A new Kempe invariant and the (non)-ergodicity of the Wang–Swendsen–Kotecký algorithm

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Abstract

We prove that for the class of three-colorable triangulations of a closed oriented surface, the degree of a four-coloring modulo 12 is an invariant under Kempe changes. We use this general result to prove that for all triangulations $T(3L, 3M)$ of the torus with $3 \leq L \leq M$, there are at least two Kempe equivalence classes. This result implies in particular that the Wang–Swendsen–Kotecký algorithm for the zero-temperature 4–state Potts antiferromagnet on these triangulations $T(3L, 3M)$ of the torus is not ergodic.

Key Words: Triangulation; Kempe chain; torus; antiferromagnetic Potts model; four-coloring of a triangulation; degree of a four-coloring; Wang–Swendsen–Kotecký algorithm; cluster algorithm.

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1 Introduction

The q-state Potts model [4, 24, 25] is certainly one of the simplest and most studied models in Statistical Mechanics. However, despite many efforts over more than 50 years, its *exact* solution (even in two dimensions) is still unknown. The ferromagnetic regime is the best understood case: there are exact (albeit not always rigorous) results for the location of the critical temperature, the order of the transition, etc. The antiferromagnetic regime is less understood, partly because universality is not expected to hold in general (in contrast with the ferromagnetic regime); in particular, critical behavior may depend on the lattice structure of the model. One interesting feature of this antiferromagnetic regime is that zero-temperature phase transition may occur for certain values of q and certain lattices: e.g., the models with $q = 2, 4$ on the triangular lattice, and $q = 3$ on the square and kagomé lattices [18, and references therein].

The standard q-state Potts model can be defined on any finite undirected graph $G = (V, E)$ with vertex set V and edge set E. On each vertex of the graph $i \in V$, we place a spin $\sigma(i) \in \{1, 2, \ldots, q\}$, where $q \geq 2$ in an integer. The spins interact via a Hamiltonian

$$H(\{\sigma\}) \;=\; -J \sum_{e=ij\in E} \delta_{\sigma(i),\sigma(j)} \,, \tag{1.1}$$

where the sum is over all edges $e \in E$, $J \in \mathbb{R}$ is the coupling constant, and $\delta_{a,b}$ is the Kronecker delta. The *Boltzmann weight* of a configuration is then $e^{-\beta H}$, where $\beta \geq 0$ is the inverse temperature. The *partition function* is the sum, taken over all configurations, of their Boltzmann weights:

$$Z_G^{\text{Potts}}(q, \beta J) \;=\; \sum_{\sigma\colon\, V \to \{1,2,\ldots,q\}} e^{-\beta H(\{\sigma\})} \,. \tag{1.2}$$

A coupling J is called *ferromagnetic* if $J \geq 0$, as it is then favored for adjacent spins to take the same value; and *antiferromagnetic* if $-\infty \leq J \leq 0$, as it is then favored for adjacent spins to take different values. The zero-temperature ($\beta \to +\infty$) limit of the antiferromagnetic ($J < 0$) Potts model has an interpretation as a coloring problem: the limit $\lim_{\beta.\to+\infty} Z_G^{\text{Potts}}(q, -\beta|J|) = P_G(q)$ is the *chromatic polynomial*, which gives the number of proper q-colorings of G. A *proper q-coloring* of G is a map $\sigma\colon V \to \{1, 2, \ldots, q\}$ such that $\sigma(i) \neq \sigma(j)$ for all pairs of adjacent vertices $ij \in E$.

For many Statistical Mechanics systems for which an exact solution is not known, Markov Chain Monte Carlo simulations [2] have become a very valuable tool to extract physical information. An necessary condition for a Markov Chain Monte Carlo algorithm to work is that it should be ergodic (or irreducible): i.e., the chain can eventually get from each state to every other state. This condition is usually easy to check at positive temperature; but in many cases, it becomes a highly non-trivial question at zero temperature in the antiferromagnetic regime.

One popular Monte Carlo algorithm for the *antiferromagnetic* q-state Potts model is the Wang–Swendsen–Kotecký (WSK) *non-local* cluster dynamics [22, 23]. At zero temperature (where we expect interesting critical phenomena), it leaves invariant the uniform measure over proper q-colorings; but its ergodicity is a non-trivial question (and not completely understood).[1] It is interesting to note that at zero temperature, the basic moves of the WSK dynamics correspond to the so-called *Kempe changes*, introduced by Kempe in his unsuccessful proof of the four-color theorem. This connection makes this problem interesting from a purely mathematical point of view.

In this paper we will address the problem of the ergodicity of the WSK algorithm for the 4–state Potts antiferromagnet on the triangular lattice. Although the Potts model can be defined on any graph G, in Statistical Mechanics one is mainly interested in "large" regular graphs embedded on the torus (to minimize finite-size effects). Therefore, we will focus on certain regular triangulations of the torus, that we will denoted as $T(3L, 3M)$ (loosely speaking the triangulation $T(3L, 3M)$ is a subset of a triangular lattice with linear size $(3L) \times (3M)$ and fully periodic boundary conditions. For a more detailed definition, see next section).

The ergodicity of the WSK algorithm for the q-state antiferromagnetic on the triangular lattice embedded on a torus is only an open question for $q = 4, 5, 6$. For $q = 2$ (the Ising model) it is trivially non-ergodic, as each WSK move is equivalent to a global spin flip; while for $q = 3$ is trivially ergodic, as there is a single allowed three-coloring modulo global color permutations. On the contrary, for $q \geq 7$ the algorithm is ergodic (See next section for more details). Among the unknown cases, $q = 4$ is the most interesting one, because the system is expected to be critical at zero temperature.

Proper 4-colorings of triangulation of the torus are rather special, as they can be regarded as maps from a sphere S^2 (using the tetrahedral representation of the spin) to an orientable surface. Therefore, one can borrow concepts from algebraic topology; in particular, the degree of a four-coloring. This approach (pioneered by Fisk [6–8]) can only deal with $q = 4$, and cannot be extended to the other two cases $q = 5, 6$.

Our first goal is to obtain a quantity that is invariant under a Kempe change (or zero-temperature WSK move), at least for a class of triangulations that includes all triangulations of the type $T(3L, 3M)$. We succeeded in proving that for any three-colorable triangulation of a closed orientable surface, the degree of a four-coloring modulo 12 is a Kempe invariant. Because any four-coloring of a closed orientable surface has a degree multiple of six, and any three-coloring has degree zero, then we conclude that WSK with $q = 4$ colors is not ergodic on any three-colorable triangulation of a closed orientable surface which admits a four-coloring with degree congruent with 6 modulo 12.

The next goal is to prove that for any triangulation $T(3L, 3M)$ of the torus, such four-coloring with degree congruent with 6 modulo 12 exists. We first proved this statement for any symmetric triangulation $T(3L, 3L)$ with $L \geq 2$. Then, we extended this result to any triangulation of the form $T(3L, 3M)$ with $L \geq 3$ and $M \geq L$, and those of the form $T(6, 6(2M + 1))$ with $M \geq 0$. Therefore, we conclude that WSK with $q = 4$ colors is generically non-ergodic on the triangulations $T(3L, 3M)$ of the torus.

The paper is organized as follows: In Section 2 we introduce our basic definitions, and review what is known in the literature about the problem of the ergodicity of the Kempe dynamics. In Section 3, we introduce the algebraic topology approach borrowed from Fisk. This section includes two main results: the proof that the degree modulo 12 is a Kempe invariant for a wide enough class of triangulations, and a complete proof of Fisk theorem [8]

[1]WSK dynamics can indeed be defined for positive temperature. In this case, it is easy to show its ergodicity on the set of *all* q-colorings of the graph G (i.e., proper and non-proper).

for the class of triangulations $T(r, s, t)$ of the torus. In the next section, we apply the new invariant to prove that WSK is non-ergodic on any triangulation $T(3L, 3L)$ with $L \geq 2$. In Section 5 we extend the later result to non-symetric triangulations of the torus $T(3L, 3M)$ with $L \geq 3$ and $M \geq L$ (and also to $T(6, 6(2M + 1))$ with $M \geq 0$). Finally, in Section 6 we present our conclusions and discuss prospects of future work.

2 Basic setup

Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E. Then for each graph G there exists a polynomial P_G with integer coefficients such that, for each $q \in \mathbb{Z}_+$, the number of proper q-colorings of G is precisely $P_G(q)$. This polynomial P_G is called the *chromatic polynomial* of G. The set of all proper q-colorings of G will be denoted as $\mathcal{C}_q = \mathcal{C}_q(G)$ (thus, $|\mathcal{C}_q(G)| = P_G(q)$).

It is far from obvious that $Z_G^{\rm Potts}(q, \beta J)$ [cf. (1.2)], which is defined separately for each positive integer q, is in fact the restriction to $q \in \mathbb{Z}_+$ of a *polynomial* in q. But this is in fact the case, and indeed we have:

Theorem 2.1 (Fortuin–Kasteleyn [9, 13] representation of the Potts model) *For every integer $q \geq 1$, we have*

$$Z_G^{\rm Potts}(q, v) \;=\; \sum_{A \subseteq E} q^{k(A)}\, v^{|A|} \;, \tag{2.1}$$

where $v = e^{\beta J} - 1$, and $k(A)$ denotes the number of connected components in the spanning subgraph (V, A).

The foregoing considerations motivate defining the *Tutte polynomial* of the graph G:

$$Z_G(q, v) \;=\; \sum_{A \subseteq E} q^{k(A)}\, v^{|A|} \;, \tag{2.2}$$

where q and v are commuting indeterminates. This polynomial is equivalent to the standard Tutte polynomial $T_G(x, y)$ after a simple change of variables. If we set $v = -1$, we obtain the *chromatic polynomial* $P_G(q) = Z_G(q, -1)$. In particular, q and v can be taken as complex variables. See [20] for a recent survey.

As explained in the Introduction, we will focus on regular triangulations embedded on the torus The class of regular triangulations of the torus with degree six is characterized by the following theorem:

Theorem 2.2 (Altschulter [1]) *Let T be a triangulation of the torus such that all vertices have degree six. Then T is one of triangulations $T(r, s, t)$, which are obtained from the $(r + 1) \times (s + 1)$ grid by adding diagonals in the squares of the grid as shown in Figure 1, and then identifying opposite sides to get a triangulation of the torus. In $T(r, s, t)$ the top and bottom rows have r edges, the left and right sides s edges. The left and right side are identified as usual; but the top and the bottom row are identified after (cyclically) shifting the top row by t edges to the right.*

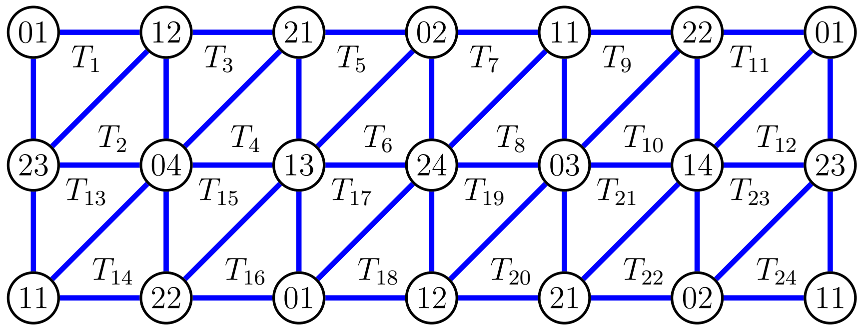

Figure 1: The triangulation $T(6,2,2) = \Delta^2 \times \partial\Delta^3$ of the torus. Each vertex x of $T(6,2,2)$ is labelled by two integers ij, where i (resp. j) corresponds to the associated vertex in Δ^2 (resp. $\partial\Delta^3$). The vertices of Δ^2 are labelled $\{0,1,2\}$, while the vertices of $\partial\Delta^3$ are labelled $\{1,2,3,4\}$. The triangulation $T(6,2,2)$ has 12 vertices, and those in the figure with the same label should be identified. We have also labelled the 24 triangular faces T_i in $T(6,2,2)$.

In Figure 1 we have displayed the triangulation $T(6,2,2)$ of the torus. We will represent these triangulations as embedded in a rectangular grid with three kinds of edges: horizontal, vertical, and diagonal.

The three-colorability of the triangulations $T(r,s,t)$ is given by the following result (whose proof is left to the reader):

Proposition 2.3 *The triangulation $T(r,s,t)$ is three-colorable if and only if $r \equiv 0 \pmod 3$ and $s - t \equiv 0 \pmod 3$.*

In Monte Carlo simulations, it is usual to consider toroidal boundary conditions with no shifting, so $t = 0$. Then, the three-colorability condition reduces to the standard result $r, s \equiv 0 \pmod 3$. In general, we will consider the following triangulations of the torus $T(3L, 3M, 0) = T(3L, 3M)$ with $L, M \geq 1$.

The unique three-coloring c_0 of $T(3L, 3M)$ can be described as:

$$c_0(x,y) \;=\; \mathrm{mod}(x+y-2,3)+1\,, \quad 1 \leq x \leq 3L\,, \quad 1 \leq y \leq 3M\,, \tag{2.3}$$

where we have explicitly used the above-described embedding of the triangulation $T(3L, 3M)$ in a square grid.

Finally, in most Monte Carlo simulations one usually considers tori of aspect ratio one: i.e., $T(3L, 3L)$. This is the class of triangulations we are most interested in from the point of view of Statistical Mechanics.

2.1 Kempe changes

Given a graph $G = (V, E)$ and $q \in \mathbb{N}$, we can define the following dynamics on $\mathcal{C}_q$: Choose uniformly at random two distinct colors $a, b \in \{1, 2, \ldots, q\}$, and let G_{ab} be the induced subgraph of G consisting of vertices $x \in V$ for which $\sigma(x) = a$ or b. Then, independently for each connected component of G_{ab}, with probability $\frac{1}{2}$ either interchange the colors a and b on it, or leave the component unchanged. This dynamics is the zero–temperature limit of the Wang–Swendsen–Kotecký (WSK) *non-local* cluster dynamics [22, 23] for the antiferromagnetic q-state Potts model. This zero–temperature Markov chain leaves invariant the uniform measure over proper q-colorings; but its ergodicity cannot be taken for granted.

The basic moves of the WSK dynamics correspond to *Kempe changes* (or K-*changes*). In each K-change, we interchange the colors a, b on a given connected component (or K-*component*) of the induced subgraph G_{ab}.

Two q-colorings $c_1, c_2 \in \mathcal{C}_q(G)$ related by a series of K-changes are *Kempe equivalent* (or K_q-*equivalent*). This (equivalence) relation is denoted as $c_1 \stackrel{q}{\sim} c_2$. The equivalence classes $\mathcal{C}_q(G)/\stackrel{q}{\sim}$ are called the *Kempe classes* (or *K_q-classes*). The number of K_q-classes of G is denoted by $\kappa(G, q)$. Then, if $\kappa(G, q) > 1$, the zero-temperature WSK dynamics is not ergodic on G for q colors.

In this paper, we will consider two q-colorings related by a *global* color permutation to be the same one. In other words, a q-coloring is actually an equivalence class of standard q-colorings modulo global color permutations. Thus, the number of (equivalence classes of) proper q-colorings is given by $P_G(q)/q!$. This convention will simplify the notation in the sequel.

2.2 The number of Kempe classes

In this section we will briefly review what it is known in the literature about the number of Kempe equivalence classes for several families of graphs. The first result implies that WSK dynamics is ergodic on any bipartite graph.[2]

Proposition 2.4 (Burton and Henley [3], Ferreira and Sokal [5], Mohar [16]) *Let G be a bipartite graph and $q \geq 2$ an integer. Then, $\kappa(G, q) = 1$.*

It is worth noting that Lubin and Sokal [14] showed that the WSK dynamics with 3 colors is not ergodic on any square–lattice grid of size $3M \times 3N$ (with M, N relatively prime) wrapped on a torus. These graphs are indeed not bipartite.

The second type of results deals with graphs of bounded maximum degree Δ, and shows that $\kappa(G, q) = 1$ whenever q is large enough.

Proposition 2.5 (Jerrum [12] and Mohar [16]) *Let Δ be the maximum degree of a graph G and let $q \geq \Delta + 1$ be an integer. Then $\kappa(G, q) = 1$. If G is connected and contains a vertex of degree $< \Delta$, then also $\kappa(G, \Delta) = 1$.*

This result implies that for any 6–regular triangulation $T = T(r, s, t)$, $\kappa(T, q) = 1$ for any $q \geq \Delta + 1 = 7$. However, the cases $q = 4, 5, 6$ are not covered by the above proposition. The case $q = 3$ is not covered either; but this one is trivial if the triangulation is *three-colorable*: the three-coloring is unique and therefore, $\kappa(T, 3) = 1$.

Finally, if we consider planar graphs the situation is better understood. Fisk [7] and Moore and Newman [17] showed that $\kappa(T, 4) = 1$ for planar 3-colorable triangulations.

[2]All the cited authors have discovered this theorem independently.

Moore and Newman's goal was to establish a height representation of the corresponding zero-temperature antiferromagnetic Potts model. One of the authors extended this result as follows:

Theorem 2.6 (Mohar [16], Theorem 4.4) *Let G be a three-colorable planar graph. Then $\kappa(G, 4) = 1$.*

Corollary 2.7 (Mohar [16], Corollary 4.5) *Let G be a planar graph and $q > \chi(G)$. Then $\kappa(G, q) = 1$.*

Indeed, none of our graphs $T(3L, 3M)$ is planar. Thus, the above results do not apply to our case. The main theorem for triangulations appears in [8]. It involves the notion of the degree of a four-coloring, whose definition is deferred to the next section.

Theorem 2.8 (Fisk [8]) *Suppose that T is a triangulation of the sphere, projective plane, or torus. If T has a three-coloring, then all four-colorings with degree divisible by* 12 *are Kempe equivalent.*

In Section 3.3, we provide a complete self-contained proof of Fisk's result when restricted to the 6-regular triangulations of the torus treated in this paper.

3 Four-colorings of triangulations of the torus

In this section we will consider four-colorings of triangulations of the torus. Most of the known results concerning this section were obtained by Fisk [6–8]. We will follow his notation hereafter.

3.1 An alternative approach to four-colorings

Fisk [6, 7] considered a definition of a four-coloring that allows to borrow concepts and results from algebraic topology. A (proper) four-coloring f of a triangulation T is a non-degenerate simplicial map

$$f \,:\, T \;\longrightarrow\; \partial\Delta^3\,, \tag{3.1}$$

where $\partial\Delta^3$ is the surface of a tetrahedron (thus, it can also be considered as a triangulation of the sphere S^2).[3] From algebraic topology [7], if T is the triangulation of an orientable closed surface (e.g., a sphere or a torus), there is an integer-valued function $\deg(f)$ determined up to a sign by f. In any practical computation, we should choose orientations for the triangulation T and the tetrahedron $\partial\Delta^3$. Then, given any triangle t of $\partial\Delta^3$ (i.e., a particular three-coloring of a triangular face), we can compute the number p (resp. n) of triangles of T mapping to t which have their orientation preserved (resp. reversed) by f. Then, the degree of the four-coloring f is defined as

$$\deg(f) \;=\; p - n\,, \tag{3.2}$$

[3] A map $f\colon T \to \partial\Delta^3$ is non-degenerate if the image of every triangle of T under f is a triangle of $\partial\Delta^3$.

and it is independent of the choice of the triangle t. For instance, the three-coloring of any triangulation has zero degree, as there are no vertices colored 4, so for $t = 124$ we have $p = n = 0$. As we are interested in equivalence classes of four-colorings modulo global color permutations, in practical computations it only makes sense to consider the absolute value of the degree: i.e., $|\deg(f)|$.

Tutte [21] proved a formula for the degree of a four-coloring modulo 2 (the parity of a four-coloring) in terms of the degrees of all vertices colored with a specific color. We write $\rho(x)$ for the degree of a vertex $x \in V$. A vertex is *even* (resp. *odd*) if its degree is even (resp. odd).

Lemma 3.1 (Tutte [21]) *Given a triangulation T of a closed orientable surface, the degree of a four-coloring f of T satisfies*

$$\deg(f) \equiv \sum_{f(x)=a} \rho(x) \pmod 2 \tag{3.3}$$

for $a = 1, 2, 3, 4$.

Proof. By definition, the degree of a four-coloring is modulo 2 equal to the number N of triangles of T mapping to a given triangle of $\partial\Delta^3$: $\deg(f) \equiv p + n \pmod 2$ and $N = p + n$. If we take a color a, which is a vertex of $\partial\Delta^3$, then there are three triangular faces of $\partial\Delta^3$ sharing this vertex a: i.e., t_1, t_2, and t_3. For each of these triangles t_i, there are N_i triangles of T mapping to t_i. Then,

$$\begin{aligned} \deg(f) &\equiv 3\deg(f) \pmod 2 \\ &\equiv N_1 + N_2 + N_3 \pmod 2 \end{aligned} \tag{3.4}$$

which is equal to the number of triangles of T with a vertex colored a. This number can indeed be written as the r.h.s. of (3.3). ∎

Lemma 3.1 implies that any Eulerian triangulation, in particular, any triangulation $T(r, s, t)$, can only have four-colorings with even degree, as every vertex $x \in V$ has even degree [i.e., $\rho(x) = 6$ for any vertex x of $T(r, s, t)$].

A natural question is how many possible values the degree of a four-coloring f can take. An answer for a restricted class of triangulations is given by the following proposition:

Proposition 3.2 (Fisk [6], Problem I.6.6 in [7]) *Let T be a triangulation of a closed orientable surface, and let f be a four-coloring of T. If T admits a three-coloring, then* $\deg(f) \equiv 0 \pmod 6$.

Proof. The idea is to mimic the proof of Theorem 4 in [7]. If T has a three-coloring h, and f is a 4-coloring of T, then we can combine these two maps and give

$$h \times f \,:\, T \longrightarrow \Delta^2 \times \partial\Delta^3\,, \tag{3.5}$$

where $\Delta^2 \times \partial\Delta^3 = T(6,2,2)$ (see Figure 1). We have the following diagram

$$\begin{tikzcd} T \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "h \times f"] & \Delta^2 \\ \partial\Delta^3 & \Delta^2 \times \partial\Delta^3 \arrow[l, "g"] \arrow[u] \end{tikzcd}$$

where g is the projection of $\Delta^2 \times \partial\Delta^3$ onto its second factor $\partial\Delta^3$. By commutativity, $\deg(f) = \deg(h \times f)\deg(g)$. As the degree of g is 6, then $\deg(f) = 6\deg(h \times f) \equiv 0 \pmod 6$. ∎

In this geometric approach to four-colorings, it is useful to introduce the concept of a Kempe region [7]. Suppose that D is a region of the triangulation T (i.e., a union of triangles of T), and that the four-coloring f uses only two colors on the boundary ∂D of D. We define a new coloring g of T that is equal to f on $T \setminus D$, and equal to $\pi(f)$ on D, where π is the permutation which interchanges the two colors *not* on ∂D. Fisk calls D a Kempe region of f, and ∂D a Kempe cycle. The coloring is *not* changed on ∂D itself. Indeed, inside a Kempe region D we find one or more Kempe components of the two colors not on ∂D. So, the new coloring is K-equivalent to f. Conversely, every K-change can be described as a change on the region consisting of all triangles containing an edge affected by the K-change.

Finally it is worth noting that Lemma 3.1 implies that the parity of a four-coloring [i.e., $\deg(f) \pmod 2$] is a Kempe invariant:

Corollary 3.3 *Given a triangulation T of a closed orientable surface, then the parity of a four-coloring of T is a Kempe invariant.*

Proof. If we consider a K-change on a region D, we take a to be one of the colors on the boundary ∂D (or one of the colors not on the Kempe component T_{bc}). Then, the parity given by (3.3) is not affected by the K-change, and therefore, it is an invariant. ∎

Unfortunately, the parity is not useful for our purposes, as we are interested in 6-regular triangulations of the torus $T(r,s,t)$. Thus, all four-colorings have even parity. In addition, in the class of three-colorable triangulations of any orientable surface, Proposition 3.2 ensures that all four-colorings have $\deg(f) \equiv 0 \pmod 6$.

3.2 A new Kempe invariant for a class of triangulations

In this section we shall consider a special class of triangulations in which every vertex is of even degree. Such a triangulation is said to be *even* (or Eulerian). Observe that every 3-colorable triangulation is even.

Tutte's lemma 3.1 implies that if we have a four-coloring f of a triangulation T and we perform a Kempe change to obtain a new four-coloring g, then

$$\deg(g) \equiv \deg(f) \pmod 2 . \tag{3.6}$$

For even triangulations this result has no useful consequences, as all four-colorings have even degree. However, for the restricted class of three-colorable triangulations of orientable surfaces we can do better.

Theorem 3.4 *Let T be a three-colorable triangulation of a closed orientable surface. If f and g are two four-colorings of T related by a Kempe change on a region R, then*

$$\deg(g) \;\equiv\; \deg(f) \pmod{12}\,. \tag{3.7}$$

Proof. We begin by noting that if T is three-colorable, then it is an even triangulation. Proposition 3.2 ensures that $\deg(f), \deg(g) \equiv 0 \pmod 6$. As in the proof of Proposition 3.2, we can combine the three-color map h with both four-colorings to define the following maps

$$F \;=\; h \times f \tag{3.8a}$$
$$G \;=\; h \times g \tag{3.8b}$$

from T onto $\Delta^2 \times \partial\Delta^3 = T(6,6,2)$, where h is the 3-coloring of T. Let us consider the following commutative diagram:

$$\begin{tikzcd}
\Delta^2 \times \partial\Delta^3 \arrow[r, "p_1"] \arrow[d, "p_2"'] & \Delta^2 & \Delta^2 \times \partial\Delta^3 \arrow[l, "p_1"'] \arrow[d, "p_2"] \\
\partial\Delta^3 & T \arrow[l, "g"] \arrow[r, "f"'] \arrow[u] \arrow[ul, "G"'] \arrow[ur, "F"] & \partial\Delta^3
\end{tikzcd}$$

Since $\deg(f) = \deg(F)\deg(p_2) = 6\deg(F)$ and $\deg(g) = 6\deg(G)$, our claim is equivalent to $\deg G \equiv \deg F \pmod 2$.

For simplicity, let us suppose that there is a Kempe region R such that its boundary ∂R is colored 3 and 4. Then, the Kempe change on R consists in swapping colors 1 and 2 on R. Let us see in detail what happens after this K-change. Consider Figure 1 for notation. Triangles in Figure 1 are labeled $T_1, \ldots, T_{24}$. We say that a triangle t in T is of *type i* with respect to the coloring f if it is mapped to T_i by the mapping F. Similarly, we consider types of triangles under g.

A triangle of type T_1 with positive (resp. negative) orientation is mapped on a triangle of type T_{24} with negative (resp. positive) orientation after we swap colors 1 and 2. We represent this correspondence as $\pm T_1 \leftrightarrow \mp T_{24}$. In fact, this K-change induces a bijection from the set of triangular faces of $T(6,2,2)$ onto itself of the form

$$\pm T_1 \;\leftrightarrow\; \mp T_{24} \tag{3.9a}$$
$$\pm T_{1+k} \;\leftrightarrow\; \mp T_{12+k}\,, \qquad 1 \le k \le 11. \tag{3.9b}$$

This correspondence can be written shortly as

$$\pm T_k \;\leftrightarrow\; \mp T_{\gamma(k)} \tag{3.10}$$

where γ is an appropriate permutation. After the K-change, the number of triangles of a given type outside R is not changed, so we have to count only the changes inside R. Let us introduce some useful notation: the total number of triangles of a given type $k \in \{1, \ldots, 24\}$ inside a region A of the triangulation T is denoted by $N_k^{(A)}$. Let $P_k^{(A)}$ (resp. $M_k^{(A)}$) denote the number of triangles of type k inside region A with positive (resp. negative) orientation. Hence,

$$N_k^{(A)} \;=\; P_k^{(A)} + M_k^{(A)}\,, \quad k = 1, 2, \ldots, 24\,, \quad A \subseteq T\,.$$

If we split the triangulation T into two regions R and $T \setminus R$, we get

$$\deg F \;=\; P_k^{(T\setminus R)} - M_k^{(T\setminus R)} + P_k^{(R)} - M_k^{(R)}\,, \quad k = 1, 2, \ldots, 24\,.$$

After the K-change we obtain a new four-coloring g. The composite coloring G is identical to F outside R. The differences can only occur inside R. The degree of G is given by:

$$\deg G \;=\; P_k^{(T\setminus R)} - M_k^{(T\setminus R)} - P_{\gamma(k)}^{(R)} + M_{\gamma(k)}^{(R)}\,, \quad k = 1, 2, \ldots, 24\,.$$

Let $\Delta \deg = \deg F - \deg G$. Then

$$\Delta \deg \;=\; P_k^{(R)} + P_{\gamma(k)}^{(R)} - (M_k^{(R)} + M_{\gamma(k)}^{(R)})\,, \quad k = 1, 2, \ldots, 24\,.$$

But this is equivalent to

$$\begin{aligned} \Delta \deg \;&\equiv\; P_k^{(R)} + P_{\gamma(k)}^{(R)} + M_k^{(R)} + M_{\gamma(k)}^{(R)} \pmod 2 \\ &\equiv\; N_k^{(R)} + N_{\gamma(k)}^{(R)} \pmod 2\,, \quad k = 1, 2, \ldots, 24. \end{aligned}$$

In particular, we have that for $k = 1, 5, 9$:

$$\begin{aligned} \Delta \deg \;&\equiv\; N_1^{(R)} + N_{24}^{(R)} \pmod 2 \\ \Delta \deg \;&\equiv\; N_5^{(R)} + N_{16}^{(R)} \pmod 2 \\ \Delta \deg \;&\equiv\; N_9^{(R)} + N_{20}^{(R)} \pmod 2 \end{aligned}$$

Summing these three equations we arrive at the formula

$$\begin{aligned} \Delta \deg \;&\equiv\; N_1^{(R)} + N_{24}^{(R)} + N_5^{(R)} + N_{16}^{(R)} + N_9^{(R)} + N_{20}^{(R)} \pmod 2 \\ &\equiv\; \#\text{ of triangles inside } R \text{ with no vertex colored } 4 \pmod 2 \\ &\equiv\; \#\text{ of triangles inside } R \text{ colored } 123 \pmod 2 \end{aligned} \tag{3.11}$$

Note that if we repeat this procedure with $k = 3, 7, 11$ we obtain a similar equation and conclude that $\Delta \deg$ has the same parity as the number of triangles inside R colored 124. On the other hand, we cannot obtain a similar formula for the triangles colored 134 or 234.

Let us go back to Eq. (3.11). All vertices colored 1 inside R belong to the interior of R (i.e., none of them lies on its boundary, as ∂R is colored $3, 4$). In addition, because the triangulation T is even, each interior vertex colored 1 belongs to an even number of triangular faces; all of them belonging to R. Let us consider one of these interior vertices colored 1, say x. If none of its neighbors is colored 4, x contributes $\rho(x)$ to $\Delta \deg$ in Eq. (3.11), which is an even number. For any neighboring vertex of x colored 4, this contribution is reduced by two. Thus, for each interior vertex colored 1, there is an even number of triangles belonging to R and colored 123. This implies that $\Delta \deg = \deg F - \deg G \equiv 0 \pmod 2$, and therefore

$$\deg f - \deg g \;=\; 6(\deg F - \deg G) \;\equiv\; 0 \pmod{12}\,,$$

as claimed. ■

Theorem 3.4 implies that a four-coloring f with degree $\deg f \equiv 6 \pmod{12}$ cannot be K-equivalent to the three-coloring h, whose degree is zero. This proves the following corollary:

Corollary 3.5 *Let T be a three-colorable triangulation of the torus. Then $\kappa(T,4) > 1$ if and only if there exists a four-coloring f with* $\deg(f) \equiv 6 \pmod{12}$.

Proof. Fisk's Theorem 2.8 together with Theorem 3.4 imply the existence of a Kempe equivalence class characterized by $\deg(g) \equiv 0 \pmod{12}$. This class includes the three-coloring. Thus, $\kappa(T,4) > 1$ if and only if there is a four-coloring f with $\deg(f) \equiv 6 \pmod{12}$. ■

By Theorem 3.4, the "if" part of Corollary 3.5 holds on arbitrary closed orientable surfaces.

The question of the ergodicity of the WSK dynamics on triangulations $T(3L, 3M)$ reduces to the existence of four-colorings of degree $\equiv 6 \pmod{12}$. If there are no such four-colorings, WSK dynamics is ergodic, while if such four-colorings exist, then WSK dynamics is non-ergodic, and the corresponding Markov chain will not converge to the uniform measure over $\mathcal{C}_4(T)$.

3.3 A complete proof of Fisk's theorem for $T(r, s, t)$

The proof of Theorem 2.8 in [8] seems to be missing some minor details, as reported in [15]. However, as far as the authors can see, Fisk's proof is complete and correct apart from these minor issues. Nevertheless, in this section we provide a self-contained proof of Fisk's result when restricted to the 6-regular triangulations of the torus treated in this paper. Another advantage of our proof is that it gives a closer insight into Kempe equivalence between 4-colorings of triangulations $T(r, s, t)$.

Theorem 3.6 *If the triangulation $T(r, s, t)$ admits a 3-coloring, then every 4-coloring of T, whose degree is divisible by 12, is K-equivalent to the 3-coloring.*

For the proof we shall consider the "non-singular structure" of 4-colorings and show that we can eliminate the "non-singular" part completely by applying K-changes and thus arrive to the 3-coloring. This will be done by a series of lemmas. But first we need some definitions.

Let f be a 4-coloring of a triangulation T. Let $xy \in E(T)$ and let xyz and xyw be the two triangles of T containing the edge xy. We say that the edge xy is *singular* (for the coloring f) if $f(z) = f(w)$, and is *non-singular* if $f(z) \neq f(w)$. Let $N(f)$ be the set of all non-singular edges, and for any distinct colors i, j, let $N_{ij} = N_{ij}(f)$ be the set of non-singular edges $xy \in N(f)$ for which $\{f(x), f(y)\} = \{i, j\}$. For a vertex x, let N_{ij}^x be the set of edges in N_{ij} that are incident with x.

From now on we assume that $T = T(r, s, t)$ is a fixed triangulation of the torus and that f is a 4-coloring of T. We also let $i, j \in \{1, \ldots, 4\}$ be distinct colors used by the 4-coloring f.

Lemma 3.7 *If x is a vertex of color $f(x) = i$, and $N_{ij}^x \neq \varnothing$, then $|N_{ij}^x| = 2$. Therefore, each N_{ij} is a union of disjoint cycles in T. If two such cycles, $C \subseteq N_{ij}$ and $C' \subseteq N_{il}$ $(j \neq l)$, cross each other at the vertex x, then there is a third cycle $C'' \subseteq N_{ik}$ $(k \neq j, l)$ passing through x and crossing both C and C' at x.*

Proof. Let us consider the possible 4-colorings around x. Up to symmetries (permutations of the colors and the dihedral symmetries of the 6-cycle), there are precisely four possibilities that are shown in Figure 2. The non-singular edges are drawn by bold solid or broken lines, and a brief inspection shows that the claims of the lemma hold. ∎

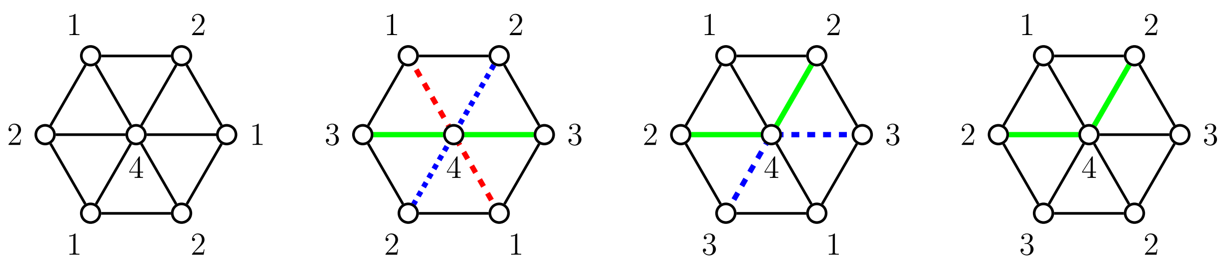

Figure 2: Non-singular edges around a vertex.

A 4-coloring f of T is said to be *non-singularly minimal* (*NS-minimal* for short) if for any two distinct colors i, j, the non-singular set N_{ij} is either empty or forms a single non-contractible cycle. The next lemma and its proof explain why such colorings are called "minimal".

Lemma 3.8 *Let f be a 4-coloring of T. Then there exists an NS-minimal 4-coloring f' of T that is K-equivalent with f and $N(f') \subseteq N(f)$.*

Proof. Let f' be a 4-coloring of T that is K-equivalent to f, such that $N(f') \subseteq N(f)$, and f' has minimum number of non-singular edges subject to these requirements. Since f has the stated conditions, f' exists.

Let us now consider an arbitrary pair of colors, say 1 and 2. If $C \subseteq N_{12}(f')$ is a contractible cycle, let R be the disk region bounded by C. By exchanging colors 3 and 4 on R (which keeps us in the same K-class), all the change in nonsingular edges is that C becomes singular. (However, note that particular sets N_{ij} may be changed.) This contradicts the minimality of $N(f')$. Therefore, every non-singular cycle in $N_{12}(f')$ is non-contractible.

Suppose that $N_{12}(f')$ contains distinct cycles C, C'. As proved above, C and C' are non-contractible. By Lemma 3.7, C and C' are disjoint, so they are homotopic and therefore together bound a cylinder region R. As above, by exchanging colors 3 and 4 on R, we get a contradiction to the minimality assumption. This completes the proof. ∎

As defined earlier, let $T_0 = \Delta^2 \times \partial\Delta^3 \approx T(6, 2, 2)$ be the 6-regular triangulation of the torus shown in Figure 3. Note that T_0 admits a 3-coloring and a non-singular 4-coloring. Its

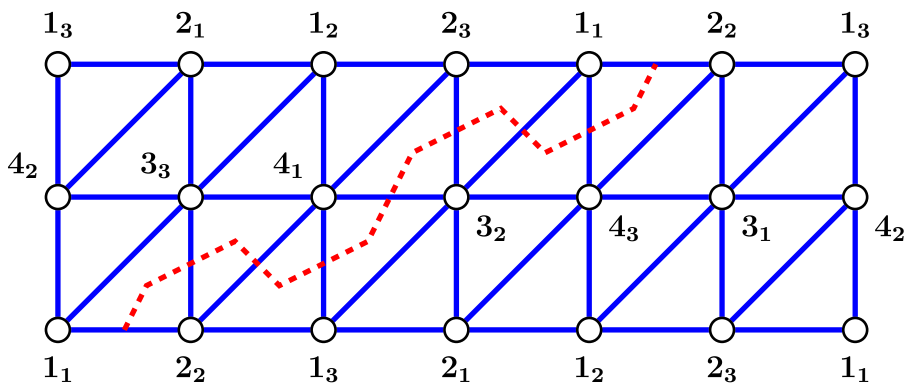

Figure 3: The triangulation $T_0 = \Delta^2 \times \partial\Delta^3 \approx T(6, 2, 2)$. The dashed line shows the sequence of triangles $(g \times f)(\gamma)$ (see text).

vertices can be labeled by pairs of colors, written as i_j, where $i \in \{1, 2, 3, 4\}$ is the color of the non-singular 4-coloring, and $j \in \{1, 2, 3\}$ is its color under the 3-coloring; see Figure 3. If the triangulation T has a 3-coloring g and a 4-coloring f, then we define a simplicial map $g\times f : T \to T_0$ by setting $(g\times f)(x) = f(x)_{g(x)} \in V(T_0)$ for every vertex x of T. If γ is a closed curve on the torus T that does not pass through the vertices of T, then γ can be described (up to homotopy) by specifying the sequence of triangles of T traversed by it. This closed sequence of triangles, $A_1, A_2, \ldots, A_N, A_1$, is uniquely determined if we cancel out possible immediate backtracking, i.e., subsequences of the form A, B, A. The mapping $g \times f$ then determines a closed sequence $B_1, B_2, \ldots, B_N, B_1$ of triangles in T_0, where $B_i = (g \times f)(A_i)$ for $i = 1, \ldots, N$. This sequence will be denoted by $(g \times f)(\gamma)$ (See Figure 3). The main property of this correspondence is that $B_i = B_{i+1}$ if and only if the edge common to A_i and A_{i+1} is singular with respect to the 4-coloring f of T, i.e. γ crosses a singular edge of f when passing from A_i to A_{i+1}.

Lemma 3.9 *Let $T = T(r, s, t)$ be a 3-colorable triangulation of the torus, and let f be an NS-minimal 4-coloring of T. If f is not the 3-coloring of T, then all non-singular cycles N_{ij} $(1 \le i < j \le 4)$ exist. Two such cycles N_{ij} and N_{kl} $(\{i, j\} \ne \{k, l\})$ are homotopic if and only if $\{i, j\} \cap \{k, l\} = \varnothing$.*

Proof. We shall use the notation introduced above. Since f is not the 3-coloring (which is unique, up to global permutations of colors), we may assume that $N_{12} \ne \varnothing$. Let γ be a simple closed curve in the torus that crosses N_{ij} precisely once and is given by the sequence of triangles $A_1, \ldots, A_N, A_1$. Let us consider the corresponding sequence $\gamma' = (g \times f)(\gamma) = B_1, B_2, \ldots, B_N, B_1$ of triangles in T_0.

Let K_{ij} be the non-singular cycle in T_0 passing through all vertices i_l and j_l, $l = 1, 2, 3$. Since γ crosses N_{12} precisely once, γ' crosses K_{12} exactly once. We may assume that it crosses K_{12} through the edge $e = 1_1 2_2$ as shown in Figure 3.

For a cycle K_{ij} we define the *algebraic crossing number* with γ' by first counting the number of consecutive triangles B_l, B_{l+1} in γ' such that B_l is "on the left" of K_{ij}, while B_{l+1} is "on the right" of it, and then subtracting the number of such pairs, where B_l is "on the right" and B_{l+1} is "on the left". (For the two "horizontal" cycles K_{12} and K_{34} we replace "left" by "bottom" and "right" by "top". All of these directions of course refer to Figure 3.) We denote this number by $\mathrm{algcr}(\gamma', K_{ij})$.

For an arbitrary edge-set $F \subseteq E(K_{ij})$, we define $\mathrm{algcr}(\gamma', F)$ in the same way, except that we only consider consecutive triangles B_l, B_{l+1} sharing the edges in F. Let $k = \mathrm{algcr}(\gamma', \{1_1 4_2, 4_2 1_3\})$. This number can be viewed as the "winding number" around the cylinder obtained from T_0 by cutting along the cycle K_{12}, cf. Figure 3. Using the fact that γ' is contained in this cylinder except for its crossing of the edge $1_1 2_2$, it is easy to see that $\mathrm{algcr}(\gamma', K_{13}) = 3k + 1$, $\mathrm{algcr}(\gamma', K_{24}) = 3k + 1$, $\mathrm{algcr}(\gamma', K_{14}) = 3k + 2$, and $\mathrm{algcr}(\gamma', K_{23}) = 3k + 2$. Moreover, $\mathrm{algcr}(\gamma', K_{12}) = \mathrm{algcr}(\gamma', K_{34}) = 1$. In particular, none of these numbers is zero (modulo 3).

Let us recall that $B_i \neq B_{i+1}$ if and only if the edge common to A_i and A_{i+1} is non-singular with respect to f. Therefore, γ' crosses an edge of K_{ij} precisely when γ crosses an edge in $N_{ij}(f)$. Therefore $\mathrm{algcr}(\gamma', K_{ij}) = \mathrm{algcr}(\gamma, N_{ij}) \neq 0$. This shows that none of the sets N_{ij} is empty.

If $\{i, j\} \cap \{k, l\} = \varnothing$, the two cycles N_{ij} and N_{kl} are disjoint. Since they are non-contractible and the surface is the torus, they are homotopic to each other. On the other hand, since $\mathrm{algcr}(\gamma, N_{13}) = \mathrm{algcr}(\gamma, N_{14}) - 1$, cycles N_{13} and N_{14} cannot be homotopic. Similarly, by starting the above proof with other cycles instead of N_{12}, we conclude that cycles N_{ij} and N_{kl} cannot be homotopic if $\{i, j\} \cap \{k, l\} \neq \varnothing$. ∎

Note that in the proof of Lemma 3.9, we did not use any assumption on the degree of the 4-coloring f. On the other hand, in our last lemma, when arguing about the degree of a 4-coloring, we will not need the existence of the 3-coloring.

Lemma 3.10 *Let f be an NS-minimal 4-coloring of T such that all non-singular cycles $N_{ij}(f)$ exist and such that two such cycles N_{ij} and N_{kl} ($\{i, j\} \neq \{k, l\}$) are homotopic if and only if $\{i, j\} \cap \{k, l\} = \varnothing$. Then the degree of f is congruent to 2 modulo 4. In particular, it is not divisible by 12.*

Proof. Let us consider cycles N_{12} and N_{13}. Since they are not homotopic, they cross at least once, and this happens at vertices of color 1. By Lemma 3.7, both these cycles are crossed by N_{14} at each such crossing point. Let us fix an orientation on the torus T and let $x \in V(T)$ be a vertex of color 1 at which N_{12}, N_{13}, N_{14} cross each other. If the local clockwise order around x is $N_{12}, N_{13}, N_{14}, N_{12}, N_{13}, N_{14}$, then we say that x is a *positive crossing point* (of color 1); if the local clockwise order is $N_{12}, N_{14}, N_{13}, N_{12}, N_{14}, N_{13}$, then x is a *negative crossing point*.

We claim that the difference of the number of positive minus the number of negative crossing points of color 1 is equal (in absolute value) to the algebraic crossing number $\mathrm{algcr}(N_{12}, N_{13})$. This is a consequence of the fact that color 4 changes sides from left to right side of N_{13}, or vice versa, every time when the curve N_{13} passes through a crossing point of color 1 or through a crossing point of color 3 (thus crossing the cycle N_{34} which is homotopic to N_{12}). We leave the details to the reader.

Since the numbers of positive and negative crossing points of color 1 are also the same for other pairs of non-singular cycles that involve color 1, we conclude that

$$|\,\mathrm{algcr}(N_{12}, N_{13})| \;=\; |\,\mathrm{algcr}(N_{12}, N_{14})| \;=\; |\,\mathrm{algcr}(N_{13}, N_{14})|\,. \tag{3.12}$$

Let us fix two simple closed curves γ, ν on the torus T, where ν is the curve corresponding to the cycle $N_{12}(f)$ and γ crosses ν precisely once. Then every closed curve α on T is homotopic to the curve which winds a times around ν, and then winds b times around γ, where a and b are integers. We say that α has *homotopy type* (a, b). The homotopy type of N_{12} is clearly (1,0). Let (a, b) and (c, d) be the homotopy types of N_{13} and N_{14}, respectively. The algebraic crossing number between closed curves is a (free) homotopy invariant and can be expressed as the determinant of the 2×2 matrix whose rows are the homotopy types of the curves (see, e.g. [26]). In particular,

$$\begin{aligned} \mathrm{algcr}(N_{12}, N_{13}) &= \pm \det \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \pm b\,, && (3.13) \\ \mathrm{algcr}(N_{12}, N_{14}) &= \pm \det \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} = \pm d\,, && (3.14) \\ \mathrm{algcr}(N_{13}, N_{14}) &= \pm \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm (ad - bc)\,. && (3.15) \end{aligned}$$

By (3.12), all three algebraic crossing numbers in (3.13)–(3.15) are equal up to the sign, so $|b| = |d| = |ad - bc|$. It follows that either $|a - c| = 1$ or $|a + c| = 1$. Here we have used the fact that $b \neq 0$, and this is true since N_{13} is not homotopic to N_{12}. A particular consequence of the above conclusion is that either a or c is even.

Suppose first that a is even. Since N_{13} is a simple curve, its homotopy type (a, b) satisfies $\gcd(a, b) = 1$ (cf. [26]). Therefore b and henceforth also d are odd.

The other case is when c is even. In that case, we derive the same conclusion as above. From this it follows that the total number of crossing points of color 1 is odd. Of course, we can repeat the same proof for crossing points of color 2 to conclude that their number is odd as well.

We are ready for the second part of the proof, where we will relate the number of crossing points and the degree of the coloring f. Let us traverse the cycle N_{12} and consider the (cyclic) sequence of all crossing points of colors 1 and 2 as they appear on N_{12}. We shall see that one can determine the degree of f just from this sequence.

Let us recall that $\deg(f)$ is equal to the difference between the number of triangles colored 123, whose orientation on the surface is 123, minus the number of such triangles whose orientation is 132. If t is such a triangle and its edge colored 12 is not in N_{12}, then there is another triangle colored 123 sharing that edge with t and having opposite orientation. The contribution of all such triangles towards the degree of f thus cancels out. On the other hand, each edge of N_{12} is contained in precisely one triangle colored 123. Consider two consecutive edges xy and yz on N_{12}. If y is not a crossing point with other non-singular curves, then one of the two triangles colored 123 and incident with these edges is oriented positively, the other one negatively, and so their contributions will cancel out. On the other hand, if y is a crossing point, then they have the same orientation. If two consecutive crossing points on N_{12} are of the same color, then the pair at one of these two crossing points is positively oriented, while the pair at the other crossing point is negatively oriented, and hence they cancel out. This has the same effect as removing two consecutive 1's or two consecutive 2's from the cyclic sequence of crossing points on N_{12}. Therefore, we may assume that the sequence of crossing points is alternating, $1212\dots12$. The number of 1's is an odd integer, say $2k + 1$, as shown in the first part of the proof. This implies that all triangles at crossing points have positive (or all have negative) orientation. Therefore, $\deg(f) = \pm 2(2k+1) \equiv 2 \pmod 4$, which we were to prove. ■

Proof of Theorem 3.6. Let f be a 4-coloring of $T = T(r,s,t)$. By Lemma 3.8 there is an NS-minimal coloring f' that is K-equivalent to f and has $N(f') \subseteq N(f)$. If f' is not the 3-coloring, then by Lemma 3.9, all six non-singular curves $N_{ij}(f')$ exist and their homotopy is as stated in the lemma. But then Lemma 3.10 implies that $\deg(f') \equiv 2 \pmod 4$. Since the K-equivalence preserves the value of the degree modulo 12 (cf. Theorem 3.4), this yields a contradiction to the assumption that the degree of f is divisible by 12. ■

4 Consequences for the triangulations $T(3L, 3L)$

A simple corollary of Proposition 3.2 and Theorem 2.8 shows that all 4-colorings of $T(3,3)$ are K-equivalent:

Corollary 4.1 $\kappa(T(3,3),4) = 1$.

Proof. The smallest (in modulus) non-zero degree for a four-coloring of an even three-colorable triangulation is 6 by Proposition 3.2. But in order to have a four-coloring f with such degree, we would need at least $6 \times 4 = 24$ triangular faces. However, the triangulation $T(3,3)$ only has $3^2 \times 2 = 18$ such faces. Then, $\deg(f) = 0$ for all four-colorings of $T(3,3)$, and Theorem 2.8 implies that $\kappa(T(3,3),4) = 1$. ■

A four-coloring f is said to be *non-singular* if all edges are non-singular with respect to f. Fisk [8] showed that the triangulation $T(r,s,t)$ has a non-singular four-coloring $c_{\rm ns}$ if and only if r,s,t are all even. In this non-singular coloring, each horizontal row uses exactly two colors. This also holds for all vertical and diagonal "straight-ahead cycles". For the triangulation $T(3L,3M)$, the non-singular coloring is given by

$$c_{\rm ns}(x,y) \;=\; \begin{cases} 1 & \text{if } x,y \equiv 1 \bmod 2 \\ 2 & \text{if } x \equiv 1 \text{ and } y \equiv 0 \bmod 2 \\ 3 & \text{if } x \equiv 0 \text{ and } y \equiv 1 \bmod 2 \\ 4 & \text{if } x,y \equiv 0 \bmod 2 \end{cases}, \quad 1 \le x \le 3L\,, \quad 1 \le y \le 3M \tag{4.1}$$

Proposition 4.2 *The triangulation $T(3L,3M)$ has a non-singular four-coloring c_{ns} if and only $L = 2\ell$ and $M = 2m$ are both even. If so, then $|\deg c_{ns}| = 18\ell m$. In particular, $\kappa(T(6\ell,6m),4) \ge 2$ if ℓ and m are both odd.*

Proof. Under the non-singular coloring, all triangles are mapped to $\partial\Delta^3$ with the same orientation. Thus, $|\deg c_{\rm ns}| = \frac{1}{4}(\#\text{triangles of } T(3L,3M)) = 18\ell m$. If ℓ and m are both odd, the degree is $\equiv 6 \bmod 12$, and now Corollary 3.5 applies. ■

The next non-trivial result shows that $\kappa(T(6,6),4) = 2$; hence WSK dynamics is non-ergodic on this triangulation.

Theorem 4.3 (with Alan Sokal) $\kappa(T(6,6),4) = 2$.

Proof. Proposition 4.2 shows that the non-singular four-coloring of $T(6,6)$ has $\deg(c_{\rm ns}) \equiv 6 \pmod{12}$ and that there are at least two Kempe equivalence classes for this triangulation. One class $\mathcal{C}_4^{(0)}$ corresponds to all colorings whose degree is a multiple of 12. The other classes contain colorings with degree $\equiv 6 \pmod{12}$.

The fact that the number of Kempe classes is exactly two can be derived as follows. Let us first observe that the maximum degree of a four-coloring of the triangulation $T(3L,3L)$ is $\lfloor 9L^2/2 \rfloor$; therefore, for $T(6,6)$ this maximum degree is 18. Thus, we should focus on all four-colorings f with $|\deg(f)| = 6, 18$, and show that they form a unique Kempe equivalence class.

There is a single four-coloring f with $|\deg(f)| = 18$: the non-singular coloring $c_{\rm ns}$ depicted in Figure 4(a). Each row (horizontal, vertical or diagonal) contains exactly two colors, and for any choice of colors a, b, the induced subgraph T_{ab} contains three parallel connected components, each of them being a cycle of length six. Then, the only non-trivial K-changes correspond to swapping colors on one of these cycles (as swapping colors simultaneously on two such cycles is equivalent to swapping colors on the third cycle and permute colors a, b globally). If we choose colors $1, 2$ and swap colors on the bottom row, we get the four-coloring f_b with degree $|\deg(f_b)| = 6$ depicted in Figure 4(b). To obtain a new coloring we should choose the other pair of colors $3, 4$, as for any other choice $(a,b) \neq (1,2)$ or $(3,4)$, the induced subgraph T_{ab} is connected, so we would not obtain a distinct coloring. Again, we only need to consider one of the three horizontal cycles of the induced subgraph T_{34}. Now we have two different choices: the second or the fourth rows from the bottom. The resulting colorings f_c, f_d are depicted respectively in Figures 4(c) and (d). Both have $|\deg(f_i)| = 6$, and all the induced subgraphs $T_{a,b}$ with $(a,b) \neq (1,2)$ or $(3,4)$ are again connected. Thus, all these colorings form a closed class $\mathcal{C}_4^{(1)}$ under K-changes; but we still need to prove that there are no additional colorings f with $|\deg f| = 6$.

To count the number of four-colorings f with $|\deg(f)| = 6$ belonging to the class $\mathcal{C}_4^{(1)}$, we can fix the colors of the three vertices of a triangular face t. Then, all we can do is (for each of the three directions – horizontal, vertical, and diagonal) to swap colors on any non-empty subset of the four cycles in the chosen direction not intersecting t. Since there are 15 non-empty subsets, we have $15 \times 3 = 45$ colorings f with $|\deg(f)| = 6$, and therefore, $|\mathcal{C}_4^{(1)}| = 46$.

Finally, we used a computer program (written in PERL) that enumerates all possible four-colorings on $T(6,6)$ and classify them according to $|\deg(f)|$. It finds 305192 proper four-colorings with zero degree, 45 colorings with $|\deg(f)| = 6$, and a single coloring with $|\deg(f)| = 18$. Therefore, $\mathcal{C}_4^{(1)}$ contains all colorings with $|\deg(f)| = 6, 18$, $\mathcal{C}_4(T(6,6)) = \mathcal{C}_4^{(0)} \cup \mathcal{C}_4^{(1)}$, and $\kappa(T(6,6),4) = 2$. Indeed, the number of all these colorings is equal to $P_{T(6,6)}(4)/4! = 305238$. ∎

Remark. The class $\mathcal{C}_4^{(0)}$ is grossly larger than $\mathcal{C}_4^{(1)}$: to be more precise, $|\mathcal{C}_4^{(1)}|/|\mathcal{C}_4^{(0)}| \approx 1.5 \times 10^{-4}$.

Let us now state a simple lemma which is the basic key in the proof of the next theorems.

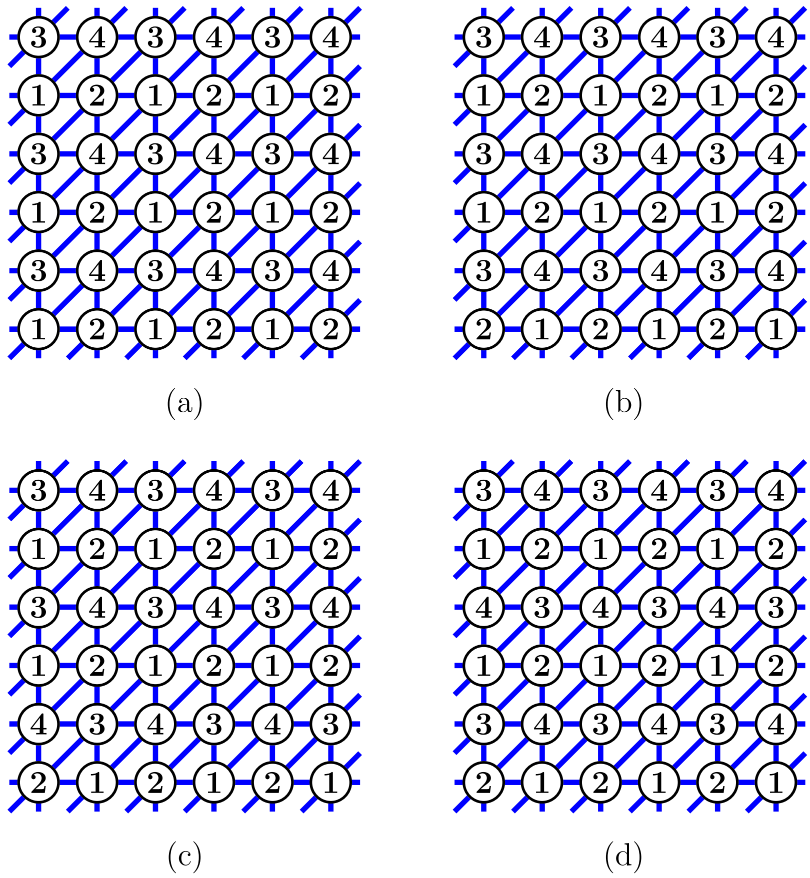

Figure 4: Four-colorings of the triangulation $T(6,6)$. (a) Coloring $c_{\rm ns}$ (4.1) with $|\deg(c_{\rm ns})| = 18$. (b) Coloring f_b obtained from $c_{\rm ns}$ by swapping colors $1,2$ on the bottom row. (c) Coloring f_c obtained from f_b by swapping colors $3,4$ on the second row from the bottom. (d) Coloring f_d obtained from f_b by swapping colors $3,4$ on the fourth row from the bottom. The coloring $c_{\rm ns}$ in (a) has $|\deg(c_{\rm ns})| = 18$; the colorings f_i in (b)–(d) have $|\deg(f_i)| = 6$.

Lemma 4.4 (a) *If there is a four-coloring f of the triangulation $T(r,s)$ with* $\deg(f) \equiv 2 \pmod 4$*, then there exists a four-coloring g of $T(3r,3s)$ with* $\deg(g) \equiv 6 \pmod{12}$.

(b) *If there is a four-coloring f of $T(3r,s)$ or $T(r,3s)$ with* $\deg(f) \equiv 2 \pmod 4$*, then there exists a four-coloring g of $T(3r,3s)$ with* $\deg(g) \equiv 6 \pmod{12}$.

(c) *If there is a four-coloring f of the triangulation $T(3r,3s)$ with* $\deg(f) \equiv 6 \pmod{12}$*, then for any odd integers p,q, there exists a four-coloring g of the triangulation $T(3rp,3sq)$ with* $\deg(g) \equiv 6 \pmod{12}$.

Proof. (a) If f is a four-coloring of $T(r,s)$, then we can obtain a four-coloring g of $T(3r,3s)$ by extending f periodically three times in each direction. If $\deg(f) = 2 + 4k$, with $k \in \mathbb{Z}$, then

$$\deg(g) \;=\; 9\deg(f) \;=\; 18 + 36k \;\equiv\; 6 \pmod{12}\,.$$

(b) The same arguments as in (a) apply here; the only difference is that the coloring of $T(3r,3s)$ is obtained from the coloring in $T(3r,s)$ (resp. $T(r,3s)$) by extending periodically the former three times in the vertical (resp. horizontal) direction. If $\deg(f) = 2 + 4k$, then the degree of the periodically extended coloring g is

$$\deg(g) \;=\; 3\deg(f) \;=\; 6 + 12k \;\equiv\; 6 \pmod{12}.$$

(c) If f is a four-coloring of $T(3r, 3s)$ with $\deg(f) \equiv 6 \pmod{12}$, then we can obtain a four-coloring g of $T(3rp, 3rq)$ by extending f periodically p times in the horizontal direction and q times in the vertical direction. If $\deg(f) = 6 + 12k$ with $k \in \mathbb{Z}$, the degree of g is

$$\deg(g) \;=\; pq\deg(f) \;=\; 6pq + 12pqk \;\equiv\; 6 \pmod{12}$$

if both p and q are odd integers. ■

4.1 Main results for $T(3L, 3L)$

Our main results for triangulations of the type $T(3L, 3L)$ can be summarized as follows:

Theorem 4.5 *For any triangulation $T(3L, 3L)$ with $L \geq 2$ there exists a four-coloring f with $\deg(f) \equiv 6 \pmod{12}$. Hence, $\kappa(T(3L, 3L), 4) > 1$. In other words, the WSK dynamics for four-colorings on $T(3L, 3L)$ is non-ergodic.*

Proof. The rest of this section is devoted to the proof of Theorem 4.5. We will show that $T(3L, 3L)$ admits a four-coloring f with $\deg(f) \equiv 6 \pmod{12}$. Then, Corollary 3.5 implies that $\kappa(T(3L, 3L), 4) > 1$ for any $L \geq 2$. The construction of f will depend on the value of L modulo 4, and we will split the proof in four cases, $L = 4k - 2, 4k - 1, 4k$, or $L = 4k + 1$, with $k \in \mathbb{N}$.

The basic strategy for all these proofs is to explicitly construct the four-coloring with the desired degree. With this aim, it is useful to fix orientations of both triangulations $T(3L, 3L)$ and $\partial\Delta^3$ in order to compute the degree of a given four-coloring (without ambiguity). We orient $T(3L, 3L)$ and $\partial\Delta^3$ in such a way that the boundaries of all triangular faces are always followed clockwise. The contribution of a triangular face t of $T(3L, 3L)$ to the degree is $+1$ (resp. -1) if the coloring is 123 (resp. 132) if we move clockwise around the boundary of t. In our figures, those faces with orientation preserved (resp. reversed) by f are depicted in light (resp. dark) gray.

The easiest case is when $L = 4k - 2$. In this case, $T(3L, 3L)$ admits the non-singular 4-coloring, whose degree is congruent to 6 modulo 12 by Proposition 4.2.

Other cases need a more elaborate construction. The common strategy is to devise an algorithm to obtain the desired four-coloring; and the main ingredient is to use the counter-diagonals of the triangulations: these counter-diagonals are orthogonal to the inclined edges of the triangulation when embedded in a square grid. They will be denoted as Dj with $1 \leq j \leq 3L$. In Figure 5 we show the triangulation $T(6, 6)$, and its six counter-diagonals Dj. As we have embedded the triangulation into a square grid, we will use Cartesian coordinates (x, y), $1 \leq x, y \leq 3L$, for labelling the vertices.

We will describe an algorithm that provides the desired coloring f. It is useful to monitor the degree of the coloring as we construct it. In particular, at a given step of the algorithm, the four-coloring f will be defined on some region R of $T = T(3L, 3L)$ (i.e., the union of all

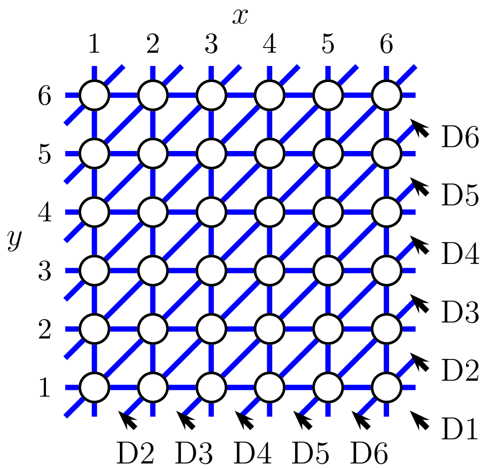

Figure 5: Notation used in the proof of Theorem 4.5. Given a triangulation $T(M, M)$ (here we depict the case $M = 6$), we label each vertex using Cartesian coordinates (x, y) $[1 \le x, y \le M]$. The arrows (pointing north-west) show the counter-diagonals Dj with $j = 1, \ldots, M$.

properly colored triangular faces of T). What we mean by the degree of f at this stage, is the contribution to the degree of f of the triangles belonging to R: $\deg(f|_R)$. Again, we will count only those triangular faces of T colored 123. Notice that at the end of the algorithm, when $R = T$, this partial degree will coincide with the standard one, $\deg(f) = \deg(f|_T)$.

Case 2: $L = 4k - 1$.

Let us consider the triangulation $T = T(12k - 3, 12k - 3)$ with $k \in \mathbb{N}$ (the case $k = 1$ will illustrate our ideas in Figures 6–7). Our goal is to obtain a four-coloring f of T with degree $\deg(f) \equiv 6 \pmod{12}$. The algorithm to obtain such a coloring consists of four steps:

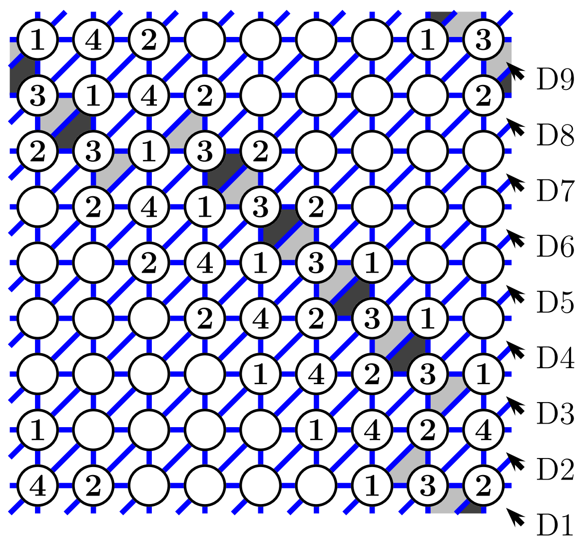

Figure 6: The 4-coloring of $T(9, 9)$ after Step 1 in the proof of the case $L = 4k - 1$.

Step 1. We start by coloring the counter-diagonal D1: we color 1 the vertices with x-coordinates $1 \leq x \leq 6k-1$; the other $6k-2$ vertices are colored 2.

On D2, we color 3 those $6k-1$ vertices with x-coordinates $3k+1 \leq x \leq 9k-1$. The other vertices on D2 are colored 4. The vertices on D$(12k-3)$ are colored 3 or 4 in such a way that the resulting coloring is proper (for each vertex, there is a unique choice).

On D3 and D$(12k-4)$, we color all vertices 1 or 2 (there is a unique choice for each vertex). The resulting coloring is depicted on Figure 6. The partial degree of f is $\deg f|_R = 4$.

Step 2. For $k > 1$, we find that there are $12k-8$ counter-diagonals to be colored and we need to sequentially color all of them but four. This can be achieved by performing the following procedure: suppose that we have already colored counter-diagonals Dj and D$(12k-j-1)$ ($j \geq 3$) using colors 1 and 2. Then, we color D$(j+1)$ and D$(12k-j-2)$ using colors 3 and 4, and D$(j+2)$ and D$(12k-j-3)$ using colors 1 and 2. As in Step 1, for each vertex there is a unique choice.

This procedure is repeated $3(k-1)$ times, so we add $12(k-1)$ counter-diagonals, and there are only four counter-diagonals not yet colored. Indeed, the last colored counter-diagonals D$(6k-3)$ and D$(6k+2)$ have colors 1 and 2, the same as it was at the end of Step 1.

Each of these $3(k-1)$ steps adds 4 to the degree of the coloring. Thus, the partial degree of f is $\deg f|_R = 4 + 12(k-1)$.

Step 3. There remain only four counter-diagonals to be colored: D$(6k-2)$, D$(6k-1)$, D$(6k)$, and D$(6k+1)$. On D$(6k-2)$, the vertices $(3k-1, 3k-1)$ and $(9k-2, 9k-3)$ only admit a single color (which is 3 for one of them, and 4 for the other one). The rest of the vertices on D$(6k-2)$ are colored 1 and 2 (again, there is a unique choice for each vertex).

We now color 3 or 4 all the vertices on D$(6k+1)$ (the choice is again unique for each vertex). The resulting coloring is depicted in Figure 7(a). The contribution to the partial degree of the new triangles is zero; the partial degree of f is given by $\deg f|_R = 4+12(k-1)$.

Step 4. On D$(6k-1)$, there are two pairs of nearby vertices which only admit a single color (which is 3 for one pair, and 4 for the other one). These vertices are located at $(3k-1, 3k)$, $(3k, 3k-1)$, $(9k-1, 9k-3)$, and $(9k-2, 9k-2)$. The other vertices on D$(6k-1)$ can be colored 3 or 4 (with only one choice for each of them). The increment of the degree after coloring these vertices is -2, thus $\deg f|_R = 2 + 12(k-1)$.

Finally, all vertices on D$(6k)$ are colored 1 and 2; and again the choice is unique for each vertex. The final coloring is depicted on Figure 7(b). The increment in the degree is 4, and therefore, the degree of the four-coloring f is

$$\deg f \;=\; 6 + 12(k-1) \;\equiv\; 6 \pmod{12} \tag{4.2}$$

This coloring f of $T(12k-3, 12k-3)$ satisfies the two needed properties: it is a proper coloring and its degree is congruent to six modulo 12.

Case 3: $L = 4k$.

Let us consider the triangulation $T = T(12k, 12k)$ with $k \in \mathbb{N}$ (we will illustrate the main steps with the case $k = 1$). Our algorithm consists of five steps:

Step 1. On the counter-diagonal D1 we color 1 the $6k$ consecutive vertices with x-coordinates $1 \leq x \leq 6k$. The other $6k$ vertices on D1 are colored 2.

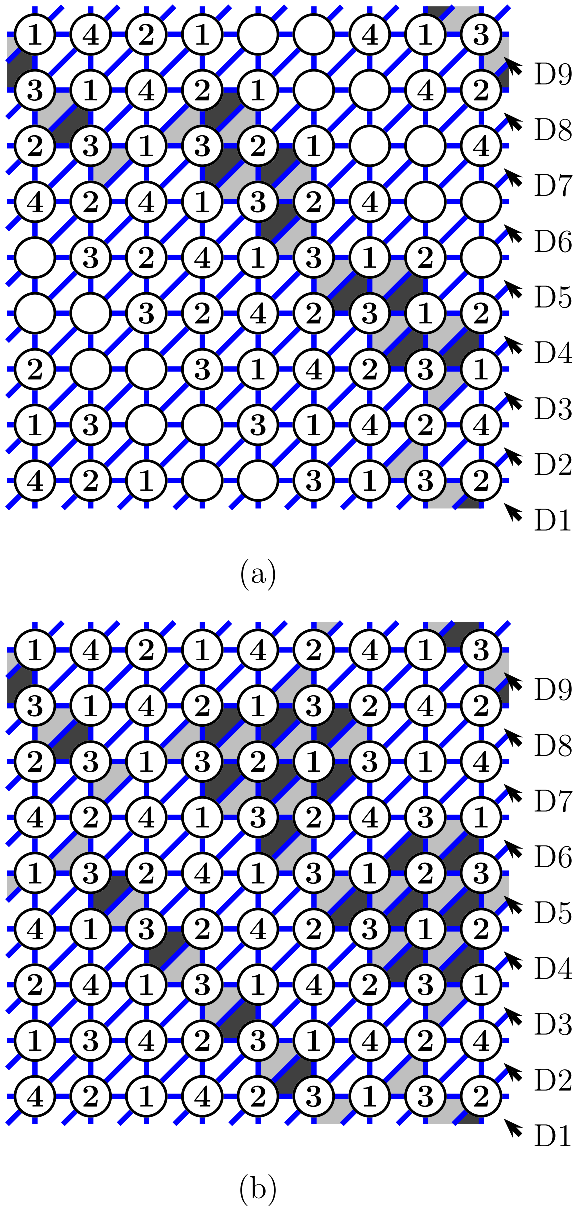

Figure 7: Four-colorings of the triangulation $T(9,9)$ after Steps 3 (a) and 4 (b) in the proof of the case $L = 4k-1$.

On D2, we color 3 the $6k$ consecutive vertices with x-coordinates $3k+2 \le x \le 9k+1$. The other vertices on D2 are colored 4. The vertices on D$(12k)$ are colored 3 or 4 in such a way that the resulting coloring is proper (for each vertex, the choice is unique).

We color all vertices on D3 and D$(12k-1)$ using colors 1 and 2. We then color D4 and D$(12k-2)$ using colors 3 and 4. Again the condition that f is proper implies that for each vertex the choice is unique. The partial degree of f is $\deg f|_R = 4$.

Step 2. For $k > 1$, we find that there are $12k-7$ counter-diagonals to be colored, and we need to sequentially color all of them but five. This can be achieved by performing the following procedure: suppose that we have already colored counter-diagonals Dj and

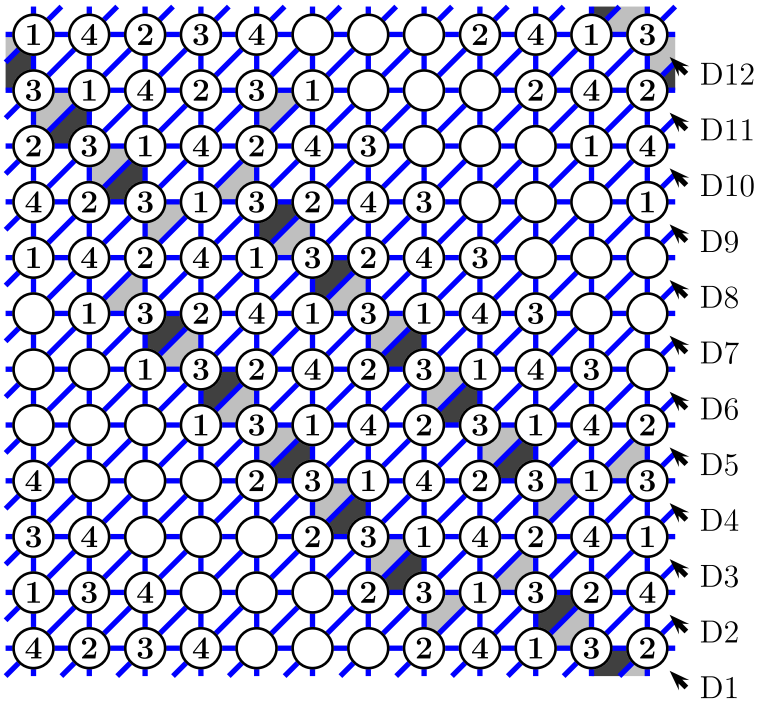

Figure 8: The 4-coloring of $T(12,12)$ after Step 3 in the case $L = 4k$.

$\mathrm{D}(12k-j-2)$ $(j \geq 4)$ using colors 3 and 4. Then, we color $\mathrm{D}(j+1)$ and $\mathrm{D}(12k-j+1)$ using 1 and 2, and then, we color $\mathrm{D}(j+2)$ and $\mathrm{D}(12k-j)$ using 3 and 4. Again, for each vertex we have only one choice. This step is repeated $3(k-1)$ times: we add $12(k-1)$ counter-diagonals, and there are only five counter-diagonals not yet colored. Indeed, the last colored counter-diagonals use colors 3 and 4, as it was at the end of Step 1.

Each of these $3(k-1)$ steps adds 4 to the degree of the coloring. Thus, the partial degree of the coloring is $\deg f|_R = 4 + 12(k-1)$.

Step 3. The last colored counter-diagonals are $\mathrm{D}(6k-2)$ and $\mathrm{D}(6k+4)$.

On $\mathrm{D}(6k-1)$, the vertices at $(6k, 12k-1)$ and $(12k, 6k-1)$ only admit one color: one of them should have color 1 and the other one 2. The rest of the vertices on $\mathrm{D}(6k-1)$ are colored 3 or 4 (again, there is a unique choice for each vertex).

We color 1 or 2 all vertices on $\mathrm{D}(6k+3)$; again there is a unique choice for each vertex. As shown in Figure 8, the contribution to the degree of these new triangles is 4; thus, the partial degree of f is $\deg f|_R = 8 + 12(k-1)$.

Step 4. On $\mathrm{D}(6k)$ the vertices at $(1, 6k-1)$, $(12k, 6k)$, $(6k+1, 12k-1)$, and $(6k, 12k)$ only admit a unique color choice: either 1 or 2. The first two vertices should be colored alike, while the last two vertices take the other color. We color the other vertices on $\mathrm{D}(6k)$ with 1 and 2 in such a way that those vertices with x-coordinate satisfying $1 \leq x < 6k$ take the same color as the vertex at $(1, 6k-1)$; the rest are colored the same as the vertex at $(6k, 12k)$.

All vertices on $\mathrm{D}(6k+1)$ are colored 3 or 4. For all of them, except for those at $(1, 6k)$ and $(6k+1, 12k)$, there is unique possibility to do so. We color 4 the vertex at $(1, 6k)$,

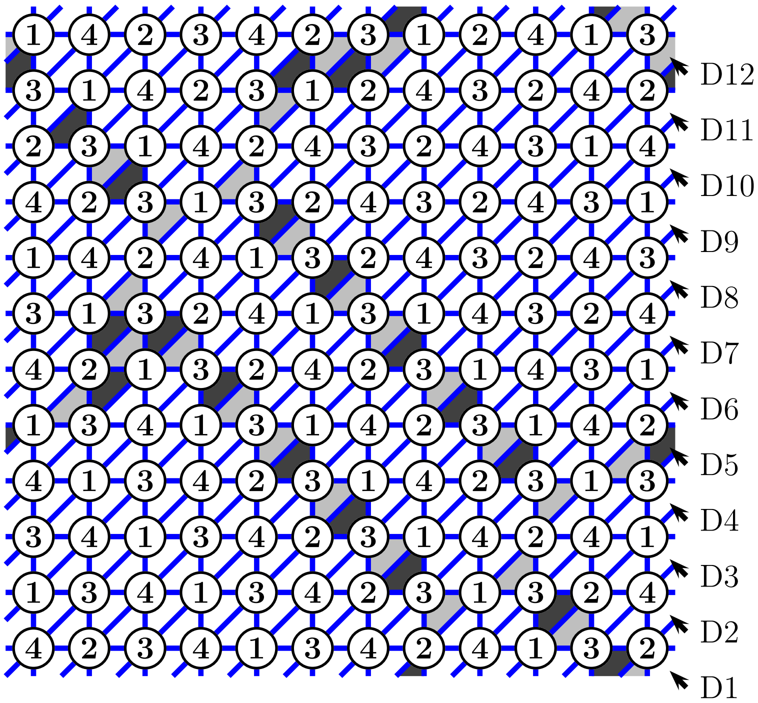

Figure 9: The 4-coloring of $T(12,12)$ after Step 5 in the case $L = 4k$.

and color 3 the vertex at $(6k+1, 12k)$. The increment of the partial degree is -2, thus $\deg f|_R = 6 + 12(k-1)$.

Step 5. Finally, on $\mathrm{D}(6k+2)$, there are two vertices which only admit a single color chosen among 1 and 2. For odd k these vertices are $(2, 6k)$ and $(6k+2, 12k)$; while for even k, these vertices are $(1, 6k+1)$ and $(6k+1, 1)$. The other vertices on $\mathrm{D}(6k+2)$ can be colored 3 and 4 (uniquely). The resulting coloring is depicted in Figure 9. In this step, the increment in the degree is zero. Therefore, the degree of the obtained four-coloring is

$$\deg f \;=\; 6 + 12(k-1) \;\equiv\; 6 \pmod{12}$$

This coloring f of $T(12k, 12k)$ is proper and its degree is congruent to six modulo 12, as claimed.

Case 4: $L = 4k+1$.

Let us consider the triangulation $T = T(12k+3, 12k+3)$ with $k \in \mathbb{N}$ (we will illustrate the main steps with the case $k = 1$).

Step 1. On D1 we color 1 the $6k+2$ consecutive vertices with x-coordinate $1 \le x \le 6k+2$. The other $6k+1$ vertices on D1 are colored 2.

On D2 we color 3 the $6k+1$ consecutive vertices with x-coordinate $3k+3 \le x \le 9k+3$. The other vertices on D2 are colored 4. We color 3 or 4 all vertices on $\mathrm{D}(12k+3)$; the choice is unique for each vertex.

We color 1 or 2 all vertices on D3, D5, D$(12k+2)$, and D$(12k)$. And we color 3 or 4 all vertices on D4 and D$(12k+1)$. In all cases, the choice is unique for each vertex.

The resulting (partial) coloring is depicted in Figure 10. The partial degree of this coloring is $\deg f|_R = 8$.

Step 2. For $k > 1$, we find that there are $12k-6$ counter-diagonals to be colored and in this step we will sequentially color all of them but six. This can be achieved by performing the following procedure: suppose that we have already colored Dj and D$(12k-j+5)$ ($j \geq 5$) using colors 1 and 2. Then, we color D$(j+1)$ and D$(12k-j+4)$ using colors 3 and 4, and D$(j+2)$ and D$(12k-j+3)$ using colors 1 and 2. Again, for each vertex the choice is unique.

This step is repeated $3(k-1)$ times; thus, we add $12(k-1)$ counter-diagonals, and there are only six counter-diagonals not yet colored. Indeed, the last colored counter-diagonals use colors 1 and 2, as it was at the end of Step 1.

Each of these $3(k-1)$ steps adds 4 to the degree of the coloring. Thus, the partial degree is $\deg f|_R = 8 + 12(k-1)$.

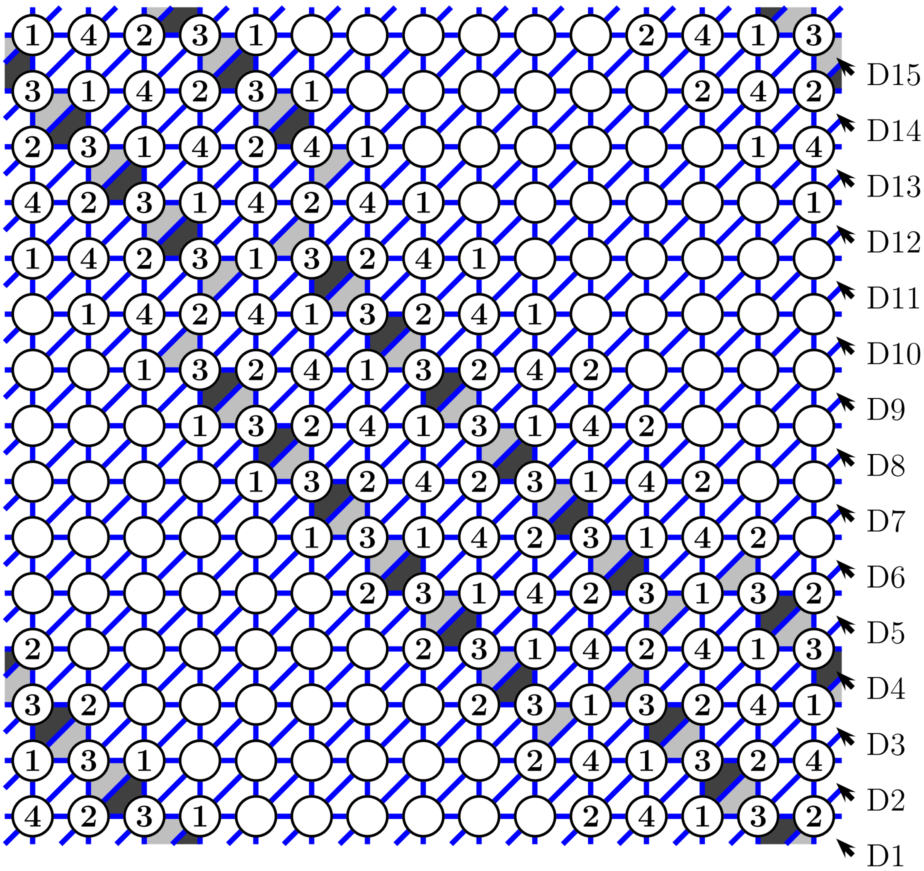

Figure 10: The 4-coloring of $T(15, 15)$ after Step 1 in the case $L = 4k+1$.

Step 3. The last colored counter-diagonals are D$(6k-1)$ and D$(6k+6)$. On D$(6k)$ the vertices at $(3k, 3k)$ and $(9k+2, 9k+1)$ only admit a single color: either 3 or 4. We color the rest of the vertices of D$(6k)$ with colors 1 and 2 (again, uniquely). On D$(6k+5)$ we perform the same procedure; here the vertices with only one color choice are located at $(3k+3, 3k)$ and $(9k+4, 9k+4)$. The contribution to the degree of the newly colored triangles is zero: the partial degree is still $\deg f|_R = 8 + 12(k-1)$.

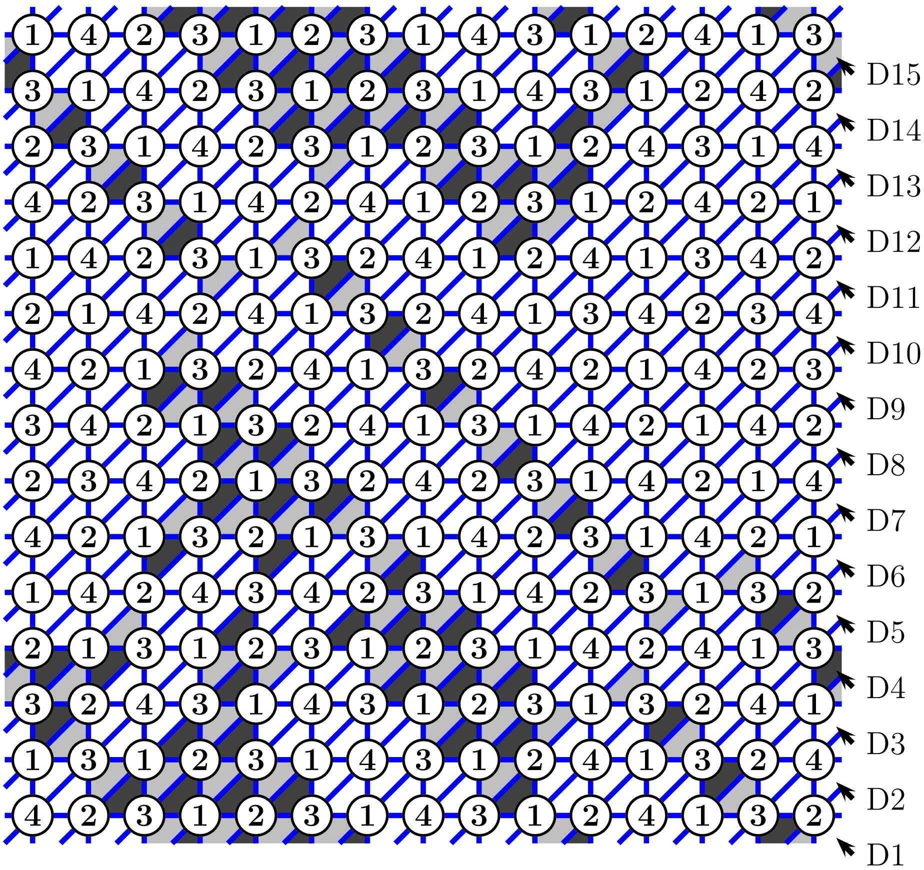

Figure 11: The 4-coloring of $T(15, 15)$ after Step 4 in the case $L = 4k + 1$.

On $\mathrm{D}(6k+1)$ there are two pairs of nearby vertices which only admit one color among 3 and 4. One pair is $(3k+1, 3k)$ and $(3k, 3k+1)$; the other one is $(9k+3, 9k+1)$ and $(9k+2, 9k+2)$. We color the other vertices on $\mathrm{D}(6k+1)$ by colors 3 and 4 while using the following rule: those with x-coordinate satisfying $3k+1 < x < 9k+2$ are colored 3 (resp. 4) if k is odd (resp. even). At the end, there are $6k+2$ and $6k+1$ vertices colored alike on $\mathrm{D}(6k+1)$.

On $\mathrm{D}(6k+4)$ we also find two pairs of vertices which only admit one color among 3 and 4: one pair is $(3k+3, 3k+1)$ and $(3k+2, 3k+2)$; the other one is $(9k+4, 9k+3)$ and $(9k+3, 9k+4)$. The other vertices on $\mathrm{D}(6k+4)$ are then colored 3 and 4 with the help of the following rules: 1) those with x-coordinate satisfying $3k+3 < x < 9k+3$ are colored 3 (resp. 4) if k is odd (resp. even); 2) the number of vertices colored 3 is the same as on $\mathrm{D}(6k+1)$. This second rule is used to determine the color of the vertex at $(3k+1, 3k+3)$.

The contribution to the partial degree of these new triangles is -4; thus, the partial degree of f is $\deg f|_R = 4 + 12(k-1)$.

Step 4. On D$(6k+2)$ there are two vertices located at $(3k, 3k+2)$ and $(9k+2, 9k+3)$ whose colors are fixed to either 1 or 2. Color with the same color as $(9k+2, 9k+3)$ the two vertices $(3k+1, 3k+1)$ and $(9k+3, 9k+2)$. At the end, there are $6k+4$ vertices having one color, and $6k+1$ having the other one.

On D$(6k+3)$ there are two vertices whose colors are fixed to either 3 or 4. There are also four additional vertices whose colors are fixed to either 1 or 2. These six vertices are located at $(3k+2, 3k+1)$, $(3k+1, 3k+2)$, $(3k, 3k+3)$, $(9k+4, 9k+2)$, $(9k+3, 9k+3)$, and $(9k+2, 9k+4)$. The other vertices on D$(6k+3)$ are colored 3 or 4 (the choice for each vertex is unique).

In Figure 11 the final coloring f is depicted. The increment in the partial degree is 2. Therefore,

$$\deg f \;=\; 6 + 12(k-1) \;\equiv\; 6 \pmod{12}.$$

The coloring f of $T(12k+3, 12k+3)$ is proper and its degree is congruent to 6 modulo 12, as claimed. This completes the proof. ∎

5 Further results for $T(3L, 3M)$

In the previous section we have proven that $T(3L, 3L)$ has at least one coloring with degree $\equiv 6 \pmod{12}$ for any $L \geq 2$, and hence $\kappa(T(3L, 3L), 4) > 1$. This result can be used for some other triangulations with aspect ratio different from 1:

Theorem 5.1 *The number of Kempe equivalence classes $\kappa(T, 4)$ is at least two for any triangulation $T(3Lp, 3Lq)$ for $L \geq 2$ and any odd integers p, q.*

Proof. Theorem 4.5 shows that there is a coloring f of $T(3L, 3L)$ for $L \geq 2$ with $\deg(f) \equiv 6 \pmod{12}$. Then, Lemma 4.4(c) proves the claimed result. ∎

In order to obtain more general results, it is convenient to prove the following simple proposition.

Proposition 5.2 *The degree of any four-coloring of any triangulation $T(L, 3)$ or $T(3, L)$ with $L \geq 1$ is zero.*

Proof. Suppose we compute the degree of a given 4-coloring c of the triangulation $T(3, L)$ by counting those triangular faces colored 123. We can focus on those sites colored 3. Let us suppose the vertex x is colored 3. Because the 4-coloring c is proper, none of the neighbors of x can be colored 3. And because the triangulation has width 3, the two neighbors along the horizontal axis are also adjacent to each other, so they have different colors, say 1 and 2. This situation is depicted in Figure 12. There are only 9 different four-colorings of the above graph, and all of them contribute zero to the degree. Therefore, the contribution of all vertices colored 3 to the degree is zero, and the claimed result is proven. ∎

The following lemma shows how to build a four-coloring of the triangulation $T(L, M+3)$ by "gluing" four-colorings of the triangulations $T(L, M)$ and $T(L, 3)$ that have the same coloring on the top row. One key point is that the degree is an invariant under this operation.

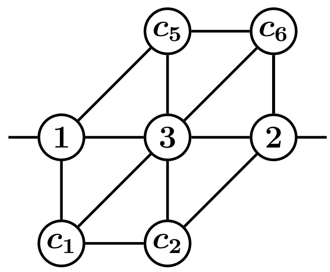

Figure 12: Subset of the triangulation $T(3, L)$ used in the proof of Proposition 5.2.

Lemma 5.3 *Let us suppose that c is a four-coloring of a triangulation $T(L, M)$ with degree d, and that the coloring on the top row is c_{top}. Let us further suppose there exists a four-coloring c' of the triangulation $T(L, 3)$ with the same coloring on the top row $c'_{\text{top}} = c_{\text{top}}$. Then, there exists a four-coloring of the triangulation $T(L, M + 3)$ with degree d.*

Proof. Because both $T(L, M)$ and $T(L, 3)$ are triangulations of a torus with the same width L, and the corresponding colorings c and c' both have the same top-row coloring c_{top}, we can obtain a four-coloring c'' of the triangulation $T(L, M + 3)$ by "gluing" together these two colorings. This is indeed a proper coloring of $T(L, M + 3)$ and its degree can be computed as $\deg(c'') = \deg(c) + \deg(c') = \deg(c) = d$, since $\deg(c') = 0$ by Proposition 5.2. ■

This lemma gives us the opportunity to devise an inductive proof that there is a four-coloring with degree 6 (mod 12) for any triangulation $T(3L, 3M)$ with $M \geq L$. The base case $L = M$ is already verified by Theorem 4.5. If we can find a proper four-coloring of the triangulation $T(3L, 3)$ with a top-row coloring equal to the top-row coloring of the coloring obtained in the proof of Theorem 4.5, then the above lemma can be used to prove the inductive step. The main issue is therefore, to prove the existence of such coloring for $T(3L, 3)$.

Theorem 5.4 *For any triangulation $T(3L, 3M)$ with any $L \geq 3$ and $M \geq L$, there exists a four-coloring f with* $\deg(f) \equiv 6 \pmod{12}$. *Consequently, the WSK dynamics for four-colorings of $T(3L, 3M)$ is non-ergodic.*

Proof. The proof is by induction on M. The base case $M = L \geq 3$ is proven by Theorem 4.5. Now suppose that there exist such colorings for all triangulations $T(3L, 3M')$ with $L \leq M' \leq M$, and we wish to prove that such configuration exists also for M. The main idea is to prove the existence of a proper four-coloring of the triangulation $T(3L, 3)$ such that its top row coloring coincides with the one obtained in the proof of the corresponding case in Theorem 4.5.

To simplify the notation we will denote by c_i the sequence of colors in the row i of $T(3L, 3)$ and by c_0 the coloring of the top row of $T(3L, 3L)$ obtained in the proof of Theorem 4.5. Of course, our goal is to have $c_0 = c_3$.

To describe a sequence of colors, we will use the following notation: $[a_1 a_2 \cdots a_s]^t$ will be the sequence of length st in which $a_1 a_2 \cdots a_s$ is repeated t times. For example, $12[34]^3 2 = 123434342$.

Our basic strategy is, as in Theorem 4.5, to explicitly construct four-colorings of $T(3L,3)$ with $L \geq 3$. The construction of such a coloring will depend on the value of L modulo 4, and we will split the proof in four cases, $L = 4k-2, 4k-1, 4k$, or $L = 4k+1$, with $k \in \mathbb{N}$.

The case $L = 4k-2$ was the easiest one in the proof of Theorem 4.5; however, in this case it is the most elaborate. Thus, we will start the proof by considering the easiest cases, and delay the most complex one to the end.

Case 1: $L = 4k-1$.

Let $t = \lfloor \frac{3k-2}{2} \rfloor$. The top-row coloring obtained from the proof of Case 2 in Theorem 4.5 can be written as

$$c_0 = c_3 = [1423]^t 1231 [3241]^t 3$$

when k is even. Then we define c_1 and c_2 as:

$$\begin{aligned} c_2 &= 3[1423]^t 142 [1324]^t 2 \\ c_1 &= 23[1423]^t 14 [2413]^t 4. \end{aligned}$$

If k is odd, then we have:

$$\begin{aligned} c_0 = c_3 &= [1423]^t 14214241 [3241]^t 3 \\ c_2 &= 3[1423]^t 1423124 [1324]^t 2 = 3[1423]^{t+1} 124 [1324]^t 2 \\ c_1 &= 23[1423]^t 14231 [3241]^t 34 = 23[1423]^{t+1} 1 [3241]^t 34. \end{aligned}$$

It is easy to verify that this gives a proper 4-coloring of $T(3L,3)$. By Proposition 5.2, it has zero degree. This completes the proof of this case.

Case 2: $L = 4k$.

As for the previous case, let $t = \lfloor \frac{3k-2}{2} \rfloor$. The top-row coloring $c_3 = c_0$ is obtained from the proof of Case 3 in Theorem 4.5. When k is even, the sought 4-coloring is defined as follows:

$$\begin{aligned} c_0 = c_3 &= [1423]^t 1431341 [3241]^t 3 \\ c_2 &= 3[1423]^t 124132 [4132]^t 4 = 3[1423]^t 12 [4132]^{t+1} 4 \\ c_1 &= 4[2314]^t 312413 [2413]^t 2 = 4[2314]^t 31 [2413]^{t+1} 2\,. \end{aligned}$$

If k is odd, then we have:

$$\begin{aligned} c_0 = c_3 &= [1423]^t 14234231241 [3241]^t 3 = [1423]^{t+1} 4231241 [3241]^t 3 \\ c_2 &= 3[1423]^t 1423423132 [4132]^t 4 = 3[1423]^{t+1} 423132 [4132]^t 4 \\ c_1 &= 4[2314]^t 2342312413 [2413]^t 2 = 4[2314]^t 234231 [2413]^{t+1} 2\,. \end{aligned}$$

Again, it is easy to verify that this gives a proper 4-coloring of $T(3L,3)$, and by Proposition 5.2, it has zero degree. This completes the proof of this case.

CASE 3: $L = 4k + 1$.

Let $t = \lfloor \frac{3k-2}{2} \rfloor$. The top-row coloring $c_3 = c_0$ is obtained from the proof of Case 4 in Theorem 4.5. When k is even, the sought 4-coloring is defined as follows:

$$\begin{aligned} c_0 = c_3 &= [1423]^t 1421423421 [3241]^t 3 \\ c_2 &= 3[1423]^t 14214213 [2413]^t 42 \\ c_1 &= 2[3142]^t 314214213 [2413]^t 4 \ = \ 2[3142]^{t+1} 14213 [2413]^t 4\,. \end{aligned}$$

If k is odd, then we have:

$$\begin{aligned} c_0 = c_3 &= [1423]^{t+1} 1231431241 [3241]^t 3 \\ c_2 &= [1423]^{t+1} 312312413 [2413]^t 42 \ = \ [1423]^{t+1} 31231 [2413]^{t+1} 42 \\ c_1 &= [2314]^{t+1} 2312312413 [2413]^t 2 \ = \ [2314]^{t+1} 2312312413 [2413]^{t+1} 2\,. \end{aligned}$$

Again, it is easy to verify that this gives a proper 4-coloring of $T(3L, 3)$, and by Proposition 5.2, it has zero degree. This completes the proof of this case.

CASE 4: $L = 4k - 2$.

We cannot use the results of the proof of Theorem 4.5, as the resulting four-coloring for $T(3L, 3L)$ is characterized by the fact that any row (horizontal, vertical or inclined) is bi-colored. Thus, we cannot obtain a four-coloring of $T(12k-6, 3)$ with a bi-colored horizontal row.

We first need to obtain a proper four-coloring f of $T(12k-6, 12k-6)$ with $\deg(f) \equiv 6 \pmod{12}$, and such as there is a proper four-coloring of $T(12k-6, 3)$ compatible with the coloring of one of the horizontal rows of f. We obtain such coloring f by a constructive proof similar to those explained in the proof of Theorem 4.5. The notation we use is the same as in Theorem 4.5.

Let us consider the triangulation $T = T(12k-6, 12k-6)$ with integer $k \geq 2$ (the case $k = 2$ will illustrate our ideas). Our goal is to obtain a four-coloring f of T with degree $\deg(f) \equiv 6 \pmod{12}$. The algorithm to obtain such a coloring consists of four steps:

Step 1. We start by coloring counter-diagonal D1: we color 1 the vertices with x-coordinates $1 \leq x \leq 6k-3$; the other $6k-3$ vertices are colored 2.

On D2, we color 3 those $6k-3$ vertices with x-coordinates $3k \leq x \leq 9k-4$. The other vertices on D2 are colored 4. The vertices on $\mathrm{D}(12k-6)$ are colored 3 or 4 in such a way that the resulting coloring is proper (for each vertex, there is a unique choice).

On D3 and $\mathrm{D}(12k-7)$, we color all vertices 1 or 2; on D4 and $\mathrm{D}(12k-8)$, we color all vertices 3 and 4, and finally, on D5 and $\mathrm{D}(12k-9)$, we color all vertices 1 and 2. In every case, there is a unique color choice for each vertex. The resulting coloring is depicted on Figure 13. The partial degree of f is $\deg f|_R = 8$.

Step 2. For $k > 2$, we find that there are $12k - 15$ counter-diagonals to be colored and we need to sequentially color all of them but nine. (Notice that this is why this algorithm does not work for $k = 1$.) This can be achieved by performing the following procedure: suppose that we have already colored counter-diagonals $\mathrm{D}j$ and $\mathrm{D}(12k-j-4)$ ($j \geq 5$) using colors 1 and 2. Then, we color $\mathrm{D}(j+1)$ and $\mathrm{D}(12k-j-5)$ using colors 3 and 4, and $\mathrm{D}(j+2)$ and $\mathrm{D}(12k-j-6)$ using colors 1 and 2. As in Step 1, for each vertex there is a unique choice.

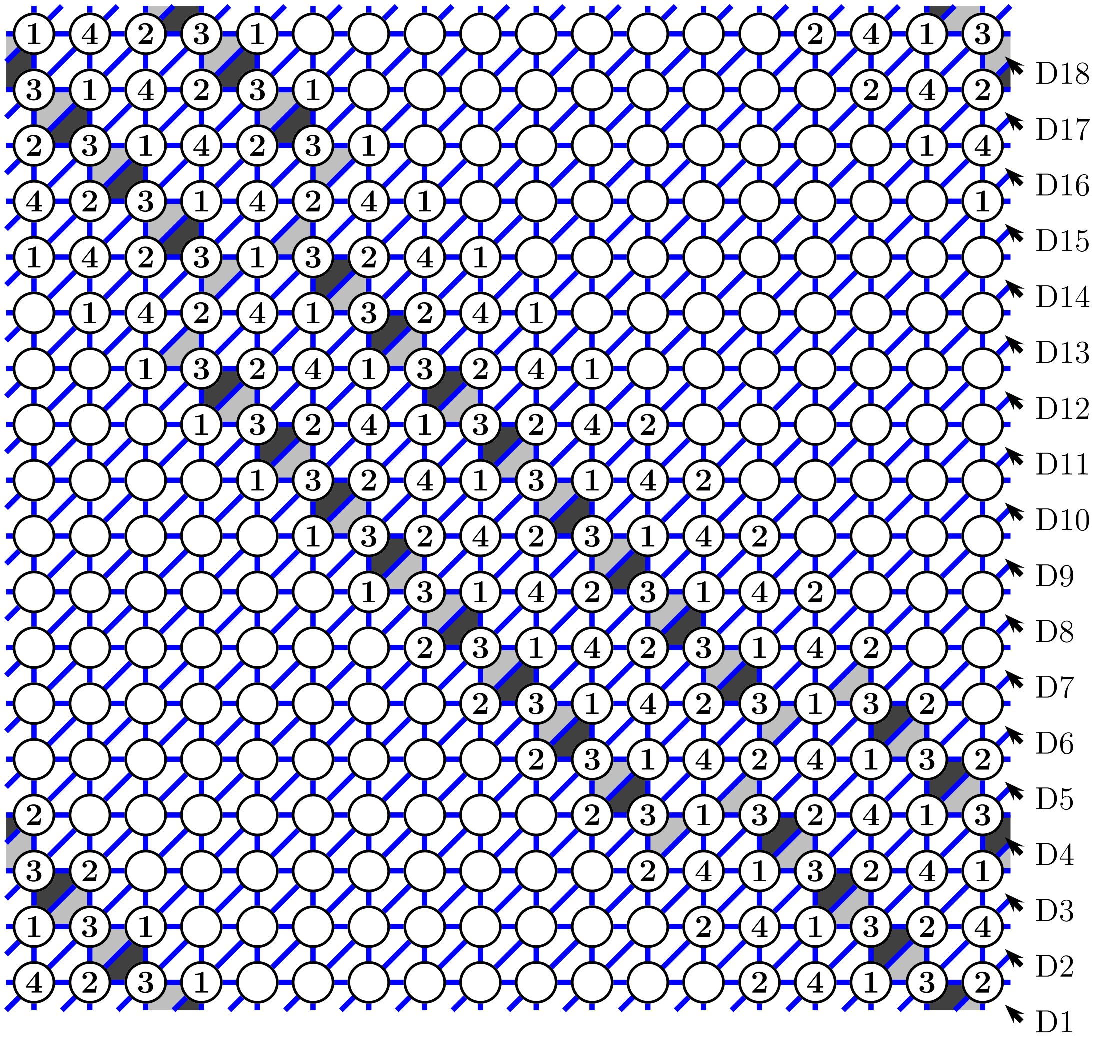

Figure 13: The 4-coloring of $T(18, 18)$ after Step 1 in the case $L = 4k - 2$.

This procedure is repeated $3(k-2)$ times, so we add $12(k-2)$ counter-diagonals, and there are only nine counter-diagonals not yet colored. Indeed, the last colored counter-diagonals $\mathrm{D}(6k-7)$ and $\mathrm{D}(6k+3)$ have colors 1 and 2, the same as it was at the end of Step 1.

Each of these $3(k-2)$ steps adds 4 to the degree of the coloring. Thus, the partial degree of f is $\deg f|_R = 8 + 12(k-2)$.

Step 3. On $\mathrm{D}(6k-6)$, the vertices $(3k-3, 3k-3)$ and $(9k-6, 9k-6)$ only admit a single color (which is 3 for one of them, and 4 for the other one). The rest of the vertices on $\mathrm{D}(6k-6)$ are colored 1 and 2 (again, there is a unique choice for each vertex).

On $\mathrm{D}(6k+2)$, there are two vertices $(3k+1, 3k+1)$ and $(9k-2, 9k-2)$ admitting a single color (again 3 or 4). The other vertices on $\mathrm{D}(6k+2)$ are colored 1 or 2 (again, the choice for each vertex is unique).

On $D(6k-5)$ there are four vertices which admit a single color $\in \{3, 4\}$: vertices $(3k-2, 3k-3)$ and $(3k-3, 3k-2)$ should be colored c_1, while $(9k-5, 9k-6)$ and $(9k-6, 9k-5)$ should be colored $c_2 \neq c_1$. The other vertices satisfying $3k-1 \leq x \leq 9k-4$ are colored c_2, and the rest of the vertices are colored c_1.

Finally, on $\mathrm{D}(6k+1)$, we also find another four vertices admitting a single color chosen from the set $\{3, 4\}$: vertices $(3k+1, 3k)$ and $(3k, 3k+1)$ should be colored c_1, while $(9k-2, 9k-3)$ and $(9k-3, 9k-2)$ should be colored $c_2 \neq c_1$. The other vertices satisfying $3k+2 \leq x \leq 9k-4$ are colored c_2, and the rest of the vertices are colored c_1.

The contribution to the partial degree of the new triangles is -4; the partial degree of f is given by $\deg f|_R = 4 + 12(k-2)$.

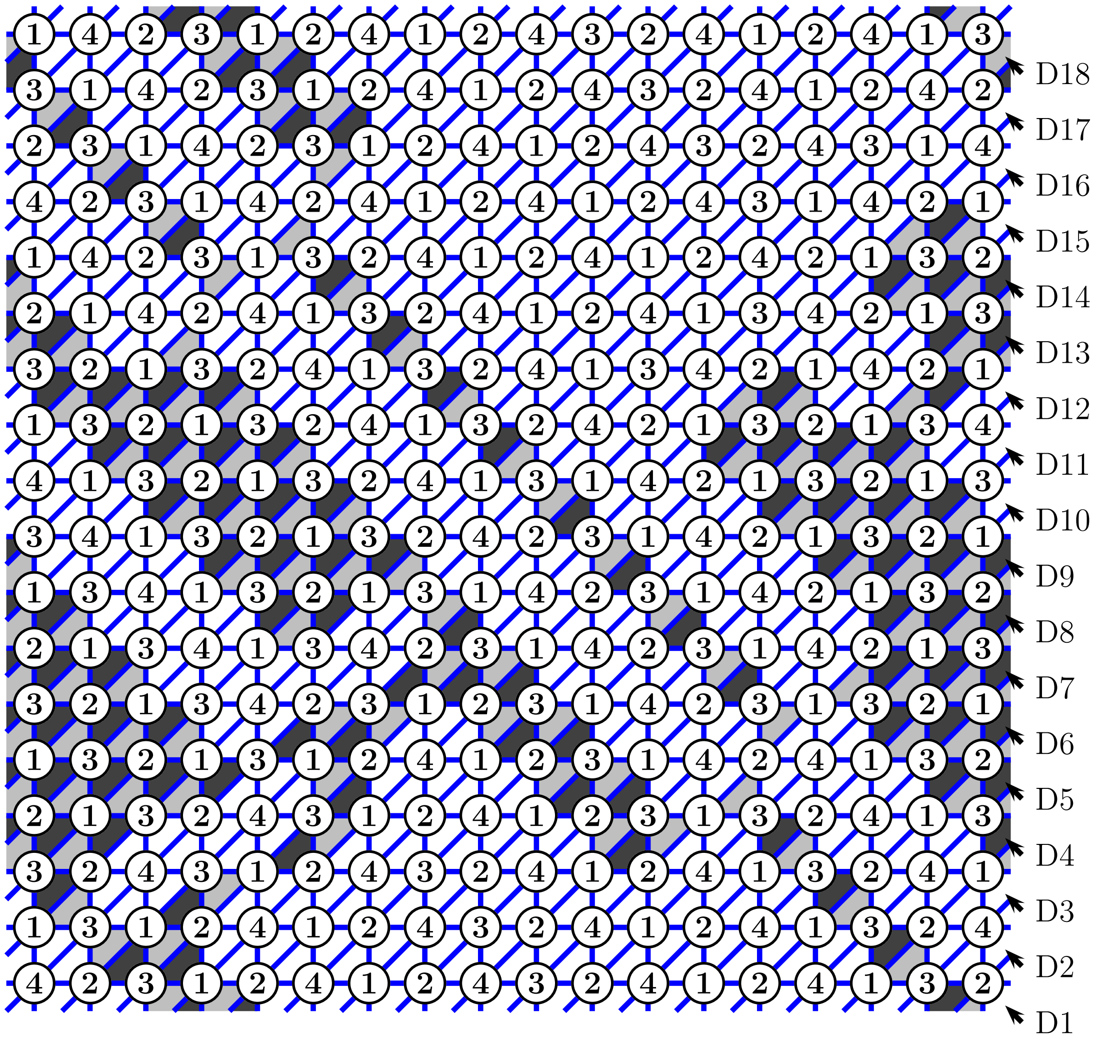

Figure 14: The 4-coloring of $T(18, 18)$ after Step 4 in the case $L = 4k - 2$.

Step 4. There are only five counter-diagonals to be colored. All vertices on $\mathrm{D}(6k-4)$ are colored 1 or 2 using the following simple rule: the vertex (x, y) is colored 1 (resp. 2) if the vertex $(x, y-1)$ is colored 4 (resp. 3). In particular, those vertices with $3k-1 \leq x \leq 9k-5$ are colored alike.

On $\mathrm{D}(6k)$ we find two vertices admitting a single color in the set $\{1, 2\}$: $(3k-1, 3k-1)$ and $(9k-2, 9k-4)$ taking respectively, colors c_1 and c_2. The vertices satisfying $3k \leq x \leq 9k-4$

are colored c_1, and the others are colored c_2.

On D$(6k-3)$ we find two vertices $(3k-1, 3k-2)$ and $(9k-4, 9k-5)$ that admit a single color from the set $\{3, 4\}$. The other vertices are colored 1 and 2 (there is a unique choice for each vertex).

On D$(6k-2)$ there are four vertices admitting a single color from the set $\{3, 4\}$: the vertices $(3k, 3k-2)$ and $(3k-1, 3k-1)$ are colored c_1, while $(9k-3, 9k-5)$ and $(9k-4, 9k-4)$ are colored $c_2 \neq c_1$. Those vertices satisfying $3k+1 \leq x \leq 9k-2$ are colored c_2, and the rest are colored c_1.

The last counter-diagonal D$(6k-1)$ contains seven vertices that admit a single color: $(3k+1, 3k-2)$, $(3k, 3k-1)$, $(3k-1, 3k)$, $(9k-1, 9k-6)$, $(9k-2, 9k-5)$, $(9k-3, 9k-4)$, and $(9k-4, 9k-3)$. The other vertices are colored 3 and 4 (there is a unique choice for each vertex).

The resulting coloring is depicted in Figure 14. The contribution to the partial degree of the new triangles is 2; the partial degree of f is given by $\deg f|_R = 6 + 12(k-2) \equiv 6 \pmod{12}$.

The above argument proves the base case of our induction. Now we have to find a four-coloring of the triangulation $T(12k-6, 3)$ with $k \geq 2$ such that it has the same top-row coloring c_3 as f (see Figure 14). We proceed as for the previous cases: let $t = \lfloor \frac{3k-6}{2} \rfloor$; the 4-coloring we need is defined as follows for k even:

$$
\begin{aligned}
c_0 = c_3 &= [1423]^{t+1}1241243241241[3241]^t 3 \\
c_2 &= 3[1423]^{t+1}12412432413[2413]^t 42 \;=\; 3[1423]^{t+1}1241243[2413]^{t+1}42 \\
c_1 &= [2314]^{t+1}2312412432413[2413]^t 4 \;=\; [2314]^{t+1}231241243[2413]^{t+1}4\,.
\end{aligned}
$$

If k is odd, then we have:

$$
\begin{aligned}
c_0 = c_3 &= [1423]^{t+1}14213213413213[2413]^{t+1} \\
c_2 &= [3142]^{t+1}314213213413[2413]^{t+1}42 \;=\; [3142]^{t+2}13213413[2413]^{t+1}42 \\
c_1 &= [2314]^{t+1}2314213213413[2413]^{t+1}4 \;=\; [2314]^{t+2}213213413[2413]^{t+1}4\,.
\end{aligned}
$$

Again, it is easy to verify that this gives a proper 4-coloring of $T(3L, 3)$, and by Proposition 5.2, it has zero degree. This completes the proof of the theorem. ∎

Theorems 4.5 and 5.4 imply that WSK is non ergodic on any triangulation $T(3L, 3M)$ with $3 \leq L \leq M$. Proposition 5.2 together with Fisk's theorem implies that WSK is ergodic on any triangulation $T(3, 3L)$. The triangulations $T(6, 3L)$ are special in the sense that WSK is ergodic depending on the value of L. In particular, WSK is not ergodic for any $T(6, 6p)$ with odd p, because of Theorem 4.5 [or Theorem 4.3] and Lemma 4.4.

By direct computer enumeration of the 299146792 proper four-colorings of $T(6, 9)$, we have checked that all of them have zero degree. We have also checked with a computer that we can transform any of these colorings into the three-coloring by a *finite* number of K–changes. Therefore we have obtained a computer–assisted proof of the following Theorem:

Proposition 5.5 $\kappa(T(6, 9), 4) = 1$

Remark. Fisk's Theorem 2.8 can be used to prove the ergodicity of the WSK on $T(6, 9)$ directly from the fact that all colorings have zero degree.

6 Summary and open problems

We have considered the question of the ergodicity of the Wang–Swendsen–Kotecký dynamics for the zero-temperature 4–state Potts antiferromagnet on triangulations $T(3L, 3M)$ of the torus. This dynamics is equivalent (for the zero-temperature case only) to that of the Kempe chains studied in Combinatorics. We have obtained two main results:

1) For the wider family of the even triangulations of the torus (which contains the triangulations $T(3L, 3M)$ as a proper subset), we find that the degree of a 4–coloring modulo 12 is invariant under Kempe changes.

2) For any triangulation $T(3L, 3M)$ of the torus with $3 \leq L \leq M$, there are at least two Kempe equivalence classes for 4 colors. In other words, the Wang–Swendsen–Kotecký dynamics with 4 colors on these triangulations is non-ergodic. For $L = 2$, we can only show that this dynamics is non-ergodic for $M = 2p$ with odd p.

In addition to their intrinsic mathematical interest, these results have a great practical importance in Statistical Mechanics. The triangular-lattice 4–state Potts antiferromagnet is believed to have a zero temperature critical point [10, and references therein]. But we *cannot* study the critical properties of this model using WSK dynamics because of the non-ergodicity of the algorithm. (This also holds for the single-site Metropolis dynamics, as it corresponds to a particular subset of moves of the WSK dynamics.) Indeed, one can simulate the 4–state Potts antiferromagnet at zero temperature using the WSK algorithm on planar graphs (e.g., a triangular grid with free boundary conditions); but surface effects cannot be eliminated, and one has to go to much larger lattice sizes to attain high–precision results. It is therefore important to devise a new Monte Carlo algorithm for this model which is ergodic at zero temperature.

There are other open problems related to the ergodicity of the Kempe dynamics. The case of four-colors on triangulations of the torus is rather special, as we can make use of concepts borrowed from Algebraic Topology. However these techniques cannot be applied to the cases of $q = 5, 6$ colors, and the ergodicity of the corresponding WSK dynamics is still an open problem.

Finally, let us mention that *at zero temperature*, the 4–state Potts model on the triangular lattice is essentially equivalent to the 3–state Potts model on the kagomé lattice. We have found that the WSK dynamics for this model also fails to be ergodic on most kagomé lattices when embedded on a torus. The details will be published elsewhere.

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