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ORIGINAL ARTICLE

Aggregative games

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Abstract
This survey presents in a historical way the main contributions to the hardcore theory of aggregative games and the applications of this model to several fields of economics, other social sciences and engineering.

Keywords Aggregative · Games · Quasi-competitiveness · Shocks

JEL Classification C7 · L1 · L2

1 Introduction

A game is aggregative when, for any player, payoffs depend on her own action and an aggregate that encapsulates all interactions in the game. Usually, this aggregate is taken to be the sum of the strategies of all players or, when the number of players is variable, the average of this sum.

The assumption that all interactions are channeled through a single number is, perhaps, a little bit extreme. In a society or in a market, I interact closely with my friends and relatives or with a small number of firms and somehow on a more anonymous way with the rest of the society.[1] The theory of aggregative games focus on anonymous interactions. This brings an enormous simplification to the players involved in the game and to the analyst using game theory as a tool: As a player, to take my optimal decision, I just need to forecast the value of the aggregate and how this aggregate changes with

[1] For a discussion of this point, see our comments in Sect. 6.

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Table 1 .

	a_i	a	$U_i(a_i, a)$	$a = f(a_1, ..a_n)$
Trade Unions	Wages	Inflation	Payoffs Un.	Inflation f(wages)
Pollution	Output	Pollution	Profit	Pollution Output
Contribution	Inputs	Pub. Good	U. Function	Prod.Pub. Good
Ppal-Agent	id	Output	id	Prod. Output
Pref Revelation	Message	Social State	id	Social Rule
Tragedy Comm.	Inputs	Environ.	Profits	Damage Function
Oligopoly	Output	Price	id	Inverse Demand
Technolog comp	Effort	Technology	id	Tech Production

my actions (of course, I should also know my preferences). As an analyst, to predict the consequences of a shift in a parameter, I do not need to disentangle the effects on this or that strategy. It suffices to focus on the aggregate. This aggregate may be an official statistic or belief that is used by every player.

It turned out that some models that economists have been using for a long time have this aggregative structure, see Table 1. So we have an additional payoff: a result obtained in the realm of aggregative games can be applied to models in different fields of economics. This point will be expanded and make more precise in Sect. 6.

The rest of the paper goes as follows. Section 2 spells the model. Section 3 presents the first results obtained in this framework. Section 4 presents the key contributions made in 1993–1994, while Sect. 5 describes the development of this area before we attain the boundary of our present knowledge. Section 6 presents some recent contributions dealing with extensions and applications both old and new. We end with a section suggesting future avenues of research.

2 Preliminaries

There are n players with actions $a_i \in A_i \subseteq \mathbb{R}_+$ $i = 1, 2, \ldots n$. Let $a = \sum_{j=1}^{n} a_j$ be the aggregate of all players' actions and $a_{-i} = a - a_i$. Preferences over actions are represented by a payoff function $U_i = U_i(a_i, a)$. An alternative representation is obtained by writing payoffs as a function of a_i and a_{-i} since $U_i(a_i, a) = U_i(a_i, a_{-i} + a_i) = V_i(a_i, a_{-i})$ say.[2] An important fact is that two-person games are aggregative.[3] Many of the results presented here are generalizable to an aggregate $\Phi(a)$ introducing suitable concavity and monotonicity assumptions on $\Phi()$ Corchón (1994, p. 154). Generalizations to multidimensional action spaces are harder, see, e.g., Okuguchi

[2] When the aggregate is represented by a general function $g(\boldsymbol{a}_{-i})$, this equivalence no longer holds, [see Jensen (2010, p. 48)].

[3] In social choice, some results for models with two agents and more than two agents are different. And in statistics and differential equations the cases of two and more than two dimensions often yield different results, see Corchón (2009), comments to Theorem 7. See also González-Maestre 2000 for this discrepancy in a IO model.

and Szidarovszky. (1990) for the case of oligopolistic firms and Jensen (2010).[4] Let $\boldsymbol{a} = (a_1, a_2, \ldots a_n)$ be a list of actions and $\boldsymbol{a}_{-i} = (a_1, a_2, \ldots a_{i-1}, a_{i+1}, \ldots a_n)$ be a list of actions of all players minus i. Note that vectors are denoted in bold to distinguish them from the sum, a.

Our main tool will be the notion of a Nash equilibrium (NE) in a normal form game in which actions are strategies. Recall that in a NE each player chooses her strategy having forecasted correctly the strategies chosen by others. Actions corresponding to a NE are denoted by $\mathbf{a}^* = (a_1^*, a_2^*, \ldots a_n^*)$. For the time being, let us stick to the classical assumptions of continuity of payoffs in actions, strict concavity in own's actions and convexity and compactness of action sets to guarantee the existence of NE, see, e.g., Friedman (1977, pp. 152–154).[5] This proof uses the concept of *Best Reply* defined as the actions that maximize a player's payoff given the actions of all other players. It is denoted by $B_i(\boldsymbol{a}_{-i})$. Under strict concavity, the collection of functions $\{B_i(\boldsymbol{a}_{-i})\}_{i=1}^n$ maps continuously $\times_{j=1}^n A_j$ into $\times_{j=1}^n A_j$ and the fixed point, whose existence is guaranteed by Brouwer fixed point theorem, is a NE.

Throughout these notes, we will assume differentiability and, as a simplification device, interiority. In a NE, first-order condition (FOC in the sequel) of payoff maximization for player i is

$$\frac{\partial U_i(a_i^*, a^*)}{\partial a_i} + \frac{\partial U_i(a_i^*, a^*)}{\partial a} = 0 \tag{1}$$

since $\mathrm{d}a/\mathrm{d}a_i = 1$. Let us denote the left hand side of (1) as $T_i(a_i^*, a^*)$. Differentiating (1), we derive the slope of the best reply

$$\frac{\mathrm{d}B_i(a_{-i}^*)}{\mathrm{d}a_{-i}} = \frac{\frac{\partial T_i(a_i,a)}{\partial a}}{-\left(\frac{\partial T_i(a_i,a)}{\partial a_i} + \frac{\partial T_i(a_i,a)}{\partial a}\right)}. \tag{2}$$

If second-order conditions of payoff maximization hold with strict inequality, the denominator of (2) is positive and the sign of the best reply is determined by the sign of the numerator which is

$$\frac{\partial T_i(a_i^*, a^*)}{\partial a_i} = \frac{\partial^2 U_i(a_i^*, a^*)}{\partial a_i \partial a} + \frac{\partial^2 U_i(a_i^*, a^*)}{\partial a^2}$$

since $\mathrm{d}a/\mathrm{d}a_{-i} = 1$.[6]

We end this section noting that when payoffs can be written as $g_i(a_i)h(a)$, aggregative games are potential games, i.e., games in which some NE (not always all NE) can be found by maximizing a single function called the potential function, see Monderer

[4] A simple case of two dimensional aggregates is presented in Sect. 5, see also Nocke and Schutz (2018). Dickson (2017) and Cornes et al. (2019).

[5] Friedman uses the more general assumption of quasi-concavity of payoffs in own's action.

[6] In more technical term, profits are a Morse function, i.e., a smooth function with non-degenerate critical points (Christensen 2019, footnote 26).

and Shapley (1996).[7] This is useful because in potential games a pure strategy NE exist without any convexity or quasi-concavity assumptions. It also implies convergence of best-reply dynamics under some additional assumptions. In our case, the potential is $h(a)\Pi_{i=1}^{n} g_i(a_i)$. FOC is

$$h(a)\frac{\mathrm{d}g_i(a_i)}{\mathrm{d}a_i}\Pi_{j\neq i}g_j(a_i) + \frac{\mathrm{d}h(a)}{\mathrm{d}a}\Pi_{i=1}^{n} g_i(a_i) = 0$$

So if in a NE all $g_i()$'s are positive, we obtain

$$h(a)\frac{\mathrm{d}g_i(a_i)}{\mathrm{d}a_i} + \frac{\mathrm{d}h(a)}{\mathrm{d}a}g_i(a_i) = 0$$

which are FOC of the maximization of $g_i(a_i)h(a)$. See Jensen (2010) for a careful analysis of the relationship between potential and aggregative games and Cheung and Lahkar (2018) for the study of large population aggregative games as potential games.

3 In the beginning

Aggregative games have been known to economists, at least, from the Cournot contribution Cournot (1838). The first person to use the concept of aggregative games was Reinhard Selten in a 1970 monograph written in German. Subsection 9.2 in the monograph is entitled "The Class of aggregative (aggregierbaren) strategic games" (p. 150). In Section 9.3 "Existence of a best equilibrium," he proves the existence of a NE. He was the first to realize that the aggregative structure simplifies the computation of a NE. From FOC of payoff maximization, he derives a mapping $a_i = R_i(a)$ and adding up over individuals he obtains

$$a = \sum_{i=1}^{n} R_i(a) \equiv R(a), \tag{3}$$

say. Fixed points of (3) are NE.[8] Then, this fixed point a is plugged into the mappings $R_i()$ to find individual actions.[9]

The second paper I am aware of, to deal with aggregative games is by Dubey et al. (1980). This paper was written at the time in which big steps forward were made

[7] An example of multiplicative separability of payoffs in a_i and a is an oligopoly with inverse demand $f(a)$ and linear costs.

[8] This idea was independently discovered by Okuguchi (1993), see Sect. 3.

[9] If all actions are positive, we are done. But when this procedure yields a negative individual action, we eliminate this agent and check that the new fixed point yields nonnegative actions for all the remaining agents. And if so, check that zero is the best reply for the agent left in the cold. If there are several agents whose actions are negative, we have to choose the agent who is more likely to be inactive. For instance in oligopoly, firms with higher costs or a bit demand are good candidates to produce zero in a NE. And repeat the procedure until you get nonnegative actions for all agents.

in implementation theory. For the benefit of the reader, a brief recap of this theory follows.

Suppose agents have utility functions defined on net trades in markets for a finite number of goods. Let z_i be the vector of net trades of agent i. The dimension of z_i is the number of goods. Preferences of i over net trades are representable by a utility function $u_i(z_i)$.[10] Each agent sends a message from a message space M_i. And "the center" (aka the planner, the designer,...) awards a net trade to i depending of the messages sent by all agents $\boldsymbol{m}$, so $z_i = f_i(\boldsymbol{m})$. The functions $\{f_i()\}_{i=1}^n$, called outcome functions, choose net trades that for each commodity are non-positive to respect feasibility. $(f_i(), M_i)_{i=1}^n$ is called a mechanism. Plugging $f_i()$ into $u_i()$ we obtain $u_i(f_i(\boldsymbol{m}))$ which can be written as $U_i(\boldsymbol{m})$ so we have a game in normal form. The idea is to choose a mechanism such that "equilibrium" messages yield allocations that are desirable, e.g., Pareto efficient, individually rational, etc.

We are familiar with a concept that yields allocations with these properties, namely Walrasian equilibrium. The problem of Walrasian equilibrium is that requires a fictitious auctioneer calling equilibrium prices. At some point, one of the creators of implementation, Leo Hurwicz asked, why do not swap the auctioneer for the center? Of course, we do not want a very complicated message space or an outcome function that is beyond comprehension for agents like me struggling with my income tax file. Suppose that we want a message space which mimics a market with messages interpreted as prices and quantities. Schmeidler (1980) presented such a mechanism. Now to avoid each agent looking at a myriad of individually set prices and quantities, we would like this game to be aggregative. This is the question addressed in the Dubey–Mas–Colell–Shubik paper.[11] And the answer is that aggregative games are unlikely to yield Pareto efficient NE when the number of agents is finite. Let us see why in a very simple model.

Under suitable concavity conditions, Pareto efficiency is found by choosing actions that maximize $\sum_{i=1}^n \alpha_i U_i(a_i, a)$. FOC are

$$\alpha_i \left(\frac{\partial U_i(a_i, a)}{\partial a_i} + \frac{\partial U_i(a_i, a)}{\partial a} \right) + \sum_{j \neq i}^{n} \alpha_j \frac{\partial U_j(a_i, a)}{\partial a} = 0, i = 1, 2, \ldots, n. \quad (4)$$

According to (1), in a NE the first two terms are zero. To ease notation let $r_j = \alpha_j \partial U_j(a_i^*, a^*)/\partial a$. Subtracting the FOC for agent 2 to FOC of agent 1, we get $r_1 - r_2 = 0$ and repeating this procedure for agents 3 and 2, etc., we get $r_1 = r_2 = \cdots r_n$ and (4) implies that all these derivatives are zero. But this means that if payoffs are concave on a, a^* is the common maximizer of the utility of all agents, a very unlikely event with a finite number of agents.[12] The result obtained by Dubey et alia is a generalization of the previous argument to multidimensional action spaces that are

[10] This is a model in which consumptions, $\boldsymbol{x}_i$, have been converted into net trades $z_i = \boldsymbol{x}_i - \boldsymbol{w}_i$ where $\boldsymbol{w}_i$ is the vector of initial endowments of i . The utility function of i is $v_i(\boldsymbol{x}_i) = v_i(z_i + \boldsymbol{w}_i) = u_i(z_i)$, say.

[11] They motivate the aggregation axiom saying that "...we would argue that it reflects well the aggregate nature of the impact of demand and supply in organized markets." (p. 346).

[12] "...but for coincidental cases, no (NE) allocation will be efficient" (Dubey et al., p. 351).

not necessarily Euclidean.[13] This result is called by Martimort and Stole (2012) the *principle of aggregate concurrence* (where concurrence means assent, this principle was already used by Lindahl) meaning that, in a NE, all agents must agree on the level of the aggregate. It is a kind of invariance property that plays an important role in some impossibility results obtained when designing optimal incentive-compatible mechanisms.

The last part of Dubey et alia deals with a large game with a continuum of agents. Now the effect of i's action on a is negligible "naturally". Therefore, each agent maximizes payoffs disregarding the strategies taken by the rest. Does it sound familiar? Yes, the resulting allocation is Walrasian *relative to the set of markets that are open* . This proviso is essential because if all my potential suppliers insist in prices that make preferable for me to consume my initial endowments, there is no way I can activate markets to fulfill its trading role so any price is good for me. The same for everybody, so a situation in which some (may be all) markets are inactive is indeed a possible outcome of a NE.[14]

Unfortunately, later work showed that the assumption of differentiability is crucial for NE to be not efficient. Dubey (1982) presented a Bertrand-type of competition model in which, NE yields Walrasian allocations for all markets that are open.[15] Corchón and Wilkie (1996) presented a mechanism with an aggregative structure yielding cost-share equilibrium allocations (a generalization of Lindahl equilibrium, also Pareto efficient) mimicking market rules. A simplified version follows. Let p_i be the Lindahl price of a public good denoted by y. The net consumption of the private good for player i is z_i. The public good is produced under constant returns with a cost cy. Feasibility requires that the sum of net contributions $-z_i$ add up to cy. A Lindahl equilibrium is a feasible allocation and a list of personalized prices $\{\bar{p}_i, \bar{z}_i, \bar{y}\}_{i=1}^n$ such that $(\bar{z}_i, \bar{y})$ maximizes $u_i(z_i, y)$ over $\bar{p}_i y = -z_i$ and y maximizes $\sum_{i=1}^n \bar{p}_i y - cy$. Now consider the following mechanism. Let $\boldsymbol{a}_i = (q_i, y_i)$ where q_i is interpreted as the personalized price proposed by agent i and y_i is the incremental value of the public good proposed by this agent. The outcome function is the following:

$$\text{If } \sum_{i=1}^n q_i \geq c, \text{ then } \quad y = \sum_{i=1}^n y_i \text{ and } z_i = -q_i \sum_{i=1}^n y_i. \; y = 0 \text{ and } z_i = 0 \text{ otherwise.}$$

The outcome function says that if the sum of intended contributions covers production costs, any agent can have as much public good as she likes (the aggregate concurrence principle mentioned before). But if production costs are not covered, sorry, there is no public good. Clearly $q_i = \bar{p}_i$, $i = 1, 2, \ldots n$, $\sum_{i=1}^n y_i = \bar{y}$ is a Nash equilibrium because any attempt to free ride paying a price below $\bar{p}_i$ will be contested by zero production of the public good. And to suggest a larger personalized price has no effect on y but on a smaller net trade of z_i . The mechanism is discontinuous, but it can be amended to be continuous, but not differentiable, at cost of some complications, see Corchón and Wilkie (1996) Sect. 4 for details.

[13] It is easy to check that the argument provided above, works for multidimensional actions.

[14] Attaining Walrasian allocations requires a more complex mechanism. Hammond (1979) proved that, in large economies, only Walrasian allocations are incentive compatible and Pareto efficient.

[15] Beviá et al. (2003) showed that this result holds in games in which coalitions can form and the relevant equilibrium notion is strong equilibrium.

4 Taking off

In 1993, Koji Okuguchi published a paper (accepted for publication in July 1992) that analyzed in "three Cournot models…. the effect of an increase in the number of firms,…effects of a change in a tax rate…(and) the aggregate provision of a pure public good". Let $f(a)$ be the inverse demand function and $c_i(a_i)$ be the cost function of firm i where actions are interpreted as outputs. Profits are $f(a)a_i - c_i(a_i)$. Assuming interiority, FOC of profit maximization is

$$\frac{\mathrm{d}f(a)}{\mathrm{d}a}a_i + f(a) - \frac{\mathrm{d}c_i(a_i)}{\mathrm{d}a_i} = 0 \tag{5}$$

Okuguchi considers an assumption introduced by Frank Hahn (1962) to study the stability of Cournot equilibrium, namely[16]

$$\frac{\mathrm{d}f(a)}{\mathrm{d}a} + a_i\frac{\mathrm{d}^2 f(a)}{\mathrm{d}a^2} < 0 \tag{6}$$

$$\frac{\mathrm{d}f(a)}{\mathrm{d}a} < \frac{\mathrm{d}^2 c_i(a_i)}{\mathrm{d}a_i^2}. \tag{7}$$

(6) and (7) say that the left hand of FOC (the marginal profit) is decreasing in a, being a_i constant and decreasing in a_i, given a. SOC of profit maximization holds because the total effect of a_i on marginal profits is decreasing in a_i. Applying the implicit function theorem, we obtain the individual output as a function of aggregate output and again adding over firms we get a continuous mapping of a over a [($R()$ in (3)] which has a fixed point. Moreover, this fixed point is unique because $R()$ is decreasing (see Okuguchi (2013) for an extension to a model with private and public firms). This fact allows Okuguchi to show that the entry of a new firm increases aggregate output because it shifts $R()$ upward and decreases outputs of incumbent firms because a and a_i must go in opposite directions to maintain FOC. This result is called *Quasi-Competitiveness of Cournot Equilibrium*. The same argument shows that an excise tax decreases both individual and aggregate output. A similar procedure is applied to a model of voluntary contributions (a_i) to a public good (a). Here, under the assumption that leisure and the public good are not substitutes, a_i and a go in opposite directions and entry increases the quantity of the public good.

In Kukushkin (1994), produced an existence theorem on aggregative games that was published next year. He proved that as long as best replies have a decreasing selection and a closed graph, there is a fixed point which is a NE of the aggregative game. With respect to the classical conditions in Kakutani's fixed point theorem, convex-valuedness is dispensed with entirely.[17] Let us see why. The correspondence in Fig. 1 depicts the best reply of player 1 (and the 45 degree line that will be used later). Now, can you draw a decreasing best reply for player 2 such that it does not intersect the

[16] The proof of Hahn was shown to be incorrect by Al-Nowaihi and Levine (1985). Okuguchi and Yamazaki (2008, 2014) provided a correct proof.

[17] This theorem generalizes a result by Novshek (1985) that only applied to Cournot equilibrium.

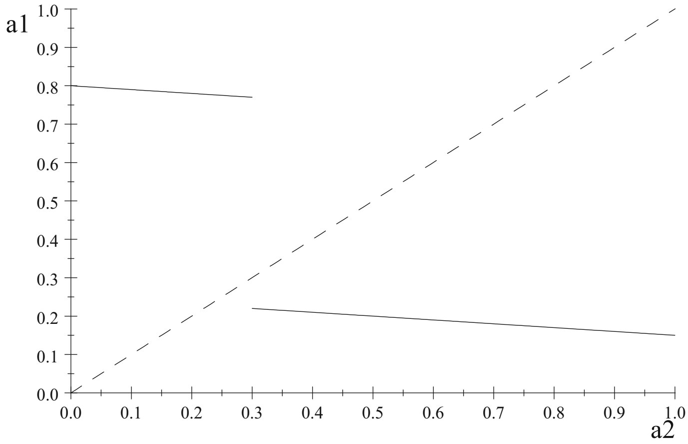

Fig. 1 Existence of equilibrium without convex-valuedness

best reply of player 1? For instance, suppose that $B_2(0) = 0.8$. For larger values of a_1 $B_2()$ must remain below $B_1()$ or jump. Suppose the former. But between $a_1 = 0.7$ and $a_1 = 0.9$ no matter how many jumps best replies must cross. The same argument works for a jumpy best reply.

But wait, if best replies admit a decreasing selection, the composition of these two selections must be increasing. And Tarski fixed point theorem makes sure that a fixed point exists (Vives 1990). The beauty of Kukushkin theorem is that it extends this result to aggregative games with more than two players with decreasing best replies. The latter requirement does not assure the existence of a fixed point in general games.[18]

My own contribution, sent to Mathematical Social Sciences in January 1993 and published in 1994 was an outgrow of my Ph.D. dissertation Corchón (1986).[19] A first version of my dissertation got a revise and resubmit and my examiners, David Ulph and John Moore insisted that my first chapter was a lukewarm, superficial survey of the literature on IO and that I should try something more ambitious. So in the revised version I "provide a general structure in which specific problems can be discussed" (p. 2 of my dissertation). I had in mind that some people refer to IO as "Industrial Disorganization" and this was my bit effort to correct this state of affairs. This chapter was left dormant until I got some spare time during a visiting to the ISI, Delhi, in January 1992 to put up something more polished.[20] I was aware that I was dealing with a whole class of games, see Table 1 taken from my 1994 paper. It is notorious the absence there of contests a subfield that did not exist back then.[21]

[18] For instance, suppose that best replies of players 2 and 3 are $a_2 = 1 - a_3$, $a_3 = 1 - a_1$. Thus, in any fixed point $a_2 = a_1$ which is the dashed line in Fig. 1 that does not intersect $B_1()$.

[19] See acknowledgments in p. 163 of my paper.

[20] The paper was summarily rejected by several major journals. The final version was written during my sabbatical at Harvard in 1994. The editor of MSS, then Hervé Moulin, and a referee provided unusually constructive and telling comments.

[21] Watts (1996) added another application, Surplus Sharing. And Okuguchi (2000) considered Oligopsony. See Sects. 5 and 6 for more recent applications.

From the analytical side, I introduced a generalization of the Hahn conditions that I termed "strong concavity" (SC) and that it says the following:

SC. For all agents, marginal payoffs—the left hand side of Eq. (1)—$T_i()$ are strictly decreasing in a_i given a and strictly decreasing in a, given a_i.[22]

A first consequence of SC is that the SOC of payoff maximization is fulfilled because marginal payoffs are decreasing in a_i. So SC is a concavity condition. It is strong because a weaker condition saying that the marginal payoff is just decreasing in a_i, given a_{-i} would suffice for this purpose. A second consequence is that for any agent a_i and a must go in opposite directions to maintain the equality in (1), i.e., a_i and a_{-i} are strategic substitutes in the terminology of Bulow et al. (1985). A third consequence is that the slope of the best reply is negative and larger than −1. This easily follows from (2) which we recall here,

$$\frac{\mathrm{d}a_i}{\mathrm{d}a_{-i}} = \frac{\frac{\partial T_i(a_i,a)}{\partial a}}{-\left(\frac{\partial T_i(a_i,a)}{\partial a_i} + \frac{\partial T_i(a_i,a)}{\partial a}\right)}. \tag{8}$$

From SC, the denominator of (8) is positive, so a simple contradiction argument shows the third consequence mentioned above namely, that best replies are contractions. And a well-known result says that these type of functions have, at most, a single fixed point, so NE is unique.[23]

Now let us deal with the quasi-competitiveness. In Fig. 2, I picture the FOC of an incumbent firm. Suppose that before entry $a = 2$ and $a_i = .4$. Suppose entry decreases a. But then all a_i must increase to maintain FOC and since a is the sum of all a_i (now augmented by the output of the entrant) this is impossible. Thus, a increases and the action of all incumbents must decrease.[24]

Note that the additive structure of a allows to incorporate easily new entrants.

To deal with how payoffs of incumbents react to entry, we need to assume that payoffs are strictly decreasing in a. Thus, an increase in the strategy of any player j other than i decreases the payoffs of i. In line with the terminology used in consumption theory, we will refer to this assumption as *strategies are substitutes*.[25] Under this assumption, the increase of a_{-i} due to entry causes a decrease in payoffs and overwhelms any possible new choice of a_i (a kind of envelope theorem).

Next, the paper considers the impact of shifts in marginal payoffs on equilibrium actions, called shocks. This shock might affect only one player—idiosyncratic—or all players—generalized. An example of the former is a technological improvement or a

[22] The extension of this condition to multidimensional strategy spaces is given in Gale and Nikaido (1965).

[23] Folmer and von Mouche (2004) generalized this uniqueness result.

[24] See Corchón (1994) for formal proofs that take care of the possibility that some FOCs hold with inequality.

[25] Do not confuse "strategies are substitutes" with strategic substitution that refers to the slope of best reply. Strategies are substitutes refers to the payoff function and is fulfilled in the Cournot model where a is total output and an increase in this variable decreases the market price which decreases profits in turn. It is also fulfilled in contests where an increase in the effort of player j decreases the probability that i obtains the prize. However, this assumption is not fulfilled in the case of price competition because an increase in the prices charged by competitors increases the payoff of firm i. Also, it does not hold in the case of contribution games in which a is the level of the public good which is assumed to be liked by the players.

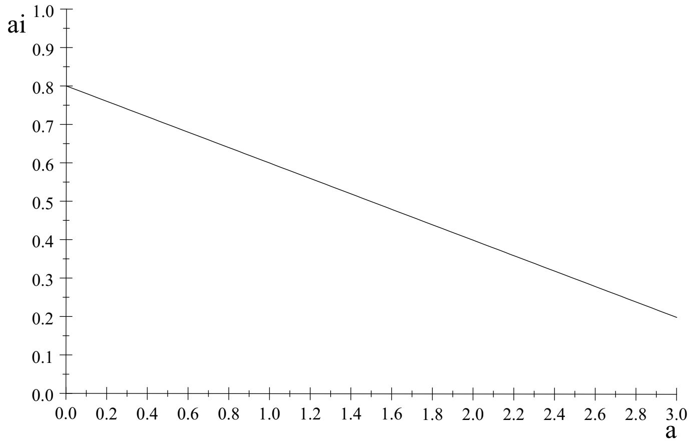

Fig. 2 First order condition of payoff maximization of an agent

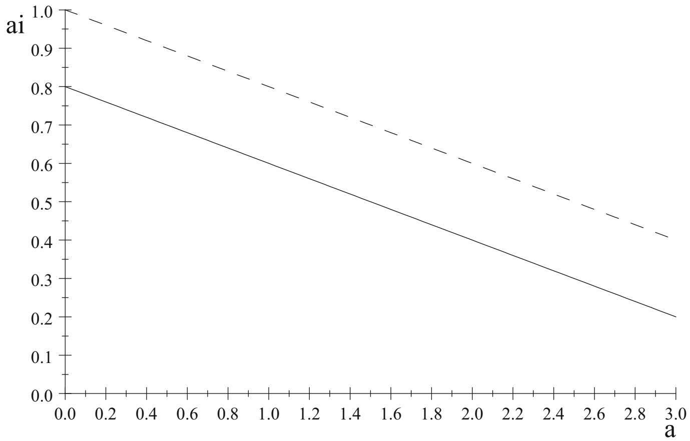

Fig. 3 Effects of a shock on the first order condition

shift in demand or in the price of a factor that only affects a firm. An example of the later is a demand shift affecting all firms.

Under SC, when a shock lifts marginal payoffs upward, a cannot decrease: In Fig. 3, we picture marginal payoffs before (solid) and after (dash) the shock. If a decreases after the shock, all players affected (like in Fig. 3) or unaffected by it (like in Fig. 2) increase their actions, so a must increase. When the shock is idiosyncratic all unaffected players decrease their actions (Fig. 2 again). If the shock affects payoffs in the same direction, given actions, they also increase in the NE after the shock. Finally, the paper presents counterexamples of all these results when SC is violated.

With the benefit of hindsight, it seems that the methods developed in my paper can tackle a more general shift. Let K be a subset of the set of players N. A shift concerning K is a shift that affects only to players in K. When K is a singleton we have an idiosyncratic shock and when $K = N$, we have a generalized shock. The two previous results would be subsumed in the following. "A shift concerning K affects a in the same direction of the shift and increases the actions and payoffs of agents in K and decreases actions and payoffs of agents outside K''.

5 Desert crossing and some applications

I continued with my research agenda by writing a book (1996) in which aggregative games were presented as the glue unifying the diverse fields of industrial organization. This approach also takes care of the fact that firms are not necessarily profit maximizers. Other motives like sales, market share, environmental concern, workers or consumers welfare, etc., have been presented by the relevant literature as legitimate goals. The presentation and writing of this book were very influenced by a book by Jim Friedman (1977). The first edition sold out its meagre first edition, and the book was reedited in hard cover in 2001.

During the nineties and the early twenty-first century, research on aggregative games was kept to a minimum. We now show three applications that were done and I am aware of: Oligopolistic competition with price setting firms, mechanism design and evolutionary equilibrium. Let us consider these applications in turn.

Firstly, d'Aspremont et al. (1991) provided a rationale for Cournot competition when firms can set prices and quantities. [26] Assume that the market price p is a function of individual prices sent by firms $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, so $p = \Psi(\boldsymbol{q})$. The function $\Psi()$ encapsulates either the working of the market or some kind of collusive device used by the oligopolist to achieve a common market price. Also, it can be thought of an outcome function in a mechanism. In addition to these individual prices, firms can set outputs denoted as before by $(a_1, a_2, \ldots, a_n)$ but in a way that they do not exceed demand (D) at the market price represented by the demand function $D = D(p)$ where $D()$ is assumed to be strictly decreasing. Thus, $a \equiv \sum_{j=1}^{n} a_j \leq D(\Psi(\boldsymbol{q}))$. This inequality can be written with equality when p exceeds the marginal cost $\mathrm{d}c_i(a_i)/\mathrm{d}a_i$ of all firms because an increase in production increases profits. In this case, payoffs for firm i are

$$pa_i - c_i(a_i) = \Psi(\boldsymbol{q}) \left(D(\Psi(\boldsymbol{q})) - \sum_{j \neq i} a_j \right) - c_i \left(D(\Psi(\boldsymbol{q})) - \sum_{j \neq i} a_j \right).$$

The game is aggregative because payoffs depend on the aggregates $\Psi(\boldsymbol{q})$ and $\sum_{j \neq i} a_j$. FOC of profit maximization is

$$\frac{\partial \Psi(\boldsymbol{q})}{\partial q_i} \left(D(\Psi(\boldsymbol{q}) - \sum_{j \neq i} a_j) \right) + \Psi(\boldsymbol{q}) \frac{\mathrm{d}D}{\mathrm{d}p} \frac{\partial \Psi(\boldsymbol{q})}{\partial q_i} - \frac{\mathrm{d}c_i(a_i)}{\mathrm{d}a_i} \frac{\mathrm{d}D}{\mathrm{d}p} \frac{\partial \Psi(\boldsymbol{q})}{\partial q_i} = 0.$$

[26] Although they did not cast their model in an aggregative game framework, we will see in a moment that their model is indeed such a game.

Taking into account that $\sum_{i=1}^{n} a_j = D(\Psi(\boldsymbol{q}))$ and if $\Psi()$ is responsive to individual prices, i.e., $\partial\Psi(\boldsymbol{q})/\partial q_i \neq 0$, the previous equation simplifies to

$$a_i + \Psi(\boldsymbol{q})\frac{\mathrm{d}D}{\mathrm{d}p} - \frac{\mathrm{d}c_i(a_i)}{\mathrm{d}a_i}\frac{\mathrm{d}D}{\mathrm{d}p} = 0. \tag{9}$$

The inverse demand function, denoted as $p = f(a)$, exists because demand is strictly decreasing. Thus, $\mathrm{d}D/\mathrm{d}p = 1/(\mathrm{d}f/\mathrm{d}a)$ and (9) yields

$$a_i\frac{\mathrm{d}f(a)}{\mathrm{d}a} + p - \frac{\mathrm{d}c_i(a_i)}{\mathrm{d}a_i} = 0$$

which are FOC of a Cournot equilibrium, see (5). This argument provides an argument for the use of Cournot equilibrium even if firms are setting prices.

The second application provides a kind of Gibbard–Satterthwaite theorem on the impossibility of achieving truthful revelation (Chichilnisky and Heal 1997). Let now utilities be a strictly concave function of a public good $y \in \mathbb{R}_+$ alone. The outcome function is $y = f(\boldsymbol{m})$ with messages denoted by $\boldsymbol{m}$ also in $\mathbb{R}^n$. Messages are interpreted as statements about own preferences, encapsulated in a single number. The function that determines the aggregate, $f()$, is a social choice function, i.e., the desired level of the public good as a function of preferences reported by players. In a NE

$$\frac{\mathrm{d}U_i(y)}{\mathrm{d}y}\frac{\partial f(\boldsymbol{m})}{\partial m_i} = 0$$

so either $\mathrm{d}U_i(y)/\mathrm{d}y = 0$, i.e., y is the most preferred level of the public good (and this agent is a dictator) or $\partial f(\boldsymbol{m})/\partial m_i = 0$ the social choice function is locally constant for this agent. If we do not want dictators, the function must be locally constant for all agents and this is hardly compatible with Pareto efficiency.

Finally, consider a static evolutionary framework. One would expect that agents having more payoffs reproduce faster than those that have smaller payoffs, so eventually they will take the field. And strategies that yield larger payoffs are likely to be copied. Schaffer (1988), using a symmetric homogeneous oligopoly model, showed that the output yielding higher profits among firms is the one that maximizes profits taking aggregate output as given. Since aggregate output and market price are in a one-to-one relationship, this is no less than price-taking behavior. The reason is that when the market price is common for all firms, those behaving as price-takers earn more profits than any other firm using a different strategy. Formally, firm i obtains larger profits than any other firm, say j, in the market, i.e.,

$$f(a)a_i - c(a_i) \geq f(a)a_j - c(a_j).$$

Since this inequality should hold for any possible output of competitors, firm i is maximizing payoffs taking the market price as given. Possajennikov (2003) and Schipper (2003) generalized this result to aggregative games showing that aggregate-taking behavior yields larger payoffs. Formally, if firm i obtains larger payoffs than any other firm, say j, and any other action a_j,

$$U(a_i, a) \geq U(a_j, a).$$

This result was generalized and expanded by Alós-Ferrer and Ania (2005). [27] See Sect. 6.7 for further developments in this area.

It is noteworthy that evolutionary considerations push agents to behave *as* if they were in a large game as discussed in Sect. 3 and Sect. 6.6.

6 The big bang

In the last ten years, we have seen an explosion in the quantity of papers devoted to aggregative games. At present, there are 1.170 results in Google Scholar for "aggregative games" with 719 in the last five years.

In this section, I indicate some new developments that either delve into the existing theory or expand the range of application of these games to uncharted territory. I leave aside important topics like contests, cooperative production or imperfectly competitive models that merit a separate treatment.[28]

6.1 Basic theory

The theory developed so far can be extended to tackle a wider set of situations. Two issues merit a special concern. (1) Under which conditions a game can be written as an aggregative game? (2) How to obtain comparative static results beyond the SC assumption. Let us take these two points in turn.

Cornes and Harley (2012) point out that games that are not aggregative at first glance can be converted into an aggregative game by cunning substitutions, see also Jensen (2010, pp. 48–49). For instance, in contests, the following payoffs are used

$$\pi_i = \frac{\phi_i(a_i)}{\sum_{j=1}^{n} \phi_j(a_j)} V - a_i.$$

The function $\phi_i()$ is the absolute impact of the actions of i in the contest. And the ratio of ϕ's is the relative impact which equals the probability that i wins a prize of value V. a_i is the cost of this action. Payoffs are expected revenue minus costs. ϕ_i's are assumed to be strictly increasing, i.e., the more effort, the more impact. Setting $y_i = \phi_i(a_i)$ we obtain

[27] They also present two new applications of aggregative games, a Diamond search model and a generalized stag-hunt model.

[28] For recent surveys on contests, see Corchón and Serena (2018) and Fu and Wu (2019). For cooperative production, see the recent survey of Beviá and Corchón (2018). For a recent entry on imperfect competition, see the Special Issue of Mathematical Social Sciences (2020).

$$\pi_i = \frac{y_i}{\sum_{j=1}^n y_j} V - \phi_i^{-1}(y_i)$$

which is aggregative.[29] Cornes and Harley generalize this observation to payoff functions $u_i(a_i, t(a))$ where $t(a) = H^{-1}(\sum_{j=1}^n \phi_j(a_j))$ and ϕ_i's are strictly increasing by setting new variables $y_i = \phi_i(a_i)$, so payoffs are now

$$u_i\big(\phi_i^{-1}(y_i), H^{-1}\big(\sum_{j=1}^n y_j\big)\big) = v_i\big(y_i, \sum_{j=1}^n y_j\big),$$

say. They show that this functional form is not only sufficient to convert this game into an aggregative one, it is also necessary (when $n > 2$, again!) for the existence of an additive aggregator. The proof uses a replacement correspondence that is reminiscent of the "cumulative reaction correspondence" introduced in the study of Cournot models by McManus (1962) and McManus (1964).

Acemoglu and Jensen (2013) were the first to venture beyond the realm of aggregative games with a unique and stable NE. First, they consider aggregative games with strategic substitutes. They prove four main results:

1. (Comparative statics of a generalized shock). A shock that hits the aggregator (a special case of a generalized shock) leads to a decrease in the smallest and largest equilibrium aggregates.
2. (Quasi-competitiveness) The entry of an additional player leads to a decrease in the smallest and largest aggregates of the existing players in a NE.
3. (Comparative statics of idiosyncratic shocks). A positive idiosyncratic shock to player i leads to an increase in the smallest and largest equilibrium strategies for player i, and to a decrease in the associated aggregates of the remaining players.
4. If, in addition, payoffs functions are monotonic in a_{-i}, a positive idiosyncratic shock causes the payoffs of the affected player to go in the direction of the shock and the payoffs of at least another player to go in the opposite direction.

In the second part of the paper, they present an alternative to strategic substitution which they call "local solvability" which says that the effect of a_i on marginal payoffs is not zero when evaluated at FOC (i.e., payoffs are a Morse function, see footnote 5). And prove results akin to 1–4 above.

Christensen (2019) extends the SC condition to one that he calls Dixit–Corchón. Under strategic complements, this condition reads

$$\frac{\partial T_i}{\partial a_i} \leq -n \frac{\partial T_i}{\partial a}.$$

where recall that T_i is marginal profits, i.e., the left hand side of (1). If all players are identical, this condition guarantees that Selten's function has a slope less than one which guarantees a unique fixed point.

[29] Actually, it is a Cournot model with an inverse demand with elasticity one and cost function $\phi_i^{-1}()$.

Under strategic substitution, he assumes that payoffs are a Morse function, so second-order conditions of payoff maximization hold with a strict inequality and we are left with

$$\frac{\partial T_i}{\partial a_i} \leq 0.$$

so the numerator of (2) is positive.

6.2 Extensions of the basic theory

Free entry in aggregative models was studied by Corchón and Fradera (2002). They show that SC suffices to prove the existence of a free entry equilibrium when players are identical. But SC is not enough to obtain the result that a generalized shock increases a in equilibrium. They provide an example in which demand increases but is also made more elastic. This intensifies competition in such a way that fewer firms find entry profitable and, as a result, industry supply is reduced. Under an additional assumption that precludes the previous example, they show the required result. Okumura (2015) proves existence of a free entry equilibrium with heterogeneous firms using SC.

Nocke and Schutz (2018) present a model of price setting multiproduct firms. They introduce and micro-foundate a class of demand functions like

$$D_i = \frac{F(p_i)}{H(\boldsymbol{p})}.$$

The values of $H()$, denoted by H represent the aggregate. These demands satisfy independence of irrelevant alternatives in the sense that D_i/D_j only depends on p_i and p_j. This class generalizes constant elasticity of substitution (CES) and multinomial logit demand functions. Clearly, the game is aggregative because profits of any firm depend on rivals' prices only through the industry-level aggregator H. Profit maximizing prices yield demands that, properly inverted, are encapsulated into a function $\Gamma(H)$. Candidates for equilibrium must solve $H = \Gamma(H)$ which yields demand and in turn prices. This is reminiscent of Selten's procedure explained in Sect. 3.

Anderson et al. (2020) work out a model with quasi-linear preferences and heterogeneous firms. They consider the demands and utility functions for which Bertrand and Cournot differentiated product oligopoly games are aggregative. They characterize the Bertrand and Cournot games where consumer welfare depends on the aggregate variable only. For example, in Bertrand games, when demand functions satisfy independence of irrelevant alternatives, as defined previously, the indirect utility function is additively separable, the corresponding game is aggregative and consumer welfare depends only on the aggregate. Their analysis shows that aggregative games can be used to obtain positive as well as normative properties of equilibria in asymmetric oligopoly models. They derive results for asymmetric oligopoly both in the short run and long run. The analysis carries to demand functions with quality as an argument. They summarize their findings by a sentence that sounds like music to my ears: "The results elucidate aggregative games as a unifying principle in the literature on merger analysis, privatization, Stackelberg leadership, and cost shocks" (id. p. 470).

6.3 Other micro-models

Dickson (2013) studies the effects of entry into a single market strategic game in which both sellers and buyers interact strategically. The model goes back to Shapley and Shubik (1977). In this model, goods are traded against "money" in trading posts. The price of a good in terms of the numeraire is determined by the ratio of the amount of the numeraire brought at each post, to the quantity of goods offered for sale at that post. Dickson shows that the market is quasi-competitive, i.e., when a new seller enters the market, the price falls and trade increases. However, existing sellers' payoffs may increase under some conditions on the elasticities of supply and demand because the entry of a new firm may move market equilibrium to a zone where demand expands far and away. This is akin to the point made in Sect. 6.2 about the effects of entry. Similarly, the entry of additional buyers increases equilibrium price, but further assumptions are needed to prove that existing buyers' payoffs decrease.

Dickson (2017) studies the case of several aggregates. Agents inside each aggregate (group) behaves like in the one aggregate model because they take as given other aggregates. To prove the existence of a NE for the whole game, he introduces an additional condition that says the marginal payoff of each group member is influenced by the actions of another group in the same direction.

Rota-Graziosi (2015) considers a model of tax competition among jurisdictions. He shows that an increase in the number of tax-competing jurisdictions decreases tax rates and tax revenues and improves the net return of capital. These results correspond to the quasi-competitiveness in a market mentioned above.

Folmer and von Mouche (2002) use aggregative games to analyze NE and Pareto efficient allocations in the acid rain game in which countries contribute (a_i) to the transnational pollution (a). In a later and more general contribution Folmer and von Mouche (2015), they show that uniqueness of NE depends on the differentiability of the damage function mapping individual pollution to social cost of aggregate pollution.

Finally, aggregative games have "invaded" the field of cooperative games. Stamatopoulos (2020) shows that the SC assumption plus other conditions implies the existence of a $\gamma-$core. And Quartieri and Ryusuke (2015) prove that, under strategic substitutability, the set of NE, the set of coalition-proof NE under strong Pareto dominance and the set of NE that are not strongly Pareto dominated by other NE are equivalent.

6.4 Macroeconomics

An early literature provided micro-foundations to macroeconomic models, specially of Keynesian flavor. Favorite models were those of monopolistic competition with a representative consumer yielding constant elasticity demands, see the surveys of Silvestre (1993) and Costa and Dixon (2011) for details. These models use an aggregative structure. It seems to me that many of the important insights obtained in this literature, like existence of unemployment and the effects of fiscal policy, could be reworked in terms of an aggregative game in which profits depend on the own price and an aggregate of all prices.

Acemoglu and Jensen (2015) present a stochastic dynamic general equilibrium model that uses some of the methods developed in their 2013 paper (see Sect. 6.1) to a framework akin to study long run macroeconomic questions. They assume that payoffs are supermodular and the graph of the constraint set is a convex sublattice. As they remark, "Economically, this means that the current decision is increasing in the last period's decision, e.g., higher past savings will increase current savings". An idiosyncratic shock increases the aggregate in the least and greatest equilibrium aggregates and causes a first-order stochastic dominance increase in the distribution of the least and greatest stationary equilibrium strategies of the agent hit by the shock.

Chakrabarti and Lahkar (2018) present a dynamic macro-model where, every now and then, the economy is hit by exogenous technological shocks. A positive shock yields a quick boom in payoffs, which is unsustainable as agents increase aggregate input to the inefficient equilibrium level. Aggregate payoff, therefore, declines. Thus, a sequence of exogenous shocks yields a sustained pattern of growth and a sequence of booms and busts.

6.5 Engineering

Very often, engineers work with shared computing systems in which there is a degree of congestion, captured by a. Besides the standard worries of game theorists on existence and comparative statics, engineers are keen on providing algorithms computing the NE in networks in polynomial time. The literature is large, and we just review two contributions. The interest reader can consult the survey of Jensen (2018) Section 7.2 for further literature on this topic.

Shi et al. (2020) study the use of a shared/buy-in computing system for research projects which consists of two items. A shared resource that all agents can use for free and buy-in computing nodes in which priority access is given to owners of those nodes, but excess idle capacity is made accessible to other users. They show that the corresponding game has a unique NE that can be computed in polynomial time. This NE is globally stable according with best response dynamics. They also show that each player can compute its best response in a simple manner.

As we saw before, NE of aggregative games is seldom efficient. In the parlance of engineers, this is the *price of anarchy*. Barreto et al. (2019) venture in the field of mechanism design by providing a mechanism with a one-dimensional strategy space whose NE is efficient but, according to the Gibbard–Satterthwaite theorem, not budget balanced. The mechanism is close to the one presented by Clarke in 1970. A more thorough investigation would be needed here because the theory of mechanism design has run a lot of mileage since Clarke's mechanism was presented (see Corchón (2009) for an exposition of later results).

6.6 Large games

As we saw in Sect. 3, these are games with a large number of players. As a consequence, the aggregate, a/n vanishes. Jensen (2018) section 7.1 provides a survey for this topic, so we will focus on contributions written after this survey was published.

Liu et al. (2020) provide additional examples of aggregative games outside the realm of economics: population dynamics, traffic analysis, communications network control and electrical system management. They take advantage of the convexifying effects of a large number of players, first noticed in economics by Aumann (1964).[30] Instead of using the continuum model, they use a result due to Shapley and Folkman which, roughly speaking, says that, under some conditions, the sum of a large number of sets is approximately convex.[31] In Liu et alia model, agents minimize the cost of using a facility $c_i(a_i, a/n)$. In a large game, the term a/n vanish so in a NE for each agent a_i^* is a best reply to the average term $\sum_{j=1}^{n} a_j^*/n$. The fixed point of best replies, whose existence is not guaranteed, is a NE. They use the Shapley–Folkman theorem to quantify the best approximation to the fixed point and provide an algorithm for a subclass of aggregative games that converges in polynomial time.

Lahkar and Mukherjee (2020) present a mechanism yielding an aggregative game which implements the unique efficient level of the public good. The trick is that the transfer to agent i is her announcement about her cost function times the marginal utility of the public good for the whole population. Since the economy is large, the amount of public good does not depend on the announcement of any agent. FOC of payoff maximization equalizes marginal utility of the public good for the population to the marginal cost so when she maximizes, she has no incentive to misreport because the only consequence of that would be that she would end up with less transfer. I conjecture that this mechanism can be of interest to engineers, see the previous subsection. In a subsequent contribution Lahkar and Mukherjee (2021), both authors consider the case of externalities.

6.7 Evolution

In this subsection, we review some contributions that single out the optimal behavior from the point of view of survival in the long run.

Sethi and Somanathan (2001) consider a model with individuals matched in small subgroups in which they interact strategically. Suppose that the population is initially composed entirely of self-regarding and altruistic individuals. Clearly, self-regarding types obtain larger payoffs and, eventually, take the field. But the authors show that this population can be successfully invaded by individuals with preferences which place negative weight on the payoffs of materialists and positive weight on the payoffs of

[30] Aumann uses a theorem due to Richter that states that the integral of a set valued correspondence is convex valued, see Aumann (1965).

[31] Arrow and Hahn (1971), pp. 396–399 and chapter 7 provide an exposition of the application of this theorem to competitive equilibrium.

sufficiently altruistic individuals. These “intelligent spiteful” agents help to maintain cooperation in small groups by being mean with the mean and kind with the kind.[32]

Lahkar (2019) shows that, in large games, the self-regarding type enjoys fitness dominance under any type distribution. Hence, in a wide class of evolutionary dynamics, all non-individualistic types are eliminated.

The last two papers are related to a point raised by Coase (1976) on the two fundamental books written by Adam Smith, *The Theory of Moral Sentiments* and *The Wealth of Nations*. In the first book, Smith distinguishes between two type of sentiments: benevolence and self-interest. He notes that societies with a small number of individuals, like families or tribes, can be organized by relying on benevolence (supplemented with the mechanism proposed by Sethi and Somanathan (1996)). But large societies cannot rely on benevolence because even if you are prepared to give all your wealth to the poor people in the world, this would hardly change the luck of the underdogs. Successful large societies have to organized on self-interest. This provides a nice introduction to the topics covered in Smith’ second book.

But what if self-interest fails? Well, in this case human-made design must take the floor. This approach is akin to what happens in medical sciences where healthy individuals (perfect markets) should not be given any treatment, but man-made medicines or procedures (mechanisms designed by us) must be used with sick individuals. Lahkar and Mukherjee (2019) present a transfer scheme to correct free market outcomes whose NE achieve an efficient state in a large population public goods game. They apply a transfer scheme to each agent equal to the externality in the game. The externality adjusted game is a potential game with a unique NE, which is the efficient state of the original game. And best response dynamic for such aggregative potential games converges to the efficient allocation.

See Jensen (2018) Section 7.4 for additional results on this topic.

6.8 Sociology

It appears that some social sciences are converging to create the powerful social science that once was dreamt by early economists. Political science and history are using game theory without apologies. Only sociology remains a little bit aloof despite the big push provided by the work of Gary Becker. But relief is on the way. A generation of young economists is uncovering the relationship between social values and the economy.[33] And aggregative games can be helpful by interpreting a as measure of the impersonal social forces and a_i as the efforts done by individuals.

In this vein, Haagsma and van Mouche (2010) analyze a game where each player’s payoff depends on his action and his social status, which is given by his rank in the actions distribution. They find that if intrinsic concerns are sufficiently important relative to status concerns, individual equilibrium actions diverge. But if status concerns are relatively important, individual actions in NE converge to a common point.

[32] In a Sethi and Somanathan (1996) paper, both authors proposed a similar mechanism to solve the *tragedy of the commons.*

[33] As an example, a recent paper by Masera and Rosenberg (2020) explains the change in the attitudes about slavery in Antebellum USA as a consequence of the expansion of territories in southern USA.

More importantly, status seeking is not always socially inefficient. When players are sufficiently heterogeneous, there is a unique NE that is Pareto efficient.

7 To boldly go where no one has gone before!

In the previous sections, I showed that aggregative games are workable and that they tackle a large range of applications. The beauty of these games is that they cover a lot of mileage with a parsimonious model. And the best is yet to come! As social sciences get closer and closer, game theory looks as one of the two models that economists can offer to make sense of the increasing quantity and quality of available data (the other model is general equilibrium). And when strategic interaction needs to be understood, Occam's razor may drive some models to the realm of aggregative games. And, by the way, Nash equilibrium has been used throughout this survey without a word about possible alternatives. But an interesting line of research would be to apply the insights obtained by behavioral economics to aggregative games and derive the corresponding comparative static results in this framework.

Let me remark that aggregative games are perhaps too bold a simplification of strategic interactions. We live in families, departments, rings of friends, market niches in which in addition to my action and "the world" we care about the actions of the close ones.[34] This suggests the following model. Suppose that each player, i, lives in a certain group—or island—denoted by the set N_i . She cares about the actions of the people in this group (including her's), denoted by the vector $\boldsymbol{a}_i$ and the state of the world a. The companions in the group could be friends or fierce competitors as it happens under oligopoly or in contests. Thus, if $N_i = \{i, j, k\}$, $\boldsymbol{a}_i$ is the vector (a_i, a_j, a_k). The model of aggregative games is a special case when $N_i = \{i\}$. Now U_i can be written as $U_i(\boldsymbol{a}_i, a)$. I will call this game Aggregative with Islands (**AwI**). A further extension would be to consider that information about agents outside own's group is incomplete.

I do not know if the **AwI** model would be workable or not but, to me, captures better the interactions that are likely to arise in small groups. As an example, suppose that all individuals inside each group are identical and NE symmetric inside each group. In this case, in any NE all terms in $\boldsymbol{a}_i$ are identical say a_i. Since we are only comparing NE, there is no harm in writing $U_i(\boldsymbol{a}_i, a) = V_i(a_i, a)$, say, so we are back to an aggregative game. And we can apply the machinery developed in previous sections. For instance, now the part of the SC assumption relating a_i uncovers the movement of all agents inside the island. It means that if all firms/contestants in an island increase their output/effort but a is kept constant, the marginal payoff of these agents shifts downward. This example is, admittedly, too simplistic. For instance, if an agent belongs to two islands, it requires that these two islands are populated by identical individuals. So this is just a start to more involved conditions allowing us to uncover the structure of **AwI** games. Also, it is not clear that agents inside the island would play a NE against other agents there. The notion of Kantian equilibrium

[34] Jackson and Rogers (2005) find that in a model of network formation, "nodes together with highly clustered link structures necessarily emerge for a wide set of parameters".

(relative to the island) may be a worthy behavioral alternative (Roemer 2019). We have a long and exciting trip in front of us. Bon voyage!

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