# Topological Indices and $f$-Polynomials on Some Graph Products 

Ricardo Abreu-Blaya ${ }^{1}$, Sergio Bermudo ${ }^{2}$, José M. Rodríguez ${ }^{3}$ (©) and Eva Tourís ${ }^{4, *}$<br>1 Facultad de Matemáticas, Universidad Autónoma de Guerrero, Ciudad Universitaria, Avenida Lázaro Cárdenas, S/N, Chilpancingo 39087, Guerrero, Mexico; rabreublaya@yahoo.es<br>2 Departamento de Economía, Métodos Cuantitativos e Historia Económica, Universidad Pablo de Olavide, Carretera de Utrera Km. 1, 41013 Sevilla, Spain; sbernav@upo.es<br>3 Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain; jomaro@math.uc3m.es<br>4 Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain<br>* Correspondence: eva.touris@uam.es; Tel.: +34-497-68-96

Abstract: We obtain inequalities involving many topological indices in classical graph products by using the $f$-polynomial. In particular, we work with lexicographic product, Cartesian sum and Cartesian product, and with first Zagreb, forgotten, inverse degree and sum lordeg indices.

Keywords: first Zagreb index; forgotten index; inverse degree index; sum lordeg index; lexicographic product; Cartesian sum; Cartesian product; polynomials in graphs

## check for updates

Citation: Abreu-Blaya, R.; Bermudo, S.; Rodríguez, J.M.; Tourís, E. Topological Indices and $f$-Polynomials on Some Graph Products. Symmetry 2021, 13, 292. https://doi.org/10.3390/ sym13020292

Academic Editor: Basil Papadopoulos
Received: 3 January 2021
Accepted: 4 February 2021
Published: 9 February 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/).

## 1. Introduction

Chemical compounds (like hydrocarbons) can be represented by means of graphs. A topological index $T$ is just a number that encapsulates some property of graphs and such that $T$ correlates with a certain molecular characteristic; therefore, it can be employed to grasp chemical properties and physical properties of chemical substances. They play an important role in mathematical chemistry, in particular, in the QSPR/QSAR (quantitative structure-property relationship/quantitative structure-activity relationship) investigations. Computational and mathematical properties of topological indices have been studied in depth for more than 50 years (see, e.g., [1-8] and the references therein). In particular, a main subject on this field is to obtain sharp bounds of topological indices.

See [9] for a review in a dialog manner discussing relevance of topological descriptors to chemical/physical properties.

One of the main topological indices is the following index, called Randić index, defined in [1] by

$$
R(G)=\sum_{u v \in E(G)} \frac{1}{\sqrt{d_{u} d_{v}}}
$$

where $G$ is a graph and $d_{w}$ is the degree of the vertex $w \in V(G)$.
Along the paper, $G=(V(G), E(G))$ will denote a (non-oriented) simple (without loops and multiple edges loops) finite graph without isolated vertices. Hence, every vertex has degree at least 1.

Two of the most popular alternatives to the Randić index are the first Zagreb and second Zagreb indices, denoted by $M_{1}$ and $M_{2}$, respectively, and defined by

$$
M_{1}(G)=\sum_{u v \in E(G)}\left(d_{u}+d_{v}\right)=\sum_{u \in V(G)} d_{u}^{2}, \quad M_{2}(G)=\sum_{u v \in E(G)} d_{u} d_{v}
$$

These two indexes are very useful in mathematical chemistry and so, they have been extensively studied, see [10-13] and the references therein. Further development of Zagreb-
type indices deals with the applications to more complex chemical objects, e.g., large carbon-based species with regular structures such as polycyclic aromatic hydrocarbons [14] and carbon nanostructures [15].

Please note that there are topological indices of different types. They may treat only vertices, only edges, or both edges and vertices of the graph to calculate an index. Thus, the first Zagreb index belongs to the first class (every index in the first class, as the first Zagreb index, also belongs to the second class).

Along this paper we obtain results for topological indices in this first class.
The so-called harmonic index, is defined in [16] by

$$
H(G)=\sum_{u v \in E(G)} \frac{2}{d_{u}+d_{v}}
$$

For more information about the properties of that index we refer to [17-22], and the book [23].

The inverse degree index ID is defined as

$$
I D(G)=\sum_{u \in V(G)} \frac{1}{d_{u}}=\sum_{u v \in E(G)}\left(\frac{1}{d_{u}^{2}}+\frac{1}{d_{v}^{2}}\right)
$$

This index first attracted attention through many conjectures obtained by the computer programme called Graffiti [16]. Since then, several authors have studied its connections with other parameters of graphs: diameter, matching number, edge-connectivity, Wiener index, etc.; also, its chemical applications have been studied by many researchers (see [24-29]).

In [30-32] the general first Zagreb and general second Zagreb indices are defined by

$$
M_{1}^{\alpha}(G)=\sum_{u \in V(G)} d_{u}^{\alpha}, \quad M_{2}^{\alpha}(G)=\sum_{u v \in E(G)}\left(d_{u} d_{v}\right)^{\alpha}
$$

Please note that the first Zagreb index $M_{1}$ is $M_{1}^{2}$, the inverse degree index $I D$ is $M_{1}^{-1}$, the forgotten index $F$ is $M_{1}^{3}, \ldots$; moreover, the Randić index $R$ is $M_{2}^{-1 / 2}$, the second Zagreb index $M_{2}$ is $M_{2}^{1}$, the modified Zagreb index is $M_{2}^{-1}, \ldots$.

The variable topological indices were introduced as a new way of characterizing heteroatoms (see [33,34]), and to assess the structural differences (see [35]). The idea behind the variable topological indices is that the parameter is determined during the regression in such a way that the error of estimate for a fixed property is minimized.

The sum lordeg index was introduced in [36]. It is defined as

$$
S L(G)=\sum_{u \in V(G)} d_{u} \sqrt{\log d_{u}}
$$

This index is interesting from an applied viewpoint since it correlates very well with the octanol-water partition coefficient for octane isomers [36], and so, it appears in numerical packages for the computation of topological indices [37]. For these reasons, in [38] is stated the open problem of finding appropriate bounds for this index.

In [39] the harmonic polynomial is introduced as

$$
H(G, x)=\sum_{u v \in E(G)} x^{d_{u}+d_{v}-1}
$$

In [40-42] several properties of the harmonic polynomial are obtained. Please note that $2 \int_{0}^{1} H(G, x) d x=H(G)$.

In [41] the inverse degree polynomial is introduced as

$$
I D(G, x)=\sum_{u \in V(G)} x^{d_{u}-1}
$$

It should be noticed that $\int_{0}^{1} I D(G, x) d x=I D(G)$. Thus, the inverse degree polynomial $I D(G, x)$ can be used to obtain information about the inverse degree index $I D(G)$ of a graph $G$.

Given any function $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$, let us define the $f$-index

$$
I_{f}(G)=\sum_{u \in V(G)} f\left(d_{u}\right)
$$

and the $f$-polynomial of $G$ by

$$
P_{f}(G, x)=\sum_{u \in V(G)} x^{1 / f\left(d_{u}\right)-1}
$$

if $x>0$. Also, let us define $P_{f}(G, 0)=\lim _{x \rightarrow 0^{+}} P_{f}(G, x)$. In particular, $P_{f}(G, x)=I D(G, x)$ if $f(t)=1 / t$. It is clear that $\int_{0}^{1} P_{f}(G, x) d x=I_{f}(G)$.

Please note that many important indices can be obtained from $I_{f}$ by choosing appropriate functions $f$ : if $f(t)=t^{2}$, then $I_{f}$ is the first Zagreb index; if $f(t)=t^{-1}$, then $I_{f}$ is the inverse degree index $I D$; if $f(t)=t^{3}$, then $I_{f}$ is the forgotten index $F$; in general, if $f(t)=t^{\alpha}$, then $I_{f}$ is the general first Zagreb index $M_{1}^{\alpha}$; if $f(t)=t \sqrt{\log t}$, then $I_{f}$ is the sum lordeg index $S L$. Thus, each theorem in this paper about $I_{f}$ is a result for each one of these indices.

The $f$-polynomial of other graph operations (e.g., join, corona product, Mycielskian and line) is studied in [43].

Operations on graphs play an important role in Mathematical Chemistry, see e.g., [44,45], since many chemical structures appear as operations of graphs: The crystal structure of sodium chloride is the Cartesian product of two path graphs. The kernel of the iron crystal structure is the join of the cube graph $Q_{3}$ and an isolated vertex. The cyclobutane is the Cartesian product of two path graphs $P_{2}$. The alkane $C_{3} H_{6}$ can be represented as the corona product of the path graph $P_{3}$ and the null graph $N_{2}$. The cyclohexane $C_{6} H_{12}$ can be represented as the corona product of the cycle graph $C_{6}$ and the null graph $N_{2}$. The carbon nanotube $T U C_{4}(m, n)$ can be seen as the Cartesian product of the path graph $P_{3}$ and the cycle graph $C_{5}$. The fence (respectively, the closed fence) is the lexicographic product of $P_{5}$ and $P_{2}$ (respectively, $C_{5}$ and $P_{2}$ ). The zigzag polyhex nanotube $T U H C_{6}[2 n, 2]$ is the generalized hierarchical product of the path graph $P_{2}$ and the cycle graph $C_{2 n}$. See [46,47].

Polynomials, in general, have lately proved to be useful in graph theory and, in particular, in Mathematical Chemistry (see [41,43,45,48-55]).

The main goal in this paper is to obtain information of many topological indices (each case of $I_{f}$ for some particular choice of $f$ ) of several graph products, from the information on topological indices of these factors, which are much easier to calculate than the products. Our approach is to obtain information about the corresponding $f$ polynomials, which are easy to calculate (see for instance Theorems 1, 11 and 17); then, we can deduce information on the $I_{f}$ index by using the formula $\int_{0}^{1} P_{f}(G, x) d x=I_{f}(G)$ (see for instance Theorems 4, 13 and 18). This is a good approach since the bounds of the $f$-polynomial of a product of two graphs allow use of analytic tools to bound the $I_{f}$ index of such a graph product, simplifying the proofs.

## 2. Background

The study of the effect of graph operations on topological indices is an active topic of research (see, e.g., [41,54-56]). We study in this section the $f$-polynomial of several graph products: lexicographic product, Cartesian sum and Cartesian product.

The Cartesian product $G_{1} \times G_{2}$ of the graphs $G_{1}$ and $G_{2}$ has the vertex set $V\left(G_{1} \times G_{2}\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and $\left(u_{i}, v_{j}\right)\left(u_{k}, v_{l}\right)$ is an edge of $G_{1} \times G_{2}$ if $u_{i}=u_{k}$ and $v_{j} v_{l} \in E\left(G_{2}\right)$, or $u_{i} u_{k} \in E\left(G_{1}\right)$ and $v_{j}=v_{l}$.

The lexicographic product $G_{1} \odot G_{2}$ of the graphs $G_{1}$ and $G_{2}$ has $V\left(G_{1}\right) \times V\left(G_{2}\right)$ as vertex set, so that two distinct vertices $\left(u_{i}, v_{j}\right),\left(u_{k}, v_{l}\right)$ of $V\left(G_{1} \odot G_{2}\right)$ are adjacent if either $u_{i} u_{k} \in E\left(G_{1}\right)$, or $u_{i}=u_{k}$ and $v_{j} v_{l} \in E\left(G_{2}\right)$.

The Cartesian sum $G_{1} \oplus G_{2}$ of the graphs $G_{1}$ and $G_{2}$ has the vertex set $V\left(G_{1} \oplus G_{2}\right)=V\left(G_{1}\right) \times V\left(G_{2}\right)$ and $\left(u_{i}, v_{j}\right)\left(u_{k}, v_{l}\right)$ is an edge of $G_{1} \oplus G_{2}$ if $u_{i} u_{k} \in E\left(G_{1}\right)$ or $v_{j} v_{l} \in E\left(G_{2}\right)$.

In [57] is defined the following Zagreb polynomial

$$
M_{1}^{*}(G, x):=\sum_{u \in V(G)} d_{u} x^{d_{u}}
$$

Please note that $x(x I D(G, x))^{\prime}=M_{1}^{*}(G, x)$.
In ([43], Propositions 1, 2 and 3) appear the following useful results.
Proposition 1. If $G$ is a graph of order $n$ and $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$, then:

- $\quad P_{f}(G, x)$ is a polynomial if and only if $1 / f\left(d_{u}\right) \in \mathbb{Z}^{+}$for every $u \in V(G)$,
- $\quad P_{f}(G, x)$ is a positive $C^{\infty}$ function on $(0, \infty)$,
- $\quad P_{f}(G, x)$ is a continuous function on $[0, \infty)$ if and only if $P_{f}(G, 0)<\infty$,
- $\quad P_{f}(G, x)$ is a continuous function on $[0, \infty)$ if and only if $f\left(d_{u}\right) \leq 1$ for every $u \in V(G)$,
- $\quad P_{f}(G, x)$ is an integrable function on $[0, A]$ for every $A>0$, and $\int_{0}^{1} P_{f}(G, x) d x=I_{f}(G)$,
- $\quad P_{f}(G, x)$ is increasing on $(0, \infty)$ if and only if $f\left(d_{u}\right) \leq 1$ for every $u \in V(G)$,
- $\quad P_{f}(G, x)$ is strictly increasing on $(0, \infty)$ if and only if $f\left(d_{u}\right) \leq 1$ for every $u \in V(G)$, and $f\left(d_{v}\right) \neq 1$ for some $v \in V(G)$,
- $\quad P_{f}(G, x)$ is convex on $(0, \infty)$ if $f\left(d_{u}\right) \in(0,1 / 2] \cup[1, \infty)$ for every $u \in V(G)$,
- $\quad P_{f}(G, x)$ is strictly convex on $(0, \infty)$ if $f\left(d_{u}\right) \in(0,1 / 2] \cup[1, \infty)$ for every $u \in V(G)$, and $f\left(d_{v}\right) \notin\{1 / 2,1\}$ for some $v \in V(G)$,
- $\quad P_{f}(G, x)$ is concave on $(0, \infty)$ if $f\left(d_{u}\right) \in[1 / 2,1]$ for every $u \in V(G)$,
- $\quad P_{f}(G, x)$ is strictly concave on $(0, \infty)$ if $f\left(d_{u}\right) \in[1 / 2,1]$ for every $u \in V(G)$, and $f\left(d_{v}\right) \notin$ $\{1 / 2,1\}$ for some $v \in V(G)$,
- $\quad P_{f}(G, 1)=n$.

If $\alpha \in \mathbb{R}$ and $f(t)=t^{\alpha}$, then Proposition 1 gives $\int_{0}^{1} P_{f}(G, x) d x=M_{1}^{\alpha}(G)$.
Proposition 2. If $G$ is a $k$-regular graph with $n$ vertices and $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$, then $P_{f}(G, x)=$ $n x^{1 / f(k)-1}$.

Next, we show the polynomial $P_{f}$ for some important graphs: $K_{n}$ (complete graph), $C_{n}$ (cycle graph), $Q_{n}$ (hypercube graph), $K_{n_{1}, n_{2}}$ (complete bipartite graph), $S_{n}$ (star graph), $P_{n}$ (path graph), and $W_{n}$ (wheel graph).

Proposition 3. If $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$, then

$$
\begin{aligned}
P_{f}\left(K_{n}, x\right)=n x^{1 / f(n-1)-1}, & P_{f}\left(C_{n}, x\right)=n x^{1 / f(2)-1}, \\
P_{f}\left(Q_{n}, x\right)=2^{n} x^{1 / f(n)-1}, & P_{f}\left(K_{n_{1}, n_{2}}, x\right)=n_{1} x^{1 / f\left(n_{2}\right)-1}+n_{2} x^{1 / f\left(n_{1}\right)-1}, \\
P_{f}\left(S_{n}, x\right)=x^{1 / f(n-1)-1}+(n-1) x^{1 / f(1)-1}, & P_{f}\left(P_{n}, x\right)=(n-2) x^{1 / f(2)-1}+2 x^{1 / f(1)-1}, \\
P_{f}\left(W_{n}, x\right)=x^{1 / f(n-1)-1}+(n-1) x^{1 / f(3)-1} . &
\end{aligned}
$$

Choose $\delta \in \mathbb{Z}^{+}$and $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+} . f$ satisfies the $\delta$-additive property $1\left(f \in A P_{1}(\delta)\right)$ if

$$
\frac{1}{f(a+b)} \geq \frac{1}{f(a)}+\frac{1}{f(b)}
$$

for every $a, b \in \mathbb{Z}^{+}$with $a, b \geq \delta$.
$f$ satisfies the $\delta$-additive property $2\left(f \in A P_{2}(\delta)\right)$ if

$$
\frac{1}{f(a+b)} \leq \frac{1}{f(a)}+\frac{1}{f(b)}
$$

for every $a, b \in \mathbb{Z}^{+}$with $a, b \geq \delta$.
$f$ satisfies the $\delta$-additive property $3\left(f \in A P_{3}(\delta)\right)$ if

$$
\frac{1}{f(a+b)} \leq \min \left\{\frac{1}{f(a)}, \frac{1}{f(b)}\right\}
$$

for every $a, b \in \mathbb{Z}^{+}$with $a, b \geq \delta$.

## 3. Inequalities for Cartesian Products

First, we prove pointwise inequalities of $P_{f}\left(G_{1} \times G_{2}, x\right)$ involving $P_{f}\left(G_{1}, x\right)$ and $P_{f}\left(G_{2}, x\right)$.

Theorem 1. Let $\delta \in \mathbb{Z}^{+}$, and let $G_{1}$ and $G_{2}$ be two graphs with minimum degree at least $\delta$. For $x \in(0,1]$, the $f$-polynomial of the Cartesian product $G_{1} \times G_{2}$ satisfies.
(1) If $f \in A P_{1}(\delta)$, then

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \leq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

(3) If $f \in A P_{3}(\delta)$ and $G_{1}$ and $G_{2}$ have $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Proof. If $(u, v) \in V\left(G_{1} \times G_{2}\right)$, we have $d_{(u, v)}=d_{u}+d_{v}$ and its corresponding monomial of the $f$-polynomial is

$$
x^{1 / f\left(d_{u}+d_{v}\right)-1} .
$$

Assume that $f \in A P_{1}(\delta)$. Since $d_{u} \geq \delta$ for every $u \in V\left(G_{1}\right) \cup V\left(G_{2}\right), f \in A P_{1}(\delta)$ and $x \in(0,1]$,

$$
\begin{aligned}
P_{f}\left(G_{1} \times G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}+d_{v}\right)-1} \leq \sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}\right)+1 / f\left(d_{v}\right)-1} \\
& =x \sum_{u \in V\left(G_{1}\right)} x^{1 / f\left(d_{u}\right)-1} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{v}\right)-1}=x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, then by a quite similar argument the result is obtained.
If $f \in A P_{3}(\delta)$, then

$$
\begin{aligned}
P_{f}\left(G_{1} \times G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}+d_{v}\right)-1} \geq \sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}\right)-1} \\
& =\sum_{u \in V\left(G_{1}\right)} x^{1 / f\left(d_{u}\right)-1} \sum_{v \in V\left(G_{2}\right)} 1=n_{2} P_{f}\left(G_{1}, x\right) .
\end{aligned}
$$

The inequality involving $P_{f}\left(G_{2}, x\right)$ is obtained in a similar way.
Remark 1. If $\delta=1$, then the condition $G_{1}$ and $G_{2}$ have minimum degree at least $\delta$, is satisfied for every graph $G_{1}, G_{2}$.

Theorem 1 has the following consequence when we consider $f(t)=t^{\alpha}$.
Theorem 2. Let $G_{1}$ and $G_{2}$ be two graphs of order $n_{1}$ and $n_{2}$, respectively, $\alpha \in \mathbb{R}$ and $f(t)=t^{\alpha}$. For $x \in(0,1]$, the $f$-polynomial of the Cartesian product $G_{1} \times G_{2}$ satisfies.
(1) If $\alpha \leq-1$, then $f \in A P_{1}(1)$ and

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \leq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

(2) If $\alpha \in[-1,0]$, then $f \in A P_{2}(1)$ and

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

(3) If $\alpha \geq 0$, then $f \in A P_{3}$ (1) and

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Proof. The inequalities are consequences of the following facts and Theorem 1.
If $\alpha \leq-1$, then $-\alpha \geq 1$ and

$$
\frac{1}{f(x+y)}=(x+y)^{-\alpha} \geq x^{-\alpha}+y^{-\alpha}=\frac{1}{f(x)}+\frac{1}{f(y)}
$$

for every $x, y>0$, and so, $f \in A P_{1}(1)$.
If $\alpha \in[-1,0]$, then $-\alpha \in[0,1]$ and

$$
\frac{1}{f(x+y)}=(x+y)^{-\alpha} \leq x^{-\alpha}+y^{-\alpha}=\frac{1}{f(x)}+\frac{1}{f(y)}
$$

for every $x, y>0$, and so, $f \in A P_{2}(1)$.
If $\alpha \geq 0$, then $1 / t^{\alpha}$ is a decreasing function on $(0, \infty)$ and

$$
\frac{1}{f(x+y)}=\frac{1}{(x+y)^{\alpha}} \leq \min \left\{\frac{1}{x^{\alpha}}, \frac{1}{y^{\alpha}}\right\}=\min \left\{\frac{1}{f(x)}, \frac{1}{f(y)}\right\}
$$

for every $x, y>0$, and so, $f \in A P_{3}(1)$.
Theorem 2 yields for the inverse degree polynomial:
Corollary 1. Let $G_{1}$ and $G_{2}$ be two graphs, the ID polynomial of the Cartesian product $G_{1} \times G_{2}$ is

$$
I D\left(G_{1} \times G_{2}, x\right)=x I D\left(G_{1}, x\right) I D\left(G_{2}, x\right)
$$

Since $f(t)=t \sqrt{\log t}$ is an increasing function on $[1, \infty), f \in A P_{3}(2)$ and Theorem 1 has the following consequence. Recall that a vertex with degree 1 is called pendant vertex. Please note that $f(t)=t \sqrt{\log t}$ is a positive function on $\mathbb{Z}^{+} \backslash\{1\}$, and so, $P_{f}(G, x)$ is well-defined for every graph $G$ without pendant vertices.

Please note that a graph has minimum degree at least 2 if and only if it does not have pendant vertices.

Theorem 3. Let $G_{1}$ and $G_{2}$ be two graphs with minimum degree at least 2 and of order $n_{1}$ and $n_{2}$, respectively. If $f(t)=t \sqrt{\log t}$, then $f \in A P_{3}(2)$ and the $f$-polynomial of the Cartesian product $G_{1} \times G_{2}$ satisfies for $x \in(0,1]$

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Next, we obtain bounds for $I_{f}\left(G_{1} \times G_{2}\right)$ by using the previous inequalities for $P_{f}\left(G_{1} \times G_{2}, x\right)$. This is a good approach since the pointwise bounds of $P_{f}\left(G_{1} \times G_{2}, x\right)$ allow use of analytic tools to bound $I_{f}\left(G_{1} \times G_{2}\right)$. We start with the case $f \in A P_{3}(\delta)$.

Theorem 4. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of order $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. If $f \in A P_{3}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \geq \max \left\{n_{2} I_{f}\left(G_{1}\right), n_{1} I_{f}\left(G_{2}\right)\right\}
$$

Proof. Theorem 1 gives

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\},
$$

for every $x \in(0,1]$. Thus, Proposition 1 leads to

$$
I_{f}\left(G_{1} \times G_{2}\right)=\int_{0}^{1} P_{f}\left(G_{1} \times G_{2}, x\right) d x \geq n_{2} \int_{0}^{1} P_{f}\left(G_{1}, x\right) d x=n_{2} I_{f}\left(G_{1}\right)
$$

The same argumentation gives $I_{f}\left(G_{1} \times G_{2}\right) \geq n_{1} I_{f}\left(G_{2}\right)$.
To deal with the cases $f \in A P_{1}(\delta)$ and $f \in A P_{2}(\delta)$, we need some technical results.
Lemma 1. [58] If $f_{1}, \ldots, f_{k}$ are non-negative convex functions on $[a, b]$, then

$$
\frac{1}{b-a} \int_{a}^{b} \prod_{i=1}^{k} f_{i}(x) d x \geq \frac{2^{k}}{k+1} \prod_{i=1}^{k} \frac{1}{b-a} \int_{a}^{b} f_{i}(x) d x
$$

Lemma 1 can be slightly improved as follows.
Proposition 4. If $f_{1}, \ldots, f_{k}$ are convex non-negative functions on $(a, b)$, then

$$
\frac{1}{b-a} \int_{a}^{b} \prod_{i=1}^{k} f_{i}(x) d x \geq \frac{2^{k}}{k+1} \prod_{i=1}^{k} \frac{1}{b-a} \int_{a}^{b} f_{i}(x) d x
$$

Proof. If $0<\varepsilon<(b-a) / 2$, then $f_{1}, \ldots, f_{k}$ are convex non-negative functions on $[a+\varepsilon, b-\varepsilon]$, and Lemma 1 gives

$$
\frac{1}{b-a-2 \varepsilon} \int_{a+\varepsilon}^{b-\varepsilon} \prod_{i=1}^{k} f_{i}(x) d x \geq \frac{2^{k}}{k+1} \prod_{i=1}^{k} \frac{1}{b-a-2 \varepsilon} \int_{a+\varepsilon}^{b-\varepsilon} f_{i}(x) d x
$$

Since $f_{1}, \ldots, f_{k}$ are non-negative on $(a, b)$, the functions $F, G:(0,(b-a) / 2) \rightarrow[0, \infty)$ defined by

$$
F(\varepsilon)=\int_{a+\varepsilon}^{b-\varepsilon} \prod_{i=1}^{k} f_{i}(x) d x, \quad G(\varepsilon)=\prod_{i=1}^{k} \int_{a+\varepsilon}^{b-\varepsilon} f_{i}(x) d x
$$

are decreasing, and so, there exist their limits as $\varepsilon \rightarrow 0^{+}$(although they can be $\infty$ ). By taking $\varepsilon \rightarrow 0^{+}$in the above inequality, we obtain the result.

Lemma 2. ([59], Corollary 5.2) If $f_{1}, \ldots, f_{k}$ are convex non-negative functions on $[a, b]$, then

$$
\int_{a}^{b} \prod_{i=1}^{k} f_{i}(x) d x \leq \frac{2}{k+1}\left(\prod_{i=1}^{k} \int_{a}^{b} f_{i}(x) d x\right)^{1 / k}\left(\prod_{i=1}^{k}\left(f_{i}(a)+f_{i}(b)\right)\right)^{1-1 / k}
$$

Since any non-negative concave function on $(a, b)$ has finite lateral limits at $a$ and $b$, ([59], Corollary 4.3) can be stated as follows.

Lemma 3. If $f_{1}, f_{2}$ are non-negative concave functions on $(a, b)$, then

$$
\begin{aligned}
\frac{2}{3} \frac{1}{b-a} \int_{a}^{b} f_{1}(x) d x \frac{1}{b-a} \int_{a}^{b} f_{2}(x) d x & \leq \frac{1}{b-a} \int_{a}^{b} f_{1}(x) f_{2}(x) d x \\
& \leq \frac{4}{3} \frac{1}{b-a} \int_{a}^{b} f_{1}(x) d x \frac{1}{b-a} \int_{a}^{b} f_{2}(x) d x
\end{aligned}
$$

With these technical results, we can deal now with the cases $f \in A P_{1}(\delta)$ and $f \in A P_{2}(\delta)$.

Theorem 5. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. If the f-polynomials of $G_{1}$ and $G_{2}$ are convex functions on $(0,1)$, then
(1) If $f \in A P_{1}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(\frac{1}{2}\left(n_{1}+P_{f}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f}\left(G_{2}, 0\right)\right)^{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \geq I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
$$

Proof. Assume first that $f \in A P_{1}(\delta)$. Theorem 1 gives

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \leq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

for every $x \in(0,1]$.
If $P_{f}\left(G_{1}, 0\right)=\infty$ or $P_{f}\left(G_{2}, 0\right)=\infty$, then (1) trivially holds.
If $P_{f}\left(G_{1}, 0\right)<\infty$ and $P_{f}\left(G_{2}, 0\right)<\infty$, then $P_{f}\left(G_{1}, x\right)$ and $P_{f}\left(G_{2}, x\right)$ are continuous functions on $[0,1]$ by Proposition 1, and so, they are convex on $[0,1]$. Proposition 1 and Lemma 2 give

$$
\begin{aligned}
I_{f}\left(G_{1} \times G_{2}\right)= & \int_{0}^{1} P_{f}\left(G_{1} \times G_{2}, x\right) d x \leq \int_{0}^{1} x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right) d x \\
\leq & \frac{1}{2}\left(\int_{0}^{1} x d x \int_{0}^{1} P_{f}\left(G_{1}, x\right) d x \int_{0}^{1} P_{f}\left(G_{2}, x\right) d x\right)^{1 / 3} \\
& \cdot\left((0+1)\left(P_{f}\left(G_{1}, 0\right)+P_{f}\left(G_{1}, 1\right)\right)\left(P_{f}\left(G_{2}, 0\right)+P_{f}\left(G_{2}, 1\right)\right)\right)^{2 / 3} \\
= & \frac{1}{2}\left(\frac{1}{2}\left(n_{1}+P_{f}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f}\left(G_{2}, 0\right)\right)^{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

Assume now that $f \in A P_{2}(\delta)$. Theorem 1 gives

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \geq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

for every $x \in(0,1]$. Since $P_{f}\left(G_{1}, x\right)$ and $P_{f}\left(G_{2}, x\right)$ are convex functions on $(0,1)$, Proposition 1 and Proposition 4 give

$$
\begin{aligned}
I_{f}\left(G_{1} \times G_{2}\right) & =\int_{0}^{1} P_{f}\left(G_{1} \times G_{2}, x\right) d x \geq \int_{0}^{1} x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right) d x \\
& \geq 2 \int_{0}^{1} x d x \int_{0}^{1} P_{f}\left(G_{1}, x\right) d x \int_{0}^{1} P_{f}\left(G_{2}, x\right) d x \\
& =I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
\end{aligned}
$$

If the $f$-polynomials of $G_{1}$ and $G_{2}$ are concave functions on $(0,1)$, we can obtain simpler bounds for $I_{f}\left(G_{1} \times G_{2}\right)$.

Theorem 6. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs with minimum degree at least $\delta$. If $f \in A P_{1}(\delta)$ and the $f$-polynomials of $G_{1}$ and $G_{2}$ are concave functions on $(0,1)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \leq \frac{4}{3} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
$$

Proof. Since $f \in A P_{1}(\delta)$, Theorem 1 gives

$$
P_{f}\left(G_{1} \times G_{2}, x\right) \leq x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right)
$$

for every $x \in(0,1]$. Since $P_{f}\left(G_{1}, x\right)$ and $P_{f}\left(G_{2}, x\right)$ are concave functions on $(0,1)$, a simple combination of Proposition 1 and Lemma 3 yields

$$
\begin{aligned}
I_{f}\left(G_{1} \times G_{2}\right) & =\int_{0}^{1} P_{f}\left(G_{1} \times G_{2}, x\right) d x \leq \int_{0}^{1} x P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right) d x \\
& \leq \int_{0}^{1} P_{f}\left(G_{1}, x\right) P_{f}\left(G_{2}, x\right) d x \leq \frac{4}{3} \int_{0}^{1} P_{f}\left(G_{1}, x\right) d x \int_{0}^{1} P_{f}\left(G_{2}, x\right) d x \\
& =\frac{4}{3} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
\end{aligned}
$$

We can obtain more bounds for $I_{f}\left(G_{1} \times G_{2}\right)$ if the image of $f$ does not intersect $(a / 2, a)$ for some constant $a>0$.

Theorem 7. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$, and $a>0$. If $f: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0, a / 2] \cup[a, \infty)$, then
(1) If $f \in A P_{1}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(\frac{a}{2}\left(n_{1}+P_{f / a}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f / a}\left(G_{2}, 0\right)\right)^{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \geq \frac{1}{a} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
$$

Proof. Let be the function $g=f / a$. Hence, $g: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0,1 / 2] \cup[1, \infty)$ and Proposition 1 gives that $P_{g}\left(G_{1}, x\right)$ and $P_{g}\left(G_{2}, x\right)$ are convex functions on $(0, \infty)$.

If $f \in A P_{1}(\delta)$, then $f / a \in A P_{1}(\delta)$ and Theorem 5 gives

$$
\begin{aligned}
\frac{1}{a} I_{f}\left(G_{1} \times G_{2}\right) & =I_{f / a}\left(G_{1} \times G_{2}\right) \\
& \leq \frac{1}{2}\left(\frac{1}{2}\left(n_{1}+P_{f / a}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f / a}\left(G_{2}, 0\right)\right)^{2} I_{f / a}\left(G_{1}\right) I_{f / a}\left(G_{2}\right)\right)^{1 / 3} \\
& =\frac{1}{2}\left(\frac{1}{2}\left(n_{1}+P_{f / a}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f / a}\left(G_{2}, 0\right)\right)^{2} \frac{1}{a} I_{f}\left(G_{1}\right) \frac{1}{a} I_{f}\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, then $f / a \in A P_{2}(\delta)$ and Theorem 5 gives

$$
\frac{1}{a} I_{f}\left(G_{1} \times G_{2}\right)=I_{f / a}\left(G_{1} \times G_{2}\right) \geq I_{f / a}\left(G_{1}\right) I_{f / a}\left(G_{2}\right)=\frac{1}{a} I_{f}\left(G_{1}\right) \frac{1}{a} I_{f}\left(G_{2}\right)
$$

The first inequality in Theorem 7 can be improved as follows.
Theorem 8. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$, and $a>0$. If $f: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0, a / 2]$, and $f \in A P_{1}(\delta)$, then

$$
I_{f}\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(\frac{a}{2} n_{1}^{2} n_{2}^{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

Proof. Theorem 7 gives

$$
I_{f}\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(\frac{a}{2}\left(n_{1}+P_{f / a}\left(G_{1}, 0\right)\right)^{2}\left(n_{2}+P_{f / a}\left(G_{2}, 0\right)\right)^{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

Since $f \leq a / 2$ on $\mathbb{Z}^{+} \cap[\delta, \infty)$, we have $a / f-1 \geq 1$ and $P_{f / a}\left(G_{1}, 0\right)=P_{f / a}\left(G_{2}, 0\right)=0$, and so, we obtain the result.

Theorems 2, 8 (with $a=2$ ), 7 (with $a=2$ ) and 4 have the following consequence for the general first Zagreb index.

Theorem 9. Let $G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and $\alpha \in \mathbb{R}$.
(1) If $\alpha \leq-1$, then

$$
M_{1}^{\alpha}\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(n_{1}^{2} n_{2}^{2} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $\alpha \in[-1,0]$, then

$$
M_{1}^{\alpha}\left(G_{1} \times G_{2}\right) \geq \frac{1}{2} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)
$$

(3) If $\alpha \geq 0$, then

$$
M_{1}^{\alpha}\left(G_{1} \times G_{2}\right) \geq \max \left\{n_{2} M_{1}^{\alpha}\left(G_{1}\right), n_{1} M_{1}^{\alpha}\left(G_{2}\right)\right\}
$$

As a rather simple consequence of the above theorem, we have for the first Zagreb, forgotten and inverse degree indices:

Corollary 2. If $G_{1}$ and $G_{2}$ are two graphs with $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
\begin{aligned}
M_{1}\left(G_{1} \times G_{2}\right) & \geq \max \left\{n_{2} M_{1}\left(G_{1}\right), n_{1} M_{1}\left(G_{2}\right)\right\}, \\
F\left(G_{1} \times G_{2}\right) & \geq \max \left\{n_{2} F\left(G_{1}\right), n_{1} F\left(G_{2}\right)\right\}, \\
\frac{1}{2} I D\left(G_{1}\right) I D\left(G_{2}\right) & \leq I D\left(G_{1} \times G_{2}\right) \leq \frac{1}{2}\left(n_{1}^{2} n_{2}^{2} I D\left(G_{1}\right) I D\left(G_{2}\right)\right)^{1 / 3} .
\end{aligned}
$$

Since $f(t)=t \sqrt{\log t} \in A P_{3}(2)$, Theorem 4 gives the following result for the $S L$ index.
Theorem 10. If $G_{1}$ and $G_{2}$ are graphs without pendant vertices and with $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
S L\left(G_{1} \times G_{2}\right) \geq \max \left\{n_{2} S L\left(G_{1}\right), n_{1} S L\left(G_{2}\right)\right\}
$$

A particular consequence of Theorems 9 and 10 is the following.
Corollary 3. If $C_{n_{1}}$ and $C_{n_{1}}$ are the cycle graphs with $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
\begin{aligned}
M_{1}\left(C_{n_{1}} \times C_{n_{2}}\right) & \geq 4 n_{1} n_{2} \\
F\left(C_{n_{1}} \times C_{n_{2}}\right) & \geq 8 n_{1} n_{2} \\
\frac{1}{8} n_{1} n_{2} \leq I D\left(C_{n_{1}} \times C_{n_{2}}\right) & \leq \frac{1}{2^{5 / 3}} n_{1} n_{2} \\
S L\left(C_{n_{1}} \times C_{n_{2}}\right) & \geq n_{1} n_{2} 2 \log 2 .
\end{aligned}
$$

## 4. Inequalities for Lexicographic Products

We start this section by proving pointwise inequalities of $P_{f}\left(G_{1} \odot G_{2}, x\right)$ involving the $f$-polynomials of $G_{1}$ and $G_{2}$.

Theorem 11. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. The f-polynomial of the lexicographic product $G_{1} \odot G_{2}$ satisfies the following inequalities for $x \in(0,1]$.
(1) If $f \in A P_{1}(\delta)$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \leq x^{n_{2}} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x\right)
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq x^{n_{2}} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x\right)
$$

(3) If $f \in A P_{3}(\delta)$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Proof. If $(u, v) \in V\left(G_{1} \odot G_{2}\right)$, then $d_{(u, v)}=n_{2} d_{u}+d_{v}$.
Assume that $f \in A P_{1}(\delta)$. Since $d_{u} \geq \delta$ for every $u \in V\left(G_{1}\right) \cup V\left(G_{2}\right)$, one can prove by induction that

$$
\frac{1}{f\left(n_{2} d_{u}+d_{v}\right)} \geq \frac{n_{2}}{f\left(d_{u}\right)}+\frac{1}{f\left(d_{v}\right)} .
$$

Since $x \in(0,1]$,

$$
\begin{aligned}
P_{f}\left(G_{1} \odot G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(n_{2} d_{u}+d_{v}\right)-1} \\
& \leq \sum_{u \in V\left(G_{1}\right)} x^{n_{2} / f\left(d_{u}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{v}\right)-1} \\
& =\sum_{u \in V\left(G_{1}\right)}\left(x^{n_{2}}\right)^{1 / f\left(d_{u}\right)-1} x^{n_{2}} P_{f}\left(G_{2}, x\right) \\
& =x^{n_{2}} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x\right) .
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, then same argument allows obtaining of the result.

If $f \in A P_{3}(\delta)$, then

$$
\begin{aligned}
P_{f}\left(G_{1} \odot G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(n_{2} d_{u}+d_{v}\right)-1} \\
& \geq \sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}\right)-1}=n_{2} P_{f}\left(G_{1}, x\right)
\end{aligned}
$$

A similar argument gives $P_{f}\left(G_{1} \odot G_{2}, x\right) \geq n_{1} P_{f}\left(G_{2}, x\right)$.
Theorems 2 and 11 have the following consequence when $f(t)=t^{\alpha}$.
Proposition 5. Let $G_{1}$ and $G_{2}$ be two graphs with $n_{1}$ and $n_{2}$ vertices, respectively, $\alpha \in \mathbb{R}$ and $f(t)=t^{\alpha}$. The $f$-polynomial of the lexicographic product $G_{1} \odot G_{2}$ satisfies the following inequalities for $x \in(0,1]$.
(1) If $\alpha \leq-1$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \leq x^{n_{2}} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x\right)
$$

(2) If $\alpha \in[-1,0]$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq x^{n_{2}} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x\right) .
$$

(3) If $\alpha \geq 0$, then

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\} .
$$

Theorem 5 gives the following equality for the inverse degree polynomial.
Corollary 4. Given two graphs $G_{1}$ and $G_{2}$, of order $n_{1}$ and $n_{2}$, respectively, the ID polynomial of the lexicographic product $G_{1} \odot G_{2}$ is

$$
I D\left(G_{1} \odot G_{2}, x\right)=x^{n_{2}} I D\left(G_{1}, x^{n_{2}}\right) I D\left(G_{2}, x\right)
$$

Since $f(t)=t \sqrt{\log t} \in A P_{3}(2)$, Theorem 11 has the following consequence.
Theorem 12. Let $G_{1}$ and $G_{2}$ be two graphs with minimum degree two and of order $n_{1}$ and $n_{2}$, respectively. If $f(t)=t \sqrt{\log t}$, then the $f$-polynomial of the lexicographic product $G_{1} \odot G_{2}$ satisfies for $x \in(0,1]$

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\} .
$$

Next, we obtain bounds for $I_{f}\left(G_{1} \odot G_{2}\right)$ by using the previous inequalities for $P_{f}\left(G_{1} \odot G_{2}, x\right)$. We start with $f \in A P_{3}(\delta)$.

Theorem 13. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of order $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. If $f \in A P_{3}(\delta)$, then

$$
I_{f}\left(G_{1} \odot G_{2}\right) \geq \max \left\{n_{2} I_{f}\left(G_{1}\right), n_{1} I_{f}\left(G_{2}\right)\right\}
$$

Proof. Theorem 11 gives

$$
P_{f}\left(G_{1} \odot G_{2}, x\right) \geq n_{2} P_{f}\left(G_{1}, x\right)
$$

for every $0 \leq x \leq 1$. Thus, Proposition 1 leads to $I_{f}\left(G_{1} \odot G_{2}\right) \geq n_{2} I_{f}\left(G_{1}\right)$. A similar argument gives the inequality $I_{f}\left(G_{1} \odot G_{2}\right) \geq n_{1} I_{f}\left(G_{2}\right)$.

We deal now with $f \in A P_{1}(\delta) \cup A P_{2}(\delta)$.

Theorem 14. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$, and $a>0$. If $f: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0, a / 2]$, then the following inequalities hold.
(1) If $f \in A P_{1}(\delta)$, then

$$
I_{f}\left(G_{1} \odot G_{2}\right) \leq \frac{1}{2}\left(\frac{n_{1}^{2} n_{2} a}{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
I_{f}\left(G_{1} \odot G_{2}\right) \geq \frac{1}{n_{2} a} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
$$

Proof. Consider the function $g=f / a$. We have $g: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0,1 / 2]$ and so, Proposition 1 implies that $P_{g}\left(G_{1}, x\right)$ and $P_{g}\left(G_{2}, x\right)$ are convex functions on the open interval $(0, \infty)$ and continuous on the closed interval $[0, \infty)$; hence, they are convex when $x \in[0,1]$.

If $f \in A P_{1}(\delta)$, then $f / a \in A P_{1}(\delta)$ and Theorem 11 implies

$$
P_{f / a}\left(G_{1} \odot G_{2}, x\right) \leq x^{n_{2}} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x\right)
$$

Notice that $f / a \leq 1 / 2$ implies $a / f-1 \geq 1$, and thus, $P_{f / a}\left(G_{i}, 0\right)=0$ and $P_{f / a}\left(G_{i}, 1\right)=n_{i}$ for $i=1,2$. Since $x$ is a convex function when $x \in[0,1]$, Lemma 2 implies

$$
\begin{aligned}
& \frac{1}{a} I_{f}\left(G_{1} \odot G_{2}\right)=I_{f / a}\left(G_{1} \odot G_{2}\right)=\int_{0}^{1} P_{f / a}\left(G_{1} \odot G_{2}, x\right) d x \\
& \leq \int_{0}^{1} x^{n_{2}} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x\right) d x \\
& \leq \frac{1}{2}\left(\int_{0}^{1} x d x \int_{0}^{1} x^{n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) d x \int_{0}^{1} P_{f / a}\left(G_{2}, x\right) d x\right)^{1 / 3}\left(1 \cdot n_{1} \cdot n_{2}\right)^{2 / 3} \\
& =\frac{1}{2}\left(\frac{1}{2} \frac{1}{n_{2}} \int_{0}^{1} P_{f / a}\left(G_{1}, t\right) d t \frac{1}{a} I_{f}\left(G_{2}\right) n_{1}^{2} n_{2}^{2}\right)^{1 / 3} \\
& =\frac{1}{2}\left(\frac{n_{1}^{2} n_{2}}{2 a^{2}} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, thus $f / a \in A P_{2}(\delta)$ and Theorem 11 implies

$$
P_{f / a}\left(G_{1} \odot G_{2}, x\right) \geq x^{n_{2}} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x\right)
$$

Thus, Lemma 1 gives

$$
\begin{aligned}
\frac{1}{a} I_{f}\left(G_{1} \odot G_{2}\right) & =I_{f / a}\left(G_{1} \odot G_{2}\right)=\int_{0}^{1} P_{f / a}\left(G_{1} \odot G_{2}, x\right) d x \\
& \geq \int_{0}^{1} x^{n_{2}} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x\right) d x \\
& \geq 2 \int_{0}^{1} x d x \int_{0}^{1} x^{n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) d x \int_{0}^{1} P_{f / a}\left(G_{2}, x\right) d x \\
& =2 \frac{1}{2} \frac{1}{n_{2} a} I_{f}\left(G_{1}\right) \frac{1}{a} I_{f}\left(G_{2}\right)=\frac{1}{n_{2} a^{2}} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
\end{aligned}
$$

Now, we deduce several inequalities for many topological indices of lexicographic products.

Theorems 2, 14 (with $a=2$ ) and 13 have the following consequence for the variable first Zagreb index.

Theorem 15. Let $G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and $\alpha \in \mathbb{R}$.
(1) If $\alpha \leq-1$, then

$$
M_{1}^{\alpha}\left(G_{1} \odot G_{2}\right) \leq \frac{1}{2}\left(n_{1}^{2} n_{2} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $\alpha \in[-1,0]$, then

$$
M_{1}^{\alpha}\left(G_{1} \odot G_{2}\right) \geq \frac{1}{2 n_{2}} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)
$$

(3) If $\alpha \geq 0$, then

$$
M_{1}^{\alpha}\left(G_{1} \odot G_{2}\right) \geq \max \left\{n_{2} M_{1}^{\alpha}\left(G_{1}\right), n_{1} M_{1}^{\alpha}\left(G_{2}\right)\right\}
$$

Theorem 15 has the following consequence for the first Zagreb, forgotten and inverse degree.

Corollary 5. If $G_{1}$ and $G_{2}$ are two graphs with $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
\begin{aligned}
M_{1}\left(G_{1} \odot G_{2}\right) & \geq \max \left\{n_{2} M_{1}\left(G_{1}\right), n_{1} M_{1}\left(G_{2}\right)\right\} \\
F\left(G_{1} \odot G_{2}\right) & \geq \max \left\{n_{2} F\left(G_{1}\right), n_{1} F\left(G_{2}\right)\right\} \\
\frac{1}{2 n_{2}} I D\left(G_{1}\right) I D\left(G_{2}\right) \leq I D\left(G_{1} \odot G_{2}\right) & \leq \frac{1}{2}\left(n_{1}^{2} n_{2} I D\left(G_{1}\right) I D\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

Since $f(t)=t \sqrt{\log t} \in A P_{3}(2)$, Theorem 13 allows deduction of the following result for the $S L$ index.

Theorem 16. If $G_{1}$ and $G_{2}$ are graphs without pendant vertices and with $n_{1}$ and $n_{2}$ vertices, respectively, then

$$
S L\left(G_{1} \odot G_{2}\right) \geq \max \left\{n_{2} S L\left(G_{1}\right), n_{1} S L\left(G_{2}\right)\right\}
$$

## 5. Inequalities for Cartesian Sums

We start this last section by proving pointwise inequalities of $P_{f}\left(G_{1} \oplus G_{2}, x\right)$ involving the $f$-polynomials of $G_{1}$ and $G_{2}$.

Theorem 17. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. For $x \in(0,1]$, the $f$-polynomial of the Cartesian sum $G_{1} \oplus G_{2}$ satisfies.
(1) If $f \in A P_{1}(\delta)$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \leq x^{n_{1}+n_{2}-1} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x^{n_{1}}\right)
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq x^{n_{1}+n_{2}-1} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x^{n_{1}}\right)
$$

(3) If $f \in A P_{3}(\delta)$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Proof. If $(u, v) \in V\left(G_{1} \oplus G_{2}\right)$, then $d_{(u, v)}=n_{2} d_{u}+n_{1} d_{v}$.
Suppose that $f \in A P_{1}(\delta)$. Since $d_{u} \geq \delta$ for every $u \in V\left(G_{1}\right) \cup V\left(G_{2}\right)$, one can prove by induction that

$$
\frac{1}{f\left(n_{2} d_{u}+n_{1} d_{v}\right)} \geq \frac{n_{2}}{f\left(d_{u}\right)}+\frac{n_{1}}{f\left(d_{v}\right)}
$$

Since $x \in(0,1]$,

$$
\begin{aligned}
P_{f}\left(G_{1} \oplus G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(n_{2} d_{u}+d_{v}\right)-1} \\
& \leq \sum_{u \in V\left(G_{1}\right)} x^{n_{2} / f\left(d_{u}\right)} \sum_{v \in V\left(G_{2}\right)} x^{n_{1} / f\left(d_{v}\right)} x^{-1} \\
& =\sum_{u \in V\left(G_{1}\right)}\left(x^{n_{2}}\right)^{1 / f\left(d_{u}\right)-1} x^{n_{2}} \sum_{v \in V\left(G_{2}\right)}\left(x^{n_{1}}\right)^{1 / f\left(d_{v}\right)-1} x^{n_{1}} x^{-1} \\
& =x^{n_{1}+n_{2}-1} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x^{n_{1}}\right)
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, then a similar argument allows obtaining of the corresponding inequality.

Suppose that $f \in A P_{3}(\delta)$. We deduce

$$
\begin{aligned}
P_{f}\left(G_{1} \oplus G_{2}, x\right) & =\sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(n_{2} d_{u}+n_{1} d_{v}\right)-1} \\
& \geq \sum_{u \in V\left(G_{1}\right)} \sum_{v \in V\left(G_{2}\right)} x^{1 / f\left(d_{u}\right)-1}=n_{2} P_{f}\left(G_{1}, x\right)
\end{aligned}
$$

A similar argument gives $P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq n_{1} P_{f}\left(G_{2}, x\right)$.
Theorems 2 and 17 have the following consequence for $f(t)=t^{\alpha}$.
Proposition 6. Let $G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, $\alpha \in \mathbb{R}$ and $f(t)=t^{\alpha}$. For $x \in(0,1]$, the $f$-polynomial of the Cartesian sum $G_{1} \oplus G_{2}$ satisfies the following inequalities for .
(1) If $\alpha \leq-1$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \leq x^{n_{1}+n_{2}-1} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x^{n_{1}}\right) .
$$

(2) If $\alpha \in[-1,0]$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq x^{n_{1}+n_{2}-1} P_{f}\left(G_{1}, x^{n_{2}}\right) P_{f}\left(G_{2}, x^{n_{1}}\right) .
$$

(3) If $\alpha \geq 0$, then

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Proposition 6 allows deduction of the following equality for the inverse degree polynomial.

Corollary 6. Given two graphs $G_{1}$ and $G_{2}$, or order $n_{1}$ and $n_{2}$, respectively, the ID polynomial of the Cartesian sum $G_{1} \oplus G_{2}$ is

$$
I D\left(G_{1} \oplus G_{2}, x\right)=x^{n_{1}+n_{2}-1} I D\left(G_{1}, x^{n_{2}}\right) I D\left(G_{2}, x^{n_{1}}\right)
$$

Since $f(t)=t \sqrt{\log t} \in A P_{3}(2)$, Theorem 17 has the following consequence.
Corollary 7. Let $G_{1}$ and $G_{2}$ be two graphs with minimum degree two and of order $n_{1}$ and $n_{2}$, respectively. If $f(t)=t \sqrt{\log t}$, then the $f$-polynomial of the Cartesian sum $G_{1} \oplus G_{2}$ satisfies for $x \in(0,1]$

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq \max \left\{n_{2} P_{f}\left(G_{1}, x\right), n_{1} P_{f}\left(G_{2}, x\right)\right\}
$$

Next, we obtain bounds for $I_{f}\left(G_{1} \oplus G_{2}\right)$ by using the previous inequalities for $P_{f}\left(G_{1} \oplus G_{2}, x\right)$. We start when $f \in A P_{3}(\delta)$.

Theorem 18. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$. If $f \in A P_{3}(\delta)$, then

$$
I_{f}\left(G_{1} \oplus G_{2}\right) \geq \max \left\{n_{2} I_{f}\left(G_{1}\right), n_{1} I_{f}\left(G_{2}\right)\right\}
$$

Proof. Theorem 17 gives

$$
P_{f}\left(G_{1} \oplus G_{2}, x\right) \geq n_{2} P_{f}\left(G_{1}, x\right)
$$

for every $0<x \leq 1$. Hence, $I_{f}\left(G_{1} \oplus G_{2}\right) \geq n_{2} I_{f}\left(G_{1}\right)$ by Proposition 1. A similar argument gives the inequality $I_{f}\left(G_{1} \oplus G_{2}\right) \geq n_{1} I_{f}\left(G_{2}\right)$.

We consider now the case $f \in A P_{1}(\delta) \cup A P_{2}(\delta)$.
Theorem 19. Let $\delta \in \mathbb{Z}^{+}, G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and minimum degree at least $\delta$, and $a>0$. If $f: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0, a / 2]$, then the following inequalities hold.
(1) If $f \in A P_{1}(\delta)$, then

$$
I_{f}\left(G_{1} \oplus G_{2}\right) \leq \frac{1}{2}\left(\frac{n_{1} n_{2} a}{2} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $f \in A P_{2}(\delta)$, then

$$
I_{f}\left(G_{1} \oplus G_{2}\right) \geq \frac{1}{n_{1} n_{2} a} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
$$

Proof. Consider the function $g=f / a$. Therefore, $g: \mathbb{Z}^{+} \cap[\delta, \infty) \rightarrow(0,1 / 2]$ and Proposition 1 implies that $P_{g}\left(G_{1}, x\right)$ and $P_{g}\left(G_{2}, x\right)$ are convex functions on the open interval $(0, \infty)$ and continuous on the closed interval $[0, \infty)$; hence, they are convex when $0 \leq x \leq 1$.

If $f \in A P_{1}(\delta)$, then the function $f / a$ belongs to $A P_{1}(\delta)$ and Theorem 17 implies

$$
P_{f / a}\left(G_{1} \oplus G_{2}, x\right) \leq x^{n_{1}+n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x^{n_{1}}\right)
$$

Notice that $f / a \leq 1 / 2$ implies $a / f-1 \geq 1$, and thus, $P_{f / a}\left(G_{i}, 0\right)=0$ and $P_{f / a}\left(G_{i}, 1\right)=n_{i}$ for $i=1,2$. Since $x$ is a convex function on the interval $[0,1]$, Lemma 2 implies

$$
\begin{aligned}
& \frac{1}{a} I_{f}\left(G_{1} \oplus G_{2}\right)=I_{f / a}\left(G_{1} \oplus G_{2}\right)=\int_{0}^{1} P_{f / a}\left(G_{1} \oplus G_{2}, x\right) d x \\
& \leq \int_{0}^{1} x^{n_{1}+n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x^{n_{1}}\right) d x \\
& \leq \frac{1}{2}\left(\int_{0}^{1} x d x \int_{0}^{1} x^{n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) d x \int_{0}^{1} x^{n_{1}-1} P_{f / a}\left(G_{2}, x^{n_{1}}\right) d x\right)^{1 / 3}\left(1 \cdot n_{1} \cdot n_{2}\right)^{2 / 3} \\
& =\frac{1}{2}\left(\frac{1}{2} \frac{1}{n_{2}} \int_{0}^{1} P_{f / a}\left(G_{1}, t\right) d t \frac{1}{n_{1}} \int_{0}^{1} P_{f / a}\left(G_{2}, t\right) d t n_{1}^{2} n_{2}^{2}\right)^{1 / 3} \\
& =\frac{1}{2}\left(\frac{n_{1} n_{2}}{2 a^{2}} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

If $f \in A P_{2}(\delta)$, thus $f / a$ belongs to $A P_{2}(\delta)$ and Theorem 17 implies

$$
P_{f / a}\left(G_{1} \oplus G_{2}, x\right) \geq x^{n_{1}+n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x^{n_{1}}\right)
$$

Thus, Lemma 1 gives

$$
\begin{aligned}
\frac{1}{a} I_{f}\left(G_{1} \oplus G_{2}\right) & =I_{f / a}\left(G_{1} \oplus G_{2}\right)=\int_{0}^{1} P_{f / a}\left(G_{1} \oplus G_{2}, x\right) d x \\
& \geq \int_{0}^{1} x^{n_{1}+n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) P_{f / a}\left(G_{2}, x^{n_{1}}\right) \cdot d x \\
& \geq 2 \int_{0}^{1} x d x \int_{0}^{1} x^{n_{2}-1} P_{f / a}\left(G_{1}, x^{n_{2}}\right) d x \int_{0}^{1} x^{n_{1}-1} P_{f / a}\left(G_{2}, x^{n_{1}}\right) d x \\
& =2 \frac{1}{2} \frac{1}{n_{2} a} I_{f}\left(G_{1}\right) \frac{1}{n_{1} a} I_{f}\left(G_{2}\right)=\frac{1}{n_{1} n_{2} a^{2}} I_{f}\left(G_{1}\right) I_{f}\left(G_{2}\right)
\end{aligned}
$$

Theorems 2, 19 (with $a=2$ ) and 18 have the following consequence for the general first Zagreb index.

Theorem 20. Let $G_{1}$ and $G_{2}$ be two graphs of orders $n_{1}$ and $n_{2}$, respectively, and $\alpha \in \mathbb{R}$.
(1) If $\alpha \leq-1$, then

$$
M_{1}^{\alpha}\left(G_{1} \oplus G_{2}\right) \leq \frac{1}{2}\left(n_{1} n_{2} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)\right)^{1 / 3}
$$

(2) If $\alpha \in[-1,0]$, then

$$
M_{1}^{\alpha}\left(G_{1} \oplus G_{2}\right) \geq \frac{1}{2 n_{1} n_{2}} M_{1}^{\alpha}\left(G_{1}\right) M_{1}^{\alpha}\left(G_{2}\right)
$$

(3) If $\alpha \geq 0$, then

$$
M_{1}^{\alpha}\left(G_{1} \oplus G_{2}\right) \geq \max \left\{n_{2} M_{1}^{\alpha}\left(G_{1}\right), n_{1} M_{1}^{\alpha}\left(G_{2}\right)\right\}
$$

Theorem 20 has the following consequence for the first Zagreb, forgotten and inverse degree indices.

Corollary 8. If $G_{1}$ and $G_{2}$ are two graphs of orders $n_{1}$ and $n_{2}$, respectively, then

$$
\begin{aligned}
M_{1}\left(G_{1} \oplus G_{2}\right) & \geq \max \left\{n_{2} M_{1}\left(G_{1}\right), n_{1} M_{1}\left(G_{2}\right)\right\} . \\
F\left(G_{1} \oplus G_{2}\right) & \geq \max \left\{n_{2} F\left(G_{1}\right), n_{1} F\left(G_{2}\right)\right\} . \\
\frac{1}{2 n_{1} n_{2}} I D\left(G_{1}\right) I D\left(G_{2}\right) \leq I D\left(G_{1} \oplus G_{2}\right) & \leq \frac{1}{2}\left(n_{1} n_{2} I D\left(G_{1}\right) I D\left(G_{2}\right)\right)^{1 / 3}
\end{aligned}
$$

Since $f(t)=t \sqrt{\log t} \in A P_{3}(2)$, Theorem 18 implies the following result for the SL index.

Theorem 21. If $G_{1}$ and $G_{2}$ are graphs without pendant vertices and of order $n_{1}$ and $n_{2}$, respectively, then

$$
S L\left(G_{1} \oplus G_{2}\right) \geq \max \left\{n_{2} S L\left(G_{1}\right), n_{1} S L\left(G_{2}\right)\right\}
$$

We obtain the following result by using the previous ideas.
Lemma 4. Let $G$ be a graph and $\Gamma$ a subgraph of $G, f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$an increasing function, and $x \in(0,1]$. Then

$$
P_{f}(\Gamma, x) \leq P_{f}(G, x), \quad I_{f}(\Gamma) \leq I_{f}(G)
$$

If $f$ is a decreasing function and $V(\Gamma)=V(G)$, then we obtain the converse inequalities.

Lemma 4 has the following consequence, relating the polynomials and indices of Cartesian products, lexicographic products and Cartesian sums.

Proposition 7. Let $G_{1}$ and $G_{2}$ be two graphs, $f: \mathbb{Z}^{+} \rightarrow \mathbb{R}^{+}$an increasing function, and $x \in(0,1]$. Then

$$
\begin{aligned}
P_{f}\left(G_{1} \times G_{2}, x\right) & \leq P_{f}\left(G_{1} \odot G_{2}, x\right) \leq P_{f}\left(G_{1} \oplus G_{2}, x\right) \\
I_{f}\left(G_{1} \times G_{2}\right) & \leq I_{f}\left(G_{1} \odot G_{2}\right) \leq I_{f}\left(G_{1} \oplus G_{2}\right)
\end{aligned}
$$

If $f$ is a decreasing function, then we obtain the converse inequalities.

## 6. Conclusions

There are several graph products which play an important role in graph theory. Three of these products are Cartesian product, lexicographic product and Cartesian sum. Many topological indices can be written as $I_{f}(G)=\sum_{u \in V(G)} f\left(d_{u}\right)$, for an appropriate choice of the function $f$ (e.g., first Zagreb, inverse degree, forgotten, general first Zagreb and sum lordeg indices). By using the $f$-polynomial $P_{f}(G, x)$ introduced in [43], we obtain in this paper several inequalities of every topological index which can be written as $I_{f}$ for a function $f$ in these classical graph products, from the information on topological indices of their factors, which are much easier to calculate than the products. These results are interesting from the theoretical viewpoint, and also from the point of view of applications since many chemical compounds can be represented by graph products (see the introduction). Our approach is to obtain information about the corresponding $f$-polynomials, which are easy to calculate (as in Theorems 1, 11 and 17); thus, we can deduce information on the $I_{f}$ index by using the formula $\int_{0}^{1} P_{f}(G, x) d x=I_{f}(G)$ (as in Theorems 4,13 and 18). This is a good approach since the bounds of the $f$-polynomial of a product of two graphs allow the use of analytic tools to bound the $I_{f}$ index of such a product, simplifying the proofs.

In [43] appear similar results for corona product and join. Consequently, two natural open questions are to study this problem for strong product and tensor product of graphs.

Author Contributions: Investigation, R.A.-B., S.B., J.M.R. and E.T.; writing-original draft preparation, R.A.-B., S.B., J.M.R. and E.T.; writing-review and editing, R.A.-B., S.B., J.M.R. and E.T.; funding acquisition, R.A.-B., J.M.R. and E.T. All authors have read and agreed to the published version of the manuscript.
Funding: This research was funded by a grant from Agencia Estatal de Investigación (PID2019-106433GB-IO0 / AEI/10.13039/501100011033), Spain.

Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.
Data Availability Statement: Not applicable.
Acknowledgments: We would like to thank the reviewers by their careful reading of the manuscript and their suggestions which have improved the presentation of this work.
Conflicts of Interest: The authors declare no conflict of interest.

## References

1. Randić, M. On characterization of molecular branching. J. Am. Chem. Soc. 1975, 97, 6609-6615. [CrossRef]
2. Gutman, I.; Furtula, B. (Eds.) Recent Results in the Theory of Randić Index; University of Kragujevac: Kragujevac, Serbia, 2008.
3. Li, X.; Gutman, I. Mathematical Aspects of Randić Type Molecular Structure Descriptors; University of Kragujevac: Kragujevac, Serbia, 2006.
4. Li, X.; Shi, Y. A survey on the Randić index. MATCH Commun. Math. Comput. Chem. 2008, 59, 127-156.
5. Rodríguez-Velázquez, J.A.; Sigarreta, J.M. On the Randić index and condicional parameters of a graph. MATCH Commun. Math. Comput. Chem. 2005, 54, 403-416.
6. Rodríguez-Velázquez, J.A.; Tomás-Andreu, J. On the Randić index of polymeric networks modelled by generalized Sierpinski graphs. MATCH Commun. Math. Comput. Chem. 2015, 74, 145-160.
7. Sigarreta, J.M. Bounds for the geometric-arithmetic index of a graph. Miskolc Math. Notes 2015, 16, 1199-1212. [CrossRef]
8. Sigarreta, J.M. Mathematical Properties of Variable Topological Indices. Symmetry 2021, 13, 43. [CrossRef]
9. Ayers, P.L.; Boyd, R.J.; Bultinck, P.; Caffarel, M.; Carbó-Dorca, R.; Causá, M.; Cioslowski, J.; Contreras-Garcia, J.; Cooper, D.L.; Coppens, P.; et al. Six questions on topology in theoretical chemistry. Comput. Theor. Chem. 2015, 1053, 2-16. [CrossRef]
10. Borovicanin, B.; Furtula, B. On extremal Zagreb indices of trees with given domination number. Appl. Math. Comput. 2016, 279, 208-218. [CrossRef]
11. Das, K.C. On comparing Zagreb indices of graphs, MATCH Commun. Math. Comput. Chem. 2010, 63, 433-440.
12. Furtula, B.; Gutman, I.; Ediz, S. On difference of Zagreb indices, Discr. Appl. Math. 2014, 178, 83-88.
13. Liu, M. A simple approach to order the first Zagreb indices of connected graphs. MATCH Commun. Math. Comput. Chem. 2010, 63, 425-432.
14. Črepnjak, M.; Pleteršek, P.Z. Correlation between heat of formation and fifth geometric-arithmetic index. Fuller. Nanot. Carbon Nanostr. 2019, 27, 559-565. [CrossRef]
15. Bultheel, A.; Ori, O. Topological modeling of 1-Pentagon carbon nanocones-Topological efficiency and magic sizes. Fuller. Nanot. Carbon Nanostr. 2018, 26, 291-302. [CrossRef]
16. Fajtlowicz, S. On conjectures of Graffiti-II. Congr. Numer. 1987, 60, 187-197.
17. Deng, H.; Balachandran, S.; Ayyaswamy, S.K.; Venkatakrishnan, Y.B. On the harmonic index and the chromatic number of a graph. Discret. Appl. Math. 2013, 161, 2740-2744. [CrossRef]
18. Favaron, O.; Mahéo, M.; Saclé, J.F. Some eigenvalue properties in graphs (conjectures of Graffiti-II). Discr. Math. 1993, 111, 197-220. [CrossRef]
19. Rodríguez, J.M.; Sigarreta, J.M. New Results on the Harmonic Index and Its Generalizations. MATCH Commun. Math. Comput. Chem. 2017, 78, 387-404.
20. Shwetha Shetty, B.; Lokesha, V.; Ranjini, P.S. On the harmonic index of graph operations. Trans. Combin. 2015, 4, 5-14.
21. Wua, R.; Tanga, Z.; Deng, H. A lower bound for the harmonic index of a graph with minimum degree at least two. Filomat 2013, 27, 51-55. [CrossRef]
22. Zhong, L.; $\mathrm{Xu}, \mathrm{K}$. Inequalities between vertex-degree-based topological Indices. MATCH Commun. Math. Comput. Chem. 2014, 71, 627-642.
23. Gutman, I.; Furtula, B.; Das, K.C.; Milovanovic, E.; Milovanovic, I. (Eds.) Bounds in Chemical Graph Theory—Basics (Three Volumes); Mathematical Chemistry Monograph No. 19; University of Kragujevac: Kragujevac, Serbia, 2017.
24. Dankelmann, P.; Hellwig, A.; Volkmann, L. Inverse degree and edge-connectivity. Discret. Math. 2008, 309, 2943-2947. [CrossRef]
25. Zhang, Z.; Zhang, J.; Lu, X. The relation of matching with inverse degree of a graph. Discret. Math. 2005, 301, 243-246. [CrossRef]
26. Erdös, P.; Pach, J.; Spencer, J. On the mean distance between points of a graph. Congr. Numer. 1988, 64, 121-124.
27. Entringer, R.C. Bounds for the average distance-inverse degree product in trees. In Combinatorics, Graph Theory, and Algorithms; Alavi, Y., Lick, D.R., Schwenk, A.J., Eds.; New Issues Press: Kalamazoo, MI, USA, 1999; pp. 335-352.
28. Rodríguez, J.M.; Sánchez, J.L.; Sigarreta, J.M. Inequalities on the inverse degree index. J. Math. Chem. 2019, 57, 1524-1542. [CrossRef]
29. Mukwembi, S. On diameter and inverse degree of a graph. Discr. Math. 2010, 310, 940-946. [CrossRef]
30. Miličević, A.; Nikolić, S. On variable Zagreb indices. Croat. Chem. Acta 2004, 77, 97-101.
31. Li, X.; Zheng, J. A unified approach to the extremal trees for different indices. MATCH Commun. Math. Comput. Chem. 2005, 54, 195-208.
32. Britto Antony Xavier, G.; Suresh, E.; Gutman, I. Counting relations for general Zagreb indices. Kragujev. J. Math. 2014, 38, 95-103. [CrossRef]
33. Randić, M. Novel graph theoretical approach to heteroatoms in QSAR. Chemom. Intel. Lab. Syst. 1991, 10, 213-227. [CrossRef]
34. Randić, M. On computation of optimal parameters for multivariate analysis of structure-property relationship. J. Chem. Inf. Comput. Sci. 1991, 31, 970-980. [CrossRef]
35. Randić, M.; Plavšić, D.; Lerš, N. Variable connectivity index for cycle-containing structures. J. Chem. Inf. Comput. Sci. 2001, 41, 657-662. [CrossRef] [PubMed]
36. Vukičević, D.; Gašperov, M. Bond additive modeling 1. Adriatic indices. Croat. Chem. Acta 2010, 83, 243-260.
37. Vasilyev, A.; Stevanović, D. MathChem: A Python package for calculating topological indices. MATCH Commun. Math. Comput. Chem. 2014, 71, 657-680.
38. Vukičević, D. Bond additive modeling 2. Mathematical properties of max-min rodeg index. Croat. Chem. Acta 2010, 83, 261-273.
39. Iranmanesh, M.A.; Saheli, M. On the harmonic index and harmonic polynomial of Caterpillars with diameter four. Iran. J. Math. Chem. 2014, 5, 35-43.
40. Carballosa, W.; Nápoles, J.E.; Rodríguez, J.M.; Rosario, O.; Sigarreta, J.M. On the properties of the harmonic polynomial. Ars Comb. 2021, 15.
41. Hernández, J.C.; Méndez-Bermúdez, J.A.; Rodríguez, J.M.; Sigarreta, J.M. Harmonic Index and Harmonic Polynomial on Graph Operations. Symmetry 2018, 10, 456. [CrossRef]
42. Nazir, R.; Sardar, S.; Zafar, S.; Zahid, Z. Edge version of harmonic index and harmonic polynomial of some classes of graphs. J. Appl. Math. Inform. 2016, 34, 479-486. [CrossRef]
43. Carballosa, W.; Rodríguez, J.M.; Sigarreta, J.M.; Vakhania, N. f-polynomial on some graph operations. Mathematics 2019, 7, 1074. [CrossRef]
44. Hua, H.; Das, K.-C.; Wang, H. On atom-bond connectivity index of graphs. J. Math. Anal. Appl. 2019, 479, 1099-1114. [CrossRef]
45. Yan, W.; Yang, B.-Y.; Yeh, Y.-N. The behavior of Wiener indices and polynomials of graphs under five graph decorations. Appl. Math. Lett. 2007, 20, 290-295. [CrossRef]
46. Cao, J.; Ali, U.; Javaid, M.; Huang, C. Zagreb Connection Indices of Molecular Graphs Based on Operations. Complexity 2020, 7385682. [CrossRef]
47. De, N. Computing Reformulated First Zagreb Index of Some Chemical Graphs as an Application of Generalized Hierarchical Product of Graphs. Open J. Math. Sci. 2018, 2, 338-350. [CrossRef]
48. Chu, Y.-M.; Imranb, M.; Qudair Baig, A.; Akhter, S.; Kamran Siddiquia, M. On M-polynomial-based topological descriptors of chemical crystal structures and their applications. Eur. Phys. J. Plus 2020, 135, 874. [CrossRef]
49. Gao, W.; Younas, M.; Farooq, A.; Mahboob, A.; Nazeer, W. M-Polynomials and Degree-Based Topological Indices of the Crystallographic Structure of Molecules. Biomolecules 2018, 8, 107. [CrossRef]
50. Kamran Siddiquia, M.; Imranb, M.; Ahmad, A. On Zagreb indices, Zagreb polynomials of some nanostar dendrimers. Appl. Math. Comput. 2016, 280, 132-139.
51. Masre, M.; Asefa Fufa, S.; Vetrík, T. Distance-based indices of complete m-ary trees. Discr. Math. Algor. Appl. 2020, 12, 2050041. [CrossRef]
52. Tratnik, N.; Zigert Pletersek, P. The edge-Hosoya polynomial of benzenoid chains. J. Math. Chem. 2019, 57, 180-189. [CrossRef]
53. Basilio-Hernandez, L.A.; Leaños, J.; Sigarreta, J.M. On the differential polynomial of a graph. Acta Math. Sin. 2019, 35, 338-354. [CrossRef]
54. Bindusree, A.R.; Naci Cangul, I.; Lokesha, V.; Sinan Cevik, A. Zagreb Polynomials of Three Graph Operators. Filomat 2016, 30, 1979-1986. [CrossRef]
55. Loghman, A. PI polynomials of product graphs. Appl. Math. Lett. 2009, 22, 975-979. [CrossRef]
56. Khalifeh, M.H.; Yousefi-Azari, H.; Ashrafi, A.R. The first and second Zagreb indices of some graph operations. Discr. Appl. Math. 2009, 157, 804-811. [CrossRef]
57. Shuxian, L. Zagreb polynomials of thorn graphs. Kragujev. J. Sci. 2001, 33, 33-38.
58. Anderson, B.J. An inequality for convex functions. Nord. Mat. Tidsk. 1958, 6, 25-26.
59. Csiszár, V.; Móri, T.F. Sharp integral inequalities for products of convex functions. J. Ineq. Pure Appl. Math. 2007, 8, 94.
