \Omega & \to & \mathbb{F}^{n\times n} \\ & \lambda & \mapsto & F(\lambda), \end{array}$$

the NLEP consists of computing scalars $\lambda_0 \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{F}^n$ (eigenvectors) such that

$$F(\lambda_0)v = 0,$$

under the regularity assumption $\det(F(\lambda)) \not\equiv 0$. Since a direct solution of NLEPs is usually not possible, they are often solved numerically via rational approximation and by solving the corresponding REP with linearizations adapted to the structure of the obtained rational matrix. The results in this chapter appear in [28] and [29].

5.1 NLEIGS pencils

For brevity of exposition, and also for recognizing the key contribution of [47], we will call *NLEIGS pencils* to the pencils introduced in [47]. The main goal of this section is to replace the vague usage of the word "linearization" in [47] by a number of rigorous results on NLEIGS pencils which, combined with the results in Sections 4.2 and 4.3, establish the precise properties enjoyed with respect to eigenvalues (and poles) of the NLEIGS pencils. We remark that NLEIGS pencils were the initial motivation for developing the theory of local linearizations, since they are not linearizations of the corresponding rational matrix according to the definitions in [2, 7]. The approach in [47] consists essentially of three steps:

(1) The matrix defining the NLEP is approximated by a rational matrix $Q_N(\lambda)$ via Hermite's interpolation in a certain compact target set $\Sigma \subset \mathbb{C}$ where the eigenvalues of interest are located.

(2) The obtained rational matrix is linearized by using a certain pencil $L_N(\lambda)$.

(3) A highly structured rational Krylov method is applied to the pencil to compute the eigenvalues of $Q_N(\lambda)$ in Σ.

Since we are interested in rational matrices and their linearizations, all the delicate details about how the rational approximations $Q_N(\lambda)$ are constructed are omitted. Such details can be found in [47]. Moreover, although [47] deals with regular rational matrices $Q_N(\lambda)$, we will not impose such condition initially in our developments.

Reference [47] uses two families of rational matrices, and corresponding pencils, depending on whether or not a certain low rank structure is present in the original NLEP. We will refer to them as the *NLEIGS basic problem* and the *NLEIGS low rank structured problem*, respectively. The NLEIGS pencils corresponding to each of these two cases will be studied from two perspectives giving rise to the four subsections included in this section. These two perspectives are considering NLEIGS pencils as block full rank pencils and, thus, as linearizations with empty state matrices, and considering them as polynomial system matrices with transfer function matrices equivalent to $Q_N(\lambda)$ everywhere except at a point ξ_N. Both perspectives allow us to state in a rigorous sense that NLEIGS pencils are linearizations of $Q_N(\lambda)$, but the one based on block full rank pencils is much simpler, does not require any hypothesis and covers fully the applications of interest in [47]. In contrast, the polynomial system matrix perspective provides more information on $Q_N(\lambda)$ but at the cost of extra hypotheses which are not imposed in [47] and that require considerable effort to check.

The families of rational matrices considered in [47] are defined in terms of the following parameters: a list of nodes $(\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$ in $\mathbb{F}$, a list of nonzero poles $(\xi_1, \xi_2, \ldots, \xi_N)$ in $\mathbb{F} \cup \{\infty\}$, and a list of nonzero scaling parameters $(\beta_0, \beta_1, \ldots, \beta_N)$ in $\mathbb{F}$. It is important to bear in mind that [47] assumes that the poles are all distinct from the nodes. However, we do not assume such property, except in a few results where it will be explicitly stated. With these parameters, the following sequence of rational scalar functions is defined:

$$b_0(\lambda) = \frac{1}{\beta_0}, \quad b_i(\lambda) = \frac{1}{\beta_0} \prod_{k=1}^{i} \frac{\lambda - \sigma_{k-1}}{\beta_k (1 - \lambda/\xi_k)}, \quad i = 1, \ldots, N. \tag{5.1}$$

Let us now define the linear scalar functions

$$g_i(\lambda) := \beta_i \left(1 - \lambda/\xi_i\right), \quad \text{and} \quad h_j(\lambda) := \lambda - \sigma_j, \tag{5.2}$$

for $i = 1, \ldots, N$, and $j = 0, \ldots, N-1$. Then, the rational functions $b_i(\lambda)$ satisfy the simple recursion

$$g_{j+1}(\lambda)\, b_{j+1}(\lambda) = h_j(\lambda)\, b_j(\lambda), \quad j = 0, 1, \ldots, N-1\,,$$

which will be useful in the sequel. Note that the rational functions $b_i(\lambda)$ could not be proper, since for any infinite pole $\xi_i = \infty$ the corresponding factor $1 - \lambda/\xi_i$ is just equal to 1, and, therefore, $b_i(\lambda)$ has a nonconstant polynomial part.

With all this information, we are in the position of introducing the first family of rational matrices considered in [47], whose elements are defined as

$$Q_N(\lambda) = b_0(\lambda)D_0 + b_1(\lambda)D_1 + \cdots + b_N(\lambda)D_N \in \mathbb{F}(\lambda)^{m\times m}, \tag{5.3}$$

where $D_0, \ldots, D_N \in \mathbb{F}^{m\times m}$ are constant matrices. In this section, all the parameters that allow us to define the considered family of rational matrices are arbitrary. However, in [47] these parameters are carefully chosen in such a way that $Q_N(\lambda)$ approximates satisfactorily the matrix defining the NLEP to be solved in a target set $\Sigma \subset \mathbb{F}$. In this scenario, it is important to stress that the poles $(\xi_1, \ldots, \xi_N)$ are always chosen outside Σ [47, p. A2852], which implies that all the zeros of $Q_N(\lambda)$ located in Σ are eigenvalues of $Q_N(\lambda)$. Thus, the REP associated with $Q_N(\lambda)$ is an explicit example of a problem in which the poles are located outside the region of interest and, then, it is not needed to compute them. Note, however, the following subtlety: though it is clear that the finite poles of $Q_N(\lambda)$ are included in the list $(\xi_1, \ldots, \xi_N)$, we can construct examples of matrices as in (5.3) for which some of the finite numbers in $(\xi_1, \ldots, \xi_N)$ are not poles due to some cancellations. Thus, all the finite numbers in $(\xi_1, \ldots, \xi_N)$ are not necessarily finite poles of $Q_N(\lambda)$ and, even more, the partial multiplicities of such poles are not immediately visible from (5.3). Despite these comments, we will call the numbers $(\xi_1, \ldots, \xi_N)$ poles, following the usage in [47].

5.1.0.1 The NLEIGS basic problem from the point of view of block full rank pencils

In order to solve the REP $Q_N(\lambda)\, y = 0$, the authors of [47] solve the generalized eigenvalue problem corresponding to the pencil

$$L_N(\lambda) = \begin{bmatrix} M_N(\lambda) \\ K_N(\lambda) \end{bmatrix}, \tag{5.4}$$

where

$$M_N(\lambda) := \begin{bmatrix} \frac{g_N(\lambda)}{\beta_N} D_0 & \frac{g_N(\lambda)}{\beta_N} D_1 & \cdots & \frac{g_N(\lambda)}{\beta_N} D_{N-2} & \frac{g_N(\lambda)}{\beta_N} D_{N-1} + \frac{h_{N-1}(\lambda)}{\beta_N} D_N \end{bmatrix},$$

$$K_N(\lambda) := \begin{bmatrix} -h_0(\lambda) & g_1(\lambda) & & & \\ & -h_1(\lambda) & g_2(\lambda) & & \\ & & \ddots & \ddots & \\ & & & -h_{N-2}(\lambda) & g_{N-1}(\lambda) \end{bmatrix} \otimes I_m.$$

In [47] the use of $L_N(\lambda)$ for solving the REP associated to $Q_N(\lambda)$ is supported by [47, Theorem 3.2], which states that $L_N(\lambda)$ is a strong linearization of the rational matrix $Q_N(\lambda)$ without specifying the exact meaning of "strong linearization" in this rational context. Moreover, the proof of [47, Theorem 3.2] consists of a reference to [5, Theorem 3.1], which is a paper dealing with strong linearizations of *polynomial* matrices in the classical sense of [44]. However, as a consequence of the results in Section 4.3, it is very easy to prove that $L_N(\lambda)$ is always a linearization of $Q_N(\lambda)$ in a set including the region of interest in [47], as well as at infinity. This is proved in Theorem 5.1.1, where the nomenclature introduced in Remark 4.3.6 is used.

Theorem 5.1.1. *Let $Q_N(\lambda)$ be the rational matrix in* (5.3) *and $L_N(\lambda)$ be the pencil in* (5.4)*. Let $\mathcal{P}_N$ and i_N be, respectively, the set of finite poles and the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_N)$. Then, the following statements hold:*

(a) *$L_N(\lambda)$ partitioned as in* (5.4) *is a block full rank pencil with only one block column associated with $Q_N(\lambda)$ in $\mathbb{F}\setminus\mathcal{P}_N$ and $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ in $\mathbb{F}\setminus\mathcal{P}_N$ with empty state matrix.*

(b) *$L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ at ∞ of grade i_N with empty state matrix.*

Proof. It is immediate to check that

$$N_N(\lambda) := \frac{1}{1 - \frac{\lambda}{\xi_N}} \begin{bmatrix} b_0(\lambda) & b_1(\lambda) & \cdots & b_{N-1}(\lambda) \end{bmatrix} \otimes I_m \tag{5.5}$$

is a rational basis dual to $K_N(\lambda)$. Note also that $K_N(\lambda)$ and $N_N(\lambda)$ have both full row rank in $\mathbb{F}\setminus\mathcal{P}_N$. In addition, an easy direct computation proves $M_N(\lambda)N_N(\lambda)^T =$

$Q_N(\lambda)$. Thus, part (a) follows from Theorem 4.4.1. Observe that (a) can also be proved from Corollary 4.3.8, since the structures of $Q_N(\lambda)$, $L_N(\lambda)$ and $N_N(\lambda)$ are particular cases of those described in that corollary.

In order to prove part (b), note first that $\operatorname{rev}_1 K_N(\lambda)$ has full row rank at 0. We now consider the rational matrix $\operatorname{rev}_{i_N-1} N_N(\lambda) = \lambda^{i_N-1} N_N\left(\frac{1}{\lambda}\right)$, which is of the form $\lambda^{i_N-1} N_N\left(\frac{1}{\lambda}\right) = \begin{bmatrix} * & \cdots & * & \frac{\lambda}{\lambda-1/\xi_N}\lambda^{i_N-1} b_{N-1}\left(\frac{1}{\lambda}\right) I_m \end{bmatrix}$, where the entries $*$ are defined at 0. Denote by i_{N-1} the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_{N-1})$. Then, $b_{N-1}\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^{i_{N-1}}} c(\lambda)$, for a certain rational function $c(\lambda)$ with $c(0) \neq 0$. Thus, we obtain that $\operatorname{rev}_{i_N-1} N_N(\lambda)$ has full row rank at 0, taking into account that $i_{N-1} = i_N$ if $\xi_N \neq \infty$, and $i_{N-1} = i_N - 1$ if $\xi_N = \infty$. Then, part (b) follows from Theorem 4.3.7. ■

Combining Theorems 5.1.1 and 4.2.6, we get that $L_N(\lambda)$ contains all the information about the finite eigenvalues of $Q_N(\lambda)$ in $\mathbb{F} \setminus \mathcal{P}_N$, including all type of multiplicities (algebraic, geometric and partial). Moreover, Proposition 4.2.15 allows us to recover the complete pole-zero structure of $Q_N(\lambda)$ at ∞ from the eigenvalue structure at 0 of $\operatorname{rev} L_N(\lambda)$, just by noting that, in this case, $t = 0$ in Proposition 4.2.15 since we are taking an empty state matrix. We stress that all these results hold for *any* rational matrix $Q_N(\lambda)$ either regular or singular. However, no information is provided on the finite poles of $Q_N(\lambda)$, and some of them could also be zeros. As explained above, this is not an issue in [47], since $\mathcal{P}_N$ is outside the target set Σ. Nevertheless, at the cost of imposing extra hypotheses, we will solve this problem in Section 5.1.0.2 for completeness and also because it is of interest for the theory of REPs.

5.1.0.2 The NLEIGS basic problem from the point of view of polynomial system matrices

As discussed previously, the approach presented in Section 5.1.0.1 to the NLEIGS pencil $L_N(\lambda)$ in (5.4) considers $L_N(\lambda)$ as a linearization with empty state matrix and, thus, it does not provide any information on the finite poles of $Q_N(\lambda)$. In order to get this information, we need to identify a convenient square regular submatrix $A_N(\lambda)$ of $L_N(\lambda)$ that may be used as state matrix. The block structure of $L_N(\lambda)$ makes it impossible to find such a matrix $A_N(\lambda)$ in a way that it includes the information of all the potential poles $(\xi_1, \ldots, \xi_N)$. This is related with the comment included in [47, p. A2849] on the fact that ξ_N plays a special role and that it is convenient to choose $\xi_N = \infty$. In what follows we will *not* assume that $\xi_N = \infty$, though the obtained results are simpler and stronger under such assumption, but we will focus on getting information on the finite poles in $(\xi_1, \ldots, \xi_{N-1})$. With this spirit, we consider the following partition of $L_N(\lambda)$ in (5.4), where $A_N(\lambda)$ will play the role of the state matrix,

$$L_N(\lambda) =: \left[\begin{array}{c|c} D_N(\lambda) & -C_N(\lambda) \\ \hline B_N(\lambda) & A_N(\lambda) \end{array}\right], \quad \text{where } D_N(\lambda) := \left(1 - \tfrac{\lambda}{\xi_N}\right) D_0, \tag{5.6}$$

and the rest of the blocks are easily described from the blocks in (5.4). With this partition, the next technical lemma reveals the transfer function matrix of $L_N(\lambda)$ and establishes necessary and sufficient conditions for $L_N(\lambda)$ to be minimal in the whole field $\mathbb{F}$. By definition, $L_N(\lambda)$ is minimal in $\mathbb{F}$ if $\left[\begin{array}{cc} B_N(\lambda_0) & A_N(\lambda_0) \end{array}\right] \in \mathbb{F}^{m(N-1)\times mN}$ and $\left[\begin{array}{cc} -C_N(\lambda_0)^T & A_N(\lambda_0)^T \end{array}\right]^T \in \mathbb{F}^{mN\times m(N-1)}$ have, respectively, full row and column rank for all $\lambda_0 \in \mathbb{F}$. The conditions in Lemma 5.1.2(b) require to evaluate the rational matrix $R_N(\lambda)$ of size $m \times m$, which for practical problems is much smaller than $m(N-1) \times mN$.

Lemma 5.1.2. *Let us consider the pencil $L_N(\lambda)$ in* (5.4) *as a polynomial system matrix with state matrix $A_N(\lambda)$, where $A_N(\lambda)$ is defined through the partition* (5.6)*, and let $Q_N(\lambda)$ be the rational matrix in* (5.3)*. Then the following statements hold:*

(a) *The transfer function matrix of $L_N(\lambda)$ is $\beta_0 \left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$.*

(b) *Let us define the rational matrix $R_N(\lambda) := (Q_N(\lambda) - b_0(\lambda)D_0)/b_N(\lambda)$, whose explicit expression is*

$$R_N(\lambda) = \sum_{j=1}^{N-1} \left(\prod_{k=j+1}^{N} \frac{g_k(\lambda)}{h_{k-1}(\lambda)} \right) D_j \,+\, D_N \in \mathbb{F}(\lambda)^{m\times m}, \tag{5.7}$$

let $\mathcal{P}_{N-1}$ be the set of finite poles in the list $(\xi_1, \xi_2, \ldots, \xi_{N-1})$, and assume $\xi_i \neq \sigma_j$, $1 \le i \le N$, $0 \le j \le N-1$. Then, $L_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if the matrix $R_N(\xi_k) \in \mathbb{F}^{m\times m}$ is nonsingular for all $\xi_k \in \mathcal{P}_{N-1}$.

Proof. Let us consider $L_N(\lambda)$ partitioned as in (5.6) and as a polynomial system matrix with state matrix $A_N(\lambda)$. The computation of the transfer function matrix of $L_N(\lambda)$ is easy because $B_N(\lambda) = \left[\begin{array}{cccc} -h_0(\lambda)I_m & 0 & \cdots & 0 \end{array}\right]^T$, which implies that only the first block column of $A_N(\lambda)^{-1}$ is needed. It is immediate to check that $\frac{1}{b_1(\lambda)g_1(\lambda)} \left[b_1(\lambda) \quad \cdots \quad b_{N-1}(\lambda)\right]^T \otimes I_m$ is that first block column. The rest of the proof of part (a) is just an elementary algebraic manipulation. For part (b), observe first that $\xi_i \neq \sigma_j$, $1 \le i \le N$ and $0 \le j \le N-1$, implies that $\left[\begin{array}{cc} B_N(\lambda_0) & A_N(\lambda_0) \end{array}\right]$ has full row rank for any $\lambda_0 \in \mathbb{F}$. On the other hand, if we define

$$Z_N(\lambda) := \left[\begin{array}{c} -C_N(\lambda) \\ A_N(\lambda) \end{array}\right], \tag{5.8}$$

then $Z_N(\lambda_0)$ has full column rank for every $\lambda_0 \in \mathbb{F} \setminus \mathcal{P}_{N-1}$, because $A_N(\lambda_0)$ is invertible in $\mathbb{F} \setminus \mathcal{P}_{N-1}$. Therefore, combining the discussion above with Definition

4.1.1, we obtain that $L_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if $Z_N(\xi_k)$ has full column rank for every $\xi_k \in \mathcal{P}_{N-1}$. The rest of the proof proceeds as follows: we will find a rational matrix $S_N(\lambda)$ such that is equivalent to $Z_N(\lambda)$ in $\mathcal{P}_{N-1}$ and has a simple structure that allows us to see that $S_N(\xi_k)$ (and, so, $Z_N(\xi_k)$) has full column rank for every $\xi_k \in \mathcal{P}_{N-1}$ if and only if $R_N(\xi_k)$ is invertible for every $\xi_k \in \mathcal{P}_{N-1}$, where $R_N(\lambda)$ is the rational matrix in (5.7). For brevity, we use the notation $g_i := g_i(\lambda)$ and $h_i := h_i(\lambda)$ for the scalar functions in (5.2). In addition, $Z_N(\lambda)$ in (5.8) is partitioned as

$$Z_N(\lambda) =: \begin{bmatrix} Z_{11}(\lambda) & Z_{12}(\lambda) \\ Z_{21}(\lambda) & Z_{22}(\lambda) \end{bmatrix}, \tag{5.9}$$

where

$$Z_{11}(\lambda) = \begin{bmatrix} \frac{g_N}{\beta_N} D_1 & \frac{g_N}{\beta_N} D_2 & \cdots & \cdots & \frac{g_N}{\beta_N} D_{N-2} \\ g_1 I_m & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad Z_{12}(\lambda) = \begin{bmatrix} \frac{g_N}{\beta_N} D_{N-1} + \frac{h_{N-1}}{\beta_N} D_N \\ 0 \end{bmatrix},$$

$$Z_{21}(\lambda) = \begin{bmatrix} -h_1 & g_2 & & & \\ & -h_2 & g_3 & & \\ & & \ddots & \ddots & \\ & & & -h_{N-3} & g_{N-2} \\ & & & & -h_{N-2} \end{bmatrix} \otimes I_m, \quad Z_{22}(\lambda) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ g_{N-1} I_m \end{bmatrix}.$$

Note that the matrix $Z_{21}(\lambda)$ is invertible in $\mathcal{P}_{N-1}$ and that the last block column of $Z_{21}(\lambda)^{-1}$ is

$$Y_{22}(\lambda) := -\left[\frac{1}{h_1} \prod_{i=2}^{N-2} \frac{g_i}{h_i}, \ \frac{1}{h_2} \prod_{i=3}^{N-2} \frac{g_i}{h_i}, \ \cdots, \ \frac{1}{h_{N-3}} \frac{g_{N-2}}{h_{N-2}}, \ \frac{1}{h_{N-2}} \right]^T \otimes I_m . \tag{5.10}$$

Next, a sequence of equivalence transformations in $\mathcal{P}_{N-1}$ are applied to $Z_N(\lambda)$. Such transformations are described by using the notation in (5.9) and (5.10), and the first one is

$$Y_N(\lambda) := \left[\begin{array}{c|c} I_{2m} & 0 \\ \hline 0 & Z_{21}(\lambda)^{-1} \end{array} \right] Z_N(\lambda) = \left[\begin{array}{c|c} Z_{11}(\lambda) & Z_{12}(\lambda) \\ \hline I_{(N-2)m} & g_{N-1} Y_{22}(\lambda) \end{array} \right].$$

The second transformation is designed to turn zero the second block row of $Z_{11}(\lambda)$ as follows

$$\begin{aligned} W_N(\lambda) &:= \operatorname{diag}\left(I_m, \begin{bmatrix} I_m & -g_1 I_m \\ 0 & I_m \end{bmatrix}, I_{(N-3)m} \right) Y_N(\lambda) \\ &= \left[\begin{array}{ccc|c} \frac{g_N}{\beta_N} D_1 & \cdots & \frac{g_N}{\beta_N} D_{N-2} & \frac{g_N}{\beta_N} D_{N-1} + \frac{h_{N-1}}{\beta_N} D_N \\ 0 & \cdots & 0 & \left(\prod_{i=1}^{N-2} \frac{g_i}{h_i} \right) g_{N-1} I_m \\ \hline & I_{(N-2)m} & & g_{N-1} Y_{22}(\lambda) \end{array} \right]. \end{aligned}$$

The third transformation turns zero the block $g_{N-1}Y_{22}(\lambda)$ of $W_N(\lambda)$ and performs a convenient scalar multiplication in its first block row. Such transformation is

$$X_N(\lambda) := \begin{bmatrix} \frac{\beta_N}{h_{N-1}} I_m & 0 \\ 0 & I_{(N-1)m} \end{bmatrix} W_N(\lambda) \left[\begin{array}{c|c} I_{(N-2)m} & -g_{N-1}Y_{22}(\lambda) \\ \hline 0 & I_m \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} \frac{g_N}{h_{N-1}} D_1 & \cdots & \frac{g_N}{h_{N-1}} D_{N-2} & R_N(\lambda) \\ 0 & \cdots & 0 & \left(\prod_{i=1}^{N-2} \frac{g_i}{h_i}\right) g_{N-1} I_m \\ \hline & I_{(N-2)m} & & 0 \end{array}\right],$$

where $R_N(\lambda)$ is the rational matrix in (5.7). The last transformation makes zero the first $N-2$ blocks of size $m \times m$ in the first block row of $X_N(\lambda)$ and yields the announced matrix $S_N(\lambda)$ equivalent to $Z_N(\lambda)$ in $\mathcal{P}_{N-1}$. More precisely,

$$S_N(\lambda) := \left[\begin{array}{cc|ccc} I_m & 0 & -\frac{g_N}{h_{N-1}} D_1 & \cdots & -\frac{g_N}{h_{N-1}} D_{N-2} \\ 0 & I_m & 0 & \cdots & 0 \\ \hline & 0 & & I_{(N-2)m} & \end{array}\right] X_N(\lambda)$$
$$= \left[\begin{array}{ccc|c} 0 & \cdots & 0 & R_N(\lambda) \\ 0 & \cdots & 0 & \left(\prod_{i=1}^{N-2} \frac{g_i}{h_i}\right) g_{N-1} I_m \\ \hline & I_{(N-2)m} & & 0 \end{array}\right].$$

The block $H(\lambda) := \left(\prod_{i=1}^{N-2} \frac{g_i}{h_i}\right) g_{N-1} I_m$ of $S_N(\lambda)$ satisfies $H(\xi_k) = 0$ for all $\xi_k \in \mathcal{P}_{N-1}$. Therefore, $S_N(\xi_k)$ (and, so, $Z_N(\xi_k)$) has full column rank for every $\xi_k \in \mathcal{P}_{N-1}$ if and only if $R_N(\xi_k)$ is invertible for all $\xi_k \in \mathcal{P}_{N-1}$, and the result is proved. ■

The constant matrix $A_N(\lambda_0)$ is invertible for any $\lambda_0 \in \mathbb{F}\backslash\mathcal{P}_{N-1}$ and, so, $L_N(\lambda)$ is minimal in $\mathbb{F}\backslash\mathcal{P}_{N-1}$. Combining this with the fact that $Q_N(\lambda)$ and $\beta_0\left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ are equivalent in $\mathbb{F}$ if $\xi_N = \infty$ or in $\mathbb{F} \setminus \{\xi_N\}$ if ξ_N is finite, we immediately obtain from Definition 4.2.1 that $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ with state matrix $A_N(\lambda)$ in $\mathbb{F}\backslash\mathcal{P}_N$, which is a result analogous to Theorem 5.1.1(a). This approach, of course, does not give any information on the finite poles of $Q_N(\lambda)$, because the finite eigenvalues of $A_N(\lambda)$ coincide with $\mathcal{P}_{N-1}$. Such information is obtained from the next result, which is the main result of this section and is a corollary of Lemma 5.1.2.

Theorem 5.1.3. *Let $Q_N(\lambda)$ be the rational matrix in (5.3), $L_N(\lambda)$ be the pencil in (5.4), $A_N(\lambda)$ be the submatrix of $L_N(\lambda)$ in (5.6), and $R_N(\lambda)$ be the rational matrix in (5.7). Consider $\mathcal{P}_{N-1}$ the set of finite poles in the list $(\xi_1, \xi_2, \dots, \xi_{N-1})$, and assume $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$. If $R_N(\xi_k) \in \mathbb{F}^{m\times m}$ is nonsingular for every $\xi_k \in \mathcal{P}_{N-1}$, then $L_N(\lambda)$ is a linearization of $Q_N(\lambda)$ with state matrix $A_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if ξ_N is finite.*

Proof. Under the hypotheses of Theorem 5.1.3, $L_N(\lambda)$ is minimal in $\mathbb{F}$. Moreover, its transfer function matrix, i.e., $\beta_0\left(1 - \frac{\lambda}{\xi_N}\right) Q_N(\lambda)$ is equivalent to $Q_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if ξ_N is finite. ■

We emphasize that the hypotheses that the constant matrices $R_N(\xi_k)$ in Theorem 5.1.3 are nonsingular are not mentioned at all in [47], but, fortunately, are generic, in the sense that they are satisfied by almost all regular rational matrices $Q_N(\lambda)$ expressed as in (5.3).

Remark 5.1.4. Under the conditions of Theorem 5.1.3, the pole elementary divisors of $Q_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if ξ_N is finite, are the elementary divisors of $A_N(\lambda)$, as a consequence of Theorem 4.2.6. These elementary divisors can be easily determined as follows: first express $A_N(\lambda) = \widehat{A}_N(\lambda) \otimes I_m$; second note that if $\widehat{S}_N(\lambda)$ is the Smith form of $\widehat{A}_N(\lambda)$, then $\widehat{S}_N(\lambda) \otimes I_m$ is the Smith form of $A_N(\lambda)$; third, use the fact that $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, to prove that the greatest common divisor of all $(N-2) \times (N-2)$ minors of $\widehat{A}_N(\lambda)$ is equal to 1, which implies, according to [39, Ch. VI], that there is only one invariant polynomial of $\widehat{S}_N(\lambda)$ different from 1 and that is equal to $p(\lambda) = c\,(1-\lambda/\xi_1)\cdots(1-\lambda/\xi_{N-1})$, where $c \in \mathbb{F}$ is a constant that makes $p(\lambda)$ monic. Finally, we get that $A_N(\lambda)$ has m invariant polynomials different from 1 all equal to $p(\lambda)$. This allows us to obtain easily the finite elementary divisors of $A_N(\lambda)$ and, thus, the finite pole elementary divisors of $Q_N(\lambda)$ (in $\mathbb{F}$ if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$ if ξ_N is finite). In particular, they are of the form $(\lambda - \xi_i)^{\nu_i}$ and, in order to obtain the partial multiplicities ν_i, we have to take into account possible repetitions in $(\xi_1, \ldots, \xi_{N-1})$. Observe that the infinite ξ_i for $i = 1, \ldots, N-1$ do not contribute at all to the finite pole elementary divisors of $Q_N(\lambda)$. Moreover, if $\xi_N = \infty$, then we can state the compact and simple result that the m denominators of the global Smith–McMillan form of $Q_N(\lambda)$ are all equal to $p(\lambda)$. However, with this choice of state matrix, there is no way of obtaining information on the pole structure of ξ_N when it is finite. This is the reason why, even if $L_N(\lambda)$ is minimal in $\mathbb{F}$, $L_N(\lambda)$ is not a linearization of $Q_N(\lambda)$ in $\mathbb{F}$.

5.1.1 The NLEIGS low rank problem

The second family of rational matrices considered in [47] comes from approximating NLEPs, $A(\lambda)x = 0$, such that the associated matrix $A(\lambda)$ is the sum of a polynomial matrix plus a matrix of the form $\sum_{i=1}^{n} C_i f_i(\lambda)$, where the constant matrices C_i have much smaller rank than the size of $A(\lambda)$ and $f_i(\lambda)$ are scalar nonlinear functions of λ. This type of NLEPs arise in several applications [46] and are approximated in [47, eq. (6.2)] by a family of rational matrices of the form

$$\widetilde{Q}_N(\lambda) = \sum_{i=0}^{p} b_i(\lambda)\,\widetilde{D}_i + \sum_{i=p+1}^{N} b_i(\lambda)\,\widetilde{L}_i\,\widetilde{U}^T \in \mathbb{F}(\lambda)^{m\times m}, \tag{5.11}$$

where $b_0(\lambda), \ldots, b_N(\lambda)$ are the scalar rational functions in (5.1), $\widetilde{D}_0, \ldots, \widetilde{D}_p \in \mathbb{F}^{m\times m}$, $\widetilde{L}_{p+1}, \ldots, \widetilde{L}_N \in \mathbb{F}^{m\times r}$ and $\widetilde{U} \in \mathbb{F}^{m\times r}$ are constant matrices, and $r \ll m$. For the functions in (5.2), let us consider the simpler notation $h_i := h_i(\lambda)$ and $g_i := g_i(\lambda)$.

5.1.1.1 The NLEIGS low rank problem from the point of view of block full rank pencils

In order to solve the REP $\widetilde{Q}_N(\lambda)y = 0$ efficiently by taking advantage of the low rank structure of $\widetilde{Q}_N(\lambda)$, the following pencil is introduced in [47, Sec. 6.4]:

$$\widetilde{L}_N(\lambda) = \begin{bmatrix} \widetilde{M}_N(\lambda) \\ \widetilde{K}_N(\lambda) \end{bmatrix}, \tag{5.12}$$

where

$$\widetilde{M}_N(\lambda) := \begin{bmatrix} \frac{g_N}{\beta_N}\widetilde{D}_0 & \frac{g_N}{\beta_N}\widetilde{D}_1 & \cdots & \frac{g_N}{\beta_N}\widetilde{D}_p & \frac{g_N}{\beta_N}\widetilde{L}_{p+1} & \cdots & \frac{g_N}{\beta_N}\widetilde{L}_{N-2} & \frac{g_N}{\beta_N}\widetilde{L}_{N-1} + \frac{h_{N-1}}{\beta_N}\widetilde{L}_N \end{bmatrix},$$

$$\widetilde{K}_N(\lambda) := \begin{bmatrix} -h_0 I_m & g_1 I_m & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -h_{p-1}I_m & g_p I_m & & & & \\ & & & -h_p\widetilde{U}^T & g_{p+1}I_r & & & \\ & & & & -h_{p+1}I_r & g_{p+2}I_r & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -h_{N-2}I_r & g_{N-1}I_r \end{bmatrix}.$$

A result analogous to Theorem 5.1.1 can be proved for the pencil $\widetilde{L}_N(\lambda)$ and the matrix $\widetilde{Q}_N(\lambda)$. This is accomplished in Theorem 5.1.5. We remark, nevertheless, that the result concerning the linearizations at ∞ is weaker in Theorem 5.1.5 than in Theorem 5.1.1. This is an unavoidable consequence of the used approach and the low rank structure of $\widetilde{Q}_N(\lambda)$.

Theorem 5.1.5. *Let $\widetilde{Q}_N(\lambda)$ be the rational matrix in* (5.11) *and $\widetilde{L}_N(\lambda)$ be the pencil in* (5.12)*. Let $\mathcal{P}_N$ and i_N be, respectively, the set of finite poles and the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_N)$. Then, the following statements hold:*

(a) *$\widetilde{L}_N(\lambda)$ partitioned as in* (5.12) *is a block full rank pencil with only one block column associated with $\widetilde{Q}_N(\lambda)$ in $\mathbb{F}\setminus\mathcal{P}_N$ and $\widetilde{L}_N(\lambda)$ is a linearization of $\widetilde{Q}_N(\lambda)$ in $\mathbb{F}\setminus\mathcal{P}_N$ with empty state matrix.*

(b) *If, in addition, the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite, then $\widetilde{L}_N(\lambda)$ is a linearization of $\widetilde{Q}_N(\lambda)$ at ∞ of grade i_N with empty state matrix.*

Proof. The proof is similar to that of Theorem 5.1.1 with some differences coming from the presence of the low rank term in $\widetilde{Q}_N(\lambda)$. It is immediate to check that

$$\widetilde{N}_N(\lambda) = \frac{1}{1-\frac{\lambda}{\xi_N}} \begin{bmatrix} b_0(\lambda)I_m & \cdots & b_p(\lambda)I_m & b_{p+1}(\lambda)\widetilde{U} & \cdots & b_{N-1}(\lambda)\widetilde{U} \end{bmatrix} \tag{5.13}$$

is a rational basis dual to $\widetilde{K}_N(\lambda)$, that $\widetilde{K}_N(\lambda)$ and $\widetilde{N}_N(\lambda)$ have both full row rank in $\mathbb{F} \setminus \mathcal{P}_N$ and that $\widetilde{M}_N(\lambda)\widetilde{N}_N(\lambda)^T = \widetilde{Q}_N(\lambda)$. Thus, part (a) follows from Theorem 4.4.1.

In order to prove part (b), note first that $\mathrm{rev}_1 \widetilde{K}_N(\lambda)$ has full row rank at 0 as a consequence of the fact that the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite. We now consider the rational matrix $\mathrm{rev}_{i_N-1} \widetilde{N}_N(\lambda) = \lambda^{i_N-1}\widetilde{N}_N\left(\frac{1}{\lambda}\right)$, which is of the form

$$\lambda^{i_N-1}\widetilde{N}_N\left(\frac{1}{\lambda}\right) = \begin{bmatrix} * & \cdots & * & \frac{\lambda}{\lambda-1/\xi_N}\lambda^{i_N-1}b_p\left(\frac{1}{\lambda}\right)I_m & * & \cdots & * \end{bmatrix},$$

where the entries $*$ are defined at 0. Denote by i_p the number of infinite poles in the list $(\xi_1, \xi_2, \ldots, \xi_p)$. Then, $b_p\left(\frac{1}{\lambda}\right) = \frac{1}{\lambda^{i_p}}\tilde{c}(\lambda)$ for a certain rational function $\tilde{c}(\lambda)$ with $\tilde{c}(0) \neq 0$. Taking into account that the poles $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1}$ are all finite, we have that $i_p = i_N$ if $\xi_N \neq \infty$, and $i_p = i_N - 1$ if $\xi_N = \infty$. Therefore, $\mathrm{rev}_{i_N-1} \widetilde{N}_N(\lambda)$ has full row rank at 0 because $\tilde{c}(0) \neq 0$. Thus, part (b) follows from Theorem 4.3.7. ■

A discussion similar to the one in the last paragraph of Section 5.1.0.1 can be developed on the basis of Theorem 5.1.5. The details are omitted for brevity. The open problem corresponding to the information of the finite poles will be solved in Section 5.1.1.2.

5.1.1.2 The NLEIGS low rank problem from the point of view of polynomial system matrices

The results in this section are the counterpart for $\widetilde{Q}_N(\lambda)$ in (5.11) and $\widetilde{L}_N(\lambda)$ in (5.12) of those presented in Section 5.1.0.2 for $Q_N(\lambda)$ and $L_N(\lambda)$. For this purpose, we consider the following partition of $\widetilde{L}_N(\lambda)$ in (5.12):

$$\left[\begin{array}{cccc|cccc} \frac{g_N}{\beta_N}\widetilde{D}_0 & \frac{g_N}{\beta_N}\widetilde{D}_1 & \cdots & \frac{g_N}{\beta_N}\widetilde{D}_p & \frac{g_N}{\beta_N}\widetilde{L}_{p+1} & \cdots & \frac{g_N}{\beta_N}\widetilde{L}_{N-2} & \frac{g_N}{\beta_N}\widetilde{L}_{N-1} + \frac{h_{N-1}}{\beta_N}\widetilde{L}_N \\ \hdashline -h_0 I_m & g_1 I_m & & & & & & \\ & \ddots & \ddots & & & & & \\ \hline & & & -h_p\widetilde{U}^T & g_{p+1}I_r & & & \\ & & & & -h_{p+1}I_r & g_{p+2}I_r & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -h_{N-2}I_r & g_{N-1}I_r \end{array}\right]$$
$$=: \left[\begin{array}{c|c} \widetilde{M}(\lambda) & -\widetilde{C}_N(\lambda) \\ \hdashline \widetilde{K}(\lambda) & 0 \\ \hline \widetilde{B}_N(\lambda) & \widetilde{A}_N(\lambda) \end{array}\right], \tag{5.14}$$

where $\widetilde{A}_N(\lambda)$ will play the role of the state matrix, and the rational basis

$$\widetilde{N}(\lambda) := \frac{1}{1-\frac{\lambda}{\xi_N}} \begin{bmatrix} b_0(\lambda) & b_1(\lambda) & \cdots & b_p(\lambda) \end{bmatrix} \otimes I_m \tag{5.15}$$

dual to $\widetilde{K}(\lambda)$. The next lemma is the counterpart of Lemma 5.1.2.

Lemma 5.1.6. *Let us consider the pencil $\widetilde{L}_N(\lambda)$ in* (5.12) *as a polynomial system matrix with state matrix $\widetilde{A}_N(\lambda)$, where $\widetilde{A}_N(\lambda)$ is defined through the partition* (5.14)*, and let $\widetilde{Q}_N(\lambda)$ be the rational matrix in* (5.11)*. Then the following statements hold:*

(a) *$\widetilde{L}_N(\lambda)$ is a block full rank linearization of $\widetilde{Q}_N(\lambda)$ in the sets where simultaneously $\widetilde{L}_N(\lambda)$ is minimal and $\widetilde{N}(\lambda)$ has full row rank.*

(b) *Let us define the rational matrix*

$$\widetilde{R}_N(\lambda) = \sum_{j=p+1}^{N-1} \left(\prod_{k=j+1}^{N} \frac{g_k(\lambda)}{h_{k-1}(\lambda)} \right) \widetilde{L}_j + \widetilde{L}_N \in \mathbb{F}(\lambda)^{m\times r}, \tag{5.16}$$

let $\widetilde{\mathcal{P}}_{N-1}$ be the set of finite poles in the list $(\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{N-1})$. Assume that $\operatorname{rank} \widetilde{U} = r$ *and $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$. Then, $\widetilde{L}_N(\lambda)$ is minimal in $\mathbb{F}$ if and only if the matrix $\widetilde{R}_N(\xi_k) \in \mathbb{F}^{m\times r}$ has full column rank for all $\xi_k \in \widetilde{\mathcal{P}}_{N-1}$.*

Proof. For part (a), we only have to check that

$$[\widetilde{M}(\lambda) + \widetilde{C}_N(\lambda)\widetilde{A}_N(\lambda)^{-1}\widetilde{B}_N(\lambda)]\widetilde{N}(\lambda)^T = \widetilde{Q}_N(\lambda).$$

For that, we take into account that the first block column of $\widetilde{A}_N(\lambda)^{-1}$ is equal to $\frac{1}{b_{p+1}(\lambda)g_{p+1}} \begin{bmatrix} b_{p+1}(\lambda) & \cdots & b_{N-1}(\lambda) \end{bmatrix}^T \otimes I_r$. The proof of part (b) is analogous to that of Lemma 5.1.2(b). ■

As a corollary of Lemma 5.1.6, we state Theorem 5.1.7 which is the main result in this section.

Theorem 5.1.7. *Let $\widetilde{Q}_N(\lambda)$ be the rational matrix in* (5.11)*, $\widetilde{L}_N(\lambda)$ be the pencil in* (5.12)*, $\widetilde{A}_N(\lambda)$ be the submatrix of $\widetilde{L}_N(\lambda)$ in* (5.14)*, and $\widetilde{R}_N(\lambda)$ be the rational matrix in* (5.16)*. Consider $\widetilde{\mathcal{P}}_{N-1}$ the set of finite poles in the list $(\xi_{p+1}, \xi_2, \ldots, \xi_{N-1})$. If $\xi_1 = \cdots = \xi_p = \infty$,* $\operatorname{rank} \widetilde{U} = r$*, $\xi_i \neq \sigma_j$, $1 \leq i \leq N$, $0 \leq j \leq N-1$, and $\widetilde{R}_N(\xi_k) \in \mathbb{F}^{m\times r}$ has full column rank for every $\xi_k \in \widetilde{\mathcal{P}}_{N-1}$, then $\widetilde{L}_N(\lambda)$ is a block full rank linearization of $\widetilde{Q}_N(\lambda)$ with state matrix $\widetilde{A}_N(\lambda)$ in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if ξ_N is finite.*

Proof. First, notice that if $\xi_1 = \cdots = \xi_p = \infty$ then the dual rational basis $\widetilde{N}(\lambda)$ in (5.15) has full row rank in $\mathbb{F}$, if $\xi_N = \infty$, or in $\mathbb{F} \setminus \{\xi_N\}$, if ξ_N is finite. In addition, the rank condition (4.15) is satisfied in $\mathbb{F}$ if $\widetilde{L}_N(\lambda)$ is minimal in $\mathbb{F}$ and $\xi_1 = \cdots = \xi_p = \infty$. By Lemma 5.1.6 and Theorem 4.4.1 the results follows. ■

Remark 5.1.8. Note that the hypothesis $\xi_1 = \cdots = \xi_p = \infty$ implies that the "no-low rank" term $\sum_{i=0}^{p} b_i(\lambda)\widetilde{D}_i$ of $\widetilde{Q}_N(\lambda)$ is a polynomial matrix, as often happens in NLEPs [47].

5.2 Linearizations for AAA rational approximations of NLEPs

In this section we study the precise properties of the linearizations for rational approximations of NLEPs in [60]. We will see that they satisfy Theorem 4.4.1 in a particular subset of $\mathbb{F}$ and in the whole field $\mathbb{F}$ under mild conditions.

An approach to obtain an automatic rational approximation for the NLEP in a region Ω is given in [60]. The authors of [60] consider the $n \times n$ nonlinear matrix $F(\lambda)$ of the NLEP written in the form

$$F(\lambda) = Q(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) g_i(\lambda) \tag{5.17}$$

with $Q(\lambda)$ a polynomial matrix, C_i and D_i $n \times n$ constant matrices, and $g_i(\lambda)$ nonlinear scalar functions. Then, a CORK linearization of $Q(\lambda)$ of those introduced in [81] is considered, and the functions $g_i(\lambda)$ are approximated by rational functions employing the adaptive Antoulas–Anderson (AAA) algorithm [73], or its set-valued generalization presented in [60]. We recall the definition of CORK linearization as given in [60].

Definition 5.2.1. *Let $G(\lambda)$ be an $n \times n$ rational matrix written as*

$$G(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda), \tag{5.18}$$

where $f_i(\lambda)$ are scalar rational functions with $f_0(\lambda) \equiv 1$, and A_i, B_i are $n \times n$ constant matrices. Define

$$f(\lambda) := [f_0(\lambda) \quad \cdots \quad f_{k-1}(\lambda)]^T,$$

and assume that the rational functions $f_i(\lambda)$ satisfy a linear relation

$$(X - \lambda Y) f(\lambda) = 0, \tag{5.19}$$

where $\operatorname{rank}(X - \lambda Y) = k - 1$ *for all* $\lambda \in \mathbb{F}$, *and* $X - \lambda Y$ *has size* $(k-1) \times k$. *Then the matrix pencil*

$$\mathcal{L}_G(\lambda) = \left[\begin{array}{ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \\ \hdashline & (X - \lambda Y) \otimes I_n & \end{array} \right]$$

is called a CORK linearization of $G(\lambda)$.

If the vector $f(\lambda)$ is polynomial then $G(\lambda)$ in (5.18) is a polynomial matrix and $\mathcal{L}_G(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{F}$, in particular, $\mathcal{L}_G(\lambda)$ is a block full rank linearization of $G(\lambda)$ in $\mathbb{F}$ with empty state matrix. However, if $f(\lambda)$ is a rational vector then $\mathcal{L}_G(\lambda)$ is not, in general, a linearization in the sense of [7], that is, in $\mathbb{F}$ but it is a linearization in a local sense [28]. More precisely, it is a block full rank linearization in all the subsets where the rational vector $f(\lambda)$ is defined, i.e., has no poles. Such result is stated in the next theorem.

Theorem 5.2.2. *Let* Ω *be a nonempty subset of* $\mathbb{F}$ *where the rational vector* $f(\lambda)$ *in* (5.19) *is defined. Then a CORK linearization* $\mathcal{L}_G(\lambda)$ *of a rational matrix* $G(\lambda)$ *as in* (5.18) *is a block full rank linearizacion of* $G(\lambda)$ *in* Ω *with empty state matrix.*

Proof. Notice that, by taking $M(\lambda) := \left[\begin{array}{ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \end{array} \right]$, $K_1(\lambda) := (X - \lambda Y) \otimes I_n$, and $K_2(\lambda)$ empty, $\mathcal{L}_G(\lambda)$ is a block full rank pencil. Moreover, $f(\lambda)^T$ is a rational basis dual to $X - \lambda Y$. Then we apply Remark 4.4.2 with $N_1(\lambda) := f(\lambda)^T \otimes I_n$ and $N_2(\lambda) = I_n$. ∎

Once CORK linearizations and some of their properties have been revised, we recall the AAA approximation of scalar functions. A given nonlinear function $g : \mathbb{F} \longrightarrow \mathbb{F}$ is approximated in [60] on a set $\Sigma \subset \mathbb{F}$ by a rational function $r(\lambda)$ in barycentric form, that is,

$$r(\lambda) = \sum_{j=1}^{m} \frac{g(z_j) w_j}{\lambda - z_j} \Big/ \sum_{j=1}^{m} \frac{w_j}{\lambda - z_j}, \tag{5.20}$$

where $z_1, \ldots, z_m$ are distinct support points and $w_1, \ldots, w_m$ are nonzero weights, that can be automatically chosen as explained in [73]. In this case, $\lim\limits_{\lambda \to z_j} r(\lambda) = g(z_j)$. It is shown in [60, Proposition 2.1] that $r(\lambda)$ in (5.20) can be written as

$$\left[g(z_1) w_1 \quad \cdots \quad g(z_m) w_m \right] \begin{bmatrix} w_1 & w_2 & \cdots & w_{m-1} & w_m \\ \lambda - z_1 & z_2 - \lambda & & & \\ & \lambda - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda & \\ & & & \lambda - z_{m-1} & z_m - \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

That is, $r(\lambda)$ can be written as a generalized state-space realization. Then, the pencil

$$P(\lambda) := \left[\begin{array}{ccccc|c} w_1 & w_2 & \cdots & w_{m-1} & w_m & -1 \\ \lambda - z_1 & z_2 - \lambda & & & & 0 \\ & \lambda - z_2 & \ddots & & & \vdots \\ & & \ddots & z_{m-1} - \lambda & & \vdots \\ & & & \lambda - z_{m-1} & z_m - \lambda & 0 \\ \hline g(z_1)w_1 & g(z_2)w_2 & \cdots & g(z_{m-1})w_{m-1} & g(z_m)w_m & 0 \end{array}\right] := \left[\begin{array}{c|c} E - \lambda F & -b \\ \hline a^T & 0 \end{array}\right] \tag{5.21}$$

is a linear polynomial system matrix of $r(\lambda)$ (i.e., with transfer function $r(\lambda)$) by considering $E-\lambda F$ as state matrix. In order to know what pole and zero information of $r(\lambda)$ we can obtain from this realization, we consider in Proposition 5.2.4 the polynomial system matrix $P(\lambda)$ and we study its minimality. First, we prove Lemma 5.2.3. In both Proposition 5.2.4 and Lemma 5.2.3, we consider $r(\lambda)$ written as the following quotient of polynomials

$$r(\lambda) := \frac{p(\lambda)}{q(\lambda)}, \tag{5.22}$$

where

$$p(\lambda) := \left(\prod_{j=1}^{m}(\lambda - z_j)\right)\left(\sum_{j=1}^{m}\frac{g(z_j)w_j}{\lambda - z_j}\right) \text{ and } q(\lambda) := \left(\prod_{j=1}^{m}(\lambda - z_j)\right)\left(\sum_{j=1}^{m}\frac{w_j}{\lambda - z_j}\right).$$

Note that the representation of the rational function (5.22) might not be irreducible. We will see that the irreducibility of (5.22) is the key property for the minimality of $P(\lambda)$.

Lemma 5.2.3. *The pencil $E - \lambda F$ in* (5.21) *is a strong block minimal bases pencil associated with the polynomial $q(\lambda)$ in* (5.22)*.*

Proof. We set

$$E - \lambda F = \left[\begin{array}{ccccc} w_1 & w_2 & \cdots & w_{m-1} & w_m \\ \hline \lambda - z_1 & z_2 - \lambda & & & \\ & \lambda - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda & \\ & & & \lambda - z_{m-1} & z_m - \lambda \end{array}\right] =: \left[\begin{array}{c} M \\ \hline K(\lambda) \end{array}\right]. \tag{5.23}$$

Since $z_1, \ldots, z_m$ are distinct, $K(\lambda_0)$ has full row rank for all $\lambda_0 \in \mathbb{F}$. In addition,

$K(\lambda)$ is row reduced because its highest row degree coefficient matrix

$$K_{hr} = \begin{bmatrix} 1 & -1 & 0 & & & \\ & 1 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -1 & 0 \\ & & & & 1 & -1 \end{bmatrix}$$

has full row rank. We conclude that $K(\lambda)$ is a minimal basis. Let us denote

$$N(\lambda) := \prod_{j=1}^{m} (\lambda - z_j) \begin{bmatrix} \frac{1}{\lambda - z_1} & \cdots & \frac{1}{\lambda - z_m} \end{bmatrix}. \tag{5.24}$$

Then, it is not difficult to prove that $N(\lambda)$ is also a minimal basis, taking again into account that $z_1, \dots, z_m$ are distinct. Moreover, since $K(\lambda)N(\lambda)^T = 0$ and $\begin{bmatrix} K(\lambda) \\ N(\lambda) \end{bmatrix}$ is a square matrix, we have that $K(\lambda)$ and $N(\lambda)$ are dual minimal bases. In addition, all the row degrees of $K(\lambda)$ are equal to 1 and the row degree of $N(\lambda)$ is equal to $m - 1$. Hence, $E - \lambda F$ is a strong block minimal bases pencil associated with the polynomial matrix $MN(\lambda)^T = q(\lambda)$. ■

$E - \lambda F$ being a strong block minimal bases pencil associated with the polynomial $q(\lambda)$ implies that $E - \lambda F$ is a (strong) linearization of $q(\lambda)$ and, in particular, that the determinant of $E - \lambda F$ is equal to $q(\lambda)$ up to a scalar multiple. This fact is used to prove the following result.

Proposition 5.2.4. *Consider the rational function $r(\lambda)$ in (5.22) and its linear polynomial system matrix $P(\lambda)$ in (5.21). Then, $P(\lambda)$ is not minimal at $\lambda_0 \in \mathbb{F}$ if and only if λ_0 is a zero of both polynomials $p(\lambda)$ and $q(\lambda)$.*

Proof. Consider $P(\lambda) = \left[\begin{array}{c|c} E - \lambda F & -b \\ \hline a^T & 0 \end{array}\right]$ as in (5.21). By Lemma 5.2.3,

$$\det(E - \lambda F) = \alpha q(\lambda) \text{ with } \alpha \neq 0. \tag{5.25}$$

In addition, since the Schur complement of $E - \lambda F$ in $P(\lambda)$ is $r(\lambda)$, we have that

$$\det(P(\lambda)) = \det(E - \lambda F) r(\lambda) = \alpha q(\lambda) \frac{p(\lambda)}{q(\lambda)} = \alpha p(\lambda). \tag{5.26}$$

Now, assume that λ_0 is a zero of both polynomials $p(\lambda)$ and $q(\lambda)$. That is, we can cancel out at least one factor of the form $(\lambda - \lambda_0)$ in both numerator and denominator of $r(\lambda)$. Then, the algebraic multiplicity of λ_0 as a zero of $r(\lambda)$ is not the same as the algebraic multiplicity of λ_0 as a zero of $P(\lambda)$. Therefore,

$P(\lambda)$ is not minimal at λ_0. For the converse, assume that $P(\lambda)$ is not minimal at λ_0. Then, rank $\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix} < m$, since rank $\begin{bmatrix} E-\lambda_0 F & -b \end{bmatrix} = m$. Actually, rank $\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix} = m-1$, as the sub-matrix

$$K(\lambda_0) = \begin{bmatrix} \lambda_0 - z_1 & z_2 - \lambda_0 & & & \\ & \lambda_0 - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda_0 & \\ & & & \lambda_0 - z_{m-1} & z_m - \lambda_0 \end{bmatrix}$$

contains a non-zero minor of order $m-1$ for all $\lambda_0 \in \mathbb{F}$, where $K(\lambda)$ is the matrix appearing in (5.23). Therefore, by using the notation $\mathcal{N}_r(\cdot)$ for the right nullspace, $\dim \mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) = 1$ and $\dim \mathcal{N}_r(K(\lambda_0)) = 1$. Then, $\mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) = \mathcal{N}_r(K(\lambda_0))$, since $\mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) \subseteq \mathcal{N}_r(K(\lambda_0))$ and both have the same dimension. Actually, $\mathcal{N}_r(K(\lambda_0)) = \text{Span}\{N(\lambda_0)^T\}$, where $N(\lambda)$ is the polynomial matrix in (5.24). Hence, $\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix} N(\lambda_0)^T = 0$ and, therefore, $\begin{bmatrix} w_1 & w_2 & \cdots & w_m \end{bmatrix} N(\lambda_0)^T = 0$ and $\begin{bmatrix} g(z_1)w_1 & g(z_2)w_2 & \cdots & g(z_m)w_m \end{bmatrix} N(\lambda_0)^T = 0$. That is, λ_0 is a root of both $q(\lambda)$ and $p(\lambda)$.

■

With these tools at hand, we go back to the original NLEP. Let $F(\lambda)$ be the nonlinear matrix function in (5.17). Then, each function $g_i(\lambda)$ is approximated in [60] on a set $\Sigma \subset \mathbb{F}$ by a rational function $r_i(\lambda)$ as in (5.20), i.e.,

$$g_i(\lambda) \approx r_i(\lambda) = \sum_{j=1}^{\ell_i} \frac{g_i(z_j^i) w_j^i}{\lambda - z_j^i} \Big/ \sum_{j=1}^{\ell_i} \frac{w_j^i}{\lambda - z_j^i},$$

where ℓ_i is the number of support points z_j^i and weights w_j^i for each $i = 1, \ldots, s$. For that, one can use the AAA algorithm on each function $g_i(\lambda)$ separately [73], or one can use the set-valued AAA algorithm in [60, Section 2.2], so that the rational approximants $r_i(\lambda)$ are all constructed simultaneously and sharing the same support points $z_j^i := z_j$ and weights $w_j^i := w_j$. By using any of the two approaches above, the following approximation of $F(\lambda)$ on Σ is obtained:

$$F(\lambda) \approx R(\lambda) := Q(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) r_i(\lambda). \tag{5.27}$$

Next, the polynomial matrix $Q(\lambda)$ is expressed in the form of (5.18), i.e., $Q(\lambda) := \sum_{i=0}^{k-1}(A_i - \lambda B_i)f_i(\lambda)$, assuming the functions $f_i(\lambda)$ are polynomials, with $f_0(\lambda) = 1$, and each $r_i(\lambda)$ is written in generalized state-space form, that is,

$$R(\lambda) = \sum_{i=0}^{k-1}(A_i - \lambda B_i)f_i(\lambda) + \sum_{i=1}^{s}(C_i - \lambda D_i)a_i^T(E_i - \lambda F_i)^{-1}b_i, \tag{5.28}$$

with $a_i = \begin{bmatrix} g_i(z_1^i)w_1^i & \cdots & g_i(z_{\ell_i}^i)w_{\ell_i}^i \end{bmatrix}^T \in \mathbb{F}^{\ell_i}$, $b_i = [1 \quad 0 \quad \cdots \quad 0]^T \in \mathbb{F}^{\ell_i}$ and $\ell_i \times \ell_i$ matrices

$$E_i = \begin{bmatrix} w_1^i & w_2^i & \cdots & w_{\ell_i-1}^i & w_{\ell_i}^i \\ -z_1^i & z_2^i & & & \\ & -z_2^i & \ddots & & \\ & & \ddots & z_{\ell_i-1}^i & \\ & & & -z_{\ell_i-1}^i & z_{\ell_i}^i \end{bmatrix} \text{ and } F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

The linearization constructed in [60] for $R(\lambda)$ is the following.

Definition 5.2.5. *[60, Definition 3.2](CORK linearization for AAA rational approximation) Let $R(\lambda)$ be a rational matrix as in* (5.28)*. Consider $b := [b_1^T \quad \cdots \quad b_s^T]^T$ and $E - \lambda F :=$* diag$(E_1 - \lambda F_1, \ldots, E_s - \lambda F_s)$*. Then a CORK linearization for $R(\lambda)$ is*

$$\mathcal{L}_R(\lambda) = \left[\begin{array}{cc|ccc} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} & & a_1^T \otimes (C_1 - \lambda D_1) & \cdots & a_s^T \otimes (C_s - \lambda D_s) \\ \hdashline (X - \lambda Y) \otimes I_n & & & 0 & \\ \hline -b \otimes I_n & 0 & & (E - \lambda F) \otimes I_n & \end{array}\right]$$

where $\left[\begin{array}{c} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} \\ \hdashline (X - \lambda Y) \otimes I_n \end{array}\right]$ *is any CORK linearization of $Q(\lambda)$.*

In particular, for the set-valued AAA approximation, all the matrices $E_i - \lambda F_i$ in (5.28) are the same for all i, as well as all the vectors b_i. Then, in [60, Remark 3.3] the following CORK linearization for AAA rational approximation is considered

$$\mathcal{L}_R^{sv}(\lambda) = \left[\begin{array}{cc|c} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} & & \sum_{i=1}^{s} a_i^T \otimes (C_i - \lambda D_i) \\ \hdashline (X - \lambda Y) \otimes I_n & & 0 \\ \hline -b_1 \otimes I_n & 0 & (E_1 - \lambda F_1) \otimes I_n \end{array}\right] \tag{5.29}$$

Notice that $\mathcal{L}_R^{sv}(\lambda)$ has size $(kn + \ell_1 n) \times (kn + \ell_1 n)$ whereas $\mathcal{L}_R(\lambda)$ in Definition 5.2.5 has size $(kn + \sum_{i=1}^{s} \ell_i n) \times (kn + \sum_{i=1}^{s} \ell_i n)$.

By using Theorem 4.4.1, we study in Theorem 5.2.6 the structure of $\mathcal{L}_R(\lambda)$ and $\mathcal{L}_R^{sv}(\lambda)$ as linearizations of $R(\lambda)$.

Theorem 5.2.6. *Let $R(\lambda)$ be a rational matrix as in* (5.28)*, and let $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$) be the matrix pencil in Definition 5.2.5 (resp., in* (5.29)*). Let $\Omega \subseteq \mathbb{F}$ be nonempty. If $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$), viewed as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$ (resp., $(E_1 - \lambda F_1) \otimes I_n$), is minimal in Ω then $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$) is a block full rank linearization of $R(\lambda)$ in Ω with state matrix $(E - \lambda F) \otimes I_n$ (resp., $(E_1 - \lambda F_1) \otimes I_n$).*

Proof. Set $M(\lambda) := \begin{bmatrix} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \end{bmatrix}$, $C(\lambda) := -[a_1^T \otimes (C_1 - \lambda D_1) \quad \cdots \quad a_s^T \otimes (C_s - \lambda D_s)]$, $B := [-b \otimes I_n \quad 0]$, $A(\lambda) := (E - \lambda F) \otimes I_n$, $K_1(\lambda) := (X - \lambda Y) \otimes I_n$, $N_1(\lambda) := (f(\lambda) \otimes I_n)^T$, and $K_2(\lambda)$ empty. $\mathcal{L}_R(\lambda)$ being minimal in Ω implies that condition (4.15) is satisfied in Ω since $BN_1(\lambda)^T = -b \otimes I_n$ because $f_0(\lambda) = 1$. Then, by Theorem 4.4.1, $\mathcal{L}_R(\lambda)$ is a linearization of $[M(\lambda) + C(\lambda)A(\lambda)^{-1}B](f(\lambda) \otimes I_n) = R(\lambda)$ in Ω with state matrix $A(\lambda)$. ■

Remark 5.2.7. Theorem 5.2.6 also holds if $f(\lambda)$ is rational but, in such a case, we need the extra hypothesis of $f(\lambda)$ being defined in Ω.

According to Theorem 5.2.6, we need minimality on $\mathcal{L}_R(\lambda)$ to be a linearization of the rational matrix $R(\lambda)$. In the following Theorem 5.2.8, we give sufficient mild conditions for $\mathcal{L}_R(\lambda)$ to be minimal in $\mathbb{F}$ in the case the rational approximants $r_i(\lambda)$ do not share the same support points and weights. That is, when the functions $g_i(\lambda)$ are approximated employing the adaptive Antoulas–Anderson (AAA) algorithm in [73] separately. For the set-valued AAA approximation [60], i.e., the rational approximants $r_i(\lambda)$ sharing the same support points and weights, the authors in [60] consider the pencil in (5.29), and we state minimality conditions for it in Theorem 5.2.9.

Theorem 5.2.8. *Assume that, for $i = 1, \ldots, s$, the rational functions $r_i(\lambda)$ in* (5.27) *are represented as in* (5.22) *and that this representation is irreducible. Let $\mathcal{L}_R(\lambda)$ be the matrix pencil in Definition 5.2.5. If the pencils $C_i - \lambda D_i$ and $E_i - \lambda F_i$ are regular for $i = 1, \ldots, s$ and the following conditions hold*

(a) *$C_i - \lambda D_i$ and $E_i - \lambda F_i$ have no finite eigenvalues in common for $i = 1, \ldots, s$, and*

(b) *$E_i - \lambda F_i$ and $E_j - \lambda F_j$ with $i \neq j$ have no finite eigenvalues in common for $i, j = 1, \ldots, s$,*

then $\mathcal{L}_R(\lambda)$, viewed as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$, is minimal in $\mathbb{F}$.

Proof. Assume first that $s = 1$. Then notice that $\mathcal{L}_R(\lambda)$ is minimal in $\mathbb{C}$ if the pencil

$$S(\lambda) := \left[\begin{array}{c|c} 0 & a_1^T \otimes (C_1 - \lambda D_1) \\ \hline -b_1 \otimes I_n & (E_1 - \lambda F_1) \otimes I_n \end{array} \right],$$

considered as a polynomial system matrix with state matrix $(E_1 - \lambda F_1) \otimes I_n$, is minimal in $\mathbb{F}$. Since $r_1(\lambda)$ is irreducible, we have, by Proposition 5.2.4, that the submatrix $\left[\begin{array}{c|c} -b_1 \otimes I_n & (E_1 - \lambda F_1) \otimes I_n \end{array}\right]$ has full row rank for all $\lambda \in \mathbb{F}$. Then we only have to prove that the submatrix $H(\lambda) := \left[\begin{array}{c} a_1^T \otimes (C_1 - \lambda D_1) \\ \hline (E_1 - \lambda F_1) \otimes I_n \end{array}\right]$ has full column rank for all $\lambda \in \mathbb{F}$. By contradiction, assume that $H(\lambda_0)$ has no full column rank for some $\lambda_0 \in \mathbb{F}$. Notice that, in such a case, λ_0 must be an eigenvalue of $E_1 - \lambda F_1$ since, otherwise, $H(\lambda_0)$ would have full column rank. In addition, there exists a nonzero vector x such that $H(\lambda_0)x = 0$. Now we write

$$H(\lambda_0)x = \begin{bmatrix} C_1 - \lambda_0 D_1 & 0 \\ 0 & I_{\ell_1 n} \end{bmatrix} \left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right] x = 0, \tag{5.30}$$

and define the vector $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right] x$, which is nonzero since $x \neq 0$ and the matrix $\left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right]$ has full column rank by Proposition 5.2.4. Moreover, by (5.30), we have that $y_2 = 0$ and, thus, $(C_1 - \lambda_0 D_1)y_1 = 0$ with $y_1 \neq 0$. Therefore, λ_0 is an eigenvalue of $C_1 - \lambda D_1$, which is a contradiction by condition (a). Finally, if $s > 1$ we have to take into account condition (b) and the result follows. ■

For the set-valued AAA approximation, the minimality conditions on $\mathcal{L}_R^{sv}(\lambda)$ are milder. We state them in the following result that is a corollary of Theorem 5.2.8.

Theorem 5.2.9. *Assume that, for $i = 1, \ldots, s$, the rational functions $r_i(\lambda)$ in* (5.27) *are represented as in* (5.22) *and that this representation is irreducible. Let $\mathcal{L}_R^{sv}(\lambda)$ be the matrix pencil in* (5.29). *If the pencils $\sum_{i=1}^{s} a_i^T \otimes (C_i - \lambda D_i)$ and $E_1 - \lambda F_1$ are regular and have no finite eigenvalues in common then $\mathcal{L}_R^{sv}(\lambda)$, viewed as a polynomial system matrix with state matrix $(E_1 - \lambda F_1) \otimes I_n$, is minimal in $\mathbb{F}$.*

Remark 5.2.10. It is clear that if we consider the set $\Omega := \{\lambda \in \mathbb{F} : E - \lambda F$ is invertible$\}$, then $\mathcal{L}_R(\lambda)$ is minimal in Ω and, by Theorem 5.2.6, $\mathcal{L}_R(\lambda)$ is a block full rank linearization of $R(\lambda)$ in Ω, with state matrix $(E - \lambda F) \otimes I_n$. However, in such a case we do not obtain any information about the poles of $R(\lambda)$ since they do not belong to Ω. For this particular choice of the set Ω, the fact that $\mathcal{L}_R(\lambda)$ is a linearization of $R(\lambda)$ in Ω can also be proved by considering $\mathcal{L}_R(\lambda)$ as a block full rank pencil of the form

$$\mathcal{L}_R(\lambda) := \left[\begin{array}{c} M(\lambda) \\ \hdashline K_1(\lambda) \end{array}\right],$$

with

$$M(\lambda) := \left[\begin{array}{ccc|ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} & a_1^T \otimes (C_1 - \lambda D_1) & \cdots & a_s^T \otimes (C_s - \lambda D_s) \end{array}\right],$$

and by applying Remark 4.4.2. For that, write $R(\lambda)$ as

$$R(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i)(f_i(\lambda) \otimes I_n) + \sum_{i=1}^{s} [a_i^T \otimes (C_i - \lambda D_i)](R_i(\lambda) \otimes I_n),$$

with $R_i(\lambda) := (E_i - \lambda F_i)^{-1} b_i$, and consider the dual rational basis of $K_1(\lambda)$

$$N_1(\lambda) := [f_0(\lambda) \quad \cdots \quad f_{k-1}(\lambda) \quad | \quad R_1(\lambda)^T \quad \cdots \quad R_s(\lambda)^T] \otimes I_n.$$

Then, $\mathcal{L}_R(\lambda)$ is a linearization of $R(\lambda)$ in Ω with empty state matrix. On the other hand, if $\mathcal{L}_R(\lambda)$ (considering the partition with state matrix $(E - \lambda F) \otimes I_n$) were minimal at those $\lambda_0 \in \mathbb{F}$ such that $E - \lambda_0 F$ is singular then $\mathcal{L}_R(\lambda)$ would be a linearization of $R(\lambda)$ in $\mathbb{F}$ with state matrix $(E - \lambda F) \otimes I_n$. That means that the zeros of $\mathcal{L}_R(\lambda)$ would be the zeros of $R(\lambda)$, and the zeros of $(E - \lambda F) \otimes I_n$ would be the poles of $R(\lambda)$, together with their partial multiplicities. This happens, for instance, under the conditions of Theorem 5.2.8.

Remark 5.2.11. In Remark 5.2.10, we consider $\mathcal{L}_R(\lambda)$ from two different points of view: as a block full rank pencil, $\begin{bmatrix} M(\lambda) \\ \hdashline K_1(\lambda) \end{bmatrix}$, and as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$. In the former case, $\mathcal{L}_R(\lambda)$ is not in general a linearization at infinity of $R(\lambda)$ since $\operatorname{rev}_1 K_1(\lambda)$ does not have full row rank at 0. In particular, Theorem 4.3.7 can not be applied and there is not always an integer g such that $\operatorname{rev}_1 \mathcal{L}_R(\lambda)$ is equivalent at 0 to $\operatorname{diag}(\operatorname{rev}_g R(\lambda), I_{k(n-1)+\sum_{i=1}^s \ell_i n})$. It is not difficult to construct examples where such a g does not exist. In the latter case, $\mathcal{L}_R(\lambda)$ is not a linearization at infinity since $\operatorname{rev}_1 \mathcal{L}_R(\lambda)$ is not minimal at 0. Both cases are due to the fact that the matrix $\operatorname{rev}_1(E - \lambda F)$ does not have full row rank at zero since F is singular.

5.2.1 Low-rank structure

Low-rank structures are exploited in [60] for constructing smaller linearizations that allow more efficient computations. In particular, a trimmed linearization is constructed if the matrix coefficients $C_i - \lambda D_i$ in (5.17) have low rank. For this purpose, write

$$C_i - \lambda D_i = (\widetilde{C}_i - \lambda \widetilde{D}_i)\widetilde{Z}_i^*, \tag{5.31}$$

with $\widetilde{C}_i, \widetilde{D}_i, \widetilde{Z}_i \in \mathbb{F}^{n \times k_i}$, and $\widetilde{Z}_i^* \widetilde{Z}_i = I_{k_i}$. In several applied problems this type of structure appears with $k_i \ll n$ [47, 60]. By using the expression (5.31) for the matrix coefficients, the matrix $R(\lambda)$ in (5.28) can be written as:

$$\begin{aligned} R(\lambda) &= \sum_{i=0}^{k-1}(A_i - \lambda B_i) f_i(\lambda) + \sum_{i=1}^{s}(\widetilde{C}_i - \lambda \widetilde{D}_i)\widetilde{Z}_i^* a_i^T (E_i - \lambda F_i)^{-1} b_i \\ &= \sum_{i=0}^{k-1}(A_i - \lambda B_i)(f_i(\lambda) \otimes I_n) + \sum_{i=1}^{s}[a_i^T \otimes (\widetilde{C}_i - \lambda \widetilde{D}_i)]((E_i - \lambda F_i)^{-1} b_i \otimes I_{k_i})\widetilde{Z}_i^*. \end{aligned} \tag{5.32}$$

Then, the trimmed linearization $\widetilde{\mathcal{L}}_R(\lambda)$ for $R(\lambda)$ constructed in [60] is the following.

Definition 5.2.12. *[60](Trimmed CORK linearization for AAA rational approximation) Let $R(\lambda)$ be a rational matrix as in* (5.32). *Consider the matrices*

$$\begin{aligned} Z &:= \begin{bmatrix} -\widetilde{Z}_1(b_1^* \otimes I_{k_1}) & \cdots & -\widetilde{Z}_s(b_s^* \otimes I_{k_s}) \end{bmatrix}, \\ E &:= \operatorname{diag}(E_1 \otimes I_{k_1}, \ldots, E_s \otimes I_{k_s}), \text{ and} \\ F &:= \operatorname{diag}(F_1 \otimes I_{k_1}, \ldots, F_s \otimes I_{k_s}). \end{aligned}$$

Then a trimmed CORK linearization for $R(\lambda)$ is

$$\widetilde{\mathcal{L}}_R(\lambda) = \left[\begin{array}{cc|c} A_0 - \lambda B_0 \ \cdots & A_{k-1} - \lambda B_{k-1} & a_1^T \otimes (\widetilde{C}_1 - \lambda \widetilde{D}_1) \ \cdots \ a_s^T \otimes (\widetilde{C}_s - \lambda \widetilde{D}_s) \\ \hdashline \multicolumn{2}{c|}{(X - \lambda Y) \otimes I_n} & 0 \\ \hline Z^* & 0 & E - \lambda F \end{array}\right],$$

where $\left[\begin{array}{c} A_0 - \lambda B_0 \ \cdots \ A_{k-1} - \lambda B_{k-1} \\ \hdashline (X - \lambda Y) \otimes I_n \end{array}\right]$ *is any CORK linearization of $Q(\lambda)$.*

Notice that the linearization $\mathcal{L}_R(\lambda)$ in Definition 5.2.5 has size $(kn + \sum_{i=1}^{s} \ell_i n) \times (kn + \sum_{i=1}^{s} \ell_i n)$ whereas the trimmed pencil $\widetilde{\mathcal{L}}_R(\lambda)$ in Definition 5.2.12 has size $(kn + \sum_{i=1}^{s} \ell_i k_i) \times (kn + \sum_{i=1}^{s} \ell_i k_i)$ with $k_i \ll n$ in several applications.

Analogous to what we did in Theorem 5.2.6, we study in Theorem 5.2.13 the structure of $\widetilde{\mathcal{L}}_R(\lambda)$ as linearization of $R(\lambda)$. The proof is omitted since it is analogous to that of Theorem 5.2.6.

Theorem 5.2.13. *Let $R(\lambda)$ be a rational matrix as in* (5.32), *and let $\widetilde{\mathcal{L}}_R(\lambda)$ be the matrix pencil in Definition 5.2.12. Let $\Omega \subseteq \mathbb{F}$ be nonempty. If $\widetilde{\mathcal{L}}_R(\lambda)$, viewed as a polynomial system matrix with state matrix $E - \lambda F$, is minimal in Ω then $\widetilde{\mathcal{L}}_R(\lambda)$ is a block full rank linearization of $R(\lambda)$ in Ω with state matrix $E - \lambda F$.*

Remark 5.2.14. As we discussed in Remark 5.2.11 for the matrix pencil $\mathcal{L}_R(\lambda)$, the trimmed CORK linearization $\widetilde{\mathcal{L}}_R(\lambda)$ is not in general a linearization at infinity of $R(\lambda)$ either. The reason is that, in this case, the matrix F is also singular and $\operatorname{rev}_1(E - \lambda F)$ has not full row rank at zero.

Chapter 6

Linearizations of arbitrary rational transfer functions

In this chapter, we construct a family of linearizations of rational matrices in the sense of the definitions introduced in Chapter 4. For that purpose, rational matrices $R(\lambda)$ are written as transfer function matrices from general realizations. That is, of the form

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda), \tag{6.1}$$

where $D(\lambda)$, $C(\lambda)$, $B(\lambda)$ and $A(\lambda)$ are arbitrary polynomial matrices of appropriate sizes. For any rational matrix $R(\lambda)$, a representation as in (6.1) always exists and is not unique. The new linearizations are constructed from linearizations of the polynomial matrices $D(\lambda)$ and $A(\lambda)$, where each of them can be represented in terms of any polynomial basis. In particular, the notion of (strong) block minimal bases pencil (recall Definition 2.5.1) will be our main tool for building the linearizations. We will only consider the so-called degenerate (strong) block minimal bases pencils, that is, (strong) block minimal bases pencils of the form

$$L(\lambda) = \begin{bmatrix} M(\lambda) \\ K(\lambda) \end{bmatrix}, \tag{6.2}$$

where $K(\lambda)$ is a minimal basis. We know by Theorem 2.5.2 that the linear polynomial matrix in (6.2) is a (strong) linearization of the polynomial matrix

$$P(\lambda) = M(\lambda)N(\lambda)^T$$

considered as a polynomial of grade $1 + \deg N(\lambda)$, where $N(\lambda)$ is a minimal basis dual to $K(\lambda)$ (recall Definition 2.3.4). If $d := \deg N(\lambda) + 1$ and $P(\lambda)$ is of size $m \times n$, then $M(\lambda)$ and $K(\lambda)$ in (6.2) are, respectively, $m \times dn$ and $(d-1)n \times dn$ linear polynomial matrices [26].

Unlike other lineariations for rational matrices, as those in [27] (Chapter 3) or in [2, 7, 79], the construction of the linearizations in this chapter do not require neither to write the corresponding rational matrix $R(\lambda)$ as the sum of its polynomial part and its strictly proper part nor to express the strictly proper rational part in state-space form. We finish the chapter by showing how to recover eigenvectors, when the rational matrix $R(\lambda)$ is regular, and minimal bases and minimal indices, when $R(\lambda)$ is singular, from those of their linearizations in this family. All the results in this chapter appear in [75].

6.1 Linearizations in a set

Throughout this chapter, $\mathbb{F}$ denotes any algebraically closed field. Theorem 6.1.2 is the main result in this section, where we construct (local) linearizations for rational matrices that are represented with general realizations as in (6.1). To prove Theorem 6.1.2, we will use Lemma 6.1.1, whose simple proof is omitted.

Lemma 6.1.1. *A polynomial matrix of the form*

$$\begin{bmatrix} X(\lambda) & A(\lambda) & Y(\lambda) & B(\lambda) \\ I_s & 0 & 0 & 0 \\ Z(\lambda) & -C(\lambda) & W(\lambda) & D(\lambda) \\ 0 & 0 & I_t & 0 \end{bmatrix}$$

is unimodularly equivalent to $\operatorname{diag}\left(\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}, I_{s+t}\right)$.

Theorem 6.1.2. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Let*

$$L_A(\lambda) = \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix} \quad \text{and} \quad L_D(\lambda) = \begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix} \tag{6.3}$$

be block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively. Let $N_A(\lambda)$ and $N_D(\lambda)$ be minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively. Consider linear polynomial matrices $M_C(\lambda)$ and $M_B(\lambda)$ such that

$$M_C(\lambda)N_A(\lambda)^T = C(\lambda) \quad \text{and} \quad M_B(\lambda)N_D(\lambda)^T = B(\lambda), \tag{6.4}$$

and the linear polynomial system matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right], \tag{6.5}$$

with state matrix $L_A(\lambda)$. If the matrices

$$\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \tag{6.6}$$

have no eigenvalues in a nonempty set $\Omega \subseteq \mathbb{F}$, then $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω.

Remark 6.1.3. Before giving the proof of Theorem 6.1.2, we recall that there exist unimodular polynomial matrices of the form

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i(\lambda) \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}, \tag{6.7}$$

for $i \in \{A, D\}$; see [26, Theorem 2.10].

Proof. Throughout the proof, we use the notation $\rho_A := \deg N_A(\lambda)$ and $\rho_D := \deg N_D(\lambda)$.

To prove that $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω, we will use the spectral characterization in Theorem 4.2.6 of linearizations in a set.

Let $P(\lambda) := \left[\begin{smallmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{smallmatrix}\right]$ be a polynomial system matrix of $R(\lambda)$. Notice that $P(\lambda)$ is minimal in Ω by hypothesis. First, we have

$$\begin{aligned} \text{pole elem. div. of } R(\lambda) \text{ in } \Omega =& \text{elem. div. of } A(\lambda) \text{ in } \Omega \\ =& \text{elem. div. of } L_A(\lambda) \text{ in } \Omega, \end{aligned}$$

since $P(\lambda)$ is minimal in Ω and $L_A(\lambda)$ is a linearization of $A(\lambda)$. Hence, the pole elementary divisors of $R(\lambda)$ in Ω are equal to the elementary divisors of $L_A(\lambda)$ in Ω.

Second, we consider Remark 6.1.3 and notice that $K_A(\lambda)\widehat{N}_A(\lambda)^T = I_{n\rho_A}$ and $K_D(\lambda)\widehat{N}_D(\lambda)^T = I_{m\rho_D}$, as this will be important in what follows. Then, multiplying $\mathcal{L}(\lambda)$ on the right by the unimodular matrix $U(\lambda) = \operatorname{diag}(U_A(\lambda)^{-1}, U_D(\lambda)^{-1})$, we get

$$\mathcal{L}(\lambda)U(\lambda) = \left[\begin{array}{cc|cc} X_{AA}(\lambda) & A(\lambda) & X_{BD}(\lambda) & B(\lambda) \\ I_{n\rho_A} & 0 & 0 & 0 \\ \hline -X_{CA}(\lambda) & -C(\lambda) & X_{DD}(\lambda) & D(\lambda) \\ 0 & 0 & I_{m\rho_D} & 0 \end{array}\right], \tag{6.8}$$

where

$$X_{ij}(\lambda) := M_i(\lambda)\widehat{N}_j(\lambda)^T, \tag{6.9}$$

with $i \in \{A, B, C, D\}$, and $j := A$ if $i \in \{A, C\}$ and $j := D$ if $i \in \{B, D\}$. Since the matrices in (6.6) have no eigenvalues in Ω, we have that $\mathcal{L}(\lambda)U(\lambda)$ is minimal in Ω and, therefore, $\mathcal{L}(\lambda)$ is minimal in Ω. By Lemma 6.1.1, $\mathcal{L}(\lambda)U(\lambda)$ is unimodularly equivalent to $\operatorname{diag}(P(\lambda), I_{n\,\rho_A+m\,\rho_D})$. Hence, $\mathcal{L}(\lambda)$ is a linearization of the polynomial system matrix $P(\lambda)$. As a consequence of this, we have

$$\begin{aligned}\text{zero elem. div. of } R(\lambda) \text{ in } \Omega =& \text{elem. div. of } P(\lambda) \text{ in } \Omega \\ =& \text{elem. div. of } \mathcal{L}(\lambda) \text{ in } \Omega,\end{aligned}$$

since $P(\lambda)$ is minimal in Ω and $\mathcal{L}(\lambda)$ is a linearization of $P(\lambda)$. Therefore, the zero elementary divisors of $R(\lambda)$ in Ω are equal to the elementary divisors of $P(\lambda)$ in Ω.

Since $\mathcal{L}(\lambda)$ is of size $(n+p+s)\times(n+m+s)$, where $s = n\rho_A + m\rho_D$, to finish the proof, it suffices to notice that

$$\begin{aligned}\operatorname{nrank}\mathcal{L}(\lambda) &= \operatorname{nrank}P(\lambda) + n\rho_A + m\rho_D && \text{(by (6.8))}\\ &= \operatorname{nrank}R(\lambda) + n + n\rho_A + m\rho_D && \text{(by (2.9))}.\end{aligned}$$

By Theorem 4.2.6, we conclude that $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω. ■

Remark 6.1.4. Notice that, in Theorem 6.1.2, we assume that the polynomial matrices $A(\lambda)$ and $C(\lambda)$ can be written in terms of the same dual minimal basis $N_A(\lambda)$. That is, that there exist pencils $M_A(\lambda)$ and $M_C(\lambda)$ such that $M_A(\lambda)N_A(\lambda)^T = A(\lambda)$ and $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$, respectively. Such pencils always exist if we consider $A(\lambda)$ and $C(\lambda)$ as polynomial matrices with the same grade g, for instance,

$$g = \max\{\deg A(\lambda), \deg C(\lambda)\},$$

see, for example, [31, Lemma 5.2]. A similar remark applies to the matrix polynomials $B(\lambda)$ and $D(\lambda)$. We can see an example in Example 6.1.5.

In Example 6.1.5, we show how to use Theorem 6.1.2 to construct linearizations of a rational matrix of the form (6.1). For simplicity, we assume that the polynomial matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are expressed in the monomial basis. But we emphasize that the construction can be easily adapted to many other polynomial bases (Chebyshev, Lagrange, Newton, etc); see Section 6.5 for an example in the Chebyshev basis.

Example 6.1.5. *Let us consider a rational matrix of the form*

$$\begin{aligned}R(\lambda) =& D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) = \\ & D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0 + \\ & \qquad (C_1\lambda + C_0)(A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)^{-1}(B_2\lambda^2 + B_1\lambda + B_0),\end{aligned}$$

where $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is regular, and $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. We will use block Kronecker pencils [26, Section 4], which are particular cases of block minimal bases pencils, to construct the linearizations $L_A(\lambda)$ and $L_D(\lambda)$

in Theorem 6.1.2. We recall that the construction of block Kronecker pencils involves a pair of dual minimal bases of the form

$$K(\lambda) = \begin{bmatrix} -I_s & I_s\lambda & 0 & \cdots & 0 \\ 0 & -I_s & I_s\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -I_s & I_s\lambda \end{bmatrix} \quad \text{and} \quad N(\lambda)^T = \begin{bmatrix} I_s\lambda^{d-1} \\ \vdots \\ I_s\lambda \\ I_s \end{bmatrix}.$$

To construct the linearization $L_A(\lambda)$ and the linear polynomial matrix $M_C(\lambda)$, we need to see both $A(\lambda)$ and $C(\lambda)$ as polynomial matrices of grade

$$\max\{\deg A(\lambda), \deg C(\lambda)\} = \max\{3, 1\} = 3.$$

Then, we can use, for example,

$$L_A(\lambda) := \begin{bmatrix} A_3\lambda + A_2 & A_1 & A_0 \\ -I_n & I_n\lambda & 0 \\ 0 & -I_n & I_n\lambda \end{bmatrix} \quad \text{and} \quad M_C(\lambda) := \begin{bmatrix} 0 & C_1 & C_0 \end{bmatrix},$$

with $N_A(\lambda) := \begin{bmatrix} I_n\lambda^2 & I_n\lambda & I_n \end{bmatrix}$. Similarly, to construct the linearization $L_D(\lambda)$ and the linear polynomial matrix $M_B(\lambda)$, we need to see both $D(\lambda)$ and $B(\lambda)$ as polynomial matrices of grade

$$\max\{\deg D(\lambda), \deg B(\lambda)\} = \max\{3, 2\} = 3.$$

Then, we can use, for instance,

$$L_D(\lambda) := \begin{bmatrix} D_3\lambda + D_2 & D_1 & D_0 \\ -I_m & I_m\lambda & 0 \\ 0 & -I_m & I_m\lambda \end{bmatrix} \quad \text{and} \quad M_B(\lambda) := \begin{bmatrix} 0 & B_2\lambda + B_1 & B_0 \end{bmatrix},$$

with $N_D(\lambda) := \begin{bmatrix} I_m\lambda^2 & I_m\lambda & I_m \end{bmatrix}$. Then the linear polynomial system matrix $\mathcal{L}(\lambda)$ in Theorem 6.1.2 is

$$\mathcal{L}(\lambda) = \left[\begin{array}{ccc|ccc} A_3\lambda + A_2 & A_1 & A_0 & 0 & B_2\lambda + B_1 & B_0 \\ -I_n & I_n\lambda & 0 & 0 & 0 & 0 \\ 0 & -I_n & I_n\lambda & 0 & 0 & 0 \\ \hline 0 & -C_1 & -C_0 & D_3\lambda + D_2 & D_1 & D_0 \\ 0 & 0 & 0 & -I_m & I_m\lambda & 0 \\ 0 & 0 & 0 & 0 & -I_m & I_m\lambda \end{array}\right].$$

Theorem 6.1.2 guarantees that $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ wherever the polynomial matrices $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix}$ do not have eigenvalues.

6.2 Linearizations at infinity

Theorem 6.2.1 is the main result of this section. It shows that, under some mild conditions, the local linearizations introduced in Section 6.1 are also linearizations at infinity of the rational matrix $R(\lambda)$.

Theorem 6.2.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Let*

$$L_A(\lambda) = \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix} \quad \text{and} \quad L_D(\lambda) = \begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$$

be strong block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively. Let $N_A(\lambda)$ and $N_D(\lambda)$ be minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and denote $\rho_A := \deg N_A(\lambda)$, $\rho_D := \deg N_D(\lambda)$. Consider linear polynomial matrices $M_C(\lambda)$ and $M_B(\lambda)$ such that

$$M_C(\lambda)N_A(\lambda)^T = C(\lambda) \quad \text{and} \quad M_B(\lambda)N_D(\lambda)^T = B(\lambda),$$

and the linear polynomial system matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right], \tag{6.10}$$

with state matrix $L_A(\lambda)$. If the matrices

$$\begin{bmatrix} \operatorname{rev}_{\rho_A+1} A(\lambda) \\ \operatorname{rev}_{\rho_A+1} C(\lambda) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \operatorname{rev}_{\rho_A+1} A(\lambda) & \operatorname{rev}_{\rho_D+1} B(\lambda) \end{bmatrix} \tag{6.11}$$

have no eigenvalues at 0, then $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at infinity of grade $\rho_D + 1$.

Remark 6.2.2. Since $L_A(\lambda)$ and $L_D(\lambda)$ are strong block minimal bases pencils, we recall that there exist unimodular polynomial matrices of the form

$$\widetilde{U}_i(\lambda) = \begin{bmatrix} \operatorname{rev}_1 K_i(\lambda) \\ \widetilde{K}_i(\lambda) \end{bmatrix} \quad \text{and} \quad \widetilde{U}_i(\lambda)^{-1} = \begin{bmatrix} \widetilde{N}_i(\lambda)^T & \operatorname{rev}_{\rho_i} N_i(\lambda)^T \end{bmatrix}, \tag{6.12}$$

for $i \in \{A, D\}$; see [26, Theorem 2.10].

Proof. To prove that $\mathcal{L}(\lambda)$ is a linearization at infinity of grade g of $R(\lambda)$, we will use the spectral characterization in Theorem 4.2.14 of linearizations at infinity.

Let $g := \rho_D + 1$. Let us consider the polynomial system matrix

$$\widetilde{P}(\lambda) := \begin{bmatrix} \operatorname{rev}_{\rho_A+1} A(\lambda) & \operatorname{rev}_{\rho_D+1} B(\lambda) \\ -\operatorname{rev}_{\rho_A+1} C(\lambda) & \operatorname{rev}_{\rho_D+1} D(\lambda) \end{bmatrix}, \tag{6.13}$$

with state matrix $\operatorname{rev}_{\rho_A+1} A(\lambda)$. We observe that the transfer function of the polynomial system matrix (6.13) is

$$\operatorname{rev}_{\rho_D+1} D(\lambda) + \operatorname{rev}_{\rho_A+1} C(\lambda) \left(\operatorname{rev}_{\rho_A+1} A(\lambda)\right)^{-1} \operatorname{rev}_{\rho_D+1} B(\lambda) = \operatorname{rev}_g R(\lambda).$$

Hence, we have

$$\begin{aligned} \text{pole elem. div. of } \operatorname{rev}_g R(\lambda) \text{ at } 0 &= \text{elem. div. of } \operatorname{rev}_{\rho_A+1} A(\lambda) \text{ at } 0 \\ &= \text{elem. div. of } \operatorname{rev}_1 L_A(\lambda) \text{ at } 0, \end{aligned}$$

since $\widetilde{P}(\lambda)$ is minimal at 0 and $L_A(\lambda)$ is a strong linearization of $A(\lambda)$. Thus, the pole elementary divisors of $\operatorname{rev}_g R(\lambda)$ at 0 are equal to the elementary divisors of $\operatorname{rev}_1 L_A(\lambda)$ at 0.

Consider the unimodular matrix $\widetilde{U}(\lambda) = \operatorname{diag}(\widetilde{U}_A(\lambda)^{-1}, \widetilde{U}_D(\lambda)^{-1})$, where $\widetilde{U}_A(\lambda)^{-1}$ and $\widetilde{U}_D(\lambda)^{-1}$ are defined in (6.12). Multiplying $\operatorname{rev}_1 \mathcal{L}(\lambda)$ on the right by $\widetilde{U}(\lambda)$, we get

$$\operatorname{rev}_1 \mathcal{L}(\lambda)\, \widetilde{U}(\lambda) = \left[\begin{array}{cc|cc} * & \operatorname{rev}_{\rho_A+1} A(\lambda) & * & \operatorname{rev}_{\rho_D+1} B(\lambda) \\ I_{n\rho_A} & 0 & 0 & 0 \\ \hline * & -\operatorname{rev}_{\rho_A+1} C(\lambda) & * & \operatorname{rev}_{\rho_D+1} D(\lambda) \\ 0 & 0 & I_{m\rho_D} & 0 \end{array}\right], \tag{6.14}$$

where $*$ denotes polynomial matrices that are not important for the argument. Since the matrices in (6.11) have no eigenvalues at 0, we have that $\operatorname{rev}_1 \mathcal{L}(\lambda)\, \widetilde{U}(\lambda)$ is minimal at 0 and, therefore, $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is minimal at 0. By Lemma 6.1.1, the matrix polynomial $\operatorname{rev}_1 \mathcal{L}(\lambda)\, \widetilde{U}(\lambda)$ is unimodularly equivalent to $\operatorname{diag}(\widetilde{P}(\lambda), I_{n\rho_A+m\rho_D})$. Hence, $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is a linearization of $\widetilde{P}(\lambda)$ at 0. Thus, we have

$$\begin{aligned} \text{zero elem. div. of } \operatorname{rev}_g R(\lambda) \text{ at } 0 &= \text{elem. div. of } \widetilde{P}(\lambda) \text{ at } 0 \\ &= \text{elem. div. of } \operatorname{rev}_1 \mathcal{L}(\lambda) \text{ at } 0, \end{aligned}$$

since $\widetilde{P}(\lambda)$ is minimal at 0 and $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is a linearization of $\widetilde{P}(\lambda)$. Therefore, the zero elementary divisors of $\operatorname{rev}_g R(\lambda)$ at 0 are equal to the elementary divisors of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ at 0.

To finish the proof, it suffices to notice that

$$\operatorname{nrank} \operatorname{rev}_1 \mathcal{L}(\lambda) = \operatorname{nrank} \widetilde{P}(\lambda) + n\rho_A + m\rho_D \qquad \text{(by (6.14))}$$

$$= \operatorname{nrank} \operatorname{rev}_g R(\lambda) + n + n\rho_A + m\rho_D \qquad \text{(by (2.9))}.$$

Conclusively, by Theorem 4.2.14, the linear polynomial system matrix $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at infinity of grade $g = \rho_D + 1$. ■

We conclude this section by noting that the linearization $\mathcal{L}(\lambda)$ constructed in Theorem 6.1.2 contains the spectral information of the rational matrix $R(\lambda)$ in a nonempty set Ω, whenever certain minimality conditions are satisfied. More precisely, if the matrix polynomials in (6.6) have no eigenvalues in Ω then the zeros in Ω of $R(\lambda)$ are the eigenvalues in Ω of $\mathcal{L}(\lambda)$, and the poles in Ω of $R(\lambda)$ are the eigenvalues in Ω of the block minimal bases pencil $L_A(\lambda)$. If $\Omega = \mathbb{F}$ then we can recover the complete information about finite poles and zeros of $R(\lambda)$ from $\mathcal{L}(\lambda)$. If, in addition, the matrix polynomials (6.11) have no eigenvalues at zero, then $\mathcal{L}(\lambda)$ is a strong linearization of grade $\rho_D + 1$ and, hence, by Proposition 4.2.15, we can also recover the complete zero and pole information of $R(\lambda)$ at infinity.

6.3 An illustrative example

Let us consider an $m \times m$ rational matrix of the form

$$R(\lambda) = D(\lambda) + f_1(\lambda)K_1 + f_2(\lambda)K_2, \tag{6.15}$$

where $D(\lambda)$ is a polynomial matrix of degree 2, $f_1(\lambda) = \frac{(\lambda^2+1)(\lambda+2)}{\lambda^2-\lambda-2}$, $f_2(\lambda) = \frac{(\lambda^2-1)\lambda^2}{\lambda+2}$, and K_1 and K_2 are constant matrices having ranks r_1 and r_2, respectively. For $i = 1, 2$, we can write $K_i = c_i b_i^T$, for some c_i and b_i both of size $m \times r_i$. Then, a realization of $R(\lambda)$ as in (6.1) is

$$R(\lambda) = D(\lambda) + \underbrace{\begin{bmatrix} c_1(\lambda^2+1) & c_2(\lambda^2-1) \end{bmatrix}}_{:=C(\lambda)} \underbrace{\begin{bmatrix} I_{r_1}(\lambda^2-\lambda-2) & 0 \\ 0 & I_{r_2}(\lambda+2) \end{bmatrix}^{-1}}_{:=A(\lambda)^{-1}} \underbrace{\begin{bmatrix} b_1^T(\lambda+2) \\ b_2^T\lambda^2 \end{bmatrix}}_{:=B(\lambda)}.$$

Notice that the polynomial matrices $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ have degree 2. Hence, we can write $A(\lambda) := A_2\lambda^2 + A_1\lambda + A_0$, $B(\lambda) := B_2\lambda^2 + B_1\lambda + B_0$, $C(\lambda) := C_2\lambda^2 + C_1\lambda + C_0$ and $D(\lambda) := D_2\lambda^2 + D_1\lambda + D_0$. Then, we set $r := r_1 + r_2$, and we define the linear polynomial system matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|cc} A_2\lambda + A_1 & A_0 & B_2\lambda + B_1 & B_0 \\ -I_r & I_r\lambda & 0 & 0 \\ \hline -C_2\lambda - C_1 & -C_0 & D_2\lambda + D_1 & D_0 \\ 0 & 0 & -I_m & I_m\lambda \end{array}\right].$$

It can be proved that condition (6.6) is satisfied for all $\lambda_0 \in \mathbb{F}$ and that condition (6.11) is also satisfied. Thus, by Theorems 6.1.2 and 6.2.1, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in $\mathbb{F}$ and also a linearization of $R(\lambda)$ at infinity of grade 2.

Notice that the rational functions $f_1(\lambda)$ and $f_2(\lambda)$ in (6.15) are not (strictly) proper. Nevertheless, we have been able to construct a strong linearization in the sense of [28] for $R(\lambda)$ without decomposing $R(\lambda)$ as the sum of its polynomial part and its strictly proper rational part and without considering a minimal state space realalization of the strictly proper part, in contrast to the methods appearing in [2, 7, 27, 79]. We emphasize that, in order to construct linearizations of rational matrices, we must take into account that a realization as in (6.1) is not unique and that the ideal goal is to consider a realization easy to build from the original expression of the corresponding rational matrix without performing many computations.

6.4 Recovery of eigenvectors, minimal bases and minimal indices

In this section, we show how to recover the elements in the right and left nullspaces of a rational matrix $R(\lambda)$ from those in the right and left nullspaces of a linearization $\mathcal{L}(\lambda)$ as in Theorem 6.1.2, as well as minimal bases and minimal indices, assuming $R(\lambda)$ is singular. In addition, we show how to recover right and left eigenvectors in the regular case. To do this, we consider the transfer function matrix $\widehat{R}(\lambda)$ of $\mathcal{L}(\lambda)$ and a polynomial system matrix $P(\lambda)$ of $R(\lambda)$, and we study the relation between the elements in their right and left nullspaces.

6.4.1 One-sided factorizations

Theorem 6.4.1 is the only result in this subsection. It establishes a relation, in terms of one-side factorizations, between a rational matrix $R(\lambda)$ and the transfer function matrix $\widehat{R}(\lambda)$ of a linearization for $R(\lambda)$ as in Theorem 6.1.2.

Theorem 6.4.1. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} \begin{matrix} M_A(\lambda) \\ K_A(\lambda) \end{matrix} & \begin{matrix} M_B(\lambda) \\ 0 \end{matrix} \\ \hline \begin{matrix} -M_C(\lambda) \\ 0 \end{matrix} & \begin{matrix} M_D(\lambda) \\ K_D(\lambda) \end{matrix} \end{array}\right],$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ are block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively, with $N_A(\lambda)$ and $N_D(\lambda)$ being minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and $M_C(\lambda)$ and $M_B(\lambda)$ being linear *polynomial matrices such that* $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ *and* $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$*, respectively. Let* $\widehat{R}(\lambda)$ *be the transfer function matrix of* $\mathcal{L}(\lambda)$*. Then, we have the following one-sided factorizations*

$$\widehat{R}(\lambda)N_D(\lambda)^T = \begin{bmatrix} R(\lambda) \\ 0 \end{bmatrix}, \tag{6.16}$$

and

$$\begin{bmatrix} I_p & -M_R(\lambda)\widehat{N}_D(\lambda)^T \end{bmatrix} \widehat{R}(\lambda) = R(\lambda)\widehat{K}_D(\lambda), \tag{6.17}$$

where $M_R(\lambda) := M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda)$*, and* $\widehat{K}_D(\lambda)$ *and* $\widehat{N}_D(\lambda)$ *are defined in* (6.7).

Proof. Notice that the matrix pencil $L_A(\lambda) = \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ is regular since $L_A(\lambda)$ is a linearization of $A(\lambda)$ and the polynomial matrix $A(\lambda)$ is regular. Then, from $L_A(\lambda)N_A(\lambda)^T = \begin{bmatrix} A(\lambda) \\ 0 \end{bmatrix}$, we obtain

$$\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = N_A(\lambda)^T A(\lambda)^{-1}. \tag{6.18}$$

Hence, the transfer function matrix of $\mathcal{L}(\lambda)$ is given by

$$\widehat{R}(\lambda) = \begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix} + \begin{bmatrix} M_C(\lambda) \\ 0 \end{bmatrix} \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} M_B(\lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda) \\ K_D(\lambda) \end{bmatrix}, \tag{6.19}$$

where we have used $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$.

Multiplying (6.19) on the right by $N_D(\lambda)^T$ yields

$$\widehat{R}(\lambda)N_D(\lambda)^T = \begin{bmatrix} D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} R(\lambda) \\ 0 \end{bmatrix},$$

where we have used $M_D(\lambda)N_D(\lambda)^T = D(\lambda)$ and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$. This establishes the right-sided factorization (6.16).

Multiplying (6.19) on the left by $\begin{bmatrix} I_p & -(M_D(\lambda)+C(\lambda)A(\lambda)^{-1}M_B(\lambda))\widehat{N}_D(\lambda)^T \end{bmatrix}$ gives

$$\begin{aligned} &\begin{bmatrix} I_p & -\left(M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda)\right)\widehat{N}_D(\lambda)^T \end{bmatrix} \widehat{R}(\lambda) = \\ &\qquad (M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda))(I_{(\rho_D+1)m} - \widehat{N}_D(\lambda)^T K_D(\lambda)) = \\ &\qquad (M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda))N_D(\lambda)^T \widehat{K}_D(\lambda) = R(\lambda)\widehat{K}_D(\lambda), \end{aligned}$$

where $\rho_D = \deg N_D(\lambda)$. This establishes the left-sided factorization (6.17). ∎

6.4.2 Recovery of minimal bases and minimal indices

In this section, we assume that the rational matrix $R(\lambda)$ is singular and show how to recover the right and left minimal indices and minimal bases of $R(\lambda)$ from those of a linearization $\mathcal{L}(\lambda)$ of $R(\lambda)$ as in Theorem 6.1.2.

We begin with Lemma 6.4.2, which establishes a bijection between the nullspaces of $R(\lambda)$ and the transfer function matrix of $\mathcal{L}(\lambda)$.

Lemma 6.4.2. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right],$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ are block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively, with $N_A(\lambda)$ and $N_D(\lambda)$ being minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and $M_C(\lambda)$ and $M_B(\lambda)$ being linear polynomial matrices such that $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$, respectively. Let $\widehat{R}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. Then the following statements hold:

(a) *The linear map*

$$\begin{aligned} M_r \,:\, \mathcal{N}_r(R) &\longrightarrow \mathcal{N}_r(\widehat{R}) \\ x(\lambda) &\longmapsto \widehat{x}(\lambda) := N_D(\lambda)^T x(\lambda) \end{aligned}$$

is a bijection between the right nullspaces of $R(\lambda)$ and $\widehat{R}(\lambda)$.

(b) *The linear map*

$$\begin{aligned} M_\ell \,:\, \mathcal{N}_\ell(R) &\longrightarrow \mathcal{N}_\ell(\widehat{R}) \\ y(\lambda)^T &\longmapsto \widehat{y}(\lambda)^T := y(\lambda)^T \begin{bmatrix} I_p & -M_R(\lambda)\widehat{N}_D(\lambda)^T \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces of $R(\lambda)$ and $\widehat{R}(\lambda)$, where $M_R(\lambda) := M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda)$ and $\widehat{N}_D(\lambda)$ is defined in (6.7).

Proof. We will prove part (a). Part (b) can be proved analogously.

That the map M_r is well-defined from the right nullspace of $R(\lambda)$ to the right nullspace $\widehat{R}(\lambda)$ follows immediately from the right-sided factorization (6.16). Notice that M_r is, in addition, linear. Moreover, by Theorem 6.1.2, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$, at least, at some point $\lambda_0 \in \mathbb{F}$. Indeed, since $A(\lambda)$ is regular, there exists $\lambda_0 \in \mathbb{F}$ such that $A(\lambda_0)$ is invertible. This implies that the realization $D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ is minimal at λ_0. Hence, we have $\dim \mathcal{N}_r(R) = \dim \mathcal{N}_r(\widehat{R})$. Thus, to show that M_r is a bijection, if suffices to show that it is injective. So, suppose $\widehat{y}(\lambda) = N_D(\lambda)^T y(\lambda) = 0$. Since $N_D(\lambda)$ is a minimal basis, $N_D(\lambda)^T$ has full column rank. Hence, $N_D(\lambda)^T y(\lambda) = 0$ implies $y(\lambda) = 0$. This establishes the injectivity of the linear map M_r. ■

Remark 6.4.3. Since the maps in Lemma 6.4.2 are bijections, they preserve linear independence and allow us to recover bases of the right (resp. left) nullspace of $R(\lambda)$ from bases of the right (resp. left) nullspace of $\widehat{R}(\lambda)$, and conversely.

By combining Lemmas 2.4.5 and 6.4.2, we obtain Theorem 6.4.4, which establishes a bijection between the nullspaces of $R(\lambda)$ and $\mathcal{L}(\lambda)$.

Theorem 6.4.4. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -\,M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right],$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ are block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively, with $N_A(\lambda)$ and $N_D(\lambda)$ being minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and $M_C(\lambda)$ and $M_B(\lambda)$ being linear polynomial matrices such that $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$, respectively. Let $\widehat{R}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. Then the following statements hold:

(a) *The linear map*

$$\begin{aligned} F_r \,:\, \mathcal{N}_r(R) &\longrightarrow \mathcal{N}_r(\mathcal{L}) \\ x(\lambda) &\longmapsto \widetilde{x}(\lambda) := \begin{bmatrix} -N_A(\lambda)^T A(\lambda)^{-1} B(\lambda) \\ N_D(\lambda)^T \end{bmatrix} x(\lambda) \end{aligned}$$

is a bijection between the right nullspaces of $R(\lambda)$ and $\mathcal{L}(\lambda)$.

(b) *The linear map*

$$F_\ell \,:\, \mathcal{N}_\ell(R) \longrightarrow \mathcal{N}_\ell(\mathcal{L})$$

$$y(\lambda)^T \longmapsto \widetilde{y}(\lambda)^T := y(\lambda)^T \begin{bmatrix} M_C(\lambda)L_A(\lambda)^{-1} & I_p & -M_R(\lambda)\widehat{N}_D(\lambda)^T \end{bmatrix}$$

is a bijection between the left nullspaces of $R(\lambda)$ *and* $\mathcal{L}(\lambda)$*, where* $L_A(\lambda) := \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$*,* $M_R(\lambda) := M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda)$ *and* $\widehat{N}_D(\lambda)$ *is defined in* (6.7).

Proof. We will prove part (a). Part (b) can be proved analogously.

Consider the linear bijections T_r and M_r in Lemma 2.4.5 and Lemma 6.4.2, respectively. Then, F_r is the composition $F_r = T_r \circ M_r$. Indeed, we have

$$\begin{aligned} F_r \,:\, \mathcal{N}_r(R) \longrightarrow \mathcal{N}_r(\widehat{R}) \quad & \longrightarrow \mathcal{N}_r(\mathcal{L}) \\ x(\lambda) \longmapsto N_D(\lambda)^T x(\lambda) \quad & \longmapsto \begin{bmatrix} -\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} M_B(\lambda) \\ 0 \end{bmatrix} N_D(\lambda)^T x(\lambda) \\ N_D(\lambda)^T x(\lambda) \end{bmatrix} = \\ & \begin{bmatrix} -N_A(\lambda)^T A(\lambda)^{-1} B(\lambda) x(\lambda) \\ N_D(\lambda)^T x(\lambda) \end{bmatrix}, \end{aligned}$$

where we have used (6.18) and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$.

■

Lemma 6.4.5 will allow us to prove that, under some minimality conditions, the linear polynomial system matrix $\mathcal{L}(\lambda)$ and its transfer function $\widehat{R}(\lambda)$ have the same right and left minimal indices.

Lemma 6.4.5. *Consider a linear polynomial system matrix*

$$L(\lambda) = \left[\begin{array}{c|c} A_1\lambda + A_0 & B_1\lambda + B_0 \\ \hline -\,(C_1\lambda + C_0) & D_1\lambda + D_0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)},$$

with state matrix $A_1\lambda + A_0 \in \mathbb{F}[\lambda]^{n\times n}$ *and transfer function matrix* $T(\lambda)$*. Then the following statements hold:*

(a) *If* $u(\lambda) = \begin{bmatrix} y(\lambda) \\ x(\lambda) \end{bmatrix} \in \mathcal{N}_r(L)$*, then* $x(\lambda) \in \mathcal{N}_r(T)$*. In addition, if* $u(\lambda)$ *is a polynomial vector and* rank $\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = n$*, then* $\deg u(\lambda) = \deg x(\lambda)$*.*

(b) *If* $v(\lambda)^T = \begin{bmatrix} \tilde{y}(\lambda)^T & \tilde{x}(\lambda)^T \end{bmatrix} \in \mathcal{N}_\ell(L)$*, then* $\tilde{x}(\lambda)^T \in \mathcal{N}_\ell(T)$*. In addition, if* $v(\lambda)$ *is a polynomial vector and* rank $\begin{bmatrix} A_1 & B_1 \end{bmatrix} = n$*, then* $\deg v(\lambda) = \deg \tilde{x}(\lambda)$*.*

Proof. The proof follows the same lines as the proof of [18, Theorem 6.8]. We only prove part (a) since part (b) follows from a similar argument. From Lemma 2.4.5, we obtain that if $u(\lambda) \in \mathcal{N}_r(L)$, then the vector $u(\lambda)$ must be of the form

$$u(\lambda) = \begin{bmatrix} y(\lambda) \\ x(\lambda) \end{bmatrix} = \begin{bmatrix} -(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)x(\lambda) \\ x(\lambda) \end{bmatrix},$$

for some $x(\lambda) \in \mathcal{N}_r(T)$, as we wanted to show.

For proving that $\deg u(\lambda) = \deg x(\lambda)$, we will show that $\deg y(\lambda) \leq \deg x(\lambda)$ by contradiction. Let us assume that $\ell := \deg y(\lambda) > \deg x(\lambda)$. Then, the vector $u(\lambda)$ must be of the form

$$u(\lambda) = \begin{bmatrix} y_\ell \\ 0 \end{bmatrix} \lambda^\ell + \text{ lower degree terms}, \quad \text{with } y_\ell \neq 0.$$

Since $u(\lambda) \in \mathcal{N}_r(L)$, we have

$$\begin{aligned} (A_1\lambda + A_0)y(\lambda) + (B_1\lambda + B_0)x(\lambda) = 0, \\ (C_1\lambda + C_0)y(\lambda) - (D_1\lambda + D_0)x(\lambda) = 0. \end{aligned}$$

Considering the highest degree terms in the left hand side of the two equations above, we obtain

$$\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} y_\ell = 0.$$

Since the matrix $\left[\begin{smallmatrix} A_1 \\ C_1 \end{smallmatrix}\right]$ has full column rank by assumption, we get $y_\ell = 0$, which contradicts our original hypothesis. ■

Remark 6.4.6. We emphasize that

$$\operatorname{rank} \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_1 & B_1 \end{bmatrix} = n \tag{6.20}$$

is the condition for a linear polynomial system matrix to be minimal at infinity in the sense of [28], which is also a necessary condition for a linear polynomial system matrix to be a linearization at infinity [28]. In Chapter 7 (or [32]) there is a procedure to reduce any linear polynomial system matrix to one satisfying condition (6.20).

Remark 6.4.7. We notice that Lemma 6.4.5 extends [18, Theorem 6.1]. But, while [18, Theorem 6.1] assumes the invertibility of A_1, we use the more general condition (6.20).

For completeness, in Lemma 6.4.8 we recall [92, Lemma 2] and different versions of it that can be analogously proved.

Lemma 6.4.8. *Let* $\begin{bmatrix} X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = 0.$

(a) *Assume that* $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ *has full column rank.*

(a1) *If X_1 has full column rank, then Y_2 also has full column rank.*

(a2) *If X_2 has full column rank, then Y_1 also has full column rank.*

(b) *Assume that* $\begin{bmatrix} X_1 & X_2 \end{bmatrix}$ *has full row rank.*

(b1) *If* Y_1 *has full row rank, then* X_2 *also has full row rank.*

(b2) *If* Y_2 *has full row rank, then* X_1 *also has full row rank.*

We are finally ready to state and prove the main results of this section, Theorems 6.4.9 and 6.4.11. These theorems show how to recover right and left minimal bases and minimal indices of rational matrices from those of their linearizations in Theorem 6.1.2.

Theorem 6.4.9 (Right minimal bases and minimal indices)**.** *Let* $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix expressed in the form* $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$*, for some regular polynomial matrix* $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ *and polynomial matrices* $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$*,* $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ *and* $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$*. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(n(1+\rho_A)+p+m\rho_D)\times(n(\rho_A+1)+m(\rho_D+1))},$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ *and* $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ *are strong block minimal bases linearizations of* $A(\lambda)$ *and* $D(\lambda)$*, respectively, with* $N_A(\lambda)$ *and* $N_D(\lambda)$ *being minimal bases dual to* $K_A(\lambda)$ *and* $K_D(\lambda)$*, respectively, and* $M_C(\lambda)$ *and* $M_B(\lambda)$ *being linear polynomial matrices such that* $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ *and* $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$*, respectively. Define* $\rho_i := \deg N_i(\lambda)$*, for* $i \in \{A, D\}$*, and let* $\widehat{R}(\lambda)$ *be the transfer function matrix of* $\mathcal{L}(\lambda)$*. If*

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = n \quad \text{for all} \quad \lambda_0 \in \mathbb{F}, \quad \text{and} \tag{6.21}$$

$$\operatorname{rank} \begin{bmatrix} \operatorname{rev}_{\rho_A+1} A(0) \\ \operatorname{rev}_{\rho_A+1} C(0) \end{bmatrix} = n, \tag{6.22}$$

then the following statements hold:

(a) *If* $\left\{ \begin{bmatrix} y_i(\lambda) \\ x_i(\lambda) \end{bmatrix} \right\}_{i=1}^{s}$ *is a right minimal basis of* $\mathcal{L}(\lambda)$*, where* $x_i(\lambda) \in \mathbb{F}[\lambda]^{(\rho_D+1)m}$*, then* $\{x_i(\lambda)\}_{i=1}^{s}$ *is a right minimal basis of* $\widehat{R}(\lambda)$*, and there exists a right minimal basis* $\{u_i(\lambda)\}_{i=1}^{s}$ *of* $R(\lambda)$ *such that* $x_i(\lambda) = N_D(\lambda)^T u_i(\lambda)$*, for* $i = 1, \ldots, s$*.*

(b) *If* $\epsilon_1 \leq \cdots \leq \epsilon_s$ *are the right minimal indices of* $\mathcal{L}(\lambda)$*, then* $\epsilon_1 \leq \cdots \leq \epsilon_s$ *are the right minimal indices of* $\widehat{R}(\lambda)$*, and* $\epsilon_1 - \rho_D \leq \cdots \leq \epsilon_s - \rho_D$ *are the right minimal indices of* $R(\lambda)$*.*

Proof. See Appendix A. ■

Remark 6.4.10. We recall that the polynomial matrix $\widehat{K}_D(\lambda)$ in (6.7) is the left polynomial matrix inverse of $N_D(\lambda)^T$. Hence, from the right minimal basis $\{x_i(\lambda)\}_{i=1}^s$ of $\widehat{R}(\lambda)$ in part (a) of Theorem 6.4.9, we can recover a right minimal basis of the rational matrix $R(\lambda)$ as $\{u_i(\lambda)\}_{i=1}^s = \{\widehat{K}_D(\lambda)x_i(\lambda)\}_{i=1}^s$.

Theorem 6.4.11 (Left minimal bases and minimal indices)**.** *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(n(1+\rho_A)+p+m\rho_D)\times(n(\rho_A+1)+m(\rho_D+1))},$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ *and* $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ *are strong block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively, with $N_A(\lambda)$ and $N_D(\lambda)$ being minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and $M_C(\lambda)$ and $M_B(\lambda)$ being linear polynomial matrices such that $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$, respectively. Define $\rho_i := \deg N_i(\lambda)$, for $i \in \{A, D\}$, and let $\widehat{R}(\lambda)$ be the transfer function matrix of $\mathcal{L}(\lambda)$. If*

$$\operatorname{rank}\begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix} = n \quad \textit{for all} \quad \lambda_0 \in \mathbb{F}, \quad \textit{and} \tag{6.23}$$

$$\operatorname{rank}\begin{bmatrix} \operatorname{rev}_{\rho_A+1} A(0) & \operatorname{rev}_{\rho_D+1} B(0) \end{bmatrix} = n, \tag{6.24}$$

then the following statements hold:

(a) *If $\{z_i(\lambda)^T\}_{i=1}^t$ is a left minimal basis of $\mathcal{L}(\lambda)$, then $z_i(\lambda)^T = \begin{bmatrix} y_i(\lambda)^T & x_i(\lambda)^T \end{bmatrix}$, for $i = 1, \ldots, s$, for some left minimal basis $\{x_i(\lambda)^T\}_{i=1}^s$ of $\widehat{R}(\lambda)$, and $x_i(\lambda)^T = \begin{bmatrix} u_i(\lambda)^T & w_i(\lambda)^T \end{bmatrix}$, for $i = 1, \ldots, s$, for some left minimal basis $\{u_i(\lambda)^T\}_{i=1}^s$ of $R(\lambda)$.*

(b) *If $\eta_1 \leq \cdots \leq \eta_t$ are the left minimal indices of $\mathcal{L}(\lambda)$, then $\eta_1 \leq \cdots \leq \eta_t$ are the left minimal indices of $\widehat{R}(\lambda)$ and $R(\lambda)$.*

Proof. See Appendix B. ■

6.4.3 Recovery of eigenvectors

In this section, we assume that the rational matrix $R(\lambda)$ is regular and show how to recover right and left eigenvectors of $R(\lambda)$ from those of a linearization $\mathcal{L}(\lambda)$ of $R(\lambda)$ as in Theorem 6.1.2. In Proposition 6.4.12 we state, without proof, analogous results to those of Theorem 6.4.4 but for the right and left nullspaces of $R(\lambda)$ evaluated at a particular value λ_0.

Proposition 6.4.12. *Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix expressed in the form $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$, for some regular polynomial matrix $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. Consider the linear polynomial matrix*

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right],$$

as in Theorem 6.1.2, where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ are block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively, with $N_A(\lambda)$ and $N_D(\lambda)$ being minimal bases dual to $K_A(\lambda)$ and $K_D(\lambda)$, respectively, and $M_C(\lambda)$ and $M_B(\lambda)$ being linear polynomial matrices such that $M_C(\lambda)N_A(\lambda)^T = C(\lambda)$ and $M_B(\lambda)N_D(\lambda)^T = B(\lambda)$, respectively. Let $\lambda_0 \in \mathbb{F}$ such that $\det A(\lambda_0) \neq 0$. Then the following statements hold:

(a) *The linear map*

$$\begin{aligned} F_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(\mathcal{L}(\lambda_0)) \\ x &\longmapsto \widetilde{x} := \begin{bmatrix} -N_A(\lambda_0)^T A(\lambda_0)^{-1} B(\lambda_0) \\ N_D(\lambda_0)^T \end{bmatrix} x \end{aligned}$$

is a bijection between the right nullspaces over $\mathbb{F}$ of $R(\lambda_0)$ and $\mathcal{L}(\lambda_0)$.

(b) *The linear map*

$$\begin{aligned} F_\ell \,:\, \mathcal{N}_\ell(R(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(\mathcal{L}(\lambda_0)) \\ y^T &\longmapsto \widetilde{y}^T := y^T \begin{bmatrix} M_C(\lambda_0)L_A(\lambda_0)^{-1} & I_p & -M_R(\lambda_0)\widehat{N}_D(\lambda_0)^T \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces over $\mathbb{F}$ of $R(\lambda_0)$ and $\mathcal{L}(\lambda_0)$, where $L_A(\lambda) := \begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$, $M_R(\lambda) := M_D(\lambda) + C(\lambda)A(\lambda)^{-1}M_B(\lambda)$ and $\widehat{N}_D(\lambda)$ is defined in (6.7).

Remark 6.4.13. Let $\mathcal{L}(\lambda)$ be as in Proposition 6.4.12, let $\lambda_0 \in \mathbb{F}$ be an eigenvalue of $\mathcal{L}(\lambda)$ such that $\det A(\lambda_0) \neq 0$, and let $\widetilde{x}$ and $\widetilde{y}^T$ be, respectively, right and left eigenvectors of $\mathcal{L}(\lambda)$ with eigenvalue λ_0.

By Proposition 6.4.12, the vector $\widetilde{y}^T$ must be of the form

$$\widetilde{y}^T = y^T \begin{bmatrix} M_C(\lambda_0)L_A(\lambda_0)^{-1} & I_p & -M_R(\lambda_0)\widehat{N}_D(\lambda_0)^T \end{bmatrix},$$

for some left eigenvector y^T of $R(\lambda)$ with eigenvalue λ_0. Hence, one can readily recover a left eigenvector y^T of $R(\lambda)$ from the middle block of $\widetilde{y}^T$. Furthermore, from Proposition 6.4.12, we get that $\widetilde{x}$ must be of the form

$$\widetilde{x} = \begin{bmatrix} -N_A(\lambda_0)^T A(\lambda_0)^{-1} B(\lambda_0) \\ N_D(\lambda_0)^T \end{bmatrix} x,$$

for some right eigenvector x of $R(\lambda)$ with eigenvalue λ_0. Since the polynomial matrix $\widehat{K}_D(\lambda)$ in (6.7) satisfies $\widehat{K}_D(\lambda) N_D(\lambda)^T = I_m$ for all $\lambda \in \mathbb{F}$, we have $\widehat{K}_D(\lambda_0) N_D(\lambda_0)^T x = x$. Thus, one can also recover a right eigenvector of $R(\lambda)$ from the right eigenvector $\widetilde{x}$ of $\mathcal{L}(\lambda)$.

6.5 Application to scalar rational equations

In this section, we show by example how the theory developed in this chapter can be used for solving (scalar) rational equations of the form

$$\frac{c(\lambda)}{a(\lambda)} = -\frac{d(\lambda)}{b(\lambda)}, \tag{6.25}$$

where $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are nonzero scalar polynomials, and where the numerators and the denominators of each rational function can be expressed in terms of different polynomial bases. For instance, let us assume that the polynomials $a(\lambda)$ and $c(\lambda)$ are written in terms of the monomial basis, that is,

$$a(\lambda) = \sum_{i=0}^{n} a_i \lambda^i \qquad \text{and} \qquad c(\lambda) = \sum_{i=0}^{n} c_i \lambda^i,$$

with $n = \max\{\deg a(\lambda), \deg c(\lambda)\}$, and that the polynomials $b(\lambda)$ and $d(\lambda)$ are written in terms of Chebyshev polynomials of the first kind $\{\phi_j(\lambda)\}_{j=0}^{\infty}$, that is,

$$b(\lambda) = \sum_{i=0}^{m} b_i \phi_i(\lambda) \qquad \text{and} \qquad d(\lambda) = \sum_{i=0}^{m} d_i \phi_i(\lambda),$$

with $m = \max\{\deg b(\lambda), \deg d(\lambda)\}$. We recall that the Chebyshev basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ satisfies the three-term recurrence relation:

$$\frac{1}{2}\phi_{j+1}(\lambda) = \lambda \phi_j(\lambda) - \frac{1}{2}\phi_{j-1}(\lambda) \quad j \geq 1 \tag{6.26}$$

where $\phi_0(\lambda) = 1$ and $\phi_1(\lambda) = \lambda$.

Notice that, outside the set of the roots of $b(\lambda)$, that is, in $\Omega := \mathbb{C} \setminus \{\lambda_0 \in \mathbb{C} : b(\lambda_0) = 0\}$, equation (6.25) is equivalent to the equation

$$r(\lambda) := d(\lambda) + c(\lambda) a(\lambda)^{-1} b(\lambda) = 0. \tag{6.27}$$

For computing the roots of (6.27), that is, the zeros that are not poles, we consider a linear polynomial system matrix of the form

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_a(\lambda) & M_b(\lambda) \\ K_a(\lambda) & 0 \\ \hline -M_c(\lambda) & M_d(\lambda) \\ 0 & K_d(\lambda) \end{array}\right],$$

where

$$M_a(\lambda) := \begin{bmatrix} a_n\lambda + a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \end{bmatrix},$$
$$M_c(\lambda) := \begin{bmatrix} c_n\lambda + c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 \end{bmatrix},$$

and

$$K_a(\lambda) := \begin{bmatrix} -1 & \lambda & 0 & \cdots & 0 \\ 0 & -1 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \lambda \end{bmatrix} \quad \text{and} \quad N_a(\lambda)^T = \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

is a pair of dual minimal bases, and

$$M_b(\lambda) := \begin{bmatrix} 2b_m\lambda + b_{m-1} & b_{m-2} - b_m & b_{m-3} & \cdots & b_1 & b_0 \end{bmatrix},$$
$$M_d(\lambda) := \begin{bmatrix} 2d_m\lambda + d_{m-1} & d_{m-2} - d_m & d_{m-3} & \cdots & d_1 & d_0 \end{bmatrix},$$

and, by (6.26),

$$K_d(\lambda) = \begin{bmatrix} -\frac{1}{2} & \lambda & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & \lambda & -\frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -\frac{1}{2} & \lambda & -\frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -1 & \lambda \end{bmatrix} \quad \text{and} \quad N_d(\lambda)^T = \begin{bmatrix} \phi_{m-1}(\lambda) \\ \phi_{m-2}(\lambda) \\ \vdots \\ \phi_1(\lambda) \\ \phi_0(\lambda) \end{bmatrix}$$

is another pair of dual minimal bases. Observe that $a(\lambda) = M_a(\lambda)N_a(\lambda)^T$, $c(\lambda) = M_c(\lambda)N_a(\lambda)^T$, $b(\lambda) = M_b(\lambda)N_d(\lambda)^T$ and $d(\lambda) = M_d(\lambda)N_d(\lambda)^T$.

It is immediate that the matrices

$$\begin{bmatrix} a(\lambda_0) \\ c(\lambda_0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a(\lambda_0) & b(\lambda_0) \end{bmatrix}$$

have full rank (equal to 1) at every $\lambda_0 \in \Omega$ that is not a root of $a(\lambda)$ and $c(\lambda)$ simultaneously. Hence, if $\frac{c(\lambda)}{a(\lambda)}$ is irreducible, i.e., $a(\lambda)$ and $c(\lambda)$ do not have roots in common, then, by Theorem 6.1.2, $\mathcal{L}(\lambda)$ is a linearization of $r(\lambda)$ in Ω. Therefore, the zeros of $\mathcal{L}(\lambda)$ in Ω are the zeros of $r(\lambda)$ in Ω.

The idea of transforming the rational problem (6.25) into an eigenvalue problem is not new [77]. An algorithm based on the Ehrlich-Aberth iteration that uses this approach can be found in [76].

Chapter 7

Strongly minimal linearizations of rational transfer functions

In this chapter we introduce the conditions for a linear polynomial system matrix

$$L(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} := \begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 \\ \lambda C_1 - C_0 & \lambda D_1 - D_0 \end{bmatrix}, \tag{7.1}$$

where $A(\lambda)$ is assumed regular, to be strongly minimal and prove that the strong minimality conditions imply the strong irreducibility conditions in [91]. We remark that, although the notions of irreducible or minimal polynomial system matrix refer to the same conditions in Definition 2.4.2, the conditions for a polynomial system matrix to be strongly irreducible or strongly minimal are different in general.

Recall that a strongly minimal polynomial system matrix, introduced in Chapter 4 (Definition 4.1.15), contains the complete finite and infinite pole and zero structures of its transfer function matrix

$$R(\lambda) := D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda).$$

We will see that strongly minimal linear polynomial system matrices preserve the minimal indices of their transfer function matrices $R(\lambda)$ as well, when $R(\lambda)$ is singular. Then, the pole structure, zero structure and null space structure of $R(\lambda)$ can be computed with the staircase algorithm and the QZ algorithm applied to $A(\lambda)$ and $L(\lambda)$. We will also show that when the strong minimality conditions are not satisfied, we can reduce the linear system matrix to one where they are satisfied, and this without modifying the transfer function matrix. Such a procedure was already derived in [87], but only for linear system matrices that were already minimal at finite points. In this chapter we extend this to arbitrary linear system matrices.

For the particular case of proper rational matrices $R_p(\lambda)$, it is known that they can be written in state-space form as $R_p(\lambda) := D + C(\lambda I - A)^{-1}B$. Then, for the corresponding state-space linear system matrix

$$L_p(\lambda) := \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix},$$

there are algorithms available in the literature to derive a minimal state-space linear system matrix from a non-minimal one, and these algorithms are based on unitary transformations only [84]. When allowing generalized state-space linear system matrices, then all the rational matrices can be realized by linear system matrices of the form

$$L_g(\lambda) := \begin{bmatrix} \lambda E - A & -B \\ C & D \end{bmatrix}, \tag{7.2}$$

where the matrix E is allowed to be singular. Moreover, when the pencils

$$\begin{bmatrix} \lambda E - A & -B \end{bmatrix}, \quad \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} \tag{7.3}$$

have, respectively, full row rank and column rank for all finite λ, then we retrieve the irreducibility or minimality conditions of Rosenbrock in (2.4.2), which imply that the finite poles of $R(\lambda) := D + C(\lambda E - A)^{-1}B$ are the finite eigenvalues of $\lambda E - A$ and the finite zeros of $R(\lambda)$ are the finite eigenvalues of $L_g(\lambda)$. It was shown in [92] that when imposing also the conditions that the pencil in (7.2) is strongly irreducible, meaning that the matrices in (7.3) have full row rank for all finite and infinite λ, then also the infinite pole and zero structure of $R(\lambda)$ can be retrieved from the infinite structure of $\lambda E - A$ and $L_g(\lambda)$, respectively, and that the left and right minimal indices of $R(\lambda)$ and $L_g(\lambda)$ are also the same. Moreover, a reduction procedure to derive a strongly irreducible generalized state-space linear system from a reducible one was also given in [84], and it is also based on unitary transformations only. In [91] these results were then extended to arbitrary polynomial system matrices, but the procedure required irreducibility tests that were more involved. In this chapter, we will show that these conditions can be simplified when the polynomial system matrix is linear. In addition, the reduction procedure presented in this chapter to derive strongly minimal linear systems only uses unitary equivalence transformations. This implies that numerical errors performed during the reduction procedure remain bounded. Since we use unitary transformations in both the reduction procedure and the computation of the eigenstructure, this guarantees that we compute the exact eigenstructure of a perturbed linear polynomial system matrix, but where the perturbation is of the order of the machine precision. The main results in this chapter appear in [32].

Recall that in Chapter 6, we construct linear polynomial system matrices from arbitrary transfer function matrices. Then, by combining the results in this chapter with those in Chapter 6, we can get strongly minimal linearizations for any of the representations of an arbitrary rational matrix considered in Chapter 6. More precisely, given a rational matrix $R(\lambda)$, we can use Theorem 6.1.2 to construct a linear polynomial system matrix of $R(\lambda)$ and, if the obtained system matrix is not strongly minimal, we can apply the reduction procedure in this chapter to obtain a strongly minimal one.

7.1 Some preliminaries

Throughout this chapter we only consider the field of complex number $\mathbb{C}$. As we explain in Chapter 2, the strictly positive structural indices in the local Smith–McMillan form of a rational matrix correspond to a zero, and the strictly negative structural indices correspond to a pole. Then, the *zero degree* is defined as the sum of all structural indices of all zeros (infinity included), and the *polar degree* is the sum of all structural indices (in absolute value) of all poles (infinity included). The polar degree of a rational matrix $R(\lambda)$ is also called as the *McMillan degree*, denoted by $\delta(R)$. The following degree sum theorem was proven in [92], and relates the McMillan degree to the other structural elements of $R(\lambda)$: to the the *zero degree* $\delta_z(R)$, to the *left nullspace degree* $\delta_\ell(R)$, that is the sum of all left minimal indices, and to the *right nullspace degree* $\delta_r(R)$, that is the sum of all right minimal indices.

Theorem 7.1.1. *Let* $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. *Then*

$$\delta(R) := \delta_p(R) = \delta_z(R) + \delta_\ell(R) + \delta_r(R).$$

Example 7.1.2. Let us consider the 2×2 rational matrix

$$R(\lambda) = \begin{bmatrix} e_5(\lambda) & 0 \\ c/\lambda & e_1(\lambda) \end{bmatrix} \tag{7.4}$$

where $e_5(\lambda)$ is a monic polynomial of degree 5 and $e_1(\lambda)$ is a monic polynomial of degree 1, with $e_5(0) \neq 0$ and $e_1(0) \neq 0$. If $c \neq 0$, the only poles are 0 and infinity, and the corresponding local Smith-McMillan forms for these two points are

$$\lambda_0 = 0 : \mathrm{diag}(\lambda^{-1}, \lambda^1), \quad \lambda_0 = \infty : \mathrm{diag}((1/\lambda)^{-5}, (1/\lambda)^{-1}),$$

indicating that $\lambda_0 = 0$ is a zero as well as a pole. The other finite zeros are the six finite roots of $e_5(\lambda)$ and $e_1(\lambda)$. The polar degree and the zero degree for this example are thus both equal to 7. When $c = 0$, the pole and zero at $\lambda = 0$ disappear and the matrix is polynomial instead of rational. The polar and zero degree are then both equal to 6.

7.2 Strong irreducibility and minimality

In this section we recall the strong irreducibility conditions in [91] for polynomial system matrices, and we recall the notion of strong minimality. Then, we study the relation between them for the case of linear system matrices. Consider a polynomial system matrix

$$S(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}, \tag{7.5}$$

where $A(\lambda)$ is assumed regular, of arbitrary degree d.

Definition 7.2.1. *A polynomial system matrix $S(\lambda)$ as in* (7.5) *is said to be* strongly controllable *and* strongly observable*, respectively, if the polynomial matrices*

$$\begin{bmatrix} A(\lambda) & -B(\lambda) & 0 \\ C(\lambda) & D(\lambda) & -I \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \\ 0 & I \end{bmatrix}, \tag{7.6}$$

have no finite or infinite zeros. If both conditions are satisfied $S(\lambda)$ is said to be strongly irreducible*.*

Let us now consider the transfer function matrix $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$ of the polynomial system matrix in (7.5). In such a case, we also say that the system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ *realizes* $R(\lambda)$. Moreover, we say that the system quadruple is *strongly irreducible* if the polynomial system matrix is *strongly irreducible.* It was shown in [91] that the pole/zero and null space structure of $R(\lambda)$ can be retrieved from a strongly irreducible system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ as follows.

Theorem 7.2.2. *If the polynomial system matrix $S(\lambda)$ in* (7.5) *is strongly irreducible, then*

1. *the zero structure of $R(\lambda)$ at finite and infinite λ is the same as the zero structure of $S(\lambda)$ at finite and infinite λ,*

2. *the pole structure of $R(\lambda)$ at finite λ is the same as the zero structure at λ of $A(\lambda)$,*

3. *the pole structure of $R(\lambda)$ at infinity is the same as the zero structure at infinity of*
$$\begin{bmatrix} A(\lambda) & -B(\lambda) & 0 \\ C(\lambda) & D(\lambda) & -I \\ 0 & I & 0 \end{bmatrix},$$

4. *the left and right minimal indices of $R(\lambda)$ and $S(\lambda)$ are the same.*

If one specializes this to the generalized state space model (7.2) one retrieves the results of [92], which are simpler and only involve the pencils $\lambda E - A$, for the finite and infinite pole structure; (7.2), for the finite and infinite zero structure and (7.3), for the definition of strongly irreducible.

We now show that the above conditions can be simplified when the system matrices are linear as in (7.1). First, we present the definition of strongly E-controllable and strongly E-observable polynomial system matrices, and based on them, we redefined the concept of strongly minimal polynomial system matrix. It is easy to see that this definition is equivalent to that in Definition 4.1.15.

Definition 7.2.3. *Let d be the degree of the polynomial system matrix $S(\lambda)$ in* (7.5)*. $S(\lambda)$ is said to be* strongly E-controllable *and* strongly E-observable*, respectively, if the polynomial matrices*

$$\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}, \tag{7.7}$$

have degree exactly d and have no finite or infinite eigenvalues. If both conditions are satisfied $S(\lambda)$ is said to be strongly minimal.

The letter E in the definition of strong E-controllability and E-observability refers to the condition of the matrices in (7.7) not having eigenvalues, finite or infinite, and emphasizes the differences with the concepts of "strong controlability, observability and irreducibility" used in [92, 91].

7.2.1 Strongly minimal linearizations

In this work we focus on linear polynomial system matrices. This means that we consider block partitioned pencils

$$L(\lambda) := \begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 \\ \lambda C_1 - C_0 & \lambda D_1 - D_0 \end{bmatrix} =: \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}, \tag{7.8}$$

where $A(\lambda)$ is regular.

7.2.1.1 Finite pole and zero structure

A linear polynomial system matrix $L(\lambda)$ as in (7.8) contains the finite zero and pole structures of its transfer function matrix $R(\lambda) := D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ provided that $L(\lambda)$ is minimal. That is, if matrices

$$\begin{bmatrix} \lambda A_1 - A_0 & -\lambda B_1 + B_0 \end{bmatrix}, \quad \begin{bmatrix} \lambda A_1 - A_0 \\ \lambda C_1 - C_0 \end{bmatrix}, \tag{7.9}$$

have, respectively, full row and column rank for all $\lambda_0 \in \mathbb{C}$. This is equivalent to state that the pencils in (7.9) do not have finite eigenvalues. If $L(\lambda)$ is minimal then we know by Rosenbrock [78] (recall Theorem 2.4.4) that the finite zero structure of $R(\lambda)$ is the same as the finite zero structure of $L(\lambda)$, and that the finite pole structure of $R(\lambda)$ is the same as the finite zero structure of $\lambda A_1 - A_0$.

Since $\dim \mathcal{N}_r(L) = \dim \mathcal{N}_r(R)$ and $\dim \mathcal{N}_\ell(L) = \dim \mathcal{N}_\ell(R)$ (recall (2.10)), we have that the number of right (resp. left) minimal indices of a minimal polynomial system matrix is equal to the number of right (resp. left) minimal indices of its transfer function matrix. However, their values may be different [92, 8].

7.2.1.2 Finite and infinite pole and zero structure

Notice that Theorem 2.4.4 does not provide information about the structure at infinity. The recovering of this structure requires the following concept introduced in Chapter 4: $L(\lambda)$ in (7.8) is minimal at infinity if the matrices

$$\begin{bmatrix} A_1 & -B_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_1 \\ C_1 \end{bmatrix} \tag{7.10}$$

have, respectively, full row and column rank. This condition is equivalent to state that the pencils in (7.9) have degree exactly 1 and do not have eigenvalues at ∞. A linear polynomial system matrix that is minimal (at finite points) and also minimal at ∞ is strongly minimal according to Definition 7.2.3. More precisely, a linear polynomial system matrix $L(\lambda)$ as in (7.8) is strongly minimal if the pencils

$$\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}, \tag{7.11}$$

have degree exactly 1 and have no finite or infinite eigenvalues. As mentioned before, the degree 1 pencils in (7.11) do not have infinite eigenvalues if and only if the matrices in (7.10) have full row and full column rank, respectively. The ranks of the matrices in (7.10) will be also called the *ranks at infinity* of the pencils in (7.7), even in the case the matrices in (7.10) do not have full ranks.

We say that strongly minimal linear polynomial system matrices are strongly minimal linearizations of their transfer function matrices.

Definition 7.2.4. *Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be a rational matrix. A linear polynomial system matrix $L(\lambda)$ as in* (7.8) *is said to be* a strongly minimal linearization of $R(\lambda)$ *if $L(\lambda)$ is strongly minimal and its transfer function matrix is $R(\lambda)$. Equivalently, $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ is said to be* a strongly minimal linear realization *of* $R(\lambda)$.

We prove in Proposition 7.2.10 that, for linear polynomial system matrices, the strong irreducibility conditions hold if the strong minimality conditions are satisfied. For this, we need to recall Lemma 1 of [92], which we give here in its transposed form. Then, we prove Theorems 7.2.6 and 7.2.7, and Proposition 7.2.10 as a corollary of them.

Lemma 7.2.5. *The zero structure at infinity of the pencil $\left[\begin{array}{c|c} \lambda K_1 - K_0 & -L_0 \end{array}\right]$ where K_1 has full column rank, is isomorphic to the zero structure at zero of the pencil $\left[\begin{array}{c|c} K_1 - \mu K_0 & -L_0 \end{array}\right]$. Moreover, if the pencil has full row normal rank, then it has no zeros at infinity, provided the constant matrix $\left[\begin{array}{c|c} K_1 & -L_0 \end{array}\right]$ has full row rank.*

Proof. The first part is proven in [92]. The second part is a direct consequence of the first part, when we evaluate at $\mu = 0$. ∎

Theorem 7.2.6. *The pencil*

$$\begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 & 0 \\ \lambda C_1 - C_0 & \lambda D_1 - D_0 & -I \end{bmatrix}, \tag{7.12}$$

where $\lambda A_1 - A_0$ *is regular, has no zeros at infinity if the pencil*

$$\begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 \end{bmatrix} \tag{7.13}$$

has no eigenvalues at infinity.

Proof. Clearly the pencils in (7.12) and (7.13) have full row normal rank since $\lambda A_1 - A_0$ is regular. We can thus apply the result of Lemma 7.2.5 as follows. If we use an invertible matrix V to "compress" the columns of the coefficient of λ in the following pencil

$$\left[\begin{array}{cc|c} \lambda A_1 - A_0 & B_0 - \lambda B_1 & 0 \\ \lambda C_1 - C_0 & \lambda D_1 - D_0 & -I \end{array}\right] \left[\begin{array}{c|c} V & 0 \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{cc|c} \lambda K_1 - K_0 & -L_0 & 0 \\ \lambda \widehat{K}_1 - \widehat{K}_0 & -\widehat{L}_0 & -I \end{array}\right],$$

such that the matrix $\begin{bmatrix} K_1 \\ \widehat{K}_1 \end{bmatrix}$ has full column rank, then this pencil has no zeros at infinity provided the constant matrix $\left[\begin{array}{cc|c} K_1 & -L_0 & 0 \\ \widehat{K}_1 & -\widehat{L}_0 & -I \end{array}\right]$ has full row rank. But if $\begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 \end{bmatrix}$ has no infinite eigenvalues, it follows that $\begin{bmatrix} A_1 & -B_1 \end{bmatrix}$ has full row rank. And since $\begin{bmatrix} A_1 & -B_1 \end{bmatrix} V = \begin{bmatrix} K_1 & 0 \end{bmatrix}$, K_1 must have full row rank as well (in fact, it is invertible). It then follows from Lemma 7.2.5 that the pencil in (7.12) has no zeros at infinity. ■

In the next theorem, we state without proof the transposed version of Theorem 7.2.6.

Theorem 7.2.7. *The pencil*

$$\begin{bmatrix} \lambda A_1 - A_0 & B_0 - \lambda B_1 \\ \lambda C_1 - C_0 & \lambda D_1 - D_0 \\ 0 & I \end{bmatrix},$$

where $\lambda A_1 - A_0$ *is regular, has no zeros at infinity if the pencil*

$$\begin{bmatrix} \lambda A_1 - A_0 \\ \lambda C_1 - C_0 \end{bmatrix} \tag{7.14}$$

has no eigenvalues at infinity.

By Theorems 7.2.6, 7.2.7 and [91], we have the next result for recovering the structure at infinity.

Theorem 7.2.8. *Let $R(\lambda)$ be the transfer function matrix of $L(\lambda)$ in* (7.8)*. If $L(\lambda)$ is minimal at ∞ then*

1. *the zero structure of $R(\lambda)$ at infinity is the same as the zero structure of $L(\lambda)$ at infinity, and*
2. *the polar structure of $R(\lambda)$ at infinity is the same as the zero structure of the pencil*
$$\begin{bmatrix} \lambda A_1 - A_0 & -\lambda B_1 & 0 \\ \lambda C_1 & \lambda D_1 & -I_m \\ 0 & I_n & 0 \end{bmatrix} \tag{7.15}$$
at infinity.

The polar structure of $R(\lambda)$ at ∞ can also be recovered without considering the extended pencil in (7.15). In particular, both the zero and polar structures of $R(\lambda)$ at infinity can be obtained from the eigenvalue structures of the pencils $L(\lambda)$ and $A(\lambda)$ at infinity as Theorem 7.2.9 shows (see Chapter 4 or [28]).

Theorem 7.2.9. *Let $R(\lambda)$ be the transfer function matrix of $L(\lambda)$ in* (7.8)*. Assume that $R(\lambda)$ has normal rank r. Let $e_1 \leq \cdots \leq e_s$ be the partial multiplicities of* $\text{rev}_1 A(\lambda)$ *at* 0 *and let $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ be the partial multiplicities of* $\text{rev}_1 L(\lambda)$ *at* 0*. If $L(\lambda)$ is minimal at ∞ then the structural indices at infinity $q_1 \leq \cdots \leq q_r$ of $R(\lambda)$ are*

$$(d_1, d_2, \ldots, d_r) = (-e_s, -e_{s-1}, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-s-u}, \widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_u) - (1, 1, \ldots, 1).$$

Notice that if a linear system matrix $L(\lambda)$ is minimal (i.e., satisfies (7.9)) and, in addition, satisfies the conditions in (7.13) and (7.14), then it is strongly minimal. By Theorems 7.2.6 and 7.2.7, we have that these conditions imply strong irreducibility on linear system matrices. We state such result in Proposition 7.2.10.

Proposition 7.2.10. *A linear system matrix as in* (7.8) *is strongly irreducible if it is strongly minimal.*

Theorems 7.2.6 and 7.2.7 and Proposition 7.2.10 can be extended to polynomial system matrices. However, we do not state these results here since we are focusing on linear system matrices.

Strongly minimal linearizations $L(\lambda)$ of a rational matrix $R(\lambda)$ have been defined with the goal of constructing pencils that allow us to recover the complete pole and zero structures of $R(\lambda)$ through Theorems 2.4.4 and 7.2.9, or 7.2.8. Surprisingly, the condition of strong minimality implies that the minimal indices of $L(\lambda)$ and $R(\lambda)$ are the same. This is proved in Theorem 7.2.11, which, together with Theorems 2.4.4 and 7.2.9, *allows us to recover the complete eigenstructure of a rational matrix from any of its strongly minimal linearizations.*

Theorem 7.2.11. *Let $L(\lambda)$ be a strongly minimal linearization of a rational matrix $R(\lambda)$. Then the left and right minimal indices of $R(\lambda)$ are the same as the left and right minimal indices of $L(\lambda)$.*

Proof. By Proposition 7.2.10, a strongly minimal linear polynomial system matrix is strongly irreducible. Then, by [91, Result 2], the left and right minimal indices of $R(\lambda)$ and $L(\lambda)$ are the same. ∎

Remark 7.2.12. It follows from Theorems 2.4.4, 7.2.8 and 7.2.11 that, if $L(\lambda)$ is a strongly minimal linearization of a rational matrix $R(\lambda)$, then

$$\delta_z(R) + \delta_\ell(R) + \delta_r(R) = \delta_z(L) + \delta_\ell(L) + \delta_r(L),$$

and then from Theorem 7.1.1 that $\delta_p(R) = \delta_p(L)$. But the only pole of $L(\lambda) := \lambda L_1 + L_0$ is the point at infinity and its polar degree $\delta_p(L)$ is equal to $\operatorname{rank} L_1$ [84, p. 126]. Therefore, the McMillan degree $\delta_p(R)$ of $R(\lambda)$ equals the rank of L_1 for any strongly minimal linearization of $R(\lambda)$, and no other pencils with the same zero structure and the same left and right minimal indices as $R(\lambda)$ can have a first order coefficient with smaller rank. Thus, strongly minimal linerizations are optimal in this sense.

7.2.2 Strongly minimal linearizations and their relation with other classes of linearizations

In this subsection we study the relation of strongly minimal linearizations with other definitions of strong linearizations of polynomial and rational matrices in the literature (recall Section 2.5).

For polynomial matrices, first we can combine Theorem 2.4.4 applied to a polynomial matrix $P(\lambda)$ and the equality of the number of the minimal indices of $L(\lambda)$ and $P(\lambda)$ with [22, Theorem 4.1] for proving that *any minimal linear polynomial system matrix of a polynomial matrix $P(\lambda)$ is always a GLR-linearization of $P(\lambda)$.* The reverse result is not true in general. Observe also that any minimal polynomial system matrix of a polynomial matrix $P(\lambda)$ must have the block $A(\lambda)$ in (7.8) unimodular, because $P(\lambda)$ does not have finite poles. However, the following example shows that strongly mininal linearizations for polynomial matrices are not, in general, GLR-strong linearizations.

Example 7.2.13. (Strongly minimal linearizations of polynomial matrices are not strong linearizations in the sense of Gohberg, Lancaster and Rodman) Consider the polynomial matrix

$$P(\lambda) = \lambda^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the partitioned pencil

$$L(\lambda) = \left[\begin{array}{c|cc} -1 & 0 & \lambda \\ \hline 0 & \lambda+1 & 0 \\ \lambda & 0 & \lambda+1 \end{array}\right].$$

The transfer function matrix of $L(\lambda)$ is $P(\lambda)$ and $L(\lambda)$ is minimal and minimal at infinity. Therefore, $L(\lambda)$ is a strongly minimal linearization of $P(\lambda)$ and also a GLR-linearization of $P(\lambda)$. However, $\text{rev}_1 L(\lambda)$ is *not* unimodularly equivalent to $\text{diag}(\text{rev}_2 P(\lambda), 1)$ and, thus, $L(\lambda)$ is not a GLR-strong linearization of $P(\lambda)$. In order to see this, observe that

$$\text{rev}_2 P(\lambda) = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\text{rev}_1 L(\lambda) = \left[\begin{array}{c|cc} -\lambda & 0 & 1 \\ \hline 0 & \lambda+1 & 0 \\ 1 & 0 & \lambda+1 \end{array}\right],$$

which makes it transparent that $\text{rev}_1 L(\lambda)$ does not have eigenvalues (or zeros) at zero, while $\text{rev}_2 P(\lambda)$ does. In general, it is possible to prove by using Theorem 7.2.9 that strongly minimal linearizations of polynomial matrices of degree larger than 1 with eigenvalues at infinity are not GLR-strong linearizations.

Despite of the fact of strongly mininal linearizations not being GLR-strong linearizations, strongly minimal linearizations $L(\lambda)$ of a polynomial matrix $P(\lambda)$ allow us to recover always the complete list of structural data of $P(\lambda)$, including its minimal indices. Moreover, the minimal indices of $L(\lambda)$ and $P(\lambda)$ are always the same in contrast with the minimal indices of GLR-strong linearizations whose values may be different to those of $P(\lambda)$. In addition, we will prove in Chapter 8 that strongly mininal linearizations allow us to preserve structures of polynomial matrices that cannot always be preserved by GLR-strong linearizations.

For rational matrices, the minimal linear polynomial system matrices of an arbitrary rational matrix $R(\lambda)$ are particular cases of the linearizations of $R(\lambda)$ defined in [6], which were introduced with the idea of combining the concept of minimal polynomial system matrix with the extension of GLR-linearizations from polynomial to rational matrices. However, strongly minimal linearizations are not always strong linearizations in the sense of [6] (Definition 2.5.3) since the first degree coefficients of their $(1,1)$-blocks are not necessarily invertible. But, as in the polynomial case, strongly minimal linearizations of a rational matrix allow us to recover always the complete list of its structural data, including its minimal indices.

7.3 Reduction to a strongly minimal linear system matrix

In this section we give an algorithm to reduce an arbitrary linear system matrix to a strongly minimal one.

Given a linear system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$, where $A(\lambda) \in \mathbb{C}(\lambda)^{d\times d}$, $B(\lambda) \in \mathbb{C}(\lambda)^{d\times n}$, $C(\lambda) \in \mathbb{C}(\lambda)^{m\times d}$, $D(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ and $A(\lambda)$ is assumed to be regular, we describe first how to obtain a strongly E-controllable quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ of smaller state dimension $(d-r)$. For that, our reduction procedure deflates finite and infinite "uncontrollable eigenvalues" by proceeding in three different steps. Then the reduction to a strongly E-observable one is dual and can be obtained by mere transposition of the system matrix and application of the first method for obtaining a strongly E-controllable system.

Step 1: We first show that there exist unitary transformations U and V that yield a decomposition of the type

$$\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 & X(\lambda)W_{13} \\ \widetilde{Y}(\lambda) & \widetilde{A}(\lambda) & -\widetilde{B}(\lambda) \\ \widetilde{Z}(\lambda) & \widetilde{C}(\lambda) & D(\lambda) \end{bmatrix}, \tag{7.16}$$

where $\widehat{W}_{11} \in \mathbb{C}^{r\times r}$ and $W_{13} \in \mathbb{C}^{r\times n}$ are constant, and $\widehat{W}_{11}$ is invertible. This will allow us in step 2 to deflate the block $X(\lambda)$ and construct a lower order model that is strongly E-controllable. In order to prove this, we start from the generalized Schur decomposition for singular pencils (see [83])

$$U \begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} W^* = \begin{bmatrix} X(\lambda) & 0 & 0 \\ Y(\lambda) & \widehat{A}(\lambda) & -\widehat{B}(\lambda) \end{bmatrix}, \tag{7.17}$$

where $X(\lambda) \in \mathbb{C}[\lambda]^{r\times r}$ is the regular part of $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$, $\widehat{A}(\lambda) \in \mathbb{C}[\lambda]^{(d-r)\times(d-r)}$, and $\begin{bmatrix} \widehat{A}(\lambda) & -\widehat{B}(\lambda) \end{bmatrix}$ has no finite or infinite eigenvalues anymore. The decomposition in (7.17) can be obtained by using unitary transformations U and W. If we partition U as $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, with $U_1 \in \mathbb{C}^{r\times d}$, then

$$U_1 \begin{bmatrix} A(\lambda) \mid -B(\lambda) \end{bmatrix} = \begin{bmatrix} X(\lambda)W_{11} & X(\lambda)W_{12} \mid X(\lambda)W_{13} \end{bmatrix},$$

where $W_{11} \in \mathbb{C}^{r\times r}$, $W_{12} \in \mathbb{C}^{r\times(d-r)}$ and $W_{13} \in \mathbb{C}^{r\times n}$ are the corresponding submatrices of W. Since $A(\lambda)$ is regular, $X(\lambda)\begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$ must be full normal rank, and hence $\begin{bmatrix} W_{11} & W_{12} \end{bmatrix}$ must be full row rank as well. Therefore, there must exist a unitary matrix V such that $\begin{bmatrix} W_{11} & W_{12} \end{bmatrix} V = \begin{bmatrix} \widehat{W}_{11} & 0 \end{bmatrix}$, where $\widehat{W}_{11}$ is invertible.

Hence, we have

$$\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 & X(\lambda)W_{13} \\ \widetilde{Y}(\lambda) & \widetilde{A}(\lambda) & -\widetilde{B}(\lambda) \\ \widetilde{Z}(\lambda) & \widetilde{C}(\lambda) & D(\lambda) \end{bmatrix},$$

where

$$W \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \widehat{W}_{11} & 0 & W_{13} \\ \widehat{W}_{21} & \widehat{W}_{22} & W_{23} \\ \widehat{W}_{31} & \widehat{W}_{32} & W_{33} \end{bmatrix}.$$

Step 2: We now define $E := -\widehat{W}_{11}^{-1}W_{13}$ and perform the following non-unitary transformation on the pencil:

$$\begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 & X(\lambda)W_{13} \\ \widetilde{Y}(\lambda) & \widetilde{A}(\lambda) & -\widetilde{B}(\lambda) \\ \widetilde{Z}(\lambda) & \widetilde{C}(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} I_r & 0 & E \\ 0 & I_{d-r} & 0 \\ 0 & 0 & I_n \end{bmatrix}$$
$$= \begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 & 0 \\ \widetilde{Y}(\lambda) & \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)E - \widetilde{B}(\lambda) \\ \widetilde{Z}(\lambda) & \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)E + D(\lambda) \end{bmatrix}.$$

We have obtained an equivalent system representation in which the $(1,1)$-block, $X(\lambda)\widehat{W}_{11}$, can be deflated since it does not contribute to the transfer function. We then obtain a smaller linear system pencil:

$$\begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)E - \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)E + D(\lambda) \end{bmatrix},$$

that has the same transfer function. One can also perform this elimination by another unitary transformation $\widetilde{W}$ constructed to eliminate W_{13}:

$$\begin{bmatrix} \widehat{W}_{11} & 0 & W_{13} \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d-r} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \end{bmatrix}, \tag{7.18}$$

implying $\widetilde{W}_{11} = \widehat{W}_{11}^*$, $\widetilde{W}_{31} = W_{13}^*$, and $\widetilde{W}_{13} = -\widehat{W}_{11}^{-1}W_{13}\widetilde{W}_{33}$. This then yields

$$\begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 & X(\lambda)W_{13} \\ \widetilde{Y}(\lambda) & \widetilde{A}(\lambda) & -\widetilde{B}(\lambda) \\ \widetilde{Z}(\lambda) & \widetilde{C}(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d-r} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix}$$

$$= \begin{bmatrix} X(\lambda) & 0 & 0 \\ \widetilde{Y}(\lambda)\widetilde{W}_{11} - \widetilde{B}(\lambda)\widetilde{W}_{31} & \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \\ \widetilde{Z}(\lambda)\widetilde{W}_{11} + D(\lambda)\widetilde{W}_{31} & \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)\widetilde{W}_{13} + D(\lambda)\widetilde{W}_{33} \end{bmatrix}.$$

Notice that the new transfer function has now changed, but only by postmultiplication by the constant matrix $\widetilde{W}_{33}$, which moreover is invertible. This follows from

$$\begin{bmatrix} E \\ I_n \end{bmatrix} \widetilde{W}_{33} = \begin{bmatrix} \widetilde{W}_{13} \\ \widetilde{W}_{33} \end{bmatrix},$$

expressing that both matrices span the null-space of the same matrix $\begin{bmatrix} \widehat{W}_{11} & W_{13} \end{bmatrix}$ and where the right hand side matrix has full rank since it has orthonormal columns. This also implies that

$$\begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)E - \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)E + D(\lambda) \end{bmatrix} \begin{bmatrix} I_{d-r} & 0 \\ 0 & \widetilde{W}_{33} \end{bmatrix} = \begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \\ \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)\widetilde{W}_{13} + D(\lambda)\widetilde{W}_{33} \end{bmatrix},$$

which shows that their Schur complements are related by the constant matrix $\widetilde{W}_{33}$.

Step 3: Finally, we show that the submatrix

$$\begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)E - \widetilde{B}(\lambda) \end{bmatrix} \begin{bmatrix} I_{d-r} & 0 \\ 0 & \widetilde{W}_{33} \end{bmatrix} = \begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \end{bmatrix},$$

has no finite or infinite eigenvalues anymore. For this, we first point out that the following product of unitary matrices has the form given below

$$W \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d-r} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix} =: \begin{bmatrix} I_r & 0 & 0 \\ 0 & \widetilde{V}_{22} & \widetilde{V}_{23} \\ 0 & \widetilde{V}_{32} & \widetilde{V}_{33} \end{bmatrix} =: \begin{bmatrix} I_r & 0 \\ 0 & \widetilde{V} \end{bmatrix}$$

because the identity (7.18) implies that the first block column equals $\begin{bmatrix} I_r & 0 & 0 \end{bmatrix}$. This then implies the equality

$$\begin{bmatrix} X(\lambda) & 0 & 0 \\ \widetilde{Y}(\lambda)\widetilde{W}_{11} - \widetilde{B}(\lambda)\widetilde{W}_{31} & \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \end{bmatrix}$$

$$= \begin{bmatrix} X(\lambda) & 0 & 0 \\ Y(\lambda) & \widehat{A}(\lambda) & -\widehat{B}(\lambda) \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & \widetilde{V} \end{bmatrix},$$

which in turn implies that $\begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \end{bmatrix}$ has no finite or infinite eigenvalues. We thus have shown that the system matrix

$$S_c(\lambda) := \begin{bmatrix} A_c(\lambda) & -B_c(\lambda) \\ C_c(\lambda) & D_c(\lambda) \end{bmatrix} := \begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{Y}(\lambda)\widetilde{W}_{13} - \widetilde{B}(\lambda)\widetilde{W}_{33} \\ \widetilde{C}(\lambda) & \widetilde{Z}(\lambda)\widetilde{W}_{13} + D(\lambda)\widetilde{W}_{33} \end{bmatrix}$$

is now strongly E-controllable and that its transfer function $R_c(\lambda)$ equals $R(\lambda)\widetilde{W}_{33}$, where $R(\lambda)$ is the transfer function of the original quadruple and $\widetilde{W}_{33}$ is invertible. We summarize the result obtained by the three-step procedure above in Theorem 7.3.1, where we denote $d-r$ by d_c, to indicate that it is the size of $A_c(\lambda)$ in the new strongly E-controllable system, and r is replaced by $d_{\overline{c}}$, so that $d = d_{\overline{c}} + d_c$.

Theorem 7.3.1. *Let $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ be a linear system quadruple, with $A(\lambda) \in \mathbb{C}[\lambda]^{d\times d}$ regular, realizing the rational matrix $R(\lambda) := C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. Then there exist unitary transformations $U, V \in \mathbb{C}^{d\times d}$ and $\widetilde{W} \in \mathbb{C}^{(d+n)\times(d+n)}$ such that the following identity holds*

$$\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix}\widetilde{W} = \begin{bmatrix} X_{\overline{c}}(\lambda) & 0 & 0 \\ Y_{\overline{c}}(\lambda) & A_c(\lambda) & -B_c(\lambda) \\ Z_{\overline{c}}(\lambda) & C_c(\lambda) & D_c(\lambda) \end{bmatrix},$$

where $\widetilde{W}$ is of the form $\widetilde{W} := \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d_c} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix} \in \mathbb{C}^{(d_{\overline{c}}+d_c+n)\times(d_{\overline{c}}+d_c+n)}$, *$d_{\overline{c}}$ is the number of (finite and infinite) eigenvalues of $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$, and $X_{\overline{c}}(\lambda) \in \mathbb{C}[\lambda]^{d_{\overline{c}}\times d_{\overline{c}}}$ is a regular pencil. Moreover,*

a) the eigenvalues of $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$ are the eigenvalues of $X_{\overline{c}}(\lambda)$,

b) $\begin{bmatrix} A_c(\lambda) & -B_c(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{d_c\times(d_c+n)}$ has no (finite or infinite) eigenvalues,

c) the quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ is a realization of the transfer function $R_c(\lambda) := R(\lambda)\widetilde{W}_{33}$, with $\widetilde{W}_{33} \in \mathbb{C}^{n\times n}$ invertible, and

d) if $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ has no finite or infinite eigenvalues, then $\begin{bmatrix} A_c(\lambda) \\ C_c(\lambda) \end{bmatrix}$ also has no finite or infinite eigenvalues.

Remark 7.3.2. Notice that conditions *b*) and *d*) in Theorem 7.3.1 imply that the system quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ is strongly minimal.

Proof. The decomposition and the three properties *a*), *b*) and *c*) were shown in the discussion above. The only part that remains to be proven is property *d*). This follows from the identity (7.16), which yields

$$\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} V = \begin{bmatrix} X(\lambda)\widehat{W}_{11} & 0 \\ \widetilde{Y}(\lambda) & A_c(\lambda) \\ \widetilde{Z}(\lambda) & C_c(\lambda) \end{bmatrix}.$$

This clearly implies that if $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ has full rank for all λ (including infinity), then so does $\begin{bmatrix} A_c(\lambda) \\ C_c(\lambda) \end{bmatrix}$. ∎

We state below a dual theorem that constructs, from an arbitrary linear system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$, a subsystem $\{A_o(\lambda), B_o(\lambda), C_o(\lambda), D_o(\lambda)\}$ where $\begin{bmatrix} A_o(\lambda) \\ C_o(\lambda) \end{bmatrix}$ has no finite or infinite eigenvalues. Its proof is obtained by applying the previous theorem on the transposed system $\{A^A(\lambda), C^A(\lambda), B^A(\lambda), D^A(\lambda)\}$ and then transposing back the result.

Theorem 7.3.3. *Let $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ be a linear system quadruple, with $A(\lambda) \in \mathbb{C}[\lambda]^{d\times d}$ regular, realizing the rational matrix $R(\lambda) := C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$. Then there exist unitary transformations $U, V \in \mathbb{C}^{d\times d}$ and $\widetilde{W} \in \mathbb{C}^{(d+m)\times(d+m)}$ such that the following identity holds*

$$\widetilde{W} \begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} X_{\overline{o}}(\lambda) & Y_{\overline{o}}(\lambda) & Z_{\overline{o}}(\lambda) \\ 0 & A_o(\lambda) & -B_o(\lambda) \\ 0 & C_o(\lambda) & D_o(\lambda) \end{bmatrix},$$

where $\widetilde{W}$ is of the form $\widetilde{W} := \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d_o} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix} \in \mathbb{C}^{(d_{\overline{o}}+d_o+m)\times(d_{\overline{o}}+d_o+m)}$, $d_{\overline{o}}$ is the number of (finite and infinite) eigenvalues of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$, and $X_{\overline{o}}(\lambda) \in \mathbb{C}[\lambda]^{d_{\overline{o}}\times d_{\overline{o}}}$ is a regular pencil. Moreover,

a) the eigenvalues of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ are the eigenvalues of $X_{\overline{o}}(\lambda)$,

b) $\begin{bmatrix} A_o(\lambda) \\ C_o(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(d_o+m)\times d_o}$ has no (finite or infinite) eigenvalues,

c) the quadruple $\{A_o(\lambda), B_o(\lambda), C_o(\lambda), D_o(\lambda)\}$ is a realization of the transfer function $R_o(\lambda) := \widetilde{W}_{33}R(\lambda)$, with $\widetilde{W}_{33} \in \mathbb{C}^{m\times m}$ invertible, and

d) if $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$ has no finite or infinite eigenvalues then $\begin{bmatrix} A_o(\lambda) & -B_o(\lambda) \end{bmatrix}$ also has no finite or infinite eigenvalues.

In order to extract from the system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ a subsystem $\{A_{co}(\lambda), B_{co}(\lambda), C_{co}(\lambda), D_{co}(\lambda)\}$ that is both strongly E-controllable and E-observable (and hence also strongly minimal), we only need to apply the above two theorems one after the other. The resulting subsystem would then be a realization of the transfer function $R_{co} = C_{co}(\lambda)A_{co}(\lambda)^{-1}B_{co}(\lambda)+D_{co}(\lambda) = W_\ell R(\lambda)W_r \in \mathbb{C}(\lambda)^{m\times n}$. Since the transfer function was changed only by left and right transformations that are constant and invertible, the left and right nullspace will be transformed by these invertible transformations, but their minimal indices will be unchanged.

7.4 Computational aspects

In this section we give a more "algorithmic" description of the procedure described in Section 7.3 to reduce a given system quadruple $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ to a strongly E-controllable quadruple $\{A_c(\lambda), B_c(\lambda), C_c(\lambda), D_c(\lambda)\}$ of smaller size. We describe the essence of the three steps that were discussed in that section.

Step 1 : Compute the staircase reduction of the submatrix $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$

$$U \begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} W^* = \left[\begin{array}{c|cc} X(\lambda) & 0 & 0 \\ \hline Y(\lambda) & \widehat{A}(\lambda) & -\widehat{B}(\lambda) \end{array}\right].$$

Step 2 : Compute the unitary matrices V and $\widetilde{W}$ to compress the first block row of W

$$\begin{bmatrix} W_{11} & W_{12} & W_{13} \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \widetilde{W}_{11} & 0 & \widetilde{W}_{13} \\ 0 & I_{d-r} & 0 \\ \widetilde{W}_{31} & 0 & \widetilde{W}_{33} \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 \end{bmatrix},$$

where V does the compression $\begin{bmatrix} W_{11} & W_{12} \end{bmatrix} V = \begin{bmatrix} \widetilde{W}_{11}^* & 0 \end{bmatrix}$ of the first two blocks and $\widetilde{W}$ does the further reduction of the first block row to $\begin{bmatrix} I_r & 0 & 0 \end{bmatrix}$.
Step 3 : Display the uncontrollable part $X(\lambda)$ using the transformations U, V and $\widetilde{W}$

$$\begin{bmatrix} U & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} \widetilde{W} = \begin{bmatrix} X_{\overline{c}}(\lambda) & 0 & 0 \\ \times & A_c(\lambda) & -B_c(\lambda) \\ \times & C_c(\lambda) & D_c(\lambda) \end{bmatrix},$$

where we have used the notations introduced in Section 7.3, and the resulting $\times$ entries are of no interest because they do not contribute to the transfer function $R_c(\lambda) := C_c(\lambda)A_c(\lambda)^{-1}B_c(\lambda) + D_c(\lambda)$.

7.5 Numerical results

We illustrate in this section that the reduction procedure presented in this chapter may improve the accuracy of computed eigenvalues, with a polynomial example and a rational one.

Example 7.5.1. We consider the 2×2 polynomial matrix

$$P(\lambda) = \begin{bmatrix} e_5(\lambda) & 0 \\ 0 & e_1(\lambda) \end{bmatrix},$$

where $e_5(\lambda)$ is a polynomial of degree 5 with coefficients $[9.6367e-01 \quad -5.4026e-07 \quad 2.6333e-01 \quad -1.1101e-04 \quad -2.9955e-04 \quad 4.4650e-02]$, ordered by descending powers of λ, and $e_1(\lambda)$ is a polynomial of degree 1 with coefficients $[-2.1886e-03 \quad -1.0000e+00]$, that were randomly chosen. Expanding this fifth order polynomial matrix as

$$P(\lambda) = P_0 + P_1\lambda + \cdots + P_5\lambda^5,$$

a linear system matrix $S_P(\lambda)$ of $P(\lambda)$ is given by the following 10×10 pencil:

$$S_P(\lambda) = \left[\begin{array}{cccc|c} I_2 & -\lambda I_2 & & & P_1 \\ & I_2 & -\lambda I_2 & & P_2 \\ & & I_2 & -\lambda I_2 & P_3 \\ & & & I_2 & P_4 + \lambda P_5 \\ \hline -\lambda I_2 & & & & P_0 \end{array}\right].$$

The six finite Smith zeros of $P(\lambda)$ are clearly those of the scalar polynomials $e_1(\lambda)$ and $e_5(\lambda)$. These are also the finite zeros of $S_P(\lambda)$, since $S_P(\lambda)$ is minimal. However, $S_P(\lambda)$ is not strongly minimal if P_5 is singular and, in fact, it has 4 eigenvalues at infinity (in the sense of [44]). But in the McMillan sense, $P(\lambda)$ has *no* infinite zeros. The deflation procedure that we derived in this chapter precisely gets rid of the *extraneous* infinite eigenvalues of $S_P(\lambda)$. The numerical tests show that the sensitivity of the true McMillan zeros also can benefit from this.

In this example we compare the roots computed by four different methods:

1. computing the roots of the scalar polynomials and appending four ∞ roots,
2. computing the generalized eigenvalues of $S_P(\lambda)$,
3. computing the roots of $QS_P(\lambda)Z$ for random orthogonal matrices Q and Z,
4. computing the roots of the *minimal* pencil obtained by our method.

The first column are the so-called "correct" eigenvalues λ_i, corresponding to the first method, the next three columns are the corresponding errors $\delta_i^{(k)} := |\lambda_i - \hat{\lambda}_i^{(k)}|$, $k = 2, 3, 4$, of the above three methods[1]. The *extraneous* eigenvalues that are deflated in our approach are put between brackets.

We notice that for the largest finite eigenvalue of the order of 10^2 the QZ algorithm applied to $S_P(\lambda)$ gets 14 digits of relative accuracy but, when deflating the four uncontrollable eigenvalues at ∞, our method recovers a relative accuracy of 16 digits.

[1]An error $\delta_i^{(k)}$ is NaN when it is the indeterminate form Inf − Inf. However, some of the eigenvalues at ∞ are computed as a large but finite number and, then, the corresponding error is Inf.

λ_i	$\delta_i^{(2)}$	$\delta_i^{(3)}$	$\delta_i^{(4)}$
-4.5811e-01	2.7756e-16	4.4409e-16	1.1102e-16
3.5076e-01 + 3.5785e-01i	9.5020e-16	1.1102e-16	4.0030e-16
3.5076e-01 - 3.5785e-01i	9.5020e-16	1.1102e-16	4.0030e-16
-1.2170e-01 + 6.2287e-01i	6.7589e-16	7.8945e-16	2.2248e-16
-1.2170e-01 - 6.2287e-01i	6.7589e-16	7.8945e-16	2.2248e-16
-4.5691e+02	2.9559e-12	2.7285e-12	5.6843e-14
Inf	NaN	NaN	(Inf)
Inf	NaN	NaN	(Inf)
Inf	NaN	NaN	(Inf)
Inf	NaN	NaN	(Inf)

Table 7.1: The correct generalized λ_i and the corresponding accuracies δ_i^k for the three different calculations

Example 7.5.2. The second example is the rational matrix $R(\lambda)$ in (7.4) with $c = 1$.

$$R(\lambda) = \begin{bmatrix} e_5(\lambda) & 0 \\ 1/\lambda & e_1(\lambda) \end{bmatrix} = P_0 + P_1\lambda + \cdots + P_5\lambda^5 + \begin{bmatrix} 0 & 0 \\ 1/\lambda & 0 \end{bmatrix},$$

by using the notation of the example above. In this case, $e_5(\lambda)$ has the row vector $[4.7865e{-}02 \;\; 1.4279e{-}04 \;\; 2.4361e{-}03 \;\; {-}1.5336e{-}02 \;\; {-}9.9155e{-}01 \;\; 1.1948e{-}01]$ as coefficients, and $e_1(\lambda)$ has the row vector $[6.5250e{-}03 \;\; 9.9997e{-}01]$. We consider the 12×12 linear system matrix

$$S_R(\lambda) = \left[\begin{array}{cccccc|c} \lambda I_2 - A & & & & & & -B \\ & I_2 & -\lambda I_2 & & & & P_1 \\ & & I_2 & -\lambda I_2 & & & P_2 \\ & & & I_2 & -\lambda I_2 & & P_3 \\ & & & & I_2 & & P_4 + \lambda P_5 \\ \hline C & -\lambda I_2 & & & & & P_0 \end{array}\right],$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a non-minimal realization of the strictly proper rational function $1/\lambda$. In fact, the matrix A in the realization triple (A, B, C) has two eigenvalues at $\lambda = 0$, of which one is uncontrollable since $1/\lambda$ only has a pole at 0 of order 1. This is an artificial example since we could have realized the strictly proper part by using a minimal triple (A, B, C) by removing the uncontrollable eigenvalue, but this is precisely what our reduction procedure does simultaneously for finite and infinite uncontrollable eigenvalues. The quantities given in the following table are defined as in the previous example, except that we added two roots at 0 corresponding to the "exact" eigenvalues.

λ_i	$\delta_i^{(2)}$	$\delta_i^{(3)}$	$\delta_i^{(4)}$
0	0	8.1752e-09	(4.5874e-16)
0	3.6752e-18	8.1752e-09	5.3729e-16
1.2028e-01	1.8041e-16	9.7145e-17	9.7145e-17
2.1135e+00	1.7764e-15	2.6645e-15	1.3323e-15
-2.1404e+00	1.7764e-15	2.2204e-15	8.8818e-16
-4.8180e-02 + 2.1412e+00i	2.3216e-15	1.7990e-15	4.0614e-15
-4.8180e-02 - 2.1412e+00i	2.3216e-15	1.7990e-15	4.0614e-15
-1.5325e+02	2.5580e-13	1.5321e-07	5.6843e-14
Inf	NaN	Inf	(Inf)
Inf	NaN	Inf	(Inf)
Inf	NaN	NaN	(NaN)
Inf	NaN	NaN	(NaN)

Table 7.2: The correct generalized λ_i and the corresponding accuracies δ_i^k for the three different calculations

In this example the QZ algorithm applied to $S_R(\lambda)$ recovers well all generalized eigenvalues. When applying the QZ algorithm to an orthogonally equivalent pencil $QS_R(\lambda)Z$ both eigenvalues at 0 gets perturbed to two roots of the order of the square root of the machine precision, which can be expected. But when deflating the uncontrollable eigenvalue at 0 part of the accuracy gets restored.

These two examples show that deflating uncontrollable eigenvalues may improve the sensitivity of the remaining eigenvalues which may improve the accuracy of their computation.

Chapter 8

Constructing strongly minimal linearizations for rational matrices from Laurent expansions: preserving structures

In this chapter we will show a procedure to construct strongly minimal linearizations (recall Definition 7.2.4) of arbitrary rational matrices $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ from their Laurent expansions around the point at infinity. Namely,

$$R(\lambda) = R_d\lambda^d + \ldots + R_1\lambda + R_0 + R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \ldots, \qquad (8.1)$$

which is convergent for sufficiently large $\lambda \in \mathbb{C}$. We will first pay special attention to the construction of strongly minimal linearizations for the particular case of polynomial matrices $P(\lambda)$, written in the form

$$P(\lambda) = P_d\lambda^d + \ldots + P_1\lambda + P_0.$$

If the rational matrix $R(\lambda)$ is square, i.e., $m = n$, and has a particular type of self-conjugate structure then the coefficients $R_i \in \mathbb{C}^{m\times m}$ of its expansion also inherit the self-conjugate structure and the poles and zeros of $R(\lambda)$ appear in self-conjugate pairs. We will also show how to construct strongly minimal linearizations preserving the structure for both polynomial and rational matrices. All the results in this chapter appear in [34].

We will consider four types of self-conjugate rational matrices, two with respect to the real line and two with respect to the imaginary axis:

- The Hermitian and skew-Hermitian rational matrices $R(\lambda)$, with respect to the real line, satisfy

$$[R(\lambda)]^* = R(\overline{\lambda}), \quad \text{and} \quad [R(\lambda)]^* = -R(\overline{\lambda}),$$

respectively. They have poles and zeros that are mirror images with respect to the real line $\mathbb{R}$, and have coefficient matrices R_i that are Hermitian (i.e. $R_i^* = R_i$) and skew-Hermitian (i.e. $R_i^* = -R_i$), respectively.

- The para-Hermitian and para-skew-Hermitian rational matrices, with respect to the imaginary axis, satisfy

$$[R(\lambda)]^* = R(-\overline{\lambda}), \quad \text{and} \quad [R(\lambda)]^* = -R(-\overline{\lambda}),$$

respectively. They have poles and zeros that are mirror images with respect to the imaginary line $\jmath\mathbb{R}$, and have scaled coefficient matrices $\jmath^i R_i$ that are Hermitian and skew-Hermitian, respectively.

Para-Hermitian and para-skew-Hermitian structures are also called alternating structures. There are equivalent definitions for real rational matrices, where all coefficient matrices R_i are real. Namely, (skew-)symmetric and para-(skew-)symmetric rational matrices. In these cases, poles and zeros appear in complex conjugate pairs as well.

We will divide the construction of strongly minimal linearizations of arbitrary and self-conjugate rational matrices in different sections. In Section 8.1, we construct strongly minimal linearizations for arbitrary polynomial matrices and, in Section 8.2, for self-conjugate polynomial matrices. In Section 8.3, we construct strongly minimal linearizations for arbitrary strictly proper rational matrices and, in Section 8.4, for self-conjugate strictly proper rational matrices. In Section 8.5 we state the results for both arbitrary and self-conjugate rational matrices, by combining the results in previous sections.

8.1 For arbitrary polynomial matrices

In this section we focus on constructing explicitly a strongly minimal linearization for any given polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ of degree $d > 1$:

$$P(\lambda) := P_0 + P_1\lambda + \cdots + P_d\lambda^d. \tag{8.2}$$

Such a strongly minimal linearization is constructed in Theorem 8.1.2 and we will prove in Section 8.2 that it inherits the structure of $P(\lambda)$, when $P(\lambda)$ possesses any of the self-conjugate structures considered in this work. The construction uses three pencils associated with $P(\lambda)$ that have appeared before in the literature. They are described in the following paragraphs.

The pencil

$$L_r(\lambda) := \left[\begin{array}{c|c} A_r(\lambda) & -B_r(\lambda) \\ \hline C_r(\lambda) & D_r(\lambda) \end{array}\right] := \left[\begin{array}{cccc|c} -I_n & \lambda I_n & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & -I_n & \lambda I_n & 0 \\ & & & -I_n & \lambda I_n \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right] \tag{8.3}$$

was used in the classical reference [87]. It is easy to see that $L_r(\lambda)$ is a linear polynomial system matrix of $P(\lambda)$, since $P(\lambda) = D_r(\lambda) + C_r(\lambda)A_r(\lambda)^{-1}B_r(\lambda)$, and that it is minimal for all finite λ. For the point at ∞, E-controllability is clearly satisfied but E-observability is only satisfied if the matrix P_d has full column rank n. Thus, $L_r(\lambda)$ is *not* a strongly minimal linearization of $P(\lambda)$ when P_d does not have full column rank. However, note that $L_r(\lambda)$ is always a GLR-strong linearization of $P(\lambda)$. This can be seen, for instance, by noting that if the two block rows in (8.3) are interchanged, we obtain one of the block Kronecker linearizations (with only one block column) associated to $P(\lambda)$ defined in [26, Section 4]. The pencil (8.3) has a structure similar to that of the classical first or row Frobenius companion form.

The pencil

$$L_c(\lambda) := \left[\begin{array}{c|c} A_c(\lambda) & -B_c(\lambda) \\ \hline C_c(\lambda) & D_c(\lambda) \end{array}\right] := \left[\begin{array}{cccc|c} -I_m & & & & \lambda P_d \\ \lambda I_m & \ddots & & & \vdots \\ & \ddots & -I_m & & \vdots \\ & & \lambda I_m & -I_m & \lambda P_2 \\ \hline 0 & \dots & 0 & \lambda I_m & \lambda P_1 + P_0 \end{array}\right] \tag{8.4}$$

is in some sense "dual" to (8.3). It is also a linear polynomial system matrix of $P(\lambda)$, since $P(\lambda) = D_c(\lambda) + C_c(\lambda)A_c(\lambda)^{-1}B_c(\lambda)$. Moreover, $L_c(\lambda)$ is strongly E-observable, but not necessarily strongly E-controllable, unless P_d has full row rank. As a consequence, $L_c(\lambda)$ is a strongly minimal linearization of $P(\lambda)$ if and only if P_d has full row rank. However, $L_c(\lambda)$ is always a GLR-strong linearization of $P(\lambda)$. The pencil (8.4) has a structure similar to that of the classical second or column Frobenius companion form.

The pencil

$$L_s(\lambda) := \left[\begin{array}{c|c} A_s(\lambda) & -B_s(\lambda) \\ \hline C_s(\lambda) & D_s(\lambda) \end{array}\right] := \left[\begin{array}{cccc|c} & & & -P_d & \lambda P_d \\ & & \iddots & \lambda P_d - P_{d-1} & \vdots \\ & -P_d & \iddots & \vdots & \vdots \\ -P_d & \lambda P_d - P_{d-1} & \dots & \lambda P_3 - P_2 & \lambda P_2 \\ \hline \lambda P_d & \dots & \dots & \lambda P_2 & \lambda P_1 + P_0 \end{array}\right] \tag{8.5}$$

was originally proposed by Lancaster in [58, pp. 58-59] for regular polynomial matrices with P_d invertible. In this chapter, we use it for arbitrary polynomial matrices, including rectangular ones. $L_s(\lambda)$ has the advantage to preserve the Hermitian or skew-Hermitian nature of the coefficients of the linearization, if $P(\lambda)$ happens to have coefficients with such properties. The pencil $L_s(\lambda)$ has been also studied more recently in [53, 63], where it is seen as one of the pencils of the standard basis of the linear space $\mathbb{DL}(P)$ of pencils related to $P(\lambda)$. It is well known that $L_s(\lambda)$ is a GLR-strong linearization of $P(\lambda)$ if and only if P_d is invertible [21, 63]. In fact, in this case, $L_s(\lambda)$ is also a strongly minimal linearization of $P(\lambda)$ since it is strongly minimal and $P(\lambda) = D_s(\lambda) + C_s(\lambda)A_s(\lambda)^{-1}B_s(\lambda)$. However, if P_d is not invertible, $L_s(\lambda)$ is not a linearization of $P(\lambda)$ in any of the senses considered in the literature and, even more, it is not a Rosenbrock polynomial system matrix of $P(\lambda)$ since $A_s(\lambda)$ is not regular. Despite of this fact, $L_s(\lambda)$ is our starting point for constructing the strongly minimal linearization of $P(\lambda)$ of interest in this work.

The constant block Hankel matrix T defined in the next equation

$$T := \begin{bmatrix} & & & P_d \\ & & \iddots & P_{d-1} \\ & P_d & \iddots & \vdots \\ P_d & P_{d-1} & \dots & P_2 \end{bmatrix} \tag{8.6}$$

plays a key role in the rest of the chapter. To begin with, it allows us to obtain the following relations

$$\begin{bmatrix} A_s(\lambda) & -B_s(\lambda) \end{bmatrix} = T \begin{bmatrix} A_r(\lambda) & -B_r(\lambda) \end{bmatrix}, \quad \begin{bmatrix} A_s(\lambda) \\ C_s(\lambda) \end{bmatrix} = \begin{bmatrix} A_c(\lambda) \\ C_c(\lambda) \end{bmatrix} T, \tag{8.7}$$

between submatrices of the pencils $L_s(\lambda)$, $L_r(\lambda)$ and $L_c(\lambda)$. The matrix T is invertible if and only if P_d is square and invertible. Otherwise, T is singular and this is the case that requires a careful analysis.

In [87], it was shown how to derive from the linear polynomial system matrix $L_r(\lambda)$ of $P(\lambda)$, a smaller linear polynomial system matrix $\widehat{L}_r(\lambda)$ that is both strongly E-controllable and E-observable, and hence strongly minimal, by using only multiplications by constant unitary matrices. This was obtained by deflating the unobservable infinite eigenvalues from the pencil $L_r(\lambda)$. Moreover, the obtained pencil $\widehat{L}_r(\lambda)$ allows us to recover the complete list of structural data of $P(\lambda)$. The reduction procedure in [87] has been extended to arbitrary linear polynomial system matrices of arbitrary rational matrices $R(\lambda)$ in Chapter 7, where it is proved that the obtained strongly minimal linear polynomial system matrix has as transfer function matrix $Q_1R(\lambda)Q_2$, where Q_1 and Q_2 are constant invertible matrices. We emphasize that the procedures in [87, 32] lead to stable and efficient numerical algorithms since both are based on unitary transformations.

We show in Theorem 8.1.2 that a procedure similar to that in [87] can be applied to $L_s(\lambda)$ in order to derive a strongly minimal linear polynomial system matrix $\widehat{L}_s(\lambda)$ of $P(\lambda)$, despite of the fact that if P_d is not square or invertible, then $L_s(\lambda)$ *is not a Rosenbrock polynomial system matrix* since $A_s(\lambda)$ is then not regular. Moreover, we remark that the procedure in Theorem 8.1.2 is much simpler than those in [87, 32] and that, as said before, it yields a polynomial system matrix whose transfer function matrix is precisely $P(\lambda)$. Before stating and proving Theorem 8.1.2, we prove the simple auxiliary Lemma 8.1.1 and introduce some other auxiliary concepts.

A rational matrix $G(\lambda) \in \mathbb{C}(\lambda)^{p\times n}$ (with $p < n$) is said to be a rational basis if its rows form a basis of the rational subspace they span, i.e., if it has full row normal rank. Two rational bases $G(\lambda) \in \mathbb{C}(\lambda)^{p\times n}$ and $H(\lambda) \in \mathbb{C}(\lambda)^{q\times n}$ are said to be dual if $p + q = n$, and $G(\lambda)\, H(\lambda)^T = 0$.

Lemma 8.1.1. *Let*

$$S(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{C}[\lambda]^{(p+m)\times(p+n)}$$

be a polynomial system matrix, where $A(\lambda)$ is assumed to be regular. Let $H(\lambda)$ be a rational basis of the form $H(\lambda) := \begin{bmatrix} M(\lambda) & I_n \end{bmatrix}$ dual to $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix}$, i.e., such that $\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} H(\lambda)^T = 0$, then $\begin{bmatrix} C(\lambda) & D(\lambda) \end{bmatrix} H(\lambda)^T$ is the transfer function of $S(\lambda)$.

Proof. The equation

$$\begin{bmatrix} A(\lambda) & -B(\lambda) \end{bmatrix} \begin{bmatrix} M(\lambda)^T \\ I_n \end{bmatrix} = 0$$

implies $A(\lambda)M(\lambda)^T = B(\lambda)$ and, since $A(\lambda)$ is regular, $M(\lambda)^T = A(\lambda)^{-1}B(\lambda)$. Thus $\begin{bmatrix} C(\lambda) & D(\lambda) \end{bmatrix} H(\lambda)^T = C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda)$. ∎

Theorem 8.1.2. *Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}$ be a polynomial matrix as in* (8.2). *Let T be the block Hankel matrix in* (8.6) *and $r := \operatorname{rank} T$. Let $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ be unitary matrices that "compress" the matrix T as follows:*

$$U^*TV = \begin{bmatrix} 0 & 0 \\ 0 & U_2^*TV_2 \end{bmatrix} =: \begin{bmatrix} 0 & 0 \\ 0 & \widehat{T} \end{bmatrix}, \tag{8.8}$$

where $\widehat{T}$ is of dimension $r \times r$ and invertible. Then, if $L_s(\lambda)$ is the matrix pencil in (8.5), *the pencil* $\operatorname{diag}(U^*, I_m)\, L_s(\lambda)\, \operatorname{diag}(V, I_n)$ *is equal to the "compressed" pencil*

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline 0 & \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] := \left[\begin{array}{c|c} U^*A_s(\lambda)V & -U^*B_s(\lambda) \\ \hline C_s(\lambda)V & D_s(\lambda) \end{array}\right], \tag{8.9}$$

and

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] \tag{8.10}$$

is a strongly minimal linearization of $P(\lambda)$*, where* $\widehat{A}_s(\lambda) \in \mathbb{C}[\lambda]^{r\times r}$ *is regular. In particular,* $P(\lambda) = \widehat{D}_s(\lambda) + \widehat{C}_s(\lambda)\widehat{A}_s(\lambda)^{-1}\widehat{B}_s(\lambda)$*.*

Proof. It follows from (8.7) and the strong E-controllability of $\left[\, A_r(\lambda) \; -B_r(\lambda) \,\right]$ that $\left[\, A_s(\lambda) \; -B_s(\lambda) \,\right]$ has rank r for all λ, infinity included, and that its left null space is spanned by the rows of U_1^*. Likewise, it follows from (8.7) and the strong E-observability of $\left[\begin{smallmatrix} A_c(\lambda) \\ C_c(\lambda) \end{smallmatrix}\right]$ that $\left[\begin{smallmatrix} A_s(\lambda) \\ C_s(\lambda) \end{smallmatrix}\right]$ has rank r for all λ, infinity included and that its right null space is spanned by the columns of V_1. This proves the compressed form (8.9).

We then prove that the $r \times r$ matrix pencil $\widehat{A}_s(\lambda)$ is regular. This follows from the identity

$$\widehat{A}_s(\lambda) = U_2^* T A_r(\lambda) V_2, \quad \text{where} \quad A_r(\lambda) = \begin{bmatrix} -I_n & \lambda I_n & & \\ & \ddots & \ddots & \\ & & -I_n & \lambda I_n \\ & & & -I_n \end{bmatrix} \tag{8.11}$$

which, for $\lambda = 0$ becomes $\widehat{A}_s(0) = -U_2^* T V_2 = -\widehat{T}$.

The fact that $\left[\, \widehat{A}_s(\lambda) \; -\widehat{B}_s(\lambda) \,\right]$ has full row rank r for all λ, ∞ included, follows from the identity

$$\left[\begin{array}{ccc} 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \end{array}\right] = \widehat{T} V_2^* \left[\begin{array}{cc} A_r(\lambda) & -B_r(\lambda) \end{array}\right] \operatorname{diag}(V, I_n).$$

The fact that $\left[\begin{smallmatrix} \widehat{A}_s(\lambda) \\ \widehat{C}_s(\lambda) \end{smallmatrix}\right]$ has full column rank r for all λ, ∞ included, follows from the dual identity

$$\begin{bmatrix} 0 \\ \widehat{A}_s(\lambda) \\ \widehat{C}_s(\lambda) \end{bmatrix} = \operatorname{diag}(U^*, I_m) \begin{bmatrix} A_c(\lambda) \\ C_c(\lambda) \end{bmatrix} U_2 \widehat{T}.$$

Together, these properties guarantee that $\widehat{L}_s(\lambda)$ is a strongly minimal linear polynomial system matrix. Its transfer function $\widehat{C}_s(\lambda)\widehat{A}_s(\lambda)^{-1}\widehat{B}_s(\lambda) + \widehat{D}_s(\lambda)$ can then be obtained from a particular dual basis $N(\lambda) \in \mathbb{C}(\lambda)^{n\times(r+n)}$ of $\left[\, \widehat{A}_s(\lambda) \; -\widehat{B}_s(\lambda) \,\right]$, by using Lemma 8.1.1. Since

$$\begin{aligned} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \end{array}\right] \operatorname{diag}(V^*, I_n) &= U^* \left[\begin{array}{cc} A_s(\lambda) & -B_s(\lambda) \end{array}\right] \\ &= U^* T \left[\begin{array}{cc} A_r(\lambda) & -B_r(\lambda) \end{array}\right], \end{aligned} \tag{8.12}$$

it follows that

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \end{array}\right] \operatorname{diag}(V^*, I_n) \begin{bmatrix} \lambda^{d-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \end{bmatrix} = 0$$

and hence that

$$\left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \end{array}\right] \operatorname{diag}(\begin{bmatrix} 0 & I_r \end{bmatrix} V^*, I_n) \begin{bmatrix} \lambda^{d-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \end{bmatrix} = 0.$$

Therefore, by setting

$$N(\lambda)^T := \operatorname{diag}(\begin{bmatrix} 0 & I_r \end{bmatrix} V^*, I_n) \begin{bmatrix} \lambda^{d-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \end{bmatrix} \in \mathbb{C}[\lambda]^{(r+n)\times n},$$

we have that $N(\lambda)$ is a dual basis of $\left[\begin{smallmatrix} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \end{smallmatrix}\right]$ with its rightmost block equal to I_n. By Lemma 8.1.1, and using the fact that

$$\widehat{C}_s(\lambda) \begin{bmatrix} 0 & I_r \end{bmatrix} V^* = \begin{bmatrix} 0 & \widehat{C}_s(\lambda) \end{bmatrix} V^* = C_s(\lambda),$$

we obtain that

$$\begin{bmatrix} \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{bmatrix} N(\lambda)^T = \begin{bmatrix} C_s(\lambda) & D_s(\lambda) \end{bmatrix} \begin{bmatrix} \lambda^{d-1} I_n \\ \vdots \\ \lambda I_n \\ I_n \end{bmatrix} = P(\lambda)$$

is the transfer function of $\widehat{L}_s(\lambda)$. ■

Remark 8.1.3. Once the unitary matrices U and V and the matrix $\widehat{T}$ in (8.8) are computed, Theorem 8.1.2 yields an efficient and stable algorithm for computing the strongly minimal linear realization $\{\widehat{A}_s(\lambda), \widehat{B}_s(\lambda), \widehat{C}_s(\lambda), \widehat{D}_s(\lambda)\}$ of $P(\lambda)$. An expensive method for computing U, V and $\widehat{T}$ is to compute the SVD of T. A cheaper method is to use the complete orthogonal decomposition in [45, Sec. 5.4.2], which amounts to compute two QR factorizations. The block Hankel structure of T (which by flipping the order of the block rows becomes block Toeplitz) allows us to use the very fast and stable method in [88, Sec. IV], which makes all the computations on $m \times n$ submatrices. The method in [88] has the additional advantage that if $r_d = \operatorname{rank}(P_d)$, then the rows of $U^*(1 : (m - r_d), 1 : m)$ and the columns of $V(1 : n, 1 : (n - r_d))$ (where we used MATLAB's notation) of the computed U and V are, respectively, unitary bases of the left and right nullspaces of P_d. Recall that these subspaces are precisely the left and right eigenspaces associated to the infinite eigenvalue of $P(\lambda)$ when $P(\lambda)$ is regular.

Remark 8.1.4. Even though the pencil $L_s(\lambda)$ in (8.5) is not a Rosenbrock polynomial system matrix and neither is a GLR-linearization of $P(\lambda)$ if P_d is rectangular or square and singular, it is easy to see, by using unimodular transformations that are well-known in the literature, that it has the same finite eigenvalues as $P(\lambda)$ with the same partial multiplicities. For this purpose, note that

$$V(\lambda)L_s(\lambda)W(\lambda) = \operatorname{diag}(-T, P(\lambda)),$$

where

$$V(\lambda) := \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & \\ \lambda^{d-1} & \cdots & \lambda^2 & \lambda & 1 \end{bmatrix} \otimes I_m, \quad W(\lambda) := \begin{bmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{d-1} \\ & 1 & \lambda & \ddots & \vdots \\ & & \ddots & \ddots & \lambda^2 \\ & & & 1 & \lambda \\ & & & & 1 \end{bmatrix} \otimes I_m.$$

Since the polynomial matrices $V(\lambda)$ and $W(\lambda)$ are unimodular, and $\operatorname{diag}(-T, P(\lambda))$ is strictly equivalent to $\operatorname{diag}(0, I_r, P(\lambda))$, this implies that $L_s(\lambda)$ is unimodularly equivalent to $\operatorname{diag}(0, I_r, P(\lambda))$. Therefore, $L_s(\lambda)$ and $P(\lambda)$ have the same finite eigenvalues with the same partial multiplicities. Of course, this also follows from Theorem 8.1.2 and the properties of strongly minimal linearizations studied in Section 7.2.2. However, note that $L_s(\lambda)$ is *not* a GLR-linearization of $P(\lambda)$ because $L_s(\lambda)$ is *not* unimodularly equivalent to $\operatorname{diag}(I, P(\lambda))$.

On the other hand, if we consider the pencil $\widehat{L}_s(\lambda)$ in (8.10), then Theorem 8.1.2 proves that $L_s(\lambda)$ is strictly equivalent to $\operatorname{diag}(0, \widehat{L}_s(\lambda))$. Combining the results above, we see that $\widehat{L}_s(\lambda)$ and $\operatorname{diag}(I_r, P(\lambda))$ have the same normal rank and the same finite eigenvalues and partial multiplicities, which implies that $\widehat{L}_s(\lambda)$ and $\operatorname{diag}(I_r, P(\lambda))$ are unimodularly equivalent [39] and, therefore, that $\widehat{L}_s(\lambda)$ is a GLR-linearization of $P(\lambda)$. This is a particular instance of the result mentioned in Section 7.2.2 that any minimal linear polynomial system matrix of $P(\lambda)$ is a GLR-linearization of $P(\lambda)$.

8.2 For self-conjugate polynomial matrices

The main purpose of this section is to prove that the strongly minimal linearization (8.10) of the matrix polynomial $P(\lambda)$ developed in Theorem 8.1.2 inherits the structure of $P(\lambda)$ for any of the four structures considered in this work.

We start by reviewing the four structures of interest in this chapter and their properties. In these definitions, note that if $P(\lambda)$ is the complex polynomial matrix in (8.2), then

$$[P(\lambda)]^* := P_0^* + P_1^*\overline{\lambda} + \cdots + P_d^*\overline{\lambda}^d.$$

The considered self-conjugate structures are:

1. Hermitian polynomial matrices, which are defined as those satisfying $[P(\lambda)]^* = P(\overline{\lambda})$. Equivalently, they are Hermitian for $\lambda \in \mathbb{R}$ or have Hermitian coefficients $P_i^* = P_i$. They have a set of eigenvalues Λ that is symmetric with respect to the real axis: $\Lambda = \overline{\Lambda}$.

2. Skew-Hermitian polynomial matrices, which are defined as those satisfying $[P(\lambda)]^* = -P(\overline{\lambda})$. Equivalently, they are skew-Hermitian for $\lambda \in \mathbb{R}$ or have skew-Hermitian coefficients $P_i^* = -P_i$. They have a set of eigenvalues Λ that is symmetric with respect to the real axis: $\Lambda = \overline{\Lambda}$.

3. Para-Hermitian polynomial matrices, which are those satisfying $[P(\lambda)]^* = P(-\overline{\lambda})$. Equivalently, they are Hermitian for $\lambda \in \jmath\mathbb{R}$, i.e., for λ on the imaginary axis, or have scaled Hermitian coefficients $P_i^* = (-1)^i P_i$. They have a set of eigenvalues Λ that is symmetric with respect to the imaginary axis: $\Lambda = -\overline{\Lambda}$.

4. Para-skew-Hermitian polynomial matrices, which are defined as those satisfying $[P(\lambda)]^* = -P(-\overline{\lambda})$. Equivalently, they are skew-Hermitian for $\lambda \in \jmath\mathbb{R}$ or have skew-Hermitian scaled coefficients $P_i^* = (-1)^{(i+1)} P_i$. They have a set of eigenvalues Λ that is symmetric with respect to the imaginary axis: $\Lambda = -\overline{\Lambda}$.

When the polynomial matrix $P(\lambda)$ has real coefficients P_i, these conditions become conditions on the transpose of each P_i, and the polynomial matrices are said to be symmetric, skew-symmetric, para-symmetric and para-skew-symmetric.

We first point out that the block Hankel matrix T defined in (8.6) and its compression (8.8) inherit particular properties from the self-conjugate structures defined above.

Lemma 8.2.1. *Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ be a polynomial matrix as in* (8.2)*. Let us define the scaling matrix $S := \mathrm{diag}((-1)^{(d-1)} I_m, \ldots, (-1)^2 I_m, -I_m)$. Then the block Hankel matrix T in* (8.6) *satisfies the following equations*

1. *for Hermitian $P(\lambda)$: $P_i^* = P_i$ and $T^* = T$,*

2. *for skew-Hermitian $P(\lambda)$: $P_i^* = -P_i$ and $T^* = -T$,*

3. *for para-Hermitian $P(\lambda)$: $P_i^* = (-1)^i P_i$ and $(ST)^* = ST$,*

4. *for para-skew-Hermitian $P(\lambda)$: $P_i^* = (-1)^{(i+1)} P_i$ and $(ST)^* = -ST$.*

The left and right transformations U and V of Theorem 8.1.2 can then be chosen as $U = V$ in the Hermitian and skew-Hermitian cases and as $U = SV$ in the para-Hermitian and para-skew-Hermitian cases.

Proof. The symmetries of the coefficient matrices P_i trivially yield the four types of symmetries of T mentioned in the Theorem. For the compression (8.8), we can then choose $U = V$ in the Hermitian and skew-Hermitian cases because T is normal, and we can choose $U = SV$ in the para-Hermitian and para-skew-Hermitian cases because ST is then normal. ∎

This implies that in the decomposition of Theorem 8.1.2, it suffices to construct a transformation V that compresses the columns of T in order to obtain a rank r factorization

$$U^*TV = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{T} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & U_2^*TV_2 \end{bmatrix} \tag{8.13}$$

where $\widehat{T}$ is $r \times r$ and is invertible. This then leads to the following theorem.

Theorem 8.2.2. *Let $P(\lambda) \in \mathbb{C}[\lambda]^{m\times m}$ be a polynomial matrix as in (8.2), with a Hermitian, skew-Hermitian, para-Hermitian or para-skew-Hermitian structure. Let U, V be the unitary matrices appearing in (8.8), where $U = V$ in the Hermitian and skew-Hermitian cases, and $U = SV$ in the para-Hermitian and para-skew-Hermitian cases with $S := \mathrm{diag}((-1)^{(d-1)} I_m, \ldots, (-1)^2 I_m, -I_m)$. Then the linear polynomial system matrix*

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right],$$

defined in Theorem 8.1.2, is a strongly minimal linear polynomial system matrix of $P(\lambda)$ with the same self-conjugate structure as $P(\lambda)$.

Proof. Let us denote the original pencil in (8.5) as

$$L_s(\lambda) = L_0 + \lambda L_1,$$

then we have the following properties in the four self-conjugate cases :

1. for Hermitian $P(\lambda)$,
$$L_0^* = L_0, \quad \text{and} \quad L_1^* = L_1,$$

2. for skew-Hermitian $P(\lambda)$,
$$L_0^* = -L_0, \quad \text{and} \quad L_1^* = -L_1,$$

3. for para-Hermitian $P(\lambda)$,

$$L_0^* \operatorname{diag}(S, I_m) = \operatorname{diag}(S, I_m) L_0, \quad \text{and} \quad L_1^* \operatorname{diag}(S, I_m) = -\operatorname{diag}(S, I_m) L_1,$$

4. for para-skew-Hermitian $P(\lambda)$,

$$L_0^* \operatorname{diag}(S, I_m) = -\operatorname{diag}(S, I_m) L_0, \quad \text{and} \quad L_1^* \operatorname{diag}(S, I_m) = \operatorname{diag}(S, I_m) L_1.$$

If we choose in the first two cases $U = V$, and in the last two cases $U = SV$ then we obtain for the transformed pair of matrices

$$\tilde{L}_0 := \operatorname{diag}(U^*, I_m) L_0 \operatorname{diag}(V, I_m), \quad \text{and} \quad \tilde{L}_1 := \operatorname{diag}(U^*, I_m) L_1 \operatorname{diag}(V, I_m)$$

the properties

1. for Hermitian $P(\lambda)$,
$$\tilde{L}_0^* = \tilde{L}_0, \quad \text{and} \quad \tilde{L}_1^* = \tilde{L}_1,$$

2. for skew-Hermitian $P(\lambda)$,
$$\tilde{L}_0^* = -\tilde{L}_0, \quad \text{and} \quad \tilde{L}_1^* = -\tilde{L}_1,$$

3. for para-Hermitian $P(\lambda)$,
$$\tilde{L}_0^* = \tilde{L}_0, \quad \text{and} \quad \tilde{L}_1^* = -\tilde{L}_1,$$

4. for para-skew-Hermitian $P(\lambda)$,
$$\tilde{L}_0^* = -\tilde{L}_0, \quad \text{and} \quad \tilde{L}_1^* = \tilde{L}_1,$$

and moreover, their first $m(d-1) - r$ columns and rows are zero because of (8.13). The pencil $\tilde{L}_s(\lambda)$ thus has the same self-conjugate structure as $P(\lambda)$, and so does the deflated pencil $\widehat{L}_s(\lambda)$. The strong minimality follows from Theorem 8.1.2. ■

Remark 8.2.3. *We remark that the procedure presented in Theorem 8.1.2 has an interpretation in terms of strongly minimal realizations of strictly proper rational matrices that have all its poles at* 0. *To see that, we apply the change of variable* $\lambda = 1/\mu$ *to the system matrix*

$$\left[\begin{array}{c|c} \lambda E - F & \lambda G \\ \hline \lambda H & 0 \end{array}\right] := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & 0 \end{array}\right]$$

and we multiply it by μ. *Then, we obtain a new linear polynomial system matrix*

$$\left[\begin{array}{c|c} E - \mu F & G \\ \hline H & 0 \end{array}\right],$$

whose transfer function matrix is

$$H(\mu F - E)^{-1}G = P_2\,\mu^{-1} + P_3\,\mu^{-2} + \cdots + P_d\,\mu^{-(d-1)}. \tag{8.14}$$

It can be proved that new system is also strongly minimal. This means that the triple $\{E - \mu F, -G, H\}$ *is a strongly minimal realization of the strictly proper transfer function in* (8.14)*, which has all its poles at* $\mu = 0$*. Moreover, the minimal degree of* $\det(E - \mu F)$ *is known to be* $r = \operatorname{rank} T$*, since, according to* (8.11)*,* $\widehat{A}_s(0) = -F = -\widehat{T}$ *is nonsingular.*

We give a general procedure for the construction of strongly minimal realizations of arbitrary strictly proper rational matrices in Section 8.3.

8.3 For arbitrary strictly proper rational matrices

For strictly proper rational matrices $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$, we represent them via a Laurent expansion around the point at infinity:

$$R_{sp}(\lambda) := R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \ldots \tag{8.15}$$

In this section, we obtain strongly minimal linearizations for such strictly proper rational matrices by using the algorithm in [48, Section 3.4], as we explain in the sequel. Let the block Hankel matrix H and shifted block Hankel matrix H_σ associated with $R_{sp}(\lambda)$ be denoted as

$$H := \begin{bmatrix} R_{-1} & R_{-2} & \ldots & R_{-k} \\ R_{-2} & & \iddots & R_{-k-1} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k} & R_{-k-1} & \ldots & R_{-2k+1} \end{bmatrix}, \quad H_\sigma := \begin{bmatrix} R_{-2} & R_{-3} & \ldots & R_{-k-1} \\ R_{-3} & & \iddots & R_{-k-2} \\ \vdots & \iddots & \iddots & \vdots \\ R_{-k-1} & R_{-k-2} & \ldots & R_{-2k} \end{bmatrix}, \tag{8.16}$$

then for sufficiently large k the rank r_f of H equals the total polar degree of the finite poles, i.e., the sum of the degrees of the denominators in the Smith McMillan form of $R_{sp}(\lambda)$ [55]. The general theory for Hankel based realizations of proper transfer function matrices in [48, Section 3.4], implies that the following Rosenbrock linear system matrix in Theorem 8.3.1 is a strongly minimal linearization for the strictly proper rational matrix $R_{sp}(\lambda)$.

Theorem 8.3.1. *Let* $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ *be a strictly proper rational matrix as in* (8.15)*. Let* H *and* H_σ *be the block Hankel matrices in* (8.16) *and* $r_f := \operatorname{rank} H$*. Let* $U := \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ *and* $V := \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *be unitary matrices such that*

$$U^*HV = \begin{bmatrix} \widehat{H} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1^*HV_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{8.17}$$

where $\widehat{H}$ *is* $r_f \times r_f$ *and invertible. Let us now partition the matrices* U_1 *and* V_1 *as follows*

$$U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}, \quad \text{and} \quad V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}, \tag{8.18}$$

where the matrices U_{11} *and* V_{11} *have dimension* $m \times r_f$ *and* $n \times r_f$*, respectively. Then*

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} U_1^* H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline U_{11}\widehat{H} & 0 \end{array}\right] \tag{8.19}$$

is a strongly minimal linearization for $R_{sp}(\lambda)$*. In particular,* $R_{sp}(\lambda) = U_{11}\widehat{H}(\lambda\widehat{H} - U_1^* H_\sigma V_1)^{-1}\widehat{H}V_{11}^*$.

8.4 For self-conjugate strictly proper rational matrices

If a strictly proper rational matrix $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ have one of the four self-conjugate structures considered in this chapter, we use Theorem 8.3.1 and the ideas developed in Chapter 3 (Remarks 3.5.6 and 3.7.4) to construct strongly mininimal self-conjugate linearizations for $R_{sp}(\lambda)$. First notice that the block Hankel matrices H and H_σ in (8.16) have the following self-conjugate property depending on that of $R_{sp}(\lambda)$.

Lemma 8.4.1. *Let* $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ *be a strictly proper rational matrix as in* (8.15)*. Let us define the scaling matrix* $S := \operatorname{diag}(-I_m, (-1)^2 I_m, \ldots, (-1)^k I_m)$*. Then the block Hankel matrices* H *and* H_σ *in* (8.16) *satisfy the following equations*

1. *for Hermitian* $R_{sp}(\lambda)$*:* $R_{-i}^* = R_{-i}$, $H^* = H$ *and* $H_\sigma^* = H_\sigma$,
2. *for skew-Hermitian* $R_{sp}(\lambda)$*:* $R_{-i}^* = -R_{-i}$, $H^* = -H$ *and* $H_\sigma^* = -H_\sigma$,
3. *for para-Hermitian* $R_{sp}(\lambda)$*:* $R_{-i}^* = (-1)^i R_{-i}$, $(SH)^* = -SH$ *and* $(SH_\sigma)^* = SH_\sigma$,
4. *for para-skew-Hermitian* $R_{sp}(\lambda)$*:* $R_{-i}^* = (-1)^{(i+1)} R_{-i}$, $(SH)^* = SH$ *and* $(SH_\sigma)^* = -SH_\sigma$.

For each of these cases, the left and right transformations U *and* V *in* (8.17) *can be chosen as* $U = V$ *in the Hermitian and skew-Hermitian cases and as* $U = SV$ *in the para-Hermitian and para-skew-Hermitian cases.*

Proof. The symmetries of the coefficient matrices R_i trivially yield the four types of symmetries of H and H_σ. For the rank compression (8.17), we can then choose $U = V$ in the Hermitian and skew-Hermitian cases because H is normal, and we can choose $U = SV$ in the para-Hermitian and para-skew-Hermitian cases because SH is then normal. ■

We can now use Lemma 8.4.1 and Theorem 8.3.1 to obtain the following strongly minimal linearizations for the four considered structures.

Theorem 8.4.2. *Let $R_{sp}(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ be a strictly proper rational matrix as in* (8.15)*, and let H and H_σ be the associated block Hankel matrices appearing in* (8.16)*. Let U, V be the unitary matrices appearing in* (8.17)*, where $U = V$ if $R_{sp}(\lambda)$ is Hermitian or skew-Hermitian, and $U = SV$ if $R_{sp}(\lambda)$ is para-Hermitian or para-skew-Hermitian, and where $S := \operatorname{diag}(-I_m, (-1)^2 I_m, \ldots, (-1)^k I_m)$. Finally, let U_1, V_1 be the $km \times r_f$ matrices formed by the first r_f columns of U and V, respectively, $\widehat{H}$ be the $r_f \times r_f$ defined in* (8.17) *and U_{11}, V_{11} be the $m \times r_f$ matrices defined in* (8.18)*.*

1. *If $R_{sp}(\lambda)$ is Hermitian, then $\widehat{H}^* = \widehat{H}$, $H_\sigma^* = H_\sigma$, and*

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} V_1^* H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline V_{11}\widehat{H} & 0 \end{array}\right]$$

is a Hermitian strongly minimal linear polynomial system matrix of $R_{sp}(\lambda)$.

2. *If $R_{sp}(\lambda)$ is skew-Hermitian, then $\widehat{H}^* = -\widehat{H}$, $H_\sigma^* = -H_\sigma$, and*

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} V_1^* H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline V_{11}\widehat{H} & 0 \end{array}\right]$$

is a skew-Hermitian strongly minimal linear polynomial system matrix of $R_{sp}(\lambda)$.

3. *If $R_{sp}(\lambda)$ is para-Hermitian, then $\widehat{H}^* = -\widehat{H}$, $(SH_\sigma)^* = SH_\sigma$, and*

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} V_1^* S H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline -\,V_{11}\widehat{H} & 0 \end{array}\right]$$

is a para-Hermitian strongly minimal linear polynomial system matrix of $R_{sp}(\lambda)$.

4. *If $R_{sp}(\lambda)$ is para-skew-Hermitian, then $\widehat{H}^* = \widehat{H}$, $(SH_\sigma)^* = -SH_\sigma$, and*

$$L_{sp}(\lambda) := \left[\begin{array}{c|c} V_1^* S H_\sigma V_1 - \lambda \widehat{H} & \widehat{H} V_{11}^* \\ \hline -\,V_{11}\widehat{H} & 0 \end{array}\right]$$

is a para-skew-Hermitian strongly minimal linear polynomial system matrix of $R_{sp}(\lambda)$.

Proof. 1. If $R_{sp}(\lambda)$ is Hermitian, then Lemma 8.4.1 and (8.17) imply that $H^* = H$, $H_\sigma^* = H_\sigma$, $U_1 = V_1$, and $U_{11} = V_{11}$. The result then follows from $\widehat{H} = V_1^* H V_1 = \widehat{H}^*$ and (8.19).

2. If $R_{sp}(\lambda)$ is skew-Hermitian, then Lemma 8.4.1 and (8.17) imply that $H^* = -H$, $H_\sigma^* = -H_\sigma$, $U_1 = V_1$, and $U_{11} = V_{11}$. The result then follows from $\widehat{H} = V_1^* H V_1 = -\widehat{H}^*$ and (8.19).

3. If $R_{sp}(\lambda)$ is para-Hermitian, then Lemma 8.4.1 and (8.17) imply that $(SH)^* = -SH$, $(SH_\sigma)^* = SH_\sigma$, $U_1 = SV_1$, and $U_{11} = -V_{11}$. The result then follows from $\widehat{H} = V_1^* SHV_1 = -\widehat{H}^*$ and (8.19).

4. If $R_{sp}(\lambda)$ is para-skew-Hermitian, then Lemma 8.4.1 and (8.17) imply that $(SH)^* = SH$, $(SH_\sigma)^* = -SH_\sigma$, $U_1 = SV_1$, and $U_{11} = -V_{11}$. The result then follows from $\widehat{H} = V_1^* SHV_1 = \widehat{H}^*$ and (8.19).

■

8.5 For arbitrary and self-conjugate rational matrices

For any given rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ we assume that we have an additive decomposition into its polynomial part $P(\lambda)$ and its strictly proper part $R_{sp}(\lambda)$ as in (2.1). That is,

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda). \tag{8.20}$$

For the Laurent expansion given in (8.1), this corresponds to

$$P(\lambda) := R_0 + R_1\lambda + \ldots + R_d\lambda^d, \quad R_{sp}(\lambda) := R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \ldots$$

In this section, we obtain strongly minimal linearizations for arbitrary and structured rational matrices by combining strongly minimal linearizations for both parts. The construction for the polynomial part was given in Sections 8.1 and 8.2, and for the strictly proper part in Sections 8.3 and 8.4. Although some partial results can be found in the literature [40, 27, 19], the derivation given here is new and more general.

Once we have strongly minimal linearizations for both the polynomial part and the strictly proper part of a given rational matrix, it is straightforward to construct a strongly minimal linearization for the sum, as shown below.

Theorem 8.5.1. *Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be an arbitrary rational matrix, i.e., regular or singular. Let $R(\lambda) = P(\lambda) + R_{sp}(\lambda)$ where $P(\lambda)$ is the polynomial part of $R(\lambda)$ and $R_{sp}(\lambda)$ is the strictly proper part of $R(\lambda)$. Let*

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right], \quad \text{and} \quad L_{sp}(\lambda) := \left[\begin{array}{c|c} A_{sp}(\lambda) & -B_{sp}(\lambda) \\ \hline C_{sp}(\lambda) & 0 \end{array}\right], \tag{8.21}$$

be strongly minimal linear polynomial system matrices of $P(\lambda)$ and $R_{sp}(\lambda)$ as described in Theorems 8.1.2 and 8.3.1, respectively. Then

$$L(\lambda) := \left[\begin{array}{cc|c} \widehat{A}_s(\lambda) & 0 & -\widehat{B}_s(\lambda) \\ 0 & A_{sp}(\lambda) & -B_{sp}(\lambda) \\ \hline \widehat{C}_s(\lambda) & C_{sp}(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] \tag{8.22}$$

is a strongly minimal linearization of $R(\lambda)$.

Proof. The transfer function of $L(\lambda)$ is clearly

$$\widehat{D}_s(\lambda) + \widehat{C}_s(\lambda)\widehat{A}_s(\lambda)^{-1}\widehat{B}_s(\lambda) + C_{sp}(\lambda)A_{sp}(\lambda)^{-1}B_{sp}(\lambda) = P(\lambda) + R_{sp}(\lambda) = R(\lambda).$$

The strong minimality of $L(\lambda)$ follows from the fact that the subsystems $\widehat{L}_s(\lambda)$ and $L_{sp}(\lambda)$ are strongly minimal and have no common poles. ■

For the construction of self-conjugate strongly minimal linearizations of structured rational matrices we proceed in the same way and obtain the following result.

Theorem 8.5.2. *Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times m}$ be an arbitrary rational matrix, i.e., regular or singular, which has one of the following structures: Hermitian, skew-Hermitian, para-Hermitian or para-skew-Hermitian. Let $R(\lambda) = P(\lambda) + R_{sp}(\lambda)$ where $P(\lambda)$ is the polynomial part of $R(\lambda)$ and $R_{sp}(\lambda)$ is the strictly proper part of $R(\lambda)$. Let*

$$\widehat{L}_s(\lambda) := \left[\begin{array}{c|c} \widehat{A}_s(\lambda) & -\widehat{B}_s(\lambda) \\ \hline \widehat{C}_s(\lambda) & \widehat{D}_s(\lambda) \end{array}\right], \quad \text{and} \quad L_{sp}(\lambda) := \left[\begin{array}{c|c} A_{sp}(\lambda) & -B_{sp}(\lambda) \\ \hline C_{sp}(\lambda) & 0 \end{array}\right], \tag{8.23}$$

be strongly minimal linear polynomial system matrices of $P(\lambda)$ and $R_{sp}(\lambda)$ as described in Theorems 8.2.2 and 8.4.2, respectively, according to the corresponding structure of $R(\lambda)$. Then

$$L(\lambda) := \left[\begin{array}{cc|c} \widehat{A}_s(\lambda) & 0 & -\widehat{B}_s(\lambda) \\ 0 & A_{sp}(\lambda) & -B_{sp}(\lambda) \\ \hline \widehat{C}_s(\lambda) & C_{sp}(\lambda) & \widehat{D}_s(\lambda) \end{array}\right] \tag{8.24}$$

is a strongly minimal linearization of $R(\lambda)$ with the same self-conjugate structure as $R(\lambda)$.

Proof. $L(\lambda)$ is a strongly minimal linearization of $R(\lambda)$ by using the same proof as that of Theorem 8.5.1. That the self-conjugate structure of $R(\lambda)$ and $L(\lambda)$ are the same, follows from the fact that the self-conjugate structures of $\widehat{L}_s(\lambda)$ and $L_{sp}(\lambda)$ coincide with that of $R(\lambda)$. ■

8.6 Algorithmic aspects

The constructive proofs given in the earlier sections in fact lead to possible algorithms for computing strongly minimal linearizations of rational matrices, provided the Laurent expansion (8.1) is given up to the term R_{-2k}. The two decompositions that are required for the construction of the linearizations are the rank factorizations of the matrices T in (8.13) and H in (8.17). Moreover, because of the symmetry in these problems, only the right transformation has to be computed in the structured cases.

It is worth pointing out also that both factorizations only require to construct unitary transformations that "compress" the rows and columns of a given matrix, which can be obtained by a QR factorization. Efficient algorithms that exploit the special block-Hankel and or block-Toeplitz structure can be found in the literature [49, 88]. We imposed no conditions of the normal rank of the rational matrix, although the self-conjugate structure imposes that they are square. There can therefore be a left and a right null space and their structural indices will be equal because of the self-conjugate nature of the transfer function and the system matrix.

Since the linear polynomial system matrices derived here are strongly minimal and structured, one can use the properties of such pencils to compute efficiently the left and right eigenvectors in the regular case, and the left and right null space structure in the singular case.

Chapter 9

Structural backward stability in rational eigenvalue problems solved via block Kronecker linearizations

In this chapter we study the backward stability of running a backward stable eigenstructure solver on a pencil $S(\lambda)$ that is a strong linearization of a rational matrix $R(\lambda)$ expressed in the form $R(\lambda) = D(\lambda) + C(\lambda I_\ell - A)^{-1}B$, where $D(\lambda)$ is a polynomial matrix and $C(\lambda I_\ell - A)^{-1}B$ is a minimal state-space realization. We consider the family of block Kronecker linearizations of $R(\lambda)$, which have the following structure

$$S(\lambda) := \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2^T(\lambda) \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix},$$

where the blocks have some specific structures. Backward stable eigenstructure solvers, such as the QZ or the staircase algorithms, applied to $S(\lambda)$ will compute the exact eigenstructure of a perturbed pencil

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda)$$

and the special structure of $S(\lambda)$ will be lost, including the zero blocks below the anti-diagonal. In order to link this perturbed pencil with a nearby rational matrix, we construct in this chapter a strictly equivalent pencil

$$\widetilde{S}(\lambda) = (I - X)\widehat{S}(\lambda)(I - Y)$$

that restores the original structure, and hence is a block Kronecker linearization of a perturbed rational matrix $\widetilde{R}(\lambda) = \widetilde{D}(\lambda) + \widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B}$, where $\widetilde{D}(\lambda)$ is a polynomial matrix with the same degree as $D(\lambda)$. Moreover, we bound appropriate norms of $\widetilde{D}(\lambda) - D(\lambda)$, $\widetilde{C} - C$, $\widetilde{A} - A$ and $\widetilde{B} - B$ in terms of an appropriate norm of $\Delta_S(\lambda)$. These bounds may be, in general, inadmissibly large, but we also introduce a scaling that allows us to make them satisfactorily tiny, by making the matrices appearing in both $S(\lambda)$ and $R(\lambda)$ have norms bounded by 1. Thus, for this scaled representation, we prove that the staircase and the QZ algorithms compute the exact eigenstructure of a rational matrix $\widetilde{R}(\lambda)$ that can be expressed in exactly the same form as $R(\lambda)$ with the parameters defining the representation very near to those of $R(\lambda)$. This shows that this approach is backward stable in a structured sense. All the results in this chapter appear in [33].

9.1 Some preliminaries

A wide family of strong linearizations called *strong block minimal bases linearizations* is proposed in [6, Theorem 5.11] for any $m \times n$ rational matrix $R(\lambda)$, inspired by previous results for polynomial matrices in [26] (recall Theorem 2.5.5). These linearizations are based on the splitting of $R(\lambda)$ into its strictly proper part $R_p(\lambda)$ and its polynomial part $D(\lambda)$ and in the representation:

$$R(\lambda) := R_p(\lambda) + D(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i, \tag{9.1}$$

where $C(\lambda I_\ell - A)^{-1}B$ is a minimal state-space realization of the strictly proper part $R_p(\lambda)$, represented in what follows by the triple $\{A, B, C\}$, and $d > 1$ is the degree of the polynomial part. Then $R(\lambda)$ is represented by the quadruple $\{\lambda I_\ell - A, B, C, D(\lambda)\}$. Since in this chapter we are analyzing perturbations related to backward errors of eigenvalue solvers of pencils with real or complex matrix coefficients, we restrict $\mathbb{F}$ to be the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

A particular case of the strong block minimal bases linearizations in [6, Theorem 5.11] of any $m \times n$ rational matrix $R(\lambda)$ represented as in (9.1) are (modulo block permutations) the pencils of the form

$$S(\lambda) := \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2^T(\lambda) \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}, \tag{9.2}$$

with

$$K_1(\lambda) := L_\epsilon(\lambda) \otimes I_n, \quad \widehat{K}_1 := \mathbf{e}_{\epsilon+1}^T \otimes I_n, \quad K_2(\lambda) := L_\eta(\lambda) \otimes I_m, \quad \widehat{K}_2 := \mathbf{e}_{\eta+1}^T \otimes I_m,$$

and where $\otimes$ denotes the Kronecker product, $\mathbf{e}_k = [0 \cdots 0\, 1]^T$ is the standard kth unit vector of dimension k and $L_k(\lambda)$ is the classical Kronecker block of dimension

$k \times (k+1)$

$$L_k(\lambda) := \begin{bmatrix} 1 & -\lambda & & & \\ & 1 & -\lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & -\lambda \end{bmatrix}.$$

Moreover, the block $M(\lambda)$ in (9.2) is related to the polynomial part $D(\lambda)$ in (9.1) by the "dual basis" vector $\Lambda_k(\lambda)$ of powers of λ,

$$\Lambda_k^T(\lambda) := \begin{bmatrix} \lambda^k & \cdots & \lambda^2 & \lambda & 1 \end{bmatrix},$$

which satisfies $L_k(\lambda)\Lambda_k(\lambda) = 0$ and also

$$D(\lambda) = (\Lambda_\eta(\lambda) \otimes I_m)^T M(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n).$$

Thus, $d = \epsilon+\eta+1$ (see [26, eq. (4.5)]). The strong linearizations (9.2) are inspired by the so-called "block Kronecker linearizations" that were introduced in [26, Section 4] for an arbitrary $m \times n$ polynomial matrix $D(\lambda)$. Therefore, we use the same name in the rational setting. The representation of $R(\lambda)$ in (9.1) and the block Kronecker linearizations $S(\lambda)$ of $R(\lambda)$ (9.2) are the two fundamental ingredients of this chapter.

As explained in [6, Section 3.1], the finite eigenvalues, together with their partial multiplicities, of $S(\lambda)$ (resp. $A - \lambda I_\ell$) coincide with the finite zeros (resp. poles) of $R(\lambda)$, together with their partial multiplicities. Moreover, the eigenvalue structure at infinity of $S(\lambda)$ allows us to obtain via a simple shift rule the pole-zero structure at infinity of $R(\lambda)$.[1] In addition, as proved in [8, Section 6], the right (resp. left) minimal indices of $S(\lambda)$ are those of $R(\lambda)$ plus ϵ (resp. η). Thus, $S(\lambda)$ comprises the complete eigenstructure of $R(\lambda)$. Observe that the application to $S(\lambda)$ of the QZ algorithm [71], in the regular case, or of the staircase algorithm [83], in the singular case, gives the zeros and the minimal indices, in the singular case, of $R(\lambda)$, but not the poles, which are in $A - \lambda I_\ell$.

It is worth mentioning that although the families of block Kronecker linearizations of polynomial [26] and rational [6] matrices are very recent, some particular examples of strong linearizations in these families appeared much earlier in the literature. For instance, it was shown in [87] that a valid "realization" for the polynomial part $D(\lambda)$ in (9.1) is given by the following minimal polynomial system matrix

$$S_D(\lambda) := \left[\begin{array}{cccc|c} I_n & -\lambda I_n & & & \\ & I_n & \ddots & & \\ & & \ddots & -\lambda I_n & \\ & & & I_n & -\lambda I_n \\ \hline \lambda D_d & \dots & \dots & \lambda D_2 & \lambda D_1 + D_0 \end{array}\right] := \left[\begin{array}{c|c} T(\lambda) & -U(\lambda) \\ \hline V(\lambda) & W(\lambda) \end{array}\right],$$

[1]More precisely, according to [6, p. 1683], if r is the normal rank of $R(\lambda)$ and $e_1 \leq \cdots \leq e_r$ are the r largest partial multiplicities at infinity of $S(\lambda)$, then $e_1 - d \leq \cdots \leq e_r - d$ are the structural indices at infinity of $R(\lambda)$.

which means that $D(\lambda) = W(\lambda) + V(\lambda)T(\lambda)^{-1}U(\lambda)$. It is easy to see that after moving the bottom block row of $S_D(\lambda)$ to the top position, a block Kronecker linearizarion of $D(\lambda)$ is obtained with $K_2(\lambda)$ empty [26, Section 4]. Combining the minimal state-space realization $C(\lambda I_\ell - A)^{-1}B$ and the polynomial system matrix $S_D(\lambda)$ yields the following minimal polynomial system matrix for the rational matrix $R(\lambda)$ in (9.1) :

$$S_R(\lambda) := \left[\begin{array}{c|cccc|c} A-\lambda I_\ell & & & & & B \\ \hline & I_n & -\lambda I_n & & & \\ & & I_n & \ddots & & \\ & & & \ddots & -\lambda I_n & \\ & & & & I_n & -\lambda I_n \\ \hline C & \lambda D_d & \dots & \dots & \lambda D_2 & \lambda D_1 + D_0 \end{array}\right] := \left[\begin{array}{c|c} T_R(\lambda) & -U_R(\lambda) \\ \hline V_R(\lambda) & W_R(\lambda) \end{array}\right],$$

i.e., $R(\lambda) = W_R(\lambda)+V_R(\lambda)T_R(\lambda)^{-1}U_R(\lambda)$. A pencil with a structure similar to $S_R(\lambda)$ can also be found in [79]. It is easy to see that, modulo some block permutations, $S_R(\lambda)$ is a particular case of the block Kronecker linearizations appearing in (9.2) for $R(\lambda)$, with $K_2(\lambda)$ empty and $\widehat{K}_2 = I_m$.

It was shown in [26] that perturbations of the block Kronecker linearizations of a polynomial matrix $D(\lambda)$ can be mapped to perturbations of the coefficients of $D(\lambda)$ without significant growth of the relative norms of the perturbations under mild assumptions that require to scale $D(\lambda)$ to have norm equal to 1 and to use linearizations with the norm of $M(\lambda)$ of the same order as the norm of $D(\lambda)$ (see [26, Corollary 5.24]). As a corollary of this perturbation result, we obtain that under such assumptions the computation of the eigenvalues and minimal indices of a polynomial matrix by applying the QZ or the staircase algorithm to one of its block Kronecker linearizations is a backward stable method from the point of view of the polynomial matrix. In this chapter we show that this can be extended to rational matrices as well, considering as coefficients of the rational matrix those in the quadruple $\{\lambda I_\ell - A, B, C, D(\lambda)\}$. However, we emphasize that the perturbation analysis for block Kronecker linearizations of rational matrices is considerably more complicated than the one in [26] and, therefore, we limit ourselves to perform a first order analysis. We also remark that the scaling needed to get satisfactory perturbation bounds is more delicate than the one in [26]. As far as we know, this is the first structural backward error analysis of this type performed in the literature for linearizations of rational matrices.

We assume throughout the chapter that $\ell > 0$ since, otherwise, $R(\lambda)$ in (9.1) is a polynomial matrix and this case was studied in [26]. Except in Section 9.6, we also assume that at least one of the parameters ϵ and η in (9.2) is larger than zero since, otherwise, none of the blocks $K_1(\lambda)$ and $K_2(\lambda)$ appears and block Kronecker linearizations collapse to much simpler pencils. Note that $\max(\eta, \epsilon) > 0$ implies that the degree d of the polynomial part $D(\lambda)$ of $R(\lambda)$ is larger than 1. The simple case $d \leq 1$ is studied in Section 9.6.

9.1.1 Norms

In order to measure perturbations, we need to introduce appropriate norms for pencils, polynomial matrices and rational matrices expressed as in (9.1). For any pair of matrices X and Y of arbitrary dimensions (that might be different), we will use the following norms

$$\|(X,Y)\|_F := \left(\|X\|_F^2 + \|Y\|_F^2\right)^{\frac{1}{2}} = \|\,[\mathrm{vec}(X)^T, \mathrm{vec}(Y)^T]\,\|_2,$$
$$\|(X,Y)\|_2 := \left(\|X\|_2^2 + \|Y\|_2^2\right)^{\frac{1}{2}},$$

where $\|X\|_F$ and $\|X\|_2$ are, respectively, the Frobenius and spectral matrix norms and $\mathrm{vec}(X)$ is the operator that stacks the columns of a matrix into one column vector [50]. For a pencil $S(\lambda) := A - \lambda B$ we define the corresponding norms via the two matrix coefficients :

$$\|S(\lambda)\|_F := \|(A,B)\|_F, \quad \|S(\lambda)\|_2 := \|(A,B)\|_2.$$

More generally, for a polynomial matrix $D(\lambda) := \sum_{i=0}^{d} D_i\lambda^i$, we will use the norm

$$\|D(\lambda)\|_F := \sqrt{\sum_{i=0}^{d} \|D_i\|_F^2},$$

and for a list of polynomial matrices $(D_1(\lambda), \ldots, D_p(\lambda))$, the norm

$$\|(D_1(\lambda), \ldots, D_p(\lambda))\|_F := \sqrt{\sum_{i=1}^{p} \|D_i(\lambda)\|_F^2}.$$

Finally, for a rational matrix $R(\lambda)$, represented by a quadruple $\{\lambda I_\ell - A, B, C, D(\lambda)\}$, as in (9.1), we use the "norm"

$$\|R(\lambda)\|_F := \|(\lambda I_\ell - A, B, C, D(\lambda))\|_F = \sqrt{\ell + \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \sum_{i=0}^{d} \|D_i\|_F^2}.$$

That is, the "norm" of a rational matrix $R(\lambda)$ is defined as the norm of an associated polynomial system matrix $P(\lambda)$, in this case,

$$\|R(\lambda)\|_F := \|P(\lambda)\|_F \quad \text{where} \quad P(\lambda) := \begin{bmatrix} \lambda I_\ell - A & -B \\ C & D(\lambda) \end{bmatrix}. \tag{9.3}$$

We remark that $\|R(\lambda)\|_F$ is not rigorously a "norm" for $R(\lambda)$ because, for instance, $R(\lambda)$ is zero if $B = 0$ and $D(\lambda) = 0$, but $\|R(\lambda)\|_F$ is not. Despite this fact, and with a clear abuse of nomenclature, we will use the terminology "norm of a rational matrix" in the sense explained above.

We describe in Section 9.2 the basic systems of matrix equations we will use in this chapter, and, in Section 9.3, some bounds for the singular values of certain matrices related to these systems of matrix equations.

9.2 Generalized Sylvester equations

In order to restore the structure of perturbed block Kronecker linearizations of rational matrices, we will need to guarantee that some matrix equations have solutions and to bound the norm of their minimal norm solution. The matrix equations that we will encounter are particular cases of the generalized Sylvester equation for $m_i \times n_i$ pencils of matrices $A_i - \lambda B_i$, $i = 1, 2$, which is the following equation in the unknowns X and Y:

$$X(A_1 - \lambda B_1) + (A_2 - \lambda B_2)Y = \Delta^a - \lambda \Delta^b. \tag{9.4}$$

It is easily seen to be equivalent to a linear system of equations, when rewriting it as

$$\begin{aligned} XA_1 + A_2Y &= \Delta^a, \\ XB_1 + B_2Y &= \Delta^b, \end{aligned}$$

or, when using Kronecker products and the vec(·) notation, as

$$\left[\begin{array}{c|c} A_1^T \otimes I_{m_2} & I_{n_1} \otimes A_2 \\ \hline B_1^T \otimes I_{m_2} & I_{n_1} \otimes B_2 \end{array}\right] \left[\begin{array}{c} \mathrm{vec}(X) \\ \hline \mathrm{vec}(Y) \end{array}\right] = \left[\begin{array}{c} \mathrm{vec}(\Delta^a) \\ \hline \mathrm{vec}(\Delta^b) \end{array}\right]. \tag{9.5}$$

The dimension of the unknowns X and Y are $m_2 \times m_1$ and $n_2 \times n_1$, respectively, and those of the right hand sides Δ^a and Δ^b are each $m_2 \times n_1$. These equations will be used in this chapter in two contexts, which we briefly recall here.

9.2.0.1 Block elimination

Let $A_i - \lambda B_i$ be two $m_i \times n_i$ pencils, $i = 1, 2$, that have respectively full column normal rank n_1 and full row normal rank m_2. Then the problem of block anti-diagonalizing the pencil $\begin{bmatrix} 0 & A_1 - \lambda B_1 \\ A_2 - \lambda B_2 & \Delta^a - \lambda \Delta^b \end{bmatrix}$, that is, finding X and Y such that

$$\begin{bmatrix} I_{m_1} & 0 \\ -X & I_{m_2} \end{bmatrix} \begin{bmatrix} 0 & A_1 - \lambda B_1 \\ A_2 - \lambda B_2 & \Delta^a - \lambda \Delta^b \end{bmatrix} \begin{bmatrix} I_{n_2} & -Y \\ 0 & I_{n_1} \end{bmatrix} = \begin{bmatrix} 0 & A_1 - \lambda B_1 \\ A_2 - \lambda B_2 & 0 \end{bmatrix}, \tag{9.6}$$

amounts to finding a solution for the generalized Sylvester equation (9.4). It is known that there exists a solution $(X, Y) \in \mathbb{F}^{m_2 \times m_1} \times \mathbb{F}^{n_2 \times n_1}$ for a *particular* right hand side $(\Delta^a, \Delta^b) \in \mathbb{F}^{m_2 \times n_1} \times \mathbb{F}^{m_2 \times n_1}$ *if and only if* the pencils

$$\begin{bmatrix} 0 & A_1 - \lambda B_1 \\ A_2 - \lambda B_2 & \Delta^a - \lambda \Delta^b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & A_1 - \lambda B_1 \\ A_2 - \lambda B_2 & 0 \end{bmatrix}$$

are strictly equivalent (i.e. have the same Kronecker structure) [23]. But in order to have a solution *for any right hand side* $\Delta^a - \lambda \Delta^b$ one requires the stronger condition that the pencils $A_1 - \lambda B_1$ and $A_2 - \lambda B_2$ have no common generalized eigenvalues (see [86]). We recall here the result proven in [86] that is relevant for our work.

Theorem 9.2.1. *[86] Let the pencils $A_i - \lambda B_i$ of dimensions $m_i \times n_i, i = 1, 2$, be respectively of full column normal rank $n_1 \leq m_1$ and of full row normal rank $m_2 \leq n_2$, and let these two pencils have no common generalized eigenvalues. Then there always exists a solution (X, Y) to the system of equations* (9.6)*, for any perturbation $\Delta^a - \lambda\Delta^b$. Moreover, the generalized eigenvalues of the pencil* (9.6) *are the union of the generalized eigenvalues of the pencils $A_i - \lambda B_i,\ i = 1, 2$.*

The system is underdetermined if either of the two inequalities $m_1 \geq n_1$ and $n_2 \geq m_2$, is strict. Under the hypotheses of Theorem 9.2.1, the system (9.5) must be compatible for any right hand side, and hence the Kronecker product matrix in the left hand side of (9.5) must have full row rank $2m_2n_1$. A bound for the minimum Frobenius-norm solution (X, Y) is then obtained in terms of the smallest singular value $\sigma_{2m_2n_1}$ of the matrix in (9.5):

$$\|(X, Y)\|_F \leq \frac{\|(\Delta^a, \Delta^b)\|_F}{\sigma_{2m_2n_1}\left(\left[\begin{array}{c|c} A_1^T \otimes I_{m_2} & I_{n_1} \otimes A_2 \\ \hline B_1^T \otimes I_{m_2} & I_{n_1} \otimes B_2 \end{array}\right]\right)}. \tag{9.7}$$

9.2.0.2 Equivalent pencils

The second problem in this chapter where a generalized Sylvester equation as in (9.4) arises is that of strictly equivalent pencils (see e.g. [39]). Let the pencils $A_i - \lambda B_i,\ i = 1, 2$, be both of dimension $m \times n$, then they are strictly equivalent if and only if there exist invertible matrices S and T such that $S(A_1 - \lambda B_1) = (A_2 - \lambda B_2)T$. Such pencils must then have the same Kronecker canonical form [39]. We are interested in finding the solution where S and T are as close as possible to the identity matrix. This can be achieved by writing the transformation matrices as

$$S = I + X, \quad T = I - Y$$

and then minimizing the Frobenius norm of the pair (X, Y). The corresponding equations are then

$$(I + X)(A_1 - \lambda B_1) = (A_2 - \lambda B_2)(I - Y)$$

or, when putting $\Delta^a - \lambda\Delta^b := (A_2 - \lambda B_2) - (A_1 - \lambda B_1)$, we finally obtain

$$X(A_1 - \lambda B_1) + (A_2 - \lambda B_2)Y = \Delta^a - \lambda\Delta^b, \tag{9.8}$$

which is again solved by using (9.5). We will use this to "restore" a slightly perturbed pencil $(A_2 - \lambda B_2) := (A_1 - \lambda B_1) + (\Delta^a - \lambda\Delta^b)$ to its original form $(A_1 - \lambda B_1)$ using a strict equivalence transformation

$$(I + X)^{-1}(A_2 - \lambda B_2)(I - Y) = A_1 - \lambda B_1 \tag{9.9}$$

that is very close to the identity, when we are sure that both pencils have the same Kronecker canonical form. The bounds for the norm of X and Y are in fact given by (9.7) for which we derive exact expressions in the next section. Notice that we can not apply Theorem 9.2.1 to prove existence of a solution for equation (9.8), since in this case both pencils must have the same generalized eigenvalues and the same normal rank. A sufficient condition for the consistency of (9.8) is that $A_1 - \lambda B_1$ and $A_2 - \lambda B_2$ have the same Kronecker canonical form.

The condition that the Kronecker canonical form of a pencil does not change under arbitrary sufficiently small perturbations only holds for very special pencils. In particular, it holds for the Kronecker product of Kronecker blocks times identity matrices, i.e., for $L_k(\lambda) \otimes I_r$. This is a consequence of the results in [89], because $L_k(\lambda) \otimes I_r$ has full-Sylvester-rank by [89, Theorem 4.3(a)] and, then, [89, Theorem 6.6] guarantees that $L_k(\lambda) \otimes I_r + (\Delta^a - \lambda\Delta^b)$ has the same Kronecker canonical form as $L_k(\lambda) \otimes I_r$ for all the perturbations (Δ^a, Δ^b) whose norms are smaller than the bounds in [89, Theorem 6.6]. Since we will solve (9.8)-(9.9) only in the case $A_1 - \lambda B_1 = L_k(\lambda) \otimes I_r$, these results prove that (9.8) has a solution for all sufficiently small perturbations (Δ^a, Δ^b) in the cases of interest in this chapter.

9.3 Singular value bounds

In the analysis of Section 9.4, we will need upper bounds for the minimum norm solutions of the generalized Sylvester equation (9.4) for pairs of pencils $(A_i - \lambda B_i)$, $i = 1, 2$, which all involve Kronecker blocks $L_k(\lambda) := E_k - \lambda F_k$, where the $k \times (k+1)$ matrices E_k and F_k are given by

$$E_k := \begin{bmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix} \quad \text{and} \quad F_k := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix}.$$

To find such upper bounds is equivalent to find lower bounds for the singular values in the denominator of the right hand side of (9.7). We consider the generalized Sylvester equations for the following list of pencil pairs with their smallest singular value of the corresponding linear maps:

1. $A_1 - \lambda B_1 = A - \lambda I_\ell$ and $A_2 - \lambda B_2 = L_\varepsilon(\lambda) \otimes I_n$:

$$\omega_1 := \sigma_{2\ell\varepsilon n} \left[\begin{array}{c|c} A^T \otimes I_{\varepsilon n} & I_\ell \otimes E_\varepsilon \otimes I_n \\ \hline I_\ell \otimes I_{\varepsilon n} & I_\ell \otimes F_\varepsilon \otimes I_n \end{array} \right]. \tag{9.10}$$

2. $A_1 - \lambda B_1 = L_\eta^T(\lambda) \otimes I_m$ and $A_2 - \lambda B_2 = A - \lambda I_\ell$:

$$\omega_2 := \sigma_{2\eta m\ell} \left[\begin{array}{c|c} E_\eta \otimes I_{m\ell} & I_{\eta m} \otimes A \\ \hline F_\eta \otimes I_{m\ell} & I_{\eta m} \otimes I_\ell \end{array} \right]. \tag{9.11}$$

3. $A_1 - \lambda B_1 = L_\eta^T(\lambda) \otimes I_m$ and $A_2 - \lambda B_2 = L_\varepsilon(\lambda) \otimes I_n$:

$$\omega_3 := \sigma_{2\eta m\varepsilon n} \left[\begin{array}{c|c} E_\eta \otimes I_{m\varepsilon n} & I_{\eta m} \otimes E_\varepsilon \otimes I_n \\ \hline F_\eta \otimes I_{m\varepsilon n} & I_{\eta m} \otimes F_\varepsilon \otimes I_n \end{array} \right]. \tag{9.12}$$

4. $A_1 - \lambda B_1 = L_k(\lambda) \otimes I_r$ and $A_2 - \lambda B_2 = L_k(\lambda) \otimes I_r$:

$$\omega_4 := \sigma_{2(k+1)rkr} \left[\begin{array}{c|c} E_k^T \otimes I_{rkr} & I_{(k+1)r} \otimes E_k \otimes I_r \\ \hline F_k^T \otimes I_{rkr} & I_{(k+1)r} \otimes F_k \otimes I_r \end{array} \right]. \tag{9.13}$$

In Lemma 9.3.1 we analyze the first problem and give a lower bound for ω_1.

Lemma 9.3.1. *Let ω_1 be the singular value in* (9.10)*. Then*

$$\omega_1 \geq \frac{1}{1 + 2\varepsilon \max(1, \|A\|_2^{\varepsilon})}. \tag{9.14}$$

Proof. It follows from the properties of singular values of Kronecker products that ω_1 is also equal to

$$\omega_1 = \sigma_{2\ell\varepsilon} \left[\begin{array}{c|c} A^T \otimes I_\varepsilon & I_\ell \otimes E_\varepsilon \\ \hline I_\ell \otimes I_\varepsilon & I_\ell \otimes F_\varepsilon \end{array} \right]$$

and using perfect shuffle permutations we also get

$$\omega_1 = \sigma_{2\varepsilon\ell} \left[\begin{array}{c|c} I_\varepsilon \otimes A^T & E_\varepsilon \otimes I_\ell \\ \hline I_\varepsilon \otimes I_\ell & F_\varepsilon \otimes I_\ell \end{array} \right].$$

The smallest singular value $\sigma_{2\varepsilon\ell}$ is larger than the smallest singular value of any $2\varepsilon\ell \times 2\varepsilon\ell$ submatrix. Let us take for this the submatrix obtained by dropping the last block column :

$$M = \left[\begin{array}{c|c} I_\varepsilon \otimes A^T & I_\varepsilon \otimes I_\ell \\ \hline I_\varepsilon \otimes I_\ell & J_\varepsilon \otimes I_\ell \end{array} \right], \quad \text{where} \quad J_\varepsilon := \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{F}^{\varepsilon \times \varepsilon}.$$

We can factorize this matrix as

$$M = \left[\begin{array}{c|c} I_\varepsilon \otimes A^T & I_\varepsilon \otimes I_\ell \\ \hline I_\varepsilon \otimes I_\ell & 0 \end{array} \right] \left[\begin{array}{c|c} I_{\varepsilon\ell} & 0 \\ \hline 0 & I_{\varepsilon\ell} - J_\varepsilon \otimes A^T \end{array} \right] \left[\begin{array}{c|c} I_\varepsilon \otimes I_\ell & J_\varepsilon \otimes I_\ell \\ \hline 0 & I_\varepsilon \otimes I_\ell \end{array} \right].$$

Therefore its inverse equals

$$M^{-1} = \left[\begin{array}{c|c} I_\varepsilon \otimes I_\ell & -J_\varepsilon \otimes I_\ell \\ \hline 0 & I_\varepsilon \otimes I_\ell \end{array} \right] \left[\begin{array}{c|c} I_{\varepsilon\ell} & 0 \\ \hline 0 & (I_{\varepsilon\ell} - J_\varepsilon \otimes A^T)^{-1} \end{array} \right] \left[\begin{array}{c|c} 0 & I_\varepsilon \otimes I_\ell \\ \hline I_\varepsilon \otimes I_\ell & -I_\varepsilon \otimes A^T \end{array} \right]$$

$$= \left[\frac{I_{\varepsilon\ell}}{0}\right] \left[\ 0 \mid I_{\varepsilon\ell}\ \right] + \left[\frac{-J_\varepsilon \otimes I_\ell}{I_{\varepsilon\ell}}\right] (I_{\varepsilon\ell} - J_\varepsilon \otimes A^T)^{-1} \left[\ I_{\varepsilon\ell} \mid -I_\varepsilon \otimes A^T\ \right].$$

It then follows that

$$\|M^{-1}\|_2 \leq 1 + \sqrt{2}\sqrt{1 + \|A\|_2^2}\left[1 + \|A\|_2 + \|A\|_2^2 + \ldots + \|A\|_2^{\varepsilon-1}\right],$$

since

$$(I_{\varepsilon\ell} - J_\varepsilon \otimes A^T)^{-1} = \sum_{i=0}^{\varepsilon-1} J_\varepsilon^i \otimes A^{iT}.$$

In particular, for $\|A\|_2 \leq 1$ we obtain the bound $\|M^{-1}\|_2 \leq 1+2\varepsilon$, while for $\|A\|_2 > 1$ we obtain the bound $\|M^{-1}\|_2 \leq 1 + 2\varepsilon\|A\|_2^\varepsilon$. This finally yields the inequality

$$\omega_1 \geq \frac{1}{1 + 2\varepsilon \max(1, \|A\|_2^\varepsilon)}.$$

■

The second generalized Sylvester equation is essentially the transposed of the first equation and the analysis is therefore completely analogous. This immediately yields Lemma 9.3.2.

Lemma 9.3.2. *Let ω_2 be the singular value in* (9.11). *Then*

$$\omega_2 \geq \frac{1}{1 + 2\eta \max(1, \|A\|_2^\eta)}. \tag{9.15}$$

The third generalized Sylvester equation was analyzed in [26] and its associated smallest singular value is exactly equal to $\omega_3 = 2\sin(\pi/(4\min(\varepsilon,\eta)+2))$ if $\varepsilon \neq \eta$, and to $2\sin(\pi/4\eta)$ if $\varepsilon = \eta$. Notice that we can assume $\min(\varepsilon,\eta) \geq 1$ since otherwise the equation is void. For $\varepsilon \neq \eta$ we then obtain $\omega_3 \geq \frac{3}{2\min(\varepsilon,\eta)+1}$ since $\sin x \geq 3x/\pi$ for $0 \leq x \leq \pi/6$, and for $\varepsilon = \eta$ we then obtain $\omega_3 \geq \frac{\sqrt{2}}{\eta}$ since $\sin x \geq 2\sqrt{2}x/\pi$ for $0 \leq x \leq \pi/4$. We have also that $2\eta = \varepsilon + \eta$ if $\varepsilon = \eta$ and $2\min(\varepsilon,\eta) + 1 \leq \varepsilon + \eta$ if $\varepsilon \neq \eta$, which finally yields the lower bound in Lemma 9.3.3 for ω_3.

Lemma 9.3.3. *Let ω_3 be the singular value in* (9.12). *Then*

$$\omega_3 \geq \frac{2\sqrt{2}}{(\varepsilon + \eta)}. \tag{9.16}$$

In Lemma 9.3.4, we give a lower bound for the smallest singular value ω_4 corresponding to the fourth generalized Sylvester equation.

Lemma 9.3.4. *Let ω_4 be the singular value in* (9.13). *Then*

$$\omega_4 \geq \frac{3}{4k-1}. \tag{9.17}$$

Proof. We prove first that $\omega_4 = 2\sin(\pi/(8k-2))$. This is obtained as follows. We can again use the properties of Kronecker products to prove that

$$\omega_4 = \sigma_{2k(k+1)} \left[\begin{array}{c|c} E_k^T \otimes I_k & I_{(k+1)} \otimes E_k \\ \hline F_k^T \otimes I_k & I_{(k+1)} \otimes F_k \end{array} \right].$$

This matrix can be transformed by row and column permutations to the direct sum of smaller matrices :

$$M_1 \oplus M_1 \oplus M_3 \oplus M_3 \oplus \cdots \oplus M_{2k-1} \oplus M_{2k-1} \oplus N_{2k},$$

see Appendix C, where the blocks

$$M_k := \begin{bmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \in \mathbb{F}^{k\times k}, \quad N_k := \begin{bmatrix} 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{bmatrix} \in \mathbb{F}^{k\times(k+1)} \qquad (9.18)$$

have as smallest singular values $2\sin\frac{\pi}{4k+2}$ and $2\sin\frac{\pi}{2k+2}$, respectively (see [26, Proof of Proposition B.4]). The smallest singular value therefore corresponds to M_{2k-1} and equals $\omega_4 = 2\sin(\pi/(8k-2))$. For $k \geq 1$, we use again that $\sin x \geq 3x/\pi$ for $0 \leq x \leq \pi/6$, to obtain the bound $\omega_4 \geq \frac{3}{4k-1}$. ∎

9.4 Restoring the structure of the linearization after perturbations

We now consider perturbations of the following block Kronecker linearization introduced in (9.2)

$$S(\lambda) := \begin{bmatrix} S_{11}(\lambda) & S_{12}(\lambda) & S_{13}(\lambda) \\ S_{21}(\lambda) & S_{22}(\lambda) & 0 \\ S_{31}(\lambda) & 0 & 0 \end{bmatrix} := \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & K_2^T(\lambda) \\ B\widehat{K}_1 & A - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}, \qquad (9.19)$$

where $S_{13}(\lambda)$ is $(\eta+1)m \times \eta m$ and has full column rank ηm, $S_{22}(\lambda)$ is $\ell \times \ell$ and is a regular pencil, $S_{31}(\lambda)$ is $\varepsilon n \times (\varepsilon+1)n$ and has full row rank εn, and where no two of these three pencils have common generalized eigenvalues. As explained in the introduction, if the state-space triple $\{A, B, C\}$ is minimal, then $S(\lambda)$ is a strong linearization of the $m \times n$ rational matrix

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + (\Lambda_\eta(\lambda) \otimes I_m)^T M(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n). \qquad (9.20)$$

Except in Section 9.6, we assume in this section that $\max(\eta, \epsilon) > 0$. This means that the degree $d = \epsilon+\eta+1$ of the polynomial part $D(\lambda) = (\Lambda_\eta(\lambda)\otimes I_m)^T M(\lambda)(\Lambda_\epsilon(\lambda)\otimes I_n)$

of $R(\lambda)$ is greater than 1 and that at least one of the blocks $K_1(\lambda)$ or $K_2(\lambda)$ is not an empty matrix. The degenerate case in which $\varepsilon = 0$ and $\eta = 0$ will be studied in Section 9.6.

Since $S(\lambda)$ is a strong linearization of $R(\lambda)$, $S(\lambda)$ has the exact eigenstructure of the finite zeros of $R(\lambda)$, and its infinite zero structure as well as its left and right null-space structure can be correctly retrieved from the pencil via simple constant shifts, as explained in the introduction. In order to compute this eigenstructure, we make use of the staircase algorithm [83], followed by the QZ algorithm [71], on $S(\lambda)$. The backward stability of these two algorithms guarantees in fact that we computed the exact eigenstructure of a slightly perturbed pencil

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \Delta_S(\lambda) := \begin{bmatrix} \Delta_{11}(\lambda) & \Delta_{12}(\lambda) & \Delta_{13}(\lambda) \\ \Delta_{21}(\lambda) & \Delta_{22}(\lambda) & \Delta_{23}(\lambda) \\ \Delta_{31}(\lambda) & \Delta_{32}(\lambda) & \Delta_{33}(\lambda) \end{bmatrix}, \tag{9.21}$$

where the pencil $\Delta_S(\lambda)$ has a norm which is much smaller than the norm of $S(\lambda)$. More precisely, $\|\Delta_S(\lambda)\|_F = O(\epsilon_M)\,\|S(\lambda)\|_F$, where ϵ_M is the machine precision of the computer. But even for very small perturbations, the structure of the pencil $S(\lambda)$ is lost, and therefore also the connection between $\widehat{S}(\lambda)$ and some rational matrix $\widehat{R}(\lambda)$ is lost. In this section, we will show that this structure can be restored, without affecting the computed eigenstructure. For this, one needs only to find a strict equivalence transformation that is close to the identity and restores the structure of $\widehat{S}(\lambda)$ to a new pencil $\widetilde{S}(\lambda)$ that is a block Kronecker linearization, with the same parameters ϵ and η as $S(\lambda)$, of a rational matrix $\widetilde{R}(\lambda)$:

$$\widetilde{S}(\lambda) := (I - X)(S(\lambda) + \Delta_S(\lambda))(I - Y) = \begin{bmatrix} \widetilde{M}(\lambda) & \widehat{K}_2^T\widetilde{C} & K_2^T(\lambda) \\ \widetilde{B}\widehat{K}_1 & \widetilde{A} - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}. \tag{9.22}$$

We will see that if $\|\Delta_S(\lambda)\|_F$ is sufficiently small, then the perturbed system triple $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$ is very close to the unperturbed minimal one $\{A, B, C\}$ and, so, $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$ is still minimal, since minimality is a generic property equivalent to the controllability matrix having full row rank and the observability matrix having full column rank [55, Chapter 6]. Observe that according to [6], or the discussion in the introduction, $\widetilde{S}(\lambda)$ is a strong linearization of the $m \times n$ rational matrix

$$\begin{aligned} \widetilde{R}(\lambda) &:= \widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B} + (\Lambda_\eta(\lambda) \otimes I_m)^T \widetilde{M}(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n) \\ &=: \widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B} + \widetilde{D}(\lambda)\,. \end{aligned} \tag{9.23}$$

Since the eigenstructures of the pencils $\widehat{S}(\lambda)$ and $\widetilde{S}(\lambda)$ are identical, the results in this section prove that the computed finite eigenvalues of $S(\lambda)$ and their partial multiplicities are the exact finite zeros and their partial multiplicities of $\widetilde{R}(\lambda)$, the computed right (resp. left) minimal indices of $S(\lambda)$ minus ϵ (resp. η) are the exact right (resp. left) minimal indices of $\widetilde{R}(\lambda)$, and, if a number ν_r of right minimal indices of $S(\lambda)$ have been computed, then the computed $n-\nu_r$ largest partial multiplicities at infinity of $S(\lambda)$ minus d are the exact structural indices at infinity of $\widetilde{R}(\lambda)$. This is a very strong backward error result for the computation of the eigenstructure of $R(\lambda)$ in the case we are able to prove that $\|\widetilde{A}-A\|_F, \|\widetilde{B}-B\|_F, \|\widetilde{C}-C\|_F$ and $\|\widetilde{D}(\lambda)-D(\lambda)\|_F$ are very small.

The restoration of the structure in $\widehat{S}(\lambda)$ will be done in three steps, each of them involving a strict equivalence transformation close to the identity:

- **Step 1:** We restore the block anti-triangular structure of the perturbed pencil $\widehat{S}(\lambda)$, i.e., the blocks (2,3), (3,2) and (3,3) are transformed to become 0.
- **Step 2:** We take care of the anti-diagonal blocks (1,3), (2,2) and (3,1), by restoring their 0 and I block matrices.
- **Step 3:** We restore the special structure of the blocks (1,2) and (2,1).

At each step k, for $k=1,2,3$, we obtain a pencil

$$\widehat{S}_k(\lambda) := (I-X_k)\widehat{S}_{k-1}(\lambda)(I-Y_k) := \widehat{S}_{k-1}(\lambda) + \Delta_k(\lambda), \tag{9.24}$$

where $\widehat{S}_0(\lambda) := \widehat{S}(\lambda)$ and $\Delta_0(\lambda) := \Delta_S(\lambda)$:

$$S(\lambda) \xrightarrow{+\Delta_0(\lambda)} \widehat{S}(\lambda) = \widehat{S}_0(\lambda) \xrightarrow{+\Delta_1(\lambda)} \widehat{S}_1(\lambda) \xrightarrow{+\Delta_2(\lambda)} \widehat{S}_2(\lambda) \xrightarrow{+\Delta_3(\lambda)} \widehat{S}_3(\lambda) = \widetilde{S}(\lambda).$$

We will compute bounds for $\|(X_k, Y_k)\|_F$ as a function of $\|\widehat{S}_{k-1}(\lambda)\|_F$, where the Frobenius norms are computed as defined in the introduction. Moreover, we define the cumulative errors:

$$\begin{aligned} \Delta_k^{old}(\lambda) &:= \sum_{i=0}^{k-1} \Delta_i(\lambda), \text{ and} \\ \Delta_k^{new}(\lambda) &:= \Delta_k^{old}(\lambda) + \Delta_k(\lambda) = \sum_{i=0}^{k} \Delta_i(\lambda), \end{aligned} \tag{9.25}$$

and we will also compute bounds for the Frobenius norm of these error pencils. In our analysis, we will assume that $\delta := \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F}$ is very small, since in practice is of the order of the machine precision ϵ_M, and we will neglect, when appropriate, terms of order larger than 1 in δ to simplify our bounds. Moreover, we will assume that δ is sufficiently small for guaranteeing that all the steps in the analysis can be performed, for instance, for guaranteeing that some perturbed matrices are invertible, etc. In particular, we have Lemma 9.4.1 for computing bounds of the growth of the cumulative errors $\Delta_k^{new}(\lambda)$.

Lemma 9.4.1. *At each step k of our method, the perturbation $\Delta_k^{new}(\lambda)$ can be bounded by*

$$\|\Delta_k^{new}(\lambda)\|_F \leq \sqrt{2}\|\widehat{S}_{k-1}(\lambda)\|_2\|(X_k, Y_k)\|_F + \|\Delta_k^{old}(\lambda)\|_F + \mathcal{O}(\delta^2),$$

assuming that $\|(X_k, Y_k)\|_F$ is of the order of $\|\Delta_S(\lambda)\|_F$.

Proof. At step k, we have $\widehat{S}_k(\lambda) = (I - X_k)\widehat{S}_{k-1}(\lambda)(I - Y_k)$. Therefore

$$\Delta_k^{new}(\lambda) = \Delta_k^{old}(\lambda) - X_k\widehat{S}_{k-1}(\lambda) - \widehat{S}_{k-1}(\lambda)Y_k + X_k\widehat{S}_{k-1}(\lambda)Y_k.$$

It then follows that the increment (up to $\mathcal{O}(\delta^2)$ terms) is given by

$$-X_kS_a - S_aY_k + \lambda(X_kS_b + S_bY_k) + \mathcal{O}(\delta^2),$$

where $S_a - \lambda S_b := \widehat{S}_{k-1}(\lambda)$. We then use the inequalities

$$\|X_kS_a + S_aY_k\|_F^2 \leq 2\|S_a\|_2^2\|(X_k, Y_k)\|_F^2, \quad \|X_kS_b + S_bY_k\|_F^2 \leq 2\|S_b\|_2^2\|(X_k, Y_k)\|_F^2$$

and the definition for $\|\widehat{S}_{k-1}(\lambda)\|_2$, to finally get the required bound. ■

9.4.1 Step 1: Restoring the block anti-triangular structure

For step 1, that is, restoring the block anti-triangular structure of $S(\lambda)$ in the perturbed matrix pencil (9.21), we apply a strict equivalence transformation of the type :

$$\begin{bmatrix} I_{(\eta+1)m} & 0 & 0 \\ -X_{21} & I_\ell & 0 \\ -X_{31} & -X_{32} & I_{\varepsilon n} \end{bmatrix} \widehat{S}(\lambda) \begin{bmatrix} I_{(\varepsilon+1)n} & -Y_{12} & -Y_{13} \\ 0 & I_\ell & -Y_{23} \\ 0 & 0 & I_{\eta m} \end{bmatrix} \tag{9.26}$$

in order to eliminate the perturbations $\Delta_{23}(\lambda)$, $\Delta_{32}(\lambda)$ and $\Delta_{33}(\lambda)$ of the error matrix pencil $\Delta_0(\lambda)$. The notation $\widehat{S}_{ij}^a - \lambda\widehat{S}_{ij}^b := \widehat{S}_{ij} := \widehat{S}_{ij}(\lambda)$ will be used in this section to refer to sub-blocks of $\widehat{S}_0(\lambda)$. Let us write down the equations that we get by setting the blocks (2,3), (3,2) and (3,3) of the matrix in (9.26) equal to zero :

$$\begin{aligned}
\Delta_{23}(\lambda) &:= \Delta_{23}^a - \lambda\Delta_{23}^b = X_{21}\widehat{S}_{13} + \widehat{S}_{21}Y_{13} + \widehat{S}_{22}Y_{23} - X_{21}\widehat{S}_{11}Y_{13} - X_{21}\widehat{S}_{12}Y_{23}, \\
\Delta_{32}(\lambda) &:= \Delta_{32}^a - \lambda\Delta_{32}^b = \widehat{S}_{31}Y_{12} + X_{31}\widehat{S}_{12} + X_{32}\widehat{S}_{22} - X_{31}\widehat{S}_{11}Y_{12} - X_{32}\widehat{S}_{21}Y_{12}, \\
\Delta_{33}(\lambda) &:= \Delta_{33}^a - \lambda\Delta_{33}^b = X_{31}\widehat{S}_{13} + \widehat{S}_{31}Y_{13} + X_{32}\Delta_{23} + \Delta_{32}Y_{23} \\
&\qquad - X_{31}\widehat{S}_{11}Y_{13} - X_{32}\widehat{S}_{21}Y_{13} - X_{31}\widehat{S}_{12}Y_{23} - X_{32}\widehat{S}_{22}Y_{23}.
\end{aligned} \tag{9.27}$$

This is a system of nonlinear matrix equations for the six matrix unknowns X_{21}, X_{31}, X_{32}, Y_{12}, Y_{13} and Y_{23}. We will show that it is consistent and that it has a solution for which the norms of the unknowns are of the order of $\|\Delta_0(\lambda)\|_F$, which implies that there are many terms in the above three equations that are of second order.

Using Kronecker product and the vec(·) notation, the system of matrix equations (9.27) can be rewritten as :

$$\underbrace{\begin{bmatrix} \mathrm{vec}(\Delta^a_{23}) \\ \mathrm{vec}(\Delta^b_{23}) \\ \mathrm{vec}(\Delta^a_{32}) \\ \mathrm{vec}(\Delta^b_{32}) \\ \mathrm{vec}(\Delta^a_{33}) \\ \mathrm{vec}(\Delta^b_{33}) \end{bmatrix}}_{:=c} = (T+\Delta T) \underbrace{\begin{bmatrix} \mathrm{vec}(X_{21}) \\ \mathrm{vec}(Y_{23}) \\ \mathrm{vec}(X_{32}) \\ \mathrm{vec}(Y_{12}) \\ \mathrm{vec}(X_{31}) \\ \mathrm{vec}(Y_{13}) \end{bmatrix}}_{:=x} - \underbrace{\begin{bmatrix} \mathrm{vec}(Z_1) \\ \mathrm{vec}(Z_2) \\ \mathrm{vec}(Z_3) \\ \mathrm{vec}(Z_4) \\ \mathrm{vec}(Z_5) \\ \mathrm{vec}(Z_6) \end{bmatrix}}_{:=z}, \tag{9.28}$$

where

$$\begin{gathered} Z_1 := X_{21}\widehat{S}^a_{11}Y_{13} + X_{21}\widehat{S}^a_{12}Y_{23}, \quad Z_2 := X_{21}\widehat{S}^b_{11}Y_{13} + X_{21}\widehat{S}^b_{12}Y_{23}, \\ Z_3 := X_{31}\widehat{S}^a_{11}Y_{12} + X_{32}\widehat{S}^a_{21}Y_{12}, \quad Z_4 := X_{31}\widehat{S}^b_{11}Y_{12} + X_{32}\widehat{S}^b_{21}Y_{12}, \\ Z_5 := X_{31}\widehat{S}^a_{11}Y_{13} + X_{32}\widehat{S}^a_{21}Y_{13} + X_{31}\widehat{S}^a_{12}Y_{23} + X_{32}\widehat{S}^a_{22}Y_{23}, \\ Z_6 := X_{31}\widehat{S}^b_{11}Y_{13} + X_{32}\widehat{S}^b_{21}Y_{13} + X_{31}\widehat{S}^b_{12}Y_{23} + X_{32}\widehat{S}^b_{22}Y_{23}, \end{gathered}$$

$$\Delta T = \begin{bmatrix} {\Delta^a_{13}}^T \otimes I_\ell & I_{\eta m} \otimes \Delta^a_{22} & 0 & 0 & 0 & I_{\eta m} \otimes \Delta^a_{21} \\ {\Delta^b_{13}}^T \otimes I_\ell & I_{\eta m} \otimes \Delta^b_{22} & 0 & 0 & 0 & I_{\eta m} \otimes \Delta^b_{21} \\ 0 & 0 & {\Delta^a_{22}}^T \otimes I_{\varepsilon n} & I_\ell \otimes \Delta^a_{31} & {\Delta^a_{12}}^T \otimes I_{\varepsilon n} & 0 \\ 0 & 0 & {\Delta^b_{22}}^T \otimes I_{\varepsilon n} & I_\ell \otimes \Delta^b_{31} & {\Delta^b_{12}}^T \otimes I_{\varepsilon n} & 0 \\ 0 & I_{\eta m} \otimes \Delta^a_{32} & {\Delta^a_{23}}^T \otimes I_{\varepsilon n} & 0 & {\Delta^a_{13}}^T \otimes I_{\varepsilon n} & I_{\eta m} \otimes \Delta^a_{31} \\ 0 & I_{\eta m} \otimes \Delta^b_{32} & {\Delta^b_{23}}^T \otimes I_{\varepsilon n} & 0 & {\Delta^b_{13}}^T \otimes I_{\varepsilon n} & I_{\eta m} \otimes \Delta^b_{31} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{11} & T_{12} & 0 & 0 & 0 & T_{16} \\ T_{21} & T_{22} & 0 & 0 & 0 & 0 \\ & & T_{33} & T_{34} & T_{35} & 0 \\ & & T_{43} & T_{44} & 0 & 0 \\ & & & & T_{55} & T_{56} \\ & & & & T_{65} & T_{66} \end{bmatrix},$$

with

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} := \left[\begin{array}{c|c} E_\eta \otimes I_{m\ell} & I_{\eta m} \otimes A \\ \hline F_\eta \otimes I_{m\ell} & I_{\eta m} \otimes I_\ell \end{array}\right], \begin{bmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{bmatrix} := \left[\begin{array}{c|c} A^T \otimes I_{\varepsilon n} & I_\ell \otimes E_\varepsilon \otimes I_n \\ \hline I_\ell \otimes I_{\varepsilon n} & I_\ell \otimes F_\varepsilon \otimes I_n \end{array}\right],$$

$$\begin{bmatrix} T_{55} & T_{56} \\ T_{65} & T_{66} \end{bmatrix} := \left[\begin{array}{c|c} E_\eta \otimes I_{m\varepsilon n} & I_{\eta m} \otimes E_\varepsilon \otimes I_n \\ \hline F_\eta \otimes I_{m\varepsilon n} & I_{\eta m} \otimes F_\varepsilon \otimes I_n \end{array}\right], \quad \left\{ \begin{array}{l} T_{16} := I_{\eta m} \otimes \mathbf{e}^T_{\varepsilon+1} \otimes B \\ T_{35} := \mathbf{e}^T_{\eta+1} \otimes C^T \otimes I_{\varepsilon n} \end{array} \right. .$$

We emphasize that the matrices in the two lines above are precisely those appearing in equations (9.11), (9.10) and (9.12), respectively.

The smallest singular value of T and the 2–norm of ΔT will be needed in the analysis of the bound for the structured backward errors. More precisely for proving that (9.27) is consistent and bounding the norm of one of its solutions. A lower bound for $\sigma_{\min}(T)$ and an upper bound for $\|\Delta T\|_2$ are given in Lemma 9.4.2 and Lemma 9.4.3, respectively.

Lemma 9.4.2. *Let T be the matrix in* (9.28). *Let $\alpha := 1 + 2\varepsilon \max(1, \|A\|_2^{\varepsilon})$, $\beta := 1 + 2\eta \max(1, \|A\|_2^{\eta})$, $\gamma := \frac{\varepsilon+\eta}{2\sqrt{2}}$ and $s := \max(\alpha, \beta, \gamma) + \gamma(\beta\|B\|_2 + \alpha\|C\|_2)$ then*

$$\sigma_{\min}(T) \geq \frac{1}{s}.$$

Proof. If we partition the matrix T as a block triangular matrix

$$T = \begin{bmatrix} T_1 & 0 & T_B \\ & T_2 & T_C \\ & & T_3 \end{bmatrix},$$

then the diagonal blocks have full row ranks because their smallest singular values are strictly larger than zero according to Lemmas 9.3.2, 9.3.1 and 9.3.3, respectively. Therefore, they are right invertible, with Moore–Penrose pseudoinverses T_i^r satisfying $T_i T_i^r = I$, for $i = 1, 2, 3$. Moreover, $\|T_1^r\|_2 = \omega_2^{-1}$, $\|T_2^r\|_2 = \omega_1^{-1}$ and $\|T_3^r\|_2 = \omega_3^{-1}$, with ω_1, ω_2 and ω_3 as in (9.10), (9.11) and (9.12). A right inverse T^r for T is given by

$$T^r = \begin{bmatrix} T_1^r & 0 & -T_1^r T_B T_3^r \\ & T_2^r & -T_2^r T_C T_3^r \\ & & T_3^r \end{bmatrix}$$

since $TT^r = I$. It then follows that the smallest singular value of T is lower bounded by $\|T^r\|_2^{-1}$. This right inverse can be written as the sum of three matrices (one of them being $\mathrm{diag}(T_1^r, T_2^r, T_3^r)$), and the 2-norm of each of them can be upper bounded using the results of Section 9.3 and the fact that $\|T_B\|_2 = \|B\|_2$ and $\|T_C\|_2 = \|C\|_2$. We then obtain the bound :

$$\begin{aligned} \sigma_{\min}(T) \geq & 1/\left[\max(\omega_1^{-1}, \omega_2^{-1}, \omega_3^{-1}) + \omega_3^{-1}(\omega_2^{-1}\|B\|_2 + \omega_1^{-1}\|C\|_2)\right] \\ \geq & 1/\left[\max(\alpha, \beta, \gamma) + \gamma(\beta\|B\|_2 + \alpha\|C\|_2)\right], \end{aligned}$$

by taking into account inequalities (9.14), (9.15), (9.16). ∎

Lemma 9.4.3. *Let ΔT be the matrix in* (9.28) *and let $\Delta_S(\lambda)$ be the pencil in* (9.21). *Then*

$$\|\Delta T\|_2 \leq \sqrt{3}\|\Delta_S(\lambda)\|_2.$$

Proof. We consider a permutation matrix P such that

$$\Delta T = \left[\begin{array}{cc|cc|cc} {\Delta_{13}^{a}}^{T} \otimes I_\ell & 0 & 0 & 0 & I_{\eta m} \otimes \Delta_{22}^{a} & I_{\eta m} \otimes \Delta_{21}^{a} \\ {\Delta_{13}^{b}}^{T} \otimes I_\ell & 0 & 0 & 0 & I_{\eta m} \otimes \Delta_{22}^{b} & I_{\eta m} \otimes \Delta_{21}^{b} \\ 0 & I_\ell \otimes \Delta_{31}^{a} & {\Delta_{22}^{a}}^{T} \otimes I_{\varepsilon n} & {\Delta_{12}^{a}}^{T} \otimes I_{\varepsilon n} & 0 & 0 \\ 0 & I_\ell \otimes \Delta_{31}^{b} & {\Delta_{22}^{b}}^{T} \otimes I_{\varepsilon n} & {\Delta_{12}^{b}}^{T} \otimes I_{\varepsilon n} & 0 & 0 \\ 0 & 0 & {\Delta_{23}^{a}}^{T} \otimes I_{\varepsilon n} & {\Delta_{13}^{a}}^{T} \otimes I_{\varepsilon n} & I_{\eta m} \otimes \Delta_{32}^{a} & I_{\eta m} \otimes \Delta_{31}^{a} \\ 0 & 0 & {\Delta_{23}^{b}}^{T} \otimes I_{\varepsilon n} & {\Delta_{13}^{b}}^{T} \otimes I_{\varepsilon n} & I_{\eta m} \otimes \Delta_{32}^{b} & I_{\eta m} \otimes \Delta_{31}^{b} \end{array}\right] P$$
$$:= [T_1|T_2|T_3]\, P.$$

Using properties of norms and Kronecker products (see [54, Chapter 4]) we have that $\|T_i\|_2 \le \|\Delta_S(\lambda)\|_2$ for $i = 1, 2, 3$. Finally, by [52, Lemma 3.5],

$$\|\Delta T\|_2 \le \sqrt{3} \max\{\|T_1\|_2, \|T_2\|_2, \|T_3\|_2\} \le \sqrt{3}\|\Delta_S(\lambda)\|_2.$$

∎

In order to prove that the system of nonlinear matrix equations (9.27) is consistent, first, we remove quadratic terms in X_{ij} and Y_{ij} of these equations and we get the following system of linear equations :

$$\begin{aligned} \Delta_{23}(\lambda) &= X_{21}\widehat{S}_{13} + \widehat{S}_{21}Y_{13} + \widehat{S}_{22}Y_{23}, \\ \Delta_{32}(\lambda) &= \widehat{S}_{31}Y_{12} + X_{31}\widehat{S}_{12} + X_{32}\widehat{S}_{22}, \\ \Delta_{33}(\lambda) &= X_{31}\widehat{S}_{13} + \widehat{S}_{31}Y_{13} + X_{32}\Delta_{23} + \Delta_{32}Y_{23}. \end{aligned}$$

This linear system of matrix equations can be rewritten as the underdetermined linear system :

$$(T + \Delta T)x = c, \tag{9.29}$$

with the same notation as in (9.28). Next we prove that (9.29) is consistent for any right hand side if ΔT is sufficiently small. From the minimum norm solution of (9.29), we obtain in Theorem 9.4.6 that there exists a solution for the quadratic system (9.28) under certain conditions and bound its norm.

Lemma 9.4.4. *Let $(T + \Delta T)x = c$ be the underdetermined linear system in* (9.29)*, and let us assume that $\sigma_{\min}(T) > \|\Delta T\|_2$. Then $(T + \Delta T)x = c$ is consistent and its minimum norm solution $(X^0, Y^0) := (X_{21}^0, X_{31}^0, X_{32}^0, Y_{12}^0, Y_{13}^0, Y_{23}^0)$ satisfies*

$$\|(X^0, Y^0)\|_F \le \frac{1}{\sigma}\|(\Delta_{23}(\lambda), \Delta_{32}(\lambda), \Delta_{33}(\lambda))\|_F,$$

where $\sigma := \sigma_{\min}(T) - \|\Delta T\|_2$.

Proof. Analogous proof as for [26, Lemma 5.6]. ∎

The notation $\sigma := \sigma_{\min}(T) - \|\Delta T\|_2$ has been chosen to remind that σ is a lower bound for the smallest singular value of $T + \Delta T$, since $\sigma_{\min}(T + \Delta T) \geq \sigma_{\min}(T) - \|\Delta T\|_2$ by Weyl's perturbation theorem for singular values [54, Theorem 3.3.16]. Lemma 9.4.5 gives a sufficient condition on $\|\Delta_S(\lambda)\|_2$ that guarantees $\sigma > 0$ and, hence, that allows us to apply Lemma 9.4.4.

Lemma 9.4.5. *Consider the real number s defined as in Lemma 9.4.2. Let T and ΔT be the matrices in* (9.29)*, and let $\Delta_S(\lambda)$ be the pencil in* (9.21)*. If $\|\Delta_S(\lambda)\|_2 < \frac{1}{2s}$ then*

$$\sigma = \sigma_{\min}(T) - \|\Delta T\|_2 > \frac{2 - \sqrt{3}}{2s} > 0.$$

Proof. If $\|\Delta_S(\lambda)\|_2 < \frac{1}{2s}$ we have, by Lemmas 9.4.2 and 9.4.3, that $\sigma_{\min}(T) - \|\Delta T\|_2 \geq \frac{1}{s} - \sqrt{3}\,\|\Delta_S(\lambda)\|_2 > \frac{2-\sqrt{3}}{2s} > 0$. ■

Theorem 9.4.6 establishes conditions in order the system of matrix equations (9.27) to have a solution as we announced. Moreover, it gives an upper bound for the Frobenius norm of this solution. We remark that Theorem 9.4.6 is similar to [26, Theorem 5.8], though the involved systems of matrix equations are very different from each other. Therefore, some details in the proof of Theorem 9.4.6 are omitted since can be found in [26].

Theorem 9.4.6. *There exists a solution $(X, Y) := (X_{21}, X_{31}, X_{32}, Y_{12}, Y_{13}, Y_{23})$ of the quadratic system of equations* (9.28) *satisfying*

$$\|(X, Y)\|_F \leq 2\frac{\theta}{\sigma},$$

whenever

$$\sigma > 0 \quad \text{and} \quad \frac{\theta\omega}{\sigma^2} < \frac{1}{4}, \tag{9.30}$$

where $\omega := \|(M(\lambda), A - \lambda I_\ell, B, C)\|_F + \|\Delta_S(\lambda)\|_F$, $\theta := \|(\Delta_{23}(\lambda), \Delta_{32}(\lambda), \Delta_{33}(\lambda))\|_F$, and $\sigma = \sigma_{\min}(T) - \|\Delta T\|_2$.

Proof. Since $\sigma > 0$, we can apply Lemma 9.4.4 and consider (X^0, Y^0) the minimum norm solution of (9.29). Let

$$x_0 := \left[\ \mathrm{vec}(X^0_{21})^T \,\mathrm{vec}(Y^0_{23})^T \,\mathrm{vec}(X^0_{32})^T \,\mathrm{vec}(Y^0_{12})^T \,\mathrm{vec}(X^0_{31})^T \,\mathrm{vec}(Y^0_{13})^T\ \right]^T.$$

Let us define the sequence $\{(X^i, Y^i) := (X^i_{21}, X^i_{31}, X^i_{32}, Y^i_{12}, Y^i_{13}, Y^i_{23})\}_{i=0}^{\infty}$ such that, for each $i > 0$, (X^i, Y^i) is the minimum norm solution of the linear system

$$(T + \Delta T)\begin{bmatrix} \mathrm{vec}(X^i_{21}) \\ \mathrm{vec}(Y^i_{23}) \\ \mathrm{vec}(X^i_{32}) \\ \mathrm{vec}(Y^i_{12}) \\ \mathrm{vec}(X^i_{31}) \\ \mathrm{vec}(Y^i_{13}) \end{bmatrix} = c + \begin{bmatrix} \mathrm{vec}(Z^{i-1}_1) \\ \mathrm{vec}(Z^{i-1}_2) \\ \mathrm{vec}(Z^{i-1}_3) \\ \mathrm{vec}(Z^{i-1}_4) \\ \mathrm{vec}(Z^{i-1}_5) \\ \mathrm{vec}(Z^{i-1}_6) \end{bmatrix}, \tag{9.31}$$

where

$$Z_1^{i-1} := X_{21}^{i-1}\widehat{S}_{11}^a Y_{13}^{i-1} + X_{21}^{i-1}\widehat{S}_{12}^a Y_{23}^{i-1}, \quad Z_2^{i-1} := X_{21}^{i-1}\widehat{S}_{11}^b Y_{13}^{i-1} + X_{21}^{i-1}\widehat{S}_{12}^b Y_{23}^{i-1},$$

$$Z_3^{i-1} := X_{31}^{i-1}\widehat{S}_{11}^a Y_{12}^{i-1} + X_{32}^{i-1}\widehat{S}_{21}^a Y_{12}^{i-1}, \quad Z_4^{i-1} := X_{31}^{i-1}\widehat{S}_{11}^b Y_{12}^{i-1} + X_{32}^{i-1}\widehat{S}_{21}^b Y_{12}^{i-1},$$

$$Z_5^{i-1} := X_{31}^{i-1}\widehat{S}_{11}^a Y_{13}^{i-1} + X_{32}^{i-1}\widehat{S}_{21}^a Y_{13}^{i-1} + X_{31}^{i-1}\widehat{S}_{12}^a Y_{23}^{i-1} + X_{32}^{i-1}\widehat{S}_{22}^a Y_{23}^{i-1}, \quad \text{and}$$

$$Z_6^{i-1} := X_{31}^{i-1}\widehat{S}_{11}^b Y_{13}^{i-1} + X_{32}^{i-1}\widehat{S}_{21}^b Y_{13}^{i-1} + X_{31}^{i-1}\widehat{S}_{12}^b Y_{23}^{i-1} + X_{32}^{i-1}\widehat{S}_{22}^b Y_{23}^{i-1}.$$

Note that the minimum norm solution of (9.31) is obtained by multiplying the right hand side of (9.31) by the Moore-Penrose pseudoinverse of $T + \Delta T$, denoted by $(T + \Delta T)^\dagger$, and that $x_0 = (T + \Delta T)^\dagger c$.

Now we assume that $\frac{\theta\omega}{\sigma^2} < \frac{1}{4}$ holds. Then we can prove that the sequence $\{(X^i, Y^i)\}_{i=0}^\infty$ converges to a solution (X, Y) of the quadratic system of equations (9.28) analogously as it is done in [26, Theorem 5.8]. For that, we have to take into account that, if $\|(X^{i-1}, Y^{i-1})\|_F \le \rho_{i-1}$, then

$$\begin{aligned}
&\|(X^i, Y^i)\|_F \\
&\le \|(X^0, Y^0)\|_F + \|(T+\Delta T)^\dagger\|_2 \left\| \begin{bmatrix} X_{21}^{i-1} & 0 \\ X_{31}^{i-1} & X_{32}^{i-1} \end{bmatrix} \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{bmatrix} \begin{bmatrix} Y_{12}^{i-1} & Y_{13}^{i-1} \\ 0 & Y_{23}^{i-1} \end{bmatrix} \right\|_F \\
&\le \rho_0 + \sigma^{-1}\rho_{i-1}^2\omega := \rho_i,
\end{aligned}$$

where $\|(X^0, Y^0)\|_F \le \theta\sigma^{-1} := \rho_0$. Therefore, we can define the same fixed point iteration as in the proof of [26, Theorem 5.8] and we obtain that the sequence is bounded, i.e., $\|(X^i, Y^i)\|_F \le \rho$, with $\rho < 2\sigma^{-1}\theta$, for all $i \ge 0$. In addition, if we define the sequence $\{C_i := (X^{i+1}, Y^{i+1}) - (X^i, Y^i)\}_{i=0}^\infty$ then

$$\begin{aligned}
\|C_i\|_F \le & \|(T+\Delta T)^\dagger\|_2 \left(\left\| \begin{bmatrix} X_{21}^{i} & 0 \\ X_{31}^{i} & X_{32}^{i} \end{bmatrix} \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{bmatrix} \begin{bmatrix} Y_{12}^{i} & Y_{13}^{i} \\ 0 & Y_{23}^{i} \end{bmatrix} \right.\right. \\
& \left.\left. - \begin{bmatrix} X_{21}^{i-1} & 0 \\ X_{31}^{i-1} & X_{32}^{i-1} \end{bmatrix} \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{bmatrix} \begin{bmatrix} Y_{12}^{i-1} & Y_{13}^{i-1} \\ 0 & Y_{23}^{i-1} \end{bmatrix} \right\|_F \right) \\
\le & \|(T+\Delta T)^\dagger\|_2 \left(\left\| \begin{bmatrix} X_{21}^{i} - X_{21}^{i-1} & 0 \\ X_{31}^{i} - X_{31}^{i-1} & X_{32}^{i} - X_{32}^{i-1} \end{bmatrix} \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{bmatrix} \begin{bmatrix} Y_{12}^{i} & Y_{13}^{i} \\ 0 & Y_{23}^{i} \end{bmatrix} \right\|_F \right. \\
& \left. + \left\| \begin{bmatrix} X_{21}^{i-1} & 0 \\ X_{31}^{i-1} & X_{32}^{i-1} \end{bmatrix} \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} \\ \widehat{S}_{21} & \widehat{S}_{22} \end{bmatrix} \begin{bmatrix} Y_{12}^{i} - Y_{12}^{i-1} & Y_{13}^{i} - Y_{13}^{i-1} \\ 0 & Y_{23}^{i} - Y_{23}^{i-1} \end{bmatrix} \right\|_F \right) \\
\le & 2\sigma^{-1}\rho\omega\|C_{i-1}\|_F.
\end{aligned}$$

The above inequality implies that $\{(X^i, Y^i)\}_{i=0}^\infty$ is a Cauchy sequence, since $2\sigma^{-1}\rho\omega < 1$. Thus, taking limits in both sides of (9.31), we see that $\{(X^i, Y^i)\}_{i=0}^\infty$ converges to a solution (X, Y) of the system of equations in (9.28) with $\|(X, Y)\|_F \le \rho$. ■

Theorem 9.4.6, together with Lemma 9.4.5, allow us to prove in Theorem 9.4.7 that there exists a solution (X, Y) of (9.27) which is of the order of the perturbation $\Delta_S(\lambda)$ whenever $\|\Delta_S(\lambda)\|_F$ is properly upper bounded.

Theorem 9.4.7. *Consider the real number s defined as in Lemma 9.4.2. Let $S(\lambda)$ be a block Kronecker linearization as in* (9.19)*, and let $\Delta_S(\lambda)$ be a perturbation of $S(\lambda)$ as in* (9.21) *such that*

$$\|\Delta_S(\lambda)\|_F < \left(\frac{2-\sqrt{3}}{4s}\right)^2 \frac{1}{1+\|(M(\lambda), A-\lambda I_\ell, B, C)\|_F}. \tag{9.32}$$

Then there exists a solution $(X, Y) := (X_{21}, X_{31}, X_{32}, Y_{12}, Y_{13}, Y_{23})$ of the quadratic system of matrix equations in (9.27) *that satisfies*

$$\|(X, Y)\|_F \leq \frac{4s\|\Delta_S(\lambda)\|_F}{2-\sqrt{3}}. \tag{9.33}$$

Proof. We have

$$\|\Delta_S(\lambda)\|_F < \left(\frac{2-\sqrt{3}}{4s}\right)^2 \frac{1}{1+\|(M(\lambda), A-\lambda I_\ell, B, C)\|_F} \leq \frac{1}{2s}$$

since $s \geq 1$. Then, by Lemma 9.4.5, $\sigma = \sigma_{\min}(T) - \|\Delta T\|_2 > \frac{2-\sqrt{3}}{2s} > 0$. In addition, using the same notation as in Theorem 9.4.6,

$$\frac{\theta\omega}{\sigma^2} \leq \frac{\|\Delta_S(\lambda)\|_F(\|(M(\lambda), A-\lambda I_\ell, B, C)\|_F + \|\Delta_S(\lambda)\|_F)}{\left(\frac{2-\sqrt{3}}{2s}\right)^2} < \frac{1}{4},$$

by (9.32). Therefore, conditions in (9.30) hold and, by Theorem 9.4.6, there exists a solution (X, Y) of the system in (9.27) satisfying

$$\|(X, Y)\|_F \leq 2\frac{\theta}{\sigma} \leq \frac{4s\|\Delta_S(\lambda)\|_F}{2-\sqrt{3}}.$$

■

After restoring the block anti-triangular structure of $S(\lambda)$, we get the perturbation error $\Delta_1^{new}(\lambda)$ defined in (9.25). The following first order bound for the norm of $\Delta_1^{new}(\lambda)$ in Corollary 9.4.8 follows from Lemma 9.4.1 and Theorem 9.4.7.

Corollary 9.4.8. *Let us define the scalar $f_1 := \frac{4\sqrt{2}s}{2-\sqrt{3}}$. Then*

$$\begin{aligned}\|\Delta_1^{new}(\lambda)\|_F &\leq [1 + f_1\|\widehat{S}_0(\lambda)\|_2]\,\|\Delta_S(\lambda)\|_F + \mathcal{O}(\delta^2)\\ &\leq [1 + f_1\|S(\lambda)\|_2]\,\|\Delta_S(\lambda)\|_F + \mathcal{O}(\delta^2).\end{aligned}$$

9.4.2 Step 2: Restoring the Kronecker blocks $K_1(\lambda)$, $K_2(\lambda)$ and the identity I_ℓ

At this stage we have obtained a pencil $\widehat{S}_1(\lambda) = S(\lambda) + \Delta_1^{new}(\lambda)$ of the type

$$\widehat{S}_1(\lambda) := \begin{bmatrix} \widehat{M}(\lambda) & \widehat{C}(\lambda) & \widehat{K}_2^T(\lambda) \\ \widehat{B}(\lambda) & \widehat{A} - \lambda \widehat{I}_\ell & 0 \\ \widehat{K}_1(\lambda) & 0 & 0 \end{bmatrix}, \tag{9.34}$$

where the zero blocks below the anti-diagonal are exact and $\widehat{S}_1(\lambda)$ is strictly equivalent to $\widehat{S}(\lambda)$. In this subsection, we will use $\Delta_{ij}^a - \lambda\Delta_{ij}^b$ to denote the corresponding blocks of the updated perturbation matrix $\Delta_1^{new}(\lambda)$. We assume that the norm of the perturbation $\Delta_1^{new}(\lambda)$ is small enough for $\widehat{K}_1(\lambda)$ and $\widehat{K}_2(\lambda)$ to be also minimal bases with row degrees all equal to 1 and the row degrees of their dual minimal bases all equal to ϵ and η, respectively [26, Corollary 5.15]. Thus, $\widehat{K}_1(\lambda)$ and $\widehat{K}_2(\lambda)$ have the same Kronecker canonical forms as $K_1(\lambda)$ and $K_2(\lambda)$, respectively, and are strictly equivalent to them. We will then perform step 2, that is, an updating block-diagonal strict equivalent transformation of the type

$$\begin{bmatrix} I_{(\eta+1)m} - X_{11} & 0 & 0 \\ 0 & I_\ell - X_{22} & 0 \\ 0 & 0 & I_{\varepsilon n} - X_{33} \end{bmatrix} \widehat{S}_1(\lambda) \begin{bmatrix} I_{(\epsilon+1)n} - Y_{11} & 0 & 0 \\ 0 & I_\ell - Y_{22} & 0 \\ 0 & 0 & I_{\eta m} - Y_{33} \end{bmatrix} \tag{9.35}$$

such that

$$(I - X_{33})\widehat{K}_1(\lambda)(I - Y_{11}) = K_1(\lambda), \quad (I - X_{11})\widehat{K}_2^T(\lambda)(I - Y_{33}) = K_2^T(\lambda),$$

and

$$(I - X_{22})\widehat{I}_\ell(I - Y_{22}) = I_\ell.$$

In the last three equations the sizes of some identity matrices are not specified for simplicity. Clearly, these three problems are independent from each other and can be treated separately.

Let us first look at the equation restoring $K_1(\lambda)$. As pointed out in Section 9.2, this can be reduced to the solution of a Sylvester equation. Let

$$\widehat{K}_1(\lambda) = K_1(\lambda) + \Delta_{K_1}(\lambda) := L_\varepsilon(\lambda) \otimes I_n + \Delta_{K_1}(\lambda) := (E_\varepsilon - \lambda F_\varepsilon) \otimes I_n + (\Delta_{31}^a - \lambda \Delta_{31}^b).$$

Then, making the change of variables $Y_{11} := Y$ and $X_{33} := X(I + X)^{-1}$, it suffices to solve

$$(K_1(\lambda) + \Delta_{K_1}(\lambda))Y + XK_1(\lambda) = \Delta_{K_1}(\lambda),$$

or, equivalently,

$$\left[\begin{array}{c|c} E_\varepsilon^T \otimes I_{n\varepsilon n} & I_{(\varepsilon+1)n} \otimes (E_\varepsilon \otimes I_n + \Delta_{31}^a) \\ \hline F_\varepsilon^T \otimes I_{n\varepsilon n} & I_{(\varepsilon+1)n} \otimes (F_\varepsilon \otimes I_n + \Delta_{31}^b) \end{array}\right] \begin{bmatrix} \mathrm{vec}(X) \\ \mathrm{vec}(Y) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(\Delta_{31}^a) \\ \mathrm{vec}(\Delta_{31}^b) \end{bmatrix}. \tag{9.36}$$

By Lemma 9.3.4, the smallest singular value of the unperturbed problem satisfies

$$\sigma_{2\varepsilon n(\varepsilon+1)n}\left[\begin{array}{c|c} E_\varepsilon^T \otimes I_{n\varepsilon n} & I_{(\varepsilon+1)n}\otimes E_\varepsilon \otimes I_n \\ \hline F_\varepsilon^T \otimes I_{n\varepsilon n} & I_{(\varepsilon+1)n}\otimes F_\varepsilon \otimes I_n \end{array}\right] \geq \frac{3}{4\varepsilon-1}.$$

Then, by using Weyl's perturbation theorem for singular values [54, Theorem 3.3.16], one obtains the following bound for the minimum norm solution of (9.36)

$$\|(X,Y)\|_F \leq \left[\frac{3}{4\varepsilon-1} - \|\Delta_{31}^a\|_2 - \|\Delta_{31}^b\|_2\right]^{-1} \|(\Delta_{31}^a,\Delta_{31}^b)\|_F,$$

assuming that the perturbation is small enough for satisfying $\frac{3}{4\varepsilon-1} - \|\Delta_{31}^a\|_2 - \|\Delta_{31}^b\|_2 > 0$. In addition,

$$\|(X_{33},Y_{11})\|_F \leq \|(X,Y)\|_F/(1-\|(X,Y)\|_F).$$

Since $\|\Delta_{31}^a\|_2$ and $\|\Delta_{31}^b\|_2$ are of the order of δ, finally yields

$$\|(X_{33},Y_{11})\|_F \leq \frac{4\varepsilon-1}{3}\|(\Delta_{31}^a,\Delta_{31}^b)\|_F + \mathcal{O}(\delta^2), \tag{9.37}$$

by neglecting quantities of the order of $\mathcal{O}(\delta^2)$.

The problem for restoring $K_2(\lambda)$ is clearly dual to the problem of $K_1(\lambda)$ and will therefore yield the bound

$$\|(X_{11},Y_{33})\|_F \leq \frac{4\eta-1}{3}\|(\Delta_{13}^a,\Delta_{13}^b)\|_F + \mathcal{O}(\delta^2). \tag{9.38}$$

The problem of restoring I_ℓ amounts to solving $(I_\ell - X_{22})(I_\ell+\Delta_{22}^b)(I_\ell - Y_{22}) = I_\ell$, with $\widehat{I}_\ell = I_\ell + \Delta_{22}^b$. There are many possible solutions. A very simple one is to take $Y_{22}=0$ and $I_\ell - X_{22} = (I_\ell+\Delta_{22}^b)^{-1}$, assuming Δ_{22}^b is small enough for the inverse to exist. This means that $X_{22} = \Delta_{22}^b + \mathcal{O}(\|\Delta_{22}^b\|_F^2)$ and

$$\|(X_{22},Y_{22})\|_F = \|\Delta_{22}^b\|_F + \mathcal{O}(\delta^2). \tag{9.39}$$

We summarize this discussion in the following Theorem.

Theorem 9.4.9. *Let the pencil $\widehat{S}_1(\lambda)$ have the block anti-triangular form given in* (9.34)*. If* $\max(\varepsilon,\eta) > 0$*, then the updating strict equivalence transformation* $(I-X)\widehat{S}_1(\lambda)(I-Y)$ *detailed in* (9.35) *exists and can be bounded by*

$$\|(X,Y)\|_F \leq \frac{4\max(\varepsilon,\eta)-1}{3}\|\Delta_1^{new}(\lambda)\|_F + \mathcal{O}(\delta^2).$$

Proof. The bound for $\|(X,Y)\|_F$ follows directly from the identity

$$\|(X,Y)\|_F^2 = \|(X_{11},Y_{33})\|_F^2 + \|(X_{22},Y_{22})\|_F^2 + \|(X_{33},Y_{11})\|_F^2,$$

from the inequality

$$\|(\Delta_{13}^a,\Delta_{13}^b)\|_F^2 + \|\Delta_{22}^b\|_F^2 + \|(\Delta_{31}^a,\Delta_{31}^b)\|_F^2 \leq \|\Delta_1^{new}(\lambda)\|_F^2$$

and from the individual inequalities (9.37), (9.38) and (9.39). ∎

The following first order bound in Corollary 9.4.10 for the norm of the perturbation error $\Delta_2^{new}(\lambda)$ follows from Lemma 9.4.1, Theorem 9.4.9 and Corollary 9.4.8.

Corollary 9.4.10. *Let us define the scalar* $f_2 := \frac{\sqrt{2}(4\max(\varepsilon,\eta)-1)}{3}$. *Then*

$$\begin{aligned}\|\Delta_2^{new}(\lambda)\|_F &\leq [1 + f_2\|\widehat{S}_1(\lambda)\|_2]\,\|\Delta_1^{new}(\lambda)\|_F + \mathcal{O}(\delta^2)\\ &\leq [1 + f_2\|S(\lambda)\|_2]\,\|\Delta_1^{new}(\lambda)\|_F + \mathcal{O}(\delta^2).\end{aligned}$$

9.4.3 Step 3: Restoring the constant B and C matrices

From steps 1 and 2, described in the previous subsections, we have obtained a pencil $\widehat{S}_2(\lambda) = S(\lambda) + \Delta_2^{new}(\lambda)$ of the type

$$\widehat{S}_2(\lambda) := \begin{bmatrix} \widehat{M}(\lambda) & \widehat{C}(\lambda) & K_2^T(\lambda) \\ \widehat{B}(\lambda) & \widehat{A} - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix} \tag{9.40}$$

strictly equivalent to $\widehat{S}(\lambda)$. We emphasize that the blocks $\widehat{M}(\lambda), \widehat{B}(\lambda), \widehat{C}(\lambda)$ and the matrix $\widehat{A}$ are obviously different in (9.40) and in (9.34). We use the same symbols for avoiding a cumbersome notation. In this subsection, we will use $\Delta_{ij}(\lambda) = \Delta_{ij}^a - \lambda\Delta_{ij}^b$ to denote the corresponding blocks of the updated perturbation matrix $\Delta_2^{new}(\lambda)$. In this third step, we will restore the pencil $\widehat{S}_2(\lambda)$ to one where the blocks

$$\widehat{B}(\lambda) = B\widehat{K}_1 + \Delta_{21}(\lambda), \quad \text{and} \quad \widehat{C}(\lambda) = \widehat{K}_2^T C + \Delta_{12}(\lambda)$$

are transformed to $\widetilde{B}\widehat{K}_1$ and $\widehat{K}_2^T\widetilde{C}$, respectively. We recall that

$$K_1(\lambda) = L_\epsilon(\lambda) \otimes I_n, \quad \widehat{K}_1 = \mathbf{e}_{\epsilon+1}^T \otimes I_n, \quad K_2(\lambda) = L_\eta(\lambda) \otimes I_m, \quad \widehat{K}_2 = \mathbf{e}_{\eta+1}^T \otimes I_m,$$

where $\mathbf{e}_k$ is the standard kth unit vector of dimension k and $L_k(\lambda)$ is the classical Kronecker block of dimension $k\times(k+1)$, as introduced below (9.2). We will construct for this a strict equivalence transformation of the type

$$\begin{aligned}&\begin{bmatrix} I_{m(\eta+1)} & -X_{12} & 0 \\ & I_\ell & -X_{23} \\ & & I_{n\varepsilon} \end{bmatrix} \widehat{S}_2(\lambda) \begin{bmatrix} I_{n(\varepsilon+1)} & & \\ -Y_{21} & I_\ell & \\ 0 & -Y_{32} & I_{m\eta} \end{bmatrix}\\ &= \begin{bmatrix} \widetilde{M}(\lambda) & \widehat{K}_2^T\widetilde{C} & K_2^T(\lambda) \\ \widetilde{B}\widehat{K}_1 & \widehat{A} - \lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}\end{aligned} \tag{9.41}$$

The problems for $\widehat{B}(\lambda)$ and $\widehat{C}(\lambda)$ can again be treated separately. Let us first focus on the subsystem

$$\begin{aligned}&\begin{bmatrix} I_{m(\eta+1)} & -X_{12} \\ & I_\ell \end{bmatrix} \begin{bmatrix} \widehat{C}(\lambda) & L_\eta^T(\lambda) \otimes I_m \\ \widehat{A} - \lambda I_\ell & 0 \end{bmatrix} \begin{bmatrix} I_\ell & \\ -Y_{32} & I_{m\eta} \end{bmatrix}\\ &= \begin{bmatrix} \mathbf{e}_{\eta+1} \otimes \widetilde{C} & L_\eta^T(\lambda) \otimes I_m \\ \widehat{A} - \lambda I_\ell & 0 \end{bmatrix}.\end{aligned}$$

If we partition the matrices X_{12}, Y_{32} and $\widehat{C}(\lambda)$ as follows :

$$X_{12} := \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_\eta \\ E_{\eta+1} \end{bmatrix}, \quad Y_{32} := \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_\eta \end{bmatrix}, \quad \widehat{C}(\lambda) := \begin{bmatrix} C_{01} \\ C_{02} \\ \vdots \\ C_{0\eta} \\ C_{0(\eta+1)} \end{bmatrix} - \begin{bmatrix} C_{11} \\ C_{12} \\ \vdots \\ C_{1\eta} \\ C_{1(\eta+1)} \end{bmatrix} \lambda,$$

where all blocks have dimension $m \times \ell$, then we need to solve the following system of equations

$$\begin{aligned} &\begin{bmatrix} E_1 & F_1 & E_2 & \dots & F_\eta & E_{\eta+1} \end{bmatrix} (I_{(2\eta+1)\ell} + N) \\ = &\begin{bmatrix} C_{11} & C_{01} & C_{12} & \dots & C_{0\eta} & C_{1(\eta+1)} \end{bmatrix}, \end{aligned}$$

where

$$I_{(2\eta+1)\ell} + N := \begin{bmatrix} I_\ell & \widehat{A} & & & & \\ & I_\ell & I_\ell & & & \\ & & I_\ell & \widehat{A} & & \\ & & & \ddots & \ddots & \\ & & & & I_\ell & I_\ell \\ & & & & & I_\ell \end{bmatrix},$$

and $\widetilde{C} := C_{0(\eta+1)} - E_{\eta+1}\widehat{A}$. Clearly

$$\| \begin{bmatrix} E_1 & F_1 & E_2 & \dots & F_\eta & E_{\eta+1} \end{bmatrix} \|_F = \|(X_{12}, Y_{32})\|_F,$$

$$\| \begin{bmatrix} C_{11} & C_{01} & C_{12} & \dots & C_{0\eta} & C_{1(\eta+1)} \end{bmatrix} \|_F \le \|\Delta_{12}(\lambda)\|_F,$$

and, since the matrix N is nilpotent with $N^{2\eta+1} = 0$,

$$(I_{(2\eta+1)\ell} + N)^{-1} = \sum_{i=0}^{2\eta} (-N)^i.$$

In addition, N has even powers N^{2i} of 2-norm $\|\widehat{A}^i\|_2 \le \|\widehat{A}\|_2^i$, whereas the odd powers N^{2i-1} have 2-norm $\max(\|\widehat{A}^{i-1}\|_2, \|\widehat{A}^i\|_2) \le \max(\|\widehat{A}\|_2^{i-1}, \|\widehat{A}\|_2^i)$. Since both of them can be bounded by $\max(1, \|\widehat{A}\|_2^i)$, it then follows that

$$\begin{aligned} \|(X_{12}, Y_{32})\|_F &\le \|\Delta_{12}(\lambda)\|_F (1 + 2\max(1, \|\widehat{A}\|_2) + \dots + 2\max(1, \|\widehat{A}\|_2^\eta)) \\ &\le [1 + 2\eta \max(1, \|\widehat{A}\|_2^\eta)] \, \|\Delta_{12}(\lambda)\|_F. \end{aligned} \tag{9.42}$$

The discussion for the $\widehat{B}(\lambda)$ block is clearly analogous and will yield the bound

$$\|(X_{23}, Y_{21})\|_F \le [1 + 2\varepsilon \max(1, \|\widehat{A}\|_2^\varepsilon)] \|\Delta_{21}(\lambda)\|_F. \tag{9.43}$$

We can thus summarize this discussion in the following Theorem.

Theorem 9.4.11. *Let the pencil* $\widehat{S}_2(\lambda)$ *have the anti-triangular form given in* (9.40). *Then the updating strict equivalence transformation* $(I-X)\widehat{S}_2(\lambda)(I-Y)$ *detailed in* (9.41) *exists and can be bounded by*

$$\|(X,Y)\|_F \leq [1 + 2\ \max(\eta,\varepsilon)\ \max(1, \|\widehat{A}\|_2^{\max(\eta,\varepsilon)})]\ \|\Delta_2^{new}(\lambda)\|_F.$$

Proof. The bound for $\|(X,Y)\|_F$ follows directly from the identity

$$\|(X,Y)\|_F^2 = \|(X_{12},Y_{32})\|_F^2 + \|(X_{23},Y_{21})\|_F^2,$$

from the inequality $\|\Delta_{12}(\lambda)\|_F^2 + \|\Delta_{21}(\lambda)\|_F^2 \leq \|\Delta_2^{new}(\lambda)\|_F^2$ and from the individual inequalities (9.42) and (9.43). ■

The following first order bound in Corollary 9.4.12 for the norm of the perturbation error $\Delta_3^{new}(\lambda)$ follows from Lemma 9.4.1, Theorem 9.4.11 and Corollaries 9.4.8 and 9.4.10.

Corollary 9.4.12. *Let us define the scalar* $f_3 := \sqrt{2}\,[1+2\max(\eta,\varepsilon)\max(1,\|\widehat{A}\|_2^{\max(\eta,\varepsilon)})]$. *Then*

$$\begin{aligned}\|\Delta_3^{new}(\lambda)\|_F &\leq [1 + f_3\|\widehat{S}_2(\lambda)\|_2]\ \|\Delta_2^{new}(\lambda)\|_F + \mathcal{O}(\delta^2)\\ &\leq [1 + f_3\|S(\lambda)\|_2]\ \|\Delta_2^{new}(\lambda)\|_F + \mathcal{O}(\delta^2).\end{aligned}$$

9.4.4 Putting it all together

In this subsection, we combine the obtained results regarding the strict equivalence transformation that restores in $\widehat{S}(\lambda)$ of (9.21) the special structure of the unperturbed block Kronecker linearization $S(\lambda)$ defined in (9.2), in such a way that the eigenstructure of $\widehat{S}(\lambda)$ can be linked to that of a particular rational matrix $\widetilde{R}(\lambda)$ as in (9.23). The final goal is to bound the norms of the differences between the quadruples $\{\lambda I_\ell - A, B, C, D(\lambda)\}$ and $\{\lambda I_\ell - \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}(\lambda)\}$ that are used for representing the unperturbed rational matrix $R(\lambda)$ and the perturbed one $\widetilde{R}(\lambda)$, respectively.

Recall that we were given the pencil $S(\lambda)$ of which we want to compute the eigenstructure, since it gives the one of the rational matrix $R(\lambda)$ in (9.20). Instead, our backward stable algorithm applied to $S(\lambda)$ computes the exact eigenstructure of a slightly perturbed pencil $\widehat{S}(\lambda)$ with additive error $\Delta_S(\lambda)$ which is induced by the eigenstructure algorithm and is bounded as :

$$\|\Delta_S(\lambda)\|_F \leq c(\ell, m\eta, n\varepsilon)\cdot\epsilon_M\cdot\|S(\lambda)\|_F,$$

where ϵ_M is the machine precision of the used computer, and $c(\ell, m\eta, n\varepsilon)$ is a moderate function depending only on the size of the matrix pencil. We then constructed in three steps a new modified block Kronecker linearization

$$\widetilde{S}(\lambda) := (I-X)\widehat{S}(\lambda)(I-Y) := (I-X_3)(I-X_2)(I-X_1)\widehat{S}(\lambda)(I-Y_1)(I-Y_2)(I-Y_3) \tag{9.44}$$

as in (9.22), strictly equivalent to $\widehat{S}(\lambda)$, where both $\|X\|_F$ and $\|Y\|_F$ are also of the order of the machine precision times some factors and such that the corresponding rational matrix $\widetilde{R}(\lambda)$ (9.23) has a similar representation as $R(\lambda)$. Since $\widehat{S}(\lambda)$ and $\widetilde{S}(\lambda)$ are strictly equivalent pencils, they have *exactly* the same eigenstructure, which implies that we have computed the exact eigenstructure of the nearby rational matrix $\widetilde{R}(\lambda)$.

For convenience, the blocks of $\widetilde{S}(\lambda)$ will be expressed in the sequel as $\widetilde{M}(\lambda) := M(\lambda) + \Delta M(\lambda)$, $\widetilde{A} := A + \Delta A$, $\widetilde{B} := B + \Delta B$ and $\widetilde{C} := C + \Delta C$. In the previous subsections, we rewrote $\widetilde{S}(\lambda)$ as an additive perturbation

$$\widetilde{S}(\lambda) = S(\lambda) + \Delta_3^{new}(\lambda)$$

and derived a first order bound for the norm of the error pencil $\Delta_3^{new}(\lambda)$ in Corollaries 9.4.8, 9.4.10 and 9.4.12 :

$$\|\Delta_3^{new}(\lambda)\|_F \leq (1 + f_1\|S(\lambda)\|_2)(1 + f_2\|S(\lambda)\|_2)(1 + f_3\|S(\lambda)\|_2)\|\Delta_S(\lambda)\|_F + \mathcal{O}(\delta^2). \tag{9.45}$$

This implies, in particular, that if $\|\Delta_S(\lambda)\|_F$ is sufficiently small, then the norms of the perturbations ΔA, ΔB and ΔC are sufficiently small to guarantee that $\widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B}$ is a minimal state-space realization, as announced. Then, according to [6], $\widetilde{S}(\lambda)$ is indeed a strong linearization of the rational matrix $\widetilde{R}(\lambda)$ in (9.23). Moreover, (9.45) also implies that if $\|\Delta_S(\lambda)\|_F$ is sufficiently small, then $\widetilde{D}(\lambda) := \sum_{i=0}^{d}(D_i + \Delta D_i)\lambda^i$ in (9.23) is a polynomial matrix with the same degree $d = \eta + \varepsilon + 1$ as the polynomial part $D(\lambda)$ of $R(\lambda)$ (recall that we are assuming that d is the degree of $D(\lambda)$ or, equivalently, that $D_d \neq 0$).

Notice that $\widetilde{R}(\lambda)$ in (9.23) is the transfer function of the following perturbed polynomial system matrix

$$P(\lambda) + \Delta P(\lambda) := \begin{bmatrix} \lambda I_\ell - A & -B \\ C & D(\lambda) \end{bmatrix} + \begin{bmatrix} -\Delta A & -\Delta B \\ \Delta C & \sum_{i=0}^{d} \Delta D_i \lambda^i \end{bmatrix}, \tag{9.46}$$

where $P(\lambda)$ is a polynomial system matrix of the original rational matrix $R(\lambda)$. Recall that $\|R(\lambda)\|_F$ is defined in (9.3) as $\|P(\lambda)\|_F$. This motivates us to define the norm of the perturbation of $R(\lambda)$ as

$$\|\Delta R(\lambda)\|_F := \|\Delta P(\lambda)\|_F = \sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}.$$

After this discussion, we present our main perturbation results in Theorems 9.4.13 and 9.4.15. The first one focuses on block Kronecker linearizations and the second one on the corresponding rational matrices.

Theorem 9.4.13. *Let $R(\lambda)$ be the $m \times n$ rational matrix in* (9.1) *and let $S(\lambda)$ be a block Kronecker linearization of $R(\lambda)$ as in* (9.2). *Let us define $\alpha := 1 + 2\varepsilon \max(1, \|A\|_2^{\varepsilon})$, $\beta := 1 + 2\eta \max(1, \|A\|_2^{\eta})$, $\gamma := \frac{\varepsilon+\eta}{2\sqrt{2}}$ and $s := \max(\alpha, \beta, \gamma) + \gamma(\beta\|B\|_2 + \alpha\|C\|_2)$. Assume that $\max(\varepsilon, \eta) > 0$ and consider the functions dependent on the initial data*

$$f_1 := f_1(\varepsilon, \eta, \|A\|_2, \|B\|_2, \|C\|_2) := \frac{4\sqrt{2}s}{2 - \sqrt{3}},$$
$$f_2 := f_2(\varepsilon, \eta) := \frac{\sqrt{2}\,(4\max(\varepsilon, \eta) - 1)}{3},$$
$$f_3 := f_3(\varepsilon, \eta, \|A\|_2) := \sqrt{2}\,[1 + 2\max(\eta, \varepsilon)\max(1, \|A\|_2^{\max(\eta,\varepsilon)})].$$

Let $\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda)$ be a perturbed pencil as in (9.21). *If $\|\Delta_S(\lambda)\|_F$ is sufficiently small, then $\widehat{S}(\lambda)$ is strictly equivalent to a block Kronecker linearization $\widetilde{S}(\lambda)$ as in* (9.22) *with the same parameters ε and η as $S(\lambda)$, i.e., the transformation* (9.44) *exists. Moreover, $\widetilde{S}(\lambda) = S(\lambda) + \Delta_3^{new}(\lambda)$ with*

$$\|\Delta_3^{new}(\lambda)\|_F \leq (1 + f_1\|S(\lambda)\|_2)(1 + f_2\|S(\lambda)\|_2)(1 + f_3\|S(\lambda)\|_2)\|\Delta_S(\lambda)\|_F + \mathcal{O}(\delta^2), \tag{9.47}$$

where $\delta := \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F}$.

Proof. This follows directly from (9.45), except that we have replaced the 2-norm of $\widehat{A}$ in f_3 in Corollary 9.4.12 by that of A, because the difference can be absorbed in the $\mathcal{O}(\delta^2)$ term. ∎

Theorem 9.4.13 does not provide directly bounds on the norms of the differences between the quadruples representing the rational matrices $R(\lambda)$ and $\widetilde{R}(\lambda)$ corresponding to the block Kronecker linearizations $S(\lambda)$ and $\widetilde{S}(\lambda)$. The reason is that the polynomial parts $D(\lambda) = (\Lambda_\eta(\lambda) \otimes I_m)^T M(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n)$ and $\widetilde{D}(\lambda) = (\Lambda_\eta(\lambda) \otimes I_m)^T \widetilde{M}(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n)$ of $R(\lambda)$ and $\widetilde{R}(\lambda)$ are not directly visible in $S(\lambda)$ and $\widetilde{S}(\lambda)$. For this reason, we will need Lemma 9.4.14, that follows from [26, Lemma 2.15, Theorem 4.4 and Lemma 5.23(b)].

Lemma 9.4.14. *Let $M(\lambda)$ be a $m(\eta+1) \times n(\varepsilon+1)$ pencil and let $\Lambda_k(\lambda) := \begin{bmatrix} \lambda^k & \cdots & \lambda & 1 \end{bmatrix}^T$. If we define the polynomial matrix $Q(\lambda)$ as*

$$Q(\lambda) := (\Lambda_\eta(\lambda) \otimes I_m)^T\, M(\lambda)\, (\Lambda_\varepsilon(\lambda) \otimes I_n), \tag{9.48}$$

then we can bound its norm as follows

$$\|Q(\lambda)\|_F \leq \sqrt{2\min(\varepsilon+1, \eta+1)}\, \|M(\lambda)\|_F.$$

Moreover, for every polynomial matrix $Q(\lambda)$ of degree at molerstewart1973t $d = \varepsilon + \eta + 1$, there exist infinitely many pencils $M(\lambda)$ satisfying (9.48). *For each of these pencils $\|M(\lambda)\|_F \geq \|Q(\lambda)\|_F/\sqrt{2d}$ and there exist pencils such that $\|Q(\lambda)\|_F = \|M(\lambda)\|_F$.*

As commented in [26], Fiedler and proper generalized Fiedler pencils (modulo permutations) of a polynomial matrix $Q(\lambda)$ satisfy $\|Q(\lambda)\|_F = \|M(\lambda)\|_F$ in Lemma 9.4.14. On the other hand, it might be worth to remind that there exist pencils $M(\lambda)$ satisfying (9.48) with norm arbitrarily larger than the norm of $Q(\lambda)$.

We are finally in the position of proving the main perturbation result of this chapter.

Theorem 9.4.15. *Let $R(\lambda) = C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i\lambda^i$ be an $m \times n$ rational matrix, where $C(\lambda I_\ell - A)^{-1}B$ is a minimal state-space realization of the strictly proper part of $R(\lambda)$, let $S(\lambda)$ be a block Kronecker linearization of $R(\lambda)$ as in* (9.2) *with* $\max(\varepsilon, \eta) > 0$, *and let f_1, f_2, f_3 be the functions defined in Theorem 9.4.13. Let $\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda)$ be a perturbed pencil as in* (9.21). *If $\|\Delta_S(\lambda)\|_F$ is sufficiently small, then $\widehat{S}(\lambda)$ is strictly equivalent to a block Kronecker linearization $\widetilde{S}(\lambda)$ as in* (9.22), *with the same parameters ε and η as $S(\lambda)$, of a rational matrix*

$$\widetilde{R}(\lambda) = \widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B} + \sum_{i=0}^{d} \widetilde{D}_i\lambda^i,$$

where $\widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B}$ is a minimal state-space realization of the strictly proper part of $\widetilde{R}(\lambda)$. Moreover, if $\widetilde{A} := A + \Delta A$, $\widetilde{B} := B + \Delta B$, $\widetilde{C} := C + \Delta C$ and $\widetilde{D}_i := D_i + \Delta D_i$, $i = 0, 1, \ldots, d$, then

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \leq K_{S,R} \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} + \mathcal{O}(\delta^2), \quad (9.49)$$

where

$$K_{S,R} := \sqrt{2\min(\varepsilon + 1, \eta + 1)}\,(1 + f_1\|S(\lambda)\|_2)(1 + f_2\|S(\lambda)\|_2)(1 + f_3\|S(\lambda)\|_2)\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$$

and $\delta = \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F}$.

Proof. Since $\widetilde{S}(\lambda)$ and $S(\lambda)$ have the same structure according to Theorem 9.4.13,

$$\Delta_3^{new}(\lambda) = \widetilde{S}(\lambda) - S(\lambda) = \begin{bmatrix} \widetilde{M}(\lambda) - M(\lambda) & \widehat{K}_2^T(\widetilde{C} - C) & 0 \\ (\widetilde{B} - B)\widehat{K}_1 & \widetilde{A} - A & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\|\Delta_3^{new}(\lambda)\|_F = \sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \|\widetilde{M}(\lambda) - M(\lambda)\|_F^2}$. Next, we combine this expression of $\|\Delta_3^{new}(\lambda)\|_F$ with $\sum_{i=0}^{d} D_i\lambda^i = (\Lambda_\eta(\lambda)\otimes I_m)^T M(\lambda)(\Lambda_\epsilon(\lambda)\otimes I_n)$, $\sum_{i=0}^{d} \widetilde{D}_i\lambda^i = (\Lambda_\eta(\lambda) \otimes I_m)^T \widetilde{M}(\lambda)(\Lambda_\epsilon(\lambda) \otimes I_n)$ and Lemma 9.4.14, and we get

$$\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^{d} \|\Delta D_i\|_F^2} \leq \sqrt{2\min(\varepsilon + 1, \eta + 1)}\,\|\Delta_3^{new}(\lambda)\|_F\,.$$

The rest of the proof follows from (9.47). ■

The strength of the new structured backward error analysis that we present in this chapter for the computation of the eigenstructure of a rational matrix $R(\lambda)$ by applying a backward stable generalized eigenvalue algorithm to a block Kronecker linearization $S(\lambda)$ of $R(\lambda)$ is that we can interpret the computed eigenstructure as the exact eigenstructure for a slightly perturbed rational matrix $\widetilde{R}(\lambda)$ corresponding to the nearby quadruple $\{\lambda I_\ell - \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}(\lambda)\}$, and that we have a bound on the error because we have a specific coordinate system in which we can describe both the original rational matrix $R(\lambda)$ and its perturbed version $\widetilde{R}(\lambda)$, namely by the quadruples $\{\lambda I_\ell - A, B, C, D(\lambda)\}$ and $\{\lambda I_\ell - \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}(\lambda)\}$. It still remains to analyze under which conditions this bound is satisfactory. This is the purpose of the next section.

9.5 Sufficient conditions for structural backward stability

The goal of this section is to establish sufficient conditions on $R(\lambda)$ and $S(\lambda)$ that guarantee that $K_{S,R}$ in (9.49) is moderate and, thus, that guarantee structural backward stability. We advance that these conditions are the following

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1 \quad \text{and} \quad \|M(\lambda)\|_F \approx \|D(\lambda)\|_F, \tag{9.50}$$

where the notation introduced in the previous section is used. Observe that the first condition is a condition on $R(\lambda)$ while the second one is on $S(\lambda)$. According to Lemma 9.4.14, the second condition can be satisfy simply by choosing an adequate block Kronecker linearization $S(\lambda)$. In addition, we will see that the conditions (9.50) are essentially necessary for $K_{S,R}$ to be moderate, though this does not mean that they are necessary for structural backward stability since (9.49) is an upper bound. For the sake of clarity, the discussion in this section focuses on identifying the key ingredients for structural backward stability instead of on providing precise bounds. There exist, obviously, rational matrices which do not satisfy the first condition in (9.50). We will discuss in Section 9.7 how to proceed in such cases.

In the first place observe that each of the essential four factors of $K_{S,R}$, that is, $(1 + f_1\|S(\lambda)\|_2)$, $(1 + f_2\|S(\lambda)\|_2)$, $(1 + f_3\|S(\lambda)\|_2)$ and $\frac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F}$, is larger than 1. This is obvious for the first three factors. For the fourth factor, it follows from the equalities

$$\begin{aligned} &\|S(\lambda)\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \|M(\lambda)\|_F^2 + \ell + 2(m\eta + n\varepsilon) \quad \text{and} \\ &\|R(\lambda)\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \|D(\lambda)\|_F^2 + \ell\,. \end{aligned} \tag{9.51}$$

To find upper bounds for the three factors $(1 + f_1\|S(\lambda)\|_2)$, $(1 + f_2\|S(\lambda)\|_2)$, $(1 + f_3\|S(\lambda)\|_2)$ of $K_{S,R}$ requires to upper bound each f_i and $\|S(\lambda)\|_2$. For this purpose, we consider Lemmas 9.5.1 and 9.5.2. Lemma 9.5.1 provides a bound on the function f_1 that allows us to identify its molerstewart1973t relevant dependencies. Moreover, Lemma 9.5.1 emphasizes the key role of $t := \max(\eta, \varepsilon)$ in our perturbation analysis. Lemma 9.5.2 bounds $\|S(\lambda)\|_2$.

Lemma 9.5.1. *Let us define* $M_a := \max(1, \|A\|_2)$*,* $M_b := \max(\|B\|_2, \|C\|_2)$ *and* $t := \max(\eta, \varepsilon) > 0$ *and consider the functions* f_1*,* f_2 *and* f_3 *in Theorem 9.4.13. Then*

$$1 \leq f_1 \leq 22(1+2\,tM_a^t)(1+\sqrt{2}\,tM_b), \;\; 1 \leq f_2 = \frac{\sqrt{2}}{3}(4t-1), \;\; 1 \leq f_3 = \sqrt{2}\,(1+2\,tM_a^t).$$

Proof. It follows by taking into account the inequalities $\gamma \leq \frac{t}{\sqrt{2}}$ and $s \leq (1 + 2tM_a^t)(1 + \sqrt{2}\,t\,M_b)$. ∎

Lemma 9.5.2. *Let* $S(\lambda)$ *be the block Kronecker linearization* (9.2)*. Then*

$$\max(1, \|A\|_2, \|B\|_2, \|C\|_2, \|M(\lambda)\|_2) \leq \|S(\lambda)\|_2$$

and

$$\|S(\lambda)\|_2 \leq \sqrt{2} + \| \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C \\ B\widehat{K}_1 & A \end{bmatrix} \|_2 \leq \sqrt{2} + \sqrt{\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \|M(\lambda)\|_F^2}.$$

Proof. The first inequality follows from the definition of the 2-norm of a pencil given in the introduction and the fact that the 2-norm of a matrix is larger than or equal to the 2-norm of any of its submatrices. The second inequality follows from applying the triangular inequality to

$$S(\lambda) = \begin{bmatrix} M(\lambda) & \widehat{K}_2^T C & 0 \\ B\widehat{K}_1 & A & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & K_2^T(\lambda) \\ 0 & -\lambda I_\ell & 0 \\ K_1(\lambda) & 0 & 0 \end{bmatrix}.$$

Note that the 2-norm of a pencil as defined in the introduction is indeed a norm and, so, the triangular inequality can be applied. ∎

We remark that Lemmas 9.5.1 and 9.5.2 imply that the conditions (9.50) are essentially necessary for $K_{S,R}$ to be moderate. This can be seen as follows. First, from Lemma 9.4.14, we have $\|M(\lambda)\|_F \geq \|D(\lambda)\|_F / \sqrt{2(\varepsilon + \eta + 1)}$. Thus, $\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \gg 1$ implies $\|S(\lambda)\|_2 \gg 1$, which in turns implies $K_{S,R} \gg 1$, since $f_i \geq 1$ for $i = 1, 2, 3$. Moreover, if $\|M(\lambda)\|_F \gg \|D(\lambda)\|_F$, then $\dfrac{\|S(\lambda)\|_F}{\|R(\lambda)\|_F} \gg 1$ may happen, according to (9.51), and $K_{S,R} \gg 1$ in that situation. We emphasize that the condition $\|M(\lambda)\|_F \approx \|D(\lambda)\|_F$ was also used in the analysis in [26, Corollary 5.24].

Next, we prove the announced result that conditions (9.50) are sufficient for $K_{S,R}$ to be moderate and, thus, for structural backward stability.

Corollary 9.5.3. *Under the hypotheses and with the notation of Theorem 9.4.15, assume, in addition, that* (9.50) *holds and let* $t := \max(\eta, \varepsilon) > 0$. *Then,*

$$K_{S,R} \leq g\, t^q \sqrt{m+n}\,,$$

where $q = 5$, *if* $\eta > 0$ *and* $\varepsilon > 0$, $q = 9/2$, *if* $\eta = 0$ *or* $\varepsilon = 0$, *and* g *is a moderate number (a constant that does not depend on* $\eta, \varepsilon, m, n, \ell$*). Moreover*

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \sum_{i=0}^d \|\Delta D_i\|_F^2}}{\|R(\lambda)\|_F} \leq g\, t^q \sqrt{m+n}\, \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} + \mathcal{O}(\delta^2)\,.$$

Proof. Note that (9.50) and Lemmas 9.5.1 and 9.5.2 imply $\|S(\lambda)\|_2 \lesssim 2 + \sqrt{2}$, $f_1 \leq g_1 t^2$, $f_2 \leq g_2 t$, and $f_3 \leq g_3 t$, with g_1, g_2, g_3 moderate numbers. Moreover, from (9.51), (9.50) and $\|R(\lambda)\|_F \geq 1$, we get that $\|S(\lambda)\|_F^2 \approx \|R(\lambda)\|_F^2 + 2(m\eta + n\varepsilon)$ and

$$\|S(\lambda)\|_F^2 \leq (1 + 2(m\eta + n\varepsilon))\, \|R(\lambda)\|_F^2 \leq 3\,(m+n)\, t\, \|R(\lambda)\|_F^2.$$

It only remains to analyze the factor $\sqrt{2\min(\varepsilon+1, \eta+1)}$ of $K_{S,R}$, which is less than or equal to $\sqrt{2(t+1)}$, if $\eta > 0$ and $\varepsilon > 0$, or equal to $\sqrt{2}$, if $\eta = 0$ or $\varepsilon = 0$. Combining all these bounds with the fact that $t \geq 1$, the result follows as a corollary of Theorem 9.4.15. ■

Remark 9.5.4. Observe that (9.50) allow $\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \ll 1$. However, since the rational matrix $R(\lambda)$ in (9.1) can be multiplied by a nonzero number without affecting at all its eigenstructure, it is natural and convenient to use as sufficient conditions

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) = 1 \quad \text{and} \quad \|M(\lambda)\|_F \approx \|D(\lambda)\|_F. \tag{9.52}$$

Such conditions would have appeared as sufficient in the analysis if we had defined the norm of $R(\lambda)$ as

$$|||R(\lambda)|||_F := \sqrt{\|A\|_F^2 + \|B\|_F^2 + \|C\|_F^2 + \sum_{i=0}^d \|D_i\|_F^2}, \tag{9.53}$$

instead as in (9.3) (observe that we have removed the ℓ summand), depending only on the free parameters of the representation of $R(\lambda)$ in (9.1). We have chosen to use (9.3) because, first, it identifies the informal "norm" of $R(\lambda)$ with the formal norm of the polynomial system matrix $P(\lambda)$ and, second, it corresponds to the particular case $E = I_\ell$ of the more general representation $R(\lambda) = C(\lambda E - A)^{-1}B + D(\lambda)$, with E nonsingular, when taking as norm the one of the corresponding polynomial system matrix. Under the conditions (9.52), it is essentially equivalent to use (9.3) or (9.53) as "norm" of $R(\lambda)$. The use of representations $R(\lambda) = C(\lambda E - A)^{-1}B + D(\lambda)$ for rational matrices is of interest in certain applications and the block Kronecker linearizations in this case are obtained just by replacing $A - \lambda I_\ell$ by $A - \lambda E$ in (9.2). We will consider the analysis of this general case in the future.

9.6 Restoring the structure when the polynomial part of the rational matrix is linear

In this section, we consider the particular case of having a rational matrix with linear polynomial part. That is, the case of having a rational matrix that can be written in the form

$$R(\lambda) = C(\lambda I_\ell - A)^{-1}B + M(\lambda),$$

where $C(\lambda I_\ell - A)^{-1}B$ is a minimal state-space realization and $M(\lambda)$ is a matrix pencil. Then $R(\lambda)$ can be strongly linearized using the following linear polynomial system matrix

$$S(\lambda) := \begin{bmatrix} M(\lambda) & C \\ B & A - \lambda I_\ell \end{bmatrix}. \tag{9.54}$$

Notice that, in this case, the linearization does not have the block anti-triangular structure as the block Kronecker linearization in (9.2) since $K_1(\lambda)$ and $K_2(\lambda)$ are empty matrices. The strong linearization (9.54) can be seen as the limit case of (9.2) when $\varepsilon = \eta = 0$.

If we compute the eigenstructure of $S(\lambda)$, the backward stability of the staircase algorithm [83] and the QZ algorithm [71] guarantees that we computed the exact eigenstructure of a slightly perturbed pencil

$$\widehat{S}(\lambda) := S(\lambda) + \Delta_S(\lambda), \quad \Delta_S(\lambda) := \begin{bmatrix} \Delta_{11}(\lambda) & \Delta_{12}(\lambda) \\ \Delta_{21}(\lambda) & \Delta_{22}(\lambda) \end{bmatrix}. \tag{9.55}$$

The structure of (9.54) is lost in (9.55) since the off-diagonal blocks of $\widehat{S}(\lambda)$ are not constant matrices and the identity block I_ℓ is not preserved by the perturbation.

Notice that restoring in $\widehat{S}(\lambda)$ the original structure of $S(\lambda)$ is much simpler than in previous sections, as we do not have to restore any anti-triangular zero block nor the minimal bases $K_1(\lambda)$ and $K_2(\lambda)$ in (9.19). We only have to take care of restoring the identity matrix I_ℓ and the constant matrices B and C to obtain in two steps a new strictly equivalent linear polynomial system matrix

$$\widetilde{S}(\lambda) := (I - X)\widehat{S}(\lambda)(I - Y) := (I - X_2)(I - X_1)\widehat{S}(\lambda)(I - Y_1)(I - Y_2) \tag{9.56}$$

of the form

$$\widetilde{S}(\lambda) := \begin{bmatrix} \widetilde{M}(\lambda) & \widetilde{C} \\ \widetilde{B} & \widetilde{A} - \lambda I_\ell \end{bmatrix}, \tag{9.57}$$

where $\widetilde{M}(\lambda) := M(\lambda) + \Delta M(\lambda)$, $\widetilde{A} := A + \Delta A$, $\widetilde{B} := B + \Delta B$ and $\widetilde{C} := C + \Delta C$. For that, we consider the discussion in Subsection 9.4.2, for restoring I_ℓ; and a simplified version of the discussion in Subsection 9.4.3, for restoring the constant matrices B and C. In particular, from the bound in (9.39) and a counterpart of Theorem 9.4.11 we get the following result.

Theorem 9.6.1. *Let $S(\lambda)$ be a minimal linear system matrix as in* (9.54)*. The transformation (X, Y) in* (9.56) *exists and we can bound the corresponding perturbation $\widetilde{S}(\lambda) - S(\lambda)$ as follows :*

$$\|\widetilde{S}(\lambda) - S(\lambda)\|_F \leq (1 + \sqrt{2}\|S(\lambda)\|_2)^2 \, \|\Delta_S(\lambda)\|_F + \mathcal{O}(\delta^2). \tag{9.58}$$

In addition, if $\|\widetilde{S}(\lambda) - S(\lambda)\|_F$ is sufficiently small, then the perturbed pencil $\widetilde{S}(\lambda)$ is a minimal linear system matrix of the rational matrix $\widetilde{R}(\lambda) = \widetilde{C}(\lambda I_\ell - \widetilde{A})^{-1}\widetilde{B} + \widetilde{M}(\lambda)$ and

$$\frac{\sqrt{\|\Delta A\|_F^2 + \|\Delta B\|_F^2 + \|\Delta C\|_F^2 + \|\Delta M(\lambda)\|_F^2}}{\|R(\lambda)\|_F} \leq (1+\sqrt{2}\|S(\lambda)\|_2)^2 \, \frac{\|\Delta_S(\lambda)\|_F}{\|S(\lambda)\|_F} + \mathcal{O}(\delta^2),$$

where $\delta = \|\Delta_S(\lambda)\|_F / \|S(\lambda)\|_F$.

The simplicity of the bound in Theorem 9.6.1 is also a consequence of $\|S(\lambda)\|_F = \|R(\lambda)\|_F$.

9.7 Scaling for obtaining structural backward stability

Once a block Kronecker linearization $S(\lambda)$ in (9.2) of $R(\lambda)$ in (9.1) satisfying

$$\|M(\lambda)\|_F \approx \|D(\lambda)\|_F$$

is chosen and the staircase or the QZ algorithm is applied to $S(\lambda)$, structural backward stability is guaranteed for the computed eigenstructure if the first condition in (9.50) holds. However, there exist rational matrices which do not satisfy

$$\max(\|A\|_F, \|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$$

and, therefore, the computation of their eigenstructure via a block Kronecker linearization might not be structurally backward stable. In this section, we study how to proceed in these cases.

Multiplication by a constant. Observe that the eigenstructure of the rational matrix $R(\lambda)$ does not change at all if it is multiplied by a positive real constant d_R. Choosing appropriately d_R, we get easily a rational matrix such that $\max(\|B\|_F, \|C\|_F, \|D(\lambda)\|_F) \leq 1$. Even more, if d_R is an integer power of 2, this multiplication can be performed without introducing any rounding error.

Diagonal scaling. The above explanation indicates that the crucial point is how to deal with rational matrices with $\|A\|_F > 1$. For this, note that when representing a rational matrix $R(\lambda)$ by a realization quadruple $\{\lambda I_\ell - A, B, C, D(\lambda)\}$, where $D(\lambda)$ is polynomial,

$$R(\lambda) := C(\lambda I_\ell - A)^{-1}B + \sum_{i=0}^{d} D_i \lambda^i,$$

one can change the coordinate system of the state-space realization $\{A, B, C\}$ of the strictly proper part of $R(\lambda)$ by a diagonal similarity scaling

$$T := \operatorname{diag}(d_1, \ldots, d_\ell) \quad \text{with} \quad d_i > 0,$$

without changing $R(\lambda)$ since

$$C(\lambda I_\ell - A)^{-1}B = CT(\lambda I_\ell - T^{-1}AT)^{-1}T^{-1}B.$$

Thus, before multiplying $R(\lambda)$ by d_R, we can choose T to balance A, i.e., to minimize its Frobenius norm under all diagonal similarities by making the 2-norms of the rows and columns of $T^{-1}AT$ become equal [74]. Moreover, at the same time, the Frobenius norms of $T^{-1}B$ and CT can be made equal by considering a positive scalar factor multiplying T. Observe, in addition, that if the entries of T are integer powers of 2, this process does not introduce rounding errors, though, in this case, the norm of $T^{-1}AT$ is only approximately minimized. However, the effects of T are limited since $\|T^{-1}AT\|_F \geq \sqrt{|\lambda_1|^2 + \cdots + |\lambda_\ell|^2}$, where $\lambda_1, \ldots, \lambda_\ell$ are the eigenvalues of A, for any invertible T, i.e., diagonal or not. Therefore, other approaches are needed for dealing with all instances of matrices A with large norms. It is important to emphasize at this point that the influence of a large norm matrix A on the bound (9.49) is huge, because it contributes to $\|S(\lambda)\|_2$, but also the factor $\|A\|_2^{\max(\eta,\varepsilon)}$ is present in both f_1 and f_3.

Change of variable. The final solution comes from changing the variable λ to

$$\widehat{\lambda} := d_\lambda \lambda$$

and from combining this with the multiplication by the constant d_R and the diagonal scaling T discussed above. Note that the change of variable transforms the zeros and the poles of $R(\lambda)$ in a very simple way, preserving their partial multiplicities, and that does not change at all its minimal indices [65, 82].

The combination of all these scalings yields a new transfer function

$$\widehat{R}(\widehat{\lambda}) := \widehat{D}(\widehat{\lambda}) + \widehat{C}(\widehat{\lambda} I_\ell - \widehat{A})^{-1}\widehat{B} := d_R R(\widehat{\lambda}/d_\lambda) \tag{9.59}$$

where

$$\widehat{A} := d_\lambda T^{-1}AT, \quad \widehat{B} := \sqrt{d_\lambda d_R}\, T^{-1}B, \quad \widehat{C} := \sqrt{d_\lambda d_R}\, CT \tag{9.60}$$

and

$$\widehat{D}_i := d_R d_\lambda^{-i} D_i, \quad \text{for all} \quad i = 0, 1, \ldots, d. \tag{9.61}$$

Then, we can choose $d_\lambda := \min(1, \|T^{-1}AT\|_F^{-1})$, such that $\widehat{A}$ has norm smaller than or equal to 1. Note that the preliminary balancing will make this step milder, in the sense that d_λ will be closer to 1. Finally, based on (9.59), we summarize the following scaling procedure for obtaining a rational matrix $\widehat{R}(\widehat{\lambda})$ with

$$\max(\|\widehat{A}\|_F, \|\widehat{B}\|_F, \|\widehat{C}\|_F, \|\widehat{D}(\widehat{\lambda})\|_F) = 1$$

from the data $\{A, B, C, D_0, D_1, \ldots, D_d\}$:

Step 1. Compute $T = \operatorname{diag} d_1, \ldots, d_\ell$ to balance A and to make equal the norms of $T^{-1}B$ and CT.

Step 2. Choose $d_\lambda := \min(1, \|T^{-1}AT\|_F^{-1})$.

Step 3. Choose

$$d_R = \frac{1}{\max(\, \|\sqrt{d_\lambda}\, T^{-1}B\|_F^2, \|\sqrt{d_\lambda}\, CT\|_F^2, \sqrt{\sum_{i=0}^{d} \|d_\lambda^{-i} D_i\|_F^2}\,)}.$$

Step 4. Compute $\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}_0, \widehat{D}_1, \ldots, \widehat{D}_d\}$ as in (9.60)-(9.61).

This process can be easily arranged to use scale factors that are all integer powers of two and, thus, can be implemented without any rounding error. Moreover, this scaling can be applied directly to the pencil $S(\lambda)$. More precisely, the pencil

$$\widehat{S}(\widehat{\lambda}) := D_\ell S(\widehat{\lambda}/d_\lambda) D_r,$$

where the left and right diagonal scalings D_ℓ and D_r are given by

$$D_\ell := \operatorname{diag} d_R^{\frac{1}{2}} d_\lambda^{-\eta} I_m, \ldots, d_R^{\frac{1}{2}} d_\lambda^{0} I_m, d_\lambda^{\frac{1}{2}} d_1^{-1}, \ldots, d_\lambda^{\frac{1}{2}} d_\ell^{-1}, d_R^{-\frac{1}{2}} d_\lambda^{\epsilon} I_n, \ldots, d_R^{-\frac{1}{2}} d_\lambda^{1} I_n,$$

$$D_r := \operatorname{diag} d_R^{\frac{1}{2}} d_\lambda^{-\epsilon} I_n, \ldots, d_R^{\frac{1}{2}} d_\lambda^{0} I_n, d_\lambda^{\frac{1}{2}} d_1, \ldots, d_\lambda^{\frac{1}{2}} d_\ell, d_R^{-\frac{1}{2}} d_\lambda^{\eta} I_m, \ldots, d_R^{-\frac{1}{2}} d_\lambda^{1} I_m,$$

is a block Kronecker linearization of the rational matrix $\widehat{R}(\widehat{\lambda})$ in (9.59).

9.8 Numerical experiments

In this section, we describe three experiments illustrating that the potential sources of structural backward instability revealed by the bound (9.49) are indeed observed in practice. More precisely, the experiments will illustrate that if a rational matrix $R(\lambda)$ as in (9.1) does not satisfy the first condition in (9.50), then the computation of the eigenstructure of $R(\lambda)$ by applying the QZ algorithm to a block Kronecker linearization $S(\lambda)$ of $R(\lambda)$ that satisfies $\|M(\lambda)\|_F = \|D(\lambda)\|_F$ is not structurally backward stable. Moreover, the experiments also illustrate that the scaling described in Section 9.7 is effective and leads to structured backward stability for the scaled rational matrices and linearizations.

A difficulty for performing fully reliable numerical experiments in this setting is that to estimate the actual global backward error for the *whole* computed eigenstructure, i.e., the left-hand side of (9.49), is a challenging optimization problem for which we do not know yet a solution. Therefore, we will limit ourselves to computing a lower bound for the backward error based on the “local” backwards errors of each computed zero of the rational matrix, as we explain below. This lower bound might severely underestimate the actual global backward error. Thus, we cannot check from our experiments the sharpness of the bound (9.49), which, on the other hand, was deduced through many potentially overestimating inequalities with the main goal of getting a bound as clear as possible instead of optimizing its sharpness.

For simplicity, we will restrict our numerical experiments to square and regular rational matrices $R(\lambda)$ with a corresponding quadruple $\{A, B, C, D(\lambda)\}$ of moderate dimensions and degree of its polynomial part: $m = n = 2$, $\ell = 5$, $d = 3$. The block Kronecker pencil we choose for our computations is

$$S(\lambda) := \begin{bmatrix} \lambda D_3 + D_2 & 0 & 0 & I_2 \\ 0 & \lambda D_1 + D_0 & C & -\lambda I_2 \\ 0 & B & A - \lambda I_\ell & 0 \\ I_2 & -\lambda I_2 & 0 & 0 \end{bmatrix},$$

which has η and ε equal to 1, size 11×11 and satisfies $\|M(\lambda)\|_F = \|D(\lambda)\|_F$. We also will look at the polynomial system matrix

$$P(\lambda) := \begin{bmatrix} A - \lambda I_\ell & B \\ C & D(\lambda) \end{bmatrix}, \quad D(\lambda) := D_0 + \lambda D_1 + \lambda^2 D_2 + \lambda^3 D_3$$

of $R(\lambda)$ because it allows us to estimate the backward errors of our algorithm as follows. We look for a rational matrix $\widetilde{R}(\lambda)$ corresponding to a quadruple $\{A + \Delta A, B + \Delta B, C + \Delta C, (D + \Delta D)(\lambda)\}$ such that all its finite zeros are exactly all the computed finite eigenvalues obtained by applying the QZ algorithm to $S(\lambda)$ and such that $\|(\, \Delta A, \Delta B, \Delta C, (\Delta D)(\lambda)\,)\|_F$ is as small as possible. As a consequence of the classical results of Rosenbrock [78], this is equivalent to find a perturbed polynomial system matrix $P(\lambda) + \Delta P(\lambda)$ of $\widetilde{R}(\lambda)$, whose finite zeros are the computed eigenvalues λ_i and such that $\|(\, \Delta A, \Delta B, \Delta C, (\Delta D)(\lambda)\,)\|_F$ is as small as possible. Therefore, $\{\Delta A, \Delta B, \Delta C, \Delta D_0, \Delta D_1, \Delta D_2, \Delta D_3\}$ must have the property that *simultaneously*, at each computed eigenvalue λ_i, the matrix

$$P(\lambda_i) + \Delta P(\lambda_i) = P(\lambda_i) + \left[\begin{array}{cc|cc|cc|cc} \Delta A & \Delta B & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta C & \Delta D_0 & 0 & \Delta D_1 & 0 & \Delta D_2 & 0 & \Delta D_3 \end{array}\right] \begin{bmatrix} I_{\ell+m} \\ \lambda_i I_{\ell+m} \\ \lambda_i^2 I_{\ell+m} \\ \lambda_i^3 I_{\ell+m} \end{bmatrix}$$

must be singular. To find the smallest possible Frobenius norm of all possible $\{\Delta A, \Delta B, \Delta C, \Delta D_0, \Delta D_1, \Delta D_2, \Delta D_3\}$ that satisfy this property *for all* computed λ_i is not obvious, however to solve this problem *for only one* computed λ_i is easy. For this purpose, let $\Delta^{(i)}$ be the minimum Frobenius norm matrix that makes $P(\lambda_i)+\Delta^{(i)}$ singular. Note that $\Delta^{(i)}$ can be computed through the singular value decomposition of $P(\lambda_i)$ and that, generically, it is a rank one matrix with Frobenius norm equal to

$\sigma_{\min} P(\lambda_i)$. Then, the linear system

$$\Delta^{(i)} := \begin{bmatrix} \Delta_{11}^{(i)} & \Delta_{12}^{(i)} \\ \Delta_{21}^{(i)} & \Delta_{22}^{(i)} \end{bmatrix} = \left[\begin{array}{cc|cc|cc|cc} \Delta A & \Delta B & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta C & \Delta D_0 & 0 & \Delta D_1 & 0 & \Delta D_2 & 0 & \Delta D_3 \end{array}\right] \begin{bmatrix} I_{\ell+m} \\ \lambda_i I_{\ell+m} \\ \lambda_i^2 I_{\ell+m} \\ \lambda_i^3 I_{\ell+m} \end{bmatrix}$$

for the unknowns $\{\Delta A, \Delta B, \Delta C, \Delta D_0, \Delta D_1, \Delta D_2, \Delta D_3\}$ is consistent and its minimum Frobenius norm solution is given by

$$\Delta A := \Delta_{11}^{(i)}, \;\; \Delta B := \Delta_{12}^{(i)}, \;\; \Delta C := \Delta_{21}^{(i)}, \;\; \Delta D_k := \Delta_{22}^{(i)} \overline{\lambda}_i^k / g(\lambda_i), \;\; k = 0, 1, 2, 3,$$

where $g(\lambda_i) := (1 + |\lambda_i|^2 + |\lambda_i|^4 + |\lambda_i|^6)$, and the Frobenius norm of this 7-tuple of matrices is given by

$$r(P, \lambda_i) := \left\| \begin{bmatrix} \Delta_{11}^{(i)} & \Delta_{12}^{(i)} \\ \Delta_{21}^{(i)} & \Delta_{22}^{(i)} / \sqrt{g(\lambda_i)} \end{bmatrix} \right\|_F .$$

This leads us to use in our experiments

$$r(P) := \max_i r(P, \lambda_i) \tag{9.62}$$

as an estimate for the structured absolute backward error induced by our algorithm, i.e., as an estimate for the numerator of the left-hand side of (9.49). We emphasize that this is a lower bound for the actual global structured backward error, since it corresponds to a rational matrix that has only one of the computed eigenvalues as a finite zero.

Experiment 1. In the first experiment, we investigate the behavior of the structured backward error for rational matrices with matrices A of increasing (large) norms, and with the rest of the matrices in the quadruple $\{A, B, C, D(\lambda)\}$ having norms of order 1. The reason why we pay first particular attention to the norm of A is because according to the bound (9.49) the influence of A should be huge because it contributes to $\|S(\lambda)\|_2$ and also to f_1 and f_3. For this purpose, we generated with the Matlab function `randn`, 7 batchs of samples of 50 random matrix-tuples $\{A, B, C, D_0, D_1, D_2, D_3\}$, and in each batch indexed with i, we multiplied the matrix A by 10^i, with i going from 1 till 7, in each of the 50 runs of each batch. In each batch, we computed the average of the absolute backward error estimators (9.62) for both the original matrix-tuples and the scaled ones after applying the procedure in Section 9.7. In Figure 9.1, we plot the results of these computations: the horizontal axis represents the index i defining each batch and the vertical axis the logarithm of the average absolute backward errors. Ideally, the absolute backward error should be of order $\epsilon_M \|R(\lambda)\|_F$, where ϵ_M is the machine precision, and, so, we also plot this magnitude for the unscaled original data taking in each batch the average of all $\|R(\lambda)\|_F$ (for the scaled data, this magnitude is always of order ϵ_M and is not plotted).

We observe that the absolute backward errors for the unscaled problem grow very strongly with the index i, i.e., with the norm of A, and that computing the zeros of a rational matrix by applying the QZ algorithm to the block Kronecker linearization $S(\lambda)$ is highly structurally backward unstable for large norms of A, as predicted by the bound (9.49). In contrast, when applying the scaling procedure described in Section 9.7, this growth is absent and we get perfect structural backward stability for the scaled rational matrix, as predicted by (9.49).

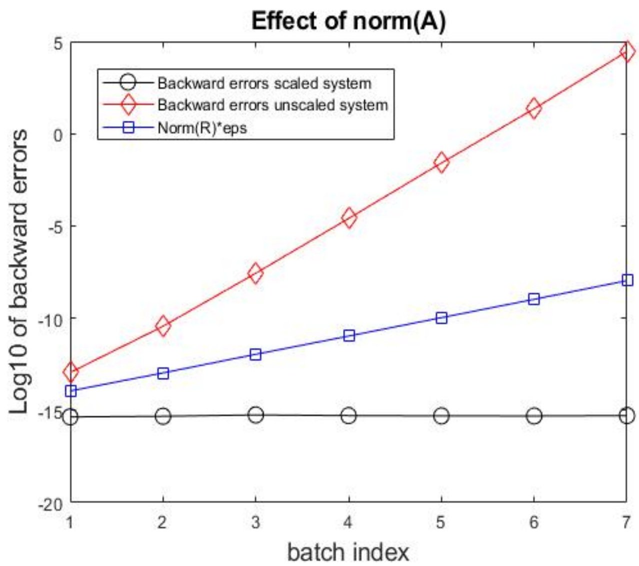

Figure 9.1: Experiment 1: behavior of absolute structured backward errors for increasing values of the norm of A.

Experiment 2. In the second experiment, we investigate the behavior of the structured backward error for rational matrices with matrices A of norms of order 1, and with the rest of the matrices in the quadruple $\{A, B, C, D(\lambda)\}$ having increasing (large) norms. The situation in this experiment is opposite to the one in the first experiment. The matrices are generated following the same pattern of the first experiment except by the fact that once the matrices $\{A, B, C, D_0, D_1, D_2, D_3\}$ are generated with `randn`, B is multiplied by $10^{i/2}$, C by $10^{i/3}$, D_1 by 10^{i}, D_2 by $10^{i/2}$ and D_3 by $10^{i/3}$, for $i = 1, \ldots, 7$. The results are plotted in Figure 9.2 and the conclusions are the same as in the first experiment and are in agreement with our analysis. However, note that the growth of the absolute backward errors of the original unscaled data is much smaller than in the first experiment. This effect is qualitatively expected from the bound (9.49), since f_3 does not depend on the norms of B, C and $D(\lambda)$, but the observed very large quantitative difference is not fully explained by (9.49). Possible reasons of this are that, as we have emphasized before, our backward error estimator is a lower bound that may underestimate severely the actual global backward error and/or that the bound in (9.49) overestimates the actual error.

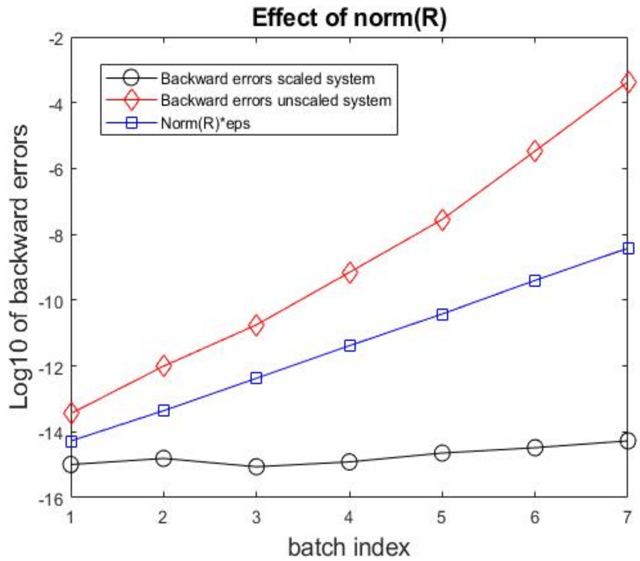

Figure 9.2: Experiment 2: behavior of absolute structured backward errors for increasing values of the norms of B, C and $D(\lambda)$.

Experiment 3. The last experiment we present combines the scalings used in the first and second experiments. That is, once the matrices $\{A, B, C, D_0, D_1, D_2, D_3\}$ are generated with `randn`, A is multiplied by the factor used in Experiment 1 and B, C, D_1, D_2, and D_3 are multiplied by the factors used in Experiment 2. Taking into account that the function f_1 appearing in the bound (9.49) includes a product of the norm of A times the norm of B and a product of the norm of A times the norm of C, we expect backward errors larger than those of Experiment 1. The results are plotted in Figure 9.3. The errors are indeed larger than those in Figure 9.1, but just a bit larger. The possible reasons of this small increment of the errors are the same as in the second experiment.

The main conclusion of this section is that our main *a priori* structured backward error bound (9.49) identifies correctly the sources of instability of computing the eigenstructure of a rational matrix by applying the QZ algorithm to its block Kronecker linearizations and that the scaling proposed in Section 9.7 leads to structural backward stability.

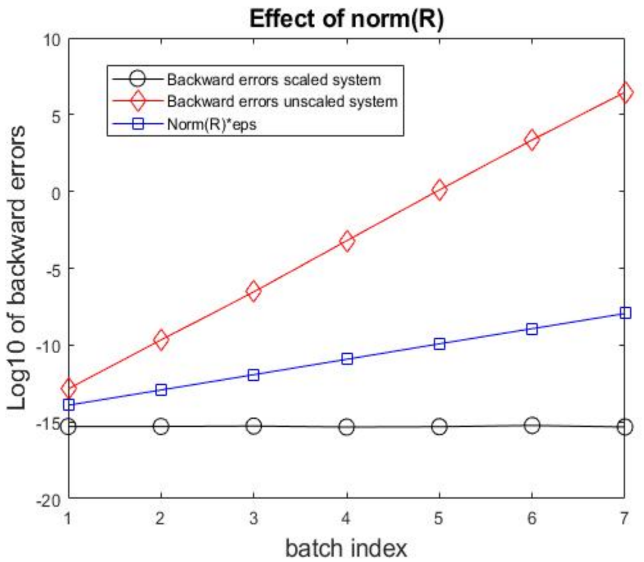

Figure 9.3: Experiment 3: behavior of absolute structured backward errors for increasing values of the norms of A, B, C and $D(\lambda)$.

Chapter 10

Conclusions, publications and open problems

In this chapter, we give some concluding remarks of the results presented in this PhD thesis and summarize the main original contributions. We also list all the papers (published or in development) that include the results obtained in this dissertation, and the conferences where many of them have been presented. Finally, we suggest a collection of related open problems and recommendations for future research directions.

10.1 Conclusions and original results

In this section, we highlight the main original results in this dissertation together with some conclusions of each chapter. In general, it is worth mentioning that as a consequence of the results of this thesis and, in particular, as a consequence of the new definition of local linearization of rational matrices in Chapter 4, most of the linearizations for rational matrices in the literature have been unified in a general framework. In particular, the theory of local linearizations of rational matrices captures all the pencils that have been used (as far as we know) in the literature for solving REPs arising from approximating NLEPs; some of them are analysed in Chapter 5. In addition, the definition of local linearizations includes the definition of linearizations and strong linearizations of arbitrary rational matrices presented in [6], just by considering as set the whole underlying field and including infinity in the strong case. But the conditions for a linearization to be strong in Chapter 4 and those in [6] are different. More precisely, given a minimal linear polynomial system matrix

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)},$$

for the linearization to be strong in [6] it is required A_1 to be invertible (recall Proposition 3.1.1). However, we do not require A_1 to be invertible but

$$\operatorname{rank}\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 & B_1 \end{bmatrix} = n,$$

which is the condition for minimality at infinity introduced in Chapter 4. A particular case of local linearizations in the strong sense are the strongly minimal linearizations in Chapter 7, for which the transfer function matrix is exactly the desired rational matrix. Strongly minimal linearizations include the generalized state-space systems originally used in [92] for linearizing rational matrices. Thus, the new local theory of linearizations for rational matrices is a flexible tool that generalizes and includes most of the previous results available in the literature in this area. Moreover, local linearizations allow us to introduce in Chapters 6 and 8 linearizations of rational matrices that can not be constructed with the previous definitions of linearization. All these results are in part possible due to the new treatment of minimality of polynomial system matrices at infinity.

We first started in **Chapter 3** by considering the definition of strong linearization for rational matrices given in [6], and we proved the key Lemma 3.1.2, which allows us to construct infinitely many strong linearizations for any rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ from any given strong linearization of $R(\lambda)$. Then Lemma 3.1.2 was applied to construct linearizations of square rational matrices $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$, written as the sum of their polynomial and strictly proper parts. Namely,

$$R(\lambda) := D(\lambda) + R_{sp}(\lambda).$$

More precisely, Lemma 3.1.2 has been combined with some of the strong linearizations constructed in [6, Theorem 5.11] for rational matrices and with strong linearizations of its polynomial part $D(\lambda)$ to create new families of strong linearizations of square rational matrices. In particular, we considered the families of strong linearizations of polynomial matrices presented in [36] to linearize the polynomial parts. These are, the ansatz spaces $\mathbb{M}_1(D)$ and $\mathbb{M}_2(D)$, where the corresponding polynomial matrix is expressed in terms of any polynomial basis satisfying a three term recurrence relation. The main results for the construction of the strong linearizations of rational matrices from $\mathbb{M}_1(D)$ and $\mathbb{M}_2(D)$ are Theorems 3.2.9 and 3.3.4, respectively. Such linearizations were called $\mathbb{M}_1$- and $\mathbb{M}_2$-strong linearizations, respectively. The recovery of the eigenvectors of the rational matrix from those of the linearizations in these families has been studied and stated in different results. For $\mathbb{M}_1$-strong linearizations, Theorems 3.4.4 and 3.4.5 state the recovery of right and left eigenvectors associated with finite eigenvalues, respectively, and Theorem 3.4.7 the recovery of right and left eigenvectors associated with infinity. For $\mathbb{M}_2$-strong linearizations, Theorems 3.4.9 and 3.4.10 state the recovery of right and left eigenvectors associated with finite eigenvalues, respectively, and Theorem

3.4.12 the recovery of right and left eigenvectors associated with infinity. Moreover, the preservation of symmetric and Hermitian structures of the rational matrix in the linearizations have been studied in Theorem 3.6.9 and Corollary 3.6.11, for the symmetric case, and in Theorem 3.7.5 and Corollary 3.7.6, for Hermitian structures. Finally, a result on how to construct strong linearizations when the polynomial part is expressed in other polynomial bases is given in Theorem 3.8.1.

We would like to emphasize that the techniques developed in this chapter could be applied to construct other families of strong linearizations of rational matrices $R(\lambda)$ expressed as the sum of its polynomial part $D(\lambda)$ and its strictly proper part $R_{sp}(\lambda)$, given a strong linearization of $D(\lambda)$ in any of the families of strong linearizations of polynomial matrices developed in the last years and a minimal state-space realization of $R_{sp}(\lambda)$. However, although any rational matrix can be expressed as the sum of its polynomial and strictly proper parts, this expression may not be easily available from the applications and/or may not be the best representation in a particular problem. Then linearizations for more general representations are presented in Chapter 6. But in Chapter 6 we used the definition of linearization for rational matrices introduced in Chapter 4, since with other definitions of linearization for rational matrices existing in the literature it is not possible to develop the results in Chapter 6.

In **Chapter 4** a theory of local linearizations of rational matrices has been carefully introduced and analysed. For that, we first developed the extension of Rosenbrock's minimal polynomial system matrices to a local sense. The notions of polynomial system matrices minimal in a set and at infinity are given in Definitions 4.1.1 and 4.1.8, respectively. For minimality at infinity we used the notion of g-reversal of rational matrices introduced in Definition 4.1.7. We studied the pole and zero information that one can obtain from polynomial system matrices minimal in a set and at infinity in Theorems 4.1.5 and 4.1.11, respectively. The study of the recovery of the invariant orders at infinity is given in Theorem 4.1.13. Polynomial system matrices that are minimal in the whole underlying field and also at infinity are called strongly minimal in Definition 4.1.15. Then we present the notions of linearizations of rational matrices in a set and also at infinity in Definitions 4.2.1 and 4.2.9, respectively, together with spectral characterizations in Theorems 4.2.6 and 4.2.14 and the recovery of the invariant orders at infinity in Proposition 4.2.15. When a linearization of a rational matrix satisfy the minimality conditions in the whole underlying field and also at infinity for a particular choice of the $g-$reversal, then we say it is a strong linearization of grade g (Definition 4.2.16). We also define and analyze the very general family of block full rank pencils and linearizations, as templates that cover many of the pencils, available in the literature, that linearize rational matrices in a target set, in the whole field and/or at infinity. In particular, block full rank pencils and linearizations in a set are introduced in Theorems 4.3.5 and 4.4.1, and block full rank pencils and linearizations at infinity in Theorems 4.3.7 and 4.4.10.

In **Chapter 5** we apply the theory of local linearizations in Chapter 4 to a number of pencils that have appeared in the influential papers [47, 60] on solving numerically NLEPs by combining rational approximations and linearizations of the resulting rational matrices. In particular, the pencils introduced in [47] have been studied in Theorems 5.1.1 and 5.1.3. In addition, in [47] trimmed linearizations are considered when the coefficients defining the NLEPs have low rank. These trimmed linearizations are studied in Theorems 5.1.5 and 5.1.7. The pencils introduced in [60] have been studied in Theorem 5.2.6 and their minimality conditions in Theorems 5.2.8 and 5.2.9. In [60] trimmed linearizations exploiting low rank structures are also considered, which we analyzed in Theorem 5.2.13. It has been emphasized throughout the chapter that the theory of local linearizations allows us to view these pencils, and to explain their properties, from rather different perspectives, by using the notions of block full rank pencils and linearizations in Chapter 4.

In **Chapter 6** rational matrices $R(\lambda)$ are expressed in the general form

$$R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda),$$

where $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are polynomial matrices with arbitrary degrees. From these representations we have constructed a family of linear polynomial matrices that, under some minimality conditions, are local linearizations for $R(\lambda)$ in the sense defined in Chapter 4. These linearizations are presented in Theorems 6.1.2 and 6.2.1. Moreover, we have showed in Theorem 6.4.12 how to recover the eigenvectors of $R(\lambda)$, when $R(\lambda)$ is regular, from those of any of the new linearizations. The recovery of right and left minimal bases and minimal indices of $R(\lambda)$, when $R(\lambda)$ is singular, is studied in Theorems 6.4.9 and 6.4.11, respectively. Finally, in Section 6.5 we apply the theory in this chapter to the solution of scalar rational equations.

We emphasize that in contrast to the construction of other families of linearizations for rational matrices, as those introduced in Chapter 3 or in the references [2, 7, 27, 79], the construction of the linearizations in this chapter do not require neither to write the corresponding rational matrix $R(\lambda)$ as the sum of its polynomial part and its strictly proper part nor to express the strictly proper rational part in state-space form, which can introduce errors that were not in the original problem.

In **Chapter 7** we considered realizations $\{A(\lambda), B(\lambda), C(\lambda), D(\lambda)\}$ for a given rational transfer function matrix $R(\lambda) = C(\lambda)A(\lambda)^{-1}B(\lambda) + D(\lambda)$, where the matrices $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are pencils, and where $A(\lambda)$ is assumed to be regular. We showed that if the corresponding linear polynomial system matrix

$$L(\lambda) := \begin{bmatrix} A(\lambda) & -B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \tag{10.1}$$

is strongly minimal then the poles, zeros and left and right null space structures of the rational matrix $R(\lambda)$ can be recovered from the generalized eigenstructures of $A(\lambda)$ and $L(\lambda)$, and $L(\lambda)$ is named as a strongly minimal linearization of $R(\lambda)$

(Definition 7.2.4). In particular, in Proposition 7.2.11 we proved that if $L(\lambda)$ is a strongly minimal linearization of a rational matrix $R(\lambda)$ then the left and right minimal indices of $R(\lambda)$ and those of $L(\lambda)$ are the same. In Subsection 7.2.2 we studied the relation of strongly minimal linearizations with other definitions of strong linearizations for polynomial and rational matrices in the literature. In addition, by Theorems 7.2.6 and 7.2.7, we showed that the strong minimality conditions imply the strong irreducibility conditions in [91]. We stated such result in Proposition 7.2.10. We also showed in Section 7.3 how to obtain such a strongly minimal linear system matrix from a non-minimal one, by applying a reduction procedure that is based on the classical staircase algorithm developed in [83]. The main results in that section are stated in Theorems 7.3.1 and 7.3.3. These results extend those previously obtained in the literature for generalized state space systems and polynomial matrices.

In **Chapter 8** we looked at strongly minimal linear polynomial system matrices for any given rational matrix $R(\lambda)$, preserving the structure whenever we have a specific type of self-conjugate structure on $R(\lambda)$. We showed that there always exist strongly minimal linear polynomial system matrices that have the same self-conjugate structure as the transfer function matrix. For that we first constructed, in Theorem 8.1.2, strongly minimal linearizations for arbitrary polynomial matrices and, in Theorem 8.2.2, for self-conjugate polynomial matrices. Then, in Theorem 8.3.1, we constructed strongly minimal linearizations for arbitrary strictly proper rational matrices and, in Theorem 8.4.2, for self-conjugate strictly proper rational matrices. Finally, in Theorems 8.5.1 and 8.5.2 we stated the results for both arbitrary and structured rational matrices. These results were known for the case of arbitrary proper transfer functions [48], but were extended in this chapter to rational matrices that are not proper. Moreover, the derivation is new and is based on arguments that are very similar for the strictly proper part and the polynomial part of the rational matrix. The proofs are also constructive and lead to efficient algorithms for their construction.

It is worth mentioning that the strong linearizations constructed in Chapter 3 preserving symmetric and Hermitian structures require the degree of the polynomial part of the rational matrix to be odd or that, when it is even, the leading matrix coefficient to be invertible. Both restrictions are inherited from the classical theory of linearizations for structured polynomial matrices and they do not appear by considering strongly minimal linearizations as we showed in this chapter. As far as we know, it is the first time that such a completely restriction free construction of structured linearizations has been achieved.

In **Chapter 9** we developed the first structured backward error analysis for an algorithm that computes the eigenstructure of a rational matrix via linearization. More precisely, the considered algorithm starts from a rational matrix expressed as in (9.1) and computes its eigenstructure by applying a backward stable generalized eigenproblem algorithm to its block Kronecker linearizations described in (9.2). The main result is stated in Theorem 9.4.15, and in Theorem 9.6.1 for the case of rational matrices with linear polynomial part. As a consequence of this analysis, we have identified the simple sufficient conditions (9.50) for global structural backward stability. In the case of rational matrices which do not satisfy these conditions, we have developed in Section 9.7 an scaling procedure that transforms the original rational matrix in another one for which structural backward stability is guaranteed. A number of numerical experiments confirming the predictions of the backward error analysis have been performed and discussed in Section 9.8. The results in Chapter 9 showed that solving numerically REPs via block Kronecker linearizations is a backward stable process in a global sense, under certain conditions involving both the representation of the rational matrix and the choice of the linearization.

10.2 Publications

The original results in Chapter 3 are contained in the following paper:

1 – F. M. Dopico, S. Marcaida, M. C. Quintana, *Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis*, Linear Algebra and its Applications, 570 (2019) 1–45.

The original results in Chapters 4 and 5 are contained in the following two papers:

2 – F. M. Dopico, S. Marcaida, M. C. Quintana, P. Van Dooren, *Local linearizations of rational matrices with application to rational approximations of nonlinear eigenvalue problems*, Linear Algebra and its Applications, 604 (2020) 441 – 475.

3 – F. M. Dopico, S. Marcaida, M. C. Quintana, P. Van Dooren, *Block full rank linearizations of rational matrices*, submitted. Available as arXiv: 2011.00955v1.

The original results in Chapter 6 are contained in the following paper:

4 – J. Pérez, M. C. Quintana, *Linearizations of rational matrices from general representations*, submitted. Available as arXiv:2003.02934v1.

The original results in Chapter 7 are contained in the following paper:

5 – F. M. Dopico, M. C. Quintana, P. Van Dooren, *Linear system matrices of rational transfer functions*, to appear in "Realization and Model Reduction of Dynamical Systems", A Festschrift to honor the 70th birthday of Thanos Antoulas", Springer-Verlag. Available as arXiv:1903.05016v2.

The original results in Chapter 8 are contained in the following paper:

6 – F. M. Dopico, M. C. Quintana, P. Van Dooren, *Strongly minimal self-conjugate linearizations for polynomial and rational matrices*, in preparation.

The original results in Chapter 9 are contained in the following paper:

7 – F. M. Dopico, M. C. Quintana, P. Van Dooren, *Structural backward stability in rational eigenvalue problems solved via block Kronecker linearizations*, submitted. Available as arXiv:2103.16395v1.

10.3 Contributions to conferences

The author of this dissertation has presented many of the original results in this thesis in several conferences. Among them, we emphasize one of the main international conferences in the area of Applied Mathematics: the International Congress on Industrial and Applied Mathematics (ICIAM), and one of the main international conferences in the area of Applied Linear Algebra: the Society for Industrial and Applied Mathematics (SIAM) Conference on Applied Linear Algebra. All the contributions are listed in what follows in reversed chronological order:

(1) May 17 - 21, 2021: SIAM Conference on Applied Linear Algebra (LA21). Virtual conference, originally scheduled in New Orleans, Louisiana, USA. Type of participation: Invited talk in minisymposium "Structured Eigenvalue Problems." Title of the work: Block Full Rank Linearizations of Rational Matrices.

(2) July 15-19, 2019: International Congress on Industrial and Applied Mathematics (ICIAM 19). Universidad de Valencia, Valencia, Spain. Type of participation: Invited talk in minisymposium "Nonlinear and multiparameter eigenvalue problems". Title of the work: Local Linearizations of Rational Matrices and their Applications to Nonlinear Eigenvalue Problems.

(3) May 29-30, 2019: Workshop "Advances in numerical linear algebra: celebrating the centenary of the birth of James H. Wilkinson". University of Manchester, Manchester, United Kingdom. Type of participation: Poster. Title of the work: On the Structure of Linearizations for Rational Approximations of Nonlinear Eigenvalue Problems.

(4) September 19, 2018: Autumn of Eigenvalues Seminar Series - Part II. Instituto de Ciencias Matematicas (ICMAT), Madrid, Spain. Type of participation: Invited talk. Title of the work: Symmetric Strong Linearizations of Symmetric Rational Matrices.

(5) May 30 - June 1, 2018: ALAMA 2018 meeting. Complejo San Juan, Sant Joan d'Alacant, Spain. Type of participation: Contributed talk. Title of the work: Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis.

(6) May 4-8, 2018: SIAM Conference on Applied Linear Algebra (LA18). Hong Kong Baptist University, Hong Kong, China. Type of participation: Invited talk in minisymposium "Polynomial and Rational Matrices". Title of the work: Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis.

(7) September 11-13, 2017: Workshop of Young Researchers in Mathematics. Universidad Complutense de Madrid, Madrid, Spain. Type of participation: Contributed talk. Title of the work: Constructing new classes of Strong Linearizations of Rational Matrices.

10.4 Open problems and future research

In this section, we discuss some open problems and directions for future research related to the results developed in this dissertation. We think that, after the results presented in this thesis and others existing in the literature, the study of the theoretical properties of linearizations of rational matrices is quite developed. However, there are few numerical stability results in this regard, in the spirit of those we presented in Chapter 9 that are limited to the family of "block Kronecker linearizations" in [6]. Therefore, the problems proposed in this section will focus more on addressing numerical properties for rational matrices and their linearizations than on theoretical results. For the case of polynomial matrices, the study of the numerical properties of their linearizations is well understood. The general idea is to extend those results for rational matrices.

Problem 1: Study the numerical properties of the linearizations introduced in this dissertation. In particular, given a linearization of a REP in a set, it is important to study the global backward stability in terms of the structure of the representation of the rational matrix defining the REP when applying a numerical method to compute the eigenvalues of the corresponding linearization and it gets perturbed, as it is done for the family of "block Kronecker linearizations" in Chapter 9. A first interesting problem inside the general Problem 1 is to extend the analysis in Chapter 9 for block Kronecker linearizations when in the representation of the rational matrix $(\lambda I_\ell - A)$ is replaced by $(\lambda E - A)$ with E nonsingular. This might help to avoid the scaling of the variable for getting backward stability in certain cases.

Now we briefly recall the families of linearizations for rational matrices introduced in this thesis:

(1) The strong linearizations of rational matrices in Chapter 3, where the corresponding rational matrix $R(\lambda)$ is written in the form $R(\lambda) = D(\lambda) + C(I_n\lambda - A)^{-1}B$ with polynomial part $D(\lambda)$ expressed in an orthogonal basis and $C(I_n\lambda - A)^{-1}B$ being a minimal state space realization.

(2) The family of block full rank linearizations in Chapter 4. Associated with such linearizations, rational matrices $R(\lambda)$ are written in the form $R(\lambda) = N_2(\lambda)[M(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$ where $M(\lambda)$, $C(\lambda)$, $A(\lambda)$ and $B(\lambda)$ are pencils and $N_1(\lambda)$ and $N_2(\lambda)$ are rational bases.

(3) The linearizations of rational matrices from general representations in Chapter 6, where rational matrices are written as $R(\lambda) = D(\lambda)+C(\lambda)A(\lambda)^{-1}B(\lambda)$, with $D(\lambda)$, $C(\lambda)$, $A(\lambda)$ and $B(\lambda)$ being arbitrary polynomial matrices.

(4) The strongly minimal linearizations in Chapter 8, where rational matrices $R(\lambda)$ are written from their Laurent expansions around the point at infinity. That is, $R(\lambda) = R_d\lambda^d + \ldots + R_1\lambda + R_0 + R_{-1}\lambda^{-1} + R_{-2}\lambda^{-2} + R_{-3}\lambda^{-3} + \ldots$.

We notice that for the families of linearizations in (1), (2) and (3) the matrix coefficients defining the representations of the rational matrix are reflected on the linearizations. Then, in principle, one might use ideas similar to those in Chapter 9 in order to recover the structure of the rational matrix in the corresponding linearization when it gets perturbed. However, the coefficients of the Laurent expansions do no appear in the strongly minimal linearizations in (4). Thus one should take into account, in addition, the numerical transformations done to build the linearizations in order to determine which rational matrix is linearized by the perturbed pencil for studying backward stability. The fact that this construction is based on unitary transformations gives us some hope of solving this difficult problem.

Problem 2: Define different condition numbers of eigenvalues λ of REPs, for different representations of the rational matrix defining the REP, that is their sensitivity to perturbations of the parameters defining the representation. For the case of polynomial matrices, definitions of condition numbers in the literature take into account that the corresponding polynomial matrix can be written in terms of different polynomial bases. For the case of rational matrices, it is important to consider as well different representations of the corresponding rational matrix and different types of perturbations. After the condition numbers are defined, the next step would be to compare the condition numbers with the standard condition numbers of the generalized eigenvalue problems associated with different linearizations by analyzing their ratios. An eigenvalue λ could then be computed stably from the linearization if the ratio is moderate. These results would determine which representations of rational matrices and which linearizations have favorable properties with respect to conditioning.

Problem 3: Define different local backward errors for particular computed pairs of eigenvalue λ and associated eigenvector x of REPs considering different representations of the corresponding rational matrix and different types of perturbations. Obtaining explicit formulas of backward errors of an approximate eigenpair (λ, x) for rational matrices can be much more involved than for polynomial matrices. This is due to the fact that in many of the representations of rational matrices appear inverses of polynomial matrices containing the pole information. After expressions or methods for computing local backward errors are obtained, the next step would be to compare the obtained local backward errors with the standard backward errors of the generalized eigenvalue problems associated with linearizations. These results would determine which representations of rational matrices and which linearizations have favorable properties with respect to local backward stability.

Problem 4: Study the stability and accuracy of the linearization-based algorithm proposed in Section 6.5 for solving (scalar) rational equations as in (6.25).

Problem 5: For rational matrices $R(\lambda)$ expressed in the general form $R(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$, the linearizations constructed in Chapter 6 are of the form

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right]$$

where $\begin{bmatrix} M_A(\lambda) \\ K_A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) \\ K_D(\lambda) \end{bmatrix}$ are (degenerate) strong block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$, respectively. A related problem is to develop the construction by considering general strong block minimal bases linearizations of $A(\lambda)$ and $D(\lambda)$ as in Definition 2.5.1. That is, strong block minimal bases linearizations of the form $\begin{bmatrix} M_A(\lambda) & K_A^2(\lambda)^T \\ K_A^1(\lambda) & 0 \end{bmatrix}$ and $\begin{bmatrix} M_D(\lambda) & K_D^2(\lambda)^T \\ K_D^1(\lambda) & 0 \end{bmatrix}$ for $A(\lambda)$ and $D(\lambda)$, respectively. In such a case, the recovery rules for minimal bases and minimal indices from the linearization would be analogous for the right and left null spaces, as well as the recovery of eigenvectors. In addition, the linearizations could also preserve structures.

Problem 6: The results in Chapter 8 are not only important for preserving structures of rational matrices but for the particular case of structured polynomial matrices. In particular, in Sections 8.1 and 8.2, we construct structured strongly minimal linearizations of structured polynomial matrices with (skew-)Hermitian and alternating structures. An open problem is to construct strongly minimal linearizations preserving other structures of polynomial matrices as (anti-)palindromic, that are also important in applications [64]. A polynomial matrix $P(\lambda)$ is palindromic if $[P(\lambda)]^* = \operatorname{rev} P(\lambda)$, and anti-palindromic if $[P(\lambda)]^* = -\operatorname{rev} P(\lambda)$.

Appendix A

Proof of Theorem 6.4.9

Proof. Let us consider a right minimal basis $\{z_i(\lambda)\}_{i=1}^s$ of $\mathcal{L}(\lambda)$ with right minimal indices $\epsilon_i = \deg z_i(\lambda)$, for $i = 1, \ldots, s$. By Lemma 2.4.5, we have that the polynomial vectors $z_i(\lambda)$ must be of the form

$$z_i(\lambda) = \begin{bmatrix} y_i(\lambda) \\ x_i(\lambda) \end{bmatrix} \quad (i = 1, \ldots, s),$$

for some basis $\{x_i(\lambda)\}_{i=1}^s$ of $\mathcal{N}_r(\widehat{R})$. We notice that the vectors $x_i(\lambda)$ must be polynomial vectors, otherwise the vectors $z_i(\lambda)$ would not be polynomial vectors.

We will prove that the polynomial basis $\{x_i(\lambda)\}_{i=1}^s$ is minimal by using Theorem 2.3.3. For this purpose, let us define the polynomial matrices

$$B(\lambda) := \begin{bmatrix} z_1(\lambda) & \cdots & z_s(\lambda) \end{bmatrix} \quad \text{and} \quad \widehat{B}(\lambda) := \begin{bmatrix} x_1(\lambda) & \cdots & x_s(\lambda) \end{bmatrix}.$$

First, let us show that $\widehat{B}(\lambda)$ has full column rank for every $\lambda_0 \in \mathbb{F}$. Consider the unimodular (and, so, invertible at every $\lambda_0 \in \mathbb{F}$) matrix $U_A(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_A(\lambda)^T & N_A(\lambda)^T \end{bmatrix}$ defined in (6.7). By (6.8), we have

$$\begin{bmatrix} M_A(\lambda_0) \\ K_A(\lambda_0) \\ -M_C(\lambda_0) \\ 0 \end{bmatrix} U_A(\lambda_0)^{-1} = \begin{bmatrix} * & A(\lambda_0) \\ I_{n\,\rho_A} & 0 \\ * & -C(\lambda_0) \\ 0 & 0 \end{bmatrix},$$

for every $\lambda_0 \in \mathbb{F}$. The equation above, together with (6.21), implies that the (constant) matrix

$$\begin{bmatrix} M_A(\lambda_0) \\ K_A(\lambda_0) \\ -M_C(\lambda_0) \\ 0 \end{bmatrix}$$

has full column rank for every $\lambda_0 \in \mathbb{F}$. Then, from $\mathcal{L}(\lambda_0)B(\lambda_0) = 0$, we obtain

$$\left[\begin{array}{c|c} M_A(\lambda_0) & M_B(\lambda_0) \\ K_A(\lambda_0) & 0 \\ -M_C(\lambda_0) & M_D(\lambda_0) \\ 0 & K_D(\lambda_0) \end{array}\right] \begin{bmatrix} * \\ \widehat{B}(\lambda_0) \end{bmatrix} = 0,$$

where $*$ indicates a constant matrix that is not important for the argument. By Lemma 6.4.8, we conclude that $\widehat{B}(\lambda_0)$ has full column rank.

Let us consider the highest column degree coefficient matrices of $B(\lambda)$ and $\widehat{B}(\lambda)$, which we denote by B_{hcd} and $\widehat{B}_{\text{hcd}}$, respectively. Let us show, next, that the matrix $\widehat{B}_{\text{hcd}}$ has full column rank. For this purpose, let us write

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|c} M_A(\lambda) & M_B(\lambda) \\ K_A(\lambda) & 0 \\ \hline -M_C(\lambda) & M_D(\lambda) \\ 0 & K_D(\lambda) \end{array}\right] =: \left[\begin{array}{c|c} M_{1A}\lambda + M_{0A} & M_{1B}\lambda + M_{0B} \\ K_{1A}\lambda + K_{0A} & 0 \\ \hline -M_{1C}\lambda - M_{0C} & M_{1D}\lambda + M_{0D} \\ 0 & K_{1D}\lambda + K_{0D} \end{array}\right]. \tag{A.1}$$

Consider the unimodular matrix $\widetilde{U}_A(\lambda)^{-1} = \begin{bmatrix} \widetilde{N}_A(\lambda)^T & \operatorname{rev}_{\rho_A} N_A(\lambda)^T \end{bmatrix}$ defined in (6.12). By (6.14), we have

$$\begin{bmatrix} \operatorname{rev}_1 M_A(0) \\ \operatorname{rev}_1 K_A(0) \\ -\operatorname{rev}_1 M_C(0) \\ 0 \end{bmatrix} \widetilde{U}_A(0)^{-1} = \begin{bmatrix} M_{1A} \\ K_{1A} \\ -M_{1C} \\ 0 \end{bmatrix} \widetilde{U}_A(0)^{-1} = \begin{bmatrix} * & \operatorname{rev}_{\rho_A+1} A(0) \\ I_{n\rho_A} & 0 \\ * & -\operatorname{rev}_{\rho_A+1} C(0) \\ 0 & 0 \end{bmatrix}.$$

The above equation, together with (6.22), implies that the matrix

$$\begin{bmatrix} M_{1A} \\ K_{1A} \\ -M_{1C} \\ 0 \end{bmatrix}$$

has full column rank. Moreover, from Lemma 6.4.5, we obtain

$$z_i(\lambda) = \begin{bmatrix} y_i(\lambda) \\ x_i(\lambda) \end{bmatrix} = \begin{bmatrix} y_{\epsilon_i} \\ x_{\epsilon_i} \end{bmatrix} \lambda^{\epsilon_i} + \text{ lower degree terms, with } x_{\epsilon_i} \neq 0.$$

Hence, from $\mathcal{L}(\lambda)z_i(\lambda) = 0$, we get

$$\begin{bmatrix} M_{1A} \\ K_{1A} \\ -M_{1C} \\ 0 \end{bmatrix} y_{\epsilon_i} + \begin{bmatrix} M_{1B} \\ 0 \\ M_{1D} \\ K_{1D} \end{bmatrix} x_{\epsilon_i} = 0,$$

which implies

$$y_{\epsilon_i} = -\underbrace{\begin{bmatrix} M_{1A} \\ K_{1A} \\ -M_{1C} \\ 0 \end{bmatrix}^{\dagger} \begin{bmatrix} M_{1B} \\ 0 \\ M_{1D} \\ K_{1D} \end{bmatrix}}_{=:E} x_{\epsilon_i} \quad (i = 1, \ldots, s),$$

where † denotes the pseudoinverse operation. Thus, we have $B_{\text{hcd}} = \begin{bmatrix} E\widehat{B}_{\text{hcd}} \\ \widehat{B}_{\text{hcd}} \end{bmatrix}$. Therefore, the matrix $\widehat{B}_{\text{hcd}}$ must have full column rank since, otherwise, the matrix B_{hcd} would not have full column rank.

By Theorem 2.3.3, we conclude that $\{x_i(\lambda)\}_{i=1}^s$ is a right minimal basis of $\widehat{R}(\lambda)$. Moreover, since $\epsilon_i = \deg z_i(\lambda) = \deg x_i(\lambda)$, for $i = 1, \ldots, s$, the right minimal indices of $\widehat{R}(\lambda)$ are equal to the right minimal indices of $\mathcal{L}(\lambda)$. This establishes the first statement in part (a) and in part (b).

Finally, let us prove that $x_i(\lambda) = N_D(\lambda)^T u_i(\lambda)$, for some right minimal basis $\{u_i(\lambda)\}_{i=1}^s$ of $R(\lambda)$. First, from Lemma 6.4.2, we get $x_i(\lambda) = N_D(\lambda)^T u_i(\lambda)$, for $i = 1, \ldots, s$, for some basis $\{u_i(\lambda)\}_{i=1}^s$ of $\mathcal{N}_r(R)$. Since $N_D(\lambda)$ is a minimal basis, by [38, Main Theorem, part 4], the vectors $u_i(\lambda)$ must be polynomial vectors.

We will show that the polynomial basis $\{u_i(\lambda)\}_{i=1}^s$ is minimal by using Theorem 2.3.3. For this purpose, let us define

$$\widetilde{B}(\lambda) := \begin{bmatrix} u_1(\lambda) & \cdots & u_s(\lambda) \end{bmatrix}.$$

Clearly, we have $\widehat{B}(\lambda) = N_D(\lambda)^T \widetilde{B}(\lambda)$. Since $\widehat{B}(\lambda_0)$ has full column rank for every $\lambda_0 \in \mathbb{F}$, because $\{x_i(\lambda)\}_{i=1}^s$ is a right minimal basis of $\widehat{R}(\lambda)$, we obtain that $\widetilde{B}(\lambda_0)$ must also have full column rank for every $\lambda_0 \in \mathbb{F}$. Next, let us denote by $\widetilde{B}_{\text{hcd}}$ the highest column degree coefficient matrix of $\widetilde{B}$. Since $N_D(\lambda)$ is a minimal basis with all its row degrees equal, the highest column degree coefficient of $N_D(\lambda)^T$ is its leading coefficient matrix, which we denote by N_{ρ_D}. Then, we have $\widehat{B}_{\text{hcd}} = N_{\rho_D} \widetilde{B}_{\text{hcd}}$. Since $\widehat{B}_{\text{hcd}}$ has full column rank, so does $\widetilde{B}_{\text{hcd}}$.

By Theorem 2.3.3, we conclude that $\{u_i(\lambda)\}_{i=1}^s$ is a right minimal basis of $R(\lambda)$. Moreover, by [38, Main Theorem, part 5] and the fact that $N_D(\lambda)$ is a minimal basis with all its row degrees equal to ρ_D, we have

$$\deg x_i(\lambda) = \rho_D + \deg u_i(\lambda) \quad (i = 1, \ldots, s),$$

which shows that the right minimal indices of $R(\lambda)$ are equal to $\epsilon_1 - \rho_D \leq \cdots \leq \epsilon_s - \rho_D$. This concludes the proof. ■

Appendix B

Proof of Theorem 6.4.11

Proof. Let us consider a left minimal basis $\{z_i(\lambda)^T\}_{i=1}^t$ of $\mathcal{L}(\lambda)$ with left minimal indices $\eta_i = \deg z_i(\lambda)$, for $i = 1, \ldots, t$. By Lemma 2.4.5, the polynomial vector $z_i(\lambda)^T$ must be of the form

$$z_i(\lambda)^T = \begin{bmatrix} y_i(\lambda)^T & x_i(\lambda)^T \end{bmatrix} \quad (i = 1, \ldots, t),$$

for some basis $\{x_i(\lambda)^T\}_{i=1}^t$ of the left nullspace of $\widehat{R}(\lambda)$. We observe the vectors $x_i(\lambda)^T$ must be polynomial vectors because the vectors $z_i(\lambda)^T$ are polynomial.

We will prove that the polynomial basis $\{x_i(\lambda)^T\}_{i=1}^t$ is a minimal basis by using the characterization in Theorem 2.3.3. With this goal in mind, let us introduce the polynomial matrices

$$X(\lambda) := \begin{bmatrix} x_1(\lambda)^T \\ \vdots \\ x_t(\lambda)^T \end{bmatrix} \quad \text{and} \quad Z(\lambda) := \begin{bmatrix} z_1(\lambda)^T \\ \vdots \\ z_t(\lambda)^T \end{bmatrix} = \begin{bmatrix} y_1(\lambda)^T & x_1(\lambda)^T \\ \vdots & \vdots \\ y_t(\lambda)^T & x_t(\lambda)^T \end{bmatrix} =: \begin{bmatrix} Y(\lambda) & X(\lambda) \end{bmatrix}.$$

Let us show, first, that the polynomial matrix $X(\lambda)$ has full row rank for every $\lambda_0 \in \mathbb{F}$. For this purpose, consider the unimodular matrices $U_A(\lambda)^{-1}$ and $U_D(\lambda)^{-1}$ defined in (6.7). By (6.8), we have that the matrix

$$\begin{bmatrix} M_A(\lambda_0) & M_B(\lambda_0) \\ K_A(\lambda_0) & 0 \end{bmatrix} \begin{bmatrix} U_A(\lambda_0)^{-1} & 0 \\ 0 & U_D(\lambda_0)^{-1} \end{bmatrix} = \begin{bmatrix} * & A(\lambda_0) & * & B(\lambda_0) \\ I_{n\,\rho_A} & 0 & 0 & 0 \end{bmatrix}$$

has full row rank for every $\lambda_0 \in \mathbb{F}$ because of (6.23). Hence, $\left[\begin{smallmatrix} M_A(\lambda_0) & M_B(\lambda_0) \\ K_A(\lambda_0) & 0 \end{smallmatrix}\right]$ has full row rank for every $\lambda_0 \in \mathbb{F}$. Then, from $Z(\lambda_0)\mathcal{L}(\lambda_0) = 0$, we obtain

$$\begin{bmatrix} Y(\lambda_0) & X(\lambda_0) \end{bmatrix} \left[\begin{array}{cc} M_A(\lambda_0) & M_B(\lambda_0) \\ K_A(\lambda_0) & 0 \\ \hline -M_C(\lambda_0) & M_D(\lambda_0) \\ 0 & K_D(\lambda) \end{array}\right] = 0.$$

By Lemma 6.4.8, we conclude that $X(\lambda_0)$ has full row rank for every $\lambda_0 \in \mathbb{F}$.

Let Z_{hrd} and X_{hrd} be the highest row degree matrix coefficients of $Z(\lambda)$ and $X(\lambda)$, respectively. Let us show that the matrix X_{hrd} has full row rank. Consider the unimodular matrices $\widetilde{U}_A(\lambda)^{-1}$ and $\widetilde{U}_D(\lambda)^{-1}$ defined in (6.12). By (6.14), we have that

$$\begin{bmatrix} \operatorname{rev}_1 M_A(0) & \operatorname{rev}_1 M_B(0) \\ \operatorname{rev}_1 K_A(0) & 0 \end{bmatrix} \begin{bmatrix} \widetilde{U}_A(0)^{-1} & 0 \\ 0 & \widetilde{U}_D(0)^{-1} \end{bmatrix} = \begin{bmatrix} * & \operatorname{rev}_{\rho_A+1} A(0) & * & \operatorname{rev}_{\rho_D+1} B(0) \\ I_{n\rho_A} & 0 & 0 & 0 \end{bmatrix}$$

has full row rank for every $\lambda_0 \in \mathbb{F}$ because of (6.24). Hence, using the notation introduced in (A.1), we have that the matrix $\left[\begin{smallmatrix} \operatorname{rev}_1 M_A(0) & \operatorname{rev}_1 M_B(0) \\ \operatorname{rev}_1 K_A(0) & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} M_{1A} & M_{1B} \\ K_{1A} & 0 \end{smallmatrix}\right]$ has full row rank. Thus, Lemma 6.4.5 implies

$$z_i(\lambda)^T = \begin{bmatrix} y_i(\lambda)^T & x_i(\lambda)^T \end{bmatrix} = \begin{bmatrix} y_{\eta_i}^T & x_{\eta_i}^T \end{bmatrix} \lambda^{\eta_i} + \text{ lower degree terms, with } x_{\eta_i} \neq 0.$$

Then, from $z_i(\lambda)^T \mathcal{L}(\lambda) = 0$, we obtain

$$\begin{bmatrix} y_{\eta_i}^T & x_{\eta_i}^T \end{bmatrix} \left[\begin{array}{cc} M_{1A} & M_{1B} \\ K_{1A} & 0 \\ \hline -M_{1C} & M_{1D} \\ 0 & K_{1D} \end{array}\right] = 0.$$

Since the matrix $\left[\begin{smallmatrix} M_{1A} & M_{1B} \\ K_{1A} & 0 \end{smallmatrix}\right]$ has full row rank, we have

$$y_{\eta_i}^T = -x_{\eta_i}^T \underbrace{\begin{bmatrix} -M_{1C} & M_{1D} \\ 0 & K_{1D} \end{bmatrix} \begin{bmatrix} M_{1A} & M_{1B} \\ K_{1A} & 0 \end{bmatrix}^\dagger}_{=:F},$$

where † indicates the pseudoinverse operation. Therefore, $Z_{\text{hrd}} = \begin{bmatrix} X_{\text{hrd}} F & X_{\text{hrd}} \end{bmatrix}$. Conclusively, the matrix X_{hrd} has full row rank because Z_{hrd} has full row rank.

From Theorem 2.3.3, we get that $\{x_i(\lambda)^T\}_{i=1}^t$ is a left minimal basis of $\widehat{R}(\lambda)$. Moreover, by Lemma 6.4.5, we have $\eta_i = \deg z_i(\lambda) = \deg x_i(\lambda)$, for $i = 1, \ldots, t$. Therefore, $\mathcal{L}(\lambda)$ and $\widehat{R}(\lambda)$ have the same left minimal indices. This establishes the first statement in part (a) and in part (b).

By Lemma 6.4.2, the vector $x_i(\lambda)^T$ must be of the form

$$x_i(\lambda)^T = \begin{bmatrix} u_i(\lambda)^T & w_i(\lambda)^T \end{bmatrix} \quad (i = 1, \ldots, t),$$

for some basis $\{u_i(\lambda)^T\}_{i=1}^t$ of the left nullspace of $R(\lambda)$. We notice that the vectors $u_i(\lambda)$ must be polynomial vectors because the $x_i(\lambda)$ vectors are polynomial. Our final goals are, first, to show that $\{u_i(\lambda)^T\}_{i=1}^T$ is a left minimal basis of $R(\lambda)$ and, second, to show that $\deg u_i(\lambda) = \eta_i$, for $i = 1, \ldots, t$.

We begin by noticing that if we combine Lemmas 2.4.5 and 6.4.2, we get that the vector $z_i(\lambda)^T$ must be of the form

$$z_i(\lambda)^T = \begin{bmatrix} \alpha_{1i}(\lambda)^T & \alpha_{2i}(\lambda)^T & u_i(\lambda)^T & w_i(\lambda)^T \end{bmatrix} \quad (i = 1, \ldots, t),$$

for some vectors $\alpha_{ji}(\lambda)$, with $j = 1, 2$, conformable with the partition of $\mathcal{L}(\lambda)$. We claim that $\{\begin{bmatrix} \alpha_{1i}(\lambda)^T & u_i(\lambda)^T \end{bmatrix}\}_{i=1}^t$ is a left minimal basis of the polynomial system matrix $P(\lambda) = \left[\begin{smallmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{smallmatrix}\right]$ with left minimal indices $\eta_1 \leq \cdots \leq \eta_t$. To see this, first, from $z_i(\lambda)^T \mathcal{L}(\lambda) = 0$, we obtain

$$\begin{bmatrix} \alpha_{1i}(\lambda)^T & u_i(\lambda)^T \end{bmatrix} \begin{bmatrix} M_A(\lambda) & M_B(\lambda) \\ -M_C(\lambda) & M_D(\lambda) \end{bmatrix} + \begin{bmatrix} \alpha_{2i}(\lambda)^T & w_i(\lambda)^T \end{bmatrix} \begin{bmatrix} K_A(\lambda) & 0 \\ 0 & K_D(\lambda) \end{bmatrix} = 0. \tag{B.1}$$

Multiplying (B.1) on the right by $\operatorname{diag}(N_A(\lambda)^T, N_D(\lambda)^T)$ yields

$$\begin{bmatrix} \alpha_{1i}(\lambda)^T & u_i(\lambda)^T \end{bmatrix} \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} = 0. \tag{B.2}$$

Hence, the polynomial vector $\begin{bmatrix} \alpha_{1i}(\lambda)^T & u_i(\lambda)^T \end{bmatrix} \in \mathcal{N}_\ell(P)$.

Next, let us consider the polynomial matrix

$$\widehat{U}(\lambda) := \begin{bmatrix} \alpha_{11}(\lambda)^T & u_1(\lambda)^T \\ \vdots & \vdots \\ \alpha_{1t}(\lambda)^T & u_t(\lambda)^T \end{bmatrix} =: \begin{bmatrix} A_1(\lambda) & U(\lambda) \end{bmatrix}.$$

From (B.2), $\widehat{U}(\lambda)P(\lambda) = 0$. Let us show that $\widehat{U}(\lambda_0)$ has full row rank for every $\lambda_0 \in \mathbb{F}$. From $Z(\lambda_0)\mathcal{L}(\lambda_0) = 0$, we obtain

$$\begin{bmatrix} \widehat{U}(\lambda_0) & * \end{bmatrix} \left[\begin{array}{cc} M_A(\lambda_0) & M_B(\lambda_0) \\ -M_C(\lambda_0) & M_D(\lambda_0) \\ \hline K_A(\lambda_0) & 0 \\ 0 & K_D(\lambda_0) \end{array}\right] = 0.$$

Since $\operatorname{diag}(K_A(\lambda_0), K_D(\lambda_0))$ has full row rank for every $\lambda_0 \in \mathbb{F}$ (because $K_A(\lambda)$ and $K_D(\lambda)$ are both minimal basis), we conclude, by Lemma 6.4.8, that the matrix $\widehat{U}(\lambda_0)$ has full row rank for every $\lambda_0 \in \mathbb{F}$.

Let us denote by $\widehat{U}_{\text{hrd}}$ the highest row degree coefficient matrix of U_{hrd}, and show that $\widehat{U}_{\text{hrd}}$ has full row rank. Since $K_A(\lambda)$ and $K_D(\lambda)$ are minimal bases with constant row degrees (equal to 1), by [38, Main Theorem, part 5], we get

$$\deg \begin{bmatrix} \alpha_{2i}(\lambda)^T & w_i(\lambda)^T \end{bmatrix} \begin{bmatrix} K_A(\lambda) & 0 \\ 0 & K_D(\lambda) \end{bmatrix} = 1 + \deg \begin{bmatrix} \alpha_{2i}(\lambda)^T & w_i(\lambda)^T \end{bmatrix}.$$

In addition, by (B.1), we have that

$$\begin{bmatrix}\alpha_{2i}(\lambda)^T & w_i(\lambda)^T\end{bmatrix}\begin{bmatrix}K_A(\lambda) & 0\\ 0 & K_D(\lambda)\end{bmatrix} = -\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix}\begin{bmatrix}M_A(\lambda) & M_B(\lambda)\\ -M_C(\lambda) & M_D(\lambda)\end{bmatrix}$$

and, therefore,

$$1 + \deg\begin{bmatrix}\alpha_{2i}(\lambda)^T & w_i(\lambda)^T\end{bmatrix} \leq 1 + \deg\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix}.$$

Hence, we have

$$\deg\begin{bmatrix}\alpha_{2i}(\lambda)^T & w_i(\lambda)^T\end{bmatrix} \leq \deg\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix}. \tag{B.3}$$

Thus, from $Z(\lambda)\mathcal{L}(\lambda) = 0$, we obtain

$$\begin{bmatrix}\widehat{U}_{\text{hrd}} & *\end{bmatrix}\left[\begin{array}{cc} M_{1A} & M_{1B}\\ -M_{1C} & M_{1D}\\ \hline K_{1A} & 0\\ 0 & K_{1D}\end{array}\right] = 0.$$

Since the matrix $\text{diag}(K_{1A}, K_{1D})$ has full row rank (because $K_A(\lambda)$ and $K_D(\lambda)$ are both minimal bases with constant row degrees), by Lemma 6.4.8, we have that the matrix $\widehat{U}_{\text{hrd}}$ has also full row rank. Conclusively, since $\dim\mathcal{N}_\ell(\mathcal{L}) = \dim\mathcal{N}_\ell(P)$ (becasue $\mathcal{L}(\lambda)$ is a linearization of $P(\lambda)$ in the classical matrix polynomial sense), Theorem 2.3.3 implies that $\{\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix}\}_{i=1}^t$ is a left minimal basis of $P(\lambda)$, as we claimed. Moreover, by (B.3), we have $\eta_i = \deg z_i(\lambda)^T = \deg\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix}$. Hence, the left minimal indices of $P(\lambda)$ are also equal to $\eta_1 \leq \cdots \leq \eta_t$.

After this small detour, we are ready to prove that $\{u_i(\lambda)^T\}_{i=1}^t$ is a minimal basis of $R(\lambda)$, with minimal indices $\eta_1 \leq \cdots \leq \eta_t$, by using Theorem 2.3.3. To this goal, let us consider the polynomial matrix

$$U(\lambda) := \begin{bmatrix}u_1(\lambda)^T\\ \vdots\\ u_t(\lambda)^T\end{bmatrix}.$$

From $\widehat{U}(\lambda_0)P(\lambda_0) = 0$, we get

$$\begin{bmatrix}A_1(\lambda_0) & U(\lambda_0)\end{bmatrix}\begin{bmatrix}A(\lambda_0) & B(\lambda_0)\\ -C(\lambda_0) & D(\lambda_0)\end{bmatrix} = 0.$$

Since the matrix $\begin{bmatrix}A(\lambda_0) & B(\lambda_0)\end{bmatrix}$ has full row rank for every $\lambda_0 \in \mathbb{F}$, Lemma 6.4.8 implies that $U(\lambda_0)$ has also full row rank for every $\lambda_0 \in \mathbb{F}$.

Let U_{hrd} denote the highest row degree coefficient matrix of $U(\lambda)$. It remains to show that U_{hrd} has full row rank. First, let us prove that

$$\deg\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix} = \deg u_i(\lambda)^T \quad (i = 1, \ldots, t). \tag{B.4}$$

By contradiction, assume

$$\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix} = \begin{bmatrix}\alpha_i^T & 0\end{bmatrix}\lambda^{\eta_i} + \text{ lower degree terms, with } \alpha_i \neq 0.$$

Since $\begin{bmatrix}\alpha_{1i}(\lambda)^T & u_i(\lambda)^T\end{bmatrix} P(\lambda) = 0$, we have that $\alpha_i^T \begin{bmatrix}\operatorname{rev}_{\rho_A+1} A(0) & \operatorname{rev}_{\rho_D+1} B(0)\end{bmatrix} = 0$. Then $\alpha_i = 0$ by (6.24), which is a contradiction. Hence $\widehat{U}_{\text{hrd}}$ must be of the form $\widehat{U}_{\text{hrd}} = \begin{bmatrix}\mathcal{A} & U_{\text{hrd}}\end{bmatrix}$, for some matrix $\mathcal{A}$. Considering again that $\widehat{U}(\lambda)P(\lambda) = 0$, we have

$$\begin{bmatrix}A_1(\lambda) & U(\lambda)\end{bmatrix}\begin{bmatrix}A(\lambda)\\ -C(\lambda)\end{bmatrix} = 0 \text{ and } \begin{bmatrix}A_1(\lambda) & U(\lambda)\end{bmatrix}\begin{bmatrix}B(\lambda)\\ D(\lambda)\end{bmatrix} = 0,$$

and, therefore,

$$\begin{bmatrix}\mathcal{A} & U_{\text{hrd}}\end{bmatrix}\begin{bmatrix}\operatorname{rev}_{\rho_A+1} A(0)\\ -\operatorname{rev}_{\rho_A+1} C(0)\end{bmatrix} = 0 \text{ and } \begin{bmatrix}\mathcal{A} & U_{\text{hrd}}\end{bmatrix}\begin{bmatrix}\operatorname{rev}_{\rho_D+1} B(0)\\ \operatorname{rev}_{\rho_D+1} D(0)\end{bmatrix} = 0.$$

That is,

$$\begin{bmatrix}\mathcal{A} & U_{\text{hrd}}\end{bmatrix}\begin{bmatrix}\operatorname{rev}_{\rho_A+1} A(0) & \operatorname{rev}_{\rho_D+1} B(0)\\ -\operatorname{rev}_{\rho_A+1} C(0) & \operatorname{rev}_{\rho_D+1} D(0)\end{bmatrix} = 0. \tag{B.5}$$

Taking into account (B.5) and condition (6.24), Lemma 6.4.8 implies that U_{hrd} has full row rank.

Thus, by Theorem 2.3.3, $\{u_i(\lambda)^T\}_{i=1}^t$ is a left minimal basis of $R(\lambda)$. Moreover, by (B.4), the left minimal indices of $R(\lambda)$ are $\eta_1 \leq \cdots \leq \eta_t$, as we wanted to prove. ■

Appendix C

Auxiliary result for Lemma 9.3.4

We prove in this appendix that the matrix

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} E_k^T \otimes I_k & I_{(k+1)} \otimes E_k \\ \hline F_k^T \otimes I_k & I_{(k+1)} \otimes F_k \end{array}\right]$$

appearing in the proof of Lemma 9.3.4 can be transformed by row and column permutations to the direct sum of the following matrices :

$$M_1 \oplus M_1 \oplus M_3 \oplus M_3 \oplus \cdots \oplus M_{2k-1} \oplus M_{2k-1} \oplus N_{2k},$$

where the blocks M_k and N_k are as defined in (9.18). Let us take for example $k = 3$, then the matrix looks like

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{ccc|cccc} I_3 & & & E_3 & & & \\ & I_3 & & & E_3 & & \\ & & I_3 & & & E_3 & \\ & & & & & & E_3 \\ \hline & & & F_3 & & & \\ I_3 & & & & F_3 & & \\ & I_3 & & & & F_3 & \\ & & I_3 & & & & F_3 \end{array}\right].$$

There are three submatrices M_1, M_3 and M_5 that take elements a, b, c and d in the respective blocks A, B, C and D, as indicated below

$$M_1 = \left[\begin{array}{c} b \end{array}\right], \quad M_3 = \left[\begin{array}{ccc} b & a & \\ & c & d \\ & & b \end{array}\right], \quad M_5 = \left[\begin{array}{ccccc} b & a & & & \\ & c & d & & \\ & & b & a & \\ & & & c & d \\ & & & & b \end{array}\right]$$

and they each start with a leading element in one of the E_3 blocks. For instance, $M_1 = [b_{10,13}]$, M_3 starts with the leading element $b_{7,9}$ in the third E_3 block, and M_5

starts with the leading element in the second E_3 block :

$$M_1 = [b_{10,13}]\,,$$

$$M_3 = \begin{bmatrix} b_{7,9} & a_{7,7} & \\ & c_{10,7} & d_{10,14} \\ & & b_{11,14} \end{bmatrix}, \quad M_5 = \begin{bmatrix} b_{4,5} & a_{4,4} & & & \\ & c_{7,4} & d_{7,10} & & \\ & & b_{8,10} & a_{8,8} & \\ & & & c_{11,8} & d_{11,15} \\ & & & & b_{12,15} \end{bmatrix}.$$

Notice that the $\begin{bmatrix} b & a \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$ pairs have the same row index and that the $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} d \\ b \end{bmatrix}$ pairs have the same column index, which explains the permutation that has to be constructed to extract the matrix. Also the transitions

$$b_{7,9} \to b_{11,14}, \quad \text{and} \quad b_{4,5} \to b_{8,10} \to b_{12,15}$$

always go down to the next diagonal element in the next E_3 block. In a similar fashion, one finds another set of submatrices M_1, M_3 and M_5 that take elements a, b, c and d in the respective blocks A, B, C and D in a different order, as indicated below

$$M_1 = \begin{bmatrix} d \end{bmatrix}, \quad M_3 = \begin{bmatrix} d & c & \\ & a & b \\ & & d \end{bmatrix}, \quad M_5 = \begin{bmatrix} d & c & & & \\ & a & b & & \\ & & d & c & \\ & & & a & b \\ & & & & d \end{bmatrix}$$

and they each start with a trailing element in one of the first three F_3 blocks. Finally, the remaining matrix N_6 takes elements in the blocks A, B, C and D in the following order

$$N_6 = \begin{bmatrix} b & a & & & & & \\ & c & d & & & & \\ & & b & a & & & \\ & & & c & d & & \\ & & & & b & a & \\ & & & & & c & d \end{bmatrix}$$

and starts with the leading element in the leading E_3 block, and ends with the trailing element in the trailing F_3 block.

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