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Undershoot and order quantity probability distributions in periodic review, reorder point, order-up-to-level inventory systems with continuous demand

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ABSTRACT

The undershoot of the reorder point in the periodic review, order-up-to-level (*R, s, S*) inventory system is known to follow a complex probability distribution which depends on the value of *S-s* (Δ) and the distribution of the demand during the review interval (*R*). We focus on the continuous demand case with full backlogging and variable lead-time. For this case, a generic formulation of the undershoot probability density function (p.d.f.) is developed. The order quantity probability distribution in (*R, s, S*) systems is the same as the undershoot probability distribution with a shift of Δ in the random variable. Therefore, the latter opens the possibility of calculating valuable managerial information such as the expected average order quantity, its standard deviation, and the probability that the order quantity is lower than or exceeds a predetermined value. Based on the proposed formulation, we derive an analytical expression of the undershoot p.d.f. (and hence the order quantity p.d.f.) for the case of gamma distributed demand, as well as a tractable approximation for the normal distributed demand. Both expressions are shown to be dependent upon two nondimensional parameters, Δ/μ_R and the coefficient of variation, with the mean demand during the review interval (μ_R) acting as a scale parameter. We thus define a nondimensional undershoot p.d.f. (NUPDF). The relevance of full nondimensionalization stems from the fact that gamma and normal NUPDF analyses can be scaled to any case of gamma and normal distributed demands. Although we focus on the inventory management viewpoint, the results for the gamma distributed case can be directly adapted for use in any renewal process.

1. Introduction

Inventory systems are essential in the day-to-day operations of warehouse facilities. For each product it is necessary to decide when to place an order and what quantity to order. Due to its importance, inventory systems have been extensively

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Table 1
Previous results regarding the undershoot p.d.f.

Reference	Distribution	Focus	Contribution
Karlin [11]	Exponential	p.d.f.	Undershoot p.d.f. is the same exponential distribution
Karlin [11]	Erlang	p.d.f.	Undershoot p.d.f. can be expressed as a combination of Erlang distributions (case a=2)
Coleman [13]	Continuous non-negative	p.d.f.	Generic expression of the n-th moment
Baxter et al. [15]	Gamma, inverse Gaussian, lognormal, truncated normal, Weibull	Mean and variance	Computation of components of expressions from [13]
Chiu, Leung, and Natarajan [17]	Non-negative, first two or three moments known	Mean	Lower and upper bounds with 2 or 3 first moments of demand distribution
Baganha, Ferrer and Pyke [18]	Discrete non-negative	p.d.f., mean and variance	Algorithm to compute undershoot p.d.f. values and mean and variance
Smith [26], Karlin [11]	Discrete/continuous non-negative	Asymptotic p.d.f.	Asymptotic undershoot p.d.f.
Peterson and Silver [27], Hill [21]	Discrete/continuous non-negative	Asymp. mean and variance	Asymptotic undershoot mean and variance
Hill [21]	Compound Poisson	Asymp. mean and variance	Case: Poisson number of demands during review period
Coleman [13]	Continuous non-negative	Asymp. p.d.f.	Generic expression of the n-th moment
Baganha, Pyke, and Ferrer [28]	Non-negative discrete, Normal, Lognormal, Poisson, Uniform, 2-point	Asymp. mean and variance	Assessment of exactitude for 33 selected cases

discussed in the academic literature, and countless specific software systems are available to support inventory managers. One of the most studied inventory systems is the single-item periodic review, reorder point, order-up-to level (R, s, S) system [1–6]. The relevance of this system stems from the fact that, under some general assumptions regarding the replenishment, carrying and shortage cost structure of an inventory system, the optimal policy that minimizes the total inventory cost is a (R, s, S) policy [7,8].

In the (R, s, S) inventory system, inventory position—defined as the stock on-hand plus stock on-order minus backorders—, is examined every R units of time. If the inventory position is at or below the reorder point (s), an order is placed so that the inventory position rises to the order-up-to level (S). The interval between placing and receiving the order is the lead time. Throughout this paper, we will consider the continuous demand distribution case with full backlogging and variable lead-time. The focus will be placed on the undershoot, which is the difference between the reorder point and the inventory position when an order is triggered at a review epoch. Silver, Naseraldin, and Bischak [9] underline the complexity of the undershoot p.d.f. and point out its dependency on both the value of $\Delta = S - s$ and the distribution of the demand during the review interval R. The order quantity and the undershoot are directly related since, by definition, the order quantity will be the undershoot plus Δ. Consequently, the order quantity is a random variable presenting the same complex probability distribution, which reflects in an ill-behaved cost function [10].

Despite the wide attention given to (R, s, S) inventory systems in the literature only a few works deal with the undershoot probability distribution as summarized in Table 1. The most common approach comes from the renewal theory. Karlin [11] gathers the basics of the renewal theory and defines the (R, s, S) inventory system as a renewal process, equating the sequence of i.i.d. non-negative demands during the review interval R with the inter-occurrence times between renewal events, and Δ with the fixed total time of the renewal process. The undershoot corresponds to the excess variable of the renewal process, also known as excess/residual life [12] or forward recurrence-time [13]. Karlin shows that when demand follows an exponential distribution (Erlang with shape parameter $a = 1$), the undershoot follows the same distribution; the author then provides the undershoot p.d.f. for the case of an Erlang with $a = 2$ and states that, in general, when demand follows an Erlang distribution, the undershoot p.d.f. can be expressed as a combination of Erlang distributions. Sahin [14] and Tijms [12] further extend the application of renewal theory to inventory systems. Coleman [13] provides a useful expression for the n-th moment of the forward recurrence time (undershoot), specifying the mean and variance. Baxter et al. [15] make use of an algorithm developed by McConalogue [16] to compute the renewal function and its integral for several distributions. The authors provide tables for a range of values of Δ and the distribution parameters, yet they do not use them to compute the undershoot mean and variance (and therefore do not make any analysis regarding the undershoot distribution). Chiu, Leung, and Natarajan [17] provide analytical boundaries for the average undershoot if the first two or three moments of non-negative demand are known. Baganha, Ferrer and Pyke [18] develop a very effective algorithm to compute the residual life distribution (undershoot p.d.f.) for discrete distributions which avoids calculating convolution integrals.

Neglecting the undershoot is known to have drastic consequences in terms of overestimation of any service level measures [19–21]. In order to take undershoots into consideration and given the intrinsic difficulty of its mathematical treatment, a usual simplification adopted in the literature consists in approximating the undershoot p.d.f. by its asymptotic distribution, defined as Δ tends to infinity [3,14]. Examples of models that consider continuous demand distributions and assume this simplification include Roberts [22], Schneider [6], Tijms and Groenevelt [4], Tempelmeier [23], and, more recently, Kiesmüller and Inderfurth [24], Zijm [25].

Karlin [11] remarks the importance of a limit theorem of the renewal theory [26] that provides the asymptotic distribution of the excess variable (undershoot). Peterson and Silver [27] use the result of Karlin to derive expressions of the undershoot mean and variance for discrete distributed demands. Hill [21] develops an alternative demonstration to derive the asymptotic expression of the undershoot mean and variance based on probability theory, as well as an expression for the undershoot mean and variance in (*R, s, S*) systems when the number of demands during the review period follows a Poisson distribution. Coleman [13] provides the expression for the n-th moment of the asymptotic distribution of the forward recurrence time (undershoot). Baganha, Pyke, and Ferrer [28] address the problem of quantifying the accuracy of this approximation. For a set of 33 selected cases from 5 demand distributions —normal, lognormal, Poisson, uniform, 2-point—, they compare the asymptotic mean and standard deviation of the undershoot with the exact values obtained using the algorithm presented in Baganha, Ferrer and Pyke [18]. In particular, the authors investigate the validity of the approximation depending on the value of Δ and the coefficient of variation, finding relevant discrepancies for low values of Δ and low coefficients of variation. It is worth noting at this point that the algorithm they use is only valid for non-negative discrete demand distributions, and hence for continuous demand distributions the computed values are approximations themselves.

In conclusion, although the undershoot p.d.f. plays a key role in (R, s, S) inventory systems, and it is directly related to the order quantity, there is a lack of knowledge regarding the undershoot p.d.f, its shape and parameter dependency. The analytical expression of the undershoot p.d.f. is only known for a few specific cases and merely the asymptotic distribution for non-negative demand distributions has been studied in some detail. From a managerial point of view, and even though there is a vast literature on (*R, s, S*) systems, basic information such as the variability of the order quantity or the probability that the order quantity exceeds a specified value are not provided. Thus, obtaining a deeper understanding of the undershoot p.d.f. in (*R, s, S*) inventory models will provide valuable and practical information. Throughout this paper we aim at this objective, based on the development of the first analytic expression of the undershoot p.d.f. (and hence the order quantity p.d.f.) valid for all the cases of continuous demand distributions.

This paper is organized as follows. In the second section we first formulate the problem and the relation between the order quantity and the undershoot, and then develop the expression for the undershoot and order quantity probability distribution functions. Results from renewal theory serve to verify the proposed expression for the case of non-negative distributions. In the third and fourth sections the expression developed is applied to perform an in-depth analysis of the undershoot p.d.f. for the cases of the two continuous probability distributions most used in inventory theory to model customer demand: gamma and normal distributions. In the third section we derive an analytic expression of the undershoot p.d.f. for the case of gamma distributed demand, and we propose a nondimensionalization that enables an in-depth study of the undershoot p.d.f. for all the possible values of the defining parameters. The analysis is complemented with an assessment of the validity of the widely used asymptotic values. In the fourth section a similar in-depth study is carried out for the case of normal distributed demand, using a tractable approximation. Finally, in the fifth section we present the main conclusions and further research.

2. Undershoot and order quantity probability distributions

Let us introduce the following notation:

R	review interval
s	reorder point
S	order-up-to-level
Δ	S - s
X_m	independent and identically distributed (i.i.d.) random variables, representing demand during review intervals m (m=1,2...)
υ_k	independent and identically distributed (i.i.d.) non-negative random variables, representing the value of the undershoot in order number k (k=1,2...)
Q_k	independent and identically distributed (i.i.d.) random variables, representing the quantity of the order number k (k=1,2...), $Q_k \geq \Delta$
L_k	Lead time of the order number k (k=1,2...)
f_R	probability density function (p.d.f.) of demand during a review interval, with mean $\mu_R > 0$ and variance σ_R^2
F_R	cumulative density function (c.d.f.) of demand during a review interval
f_{nR}	p.d.f. of demand during n consecutive review intervals
g_{nR}	p.d.f. of demand during n consecutive review intervals since the last ordering without reaching the reorder level in the previous n-1 review epochs
$f_u^{(\Delta)}$	p.d.f. of the undershoot for a given value of Δ, with mean $\mu_u^{(\Delta)}$ and variance $\sigma_u^{(\Delta)\,2}$
$F_u^{(\Delta)}$	c.d.f. of the undershoot for a given value of Δ
$f_Q^{(\Delta)}$	p.d.f. of the order quantity for a given value of Δ, mean $\mu_Q^{(\Delta)}$ and variance $\sigma_Q^{(\Delta)\,2}$
$F_Q^{(\Delta)}$	c.d.f. of the order quantity for a given value of Δ
$\Theta_{nR}^{k}(x)$	$\int_x^{\infty}(y-x)^k \cdot f_{nR}(y)dy$ ($Note:\ \Theta_R^k(0)=E(X_m{}^k)$)

We consider the classic (*R, s, S*) inventory system depicted in Fig. 1, restricting ourselves to the case in which customer demand follows a continuous probability density function (p.d.f.) with positive mean. When customer demand cannot be met from inventory on stock, demand will be backordered; i.e., backorders at review epochs will account for the undershoot. In what follows, the review interval R is taken as the unit of time without loss of generality. It is important to note that lead-

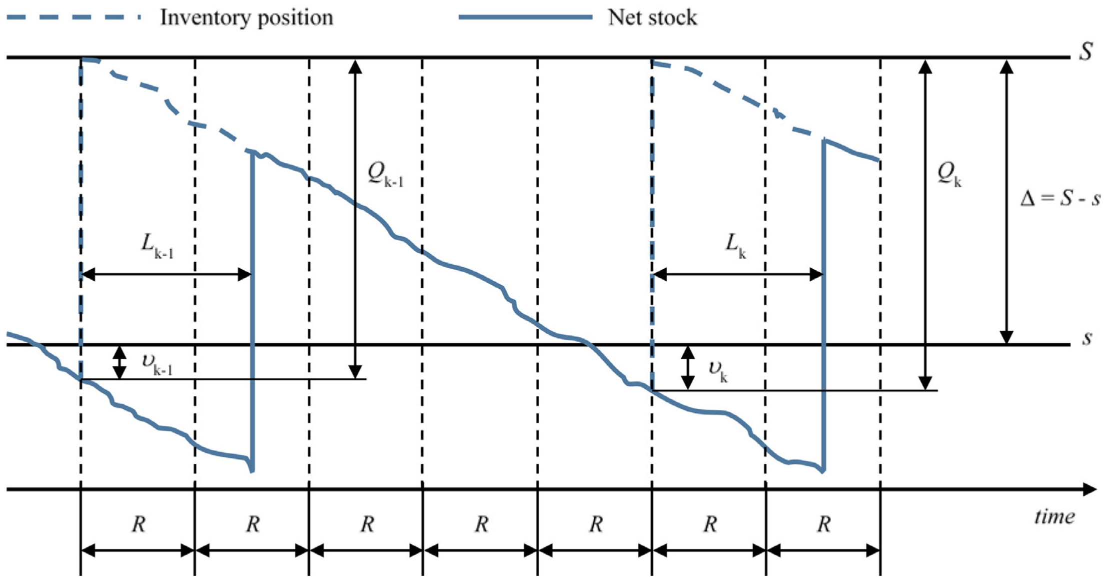

Fig. 1. (*R, s, S*) inventory system.

time variability does not affect the undershoot p.d.f., since the latter is defined based on the inventory position. Inventory position rises up to *S* at review epochs regardless of when the order placed is to be received. Therefore, we may assume that the lead-time can vary from order to order (L_k) and, unlike what is commonly assumed when setting the inventory parameters [4], order crossing is allowed.

2.1. Relation between the undershoot and order quantity probability distributions

As aforementioned and illustrated in Fig. 1, by definition, when an order is triggered in a (R, s, S) inventory system, its size will be the undershoot plus $\Delta = S - s$. Thus, the undershoot of the reorder point and the order quantity are interrelated variables as expressed in (1).

$$Q_k = \Delta + \upsilon_k \tag{1}$$

Likewise, the probability for the undershoot to lie in the interval $(\upsilon, \upsilon + d\upsilon)$, $\upsilon \geq 0$, will be equivalent to the probability for the order quantity to lie in the interval $(\Delta+\upsilon, \Delta+\upsilon + d\upsilon)$. Therefore, there is a direct relation between the undershoot p.d.f. and the order quantity p.d.f., as well as between their respective means and standard deviations (2).

$$f_Q^{(\Delta)}(\Delta + \upsilon) = f_u^{(\Delta)}(\upsilon), \ \mu_Q^{(\Delta)} = \Delta + \mu_u^{(\Delta)}, \ \sigma_Q^{(\Delta)} = \sigma_u^{(\Delta)} \tag{2}$$

Consequently, in what follows we will focus on the development and analysis of the undershoot p.d.f., since its results can be directly extended to the order quantity distribution and interpreted accordingly to Eqs. (1) and (2).

2.2. Undershoot probability distribution

To build the undershoot p.d.f., $f_u^{(\Delta)}$, we express the probability for the undershoot to lie in a specific infinitesimal interval at each review epoch since the last ordering, taking that last ordering moment as the origin of time (Fig. 2):

- At the end of the first review period since the last ordering, if demand is greater than or equal to Δ, it is immediate to see that the probability for the undershoot to lie in the interval $(\upsilon, \upsilon + d\upsilon)$, $\upsilon \geq 0$, will be given by $f_R(\Delta + \upsilon)d\upsilon$.
- At review epoch $n+1$, the probability for the undershoot to lie in a specific interval $(\upsilon, \upsilon + d\upsilon)$ will depend on both the cumulative demand over the first n periods without reaching the reorder level (y_n) and the demand over period $n+1$. Therefore, for a cumulative demand y_n, undershoot will take a value of υ when demand over period $n+1$ is $\Delta+\upsilon - y_n$. Integrating over all valid values of y_n, i.e. $y_n \in (-\infty, \Delta)$, leads to an expression of the probability for the undershoot to lie in the interval $(\upsilon, \upsilon + d\upsilon)$, $\upsilon \geq 0$:

$$\Pr\left(\upsilon_k \in (\upsilon, \upsilon + d\upsilon) \ in\ review\ n+1\right) = \left(\int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot f_R(\Delta + \upsilon - y_n) dy_n\right) d\upsilon. \tag{3}$$

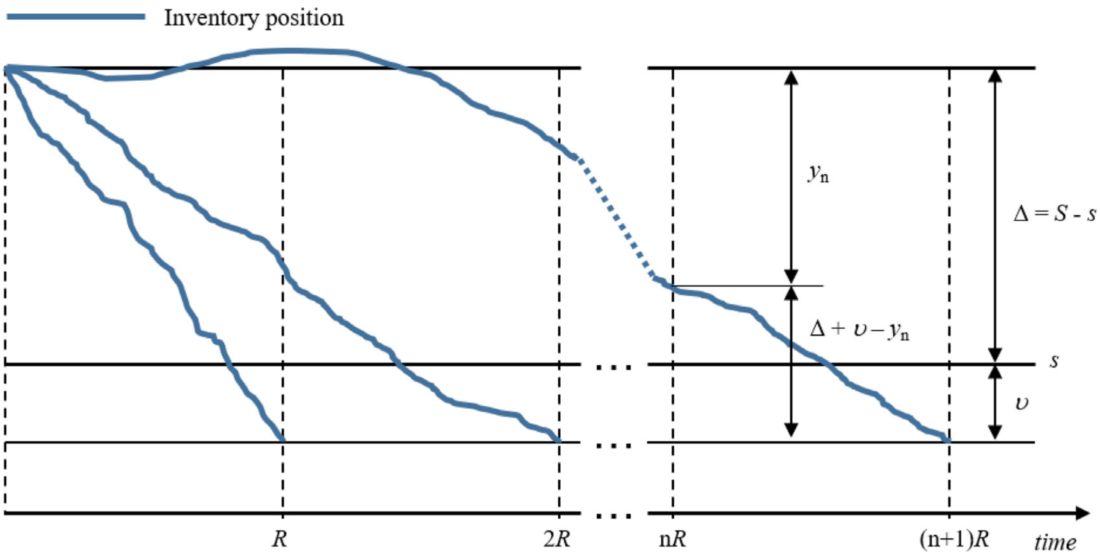

Fig. 2. Development of the undershoot probability density function.

Expression (3) is based on the convolution of g_{nR} and f_R. Due to the probability of negative values (see longest trajectory in Fig. 2), not having crossed the reorder level at review n does not imply not having crossed the reorder level in all the previous review epochs since the last ordering. Hence, in general, $g_{nR}(x)$ does not equate to $f_{nR}(x)$. By definition, g_{nR} is the n-fold partial convolution of f_R as recursively formulated in (4) for $n>1$ and with $g_R(y_1) \equiv f_R(y_1)$,

$$g_{nR}(y_n) = \int_{-\infty}^{\Delta} g_{(n-1)R}(y_{n-1}) \cdot f_R(y_n - y_{n-1}) dy_{n-1}. \tag{4}$$

The undershoot p.d.f. is then expressed as the summation of the undershoot probability expressions for each period:

$$f_u^{(\Delta)}(\upsilon) = f_R(\Delta + \upsilon) + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot f_R(\Delta + \upsilon - y_n) dy_n. \tag{5}$$

With some algebra (see Appendix A for details), we find the following generic expressions for the mean and variance of the undershoot distribution that show similar structure:

$$\mu_u^{(\Delta)} = \int_0^{\infty} \upsilon \cdot f_u^{(\Delta)}(\upsilon) d\upsilon = \Theta_R^1(\Delta) + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \Theta_R^1(\Delta - y_n) dy_n,$$

$$\sigma_u^{(\Delta)2} = \int_0^{\infty} \left(\upsilon - \mu_u^{(\Delta)}\right)^2 \cdot f_u^{(\Delta)}(\upsilon) d\upsilon = \Psi_R(\Delta) + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \Psi_R(\Delta - y_n) dy_n$$

$$\text{with } \Psi_R(x) = \Theta_R^2(x) - 2\mu_u^{(\Delta)} \Theta_R^1(x) + \mu_u^{(\Delta)2} \Theta_R^0(x). \tag{6}$$

Evolution of the inventory position in the (*R, s, S*) inventory system has its equivalent in the classic random walk stochastic process. Specifically, it corresponds to the general one-dimensional random walk in discrete time with one absorbing barrier [29]. Crossing the absorbing barrier will be equivalent to crossing the reorder point [30]. Therefore, results from random walk theory may be used to provide insights on the undershoot p.d.f. In particular, Cox and Miller [29] show that, for one absorbing barrier random walks with positive mean step, the absorption will be certain. Consequently, in a (R, s, S) inventory system with positive mean demand it can be assured that, after placing an order, inventory position will eventually cross the reorder point. It can then be concluded that $f_u^{(\Delta)}$ in (5) is a complete p.d.f., since probability of crossing the reorder point can be expressed as the probability of the undershoot taking on any value:

$$Pr(crossing) = \int_0^{\infty} f_u^{(\Delta)}(\upsilon) d\upsilon \rightarrow \int_0^{\infty} f_u^{(\Delta)}(\upsilon) d\upsilon = 1.$$

For a large value of Δ, the number of steps until the passage time (N), i.e. the number of review periods until crossing the reorder point, will be a random variable with the following approximate values of mean and variance [29]:

$$E(N) \cong \frac{\Delta}{\mu} \left(E(N) \geq \frac{\Delta}{\mu}\right), \; V(N) \cong \Delta \frac{\sigma^2}{\mu^3}. \tag{7}$$

Practical implications of the above results in terms of the undershoot p.d.f. tractability are significant. The infinite summation in (5) can be approached by a virtually equivalent finite summation, since even in the less favorable case (large Δ) there will be an interval for the index $n \in (\lfloor E(N) - c \cdot \sqrt{V(N)} \rfloor, \lceil E(N) + c \cdot \sqrt{V(N)} \rceil)$ with a tractable c value that will yield high precision.

2.3. Non-negative demand distributions: validation and results from renewal theory

For the case of non-negative demand distributions, lower limit of integrals in (3–5) becomes zero since there is no possibility of negative cumulative demands. On the other hand, due to the non-negativity of X_m, not having crossed the reorder level at review epoch n implicitly denotes not having crossed the reorder level in all previous periods, i.e., $g_{nR}(x)$ equates $f_{nR}(x)$. This fact noticeably simplifies handling the undershoot p.d.f. for the probability distributions with tractable convolution expressions. Therefore, the undershoot p.d.f. expression will be

$$f_u^{(\Delta)}(\upsilon) = f_R(\Delta + \upsilon) + \sum_{n=1}^{\infty} \int_0^{\Delta} f_{nR}(y_n) \cdot f_R(\Delta + \upsilon - y_n) dy_n. \tag{8}$$

As pointed out in the introduction section, the (*R, s, S*) inventory system with non-negative demands is equivalent to a renewal process. In what follows, only a few important results and necessary definitions from renewal theory will be outlined in order to study the undershoot distribution; for a rigorous treatment, we refer to [31–34]. In general, a sequence of non-negative i.i.d. random variables Y_n defines an ordinary renewal process. In renewal theory, the variables Y_n typically represent interocurrence times between renewals or events related to some probability problem. The counter of the number of renewals during a prespecified time *t*, N_t, is the main random variable associated with the renewal process. The counting process $\{N_\mathrm{t}, t \geq 0\}$ is the renewal process. Of special importance are the so-called renewal function, defined by $H(t) = \mathrm{E}(N_\mathrm{t})$, and its first derivative, the renewal density function, that "specifies the mean number of renewals to be expected in a narrow interval near *t*" [31]. It is well known that

$$H(x) = \sum_{n=1}^{\infty} F_{nR}(x) \to h(x) = \sum_{n=1}^{\infty} f_{nR}(x). \tag{9}$$

Expressions for the excess-life or forward recurrence-time given in the renewal theory literature differ from (8) although they are equivalent (for example [14,15,31]). Using the same notation as in (8) (with Δ playing the renewal role of the time *t*), the p.d.f. of the excess-life (undershoot) in terms of the renewal theory is given by

$$f_u^{(\Delta)}(\upsilon) = f_R(\Delta + \upsilon) + \int_0^{\Delta} h(y_n) \cdot f_R(\Delta + \upsilon - y_n) dy_n. \tag{10}$$

Substituting $h(y_n)$ by its definition of Eq. (9), leads to expression (8). Therefore, for non-negative distributions, the analytic generic expression of the undershoot p.d.f. (5) developed in the previous subsection leads to the known expression of the excess life p.d.f. from renewal theory.

There are also additional results from renewal theory that will be used in the analysis performed in this paper. Coleman [13] provides an expression for the *n*-th moment of the excess-life. In particular, this author obtains the following expressions for the undershoot mean and variance:

$$\begin{aligned} \mu_u^{(\Delta)} &= \mu_R(1 + H(\Delta)) - \Delta \\ \sigma_u^{(\Delta)2} &= \Theta_R^2(0) \cdot (1 + H(\Delta)) - \mu_R^2 \cdot (1 + H(\Delta))^2 + 2\mu_R \left(\Delta \cdot H(\Delta) - \int_0^{\Delta} H(\eta) d\eta \right) \end{aligned} \tag{11}$$

As stressed in the introduction, the complexity of dealing with the undershoot distribution yields to the extensive use of its asymptotic expression ($\Delta \to \infty$) [31]:

$$f_u^{(\infty)}(\upsilon) = \frac{1 - F_R(\upsilon)}{\mu_R}, \ \mu_u^{(\infty)} = \frac{\Theta_R^2(0)}{2\mu_R}, \ \sigma_u^{(\infty)2} = \frac{\Theta_R^3(0)}{3\mu_R} - \mu_u^{(\infty)2}. \tag{12}$$

3. Gamma distributed demand

Among the continuous probability distributions, many authors have underlined the gamma distribution as particularly suitable to model demand behavior [35–40]. To this regard, Burgin [35] highlights some of its advantages: defined only for non-negative values; easily convolvable; mathematically tractable; and ranging, according to values of a parameter, from a monotonic decreasing function, through unimodal distributions skewed to the right, to normal type distributions.

In this section we will study the undershoot p.d.f when demand during a review interval *R* follows a gamma distribution f_R^G with shape parameter *a* and scale parameter *b*. Superscript *G* will be used to denote the gamma distribution:

$$f_R^G(x \mid a, \ b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{\frac{-x}{b}}, \ x > 0; \ \mu_R = a \cdot b, \ \sigma_R^2 = a \cdot b^2. \tag{13}$$

We can express the parameters of the gamma distribution as function of the mean and the coefficient of variation (CV_R):

$$a = \frac{{\mu_R}^2}{\sigma_R^2} = \frac{1}{{CV_R}^2},\ b = \frac{\sigma_R^2}{\mu_R} = \mu_R \cdot {CV_R}^2 \Rightarrow f_R^G(x \mid a,\ b) = f_R^G\left(x \mid 1/{CV_R}^2,\ \mu_R \cdot {CV_R}^2\right). \tag{14}$$

If the shape parameter a=1 (CV_R=1) then the gamma distribution is an exponential distribution. When a is an integer, then the gamma distribution is an Erlang distribution (which is the sum of a independent exponential distributions with the same parameter b). Besides the exponential distribution, there are only two Erlang distributions with CV_R>0.5, corresponding to a=2 ($CV_R = \sqrt{2}/2$) and a =3 ($CV_R = \sqrt{3}/3$). Therefore, the Erlang distribution is very limited as an alternative distribution to model non-negative demand.

3.1. Nondimensional undershoot p.d.f

As stressed by Burgin [35], one advantage of the gamma distribution is that it is easily convolvable. If demand during a review interval is gamma distributed with parameters (a, b), then the demand during t review intervals will be gamma distributed with parameters ($t\cdot a$, b); moreover, since gamma distribution is non-negative, it will also correspond to the distribution of demand during t review intervals without reaching the reorder point.

This fact enables us to solve the convolution integral in (8) (see Appendix B.1), leading to the following undershoot p.d.f. analytic expression (15), where F^B denotes the beta cumulative distribution function (equal to the regularized incomplete beta function) and (a, b) are the parameters of f_R^G:

$$f_u^{(\Delta)}(\upsilon \mid a, b) = f_R^G(\Delta + \upsilon \mid a, b) + \sum_{n=1}^{\infty} f_{(n+1)R}^G(\Delta + \upsilon \mid (n+1)\cdot a, b) \cdot F^B\left(\frac{\Delta}{\Delta + \upsilon} \mid n \cdot a, a\right). \tag{15}$$

To the best of our knowledge, Eq. (15) is the first tractable analytical expression of the undershoot p.d.f. for the case of gamma distributed demand. Beta distribution is available in common mathematical software, including spreadsheets. Hence, expression (15) can be conveniently calculated. To set the upper limit of the summation ($n_{\max}$) it is possible to define the condition that the probability of not having crossed the reorder point at the final review epoch $n_{\max}$ (equal to $F_{n_{max}R}^G(\Delta)$) has to be as low as needed to achieve the desired accuracy ($< \varepsilon$).

$$Pr(not\ having\ crossed\ s\ at\ review\ epoch\ n_{max}) = F_{n_{max}R}^G(\Delta) < \varepsilon.$$

Through some basic calculations, an essential property of the undershoot p.d.f. with gamma distributed demand may be derived from (15). We can express it in terms of a nondimensional undershoot p.d.f. ($f_u^{(\Delta/\mu_R)}$, NUPDF) with μ_R acting as a scale factor. The NUPDF is a function of a nondimensional variable υ/μ_R and is characterized by two nondimensional parameters: Δ/μ_R and CV_R. The relationship between the undershoot p.d.f. and the NUPDF verifies Eq. (16) (see Appendix B.2).

$$f_u^{(\Delta)}\left(\upsilon \mid a = 1/{CV_R}^2, b = \mu_R \cdot {CV_R}^2\right) = \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R} \mid 1/{CV_R}^2, {CV_R}^2\right)$$
$$\mu_u^{(\Delta)} = \mu_R \cdot \mu_u^{(\Delta/\mu_R)},\ \sigma_u^{(\Delta)} = \mu_R \cdot \sigma_u^{(\Delta/\mu_R)} \tag{16}$$

Thus, it is worth highlighting that by studying the nondimensional undershoot p.d.f. we will obtain results of general validity. For any combination of (Δ, μ_R,σ_R) it will be possible to find the values of the corresponding specific undershoot p.d.f. by just applying the scale parameter μ_R to the nondimensional distribution (Δ/μ_R, CV_R).

With the aid of Matlab® software, used throughout this work, we analyze the NUPDF for a representative number of values of CV_R. Fig. 3a shows the case CV_R =0.1 as function of the variable υ/μ_R ($0 \le \upsilon/\mu_R \le 1.5$) and the parameter Δ/μ_R ($0 \le \Delta/\mu_R \le 10$). Fig. 3b and c represent the NUPFD for Δ/μ_R ={0.5, 1.5,..., 9.5} and Δ/μ_R ={1, 2,...,10} respectively, corresponding to the 3D lines highlighted in Fig. 3a. Remarkably, NUPDF presents an oscillatory behavior with the values of Δ/μ_R, alternating between unimodal (see Fig. 3b) and bimodal (see Fig. 3c) shapes (with a continuum in between (Fig. 3a)).

It becomes evident from Fig. 3b and c that approximating the undershoot distribution by the asymptotic distribution will be highly inaccurate for a considerable range of values of Δ/μ_R. On the other hand, from the managerial perspective and for low values of CV_R, it is advisable to avoid close to integer values of Δ/μ_R since the order quantities will concentrate on two separated values (bimodal distribution). From the day-to-day managerial point of view, it is preferable to operate a system in which the order quantities are as homogeneous as possible.

Fig. 4 provides insights into the oscillatory behavior of NUPDF as Δ/μ_R increases. In general, in a (R, s, S) model the limit case of Δ=0 (s=S) represents the well-known periodic review (R, S) system [25,41,42]. If the demand follows a non-negative distribution, there will be an order in each review epoch equal to the demand of the last review period. Hence, the undershoot p.d.f. will be identical to the demand distribution ($f_u^{(\Delta=0)}$=f_R^G). Regarding the NUPDF, as Δ/μ_R begins to increase, there is a translation of the NUPDF equal to the variation of Δ/μ_R. For the case CV_R=0.1, the demand is mostly concentrated around the mean value, and the probability that the demand during a review period is lower than half the mean μ_R is neglectable

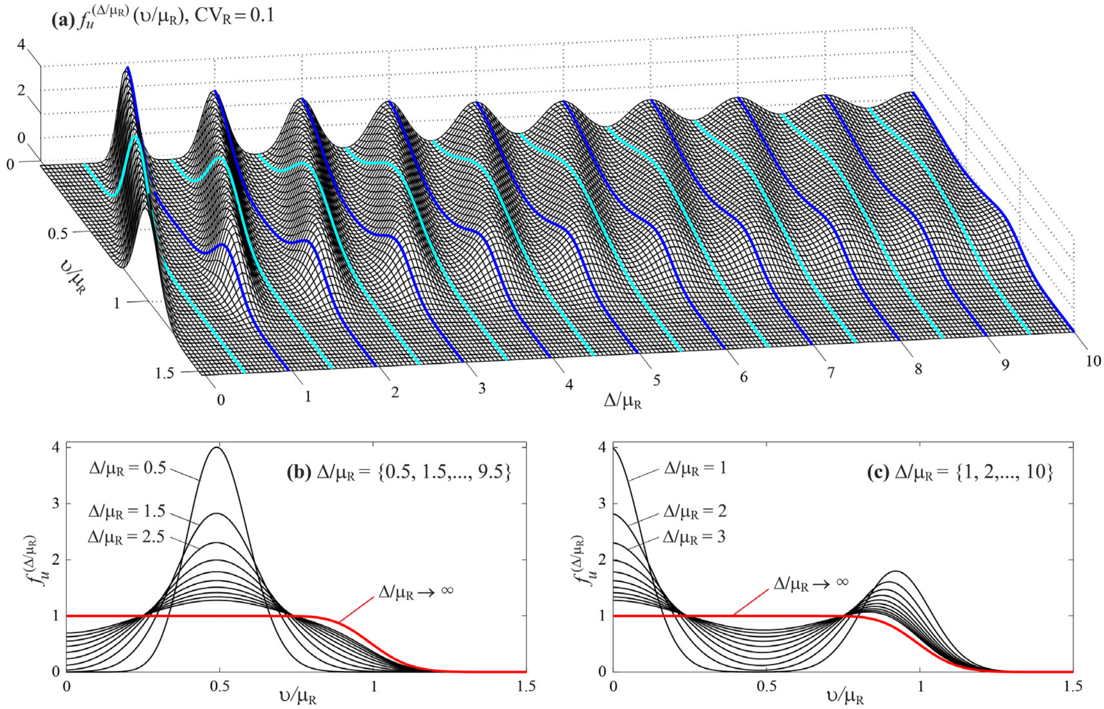

Fig. 3. NUPDF with CV_R =0.1 (a) $0 \le \Delta/\mu_R \le 10$, (b) Δ/μ_R ={0.5, 1.5,..., 9.5} (c) Δ/μ_R ={1, 2,...,10}.

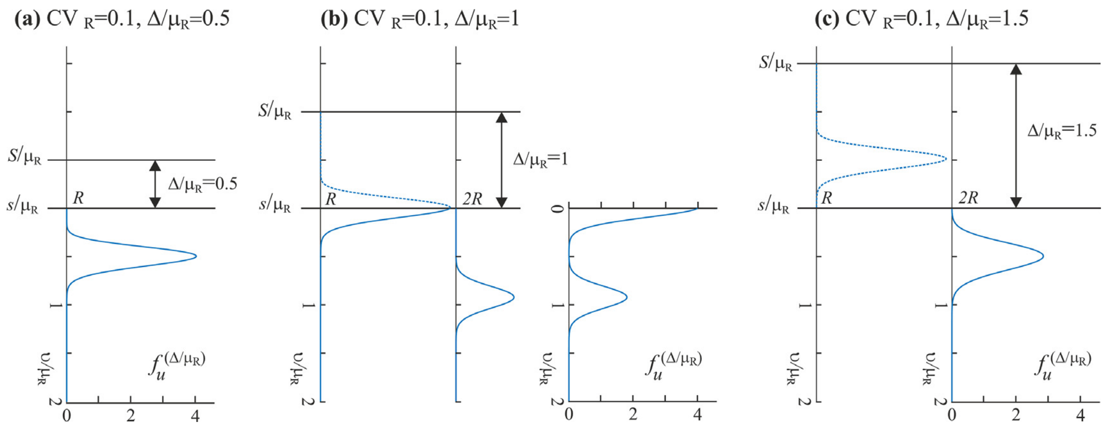

Fig. 4. Oscillatory behavior of the NUPDF.

($F_R^G(0.5 \cdot \mu_R) < 10^{-9}$). This implies that if we consider the case of Δ/μ_R=0.5, all the possible trajectories of the inventory position as conceptualized in Fig. 2 will cross the reorder level in the first review period (not crossing would imply a demand lower than half μ_R). This fact is represented schematically in Fig. 4a. The vertical axis represents the inventory position at review epoch R. The line below the nondimensional reorder level (s/μ_R) shows the distribution of the end of the trajectories of the inventory position (the lower horizontal axis includes the scale of the distribution). In this particular case, the NUPDF will have the same shape as the demand p.d.f. with a translation equal to Δ/μ_R, i.e., $f_u^{(\Delta/\mu_R=0.5)}(^{\upsilon}/_{\mu_R}) = f_R^G(^{\upsilon}/_{\mu_R} + 0.5)$.

If Δ/μ_R increases, the probability of not crossing the reorder level at the first review period also rises. Fig. 4b represents the distribution of the inventory position trajectories at the end of the first and second review periods for the case Δ/μ_R=1. The dotted line represents the distribution of trajectories that end above the reorder level at R and will cross the reorder level in the second review period. The NUPDF will be the summation of the values of the continuous lines at R and $2R$, leading to a bimodal distribution. Finally, Fig. 4c represents the case Δ/μ_R=1.5. Now, all the trajectories cross the reorder

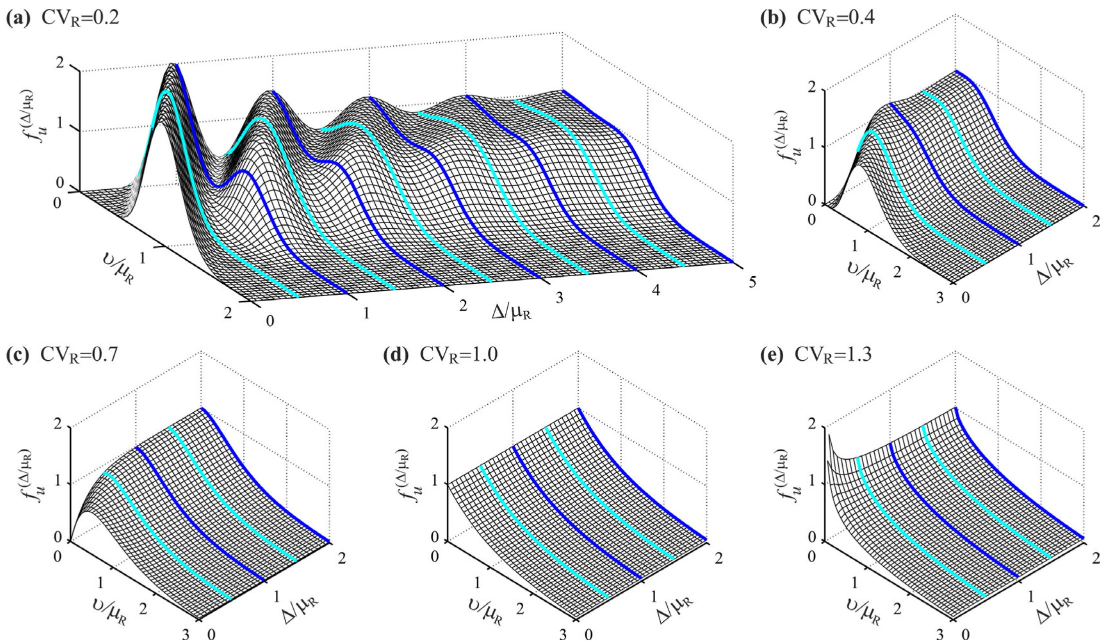

Fig. 5. Nondimensional undershoot p.d.f. with gamma distributed demand (a) CV_R =0.2, (b) CV_R =0.4, (c) CV_R =0.7, (d) CV_R =1.0, (e) CV_R =1.3.

level in the second review period ($F_R^G(1.5 \cdot \mu_R) \approx 1$ and $F_{2R}^G(1.5 \cdot \mu_R) \approx 0$). The NUPDF is unimodal again, but the variance increases. This is expected since now the shape of NUPDF is equal to the shape of the distribution during two review periods and, therefore, the variance doubles the variance of the demand during one review period. Thus, we can observe the origin of the alternation between unimodal and bimodal shapes, as well as how the oscillation damps out due to the increase of the demand variance. As Δ/μ_R increases, the oscillation decays until finally converging to the asymptotic distribution, which is very similar to a uniform distribution with a smooth tale (see Fig. 3).

Coherently, the damping of the oscillation also strengthens with CV_R as shown in Fig. 5. For $CV_R = 0.2$ there is still a significant oscillation (Fig. 5a), which decays progressively as CV_R rises (Fig. 5b); it becomes almost imperceptible for $CV_R = 0.7$ (Fig. 5c) until completely vanishing for $CV_R = 1.0$ (Fig. 5d). The shape of the distribution changes in the latter case: NUPDF does not vary with the value of Δ/μ_R, which is a result to be expected since for $CV_R = 1.0$ the gamma distribution is in fact an exponential distribution and, as mentioned in the introduction, it is known that the undershoot p.d.f. will be that same exponential distribution (Karlin, 1958). For CV_R >1.0, NUPDF is a monotonically decreasing function for all values of Δ/μ_R.

3.2. NUPDF mean and standard deviation

Through expressions (11), we can analyze the behavior of NUPDF mean and standard deviation. Fig. 6 plots their evolution with Δ/μ_R for $CV_R = \{0.1, 0.2, \ldots, 1.5\}$. As aforementioned, if Δ=0 then $f_u^{(0)}$=f_R^G. Correspondingly, all the lines in Fig. 6a start from the value $\mu_u^{(\Delta/\mu_R)}$=1 ($\mu_u^{(0)}$=μ_R in general) and the lines in Fig. 6b start from $\sigma_u^{(\Delta/\mu_R)}$=$CV_R$ ($\sigma_u^{(0)}$ =σ_R in general). For CV_R=1 the nondimensional gamma distribution (a=0, b=1) is an exponential with mean and standard deviation equal to 1 for all values of Δ/μ_R. The asymptotic formulae from Eq. (12) lead to expressions in which for CV_R<1 both $\mu_u^{(\infty/\mu_R)} < 1$ and $\sigma_u^{(\infty/\mu_R)} < 1$ whereas for CV_R>1 both asymptotic values are also higher than one. As expected in view of the shape of the density function, undershoot mean shows a strong oscillatory behavior with Δ/μ_R for low values of CV_R; oscillation decays progressively as CV_R rises. Oscillation is also present in the undershoot deviation. In the only precedent of this analysis, Baganha, Pyke, and Ferrer [28] include an approximate representation of the evolution of the undershoot mean with Δ for two selected cases of Erlang distributed demand (a=3, b=1, CV_R=$\sqrt{3}/3$; a=9, b=27, CV_R=1/3).

3.3. Practical application

Nondimensionalization allows providing extensively applicable tables of the undershoot mean and standard deviation. Tables for $CV_R = \{0.1, 0.2, \ldots, 1.5\}$ and a wide range of values of Δ/μ_R are included in Appendix D[1]. We show the practical application of the results presented so far through an illustrative case composed of four linked set of calculations.

[1] Appendix D is provided as supplementary material.

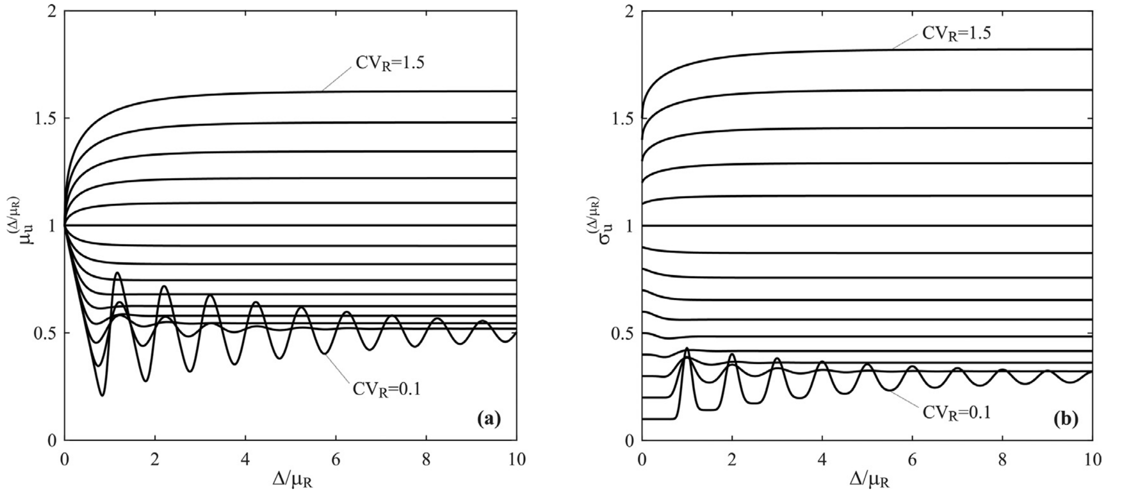

Fig. 6. Gamma distributed demand: NUPDF mean (a) and deviation (b) for $0 \leq \Delta/\mu_R \leq 10$ and $CV_R = \{0.1, 0.2,..., 1.5\}$.

*Numerical illustration 1***.** Consider a company using a (*R, s, S*) inventory system to manage the replenishment of a product with gamma distributed demand. The system is defined by the following set of parameter values: *R*=1 week, *s*=40 units, *S*=100 units; the values of the mean and standard deviation of the weekly demand are also known: μ_R=30; σ_R=3. The company wants to know the expected average order size as well as the standard deviation from this value.

First, we analyze the nondimensional problem and then we scale the results:

- $CV_R = \frac{\sigma_R}{\mu_R} = 0.1$; and $\frac{\Delta}{\mu_R} = \frac{S-s}{\mu_R} = \frac{100-40}{30} = 2.0$
- From Table D.1 of supplementary material, alternatively with the aid of mathematical software, we obtain the nondimensional value of the undershoot mean:

$$\mu_u^{(\Delta/\mu_R)} = 0.50940$$

- The undershoot mean for this case will thus be:

$$\mu_u^{(\Delta)} = \mu_R \cdot \mu_u^{(\Delta/\mu_R)} = 30 \cdot 0.50940 = 15.282$$

- The undershoot standard deviation can be obtained in a similar manner from Table D.2:

$$\sigma_u^{(\Delta)} = \mu_R \cdot \sigma_u^{(\Delta/\mu_R)} = 30 \cdot 0.40276 = 12.083$$

- Finally, the expected average order size and its standard deviation will be:

$$\mu_Q^{(\Delta)} = \Delta + \mu_u^{(\Delta)} = 60 + 15.282 = 75.282$$

$$\sigma_Q^{(\Delta)} = \sigma_u^{(\Delta)} = 12.083$$

*Numerical illustration 2***.** The representation of Fig. 6 is also very practical for inventory managers when dealing with systems with low values of CV_R as it immediately illustrates the impact of a small increase or decrease of the value for Δ/μ_R on the average and variability of the order size. For the numerical example, we can see in Fig. 6b that it might be advisable to change the value of Δ/μ_R in order to reduce significantly the standard deviation of both the undershoot and the order quantity. In Fig. 6a we can see that lowering Δ/μ_R will decrease the average order quantity whereas the opposite happens if Δ/μ_R increases. When Δ/μ is reduced to 1.7 for instance, then $\Delta = 51$ such that $\mu_u^{(\Delta)} = 30 \cdot 0.31342 = 9.403$, $\sigma_u^{(\Delta)} = 30 \cdot 0.15540 = 4.662$, $\mu_Q^{(\Delta)} = 60.403$ and $\sigma_Q^{(\Delta)} = \sigma_u^{(\Delta)}$. It will be necessary to take the relevant costs in consideration before deciding to change the policy, but the example shows how the plots of Fig. 6 can guide a marked reduction (>60%) in the standard deviation of the undershoot and the order quantity with a 20% reduction in the order size.

*Numerical illustration 3***.** Having the order quantity p.d.f. also allows to calculate the probability that the order quantity is lower than or exceeds a predetermined value as well as probability intervals for Q_k. As stressed in the previous subsection, for the case of the numerical illustration (CV_R=0.1) the shape of the order quantity p.d.f. alternates between unimodal and bimodal shapes. Symmetric probability intervals show the impact of this fluctuation. We calculate with Matlab® representative examples of p-probability intervals of the NUPDF (using the inverse $F_u^{(\Delta/\mu_R)}$ (Eq. B.1)) for the initial Δ=60 (Δ/μ_R =2.0) and the modified value Δ=51 (Δ/μ_R =1.7).

- 50% and 80% symmetric probability interval for υ_k with CV_R=0.1 and Δ=60:

$$p = 50\% : \left[F_u^{(\Delta)^{-1}}(0.25),\ F_u^{(\Delta)^{-1}}(0.75)\right] = [2.935,\ 26.864],$$

$$p = 80\% : \left[F_u^{(\Delta)^{-1}}(0.1),\ F_u^{(\Delta)^{-1}}(0.9)\right] = [1.085,\ 29.967].$$

- Then, we add Δ to obtain the desired result for Q_k (Eq. (1)).

$$p = 50\% : [\Delta + 2.935,\ \Delta + 26.864] = [62.935,\ 86.864] \Rightarrow Amplitude = 23.929,$$

$$p = 80\% : [\Delta + 1.085,\ \Delta + 29.967] = [61.085,\ 89.967] \Rightarrow Amplitude = 28.882.$$

- Likewise for Δ=51:

$$p = 50\% : \left[\Delta + F_u^{(\Delta)^{-1}}(0.25),\ \Delta + F_u^{(\Delta)^{-1}}(0.75)\right] = [57.258,\ 62.992\ ] \Rightarrow Amplitude = 5.734,$$

$$p = 80\% : \left[\Delta + F_u^{(\Delta)^{-1}}(0.1),\ \Delta + F_u^{(\Delta)^{-1}}(0.9)\right] = [54.923,\ 65.860] \Rightarrow Amplitude = 10.937.$$

We can see the relevance of the change in the p.d.f. shape. For the first case ($\Delta/\mu_R = 2.0$), due to the bimodality (Fig. 3c), it is very unlikely for an order quantity to lie close to the mean, thus making it necessary to extend the probability intervals significantly. For the second case ($\Delta/\mu_R = 1.7$), the intervals are markedly narrowed, which, for managerial purposes, may yield a more desirable behavior. It is very difficult to quantify in terms of cost a penalty for the variability of the order quantity, and consequently, it is not considered in the standard cost functions [43]. Yet, stability in the orders is very valuable for practitioners since it facilitates the day-to-day operations as well as the relationships with the suppliers.

*Numerical illustration 4***.** Finally, let us consider that the lead-time of the supplier follows a continuous uniform distribution in the interval (0, R). This implies that once the order is placed, it will arrive before the next review epoch. Therefore, at each review epoch there will be no pending orders and the inventory position will be the same as the inventory on hand. With the undershoot p.d.f. we can directly calculate the cycle service level (CSL), which is defined as the “fraction of cycles in which a stockout does not occur” [42]. For a stockout to occur it will be necessary for the undershoot to be higher than the reorder level. Hence, it is verified that

$$CSL = 1 - F_u^{(\Delta)}(s).$$

- For the initial case (Δ/μ_R =2.0) Δ=60 the CSL will be:

$$CSL = 1 - F_u^{(\Delta)}(40) = 99,989\%$$

- For the second case (Δ/μ_R =1.7) Δ=51 the CSL will be:

$$CSL = 1 - F_u^{(\Delta)}(40) = 99,999\%$$

It is important to note that, even without changing the value of Δ/μ_R, if the reorder level is lowered then the CSL will be relevantly affected. For example, if $s = 20$ units, all the calculations of the previous three numerical illustrations remain the same whereas the CSL changes significantly. This time, the change in the value of Δ/μ_R has a drastic positive impact in the CSL:

- For the initial case (Δ/μ_R =2.0) Δ=60 the CSL will be:

$$CSL = 1 - F_u^{(\Delta)}(20) = 51,436\%$$

- For the second case (Δ/μ_R =1.7) Δ=51 the CSL will be:

$$CSL = 1 - F_u^{(\Delta)}(20) = 97,992\%$$

3.4. Validity of the asymptotic approximation

As underlined in the introduction, one recurrent question in the literature regarding the undershoot distribution is the practical validity of using the asymptotic approximation. Baganha, Pyke and Ferrer [28] provide generic indications for several discrete distributions: the approximation can lead to high deviations for low coefficients of variation and low values of Δ/μ_R. Although the indications align qualitatively with other results from the literature, there has been a controversy regarding the quantification of the range of validity of the approximation. Even more, there is a lack of a quantitative study for continuous distributions since the algorithm used in [28] is only valid for non-negative discrete distributions. As stressed

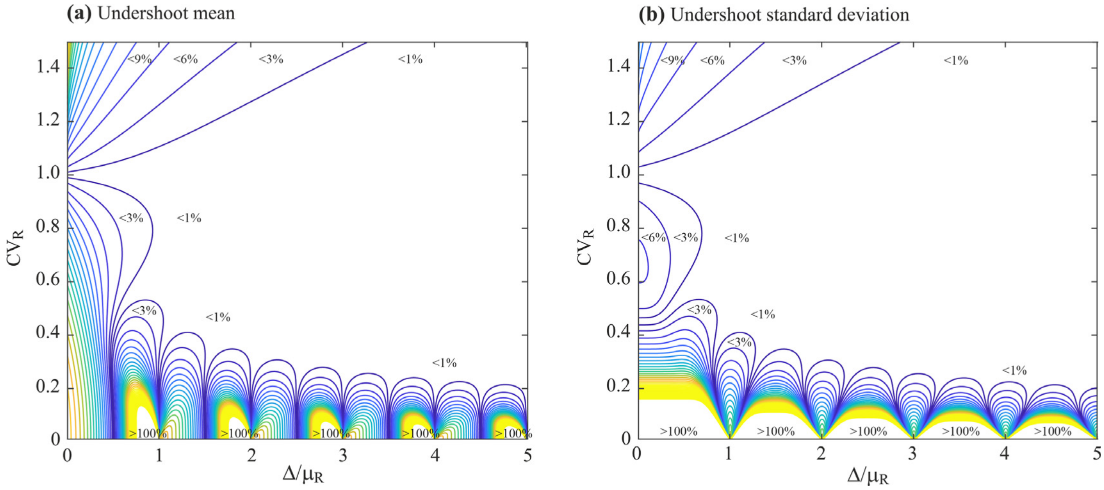

Fig. 7. Gamma distributed demand: Absolute percentage error of the asymptotic approximation, $0 \leq \Delta/\mu_R \leq 5$ and $0 < CV_R \leq 1.5$.

by Bijvank [3], many authors refer to a validity of the asymptotic approximation for $\Delta/\mu_R > 1.5$; others just point out that order quantity should be sufficiently large without a quantification [24,25,44]. With the exact values of NUPDF mean and standard deviation obtained it is possible to perform a thorough analysis of the accuracy of the asymptotic approximation and, taking advantage of the nondimensionalization, provide values of the absolute percentage errors (APE) applicable to any gamma distributed demand case.

$$APE_\mu = \frac{\left|\mu_u^{(\Delta)} - \mu_u^{(\infty)}\right|}{\mu_u^{(\Delta)}} \cdot 100 = \frac{\left|\mu_u^{(\Delta/\mu_R)} - \mu_u^{(\infty/\mu_R)}\right|}{\mu_u^{(\Delta/\mu_R)}} \cdot 100$$
$$APE_\sigma = \frac{\left|\sigma_u^{(\Delta)} - \sigma_u^{(\infty)}\right|}{\sigma_u^{(\Delta)}} \cdot 100 = \frac{\left|\sigma_u^{(\Delta/\mu_R)} - \sigma_u^{(\infty/\mu_R)}\right|}{\sigma_u^{(\Delta/\mu_R)}} \cdot 100 \quad (17)$$

Resulting absolute percentage errors are represented in the contour plots of Fig. 7. The most internal line delimits the area in which the approximation deviates less than 1% from the true value and is therefore highly accurate. The rest of the lines correspond to multiples of 3% of APE ending in 99%. The last line defines areas with higher than 100% deviation.

Baganha, Pyke and Ferrer [28] draw attention to the fact that low true values of undershoot mean and standard deviation lead to very high percentage errors, whereas in some cases its impact in terms of deviation in units might be of much less practical importance. Absolute percentage error is the standard measure to provide information about the quality of an approximation; e.g., a deviation of 1 unit when the true value is 4 units (25% error) will be in general clearly better than when the true value is only 2 units (50% error). Nevertheless, in the case of the undershoot and from the inventory management perspective, typically the extra cost will be directly related to the value of the absolute deviation and not to the percentage error. Besides, the relevance of the deviations will depend directly on their magnitude related to the demand of a review period (e.g., 5 units of undershoot will be highly relevant if demand over a review period is 20 units, and totally neglectable if demand is 10.000 units). Thus, we propose the absolute percentage nondimensional deviation (APND) of Eqs. (18) as a complementary measure to assess the accuracy of the asymptotic approximation.

$$APND_\mu = \frac{\left|\mu_u^{(\Delta)} - \mu_u^{(\infty)}\right|}{\mu_R} \cdot 100 = \left|\mu_u^{(\Delta/\mu_R)} - \mu_u^{(\infty/\mu_R)}\right| \cdot 100$$
$$APND_\sigma = \frac{\left|\sigma_u^{(\Delta)} - \sigma_u^{(\infty)}\right|}{\mu_R} \cdot 100 = \left|\sigma_u^{(\Delta/\mu_R)} - \sigma_u^{(\infty/\mu_R)}\right| \cdot 100 \quad (18)$$

*Numerical illustration***.** Let us consider the same base case of 3.3, with $CV_R = 0.1$ and the two variations considered to reduce the order quantity variability: $\Delta/\mu_R = 1.7$ ($\Delta = 51$) and $\Delta/\mu_R = 2.3$ ($\Delta = 69$). Both cases will result in very different absolute percentage errors, whereas having similar absolute deviations in units (and therefore of much more analogous practical impact as the difference in APE suggests).

- $APE_{\mu 1} = \frac{|0.31342 - 0.50500|}{0.31342} \cdot 100 = 61.13\%$; $APE_{\mu 2} = \frac{|0.67967 - 0.50500|}{0.67967} \cdot 100 = 25.70\%$
- $$APND_{\mu 1} = |0.31342 - 0.50500| \cdot 100 = 19.16\% \Rightarrow 5.75\ units$$
  $$APND_{\mu 2} = |0.71819 - 0.50500| \cdot 100 = 17.47\% \Rightarrow 5.24\ units$$

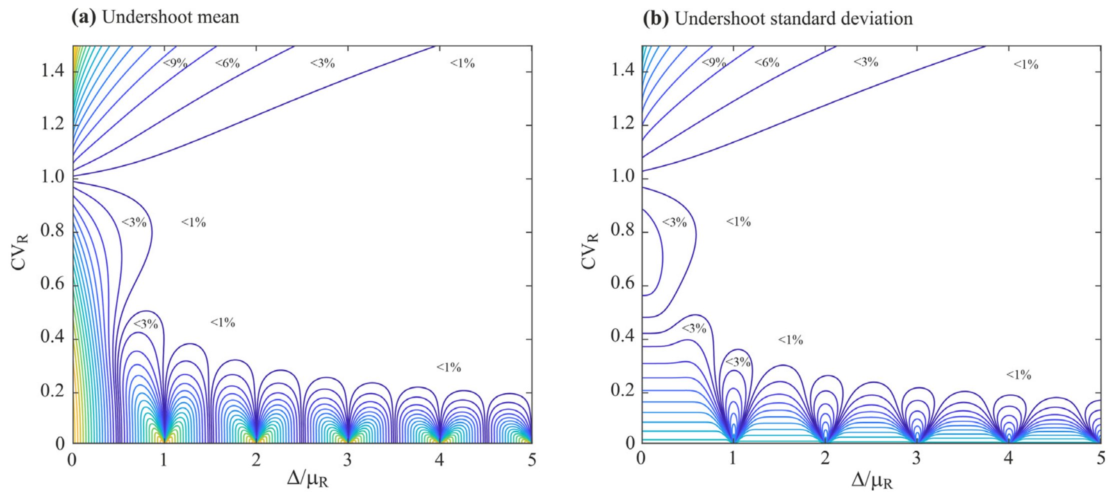

Fig. 8. Gamma distributed demand: Absolute percentage nondimensional deviation of the asymptotic approximation $0 \le \Delta/\mu_R \le 5$ and $0 < CV_R \le 1.5$.

From the managerial viewpoint, a deviation in terms of percentage of the average demand provides a direct insight of the relevance of the undershoot. Fig. 8 shows the resulting plots of the proposed deviation measure following the same pattern of Fig. 7. Comparing Figs. 7a and 8a, in the latter we can observe the symmetry of the overestimations and underestimations close to the integer values of Δ/μ_R and low CV_R. On the other hand, in Fig. 8b there is a homogeneity in the progression of the deviation as CV_R decreases, and the deviations for low CV_R are relevant but not so marked as shown in Fig. 7b.

4. Normal distribution

Although some authors criticize the use of the normal distribution to model customer demand due to the probability of negative values, it is by far the most commonly used by practitioners [20,41,42,45,46]. Its use is standard practice in operations management, and it is present in all commercial inventory management software packages. On the other hand, with the spread of electronic commerce in the retail sector, the importance of product returns is gaining importance [47], and returns can be interpreted as negative demand [9].

For the case of normal distributed demand, undershoot p.d.f. is defined by the generic expression (5), with mean and standard deviation given by (6). From now on, superscript N will be used to denote the normal distribution.

$$f_u^{(\Delta)}(\upsilon) = f_R^N(\Delta+\upsilon) + \sum_{n=1}^{\infty}\int_{-\infty}^{\Delta} g_{nR}(y_n)\cdot f_R^N(\Delta+\upsilon-y_n)dy_n. \tag{19}$$

4.1. Undershoot p.d.f. computation

The fact that demand may be negative in some periods introduces high complexity in the computation of $g_{nR}(x)$ and, consequently, of $f_u^{(\Delta)}(\upsilon)$. For the case of normal distributed demand, $g_{nR}(x)$ does not equate to $f_{nR}(x)$ as it did in the case of gamma distributed demand. This problem is similar to the one studied by Miltenburg and Silver [30] in the context of coordinated replenishments; in this work the authors conclude that there is no solution for the convolution integrals equivalent to the ones in (19) and adopt a tractable simplification for which they provide valuable insights. With the aid of current mathematical software, it is possible to find an accurate estimation valid for a wide range of values of CV_R and therefore to quantify the exactitude of the practical simplifications. This analysis is the first attempt to estimate the effect of probability of negative values in the undershoot p.d.f.

As shown in Appendix C.2, it is possible to approximate g_{nR} by demand during n consecutive review intervals since the last ordering without reaching the reorder level in the j preceding review epochs ($0 \le j < n$ -1). The exactitude of the approximation rises with the value of j. For j=0, the approximation consists in taking $g_{nR}(x) \cong f_{nR}^N(x)$ (which is the simplification adopted in [30]). For $j \ge n$ -1, $g_{nR}(x)$ will take its true value.

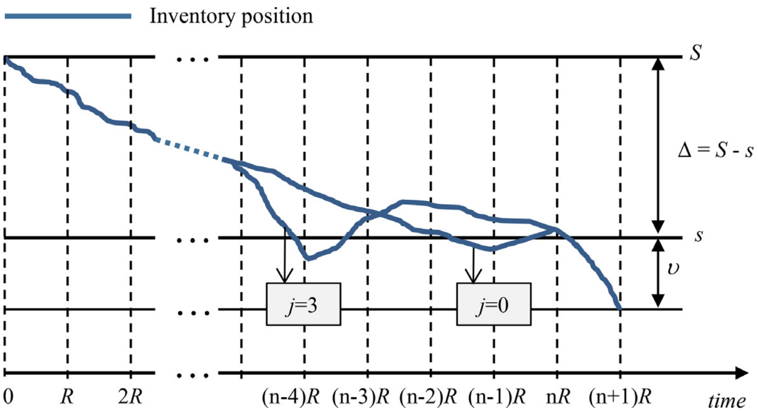

Fig. 9. Illustrative examples of inventory position paths for $j = 0$ and $j = 3$.

The procedure is as follows:

- for $n \leq j + 1$, true value of $g_{nR}(x)$ is directly calculated according to (4), which leads to

$$\begin{aligned} & g_R(y_1) = f_R^N(y_1), \\ & g_{2R}(y_2) = f_{2R}^N(y_2) \cdot F^N(\Delta \mid \tfrac{1}{2}y_2, \tfrac{1}{2}\sigma^2), \\ & g_{nR}(y_n) = \\ & f_{nR}^N(y_n) \cdot \textstyle\int_{-\infty}^{\Delta} \int_{-\infty}^{\Delta} \ldots \int_{-\infty}^{\Delta} \left[\prod_{i=3}^{n} f^N(y_{i-1} \mid \tfrac{i-1}{i}y_i, \tfrac{i-1}{i}\sigma^2) \right] \cdot F^N(\Delta \mid \tfrac{1}{2}y_2, \tfrac{1}{2}\sigma^2) dy_{n-1} \ldots dy_3 dy_2, \ n > 2; \end{aligned} \tag{20}$$

- for $n > j + 1$, the approximation $g_{(n-j)R}(x) \cong f_{(n-j)R}^N(x)$ is adopted, which, through (4), leads to

$$\begin{aligned} & g_{nR}(y_n) \cong f_{nR}^N(y_n), \ j = 0, \\ & g_{nR}(y_n) \cong f_{nR}^N(y_n) \cdot F^N(\Delta \mid \tfrac{n-1}{n}y_n, \tfrac{n-1}{n}\sigma^2), \ j = 1, \\ & g_{nR}(y_n) \cong f_{nR}^N(y_n) \cdot \textstyle\int_{-\infty}^{\Delta} \int_{-\infty}^{\Delta} \ldots \int_{-\infty}^{\Delta} \left[\prod_{i=n-j+2}^{n} f^N(y_{i-1} \mid \tfrac{i-1}{i}y_i, \tfrac{i-1}{i}\sigma^2) \right] \cdot F^N\Big(\Delta \mid \tfrac{n-j}{n-j+1} \\ & y_{n-j+1}, \tfrac{n-j}{n-j+1}\sigma^2\Big) dy_{n-1} \ldots dy_{n-j+2} dy_{n-j+1}, \ j > 1. \end{aligned} \tag{21}$$

Fig. 9 shows two selected illustrative trajectories of inventory position that should not be included in the calculation of the probability of the undershoot yet they are included respectively when taking j=0 and j=3. In the trajectory with the label j=0, which would not be included for j>0, the inventory position crosses the reorder level at review epoch n-1 (at review epoch n-4 for the trajectory with label j=3). The limits of the convolution integral in Eq. (19) force that in the review epoch (n) immediately prior to the undershoot occurrence (review epoch $n+1$) the inventory position will be above the reorder level. Thus, a value of j implies that the inventory position has not crossed the reorder level in the $j+1$ preceding review epochs. Incrementing the value of j will result in a better approximation of $f_u^{(\Delta)}(\upsilon)$, but, in parallel, the order of integration alongside with the complexity of the calculations will increase. As we will detail in Section 4.4, on the one hand it is possible to find a tractable value of j that yields an extremely accurate approximation for a representative range of values and on the other it is possible to determine the error made when taking a low value of j.

4.2. Nondimensional undershoot p.d.f

As in the case of gamma distributed demand, we can define a NUPDF ($f_u^{(\Delta/\mu_R)}$) for the case of normal distributed demand that satisfies equations equivalent to (16). Any normal distribution verifies $f^N(x \mid \mu, \sigma^2) = 1/\mu \cdot f^N(x/\mu \mid 1, (\sigma/\mu)^2)$; applying this property to Eqs. (4) and (19) yields results that are identical to the case of gamma distributed demand (see Appendix C.3 for details). Again, this fact is highly relevant since it allows focusing on NUPDF and analyzing its behavior with the two nondimensional parameters without any loss of generality.

We use the proposed approximation of $g_{nR}(x)$ to proceed in the same manner as in the previous section and study the NUPDF. For low values of CV_R, the normal and gamma p.d.f. are very alike and, consequently, so are their NUPDFs. In particular, we do not include the study of the case $CV_R = 0.1$ due to its great similarity with Fig. 3. The results of the analysis

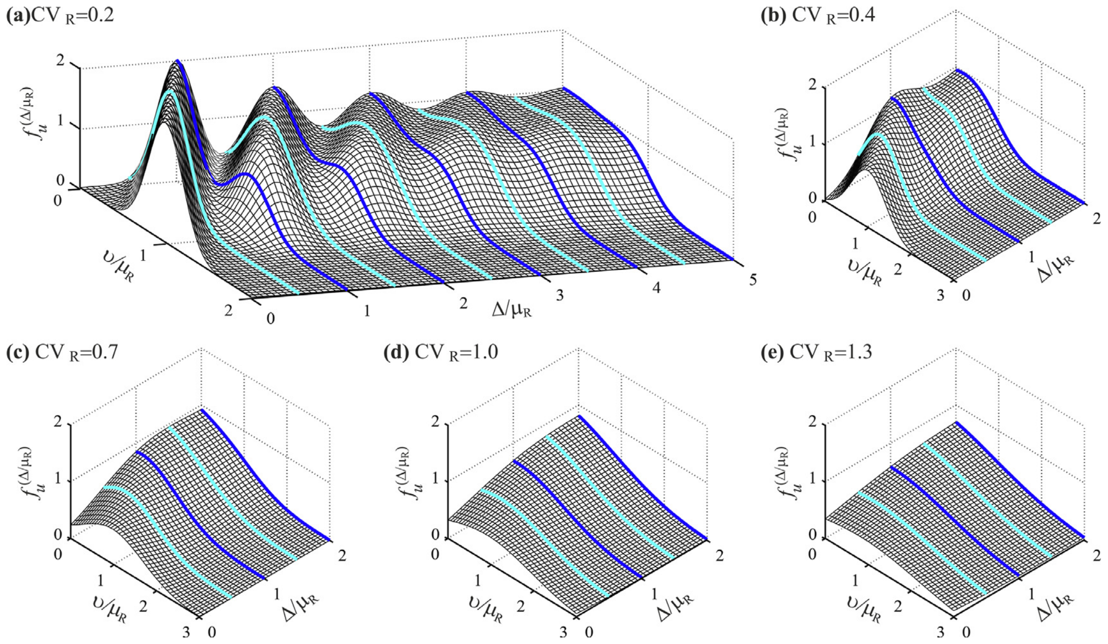

Fig. 10. Nondimensional undershoot p.d.f. with normal distributed demand (a) $CV_R = 0.2$, (b) $CV_R = 0.4$, (c) $CV_R = 0.7$, (d) $CV_R = 1.0$, (e) $CV_R = 1.3$.

taking $j = 4$ are summarized in Fig. 10, which parallels Fig. 5 and shows the NUPDF as function of υ/μ_R and the parameter Δ/μ_R for five representative values of CV_R. For $CV_R = 0.2$ (Fig. 10a), there are only slight differences between gamma and normal distributed cases. Distinctions begin to be identifiable for $CV_R = 0.4$ (Fig. 10b) and become progressively significant as CV_R increases (Fig. 10c–e). Compared to Fig. 5, although in Fig. 10 it may also be observed how the damping rate of the oscillation increases with CV_R, there is a substantial variation in the evolution of the shape of NUPDF for CV_R>0.4. The use of normal distribution to model demand behavior for high values of CV_R is not advised due to the high probability of negative values yet performing the same analysis as in the gamma distributed case allows to provide valuable insights about the behavior of the undershoot (and hence, the order quantity) distribution.

4.3. NUPDF mean and standard deviation

The differences and similarities between the NUPDF in the cases of normal and gamma distributed demands is reflected in the evolution of the NUPDF mean and standard deviation plotted in Fig. 11 compared to the evolution shown in Fig. 6. The values are calculated for $0 \le \Delta/\mu_R \le 10$ and CV_R={0.1, 0.2,..., 1.5} with expressions (6) (see Appendix C.4) and taking j=3 in the approximation of $g_{nR}(x)$. Coherently with the results of the previous subsection, there is a great parallelism between the gamma and normal distributed cases for low values of CV_R. As CV_R increases the divergence between cases becomes progressively higher, the shape of the lines differs noticeably although maintaining a similarity in the fast convergence to asymptotic values of both the mean and standard deviation. The asymptotic values of the NUPDF mean (Fig. 11a) are lower than the initial values ($\Delta/\mu_R = 0$). In this case, the initial values are not equal to 1 as they were in Fig. 6a., which is due to the probability of negative demands. The initial values of the NUPDF standard deviation (Fig. 11b) are not equal to CV_R as they were in Fig. 6b, and in general, the values of NUPDF standard deviation for high CV_R are much lower as compared to the gamma distributed case.

4.4. Assessment of $g_{nR}(x)$ *approximation*

The $g_{nR}(x)$ approximation opens the possibility to calculate highly accurate values of the undershoot pdf, its mean and standard deviation. Yet, computation times increase significantly with the value of j, since the order of integration rises accordingly. Transformations detailed in Appendix C.2 enable us to compute the values of NUPDF mean and standard deviation up to $j = 2$ very fast (tenths of seconds), $j = 3$ in short times (seconds) and $j = 4$ in affordable times (minutes) using one Intel® core i7 processor. Computation times depend on different aspects related to numerical integrations besides the defining parameters; they grow with CV_R in general, because the probability of crossing the reorder level prior to the last j+1 review epochs will be higher, and also with Δ/μ_R, as the average number of review periods until crossing the reorder level increases.

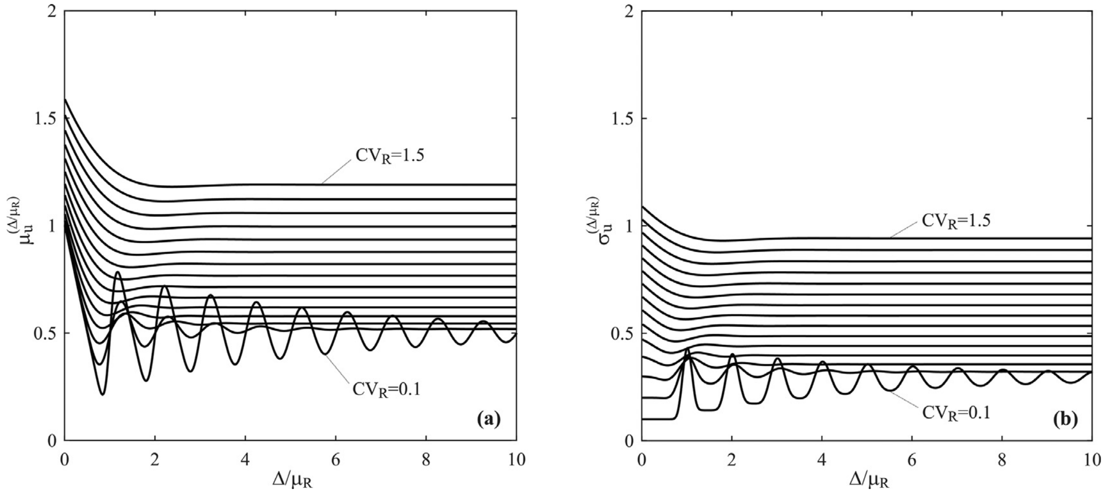

Fig. 11. Normal distributed demand: NUPDF mean (a) and deviation (b) for $0 \leq \Delta/\mu_R \leq 10$ and $CV_R = \{0.1, 0.2, \ldots, 1.5\}$.

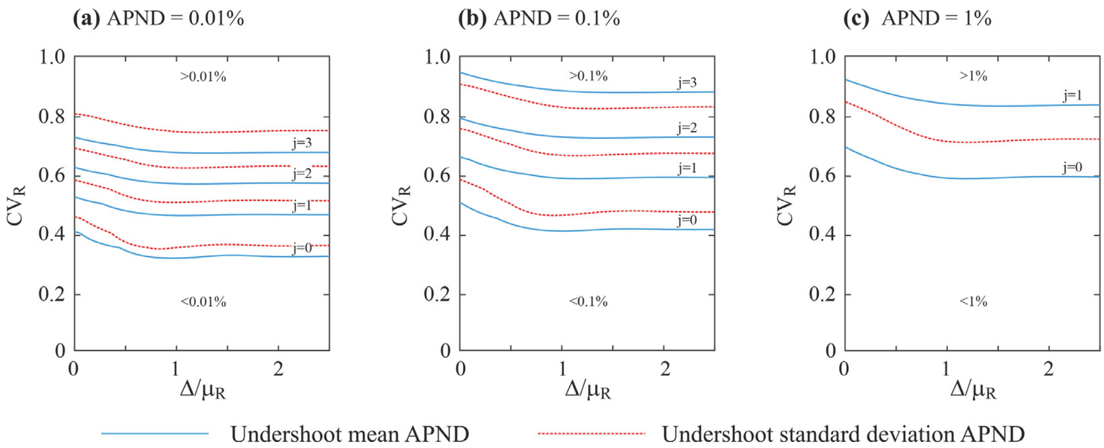

Fig. 12. Normal distributed demand: APND of the undershoot mean and standard deviation computed using j={0, 1, 2, 3} with respect to j=4 for $0 \leq \Delta/\mu_R \leq 2.5$ and $0 < CV_R \leq 1.0$.

The simplification j=0 (Miltenburg and Silver [30]), is assumed to yield good results as long as the probability of negative values is very low. But there has been no attempt to determine neither the range of validity nor the exactitude of this simplification. Therefore, it is very useful to assess not only the achievable accuracy of the $g_{nR}(x)$ approximation, but also the evolution of the accuracy as j increases as well as and the range of values of the nondimensional parameters for which each value of j provides accurate approximations. With this aim, we calculate the values of NUPDF mean and standard deviation for j={0, 1, 2, 3} and compare them with the corresponding results obtained for j=4. In order to establish the relevance of the value of j, the comparison is made using the absolute percentage nondimensional deviation (APND, see Section 4.3), which measures the absolute deviation as a percentage of μ_R. The calculation is similar to Eq. (18), but this time the values obtained for j={0, 1, 2, 3} are subtracted from the values obtained for j=4. Results are summarized in the three plots of Fig. 12. The lines in each plot represent, for each value of j, the border that delimits the areas in which the APND is either lower or higher than three representative thresholds: 0.01% (Fig. 12a), 0.1% (Fig. 12b) and 1% (Fig. 12c). The continuous lines correspond to the NUPDF mean, and the dotted lines to the standard deviation.

We observe that great level of exactitude is achieved, even for the lowest values of j, for a representative and practical range of CV_R. It is also noticeable that, for each value of j, the APND of the standard deviation is lower than the APND of the undershoot mean. On the other hand, in all the cases the threshold lines converge with a very slight fluctuation to an asymptotic value that is associated to a specific CV_R. Therefore, there is an interest in complementing the analysis with a detailed study of the evolution of the asymptotic values of the NUPDF mean and standard deviation as j increases. Table 2a and b show the asymptotic results (Δ/μ_R=100) for CV_R={0.1, 0.2,..., 1.0} and j={0, 1, 2, 3, 4}. Values obtained with

Table 2a
Evolution of the approximation to the asymptotic NUPDF mean with the parameter j

CV_R	0.1		0.2		0.3		0.4		0.5	
	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND
Renewal asymptotic	0.50500	0.000%	0.52000	0.000%	0.54500	0.003%	0.58000	0.082%	0.62500	0.451%
j=0	0.50500	0.000%	0.52000	0.000%	0.54500	0.003%	0.57990	0.072%	0.62428	0.379%
j=1	0.50500	0.000%	0.52000	0.000%	0.54497	0.000%	0.57920	0.002%	0.62070	0.021%
j=2	0.50500	0.000%	0.52000	0.000%	0.54497	0.000%	0.57918	0.000%	0.62051	0.002%
j=3	0.50500	0.000%	0.52000	0.000%	0.54497	0.000%	0.57918	0.000%	0.62049	0.000%
j=4	0.50500		0.52000		0.54497		0.57918		0.62049	
CV_R	0.6		0.7		0.8		0.9		1.00	
	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND	$\mu_u^{(\infty/\mu_R)}$	APND
Renewal asymptotic	0.68000	1.344%	0.74500	2.920%	0.82000	5.271%	0.90500	8.441%	1.00000	12.446%
j=0	0.67735	1.079%	0.73829	2.249%	0.80643	3.914%	0.88124	6.065%	0.96233	8.679%
j=1	0.66769	0.113%	0.71928	0.348%	0.77513	0.784%	0.83507	1.448%	0.89902	2.348%
j=2	0.66672	0.016%	0.71651	0.071%	0.76930	0.201%	0.82490	0.431%	0.88329	0.775%
j=3	0.66658	0.002%	0.71594	0.014%	0.76776	0.047%	0.82174	0.115%	0.87778	0.224%
j=4	0.66656		0.71580		0.76729		0.82059		0.87554	

Table 2b
Evolution of the approximation to the asymptotic NUPDF std. deviation with the parameter j

CV_R	0.1		0.2		0.3		0.4		0.5	
	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND
Renewal asymptotic	0,29717	0,000%	0,32083	0,000%	0,35540	0,001%	0,39615	0,025%	0,43899	0,175%
j=0	0,29717	0,000%	0,32083	0,000%	0,35542	0,001%	0,39666	0,026%	0,44214	0,140%
j=1	0,29717	0,000%	0,32083	0,000%	0,35541	0,000%	0,39640	0,000%	0,44082	0,008%
j=2	0,29717	0,000%	0,32083	0,000%	0,35541	0,000%	0,39640	0,000%	0,44075	0,001%
j=3	0,29717	0,000%	0,32083	0,000%	0,35541	0,000%	0,39640	0,000%	0,44074	0,000%
j=4	0,29717		0,32083		0,35541		0,39640		0,44074	
CV_R	0.6		0.7		0.8		0.9		1.00	
	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND	$\sigma_u^{(\infty/\mu_R)}$	APND
Renewal asymptotic	0,48056	0,634%	0,51798	1,614%	0,54857	3,350%	0,56948	6,113%	0,57735	10,235%
j=0	0,49100	0,410%	0,54295	0,883%	0,59795	1,588%	0,65609	2,548%	0,71752	3,782%
j=1	0,48731	0,041%	0,53544	0,132%	0,58510	0,303%	0,63632	0,571%	0,68918	0,948%
j=2	0,48696	0,006%	0,53439	0,027%	0,58284	0,077%	0,63228	0,167%	0,68276	0,306%
j=3	0,48691	0,001%	0,53418	0,006%	0,58225	0,018%	0,63105	0,044%	0,68058	0,088%
j=4	0,48690		0,53412		0,58207		0,63061		0,67970	

the renewal asymptotic expressions (8), which assume non-negativity in the distribution, are also included in the first row. For each value, the APND with respect to j=4 is shown in the adjacent column.

Results from Table 2a and b provide very good insight in how the accuracy of $g_{nR}(x)$ approximation improves as j increases, as well as of the benefits of each increment of j depending on the value of CV_R. For the lowest values of CV_R (0.1, 0.2), even the renewal formulas give the same value as when using $j = 4$. The first slight deviation from the true value appears for $CV_R = 0.3$ both for the renewal and $j = 0$ approximations, whereas $j = 3$ and $j = 4$ approximations begin to differ for $CV_R = 0.6$. We can infer that $j = 4$ gives an exact approximation (<0.001%) for $CV_R<0.6$ and very good approximations even for high values of CV_R. Additionally, estimations of both the mean and standard deviation diminish with the increase of j, whereas renewal formulas overestimate the undershoot mean (more than $j = 0$) but underestimate the standard deviation.

Implications of the analysis carried out in this subsection are highly relevant:

- Adopting the simplification j=0, which implies taking $g_{nR}(x) \cong f_{nR}^{N}(x)$, yields highly accurate results (<0.01% of μ_R) for $CV_R<0.3$ whereas good approximations (<1% of μ_R) are achieved for a practical range of $CV_R<0.6$. Although great level of accuracy was to be expected due to the low probability of negative values, results determine the impact of negative values of the normal distribution and provide a confirmation of the hypothesis of validity for $CV_R<0.3$ as well.
- Since calculations up to j=2 are very fast using mathematical software, it is possible to obtain highly accurate values for the most applicable range of CV_R with a very low computational effort.
- If some specific application requires accuracy for high values of CV_R, it is also possible to take j=4 and obtain high quality results in acceptable times.

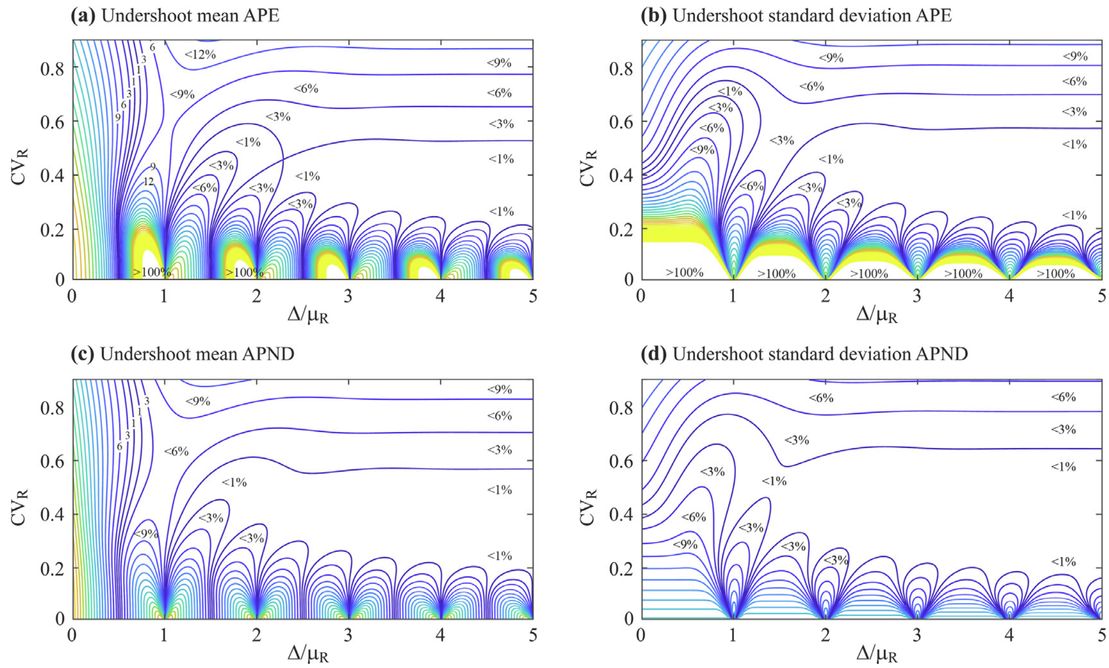

Fig. 13. Normal distributed demand: Absolute percentage nondimensional deviation of the renewal asymptotic approximation $0 \leq \Delta/\mu_R \leq 5$ and $0 < CV_R \leq 0.9$.

Finally, as in the case of the gamma distributed demand, we take advantage of the nondimensionalization to include in Appendix D[2] tables of the NUPDF mean (Table D.3) and standard deviation (Table D.4) of general validity. Exact values with five decimal places are calculated with $j = 4$ for CV_R={0.1, 0.2,..., 0.5}, which is a range of particular interest in inventory management.

4.5. Validity of the renewal asymptotic approximation

In the attempt to assess the validity of the renewal asymptotic approximation in normal distributed demand systems, Baganha, Pyke, and Ferrer [28] assume non-negativity and discretize the normal distribution to calculate the undershoot mean and standard deviation for a set of cases (9 for the mean and 7 for the standard deviation). Noteworthy, two of the cases correspond to the same non-dimensional cases. The results of the equivalent cases differ slightly due to the simplifications made in the calculations (non-negativity and discretization).

When using the renewal formulas in the normal distributed case, there is an additional source of error: as Table 2 shows, the renewal asymptotic values do not correspond to the true ones. Therefore, unlike in the case of gamma distributed demand, there is a bias when approximating the lines of Fig. 11 by constant value horizontal lines. The representation of both the APE (Fig. 13a and b) and APND (Fig. 13c and d) confirms that the error of the renewal asymptotic approximation is noticeably higher in this case. The area in which the APE or the APND is <1% is much smaller compared with the Gamma distribution. For low values of CV_R there are similarities with Figs. 6 and 7, but differences become marked and intensify as CV_R increases. On the other hand, for CV_R<0.2, the errors are relevant even for high values of Δ/μ_R (large order quantities).

5. Conclusions and future research

A generic expression for the undershoot p.d.f. and order quantity p.d.f. in the (*R, s, S*) inventory system with continuous demand has been developed. The analyses carried out has shown the interest from different viewpoints as well as varied applications.

For the case of gamma distributed demand, a tractable analytical expression of the undershoot p.d.f. has been derived, thus adding this case to the reduced set of known undershoot distributions. The obtained expression is shown to depend upon two nondimensional parameters, Δ/μ_R and CV_R, plus a nondimensional variable. This fact is highly relevant for it allows us to define and study the nondimensional undershoot p.d.f. (NUPDF) and obtain results with general validity for any combination of parameters of the gamma distribution. For low CV_R values, the distribution alternates between unimodal and

[2] Appendix D is provided as supplementary material.

bimodal shapes and shows highly oscillatory behavior with Δ/μ_R. When CV_R increases, the oscillation damps out until fully vanishing. For $CV_R>1$ the undershoot p.d.f. is a monotonically decreasing function for all values of Δ/μ_R. The undershoot mean and standard deviation also show a similar oscillatory behavior with Δ/μ_R. The analysis yields for the first time a detailed complete characterization of the range of validity of the asymptotic approximation widely used in the literature. We calculate the absolute percentage error and propose a new measure, the absolute percentage nondimensional deviation, which measures the deviation as a percentage of the average demand over one review period. This measure gives an immediate idea of the impact of the error in units of product. One additional advantage of the nondimensionalization is that we can produce simple tables for both the NUPDF mean and standard deviation applicable to all the cases of gamma distributed demand. It is worth highlighting that the results for the gamma distribution case can be directly adapted for use in any renewal process.

The generic expression of the undershoot p.d.f. becomes non-tractable for the case of normal distributed demand, due to the convolution integrals involved. To overcome this difficulty a tractable approximation has been proposed and applied. It has been shown that it yields high precision values of the undershoot p.d.f., as well as its mean and standard deviation, for practical values of CV_R ($CV_R<0.6$); for high values of CV_R it also produces good approximations. To the best of our knowledge, it is the first successful attempt to complete these calculations without assuming non-negativity. For the normal distribution case, it is also possible to define a NUPDF with general validity. We provide simple tables of NUPDF mean and standard deviation for $CV_R=\{0.1, 0.2,\ldots, 0.5\}$. Regarding the shape of the undershoot p.d.f., it is highly similar to the case of gamma distributed demand for low CV_R values. As the probability of negative demand increases (higher CV_R), differences between the gamma and normal cases arise and become progressively significant. For $CV_R>0.5$ evolution of NUPDF with CV_R changes substantially. Finally, as in the case of gamma distributed demand, the range of validity of the renewal asymptotic approximation has been determined. In this case the approximation has an additional error bias, since the asymptotic values differ from the renewal formulas due to the probability of negative demand, and in general has less validity as compared to the gamma distributed case.

The results of the analysis for the order quantity distribution are of direct practical interest: provide useful information about the expected average order size of a (*R, s, S*) system, its variability, and the probability intervals for the order quantity. Besides, knowing the evolution of the distribution, mean and standard deviation with the value of $\Delta=S\text{-}s$, opens the possibility to reduce variability and find stable order size policies as well as to anticipate the consequences of a certain set of parameter values in terms of order variability.

The results presented in this paper open new research possibilities, in particular:

- Explore the impact of taking the asymptotic value of the undershoot mean for the cases of gamma and normal distributed demands in the inventory models that include this approximation.
- Extend the nondimensionalization to the determination of the parameters of the inventory models with gamma and normal distributed demands as in [9].
- Consider the probabilistic treatment of constraints regarding thresholds of the order quantity. In some real settings, it might be advisable to avoid order quantities higher than a predetermined value. With the order quantity distribution it is possible to deal with the constraint in statistical terms (for instance, the probability of ordering a quantity higher than a predetermined value should be lower than 1%).

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.apm.2020.09.014.

Appendix A. Undershoot mean and variance

A.1. Undershoot Mean

By definition,

$$\mu_u^{(\Delta)} = \int_0^\infty \upsilon \cdot f_R(\Delta+\upsilon)d\upsilon + \int_0^\infty \upsilon \cdot \sum_{n=1}^\infty \int_{-\infty}^\Delta g_{nR}(y_n)\cdot f_R(\Delta+\upsilon-y_n)dy_n d\upsilon$$
$$= \int_0^\infty \upsilon \cdot f_R(\Delta+\upsilon)d\upsilon + \sum_{n=1}^\infty \int_{-\infty}^\Delta g_{nR}(y_n)\cdot\left(\int_0^\infty \upsilon\cdot f_R(\Delta+\upsilon-y_n)d\upsilon\right)dy_n.$$

Changing the variable υ in the n+1 integrals by, respectively, $\upsilon = \zeta_0 - \Delta,\ d\upsilon = d\zeta_0$ in the first term and $\upsilon = \zeta_n - \Delta + y_n,\ d\upsilon = d\zeta_n,\ n \geq 1,$ in the n integrals of the parenthesis inside the summation leads to

$$\begin{aligned}\mu_u^{(\Delta)} &= \int_{\Delta}^{\infty} (\zeta_0 - \Delta) \cdot f_R(\zeta_0) d\zeta_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \left(\int_{\Delta - y_n}^{\infty} (\zeta_n - (\Delta - y_n)) \cdot f_R(\zeta_n) d\zeta_n \right) dy_n \\ &= \Theta_R^1(\Delta) + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \Theta_R^1(\Delta - y_n) dy_n.\end{aligned}$$

A.2. Undershoot Variance

By definition,

$$\sigma_u^{(\Delta)2} = \int_0^{\infty} \left(\upsilon - \mu_u^{(\Delta)}\right)^2 f_R(\Delta + \upsilon) d\upsilon + \int_0^{\infty} \left(\upsilon - \mu_u^{(\Delta)}\right)^2 \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot f_R(\Delta + \upsilon - y_n) dy_n d\upsilon.$$

Expanding the binomial and applying the same transformations as before yields

$$\begin{aligned}\sigma_u^{(\Delta)2} &= \left(\Theta_R^2(\Delta) - 2\mu_u^{(\Delta)}\Theta_R^1(\Delta) + \mu_u^{(\Delta)2}\Theta_R^0(\Delta)\right) \\ &\quad + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \left(\Theta_R^2(\Delta - y_n) - 2\mu_u^{(\Delta)}\Theta_R^1(\Delta - y_n) + \mu_u^{(\Delta)2}\Theta_R^0(\Delta - y_n)\right) dy_n.\end{aligned}$$

Defining $\Psi_R(x) = \Theta_R^2(x) - 2\mu_u^{(\Delta)}\Theta_R^1(x) + \mu_u^{(\Delta)\,2}\Theta_R^0(x)$, leads to

$$\sigma_u^{(\Delta)^2} = \Psi_R(\Delta) + \sum_{n=1}^{\infty} \int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot \Psi_R(\Delta - y_n) dy_n.$$

Appendix B. Gamma distribution calculations

B.1. Undershoot p.d.f. with gamma distributed demand

$$f_u^{(\Delta)}(\upsilon) = f_R^G(\Delta + \upsilon) + \sum_{n=1}^{\infty} \int_0^{\Delta} f_{nR}^G(y_n) \cdot f_R^G(\Delta + \upsilon - y_n) dy_n.$$

We can solve the convolution integral as follows.

$$\Pr\left(\upsilon \in (\upsilon_0, \upsilon_0 + d\upsilon)\ \textit{in review}\ n+1\right) = \int_0^{\Delta} f_{nR}^G(\eta) \cdot f_R^G(\Delta + \upsilon - \eta) d\eta$$

$$\begin{aligned}&= \int_0^{\Delta} \frac{1}{b^{n\cdot a}\Gamma(n\cdot a)} \eta^{n\cdot a - 1} e^{\frac{-\eta}{b}} \cdot \frac{1}{b^a\Gamma(a)} (\Delta + \upsilon - \eta)^{a-1} e^{\frac{-(\Delta+\upsilon-\eta)}{b}} d\eta \\ &= \int_0^{\Delta} \frac{1}{b^{(n+1)\cdot a} \cdot \Gamma(n\cdot a) \cdot \Gamma(a)} \eta^{n\cdot a - 1} (\Delta + \upsilon - \eta)^{a-1} e^{\frac{-(\Delta+\upsilon)}{b}} d\eta\end{aligned}$$

Change of variable $\eta = (\Delta + \upsilon)t$ leads to:

$$\begin{aligned}&\int_0^{\Delta/\Delta+\upsilon} \frac{1}{b^{(n+1)\cdot a} \cdot \Gamma(n\cdot a) \cdot \Gamma(a)} (\Delta + \upsilon)^{(n+1)\cdot a - 1} \cdot t^{n\cdot a - 1} (1 - t)^{a-1} \cdot e^{\frac{-(\Delta+\upsilon)}{b}} dt \\ &= \frac{(\Delta + \upsilon)^{(n+1)\cdot a - 1} \cdot e^{\frac{-(\Delta+\upsilon)}{b}}}{b^{(n+1)\cdot a} \cdot \Gamma((n+1)\cdot a)} \int_0^{\Delta/\Delta+\upsilon} \frac{\Gamma((n+1)\cdot a)}{\Gamma(n\cdot a)\cdot\Gamma(a)} t^{n\cdot a - 1}(1-t)^{a-1} dt = f_{(n+1)R}^G(\Delta + \upsilon) \cdot F^B\left(\frac{\Delta}{\Delta + \upsilon} \mid n\cdot a, a\right)\end{aligned}$$

Hence, the undershoot p.d.f. with gamma distributed demand will be:

$$f_u^{(\Delta)}(\upsilon) = f_R^G(\Delta + \upsilon) + \sum_{n=1}^{\infty} f_{(n+1)R}^G(\Delta + \upsilon) \cdot F^B\left(\frac{\Delta}{\Delta + \upsilon} \mid n\cdot a, a\right).$$

B.2. Nondimensional undershoot p.d.f. (NUPDF) with gamma distributed demand

In general, for any gamma distribution $f^G(x \mid a, b) = 1/p \cdot f^G(x/p \mid a, b/p)$. Applying this property to Eq. (15) and expressing the gamma parameters as in (14) leads to:

$$f_u^{(\Delta)}\left(\upsilon \mid a = 1/CV_R{}^2,\ b = \mu_R \cdot CV_R{}^2\right) = \frac{1}{\mu_R} \cdot f_R^G\left(\frac{\Delta}{\mu_R} + \frac{\upsilon}{\mu_R} \mid 1/CV_R{}^2,\ CV_R{}^2\right)$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\mu_R} \cdot f_{(n+1)R}^G\left(\frac{\Delta}{\mu_R} + \frac{\upsilon}{\mu_R} \mid (n+1) \cdot 1/CV_R{}^2,\ CV_R{}^2\right) \cdot F^B\left(\frac{\Delta/\mu_R}{\Delta/\mu_R + \upsilon/\mu_R} \mid n \cdot 1/CV_R{}^2, 1/CV_R{}^2\right)$$
$$= \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R} \mid a' = 1/CV_R{}^2,\ b' = CV_R{}^2\right)$$

For the undershoot c.d.f., the probability for the undershoot to take a value lower than or equal to υ with parameters (a, b) is the same as the cumulative probability of υ/μ_R with parameters (a, b/μ_R).

$$F_u^{(\Delta)}(\upsilon \mid a,\ b) = \int_0^{\upsilon} f_u^{(\Delta)}(\upsilon \mid a,\ b) d\upsilon = \int_0^{\upsilon} \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R} \mid a' = a,\ b' = \frac{b}{\mu_R}\right) d\upsilon$$
$$= \int_0^{\frac{\upsilon}{\mu_R}} \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R} \mid a' = a,\ b' = \frac{b}{\mu_R}\right) \cdot \mu_R \cdot d\left(\frac{\upsilon}{\mu_R}\right) = F_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R} \mid a' = a,\ b' = \frac{b}{\mu_R}\right) \tag{B.1}$$

Likewise, for the mean and variance of the undershoot distribution:

$$\mu_u^{(\Delta)} = \int_0^{\infty} \upsilon \cdot f_u^{(\Delta)}(\upsilon) d\upsilon = \int_0^{\infty} \left(\mu_R \cdot \frac{\upsilon}{\mu_R}\right) \cdot \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R}\right) \cdot \mu_R d\left(\frac{\upsilon}{\mu_R}\right)$$
$$= \mu_R \cdot \int_0^{\infty} \frac{\upsilon}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R}\right) d\left(\frac{\upsilon}{\mu_R}\right) = \mu_R \cdot \mu_u^{(\Delta/\mu_R)};$$

$$\sigma_u^{(\Delta)2} = \int_0^{\infty} \left(\upsilon - \mu_u^{(\Delta)}\right)^2 f_u^{(\Delta)}(\upsilon) d\upsilon = \int_0^{\infty} \left(\mu_R \cdot \frac{\upsilon}{\mu_R} - \mu_R \cdot \mu_u^{(\Delta/\mu_R)}\right)^2 \cdot \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R}\right) \cdot \mu_R d\left(\frac{\upsilon}{\mu_R}\right)$$
$$= \mu_R{}^2 \cdot \int_0^{\infty} \left(\frac{\upsilon}{\mu_R} - \mu_u^{(\Delta/\mu_R)}\right)^2 f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R}\right) d\left(\frac{\upsilon}{\mu_R}\right) = \mu_R{}^2 \cdot \sigma_u^{(\Delta/\mu_R)2}.$$

Appendix C. Normal distribution calculations

C.1. Auxiliary convolution integral

Let us consider $f^N(x \mid \mu, \sigma^2)$ normal p.d.f. with mean μ and variance σ^2, and $\varphi(x)$ the standard normal p.d.f. $F^N(x \mid \mu, \sigma^2)$ and $\Phi(x)$ will be respectively the cumulative distribution functions of $f^N(x)$ and $\varphi(x)$:

$$f^N\left(x \mid \mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sigma}\,\varphi\left(\frac{x-\mu}{\sigma}\right),\ F^N\left(x \mid \mu, \sigma^2\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

We can solve the following convolution integral:

$$I = \int_{-\infty}^{\Delta} f^N(x \mid n\mu, n\sigma^2) \cdot f^N(y - x \mid \mu, \sigma^2) dx = \int_{-\infty}^{\Delta} \frac{1}{\sqrt{2\pi n}\sigma} e^{-\frac{1}{2}\frac{(x-n\mu)^2}{n\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(y-x-\mu)^2}{\sigma^2}} dx.$$

Completing squares of the exponents, we can conveniently group the terms of the integral,

$$\frac{(x-n\mu)^2}{n\sigma^2} + \frac{(y-x-\mu)^2}{\sigma^2} = \frac{(y-(n+1)\mu)^2}{(n+1)\sigma^2} + \frac{\left(x - \frac{n}{n+1}y\right)^2}{\frac{n}{n+1}\sigma^2}.$$

Finally, substituting in the integral and making some adjustments [30],

$$I = f^N(y \mid (n+1)\mu, (n+1)\sigma^2) \cdot \int_{-\infty}^{\Delta} f^N(x \mid \frac{n}{n+1}y, \frac{n}{n+1}\sigma^2) dx$$
$$= f^N(y \mid (n+1)\mu, (n+1)\sigma^2) \cdot F^N(\Delta \mid \frac{n}{n+1}y, \frac{n}{n+1}\sigma^2). \tag{C.1}$$

C.2. Expression $g_{nR}(y_n)$

Applying the recurrent formula (4) and using Eq. (C.1), yields an expression for g_{nR}:

$$g_R(y_1) = f_R^N(y_1),$$

$$g_{2R}(y_2) = \int_{-\infty}^{\Delta} f_R^N(y_1) \cdot f_R(y_2 - y_1)dy_1 = f_{2R}^N(y_2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2),$$

$$\begin{aligned} g_{3R}(y_3) &= \int_{-\infty}^{\Delta} f_{2R}^N(y_2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2) \cdot f_R(y_3 - y_2)dy_2 \\ &= f_{3R}^N(y_3) \cdot \int_{-\infty}^{\Delta} f^N(y_2 \mid \frac{2}{3}y_3, \frac{2}{3}\sigma^2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2)dy_2, \end{aligned}$$

$$\begin{aligned} g_{4R}(y_4) &= \int_{-\infty}^{\Delta} f_{3R}^N(y_3) \cdot \left(\int_{-\infty}^{\Delta} f^N(y_2 \mid \frac{2}{3}y_3, \frac{2}{3}\sigma^2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2)dy_2\right) \cdot f_R(y_4 - y_3)dy_3 \\ &= f_{4R}^N(y_4) \cdot \int_{-\infty}^{\Delta} f^N(y_3 \mid \frac{3}{4}y_4, \frac{3}{4}\sigma^2) \cdot \left(\int_{-\infty}^{\Delta} f^N(y_2 \mid \frac{2}{3}y_3, \frac{2}{3}\sigma^2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2)dy_2\right)dy_3 \\ &= f_{4R}^N(y_n) \cdot \int_{-\infty}^{\Delta}\int_{-\infty}^{\Delta} f^N(y_2 \mid \frac{2}{3}y_3, \frac{2}{3}\sigma^2) \cdot f^N(y_3 \mid \frac{3}{4}y_4, \frac{3}{4}\sigma^2) \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2)dy_3dy_2, \\ &\dots \end{aligned}$$

$$g_{nR}(y_n) = f_{nR}^N(y_n) \cdot \int_{-\infty}^{\Delta}\int_{-\infty}^{\Delta}\cdots\int_{-\infty}^{\Delta}\left[\prod_{i=3}^{n} f^N(y_{i-1} \mid \frac{i-1}{i}y_i, \frac{i-1}{i}\sigma^2)\right] \cdot F^N(\Delta \mid \frac{1}{2}y_2, \frac{1}{2}\sigma^2)dy_{n-1}\dots dy_3dy_2,\ n > 2$$

In order to approximate g_{nR} by demand during n consecutive review intervals since the last ordering without reaching the reorder level in the j preceding review epochs:

- for $n \le j + 1$, we use the former expressions of g_{nR};
- for $n > j + 1$, we use the recurrent formula of Eq. (4) making the approximation $g_{(n-j)R} \cong f_{(n-j)R}^N$, which leads to:

$$\begin{aligned} g_{nR}(y_n) \cong f_{nR}^N(y_n) \cdot \int_{-\infty}^{\Delta}\int_{-\infty}^{\Delta}\cdots\int_{-\infty}^{\Delta}\left[\prod_{i=n-j+2}^{n} f^N(y_{i-1} \mid \frac{i-1}{i}y_i, \frac{i-1}{i}\sigma^2)\right] \\ \cdot F^N(\Delta \mid \frac{n-j}{n-j+1}y_{n-j+1}, \frac{n-j}{n-j+1}\sigma^2)dy_{n-1}\dots dy_{n-j+2}dy_{n-j+1},\ j > 1. \end{aligned}$$

C.3. Nondimensional undershoot p.d.f. (NUPDF) with normal distributed demand.

For any normal distribution $f^N(x \mid \mu,\sigma^2) = 1/\mu \cdot f^N(x/\mu \mid 1,(\sigma/\mu)^2)$. Applying this property to Eq. (17) leads to:

$$f_u^{(\Delta)}(\upsilon) = \frac{1}{\mu_R} \cdot f_R^N\left(\frac{\Delta}{\mu_R} + \frac{\upsilon}{\mu_R}\right) + \sum_{n=1}^{\infty}\int_{-\infty}^{\Delta} g_{nR}(y_n) \cdot f_R^N\left(\frac{\Delta}{\mu_R} + \frac{\upsilon}{\mu_R} - \frac{y_n}{\mu_R}\right)d\left(\frac{y_n}{\mu_R}\right).$$

Taking the definition of $g_{nR}(y_n)$ (4) and following the same transformation:

$$g_R(y_1) = 1/\mu_R \cdot f_R(y_1/\mu_R) = 1/\mu_R \cdot g_R(y_1/\mu_R),$$

$$g_{2R}(y_2) = \int_{-\infty}^{\Delta} 1/\mu_R \cdot g_R(y_1/\mu_R) \cdot f_R(y_2/\mu_R - y_1/\mu_R)d(y_2/\mu_R) = 1/\mu_R \cdot g_{2R}(y_2/\mu_R),$$

...

$$g_{nR}(y_2) = 1/\mu_R \cdot g_{nR}(y_n/\mu_R).$$

Thus,

$$f_u^{(\Delta)}(\upsilon) = \frac{1}{\mu_R} \cdot f_u^{(\Delta/\mu_R)}\left(\frac{\upsilon}{\mu_R}\right)$$

For the mean and variance, the demonstration is identical to the case of gamma distributed demand (B.2) except for the lower limit of integration which is -∞ in this case.

$$\mu_u^{(\Delta)} = \mu_R \cdot \mu_u^{(\Delta/\mu_R)},$$

$$\sigma_u^{(\Delta)2} = {\mu_R}^2 \cdot \sigma_u^{(\Delta/\mu_R)2}.$$

C.4. Undershoot mean and standard deviation

Through standard integration, we express $\Theta_R^0(x)$, $\Theta_R^1(x)$, $\Theta_R^2(x)$ and $\Psi_R(x)$ in terms of

$k = (x - \mu)/\sigma$.

$$\Theta_R^0(x) = 1 - \Phi(k) = \Phi(-k),$$

$$\Theta_R^1(x) = \sigma(\varphi(k) - k \cdot \Phi(-k)),$$

$$\Theta_R^2(x) = \sigma^2\left[\left(1 + k^2\right) \cdot \Phi(-k) - k \cdot \varphi(k)\right],$$

$$\Psi_R(x) = \Theta_R^2(x) - 2\mu_u^{(\Delta)}\Theta_R^1(x) + \mu_u^{(\Delta)2}\Theta_R^0(x) = \left[\sigma^2 + \left(\sigma k + \mu_u^{(\Delta)}\right)^2\right] \cdot \Phi(-k) - \left(\sigma^2 k + 2\mu_u^{(\Delta)}\sigma\right)\varphi(k).$$

To calculate the undershoot mean and standard deviation we substitute $\Theta_R^1(\Delta - y_n)$ and $\Psi_R(\Delta - y_n)$ in (6), computing g_{nR} according to C.2.

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