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Some results on optimally exercising American put options for time-inhomogeneous processes

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Abstract

We solve the finite-horizon, discounted, Mayer optimal stopping problem, with the gain function coming for exercising an American put option, and the underlying process modeled by a diffusion with constant volatility and a time-dependent drift satisfying certain regularity conditions. Both the corresponding value function and optimal stopping boundary are proved to be Lipschitz continuous away from the terminal time. The optimal stopping boundary is characterized as the unique solution, up to mild regularity conditions, of the free-boundary equation. When the underlying process has Gaussian marginal distributions, more tractable expressions for the pricing formula and free-boundary equation are provided. Finally, we check that an Ornstein–Uhlenbeck process with time-dependent parameters fulfills the required conditions assumed throughout the paper.

Keywords: American put option; Free-boundary problem; Optimal stopping; Ornstein–Uhlenbeck; Time-inhomogeneity.

1 Introduction

Let $X = \{X_s\}_{s\geq 0}^T$ be a stochastic process satisfying the stochastic differential equation

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}s + \sigma \mathrm{d}W_s, \quad 0 \leq s \leq T, \tag{1}$$

in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq 0}^T)$, where $\{\mathcal{F}_s\}_{s\geq 0}^T$ is the natural filtration of the underlying standard Brownian motion $\{W_s\}_{s\geq 0}^T$. In (1), $\mu : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a time-inhomogeneous drift function with regularity later specified and $\sigma > 0$ is the constant-diffusion coefficient.

Consider the finite-horizon, discounted, Mayer Optimal Stopping Problem (OSP)

$$V(t, x) = \sup_{0\leq\tau\leq T-t} \mathbb{E}_{t,x}\left[e^{-\lambda\tau} G(X_{t+\tau})\right], \tag{2}$$

where V is the value function, $G(x) = (A - x)^+$, for some $A \in \mathbb{R}$, is the gain function, and $\lambda \geq 0$ is the discounting rate. The supremum above is taken under all random times τ such that $t + \tau$ is a stopping time in $\{\mathcal{F}_s\}_{s\geq 0}^T$ and $\mathbb{E}_{t,x}$ represents the expectation under the probability measure $\mathbb{P}_{t,x}$ defined as $\mathbb{P}_{t,x}(\cdot) = \mathbb{P}(\cdot \mid X_t = x)$. In what remains, we will refer to τ as a stopping time while keeping in mind that $t + \tau$ is the actual stopping time in the filtration $\{\mathcal{F}_s\}_{s\geq 0}^T$.

The particular form of G implies that

$$-(y - x)^- \leq G(x) - G(y) \leq (y - x)^+, \tag{3}$$

for all $x, y \in \mathbb{R}$, inequalities which will be recurrently used throughout the paper.

It is useful to keep track of the condition $X_t = x$ in a way that does not change the probability measure whenever t or x change. To do so, we denote the process $X^{t,x} = \{X_s^{t,x}\}_{s\geq 0}^{T-t}$ in the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq 0}^{T-t})$ such that

$$\mathrm{Law}(\{X_s^{t,x}\}_{s\geq 0}^{T-t}, \mathbb{P}) = \mathrm{Law}(\{X_{t+s}\}_{s\geq 0}^{T-t}, \mathbb{P}_{t,x}). \tag{4}$$

Notice that $X_0^{t,x} = x$ $\mathbb{P}$-a.s.

We can now define the processes $\partial_t X^{t,x} = \{\partial_t X_s^{t,x}\}_s^{T-t}$ and $\partial_x X^{t,x} = \{\partial_x X_s^{t,x}\}_s^{T-t}$, as the $\mathbb{P}$-a.s. limits

$$\partial_t X_s^{t,x} := \lim_{\varepsilon \to 0} \left(X_s^{t+\varepsilon,x} - X_s^{t,x} \right) \varepsilon^{-1}, \quad \partial_x X_s^{t,x} := \lim_{\varepsilon \to 0} \left(X_s^{t,x+\varepsilon} - X_s^{t,x} \right) \varepsilon^{-1},$$

which, according to (1), take the forms

$$\partial_t X_s^{t,x} = \int_0^s \left(\partial_t \mu(t+u, X_u^{t,x}) + \partial_x \mu(t+u, X_u^{t,x}) \partial_t X_u^{t,x} \right) \mathrm{d}u \tag{5}$$

and

$$\partial_x X_s^{t,x} = 1 + \int_0^s \partial_x \mu(t+u, X_u^{t,x}) \partial_x X_u^{t,x} \, \mathrm{d}u, \tag{6}$$

where we use ∂_t and ∂_x to represent, respectively, the partial derivatives with respect to time and space. Be aware of the distinction between the differential operators ∂_t and ∂_x and the processes $\partial_t X_s^{t,x}$ and $\partial_t X_s^{t,x}$. Note that Itô's formula yields from (6) the following form for $\partial_x X_s^{t,x}$,

$$\partial_x X_s^{t,x} = \exp\left\{ \int_0^s \partial_x \mu(t+u, X_u^{t,x}) \, \mathrm{d}u \right\}. \tag{7}$$

Also, it follows from (1) that the infinitesimal generator $\mathbb{L}$ of the process $\{(t, X_t)\}_{t \geq 0}^T$ is given by

$$(\mathbb{L}f)(t,x) = \partial_t f(t,x) + \mu(t,x) \partial_x f(t,x) + \frac{\sigma^2}{2} \partial_{xx} f(t,x), \tag{8}$$

with ∂_{xx} standing for a shorthand of $\partial_x \partial_x$.

Denote by $D := \{V = G\}$ and $C := D^c = \{V > G\}$ the so-called stopping set and continuation set respectively. If

$$\mathbb{E}_{t,x} \left[\sup_{0 \leq s \leq T-t} e^{-\lambda s} G(X_{t+s}) \right] < \infty \tag{9}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$, then we can guarantee that, under $\mathbb{P}_{t,x}$, the first hitting time of $\{X_{t+s}\}_{s \geq 0}^{T-t}$ into D, denoted by $\tau^* = \tau^*(t,x)$, is optimal in (2) (see, e.g., Karatzas and Shreve (1998, Appendix D) and Peskir and Shiryaev (2006, Chapter 1)), meaning that

$$V(t,x) = \mathbb{E}_{t,x} \left[e^{-\lambda \tau^*} G(X_{t+\tau^*}) \right]. \tag{10}$$

Moreover, if there is another Optimal Stopping Time (OST) τ, then $\tau^* \leq \tau$ $\mathbb{P}_{t,x}$-a.s. Solving the OSP (2) means to provide tractable expressions for both the value function V and the OST τ^*.

The boundary of D (or C), denoted by ∂D (or ∂C), is called the Optimal Stopping Boundary (OSB). It turns out that both the value function V and the OSB ∂D are the solution of a Stefan problem with the infinitesimal generator $\mathbb{L}$ as the differential operator acting on V in C (see Peskir and Shiryaev (2006) for more on the relation between OSPs and free-boundary problems), and therefore the OSB is also referred to as the free-boundary. If the OSB can be depicted by the graph of a function $b : [0,T] \to \mathbb{R}$, that is $\partial D = \{(t, b(t)) : t \in [0,T]\}$, then b is referred as the OSB too.

Finally, it is convenient to recall the martingale and supermartingale properties of V,

$$\mathbb{E}_{t,x} \left[V(t + \tau^* \wedge s, X_{t+\tau^* \wedge s}) \right] = V(t,x), \tag{11}$$

$$\mathbb{E}_{t,x} \left[V(t+s, X_{t+s}) \right] \leq V(t,x), \tag{12}$$

for all $0 \leq s < T-t$, as they will be often used to prove results in Section 2.

2 Regularities of the boundary and the value function

In this section we state and prove regularity conditions about the OSB and value function. These conditions allow obtaining a solution for the OSP (2), later addressed in Section 3, by using an extension of the Itô's formula to derive a characterization of the OSB via a Volterra integral equation. Obviously, these results are subject to certain properties of the underlying process $X^{t,x}$. For the sake of the clarity of exposition, we collect those required properties in the following list of assumptions, which will be hold true henceforth.

A2.1 There exists a function $u : [0,T] \to \mathbb{R}$ such that $\mu(t,x) \geq 0$ for all $x \leq u(t)$ and $\mu(t,x) \leq 0$ for all $x \geq u(t)$.

A2.2 $x \mapsto \mu(t,x)$ and $x \mapsto \partial_t\mu(t,x)$ are Lipschitz continuous for all $t \in [0,T]$, with Lipschitz constant uniform in t. Additionally, there exists a continuous function $\bar{\mu} : [0,T] \to \mathbb{R}$ such that

$$\partial_x\mu(t,x) \leq \bar{\mu}(t) < 0, \tag{13}$$

for all $x \in \mathbb{R}$.

Assumption A2.1 has the sole purpose of supporting a comparison argument between X and a reflected Brownian motion, in order to find a lower bound for the free-boundary (last part of the proof of Proposition 1). The purpose of A2.2 is manifold. Firstly, a Lipschitz-continuous drift guarantees (see (37) from Lemma 1) that

$$\mathbb{E}_{t,x}\left[\sup_{0\leq s\leq T-t} e^{-\lambda s}G(X_{t+s})\right] < \mathbb{E}_{t,x}\left[\sup_{0\leq s\leq T-t} |X_{t+s}|\right] < \infty,$$

on which relies the optimality of $\tau^*(t,x)$ and properties (11) and (12). Also, the Lipschitz continuity of μ is sufficient (but not necessary) to prove that (9) holds true, and to invoke standard results on partial differential equation's theory that provide the smoothness of V in C (see (ii) in Proposition 2). Moreover, by adding the Lipschitz continuity of $\partial_t\mu$ and relation (13), we are able to obtain bounds on several characteristics of $X^{t,x}$ (see Lemma 1). These bounds are specially useful to get the Lipschitz continuity of V and the OSB.

Before proving properties of V and the OSB, we shed light on the geometry of the stopping and continuation regions in the following proposition, which entails that the stopping set lies below the continuation set and that the boundary between them can be seen as the graph of a function. It also provides upper and lower bounds for such a function.

Proposition 1 (Boundary existence and shape of the stopping set)**.**
There exists a function $b : [0,T] \to \mathbb{R}$ such that $\infty < b(t) < A$ for all $t \in [0,T)$, and $D = \{(t,x) \in [0,T] \times \mathbb{R} : x \leq b(t)\}$.

Proof. ($b(t) < A$, $t \in [0,T)$) Let $(t,x) \in \mathcal{A}^+ := [0,T) \times [A,\infty)$ and define the stopping time $\tau_\delta := \inf\{s \in [0,T-t] : X_{t+s} \leq A - \delta\}$ for some $\delta > 0$, assuming that $\inf\{\emptyset\} = T - t$. Then, since $\tau_\delta \leq T - t$ and $p := \mathbb{P}(\tau_\delta < T - t) > 0$,

$$V(t,x) \geq \mathbb{E}_{t,x}\left[e^{-\lambda\tau_\delta}G(X_{t+\tau_\delta})\right] \geq e^{-\lambda(T-t)}p\delta > 0 = G(x),$$

that is, $(t,x) \in C$. Therefore, $\mathcal{A}^+ \subset C$.

($D = \{(t,x) : x \leq b(t)\}$) Define $b(t) := \sup\{x : (t,x) \in D\}$. We have already proved that $b(t) < A$ for all $t \in [0,T)$. Additionally, since $X_s^{t,x}$ increases ($\partial_x X_s^{t,x} \geq 0$ for all s according to (7)) and G decreases, both as functions of x, for all $s \in [0,T-t)$, we can ensure that D lies below the curve b. Moreover, since D is closed, $(t,b(t)) \in D$ for all $t \in [0,T)$, which guarantees that D has the claimed shape.

$(\infty < b(t))$ Consider the function u from A2.1 and define $m := \inf_{t\in[0,T)}\{u(t)\}$. Notice that

$$X_t \geq u(t) - |u(t) - X_t| \geq u(t) - |u(t) - \sigma B_t| \geq m - |m - \sigma B_t|,$$

where $B_t = X_0 + W_t$ and $\{W_t\}_{t\geq 0}$ is the underlying standard Brownian motion in (1). The second inequality holds since the drift of the reflection of X with respect to u is always higher than the drift of the reflection of the Brownian motion with respect to u, and therefore we can ensure that the first process is greater than the last one pathwise $\mathbb{P}$-a.s. (see Ikeda and Watanabe (1977, Theorem 1.1)). The above inequalities guarantee that the value function (2) is lower than the value function associated to a reflected (with respect to m) Brownian motion, and hence the respective OSBs hold the reverse inequality. Additionally, it is easy to show that the free-boundary for the discounted OSP with the gain function G and m-reflected Brownian motion is finite. Actually, one can obtain explicitly the OSB, which is constant, by directly solving the associated free-boundary problem. Therefore b is bounded from below. □

The value function satisfies the regularity properties listed in the next proposition. Besides the results *per se*, the method used to get the Lipschitz continuity of V, based on its martingale and supermartingale properties, (11) and (12) respectively, is a powerful, go-to technique in solving OSPs framed in Markovian processes. The work of De Angelis and Stabile (2019) also provides Lipschitz continuity of the value function in the same fashion for smooth gain functions and time-homogeneous high-dimensional diffusions. For a neat summary on further developments on the regularity of the value function, see De Angelis and Peskir (2020), which proves $V \in \mathcal{C}^1([0,T]\times\mathbb{R})$, and references therein.

Proposition 2 (Regularity of V).
The value function V satisfies the following properties:

(i) V is Lipschitz continuous on $I\times\mathbb{R}$ for all compacts $I \in [0,T)$.

(ii) V is $\mathcal{C}^{1,2}$ on C and on D, and $\mathbb{L}V = \lambda V$ on C.

(iii) $x \mapsto V(t,x)$ is decreasing for all $t\in[0,T)$. Moreover, for $x \neq b(t)$,

$$0 \geq \partial_x V(t,x) \geq -\mathbb{E}\left[e^{-\lambda\tau^*}\partial_x X_{\tau^*}^{t,x}\right], \tag{14}$$

where $\tau^* = \tau^*(t,x)$. Additionally,

$$\partial_t V(t,x) \leq \mathbb{E}\left[\left|\partial_t X_{\tau^*}^{t,x}\right|\right], \tag{15}$$

$$\partial_t V(t,x) \geq -\mathbb{E}\left[\left|\partial_t X_{\tau^*}^{t,x}\right|\right] - \mathbb{P}\left(\tau^* = T-t\right)\left(L + \frac{\sigma^2}{2}\right), \tag{16}$$

for some positive constant L.

Proof. (i) Fix $(t,x)\in I\times K$. Due to the martingale and supermartingale properties of V, the fact that $V \geq G$ on $[0,T)\times\mathbb{R}$ and $V = G$ on D, and inequality (3), it follows that, for $\delta > 0$ small enough and $\tau^* = \tau^*(t,x)$,

$$\begin{aligned}
V(t,x) - V(t-\delta,x) &\leq \mathbb{E}\left[V(t+\tau^*, X_{\tau^*}^{t,x})\right] - \mathbb{E}_{t-\delta,x}\left[V(t-\delta+\tau^*, X_{\tau^*}^{t-\delta,x})\right] \\
&\leq \mathbb{E}\left[G\left(X_{\tau^*}^{t,x}\right) - G\left(X_{\tau^*}^{t-\delta,x}\right)\right] \\
&\leq \mathbb{E}\left[\left(X_{\tau^*}^{t-\delta,x} - X_{\tau^*}^{t,x}\right)^+\right] \\
&= \delta\mathbb{E}\left[\left(-\partial_t X_{\tau^*}^{t_\delta,x}\right)^+\right]
\end{aligned}$$

$$\leq \delta L_{x,I}. \tag{17}$$

The last equality holds for some $t_\delta \in (t-\delta, t) \subset I$ due to the mean value theorem. In (17) and, thereafter,

$$L_{x,I} := \sup_{t\in I} \mathbb{E}\left[\sup_{s\leq T-t} \left|\partial_t X_s^{t,x}\right|\right].$$

Besides, applying similar arguments as the ones used in (17), and noticing that $\tau^* \wedge (T-t-\delta)$ is admissible for $V(t-\delta, x)$, we have that

$$\begin{aligned}
&V(t+\delta, x) - V(t,x)\\
&\quad\geq \mathbb{E}\left[V\left(t+\delta+\tau^*\wedge(T-t-\delta), X^{t+\delta,x}_{\tau^*\wedge(T-t-\delta)}\right)\right] - \mathbb{E}\left[V\left(t+\tau^*, X^{t,x}_{\tau^*}\right)\right]\\
&\quad= \mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(V\left(t+\delta+\tau^*, X^{t+\delta,x}_{\tau^*}\right) - V\left(t+\tau^*, X^{t,x}_{\tau^*}\right)\right)\right]\\
&\qquad+ \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(V\left(T, X^{t+\delta,x}_{T-t-\delta}\right) - V\left(t+\tau^*, X^{t,x}_{\tau^*}\right)\right)\right]\\
&\quad\geq \mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(G\left(X^{t+\delta,x}_{\tau^*}\right) - G\left(X^{t,x}_{\tau^*}\right)\right)\right]\\
&\qquad+ \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t+\delta,x}_{T-t-\delta}) - V(t+\tau^*, X^{t,x}_{\tau^*})\right)\right]\\
&\quad\geq -\mathbb{E}\left[\mathbb{1}(\tau^* \leq T-t-\delta)\left(X^{t,x}_{\tau^*} - X^{t+\delta,x}_{\tau^*}\right)^-\right]\\
&\qquad+ \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t+\delta,x}_{T-t-\delta}) - \mathbb{E}\left[V(t+\tau^*, X^{t,x}_{T-t-\delta+\tau^*\circ\theta_{T-t-\delta}}) \mid \mathcal{F}_{T-t-\delta}\right]\right)\right]\\
&\quad\geq -\delta L_{x,I}\mathbb{P}\left(\tau^* \leq T-t-\delta\right) + \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t+\delta,x}_{T-t-\delta}) - V(T-\delta, X^{t,x}_{T-t-\delta})\right)\right],
\end{aligned} \tag{18}$$

where θ is the shift operator in the space of real-valued functions with domain $[0,\infty]$. By the definition of V in (2), using the Itô–Tanaka formula, and acknowledging that $\left(T-\delta, X^{t,x}_{T-t-\delta}\right) \in C$ in the set $\{\tau^* > T-t-\delta\}$, we derive the following inequality for $\rho^* = \tau^* \circ \theta_{T-t-\delta}$ in $\{\tau^* > T-t-\delta\}$,

$$\begin{aligned}
&V(T-\delta, X^{t,x}_{T-t-\delta}) - G(X^{t,x}_{T-t-\delta})\\
&\quad= \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[e^{-\lambda\rho^*} G(X_{T-\delta+\rho^*})\right] - G(X^{t,x}_{T-t-\delta})\\
&\quad= \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[\frac{1}{2}\int_0^{\rho^*} \mathbb{1}(X_{T-\delta+u} = A)\,\mathrm{d}l^A_u(X_{T-\delta+\cdot})\right]\\
&\qquad- \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[\int_0^{\rho^*} e^{-\lambda u}\mathbb{1}\left(X_{T-\delta+u} < A\right)\left(\lambda\left(A - X_{T-\delta+u}\right) + \mu(T-\delta+u, X_{T-\delta+u})\right)\mathrm{d}u\right]\\
&\quad\leq \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[\frac{1}{2}\int_0^{\rho^*} \mathbb{1}(X_{T-\delta+u} = A)\,\mathrm{d}l^A_u(X_{T-\delta+\cdot})\right]\\
&\qquad- \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[\int_0^{\rho^*} \mathbb{1}(X_{T-\delta+u} < A)\mu(T-\delta+u, X_{T-\delta+u})\,\mathrm{d}u\right]\\
&\quad\leq \delta\frac{\sigma^2}{2} + \mathbb{E}_{T-\delta, X^{t,x}_{T-t-\delta}}\left[\int_0^{\rho^*} -\mathbb{1}(X_{T-\delta+u} < A)\mu(T-\delta+u, X_{T-\delta+u})\,\mathrm{d}u\right]\\
&\quad\leq \delta\left(\frac{\sigma^2}{2} + L\right),
\end{aligned} \tag{19}$$

where $L = \max\{|\mu(t,x)| : 0 \leq t \leq T, b(t) \leq x \leq A\} < \infty$ and $l^A_s(X)$ is the local time of the process X at A and up to time s, that is,

$$l^A_s(X) = \lim_{h\downarrow 0}\int_0^s \mathbb{1}(A-h \leq X_u \leq A+h)\,\mathrm{d}\langle X, X\rangle_u.$$

Plugging (19) into (18), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t+\delta,x}_{T-t-\delta}) - V(T-\delta, X^{t,x}_{T-t-\delta})\right)\right] \\
&\geq \mathbb{E}\left[\mathbb{1}(\tau^* > T-t-\delta)\left(G(X^{t+\delta,x}_{T-t-\delta}) - G(X^{t,x}_{T-t-\delta})\right)\right] - \delta\mathbb{P}\left(\tau^* > T-t-\delta\right)\left(L + \frac{\sigma^2}{2}\right) \\
&\geq -\delta\left(L_{x,I} + L + \frac{\sigma^2}{2}\right)\mathbb{P}\left(\tau^* > T-t-\delta\right). \qquad (20)
\end{aligned}
$$

Therefore,

$$
V(t+\delta, x) - V(t,x) \geq -\delta\left(L_{x,I} + \mathbb{P}\left(\tau^* > T-t-\delta\right)\left(L + \frac{\sigma^2}{2}\right)\right). \qquad (21)
$$

Now consider $\tau_\delta = \tau^*(t-\delta, x)$ for $0 \leq \delta \leq t \leq T$, and notice that $\tau_\delta \wedge (T-t)$ is admissible for $V(t,x)$. Then, arguing as in (18), it follows that

$$
\begin{aligned}
V(t,x) - V(t-\delta, x) \geq &-\mathbb{E}\left[\mathbb{1}(\tau_\delta \leq T-t)\left(X^{t-\delta,x}_{\tau_\delta} - X^{t,x}_{\tau_\delta}\right)^-\right] \\
&+ \mathbb{E}\left[\mathbb{1}(\tau_\delta > T-t)\left(G(X^{t,x}_{T-t}) - V(t-\delta+\tau_\delta, X^{t-\delta,x}_{\tau_\delta})\right)\right],
\end{aligned}
$$

with

$$
\mathbb{E}\left[\mathbb{1}(\tau_\delta \leq T-t)\left(X^{t-\delta,x}_{\tau_\delta} - X^{t,x}_{\tau_\delta}\right)^-\right] \leq \delta L_{x,I}
$$

and

$$
\mathbb{E}\left[\mathbb{1}(\tau_\delta > T-t)\left(G(X^{t,x}_{T-t}) - V(t-\delta+\tau_\delta, X^{t-\delta,x}_{\tau_\delta})\right)\right] \geq -\delta\left(L_{x,I} + L + \frac{\sigma^2}{2}\right).
$$

Additionally, for $0 \leq \delta \leq T-t$ and $\tau^\delta = \tau^*(t+\delta, x)$ one gets the following by proceeding as in (17),

$$
V(t+\delta, x) - V(t,x) \leq \delta L_{x,I},
$$

So far, as $L_{x,I}$ is finite (see Lemma 1), we have proved that, for any $x \in \mathbb{R}$, $t \mapsto V(t,x)$ is Lipschitz continuous in the compact $I \in [0,T)$. We will now prove that V is also Lipschitz continuous with respect to $x \in \mathbb{R}$ for all $t \in I$, which will complete the proof.

Since G is decreasing and $x \mapsto X^{t,x}_s$ is increasing ($\partial_x X^{t,x}_s \geq 0$) for all $s \in [0, T-t)$ and $t \in [0,T)$, then $x \mapsto V(t,x)$ is decreasing for all $t \in [0,T)$. Fix $(t,x) \in [0,T) \times \mathbb{R}$ and $\delta > 0$. Consider $\tau^* = \tau^*(t,x)$, and combine (2), (10), and (3), to get

$$
\begin{aligned}
0 \geq V(t, x+\delta) - V(t,x) &\geq \mathbb{E}_{t,x+\delta}\left[e^{-\lambda\tau^*}G(X_{t+\tau^*})\right] - \mathbb{E}_{t,x}\left[e^{-\lambda\tau^*}G(X_{t+\tau^*})\right] \\
&= \mathbb{E}\left[e^{-\lambda\tau^*}\left(G(X^{t,x+\delta}_{\tau^*}) - G(X^{t,x}_{\tau^*})\right)\right] \\
&\geq -\mathbb{E}\left[e^{-\lambda\tau^*}\left(X^{t,x}_{\tau^*} - X^{t,x+\delta}_{\tau^*}\right)^-\right] \\
&= -\delta\mathbb{E}\left[e^{-\lambda\tau^*}\left(-\partial_x X^{t,x_\delta}_{\tau^*}\right)^-\right] \\
&= -\delta\mathbb{E}\left[e^{-\lambda\tau^*}\partial_x X^{t,x_\delta}_{\tau^*}\right] \qquad (22) \\
&\geq -\delta,
\end{aligned}
$$

where $x_\delta \in (x, x+\delta)$ comes after applying the mean value theorem. In the last inequality we used that $\partial_x X^{t,x}_s \leq 1$ after $\partial_x \mu < 0$ and representation (7).

Arguing similarly, with $\tau_\delta = \tau^*(t, x - \delta)$,

$$0 \geq V(t,x) - V(t,x-\delta) \geq -\delta \mathbb{E}\left[e^{-\lambda \tau_\delta} \partial_x X_{\tau_\delta}^{t,x_\delta}\right] \geq -\delta.$$

Then, $x \mapsto V(t,x)$ is Lipschitz continuous for all $t \in I$, which alongside the Lipschitz continuity of $t \mapsto V(t,x)$ in I for all $x \in \mathbb{R}$, allows us to conclude that V is Lipschitz continuous on $I \times \mathbb{R}$, that is, there exists a constant $L_I > 0$ such that

$$|V(t_1,x_1) - V(t_2,x_2)| \leq L_I(|t_1 - t_2| + |x_1 - x_2|),$$

for all $(t_1,x_1),(t_2,x_2) \in I \times \mathbb{R}$.

(ii) The fact that $\mathbb{L}V = \lambda V$ on C comes right after the strong Markov property of (t, X_t) (see Peskir and Shiryaev (2006, Section 7.1) for more details).

Also, since V is continuous in C (see *(i)*), μ in (1) is Lipschitz continuous (actually, it suffices to require local α-Hölder continuity) in $[0,T] \times \mathbb{R}$, and σ is constant, then, one can borrow a classic result from parabolic partial differential equation's theory (Friedman, 1964, Section 3, Theorem 9) to guarantee that, for an open rectangle $R \subset C$, the first initial-boundary value problem

$$\mathbb{L}f - \lambda f = 0 \qquad \text{in } R, \tag{23}$$

$$f = V \qquad \text{on } \partial R, \tag{24}$$

has a unique solution $f \in \mathcal{C}^{1,2}(R)$. Therefore, we can use Itô's formula on $f(X_{t+s})$ at $s = \tau_{R^c}$, that is, the first time X_{t+s} exits R, and then take $\mathbb{P}_{t,x}$-expectation with $x \in R$, which guarantees the vanishing of the martingale term and yields, together with (23) and (24), the equality $\mathbb{E}_{t,x}[V(X_{t+\tau_{R^c}})] = f(t,x)$. Finally, notice that, due to the strong Markov property, $\mathbb{E}_{t,x}[V(X_{t+\tau_{R^c}})] = V(t,x)$.

(iii) To show that $x \mapsto V(t,x)$ is decreasing for all $t \in [0,T)$ it is enough to prove (14). For $(t,x) \in [0,T) \times \mathbb{R}$ such that $x \neq b(t)$, (14) follows after recalling that V is differentiable with respect to x in C and D, dividing by δ in (22), and taking $\delta \to 0$, while using the dominated convergence theorem. The same procedure used in (17) and (21) yields (15) and (16). □

So far we have nearly no information about the function b giving the free-boundary, besides its existence and a vague idea on how it shapes the stopping and continuation regions. Smoothness of the free-boundary is essential to prove the smooth-fit condition, in which relies the uniqueness of the value function V solving the OSP. The works of De Angelis (2015), Peskir (2019), and De Angelis and Stabile (2019) are a good compendium on the smoothness of the OSB. For time-homogeneous processes and smooth gain functions, De Angelis (2015) provides the continuity of the free-boundary for one-dimensional processes with locally Lipschitz continuous drift and volatility. The two-dimensional case (including time-space diffusions) is addressed by Peskir (2019) in a fairly general setting, which proves the impossibility of first-type discontinuities of the OSB for Mayer–Lagrange OSPs, yet without addressing second-type discontinuities. Unfortunately, this is not enough to prove the smooth-fit condition (piecewise monotonicity is sufficient; see Proposition 4). De Angelis and Stabile (2019) goes further in proving the Lipschitz continuity of the free-boundary in a higher dimensional framework, also for Mayer–Lagrange OSPs, although only tackling time-homogeneous processes and imposing restrictive conditions. For example, the gain function is assumed to be $\mathcal{C}^3$ with respect to the spatial coordinate, which is not satisfied by our gain function G.

Inspired by the methodology displayed in De Angelis and Stabile (2019), yet dealing with a non-differentiable gain function and time-inhomogeneous underlying process, the next proposition proves the Lipschitz continuity of the OSB.

Proposition 3 (Lipschitz continuity of b).
The OSB b is Lipschitz continuous on any closed interval $I \subset [0,T)$.

Proof. Consider the function $W(t,x) = V(t,x) - G(x)$ and the closed interval $I \subset [0,T)$. Propositions 1 and 2 guarantee that W is continuous and $W(t,A) > 0$ for all $t \in I$. Hence, there exists $a > 0$ such that $W(t,A) \geq a$ for all $t \in I$. Therefore, for all δ such that $0 < \delta \leq a$, the equation $W(t,x) = \delta$ has a solution in C for all $t \in I$. Moreover, this solution is unique for each t, hence we can denote it by $b_\delta(t)$, where $b_\delta : I \to (b(t), A]$. The uniqueness comes after (14) from Proposition 2 alongside the fact that $\partial_x \mu < 0$ (see A2.2) and representation (7), which give us $\partial_x W > 0$ on C. The continuity of b_δ on I comes from the continuity of W. Furthermore, the implicit function theorem guarantees that b_δ is differentiable and

$$b_\delta'(t) \geq -\partial_t W(t, b_\delta(t))/\partial_x W(t, b_\delta(t)). \tag{25}$$

Notice that b_δ is decreasing in δ and therefore it converges pointwise to some limit function b_0, which satisfies $b_0 > b$ in I as $b_\delta > b$ for all δ. In addition, since $W(t, b_\delta(t)) = \delta$ and W is continuous, it follows that $W(t, b_0(t)) = 0$ after taking $\delta \downarrow 0$, which means that $b_0 < b$ in I and hence $b_0 = b$ in I.

Recalling (14), we get the following lower bound for $\partial_x W(t, x_\delta)$ based on (38) from Lemma 1, for $x_\delta = b_\delta(t)$ and $\tau_\delta = \tau^*(t, x_\delta)$,

$$\partial_x W(t, x_\delta) \geq M^{(2)}_{T-\underline{t}} \mathbb{E}\left[\tau_\delta\right], \tag{26}$$

where $\underline{t} = \inf\{t : t \in I\}$.

Also, arguing as we did in Lemma 1 to derive (39), alongside (15), it readily follows that

$$\begin{aligned} \partial_t W(t, x_\delta) &\leq \mathbb{E}\left[\left|\partial_t X^{t,x_\delta}_{\tau_\delta}\right|\right] \\ &\leq L\left(1 + (T - \underline{t})e^{L(T-\underline{t})}\right)\mathbb{E}\left[\int_0^{\tau_\delta} \left(\left|X^{t,x_\delta}_u\right| + 1\right)\,\mathrm{d}u\right] \end{aligned} \tag{27}$$

for some positive constant L coming from the Lipschitz continuity of $\partial_t \mu$.

Hence, taking into account (25), (26), and (27), we get that

$$b_\delta'(t) \geq -K_{\underline{t}} \frac{\mathbb{E}\left[\int_0^{\tau_\delta}\left(\left|X^{t,x_\delta}_u\right| + 1\right)\,\mathrm{d}u\right]}{\mathbb{E}\left[\tau_\delta\right]} \tag{28}$$

for a positive constant $K_{\underline{t}}$ independent from δ.

Consider the stopping time $\tau_r = \inf\{s \geq 0 : X_{t+s} \notin I \times (-\infty, r)\}$. Due to the tower property of conditional expectation and the strong Markov property, we have that

$$\begin{aligned} m(t,x) &:= \mathbb{E}\left[\int_0^{\tau^*} \left(\left|X^{t,x}_u\right| + 1\right)\,\mathrm{d}u\right] \\ &= \mathbb{E}_{t,x}\left[\int_0^{\tau^*\wedge\tau_r} \left(|X_{t+u}| + 1\right)\mathrm{d}u + \mathbb{1}(\tau_r \leq \tau^*)\int_{\tau_r}^{\tau^*} \left(|X_{t+u}| + 1\right)\mathrm{d}u\right] \\ &= \mathbb{E}_{t,x}\left[\int_0^{\tau^*\wedge\tau_r} \left(|X_{t+u}| + 1\right)\mathrm{d}u + \mathbb{1}(\tau_r \leq \tau^*)\mathbb{E}_{t,x}\left[\int_{\tau_r}^{\tau_r + \tau^*\circ\theta_{\tau_r}} \left(|X_{t+u}| + 1\right)\mathrm{d}u \middle| \mathcal{F}_{\tau_r}\right]\right] \\ &= \mathbb{E}_{t,x}\left[\int_0^{\tau^*\wedge\tau_r} \left(|X_{t+u}| + 1\right)\mathrm{d}u + \mathbb{1}(\tau_r \leq \tau^*)\mathbb{E}_{t+\tau_r, X_{t+\tau_r}}\left[\int_0^{\tau^*} \left(|X_{t+u}| + 1\right)\mathrm{d}u\right]\right] \\ &= \mathbb{E}_{t,x}\left[\int_0^{\tau^*\wedge\tau_r} \left(|X_{t+u}| + 1\right)\mathrm{d}u + \mathbb{1}(\tau_r \leq \tau^*) m\left(t + \tau_r, X_{t+\tau_r}\right)\right], \end{aligned} \tag{29}$$

where $\tau^* = \tau^*(t,x)$ and θ is the shift operator in the canonical space. Notice that, for $x_\delta < r$, $\left(t + \tau_r, X^{t,x_\delta}_{t+\tau_r}\right) \in \Gamma_t := \{(t, \bar{t}) \times \{r\}\} \cup \{\bar{t} \times (b(\bar{t}), r]\}$, with $\bar{t} = \sup\{t : t \in I\}$. Then,

$$m\left(t + \tau_r, X_{t+\tau_r}\right) \leq \sup_{(s,y)\in\Gamma_t} m\left(s, y\right)$$

$$
\begin{aligned}
&= \sup_{(s,y)\in\Gamma_t} \mathbb{E}\left[\int_0^{\tau^*(s,y)} \left(|X_u^{s,y}|+1\right) \mathrm{d}u\right] \\
&\le T \sup_{(s,y)\in\Gamma_t} \sqrt{\mathbb{E}\left[\sup_{u\le T-s} \left(|X_u^{s,y}|+1\right)^2\right]} \\
&\le T \sup_{(s,y)\in\Gamma_t} \sqrt{M_{T-s}^{(1)}\left(|y|^2+1\right)} \\
&\le T\sqrt{M_T^{(1)}\left(\max\{|b(\bar{t})|^2, |r|^2\}+1\right)},
\end{aligned} \tag{30}
$$

where we used the Cauchy–Schwartz inequality and (37) from Lemma 1. Also, for $u \le \tau_\delta \wedge \tau_r$, $|X_{t+u}| \le \max\{|\underline{b}|, r\}$, with $\underline{b} = \inf\{b(t) : t \in [0,T]\}$. Notice that, in the same fashion we derived (29), we can also get

$$
\mathbb{E}\left[\tau_\delta\right] = \mathbb{E}_{t,x_\delta}\left[(\tau_r \wedge \tau_\delta) + \mathbb{1}(\tau_r \le \tau_\delta)\mathbb{E}_{t+\tau_r, X_{t+\tau_r}}\left[\tau^*\right]\right]. \tag{31}
$$

Therefore, putting together equations (28) to (31), we obtain

$$
\begin{aligned}
b_\delta'(t) &\ge -K\frac{\mathbb{E}_{t,x_\delta}\left[(\tau_\delta\wedge\tau_r) + \mathbb{1}(\tau_r\le\tau_\delta)\right]}{\mathbb{E}\left[\tau_\delta\right]} \\
&= -K\frac{\mathbb{E}_{t,x_\delta}\left[(\tau_\delta\wedge\tau_r) + \mathbb{1}(\tau_r\le\tau_\delta)\right]}{\mathbb{E}_{t,x_\delta}\left[(\tau_\delta\wedge\tau_r) + \mathbb{1}(\tau_r\le\tau_\delta)\mathbb{E}_{t+\tau_r,X_{t+\tau_r}}\left[\tau^*\right]\right]} \\
&\ge -K\left(1+\frac{\mathbb{P}_{t,x_\delta}\left(\tau_r\le\tau_\delta\right)}{\mathbb{E}_{t,x_\delta}\left[(\tau_\delta\wedge\tau_r) + \mathbb{1}(\tau_r\le\tau_\delta)\mathbb{E}_{t+\tau_r,X_{t+\tau_r}}\left[\tau^*\right]\right]}\right) \\
&\ge -K\left(1+\frac{\mathbb{P}_{t,x_\delta}\left(\tau_r\le\tau_\delta, \tau_r=\bar{t}-t\right)}{\mathbb{E}_{t,x_\delta}\left[(\tau_\delta\wedge\tau_r)\right]} + \frac{\mathbb{P}_{t,x_\delta}\left(\tau_r\le\tau_\delta, \tau_r<\bar{t}-t\right)}{\mathbb{E}_{t,x_\delta}\left[\mathbb{1}(\tau_r\le\tau_\delta)\mathbb{E}_{t+\tau_r,X_{t+\tau_r}}\left[\tau^*\right]\right]}\right) \\
&\ge -K\left(1+\frac{\mathbb{P}_{t,x_\delta}\left(\tau_r\le\tau_\delta, \tau_r=\bar{t}-t\right)}{\mathbb{E}_{t,x_\delta}\left[\mathbb{1}(\tau_r\le\tau_\delta, \tau_r=\bar{t}-t)(\tau_\delta\wedge\tau_r)\right]} + \frac{\mathbb{P}_{t,x_\delta}\left(\tau_r\le\tau_\delta, \tau_r<\bar{t}-t\right)}{\mathbb{E}_{t,x_\delta}\left[\mathbb{1}(\tau_r\le\tau_\delta, \tau_r<\bar{t}-t)\mathbb{E}_{t+\tau_r,X_{t+\tau_r}}\left[\tau^*\right]\right]}\right) \\
&\ge -K\left(1+\frac{1}{\bar{t}-t}+\frac{1}{\inf_{s\in(t,\bar{t})}\mathbb{E}_{s,r}\left[\tau^*\right]}\right),
\end{aligned} \tag{32}
$$

where, in the last inequality, we used the fact that, in the set $\{\tau_r \le \tau_\delta, \tau_r < \bar{t}-t\}$, $X_{t+\tau_r} = r$ under $\mathbb{P}_{t,x_\delta}$ for $x_\delta < r$.

Consider now an open set $\mathcal{B}$ and a compact set $\mathcal{K}$ such that $\{(t,\bar{t})\times\{r\}\} \subset \mathcal{K} \subset \mathcal{B} \subset C$, and denote by $\tau_{\mathcal{B}^c}$ the first time $X_u^{s,r}$ exits $\mathcal{B}$ for some $s \in (t,\bar{t})$. Pick a function ϕ such that $\mathbb{L}\phi \le M_{\mathcal{B}}$ in $\mathcal{B}$ for a positive constant $M_{\mathcal{B}}$, $\phi \le 1$ in $\mathcal{K}$, and $\phi > 2$ in $\mathcal{B}^c$. Then,

$$
\begin{aligned}
\mathbb{E}_{s,r}\left[\tau^*\right] \ge \mathbb{E}_{s,r}\left[\tau_{\mathcal{B}^c}\right] = \mathbb{E}_{s,r}\left[\int_0^{\tau_{\mathcal{B}^c}} \mathrm{d}u\right] &\ge M_{\mathcal{B}}^{-1}\mathbb{E}_{s,r}\left[\int_0^{\tau_{\mathcal{B}^c}} \mathbb{L}\phi\,\mathrm{d}u\right] \\
&= M_{\mathcal{B}}^{-1}\left(\mathbb{E}_{s,r}\left[\phi\left(s+\tau_{\mathcal{B}}, X_{s+\tau_{\mathcal{B}}}\right)\right] - \phi(s,r)\right) \\
&\ge M_{\mathcal{B}}^{-1} > 0.
\end{aligned}
$$

Therefore, going back to (32), we have that, for all $t \in I' = [\underline{t}, \bar{t}-\varepsilon]$ and $\varepsilon > 0$ small enough,

$$
b_\delta'(t) \ge -K\left(1+\varepsilon^{-1}+M_{\mathcal{B}}\right). \tag{33}
$$

To find a bound (uniform with respect to δ and for all $t \in I'$) in the opposite sense, consider (16) and (27), and use the Markov inequality to get

$$
\partial_t W(t,x_\delta) \ge -L\left(1+(T-\underline{t})e^{L(T-\underline{t})}\right)\mathbb{E}\left[\int_0^{\tau_\delta}\left(\left|X_u^{t,x_\delta}\right|+1\right)\mathrm{d}u\right]
$$

$$- (T-t)^{-1}\left(\widehat{L}+\frac{\sigma^2}{2}\right)\mathbb{E}\left[\tau_\delta\right] \tag{34}$$

for some positive constant $\widehat{L}$. Hence, relying on the same arguments as the ones used to get (33), but with (34) instead of (27), we obtain

$$b_\delta'(t) \leq K\left(1+\varepsilon^{-1}+M_{\mathcal{B}}\right)+(T-\bar{t})^{-1}\left(\widehat{L}+\frac{\sigma^2}{2}\right). \tag{35}$$

Finally, we have proved that $|b_\delta'(t)|$ is bounded by a constant, uniformly in δ and for all $t \in I' = (\underline{t}, \bar{t}-\varepsilon)$, and then we are able to apply the Arzelà–Ascoly's theorem to prove that b_δ converges uniformly to b in I' as $\delta \to 0$, which implies that b is differentiable in I'. Since $\varepsilon > 0$ and I can be chosen arbitrarily, then we conclude that b is differentiable anywhere away from T. □

The so-called principle of smooth fit provides an extra condition to be satisfied by the value function at the free-boundary, which offsets the added unknown of not having a fixed boundary as in initial-value partial differential equations, and hence it gives uniqueness of solution for the associated free-boundary problem with $\mathbb{L}V = \lambda V$ in C and $V = G$ on D. Roughly speaking, the smooth-fit condition is fulfilled whenever the boundary is (probabilistic) *regular* for the underlying process, that is, if after starting at a point $(t, x) \in \partial C$, the process enters D immediately $\mathbb{P}_{t,x}$-a.s. This type of regularity of the OSB is usually obtained provided smooth dependency of the process with respect to its initial values, and of the gain function around the OSB. In Peskir and Shiryaev (2006, Section 9.1), two methods for proving the smooth-fit condition, that could be tailored to specific cases, are exposed. Lemma 5.1 in De Angelis and Ekström (2017) proves the regularity of b whenever the boundary is monotone. The work of De Angelis and Peskir (2020) derives the principle of smooth fit (both vertically and horizontally) in terms of different kinds of regularities of the OSB for strong Feller/Markov processes.

We obtain the smooth-fit condition for our setting in the next proposition by relying on the work of Cox and Peskir (2015), which implicitly shows that the condition holds for piecewise monotone boundaries for general recurrent diffusions.

Proposition 4 (Smooth-fit condition)**.**
The smooth-fit condition holds, i.e., $\partial_x V(t, b(t)) = -1$ *for all* $t \in [0, T)$.

Proof. Take a point $(t, b(t))$ for some $t \in [0, T)$ and consider $\delta > 0$. Since $(t, b(t)) \in D$ and $(t, b(t) + \delta) \in C$, we get that $\delta^{-1}(V(t, b(t) + \delta) - V(t, b(t))) \geq \delta^{-1}(G(b(t) + \delta) - G(b(t))) = -1$. Therefore, $\partial_x^+ V(t, b(t)) \geq -1$.

Besides, reasoning as in (22), we get that

$$\delta^{-1}(V(t,b(t)+\delta)-V(t,b(t))) \leq -\mathbb{E}\left[e^{-\lambda\tau_\delta}\partial_x X_{\tau_\delta}^{t,x_\delta}\right] \leq \sup_{x\in(b(t),A)}\mathbb{E}\left[\sup_{s\leq T-t}\partial_x X_s^{t,x}\right] < \infty,$$

where $\tau_\delta = \tau^*(t, b(t) + \delta)$ and $x_\delta \in (b(t), b(t) + \delta)$. The supremum is finite (actually, it is lower than 1) since $\partial_x \mu < 0$. Hence, we can apply the dominated convergence theorem to obtain

$$\partial_x^+ V(t,b(t)) \leq -\mathbb{E}\left[e^{-\lambda\tau_0}\partial_x X_{\tau_0}^{t,b(t)}\right], \tag{36}$$

with $\tau_0 := \lim_{\delta\to 0}\tau_\delta$ (τ_0 is well-defined since the sequence τ_δ decreases with respect to δ). Since b is continuous and piecewise monotone (b is Lipschitz continuous by Proposition 3), we can guarantee that $\tau_0 = \tau^*(t, b(t)) = 0$ $\mathbb{P}$-a.s. (Cox and Peskir, 2015, Corollary 8), and therefore $\partial_x^+ V(t, b(t)) \leq -1$.

Finally, the smooth-fit condition arises by recalling that $\partial_x^- V(t, b(t)) = -1$ since $V = G$ on D. □

Lemma 1. *(Some useful bounds)*
Under A2.2, the following inequalities hold for positive constants $M_s^{(i)}$*,* $i = 1, 2, 3$*, and* $(t, x) \in [0, T) \times \mathbb{R}$*,* $s \in (0, T - t]$*,*

$$\mathbb{E}\left[\sup_{u\le s}\left(\left|X_u^{t,x}\right|+1\right)^2\right] \le M_s^{(1)}\left(|x|^2+1\right), \tag{37}$$

$$\partial_x X_s^{t,x} \le 1 - M_s^{(2)} s, \tag{38}$$

$$\mathbb{E}\left[\sup_{u\le s}\left|\partial_t X_u^{t,x}\right|\right] \le M_s^{(3)}\left(|x|+1\right). \tag{39}$$

Moreover, $s \mapsto M_s^{(i)}$ *increases for* $i = 1, 3$ *and decreases for* $i = 2$*.*

Proof. Due to the Lipschitz continuity of $x \mapsto \mu(t, x)$ and the fact that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have that

$$\begin{aligned}\left(\left|X_s^{t,x}\right|+1\right)^2 &\le 3\left(|x|+1\right)^2 + 3\int_0^s \left(\mu(t+r, X_r^{t,x})\right)^2 \mathrm{d}r + 3\sigma^2\left(W_s\right)^2 \\ &\le 3\left(|x|+1\right)^2 + 3L\int_0^s \left(\left|X_r^{t,x}\right|+1\right)^2 \mathrm{d}r + 3\sigma^2\left(W_s\right)^2\end{aligned}$$

for some positive constant L. Hence, using the maximal inequalities in Theorem 14.13 (d) from Schilling et al. (2012), it follows that

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le s}\left(\left|X_u^{t,x}\right|+1\right)^2\right] &\le 3\left(|x|+1\right)^2 + 3L\int_0^s \mathbb{E}\left[\left(\left|X_r^{t,x}\right|+1\right)^2\right]\mathrm{d}r + 3\sigma^2\mathbb{E}\left[\sup_{u\le s}\left(W_u\right)^2\right] \\ &\le 3\left(|x|+1\right)^2 + 12\sigma^2 s + 3L\int_0^s \mathbb{E}\left[\sup_{u\le r}\left(\left|X_u^{t,x}\right|+1\right)^2\right]\mathrm{d}r.\end{aligned}$$

Therefore, Gronwall's inequality (Schilling et al., 2012, Theorem A.43) guarantees that

$$\begin{aligned}\mathbb{E}\left[\sup_{u\le s}\left(\left|X_u^{t,x}\right|+1\right)^2\right] &\le 3\left(|x|+1\right)^2 + 12\sigma^2 s + 3L\int_0^s \left(3\left(|x|+1\right)^2 + 12\sigma^2 u\right)e^{3L(s-u)}\,\mathrm{d}u \\ &\le 3\left(|x|+1\right)^2 + 12\sigma^2 s + 3Le^{3Ls}\left(3\left(|x|+1\right)^2 s + 6\sigma^2 s^2\right),\end{aligned}$$

from which follows (37).

To obtain (38), we use the fact that μ is Lipschitz continuous and $\partial_x\mu < 0$, alongside representation (7) and (14), which lead to

$$\begin{aligned}1 - \partial_x X_s^{t,x} &= \int_0^s -\partial_x\mu(t+u, X_u^{t,x})\partial_x X_u^{t,x}\,\mathrm{d}u \\ &= \int_0^s -\partial_x\mu(t+u, X_u^{t,x})\exp\left\{\int_0^u \partial_x\mu(t+r, X_r^{t,x})\,\mathrm{d}r\right\}\mathrm{d}u \\ &\ge \exp\left\{\int_0^s \partial_x\mu(t+u, X_u^{t,x})\,\mathrm{d}u\right\}\int_0^s -\partial_x\mu(t+u, X_u^{t,x})\,\mathrm{d}u \\ &\ge e^{-Ls}\int_0^s -\partial_x\mu(t+u, X_u^{t,x})\,\mathrm{d}u \\ &\ge e^{-Ls}\min_{u\in[0,T]}\left\{-\bar{\mu}(u)\right\}s,\end{aligned}$$

where L is the Lipschitz constant.

Let us prove now (39). To do so, notice first that (5) implies

$$\left|\partial_t X_s^{t,x}\right| \le a_1(s,t,x) + \int_0^s a_2(u,t,x)\left|\partial_t X_u^{t,x}\right|\,\mathrm{d}u,$$

with

$$a_1(s,t,x) = \int_0^s \left|\partial_t \mu(t+u, X_u^{t,x})\right| \mathrm{d}u, \quad a_2(u,t,x) = \left|\partial_x \mu(t+u, X_u^{t,x})\right|.$$

Therefore, an application of the Gronwall's inequality yields

$$\left|\partial_t X_s^{t,x}\right| \le a_1(s,t,x) + \int_0^s a_1(u,t,x)a_2(u,t,x)\exp\left\{\int_u^s a_2(r,t,x)\,\mathrm{d}r\right\}\mathrm{d}u.$$

Hence, due to the Lipschitz continuity of μ and $\partial_t\mu$, and using (37) alongside the Cauchy–Schwartz inequality, we have that

$$\begin{aligned}
\mathbb{E}\left[\sup_{u\le s}\left|\partial_t X_u^{t,x}\right|\right] &\le \mathbb{E}\left[a_1(s,t,x)\left(1+\int_0^s a_2(u,t,x)\,\mathrm{d}u\exp\left\{\int_0^s a_2(u,t,x)\,\mathrm{d}u\right\}\right)\right] \\
&\le L\mathbb{E}\left[\int_0^s \left(\left|X_u^{t,x}\right|+1\right)\mathrm{d}u\left(1+se^{Ls}\right)\right] \\
&\le L\left(1+se^{Ls}\right)s\sqrt{\mathbb{E}\left[\sup_{u\le s}\left(\left|X_u^{t,x}\right|+1\right)^2\right]} \\
&\le L\left(1+se^{Ls}\right)s\sqrt{M_s^{(1)}\left(|x|^2+1\right)},
\end{aligned}$$

which entails (39). □

3 The pricing formula and the free-boundary equation

Propositions 1–4 allow applying an extension of the Itô's formula (D'Auria et al., 2020, Lemma A2) to $e^{-\lambda s}V(t+s, X_{t+s})$ which, after setting $s = T-t$, taking $\mathbb{P}_{t,x}$-expectation, and shifting the integrating variable t units, yields

$$V(t,x) = e^{-\lambda(T-t)}\mathbb{E}_{t,x}\left[G(X_T)\right] - \int_t^T \mathbb{E}_{t,x}\left[e^{-\lambda(u-t)}(\mathbb{L}V - \lambda V)(u,\ X_u)\right]\mathrm{d}u, \tag{40}$$

where the martingale term is vanished after taking $\mathbb{P}_{t,x}$-expectation and the local time term does not appear due to the smooth-fit condition. Recall that $\mathbb{L}V = \lambda V$ on C and $V = G$ on D. Therefore, (40) turns into the pricing formula

$$\begin{aligned}
V(t,x) &= e^{-\lambda(T-t)}\mathbb{E}_{t,x}\left[(A - X_T)\,\mathbb{1}\left(X_T \le A\right)\right] \\
&\quad + \int_t^T \mathbb{E}_{t,x}\left[e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu(u, X_u)\right)\mathbb{1}\left(X_u \le b(u)\right)\right]\mathrm{d}u \\
&= e^{-\lambda(T-t)}\int_{-\infty}^A (A-y)f(T,y \mid t,x)\mathrm{d}y \\
&\quad + \int_t^T\int_{-\infty}^{b(u)} e^{-\lambda(u-t)}\left(\lambda\left(A-y\right)+\mu(u,y)\right)f(u,y\mid t,x)\mathrm{d}y\,\mathrm{d}u,
\end{aligned} \tag{41}$$

where $f(s,y \mid t,x) = \partial_y\mathbb{P}_{t,x}\left(X_s \le y\right)$ is the transition density of X under $\mathbb{P}_{t,x}$. The first term of the right hand sum in (41) stands for the price of the European put option written on the same asset and expiring on the same date, while the second term, called the *early exercise premium*, represents the cost of having the added flexibility of being able to exercise the option before the expiration time T.

By taking $x \uparrow b(t)$ in (41), we get the free-boundary equation

$$b(t) = A - e^{-\lambda(T-t)}\mathbb{E}_{t,b(t)}\left[(A - X_T)\,\mathbb{1}\left(X_T \le A\right)\right] \tag{42}$$

$$
\begin{aligned}
&- \int_t^T \mathbb{E}_{t,b(t)} \left[e^{-\lambda(u-t)} \left(\lambda \left(A - X_u \right) + \mu(u, X_u) \right) \mathbb{1} \left(X_u \leq b(u) \right) \right] \mathrm{d}u \\
&= A - e^{-\lambda(T-t)} \int_{-\infty}^{A} (A-y) f(T, y \mid t, b(t)) \mathrm{d}y \\
&- \int_t^T \int_{-\infty}^{b(u)} e^{-\lambda(u-t)} \left(\lambda \left(A - y \right) + \mu(u, y) \right) f(u, y \mid t, b(t)) \mathrm{d}y \, \mathrm{d}u.
\end{aligned}
$$

It turns out that there exists a unique solution for (42) up to certain mild conditions. We include a proof of such an assertion in the next theorem, by adapting to our own framework the methodology used by Peskir (2005, Theorem 3.1), who firstly proved the uniqueness of solution of the free-boundary equation for an American put option when the underlying process is a geometric Brownian motion.

Theorem 1. *If the function $x \mapsto X_s^{t,x}$ is twice continuously differentiable for all $s \in [0, T]$, then the integral equation (42) has unique solution among the class of continuous functions $c : [0, T] \to \mathbb{R}$ of bounded variation, and such that $c(t) < A$ for all $t \in [0, T)$.*

Proof. Suppose there exists a function $c : [0, T] \to \mathbb{R}$ solving the integral equation (42), and define V^c as in (41) but with c instead of b. Examining the representation (41) we can conclude that the integrand is twice continuously differentiable with respect to x. Therefore, we can obtain $\partial_x V^c$ and $\partial_{xx} V^c$ and guarantee their continuity on $[0, T] \times R$ by differentiating inside the integral symbol.

Let us compute the operator $\mathbb{L}$ acting on V^c, which by definition takes the form

$$
(\mathbb{L}V^c)(t, x) = \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x} \left[V^c \left(t+h, X_{t+h} \right) \right] - V^c(t, x)}{h}.
$$

Define the function

$$
I(t, u) = e^{-\lambda(u-t)} \left(\lambda \left(A - X_u \right) + \mu \left(u, X_u \right) \right) \mathbb{1}(X_u \leq c(u)) \tag{43}
$$

such that, according to (41), and due to the Markovianity of X,

$$
\begin{aligned}
\mathbb{E}_{t,x} \left[V^c \left(t+h, X_{t+h} \right) \right] &= \mathbb{E}_{t,x} \left[\mathbb{E}_{t+h, X_{t+h}} \left[e^{-\lambda(T-t-h)} (A - X_T)^+ + \int_{t+h}^T I(t+h, u) \, \mathrm{d}u \right] \right] \\
&= \mathbb{E}_{t,x} \left[\mathbb{E}_{t,x} \left[e^{-\lambda(T-t-h)} (A - X_T)^+ + \int_{t+h}^T I(t+h, u) \, \mathrm{d}u \;\middle|\; \mathcal{F}_h \right] \right] \\
&= \mathbb{E}_{t,x} \left[e^{-\lambda(T-t-h)} (A - X_T)^+ + \int_{t+h}^T I(t+h, u) \, \mathrm{d}u \right] \\
&= \mathbb{E}_{t,x} \left[e^{\lambda h} \left(e^{-\lambda(T-t)} (A - X_T)^+ + \int_{t+h}^T I(t, u) \, \mathrm{d}u \right) \right].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\mathbb{L}V^c)(t, x) \\
&= \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x} \left[e^{\lambda h} \left(e^{-\lambda(T-t)} (A - X_T)^+ + \int_{t+h}^T I(t, u) \, \mathrm{d}u \right) \right] - \mathbb{E}_{t,x} \left[e^{-\lambda(T-t)} (A - X_T)^+ + \int_t^T I(t, u) \, \mathrm{d}u \right]}{h} \\
&= \lim_{h \downarrow 0} \frac{\mathbb{E}_{t,x} \left[\left(e^{\lambda h} - 1 \right) \left(e^{-\lambda(T-t)} (A - X_T)^+ + \int_t^T I(t, u) \, \mathrm{d}u \right) \right] - \mathbb{E}_{t,x} \left[e^{\lambda h} \int_t^{t+h} I(t, u) \, \mathrm{d}u \right]}{h} \\
&= \lim_{h \downarrow 0} \frac{e^{\lambda h} - 1}{h} \mathbb{E}_{t,x} \left[e^{-\lambda(T-t)} (A - X_T)^+ + \int_t^T I(t, u) \, \mathrm{d}u \right] - \lim_{h \downarrow 0} \mathbb{E}_{t,x} \left[e^{\lambda h} \frac{1}{h} \int_t^{t+h} I(t, u) \, \mathrm{d}u \right]
\end{aligned}
$$

$$= \lambda V^c(t,x) - (\lambda (A-x) - \mu (t,x))\, \mathbb{1}(x \le c(u)).$$

Now define the sets

$$C_1 := \{(t,x) \in [0,T) \times \mathbb{R} : x > c(t)\}, \quad C_2 := \{(t,x) \in [0,T) \times \mathbb{R} : x < c(t)\},$$

alongside the function $F^{(t)}(s,x) := e^{-\lambda s} V^c(t+s,x)$ with $s \in [0, T-t)$, $x \in \mathbb{R}$, and consider

$$C_1^t := \{(s,x) \in C_1 : t \le s < T\}, \quad C_2^t := \{(s,x) \in C_2 : t \le s < T\}.$$

Notice that $F^{(t)}$ satisfies the hypothesis of Lemma A2 in D'Auria et al. (2020), with $C = C_1^t$ and $D^\circ = C_2^t$: $F^{(t)}$, $\partial_x F^{(t)}$, and $\partial_{xx} F^{(t)}$ are continuous on $[0,T) \times \mathbb{R}$; $F^{(t)}$ is $\mathcal{C}^{1,2}$ on C_1^t and C_2^t; c is a continuous function of bounded variation; and $(\mathbb{L}F^{(t)})(s,x) = -(\lambda (A-x) - \mu (t,x))\, \mathbb{1}(x \le c(u))$ is locally bounded on $C_1^t \cup C_2^t$. Thereby, we can validate the following change-of-variable formula, which is missing the local time term due to the continuity of F_x on $[0,T) \times \mathbb{R}$:

$$e^{-\lambda s} V^c(t+s,\ X_{t+s}) = V^c(t,x) - \int_t^{t+s} I(t,u)\, \mathrm{d}u + M_s^{(1)}, \tag{44}$$

with $M_s^{(1)} = \int_t^{t+s} e^{-\lambda(u-t)} \sigma \partial_x V^c(u, X_u)\, \mathrm{d}B_u$. Notice that $(M_s^{(1)})_{s \ge 0}^{T-t}$ is a martingale under $\mathbb{P}_{t,x}$.

Similarly, we can apply Lemma A2 from D'Auria et al. (2020), but plugging in the function $F(s,x) = e^{-\lambda s} G(X_{t+s})$, and taking $C = \{(s,x) \in [0, T-t) \times \mathbb{R} : x > A\}$ and $D^\circ = \{(s,x) \in [0, T-t) \times \mathbb{R} : x < A\}$. We thereby get

$$\begin{aligned} e^{-\lambda s} G(X_{t+s}) = G(x) - \int_t^{t+s} I_A(t,u)\, \mathrm{d}u \\ - M_s^{(2)} + \frac{1}{2} \int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u = A)\, \mathrm{d}l_u^A(X_{t+\cdot}), \end{aligned} \tag{45}$$

where $M_s^{(2)} = \sigma \int_t^{t+s} e^{-\lambda(u-t)} \mathbb{1}(X_u < A)\, \mathrm{d}B_u$ is a martingale under $\mathbb{P}_{t,x}$ and I_A is defined as in (43) but with A instead of $c(u)$.

Consider the stopping time

$$\rho_c := \inf \{0 \le s \le T-t : X_{t+s} \ge c(t+s)\}. \tag{46}$$

Fix (t,x) such that $x \le c(t)$. Since we are assuming $c(t) < A$ for all $t \in [0,T)$, we can guarantee that $\mathbb{P}_{t,x}(X_{t+s} \le c(t+s)) = \mathbb{P}_{t,x}(X_{t+s} < A) = 1$ for all $s \in [0, \rho_c)$, and hence $I(t,u) = I_A(t,u)$ and $\int_t^{t+\rho_c} e^{-\lambda(u-t)} \mathbb{1}(X_u = A)\, \mathrm{d}l_s^A(X) = 0$ under $\mathbb{P}_{t,x}$. Recall that $V^c(t, c(t)) = G(c(t))$ for all $t \in [0,T)$ as c solves (42). Also, $V^c(T, X_T) = G(X_T)$. Hence, $V^c(t+\rho_c, X_{t+\rho_c}) = G(X_{t+\rho_c})$. We are able now to derive the following relation from equations (44) and (45),

$$\begin{aligned} V^c(t,x) &= \mathbb{E}_{t,x}[e^{-\lambda \rho_c} V^c(t+\rho_c,\ X_{t+\rho_c})] + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} I(t,u)\, \mathrm{d}u\right] \\ &= \mathbb{E}_{t,x}\left[e^{-\lambda \rho_c} G(X_{t+\rho_c})\right] + \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} I_A(t,u)\, \mathrm{d}u\right] \\ &= G(x). \end{aligned}$$

Therefore, we have proved that $V^c = G$ on $\bar{C}_2$.

Now, define the stopping time

$$\tau_c := \inf\{0 \le s \le T-t : X_{t+s} \le c(t+s)\}$$

and plug it into (44) to obtain the expression

$$V^c(t,x) = e^{-\lambda\tau_c} V^c(t+\tau_c,\ X_{t+\tau_c}) + \int_t^{t+\tau_c} I(t,u)\,\mathrm{d}u - M^{(1)}_{\tau_c}.$$

Notice that, due to the definition of τ_c, $\mathbb{1}(X_{t+u} \leq c(t+u)) = 0$ for all $0 \leq u < \tau_c$ whenever $\tau_c > 0$ (the case $\tau_c = 0$ is trivial). Therefore, the following formula comes after taking $\mathbb{P}_{t,x}$-expectation in the above equation and considering that $V^c = G$ on $\bar{C}_2$:

$$V^c(t,x) = \mathbb{E}_{t,x}[e^{-\lambda\tau_c} V^c(t+\tau_c, X_{t+\tau_c})] = \mathbb{E}_{t,x}\left[e^{-\lambda\tau_c} G(X_{t+\tau_c})\right],$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Recalling the definition of V from (2), the above equality leads to

$$V^c(t,x) \leq V(t,x), \tag{47}$$

for all $(t,x) \in [0,T) \times \mathbb{R}$.

Take $(t,x) \in C_2$ satisfying $x < b(t)$, and consider the stopping time ρ_b defined as

$$\rho_b := \inf\left\{0 \leq s \leq T-t : X_{t+s} \geq b(t+s)\right\}.$$

Since $V = G$ on D, the following equality holds due to (41) and after noticing that $\mathbb{P}_{t,x}(X_{t+u} \leq b(t+u)) = 1$ for all $0 \leq u < \rho_b$,

$$\mathbb{E}_{t,x}[e^{-\lambda\rho_b} V(t+\rho_b, X_{t+\rho_b})] = G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu\left(u, X_u\right)\right)\,\mathrm{d}u\right].$$

Additionally, after replacing s with ρ_b at (44) and recalling that $V = G$ on $\bar{C}_2$, we get that

$$\begin{aligned}&\mathbb{E}_{t,x}[e^{-\lambda\rho_b} V(t+\rho_b, X_{t+\rho_b})]\\ &\qquad = G(x) - \mathbb{E}_{t,x}\left[\int_t^{t+\rho_c} e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu\left(u, X_u\right)\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right].\end{aligned}$$

Therefore, we can use (47) to merge the two previous equalities into

$$\begin{aligned}&\mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu\left(u, X_u\right)\right)\mathbb{1}(X_u \leq c(u))\,\mathrm{d}u\right]\\ &\qquad \geq \mathbb{E}_{t,x}\left[\int_t^{t+\rho_b} e^{-\lambda(u-t)}\left(\lambda\left(A - X_u\right) + \mu\left(u, X_u\right)\right)\,\mathrm{d}u\right],\end{aligned}$$

meaning that $b(t) \leq c(t)$ for all $t \in [0,T]$ since c is continuous.

Suppose there exists a point $t \in (0,T)$ such that $b(t) < c(t)$ and fix $x \in (b(t), c(t))$. Consider the stopping time

$$\tau_b := \inf\{0 \leq u \leq T-t : X_{t+u} \leq b(t+u)\},$$

plug it into both (41) and (44) by replacing s, and then take $\mathbb{P}_{t,x}$-expectation to obtain

$$\mathbb{E}_{t,x}[e^{-\lambda\tau_b} V^c(t+\tau_b, X_{t+\tau_b})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_b} G(X_{t+\tau_b})] = V^c(t,x) - \mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} I(t,u)\,\mathrm{d}u\right]$$

and

$$\mathbb{E}_{t,x}[e^{-\lambda\tau_b} V(t+\tau_b, X_{t+\tau_b})] = \mathbb{E}_{t,x}[e^{-\lambda\tau_b} G(X_{t+\tau_b})] = V(t,x).$$

Thus, from (47), we get

$$\mathbb{E}_{t,x}\left[\int_t^{t+\tau_b} I(t,u)\,\mathrm{d}u\right] \leq 0.$$

Using the fact that $x > b(t)$ and the time-continuity of the process X, we can state that $\tau_b > 0$. Therefore, the previous inequality can only happen if $\mathbb{1}(X_s \leq c(s)) = 0$ for all $t \leq s \leq t+\tau_b$, meaning that $b(s) \geq c(s)$ for all $t \leq s \leq t+\tau_b$, which contradicts the assumption $b(t) < c(t)$. □

3.1 Gaussian marginal distributions

We can provide a more tractable expression for both the pricing formula (41) and the free-boundary equation (42) under the assumption that the underlying process has Gaussian marginal distributions.

A3.1 X_u is a Gaussian random variable with mean $\nu(u-t,x)$ and variance $\gamma^2(u-t)$ for all $u \in [0,T]$, under $\mathbb{P}_{t,x}$, where $\nu : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\gamma^2 : [0,T] \to \mathbb{R}^+$.

Equation (41) can be simplified by using the fact that, since $X_u \sim \mathcal{N}(\nu(u-t,x), \gamma^2(u-t))$ under $\mathbb{P}_{t,x}$ for $u \in [t,T]$, then $\mathbb{E}_{t,x}[X_u \mathbb{1}(X_u \le a)] = \nu(u-t,x)\Phi(\tilde{a}) - \gamma(u-t)\phi(\tilde{a})$, where ϕ and Φ are the density and distribution of a standard normal random variable, and $\tilde{a} = (a - \nu(u-t,x))/\gamma(u-t)$. Thereby, we get the following form of the pricing formula (41),

$$
\begin{aligned}
V(t,x) &= K_\lambda(T-t,x,A) + \lambda \int_t^T K_\lambda(u-t,x,b(u))\,\mathrm{d}u \\
&\quad + \int_t^T e^{-\lambda(u-t)} \left(\int_0^{b(u)} \mu(u,r)\gamma^{-1}(u-t)\phi\left(\frac{r-\nu(u-t,x)}{\gamma(u-t)}\right) \mathrm{d}r \right) \mathrm{d}u,
\end{aligned}
\tag{48}
$$

where

$$
K_\lambda(t,x_1,x_2) = e^{-\lambda t}\left((A - \nu(t,x_1))\,\Phi\left(\frac{x_2-\nu(t,x_1)}{\gamma(t)}\right) + \gamma(t)\phi\left(\frac{x_2-\nu(t,x_1)}{\gamma(t)}\right)\right). \tag{49}
$$

By taking $x \uparrow b(t)$ in (48), we can derive the free-boundary equation

$$
\begin{aligned}
b(t) &= A - K_\lambda(T-t,b(t),A) - \lambda \int_t^T K_\lambda(u-t,b(t),b(u))\,\mathrm{d}u \\
&\quad - \int_t^T e^{-\lambda(u-t)} \left(\int_0^{b(u)} \mu(u,r)\gamma^{-1}(u-t)\phi\left(\frac{r-\nu(u-t,b(t))}{\gamma(u-t)}\right) \mathrm{d}r \right) \mathrm{d}u.
\end{aligned}
\tag{50}
$$

4 Example

Let $\mu(t,x)$ in (1) take the form

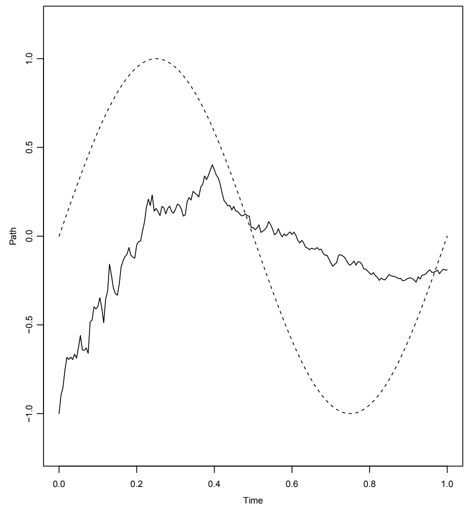
$$
\mu(t,x) = \theta(t)(\alpha(t) - x), \tag{51}
$$

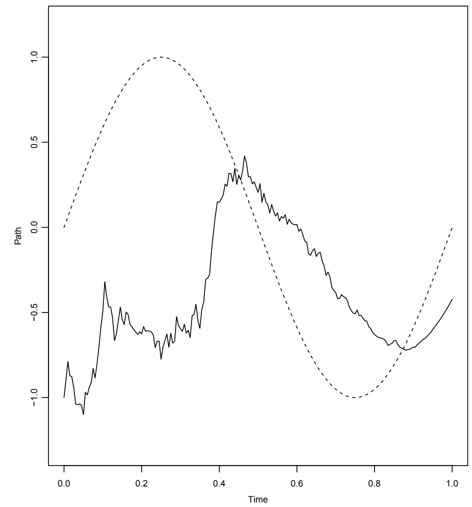
that is, $\{X_t\}_{t\ge 0}^T$ is an Ornstein–Uhlenbeck process with time-dependent parameters $\theta : [0,T] \to \mathbb{R}$ and $\alpha : [0,T] \to \mathbb{R}$, and constant volatility $\sigma > 0$. Such a process has been studied, among others, in Albano and Giorno (2020), Deng et al. (2016), and Palamarchuk (2018). It has the integral representation

$$
X_t = xe^{-\int_0^t \theta(u)\,\mathrm{d}u} + \int_0^t \alpha(u)\theta(u)e^{-\int_u^t \theta(r)\,\mathrm{d}r}\,\mathrm{d}u + \sigma \int_0^t e^{-\int_u^t \theta(r)\,\mathrm{d}r}\,\mathrm{d}W_u, \tag{52}
$$

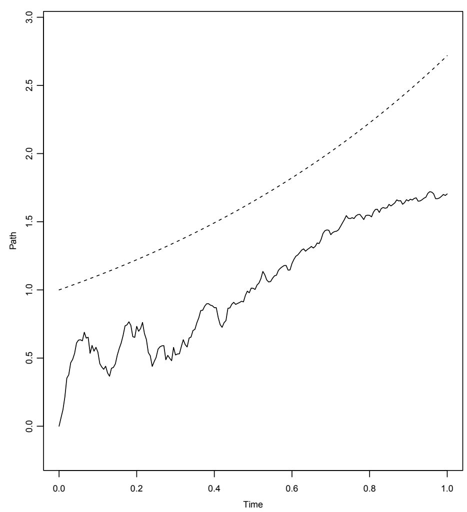
with $0 \le t \le T$.

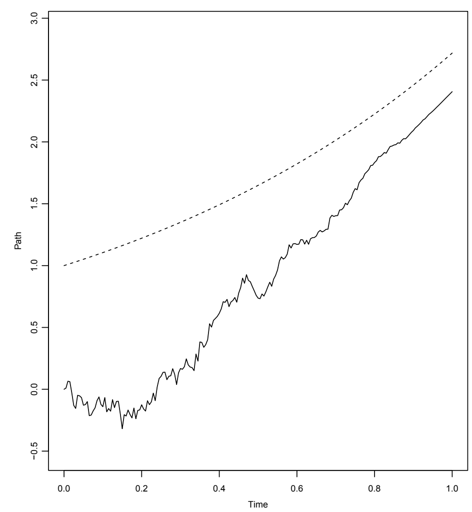
Notice that, depending on whether $\theta(t)$ is positive or negative, the process gets pulled towards or pushed away from $\alpha(t)$ with a strength that depends on $|\theta(t)|$. Figure 1 illustrates this behaviour for $\theta > 0$.





(a) $\theta(t) = \frac{10}{1+9t}, \quad \alpha(t) = \sin(2t\pi t), \quad X_0 = -1.$ (b) $\theta(t) = 10t, \quad \alpha(t) = \sin(2t\pi t), \quad X_0 = -1.$





(c) $\theta(t) = \frac{10}{1+9t}, \quad \alpha(t) = e^t, \quad X_0 = 0.$ (d) $\theta(t) = 10t, \quad \alpha(t) = e^t, \quad X_0 = 0.$

Figure 1: Time-dependent Ornstein–Uhlenbeck paths (solid coloured lines). Dashed lines represent the attracting curves α for situations with temporally decreasing (left column) and increasing (right) attraction strength. The volatility is $\sigma = 1$.

Proposition 5.
Let $X = \{X_t\}_{t\geq 0}^T$ be a process that satisfies (1) *with μ as in* (51)*. Assume the real functions θ and α are continuously differentiable, and such that $\theta(t) > 0$ for all $t \in [0, T]$. Then, X satisfies A2.1, A2.2, and A3.1.*

Proof. Assumption A2.1 is fulfilled by taking $u(t) = \alpha(t)$. To satisfy A2.2 it is enough to take $\bar{\mu} = -\theta$ and notice that the Lipschitz continuity of both μ and $\partial_t \mu$ follows straightforwardly from (51). Finally, the Gaussianity of X arises from (52), with marginal mean and variance respectively given by

$$\begin{aligned}
\nu(t, x) &= x e^{-\int_0^t \theta(u)\,\mathrm{d}u} + \int_0^t \alpha(u)\theta(u) e^{-\int_u^t \theta(r)\,\mathrm{d}r}\,\mathrm{d}u, \\
\gamma^2(t) &= \sigma^2 \int_0^t e^{-2\int_u^t \theta(r)\,\mathrm{d}r}\,\mathrm{d}u.
\end{aligned}$$

$\square$

Finally, it is worthwhile to notice that, due to the linearity of $x \mapsto \mu(t,x)$, the free-boundary equation (50) can be simplified even further, taking the form

$$b(t) = A - K_\lambda(T-t, b(t), A) - \int_t^T K_\lambda^{\mathrm{OU}}(u-t, b(t), b(u))\,\mathrm{d}u, \tag{53}$$

with K_λ as in (49) and

$$\begin{aligned} K_\lambda^{\mathrm{OU}}(t, x_1, x_2) = e^{-\lambda t}\Bigg(&(\lambda A + \theta(t)\alpha(t) - \nu(t,x_1)(\lambda + \theta(t))\,\Phi\left(\frac{x_2 - \nu(t,x_1)}{\gamma(t)}\right) \\ &+ \gamma(t)(\lambda+\theta(t))\phi\left(\frac{x_2 - \nu(t,x_1)}{\gamma(t)}\right)\Bigg). \end{aligned}$$

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