New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark

OPEN ACCESS

RECEIVED
20 October 2020

REVISED
25 January 2021

ACCEPTED FOR PUBLICATION
12 February 2021

PUBLISHED
30 March 2021

PAPER

Measurement outcomes that do not occur and their role in entanglement transformations

Martin Hebenstreit[1], **Matthias Englbrecht**[1,*], **Cornelia Spee**[2,3], **Julio I. de Vicente**[4] and **Barbara Kraus**[1]

1 Institute for Theoretical Physics, University of Innsbruck, Technikerstr. 21A, 6020 Innsbruck, Austria
2 Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, 1090 Vienna, Austria
3 Naturwissenschaftlich-Technische Fakultät, Universität Siegen, Walter-Flex-Straße 3, 57068 Siegen, Germany
4 Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, E-28911, Leganés (Madrid), Spain
* Author to whom any correspondence should be addressed.

E-mail: Matthias.Englbrecht@uibk.ac.at

Keywords: separable operations, local operations and classical communication, pure state transformations

Abstract

The characterization of transformations among entangled pure states via local operations assisted by classical communication (LOCC) is a crucial problem in quantum information theory for both theoretical and practical reasons. As LOCC has a highly intricate structure, sometimes the larger set of separable (SEP) maps is considered, which has a mathematically much simpler description. In the literature, mainly SEP maps consisting of invertible Kraus operators have been taken into account. In this paper we show that the consideration of those maps is not sufficient when deciding whether a state can be mapped to another via general SEP transformations. This is done by providing explicit examples of transformations among pure three- and five-qubit states, which are feasible via SEP maps containing singular Kraus operators, however, not possible via SEP maps containing solely regular Kraus operators. The key point that allows to construct the SEP maps is to introduce projective measurements that occur with probability zero on the input state. The fact that it is not sufficient to consider SEP maps composed out of regular Kraus operators even in the case of pure state transformations, also affects the results on LOCC transformations among pure states. However, we show that non-invertible Kraus operators do not help in state transformations under LOCC with finitely many rounds of classical communication, i.e. the necessary and sufficient condition for SEP transformations with invertible Kraus operators is still a necessary condition for convertibility under finite-round LOCC. Moreover, we show that the results on transformations via SEP that are not possible with LOCC (including infinitely many rounds of classical communication) presented in Hebenstreit *et al* 2016 *Phys. Rev. A* **93**, 012339 are not affected.

1. Introduction

Understanding the entanglement properties of multipartite quantum systems plays a major role in both quantum information theory and condensed-matter physics. On the one hand, this allows to derive protocols for quantum communication such as secret sharing [1] and schemes for quantum computation such as measurement-based computation [2] to cite some examples. On the other hand, the entanglement structure of many-body systems can be used to characterize phase transitions [3] and to devise schemes for numerical simulation using tensor network states [4]. In general, entanglement is considered to be one of the non-classical ingredients that allows quantum technologies to outperform their classical counterparts. For this reason, a resource theory of entanglement has been developed over the last two decades [5]. This theory provides a rigorous framework that makes it possible to qualify and quantify this resource and to understand the fundamental possibilities and limitations behind its manipulation. However, many questions that have been long answered for bipartite systems turn out to be much more difficult when more parties are taken into account. Besides its fundamental interest, advancing further the resource theory of entanglement in the multipartite regime might lead to new genuinely many-body applications of quantum information theory.

Entanglement theory is formulated as a resource theory [6]. Such theories are built from the notion of the so-called free operations, which, due to the physical setting, are easily implementable and are therefore considered to be accessible at no cost. States that cannot be prepared with free operations acquire the status of a resource, in the sense that they might allow to overcome the limitations of what is possible by means of the free operations alone. Furthermore, the notion of free operations allows to define an operational partial order in the set of resource states: if there exists a free operation Λ such that $\Lambda(\rho) = \sigma$, then ρ is not less resourceful than σ. This is because any protocol that can be successfully implemented in this scenario (i.e. with free operations) starting from σ can also be implemented successfully starting from ρ. Functionals that preserve this ordering are considered to be resource quantifiers. Entanglement is a resource shared by different possibly space-separated parties. In this context, local operations assisted by classical communication (LOCC) arise as a natural and operationally motivated choice of free operations. LOCC maps are built from local completely positive, trace preserving (CPTP) maps which the parties can correlate by exchanging classical communication. On the one hand, understanding LOCC allows to order and quantify the set of entangled states and to identify those that are potentially more useful. On the other hand, it provides protocols for the manipulation of this resource in practice.

A milestone result in this context is Nielsen's theorem [7], which characterizes LOCC convertibility among pure bipartite states in terms of majorization. Unfortunately, the extension of Nielsen's theorem to the multipartite case is not straightforward at all. The mathematical characterization of the set of LOCC maps and LOCC transformations is extremely complicated due to the intricacies that arise when considering a potentially unbounded number of rounds of classical communication [8]. Indeed, it is known that in contrast to bipartite pure state transformations [9], no simplification can be placed on the number of rounds of classical communication that is sufficient to consider in general [10, 11]. Notwithstanding, several different works over the last years have led to considerable progress in our understanding of the rich entanglement structure of pure multipartite states. Reference [12] characterizes when pure multipartite qubit-states are related by local unitary (LU) transformations. Since LUs are invertible LOCC transformations, this defines equivalence classes of states with the same entanglement [13]. Reference [14] introduces the notion of stochastic-LOCC (SLOCC) classes, which provides a coarse-grained classification of states with different entanglement properties. In more detail, two pure states are said to be in the same SLOCC class if they can be interconverted with non-vanishing probability by probabilistic LOCC. Thus, although this classification is based on an equivalence relation and, therefore, provides no sense of ordering, it tells us that LOCC manipulation can only occur within these classes. Indeed, LOCC convertibility has been later characterized within SLOCC classes with a simple mathematical structure such as the GHZ [15] or the W [16] family. Another fruitful approach is to consider inner or outer approximations of the set of LOCC maps with a mathematically more tractable set of maps within a fixed SLOCC class. A natural and physically motivated inner approximation to LOCC is $\mathrm{LOCC}_{\mathbb{N}}$, the set of LOCC maps implementable with a finite number of rounds of classical communication. The fact that such protocols have to terminate has allowed to characterize all states that are reachable by this class of transformations within a given (generic) SLOCC class and has allowed to identify multipartite protocols which cannot be boiled down to a concatenation of deterministic one-round protocols as in the bipartite and the aforementioned multipartite case [17]. A particularly useful superset of LOCC is that of separable (set of separable (SEP)) maps, which are those CPTP maps that admit a Kraus decomposition in which all Kraus operators factorize in tensor products for each party [18]. Although it is known that the inclusion is strict, instances of protocols in which SEP outperforms LOCC are rare [19] and, moreover, for certain tasks such as bipartite pure-state transformations they are known to be effectively the same [20]. In [21] transformations among multipartite pure, fully entangled states (i.e. states for which the local density matrices are of full rank) within the same SLOCC class have been considered. There, a necessary and sufficient condition for the existence of a SEP map which transforms the pure initial to the pure final state has been provided. However, there has been a constraint on the SEP map, which has been overlooked so far [22]. The criterion holds for SEP maps, whose Kraus operators are invertible. In the following we refer to this set of CPTP maps as SEP_1. Until now (with the exception of [23, 24]) SEP_1 has been considered as a superset of LOCC. The main reason why singular Kraus operators have not been considered (in the context of LOCC) is that they map the initial state into a state which is no longer in the same SLOCC class as the final state. However, the fact that the initial state could be annihilated by the Kraus operator has been ignored. Due to that, the condition on the existence of a SEP_1 map has been subsequently used to characterize LOCC convertibility among pure multipartite fully entangled states in several general systems such as three-qubit states, four-qubit states and three-qutrit states

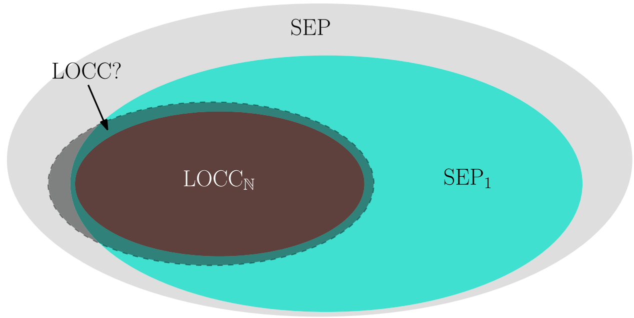

Figure 1. Summary of the inclusion relations of possible pure state transformations among fully entangled states. As explained in section 2, we use the subscript T to indicate that a relation should be understood in this sense. Surprisingly, there exist pure state transformations that are possible via SEP, but not via SEP$_1$, i.e. when considering only regular Kraus operators (see section 4.1). Hence, SEP$_1$ $\subsetneq_T$ SEP. It is an open problem whether there exist pure state transformations among fully entangled states that are possible via LOCC, but not via LOCC$_{\mathbb{N}}$. There do exist pure state transformations that are possible via SEP, but not via LOCC (which is proven here by adapting the proof of [28], in which SEP was considered to coincide with SEP$_1$). In fact, these examples show that SEP$_1$ $\not\subseteq_T$ LOCC. It is currently not clear, whether there exist LOCC transformations which are not possible via SEP$_1$. In this article, we answer a related question by showing that finite round LOCC transformations among fully entangled states are always possible via SEP$_1$, i.e. LOCC$_{\mathbb{N}}$$\subseteq_TSEP_1$.

[25–28]. In [23, 24], however, it has been proven that generic pure fully entangled states, i.e. almost all fully entangled states, of more than three parties with arbitrary equal local dimension are isolated, i.e. they cannot be obtained from nor transformed to inequivalent pure fully entangled states by SEP and, hence, by LOCC.

In this work we explore the differences in what comes to fully-entangled pure-state transformations between SEP$_1$ and SEP and its consequences for deciding LOCC convertibility. Remarkably, we show that necessary and sufficient conditions for SEP$_1$ convertibility are only sufficient for SEP convertibility. Note that this implies that SEP$_1$ is not necessarily a superset of LOCC. In order to prove this, we construct explicit examples of SEP transformations which are infeasible via SEP$_1$. Interestingly, these instances exist for systems of very small size and dimension such as three-qubit and five-qubit states. The crucial observation behind these constructions is that SEP transformations can contain, in contrast to SEP$_1$, projective Kraus operators which annihilate the initial state. Stated more operationally, since one can see that non-invertible Kraus operators which occur with non-zero probability do not need to be taken into account, the difference is given by measurement operators whose outcomes have zero probability when applied to the initial state. This does not only shed light on the role of the outcomes that cannot occur but, as explained above, it is important to decide how to interpret results that have been obtained previously based on the condition of [21]. Importantly, we show here that for LOCC$_{\mathbb{N}}$ transformations among fully entangled states non-invertible Kraus operators do not need to be taken into account. In other words, the necessary and sufficient condition for SEP$_1$ convertibility remains a necessary condition for LOCC$_{\mathbb{N}}$ convertibility. Furthermore, we will also provide a general condition under which the conditions for the existence of a SEP state transformation coincide with those for the existence of a SEP$_1$ map. This is used to show that the examples given in [28] using the SEP$_1$ condition indeed provide pure state transformations which are possible via SEP but not via LOCC. On the other hand, the question of whether LOCC transformations in this context are only possible if they can be be implemented by SEP$_1$ remains unanswered, i.e. it is not clear whether the necessary and sufficient condition for SEP$_1$ convertibility is also a necessary condition for LOCC convertibility if one allows infinitely many rounds of classical communication. Figure 1 summarizes the relation between aforementioned sets of pure state transformations incorporating the findings of this paper.

Our results go far beyond mere mathematical statements on the inclusion of different sets of operations and the extension of a previously obtained condition. They form a profound basis on which LOCC convertibility can be decided in a systematic fashion in the future. Under the practical constraint of finitely many rounds of classical communication we show here that the condition observed in [21] is indeed necessary for LOCC convertibility. However, when addressing questions concerning the entanglement of states the possibility of infinitely many classical communications rounds has to be taken into account. It is not yet clear whether SEP$_1$ can be considered to be a superset of LOCC. Therefore, it is necessary to include singular Kraus operators when studying entanglement with the use of separable maps as we show here that SEP and SEP$_1$ do not coincide, even for pure states transformations. Moreover, we can reassure on the examples of pure state transformations that are possible via SEP but not via LOCC found in [28] that the separation between SEP and SEP$_1$ does not imply that previously obtained results are obsolete.

The structure of this paper is as follows. We will first define our notation. Then we will review the result of [21] and provide necessary and sufficient conditions for transformations via SEP. We will then discuss the relations among the different separable classes of operations. In particular, we will provide examples for transformations that are only possible if singular Kraus operators are taken into account and we will show that $\mathrm{LOCC}_{\mathbb{N}}$ transformations among fully entangled pure states are included in SEP_1. Moreover, we will derive a sufficient condition for SEP transformations to be implementable via SEP_1 and we will provide an adaptation of the proof for the examples of [28]. Finally we will give a conclusion and an outlook.

2. Notation and preliminaries

In this work we consider pure states of an arbitrary number of parties n and arbitrary local dimensions $\{d_i\}$ which are fully entangled. That is, states $|\psi\rangle$ in the Hilbert space $\mathcal{H} = \bigotimes_{i=1}^n \mathbb{C}^{d_i}$ such that the reduced density matrix for each party i, ρ_i, fulfills $\mathrm{rank}\rho_i = d_i$. We call a state critical if $\rho_i \propto \mathbb{1}$ for all parties i. We will consider transformations among fully entangled states, which are given by CPTP maps $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators acting on H [34]. Every CPTP map admits a Kraus decomposition, i.e. it can be written as

$$\Lambda(X) = \sum_i K_i X K_i^\dagger, \tag{1}$$

for some set of operators $\{K_i\} \subset \mathcal{B}(\mathcal{H})$ fulfilling $\sum_i K_i^\dagger K_i = \mathbb{1}$ referred to as Kraus operators. As mentioned above, we will be interested in transformations among fully entangled pure states with particular subsets of the set of CPTP maps: SEP, SEP_1, $\mathrm{LOCC}_{\mathbb{N}}$ and LOCC, whose definitions we provide in the following.

A CPTP map Λ is said to be in SEP if it admits a Kraus decomposition with Kraus operators $\{K_i\}$ such that for all i $K_i = \bigotimes_{j=1}^n K_i^{(j)}$ with $K_i^{(j)} \in M(d_j, \mathbb{C})$ $\forall j$, the last symbol referring to square matrices of size d_j with complex entries. A SEP map Λ is said to be in SEP_1 if there exists a Kraus decomposition of the above form which fulfills moreover that $K_i^{(j)} \in GL(d_j, \mathbb{C})$ $\forall i, j$, i.e. every Kraus operator is regular. A CPTP map Λ is said to be in LOCC if spatially separated parties can implement it using local generalized measurements and classical communication. If an LOCC map can be realized with finitely many rounds of classical communication, it is said to be in $\mathrm{LOCC}_{\mathbb{N}}$. Formal definitions of LOCC and $\mathrm{LOCC}_{\mathbb{N}}$ are provided in appendix A.

Whenever there exists a map Λ in SEP, SEP_1 or $\mathrm{LOCC}_{\mathbb{N}}$ such that $\Lambda(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$, we say that $|\psi\rangle$ can be converted into $|\phi\rangle$ by SEP, SEP_1 or $\mathrm{LOCC}_{\mathbb{N}}$ operations (the analogous claim for LOCC with infinitely many rounds of classical communication can be made by means of a limiting procedure, cf appendix A). From the definitions it should be clear that $\mathrm{SEP}_1 \subset \mathrm{SEP}$ and $\mathrm{LOCC}_{\mathbb{N}} \subset \mathrm{LOCC} \subset \mathrm{SEP}$. However, we want to understand here whether the inclusion $X \subset Y$ translates into the existence of a transformation among fully entangled pure states by the operations given by Y but not by the operations given by X or whether both sets of transformation are equally powerful in this context. For this, we write $X =_T Y$ if whenever there exists a map Λ in Y that transforms $|\psi\rangle$ into $|\phi\rangle$, there exists a map Λ' in X that transforms $|\psi\rangle$ into $|\phi\rangle$ and vice versa. On the other hand, we write $X \subsetneq_T Y$ if $X \subset Y$ and for some states $|\psi\rangle$ and $|\phi\rangle$ the conversion $|\psi\rangle$ into $|\phi\rangle$ is possible within Y but there exists no map in X that transforms $|\psi\rangle$ into $|\phi\rangle$.

As explained in the introduction the considered transformations can only occur within SLOCC classes. Two states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ are said to be in the same SLOCC class if $|\phi\rangle = \bigotimes_{i=1}^n g_i|\psi\rangle$ with $g_i \in GL(d_i, \mathbb{C})$. We will consider for each SLOCC class a representative which we will refer to as $|\psi\rangle$. Other states in the SLOCC class are then identified by regular local operators acting on $|\psi\rangle$. Usually, we will use $g\,|\psi\rangle$ with $g = \otimes_i g_i$ to denote the initial state and $h\,|\psi\rangle$ with $h = \otimes_i h_i$ as the final state of a potential state transformation. Here, g_i and h_i are regular operators which reflects that we are interested in transformations among fully entangled states. Moreover, we will use the notation $G = g^\dagger g$ and $H = h^\dagger h$.

The stabilizer (or symmetry group) of $|\psi\rangle$, i.e. the set of local invertible operators leaving $|\psi\rangle$ invariant, will be denoted by $\mathcal{S}_\psi$. More precisely, we have that

$$\mathcal{S}_\psi = \left\{S : S\,|\psi\rangle = |\psi\rangle\,, S = S^{(1)} \otimes \cdots \otimes S^{(n)}, S^{(i)} \in GL(d_i, \mathbb{C})\right\}. \tag{2}$$

Furthermore, we will denote by $\mathcal{N}_\psi$ the set of local operators which annihilate the state $|\psi\rangle$, i.e.

$$\mathcal{N}_\psi = \left\{N : N\,|\psi\rangle = 0, N = N^{(1)} \otimes \cdots \otimes N^{(n)}, N^{(i)} \in M(d_i, \mathbb{C})\right\}. \tag{3}$$

As we will see, the stabilizer and the set annihilating the representative define which state transformations are possible via SEP. In the next section we will discuss in detail the necessary and sufficient condition for such transformations, as well as the condition introduced previously in [21].

3. State transformations

In [21] state transformation via separable maps which only involve regular matrices as Kraus operators have been considered. In particular, the following necessary and sufficient condition for the existence of transformations among pure states via SEP_1 has been shown [21].

Theorem 1. ([21]). *The state $g|\psi\rangle$ can be transformed to $h|\psi\rangle$ via SEP_1 if and only if there exists a finite set of probabilities $\{p_k\}$, i.e. $p_k \geqslant 0$, $\sum_k p_k = 1$, and symmetries $\{S_k\} \subseteq \mathcal{S}_\psi$ such that*

$$\sum_k p_k S_k^\dagger H S_k = rG, \tag{4}$$

where $r = \|h|\psi\rangle\|^2/\|g|\psi\rangle\|^2$.

It is currently unclear whether a pure state transformation that is possible via LOCC is always possible via SEP_1. However, we will show in the following that there exist state transformations via SEP which are impossible via SEP_1 and therefore SEP is strictly larger than SEP_1, i.e. $\mathrm{SEP}_1 \subsetneq_T \mathrm{SEP}$. In order to see this, let us note that the Kraus operators occurring in a separable map might also annihilate the initial state, leading to more general maps. That is, operators M_k, with $M_k g|\psi\rangle = 0$ need to be taken into account. Hence, we have the following theorem characterizing SEP transformations.

Theorem 2. *The state $g|\psi\rangle$ can be transformed to $h|\psi\rangle$ via SEP if and only if there exists a finite set of probabilities $\{p_k\}$, i.e. $p_k \geqslant 0$, $\sum_k p_k = 1$, symmetries $\{S_k\} \subseteq \mathcal{S}_\psi$, and local singular matrices $N_q \in \mathcal{N}_{g\psi}$ such that*

$$\frac{1}{r}\sum_k p_k S_k^\dagger H S_k + g^\dagger \sum_q N_q^\dagger N_q g = G, \tag{5}$$

where $r = \|h|\psi\rangle\|^2/\|g|\psi\rangle\|^2$.

Proof. The proof of this theorem is analogous to the proof of theorem 1 presented in [21]. However, here non-invertible matrices have to be taken into account. We will first show that equation (5) necessarily holds, if the transformation is possible via SEP. Let M_k (N_q) denote those Kraus operators, which reach the final state with non-vanishing probability (annihilate the initial state) respectively, i.e.

$$M_k g|\psi\rangle/n_1 = \sqrt{p_k}\, h|\psi\rangle/n_2, \tag{6}$$

$$N_q g|\psi\rangle = 0 \tag{7}$$

where $p_k > 0$ and $n_1 = \|g|\psi\rangle\|$, $n_2 = \|h|\psi\rangle\|$. Note that only finitely many measurement operators have to be taken into account (even if the stabilizer contains infinitely many elements) due to Caratheodory's theorem. The first equation leads to $M_k = \sqrt{p_k} n_1/n_2 h S_k g^{-1}$ where S_k is an element of the stabilizer $\mathcal{S}_\psi$ of $|\psi\rangle$. The completeness relation, $\sum_k M_k^\dagger M_k + \sum_q N_q^\dagger N_q = \mathbb{1}$ is hence equivalent to

$$\frac{1}{r}\sum_k p_k S_k^\dagger H S_k + g^\dagger \sum_q N_q^\dagger N_q g = G, \tag{8}$$

which proves that equation (5) has to be necessarily satisfied. That this condition is sufficient follows using the argument above in the reverse order. □

As we will see in the following there exist separable transformations which solely become possible when taking Kraus operators with vanishing probability into account. Note, however, that the results presented in [23, 24], where it has been shown that almost all n-qudit states possess only the trivial stabilizer and are hence not convertible into any other state are not affected, as already proven in [23].

4. Relations among classes of separable operations

Whereas it is currently not clear whether pure state transformations that are possible via LOCC are always possible via SEP_1, we will show here that SEP_1 does not coincide with SEP. Furthermore, we will show that a pure state transformation among fully entangled states that is possible via $\mathrm{LOCC}_\mathbb{N}$ is necessarily possible via SEP_1 and therefore, any such pure state transformation via $\mathrm{LOCC}_\mathbb{N}$ necessarily has to obey the conditions in theorem 1. Moreover, we will derive sufficient conditions for which pure state transformations that are possible via SEP coincide with those via SEP_1. Finally, we will revisit the example presented in [28] of SEP_1 pure state transformations which cannot be realized via LOCC (taking infinitely many rounds into account). We show that the statement remains true if one takes into account that there may be more transformations possible via SEP than SEP_1, implying that these are indeed examples of pure state transformations which can be achieved with SEP, however not with LOCC.

4.1. Examples of SEP transformation that are not possible via SEP_1

Let us start by presenting two distinguished examples of state transformations which are possible via SEP, but not via SEP_1. The first example is notable because the considered initial state has solely unitary stabilizer. The second example is found among three-qubit states and thus within the smallest possible multipartite quantum system [35].

Let us first consider the five-qubit ring graph state, $|\psi\rangle$. A graph state is a special type of stabilizer state. For an introduction to graph states and stabilizer states we refer the reader to [30]. The Pauli stabilizer of the state $|\psi\rangle$ is generated by $A_i = Z_{i-1}X_iZ_{i+1}$, for $1 \leqslant i \leqslant 5$ and $Z_0 = Z_5, Z_6 = Z_1$. For this state we find that $\mathcal{S}_\psi = \langle \{A_i\}_{i=1}^5 \rangle$. To show this statement we use that if a critical state has finitely many unitary symmetries, then it has no other regular symmetries [31] and that any graph state is a critical state. Considering the reduced density operators of three qubits it is straightforward to show that all LU symmetries of $|\psi\rangle$ are contained in its Pauli stabilizer and thus that there are only finitely many symmetries (for details see appendix B). We then consider the state transformation from $|\psi\rangle$ to a state $h\,|\psi\rangle$. Hence, we have that $G = \mathbb{1}$ and h will be specified below. Using that the Pauli stabilizer is abelian, we obtain that equation (4) is fulfilled only if $\mathrm{tr}(HP) = 0$ for any non-trivial element P of the Pauli stabilizer. Choosing $H = h^\dagger h = (1/2\mathbb{1} + aZ) \otimes (1/2\mathbb{1} + aX) \otimes (1/2\mathbb{1} + aZ) \otimes 1/2\mathbb{1} \otimes 1/2\mathbb{1}$, for some $a \in (0, 1/2)$, it holds that $\mathrm{tr}(HA_2) \neq 0$ and therefore the transformation is not possible via SEP_1.

We construct now the SEP map which transforms $|\psi\rangle$ into $h\,|\psi\rangle$. In order to do so, we use the following projectors, which annihilate the initial state,

$$Q_1 = \frac{1}{8}(\mathbb{1} + Z) \otimes (\mathbb{1} + X) \otimes (\mathbb{1} - Z) \otimes \mathbb{1}^{\otimes 2} \tag{9}$$

$$Q_2 = \frac{1}{8}(\mathbb{1} + Z) \otimes (\mathbb{1} - X) \otimes (\mathbb{1} + Z) \otimes \mathbb{1}^{\otimes 2} \tag{10}$$

$$Q_3 = \frac{1}{8}(\mathbb{1} - Z) \otimes (\mathbb{1} + X) \otimes (\mathbb{1} + Z) \otimes \mathbb{1}^{\otimes 2} \tag{11}$$

$$Q_4 = \frac{1}{8}(\mathbb{1} - Z) \otimes (\mathbb{1} - X) \otimes (\mathbb{1} - Z) \otimes \mathbb{1}^{\otimes 2}. \tag{12}$$

The Kraus operators for the separable map are then given by: $M_i = a_1 h Q_i$, for $i = 1, 2, 3$ and with $a_1 = 2\sqrt{2a^3/((1/2 + a)^2(1/2 - a)(1/8 + a^3))}$, $M_4 = 2\sqrt{2a^3/((1/2 - a)^3(1/8 + a^3))}hQ_4$, and $M_5 = \sqrt{1/(1/8 + a^3)}h$; $M_6 = M_5A_1, M_7 = M_5A_3, M_8 = M_5A_1A_3$. It is straightforward to verify the completeness relation $\sum_k M_k^\dagger M_k = \mathbb{1}$ and that the separable map corresponding to these Kraus operators indeed implements the transformation.

State transformations which are possible via SEP, but not via SEP_1, can be also found among three-qubit states. The following example is an adaption of the five-qubit example presented above. Here, we consider a transformation from the three-qubit ring graph state $|\psi\rangle$, which is LU equivalent to the three-qubit GHZ state, to $h_1 \otimes h_2 \otimes h_3\,|\psi\rangle$. As before, $G = \mathbb{1}$ and we choose h such that $H = h^\dagger h = (1/2\mathbb{1} + aZ) \otimes (1/2\mathbb{1} + aX) \otimes (1/2\mathbb{1} + aZ)$, for some $a \in (0, 1/2)$. The stabilizer of the considered representative, $|\psi\rangle$, contains (by definition) the operators $A_i = Z_{i-1}X_iZ_{i+1}$, for $1 \leqslant i \leqslant 3$ and $Z_0 = Z_3, Z_4 = Z_1$ and products thereof. However, in contrast to the five-qubit state above, the state considered here does possess additional symmetries, i.e. more than its Pauli stabilizer. Hence, in order to show that the considered transformation is not possible via SEP_1, we cannot use the same argument as above. However, we can resort to previous results on SEP_1 transformations among three-qubit states [25], instead. In order to do so, we write the final state in the standard form introduced in [25]. One obtains that the final state is up to local unitaries of the form

$$h_x \otimes h_x \otimes h_x\,|\mathrm{GHZ}\rangle\,, \tag{13}$$

with $h_x^\dagger h_x \propto 1/2\mathbb{1} + aX$. As has been shown in [25] it is not possible to reach states of the form above via SEP_1.

Let us now show that the inclusion of singular Kraus operators allows to derive a map in SEP, which maps $|\psi\rangle$ into $h\,|\psi\rangle$. We use the projectors Q_1', Q_2', Q_3', Q_4' defined such that $Q_j' \otimes \mathbb{1}^{\otimes 2} = Q_j$, for Q_j as in equations (9)–(12). Any of these operators annihilates the initial state. The Kraus operators for the separable map then take a similar form as in the previous example, namely: $M_i = a_1 h Q_i'$, for $i = 1, 2, 3$ and with $a_1 = \sqrt{2a^3/((1/2 + a)^2(1/2 - a)(1/8 + a^3))}$, $M_4 = \sqrt{2a^3/((1/2 - a)^3(1/8 + a^3))}hQ_4'$, and $M_5 = (1/2)\sqrt{1/(1/8 + a^3)}h$; $M_6 = M_5A_1, M_7 = M_5A_3, M_8 = M_5A_1A_3$. Again it is straightforward to verify the completeness relation $\sum_k M_k^\dagger M_k = \mathbb{1}$ and that the separable map corresponding to these Kraus operators indeed implements the transformation. Hence, already for three qubits one can observe a difference among these sets of operations. In the next section we will see that finite-round LOCC transformations among pure states are contained in SEP_1.

4.2. State transformations using finitely many rounds of communication

Finite-round LOCC protocols constitute a subset of LOCC that is of particular practical relevance. In this subsection we will show that there exists an $\mathrm{LOCC}_{\mathbb{N}}$ transformation among fully entangled states only if equation (4) holds, as stated in the following lemma.

Lemma 3. *If there exists a map in* $\mathrm{LOCC}_{\mathbb{N}}$ *which transforms a pure, fully entangled state into another, then there also exists a map in* SEP_1 *which accomplishes this transformation, i.e.* $\mathrm{LOCC}_{\mathbb{N}} \subseteq_T \mathrm{SEP}_1$.

Proof. First, note that if an $\mathrm{LOCC}_{\mathbb{N}}$ protocol is solely composed of measurements with regular measurement operators, then a Kraus decomposition of the map containing only local invertible Kraus operators exists. In this case, operators N_q which contain singular matrices are thus not present in equation (5). Let us now show that the case, in which measurements including a singular measurement operator are performed, cannot occur. In order to see this, note that any local operator that annihilates a fully entangled state must be singular at not less than two sites. Any local operator that is singular at only one site, acting on a fully entangled state, thus yields a state with strictly positive norm, which, moreover, must have a rank deficient reduced density matrix at some site. Let us now consider the first round in which one of the parties implements a measurement containing a singular measurement operator. Due to the considerations above, the resulting state corresponding to the singular measurement operator occurs with a strictly positive probability and, furthermore, the resulting state is no longer in the same SLOCC class as the final state. Hence, it is impossible to transform this state via LOCC into the final state [36]. Hence, there is always a non-vanishing probability to obtain a state which is not in the same SLOCC class as the target state, which shows that it is impossible to deterministically transform one fully entangled state into another utilizing in any step of an $\mathrm{LOCC}_{\mathbb{N}}$ protocol a singular matrix. This completes the proof. □

Whether the same holds true also when one includes the possibility of infinitely many rounds of classical communication is currently not clear. When dealing with infinite round protocols, many reasonings that apply to finite round protocols do not hold any more. Hence, the investigation of these protocols is more complicated. In particular, the proof of lemma 3 cannot be straightforwardly generalized to cover LOCC protocols with infinitely many rounds of classical communication. This is because such protocols could in principle implement a SEP transformation with non-invertible Kraus operators through a sequence of $\mathrm{LOCC}_{\mathbb{N}}$ maps $\{\Lambda_i\}$ in which every Λ_i has invertible Kraus operators. In fact, notice that if $|\psi\rangle$ cannot be transformed into $|\phi\rangle$ by SEP_1, this does not forbid the existence of a sequence of SEP_1 maps $\{\Lambda_i\}$ such that $\|\Lambda_i(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|\| \to 0$ as $i \to \infty$.

4.3. States with unitary stabilizer

In this section we will focus on state transformations within an SLOCC class for which a representative with solely unitary local symmetries can be found. It has been shown that whenever there exists a state in an SLOCC class which has a finite stabilizer, then there exists a state in the same SLOCC class which has a unitary stabilizer [21, proposition 5]. Moreover, in case a critical state exists, this state is the critical state [21, proposition 6]. We derive a necessary condition for SEP-transformations to be possible as well as a sufficient condition under which singular Kraus operators need not be taken into account, as stated in the following lemma.

Lemma 4. *Let* $\mathcal{S}_\psi$ *be unitary. Consider an initial state* $g\,|\psi\rangle$ *and a final state* $h\,|\psi\rangle$*, where we choose w.l.o.g.* $\|g\,|\psi\rangle\,\| = \|h\,|\psi\rangle\,\| = 1$*. Then, if* $g\,|\psi\rangle$ *can be transformed into* $h\,|\psi\rangle$ *via SEP, it necessarily holds that* $\mathrm{tr}(G) \geqslant \mathrm{tr}(H)$*. Moreover, in case* $\mathrm{tr}(G) = \mathrm{tr}(H)$*, the SEP transformation is possible if and only if equation (4) holds, i.e. if and only if a* SEP_1 *transformation is possible.*

Proof. Consider theorem 2 for states with unitary stabilizers, i.e. $\mathcal{S}_\psi \subset U(d_1) \otimes \cdots \otimes U(d_n)$. Taking the trace of equation (5) and using that $r = 1$ we obtain

$$\mathrm{tr}(H) + p = \mathrm{tr}(G), \tag{14}$$

where $p = \mathrm{tr}(g^\dagger \sum_q N_q^\dagger N_q g)$. Note that $p \geqslant 0$. The assertion follows from the fact that $p = 0$ iff $N_q = 0\,\forall q$, as the trace of positive operators is positive and as g is regular. □

Hence, for normalized initial and final states, projective measurements need not be taken into account as long as $\mathrm{tr}(H) = \mathrm{tr}(G)$ (in case the stabilizer is unitary). Note that in the examples presented in section 4.1, $\mathrm{tr}(H) = \mathrm{tr}(G)$ is obviously not fulfilled, when one normalizes the states.

In the following we will use lemma 4 to provide an adaptation of the proof of the examples of pure state transformation that are possible via SEP_1 but not via LOCC given in [28]. In particular, we will take into account that there might exist transformations that can be implemented via SEP but not via SEP_1.

4.4. Examples of pure state transformations that are possible via SEP, but not via LOCC

In [28], some of us have considered examples of pure state transformations that are possible via SEP, but not (infinite-round) LOCC. There, however, restricted LOCC operations have been considered, as SEP_1 was considered to be a superset of LOCC, instead of SEP. We will first briefly review the examples, as well as the main idea of the proof. Then, we will present an adaptation of the proof to show that, indeed, these examples of state transformations are possible via SEP, but not LOCC. Let us mention here that these examples further show that $\mathrm{SEP}_1 \not\subseteq_T \mathrm{LOCC}$.

Let $|\psi\rangle$ denote the 3 qutrit seed states presented in [28]. As shown in [28] (see also [32]), we have that $\mathcal{S}_\psi$ contains only (nine) unitary elements. We consider the transformation from $|\psi\rangle$ to $h\,|\psi\rangle$ (both normalized), where $h = h_1 \otimes h_2 \otimes \mathbb{1}$ as given in lemma 9 in [28]. In order to keep the flow of reading, we defer recalling additional details on h and $|\psi\rangle$ to appendix C, as they are unnecessary to follow the subsequent reasoning. However, it will become important that $\mathrm{tr}(H) = \mathrm{tr}(\mathbb{1})$ for $\|h\,|\psi\rangle\,\| = 1$, as can be easily verified. As shown in [28] $|\psi\rangle$ can be mapped to $h\,|\psi\rangle$ via SEP_1 (and therefore also via SEP). However, the proof that the transformation is not possible via LOCC has to be adapted. The reason for that is that in [28], we have argued that the transformation is not possible via LOCC as

(a) $|\psi\rangle$ cannot be transformed to $h\,|\psi\rangle$ in a single round of classical communication (not even probabilistically) and

(b) $|\psi\rangle$ is the only state that can be transformed to $h\,|\psi\rangle$ via SEP_1.

However, statement (b) might no longer hold for SEP if one takes operators N_q into account, i.e. if one considers the most general SEP operations.

Let us now assume that there exists an LOCC protocol transforming $|\psi\rangle$ into $h\,|\psi\rangle$ and show a contradiction. The LOCC protocol must be non-trivial, hence there must exist a first round, in which a non-trivial measurement is performed. The state at hand before this round is still (LU-equivalent to) $|\psi\rangle$ and all intermediate states afterward are of the form $g_i\,|\psi\rangle$, where g_i acts trivially on all parties but i, and are normalized such that $\|g_i\,|\psi\rangle\,\| = 1$. It is important to note that for any such g_i, it can be shown that $\mathrm{tr}G_i = \mathrm{tr}\,\mathbb{1} = \mathrm{tr}\,H$. Here, the first equality follows from the fact that g_i acts trivially on all but one parties and the special form of $|\psi\rangle$. The second equality follows from the special form of the considered h and $|\psi\rangle$ as in [28] (see also appendix C). As the protocol must be deterministic, all intermediate states $g_i\,|\psi\rangle$ must be convertible to $h\,|\psi\rangle$ via LOCC and thus via SEP. As all the conditions for lemma 4 are satisfied, this lemma implies that g_i and h must satisfy equation (4). However, in [28] it is shown that the only state which fulfills (up to LU) this condition is $|\psi\rangle$ itself. Hence, all $g_i\,|\psi\rangle$ are LU-equivalent to $|\psi\rangle$. This contradicts the fact that we were considering a non-trivial round and proves that these transformations cannot be implemented via LOCC.

5. Conclusion and outlook

In this work we considered state transformations among pure fully entangled states via separable maps and certain subsets of SEP. In particular, we showed that for the most general transformation via SEP, it is essential to include Kraus operators that occur with zero probability when applied to the initial state as there exist state transformations which are not possible otherwise. This can already be observed in the three qubit and five-qubit scenario. Moreover, we proved that finite-round LOCC protocols do neither require nor even allow for local measurements containing singular measurement operators in case the initial and the final state are fully entangled. In case the stabilizer is unitary we found a necessary condition for the existence of pure state transformations via SEP that is independent of the stabilizer. Moreover, we found constraints under which the existence of pure state transformations via SEP coincides with those via SEP_1. The latter we used to prove that the examples given in [28] indeed correspond to pure state transformations which are possible via SEP and not via LOCC (including infinitely many rounds of classical communication). The main open question is whether $\mathrm{LOCC} \subseteq_T \mathrm{SEP}_1$ holds or not. The answer to it would not only shed light on how results of previous works need to be interpreted but in case it is negative, it would also show that there are pure state transformations which only become possible if infinite rounds of classical communication are utilized.

Acknowledgments

BK thanks R Brieger and D Sauerwein for discussions related to the characterization of the local unitary symmetries of special graph states. ME, MH, and BK acknowledge financial support from the Austrian Science Fund (FWF) grant DK-ALM: W1259-N27 and the SFB BeyondC (Grant No. F7107). Furthermore, ME and BK acknowledge support of the Austrian Academy of Sciences via the Innovation Fund 'Research, Science and Society' as well as support from the Austrian Science Fund (FWF) grant FG5-L. CS acknowledges support by the Austrian Science Fund (FWF): J 4258-N27 and the ERC (Consolidator Grant 683107/TempoQ). JIdV acknowledges financial support by the Spanish MINECO through Grants MTM2017-84098-P and MTM2017-88385-P and by the Comunidad de Madrid through Grant QUITEMAD-CM P2018/TCS-4342.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Formal details on LOCC

In this section we give precise definitions of the terms LOCC and $\text{LOCC}_{\mathbb{N}}$. A CPTP map Λ is said to be in $\text{LOCC}_{\mathbb{N}}$ with m rounds of classical communication if it admits a Kraus decomposition with Kraus operators $\{K_i\}$ in which the index i can be decomposed as a multi-index $i = (i_1, \dots, i_m)$ so that

$$K_i = K_{(i_1,\dots,i_m)} = \prod_{k=1}^{m} L_{i_k}(\{i_j\}_{j<k}), \tag{A1}$$

where the product should be understood from right to left (e.g. $\prod_{k=1}^{2} L_{i_k} = L_{i_2}L_{i_1}$) and

$$\begin{aligned} L_{i_k}(\{i_j\}_{j<k}) = U_{i_k}^{(1)}(\{i_j\}_{j<k}) \otimes \cdots \otimes U_{i_k}^{(s_k-1)}(\{i_j\}_{j<k}) \otimes P_{i_k}^{(s_k)}(\{i_j\}_{j<k}) \\ \otimes U_{i_k}^{(s_k+1)}(\{i_j\}_{j<k}) \otimes \cdots \otimes U_{i_k}^{(n)}(\{i_j\}_{j<k}), \end{aligned} \tag{A2}$$

where all the matrices labeled with U are unitary, $s_k = s_k(\{i_j\}_{j<k})$ and

$$\sum_{i_k} (P_{i_k}^{(s_k)}(\{i_j\}_{j<k}))^{\dagger} P_{i_k}^{(s_k)}(\{i_j\}_{j<k}) = \mathbb{1} \tag{A3}$$

for all the possible values of $\{i_j\}_{j<k}$. That is, every element of the multi-index i_k corresponds to a round, in which party s_k implements a generalized measurement with measurement operators $\{P_{i_k}\}$. The identity of this party and the particular map he/she implements depend on all previous values of the elements of the multi-index $\{i_j\}_{j<k}$, which are known to every party through the use of classical communication. Then, party s_k transmits to all other parties the precise outcome i_k he/she obtains implementing the generalized measurement. Based on this value and all previous values of the elements of the multi-index, the remaining parties implement a unitary transformation to their share of the state, which concludes the round.

In order to define the set LOCC allowing for infinitely many rounds of classical communication, we first need to introduce the notion of composable $\text{LOCC}_{\mathbb{N}}$ maps. An $\text{LOCC}_{\mathbb{N}}$ map Λ with m rounds of classical communication and an $\text{LOCC}_{\mathbb{N}}$ map Λ' with $m+1$ rounds of classical communication are said to be composable if they admit a Kraus decomposition as above with respective Kraus operators $\{K_i\}$ and $\{K_i'\}$ such that

$$K'_{(i_1,\dots,i_{m+1})} = L_{i_{m+1}}(\{i_j\}_{j<m+1}) K_{(i_1,\dots,i_m)}, \tag{A4}$$

where $L_{i_{m+1}}(\{i_j\}_{j<m+1})$ can be written as in equation (A2). Thus, a CPTP map Λ is said to be in LOCC if it is in $\text{LOCC}_{\mathbb{N}}$ or if it is the limit of a sequence of $\text{LOCC}_{\mathbb{N}}$ maps $\{\Lambda_i\}$ in which the maps Λ_i and Λ_{i+1} are composable $\forall i$ and in the latter case we say that $|\psi\rangle$ can be converted into $|\phi\rangle$ if $\lim_{i\to\infty} \|\Lambda_i(|\psi\rangle\langle\psi|) - |\phi\rangle\langle\phi|\| = 0$ in any matrix norm $\|\cdot\|$ of choice.

Appendix B. Local stabilizer of the 5 qubit ring graph

Let $|\psi\rangle$ be the five-qubit ring graph state and let $T_{\psi} = \langle\{A_i\}_i\rangle$ be its Pauli stabilizer. For an introduction to graph states see [30]. We show here that $\mathcal{S}_{\psi} = T_{\psi}$. For a more general form of this proof see [33].

First we use that $|\psi\rangle$ is a connected graph state and thus a critical state. For critical states it holds that if the number of unitary elements in $\mathcal{S}_\psi$ is finite, then these are the only elements of $\mathcal{S}_\psi$ [31]. Hence, showing that any unitary element of $\mathcal{S}_\psi$ is an element of T_ψ (which is a finite group) implies the statement. In order to see that, note that for a graph state, $|\psi\rangle$ it holds that $\rho \equiv |\psi\rangle\langle\psi| \propto \sum_{T\in T_\psi} T$. Taking the partial trace over system 4, 5 and over system 3, 4, the condition

$$U\rho U^\dagger = \rho, \tag{B1}$$

with $U = U_1 \otimes \cdots \otimes U_5$ implies that

$$U_1 Z U_1^\dagger \otimes U_2 X U_2^\dagger \otimes U_3 Z U_3^\dagger = Z \otimes X \otimes Z \tag{B2}$$

$$U_1 X U_1^\dagger \otimes U_2 Z U_2^\dagger \otimes U_5 Z U_5^\dagger = X \otimes Z \otimes Z. \tag{B3}$$

Hence, U_1 has to leave X and Z invariant under conjugation (up to a proportionality factor). It is straightforward to see that this implies that $U_1 \in \langle X, Z\rangle$ (up to a phase factor). A similar argument holds for any other U_j $(j \neq 1)$ due to the symmetry of the state. Next, we show that there exists no Pauli operator, $\sigma_1 \otimes \cdots \otimes \sigma_5 \notin T_\psi$, which is a symmetry of the graph state. To demonstrate this, we note that the action of any Pauli operator on a graph state coincides with the action of an operator $Z^{\vec{k}}$, with $k_i \in \{0, 1\}$ (up to some phase) (see e.g. [30]), i.e.

$$U|\psi\rangle = \mathrm{e}^{\mathrm{i}\gamma}\sigma_1 \otimes \cdots \otimes \sigma_5 |\psi\rangle = \mathrm{e}^{\mathrm{i}\tilde{\gamma}} Z^{\vec{k}} |\psi\rangle \stackrel{!}{=} |\psi\rangle . \tag{B4}$$

For $\vec{k} \neq \vec{0}$ we have that $|\psi\rangle$ is orthogonal to $Z^{\vec{k}}|\psi\rangle$ (see e.g. [30]) and thus for equation (B4) to hold necessarily $U \in T_\psi$.

Appendix C. Details on the states considered in section 4.4

In this appendix we recall some details on the states $|\psi\rangle$ and $h|\psi\rangle$ from [28], which we use in section 4.4 in the main text.

The state $|\psi\rangle$ is defined as

$$|\psi\rangle = a\left(|000\rangle + |111\rangle + |222\rangle\right) + b\left(|012\rangle + |201\rangle + |120\rangle\right) + c\left(|021\rangle + |210\rangle + |102\rangle\right), \tag{C1}$$

for some $a, b, c \in \mathbb{C}$ s.t. $|\psi\rangle$ is normalized. Excluding certain particular choices a, b, c which are listed below equation (4) in [28], the stabilizer of $|\psi\rangle$ is given by a discrete set of nine unitary operators, as mentioned in the main text.

The operator h which we use in section 4.4 is given in lemma 9 of [28]. Let us recall here the definition of h. To do so, we make use of the nine generalized Pauli matrices of dimension 3, which form a basis for 3×3 matrices. As mentioned in the main text, the considered operator h acts non-trivially on two sites, i.e. $h = h_1 \otimes h_2 \otimes \mathbb{1}$. Decomposing the operators $H_1 = h_1^\dagger h_1$ and $H_2 = h_2^\dagger h_2$ into the basis of the nine generalized Pauli matrices and disregarding the $\mathbb{1}$ component, we have eight components for H_1 and H_2, respectively (note that some of the components are not independent as H_1, H_2 must be Hermitian). See also equations (9), (10) and (13) in [28]. We now have that considering H_1, four of the components must vanish while the remaining four components must not vanish. The same is true for H_2. Moreover, there is a constraint specifying which of the components are the vanishing ones. Namely, the components that vanish for H_1 must not vanish for H_2 and vice versa. This completes the definition of h [28]. As stated in the main text, considering the normalization $\langle\psi|\psi\rangle = \langle\psi| h^\dagger h |\psi\rangle = 1$, the specific form of $|\psi\rangle$, as well as the specific form of h it may be easily verified that $\mathrm{tr}\, H = \mathrm{tr}\, \mathbb{1}$.

ORCID iDs

Martin Hebenstreit https://orcid.org/0000-0002-5841-7082
Julio I. de Vicente https://orcid.org/0000-0002-6508-5709
Barbara Kraus https://orcid.org/0000-0001-7246-6385

References

[1] Hillery M, Bužek V and Berthiaume A 1999 *Phys. Rev.* A **59** 1829
Gottesman D 2000 *Phys. Rev.* A **61** 042311
[2] Raussendorf R and Briegel H J 2001 *Phys. Rev. Lett.* **86** 5188

[3] For a review see e.g. Amico L, Fazio R, Osterloh A and Vedral V 2008 *Rev. Mod. Phys.* **80** 517
[4] See e.g. Orús R 2014 *Ann. Phys., NY* **349** 117
[5] See e. g. Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[6] Chitambar E and Gour G 2019 *Rev. Mod. Phys.* **91** 025001
[7] Nielsen M A 1999 *Phys. Rev. Lett.* **83** 436
[8] Chitambar E, Leung D, Mančinska L, Ozols M and Winter A 2014 *Commun. Math. Phys.* **328** 303
[9] Lo H-K and Popescu S 2001 *Phys. Rev.* A **63** 022301
[10] Chitambar E 2011 *Phys. Rev. Lett.* **107** 190502
[11] Chitambar E and Hsieh M-H 2017 *Nat. Commun.* **8** 2086
[12] Kraus B 2010 *Phys. Rev. Lett.* **104** 020504
Kraus B 2010 *Phys. Rev.* A **82** 032121
[13] Gingrich R 2002 *Phys. Rev.* A **65** 052302
[14] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[15] Turgut S, Gül Y and Pak N K 2010 *Phys. Rev.* A **81** 012317
[16] Kintas S and Turgut S 2010 *J. Math. Phys.* **51** 092202
[17] Spee C, de Vicente J I, Sauerwein D and Kraus B 2017 *Phys. Rev. Lett.* **118** 040503
de Vicente J I, Spee C, Sauerwein D and Kraus B 2017 *Phys. Rev.* A **95** 012323
[18] Rains E M 1997 arXiv:quant-ph/9707002
[19] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Rev.* A **59** 1070
Kleinmann M, Kampermann H and Bruß D 2011 *Phys. Rev.* A **84** 042326
[20] Gheorghiu V and Griffiths R B 2008 *Phys. Rev.* A **78** 020304(R)
[21] Gour G and Wallach N R 2011 *New J. Phys.* **13** 073013
[22] Gour G and Wallach N R 2019 *New J. Phys.* **21** 109502
[23] Gour G, Kraus B and Wallach N R 2017 *J. Math. Phys.* **58** 092204
[24] Sauerwein D, Wallach N R, Gour G and Kraus B 2018 *Phys. Rev.* X **8** 031020
[25] de Vicente J I, Spee C and Kraus B 2013 *Phys. Rev. Lett.* **111** 110502
[26] Sauerwein D, Schwaiger K, Cuquet M, de Vicente J I and Kraus B 2015 *Phys. Rev.* A **92** 062340
[27] Spee C, de Vicente J I and Kraus B 2016 *J. Math. Phys.* **57** 052201
[28] Hebenstreit M, Spee C and Kraus B 2016 *Phys. Rev.* A **93** 012339
[29] Englbrecht M and Kraus B 2020 *Phys. Rev.* A **101** 062302
[30] Hein M, Dür W, Eisert J, Raussendorf R, Van den Nest M and Briegel H-J 2006 arXiv:quant-ph/0602096
[31] Wallach N R 2017 *Geometric Invariant Theory: Over the Real and Complex Numbers* (Berlin: Springer)
[32] Briand E, Luque J-G, Thibon J-Y and Verstraete F 2004 *J. Math. Phys.* **45** 4855
[33] Brieger R, Kraus B and Sauerwein D 2018 *Master Theses* University of Innsbruck
[34] Of course, one can also consider maps with different input and output Hilbert spaces but here we only analyze transformations among states with the same number of parties and local dimensions and we will take this into account in all forthcoming definitions.
[35] A construction to obtain more examples of state transformations which are possible via SEP, but not via SEP_1, can be found in [29].
[36] Note that it is impossible that this branch gets completely annihilated as any subsequent measurement (by any party) has to be complete.